ON AN ERDŐS PROBLEM

JAYANT SHARMA

Abstract. The Erdős-Straus conjecture was proposed by Paul Erdős and Ernő Straus in 1948. The conjecture asks wether, for every $n \geq 2, \in \mathbb{N}$, we can express,

$$\frac{4}{n} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}$$

Such that, x, y and z are natural numbers. In this paper, we aim to investigate the conjecture, but generalize it to any k number of fractions. That is, *for any natural number k, can we express $\frac{2(k-1)}{n}$ as the sum of reciprocals of k natural numbers.* For $k = 1$ the solution is when $x_1 = x_2 = \frac{n}{2}$, and for $k = 2$, the conjecture reduces to the famous *Erdős-Straus Conjecture*.

In this paper, by the means of *Prime Factor Anlyasis* we show that the conjecture is true, for all $k \in \mathbb{N}$.

Contents

1. Preliminaries

We begin by formally stating the generalized form of the Erdős–Straus-type conjecture we propose:

Conjecture I (Generalized Erdős–Straus Form)**.** *For any integer $k \in \mathbb{N}$, the rational expression $\frac{2k}{n}$ can be represented as a sum of $(k+1)$ unit fractions. That is, for every pair $(k, n) \in \mathbb{N}^2$ with $n \geq 2$, there exists a set of natural numbers $\{x_0, x_1, \ldots, x_k\} \subset \mathbb{N}$ such that:*

$$\frac{2k}{n} = \frac{1}{x_0} + \frac{1}{x_1} + \cdots + \frac{1}{x_k} = \sum_{i=0}^{k} \frac{1}{x_i}.$$

We now justify the necessity of the condition $n \geq 2$ by establishing the following lemma:

Lemma I. *For every $k \in \mathbb{N}$, the equation in Conjecture 1 admits a solution when $n = 2$.*

Proof. Substituting $n = 2$ into the conjectured identity, we aim to express:

$$\frac{2k}{2} = k$$

as a sum of $k + 1$ unit fractions of natural numbers.

We construct the following multiset S of size $k + 1$:

$$S = \underbrace{\{1, 1, \ldots, 1\}}_{k-1 \text{ times}} \cup \{2, 2\}.$$

Clearly, the sum of the reciprocals of the elements in S is:

$$\left(\sum_{i=1}^{k-1} \frac{1}{1}\right) + \frac{1}{2} + \frac{1}{2} = (k-1) + \frac{1}{2} + \frac{1}{2} = k.$$

Hence, the equality

$$k = \sum_{i=0}^{k} \frac{1}{x_i}$$

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holds true for $x_i \in S$, satisfying the conjecture for $n = 2$. This completes the proof. *(end of proof)*

One may naturally ask that, *if by Lemma 1, the conjecture holds for $n = 2$, then does it also holds for any $n \geq 2, \in \mathbb{N}$?* This is specifically what builds the *Conjecture 1*.