ON THE ENTROPY THEORY

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Abstract. This paper discusses the relevance of entropy theory and it's applications.

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1. Motivation

To introduce the underlying theory, let us begin with the set of prime numbers, denoted by $\mathbb{P}$. This set can be systematically constructed using the *Inclusion-Exclusion Principle*, most notably realized through the *Sieve of Eratosthenes*. Via such combinatorial means, $\mathbb{P}$ appears deterministic and fully describable.

However, when we shift focus to the realm of *arithmetic operations*—addition, multiplication, exponentiation, and their respective inverses such as subtraction, division, and logarithms—we encounter a striking limitation. It is intuitively evident that:

> *The set $\mathbb{P}$ cannot be fully described or generated using only arithmetic operations.*

This leads to a profound observation:

> *The amount of information required to describe a system is not absolute—it depends on the set of operations we are allowed to use.*

A system may appear incompressibly complex (requiring infinite information) within a restricted operational framework. Yet, by expanding or altering the permissible operations, the same system might become constructible using finite information.

Thus, we may posit:

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> **The "degree of disorder" (or entropy) in a system's evolution is relative to the operational framework used to interpret it.** That is, the less sufficient the set of allowed operations is to encode or generate a system, the higher its apparent entropy.

By examining the entropy of mathematical systems relative to their operational constraints, we may uncover deep, intrinsic patterns hidden within arithmetic itself. This motivates the development of a formal *Entropy Theory of Arithmetic*, which aims to quantify how "constructible" or "disordered" a structure is, with respect to the set of operations available.

The illustrations presented above are not formal axioms or foundational results of the theory; rather, they serve as preliminary intuitions—an initial spark—that inspired the development of the entropy-theoretic perspective.

2. Preliminaries

In this section, we establish foundational axiomatic structures necessary for the development of our theory.

2.1. Mathematical Evolution.

We define a *Mathematical Evolution* as the process by which an object evolves into something distinct from its original form. This evolution may occur through various mechanisms, such as:

- the iterative development of operations (e.g., the evolution of addition into multiplication, and multiplication into exponentiation),
- the introduction of inverse operations (e.g., multiplication evolving into division),
- the transformation of functions (e.g., a domain evolving into its range), or
- the aggregation of individual elements into a collective structure (e.g., a set).

Formally, given an object a that evolves into an object b under the influence of a governing mechanism or principle c, we refer to this process as a *Mathematical Evolution*, and denote it by:

$$a \xrightarrow{c} b$$

We call c the *Transform* of the evolution—it encapsulates the rules, information, or structure under which the evolution occurs.

2.2. Mathematical Blocks.

Given an evolution of the form:

$$a \xrightarrow{c} b$$

we refer to a as the *Initial Block* and b as the *Final Block* of the evolution. The intermediate structure c remains the *Transform*, as previously defined.

Mathematical Blocks may consist of individual values, sets of values, or any well-defined mathematical object capable of undergoing a Mathematical Evolution. The Transform may be an operation (e.g., addition, subtraction, exponentiation, logarithm), a function, or any abstract rule facilitating the transition from one block to another.

For generality and clarity, we adopt the notation:

$$a \xrightarrow{c} b$$

to represent the evolution from an Initial Block a to a Final Block b via Transform c.

2.3. Mathematical Framework.

We define a *Mathematical Framework*, denoted by $\mathcal{F}$, as a constraint imposed on the admissibility of *Transforms* within a given mathematical context.

Formally, every framework $\mathcal{F}$ is associated with a set of permitted transforms, denoted by $\mathcal{T}_{\mathcal{F}}$. That is, any Mathematical Evolution e is said to be valid within the framework $\mathcal{F}$ if and only if the transform $\mathcal{T}$ governing e satisfies:

$$\mathcal{T} \in \mathcal{T}_{\mathcal{F}}.$$

In such a case, we write $e \in \mathcal{F}$ to indicate that the evolution e is permitted under the constraints of the framework $\mathcal{F}$.

2.4. **Entropy.** Let an evolution e be defined by

$$e : a \xrightarrow{c} b, \quad \text{with } e \in \mathcal{F}.$$

Since the transform c is permitted under the framework $\mathcal{F}$—i.e., $c \in \mathcal{T}_{\mathcal{F}}$—the evolution e is realizable within $\mathcal{F}$. We call such an evolution a *feasible evolution* with respect to the framework $\mathcal{F}$.

Now consider a different framework $\mathcal{F}'$ such that $c \notin \mathcal{T}_{\mathcal{F}'}$, or more generally, any fundamental component required to constitute c is not permitted by $\mathcal{F}'$. Then, within this new framework, the evolution e cannot be realized, as no admissible transform exists to connect a to b. In this case, we say that e is a *non-feasible evolution* with respect to $\mathcal{F}'$.

This feasibility—or lack thereof—naturally leads us to define a notion of *disorder* associated with an evolution in a given framework. If an evolution e is feasible within a framework $\mathcal{F}$, we say that the disorder of e in $\mathcal{F}$ is zero. On the other hand, if e is non-feasible within a framework $\mathcal{F}'$, we say that the disorder of e in $\mathcal{F}'$ is infinite.

We now define the concept of *entropy*:

Definition I. *Let e be an evolution and $\mathcal{F}$ a framework. The* entropy *of e with respect to $\mathcal{F}$, denoted* $\mathrm{Ent}_{\mathcal{F}}(e)$*, is defined as the amount of disorder in the realization of e under $\mathcal{F}$. That is,*

$$\mathrm{Ent}_{\mathcal{F}}(e) = \begin{cases} 0, & \textit{if } e \textit{ is feasible under } \mathcal{F}, \\ \infty, & \textit{if } e \textit{ is non-feasible under } \mathcal{F}. \end{cases}$$

Intuitively, the quantity $\mathrm{Ent}_{\mathcal{F}}(e)$ measures the extent to which an evolution e appears disordered or chaotic within a given framework $\mathcal{F}$. It captures the perceived irregularity or incompatibility between the evolution and the structural constraints imposed by $\mathcal{F}$.

For instance, consider the sequence of prime numbers. When viewed under the arithmetic framework (i.e., one based on standard operations such as addition, multiplication, and so on), the mapping from natural numbers to their corresponding primes appears highly erratic and non-linear. This is because the framework required to systematically generate prime numbers—mapping an index n to its n-th prime p_n—is fundamentally misaligned with the framework of ordinary arithmetic operations. As a result, the entropy $\mathrm{Ent}_{\mathcal{A}}(e)$, where $\mathcal{A}$ denotes the arithmetic framework, is infinite for such an evolution e.

This highlights how entropy is not an intrinsic property of the evolution alone, but a relative measure dependent on the choice of framework.

3. Relative Entropy of Frameworks

3.1. **Comparable Frameworks.** Two frameworks are said to be *comparable* if their associated transform sets satisfy either of the following relations:

$$\mathcal{T}_{\mathcal{F}} \subseteq \mathcal{T}_{\mathcal{G}} \quad \text{or} \quad \mathcal{T}_{\mathcal{F}} \supseteq \mathcal{T}_{\mathcal{G}}$$

In such cases, one of the frameworks permits a strictly larger or equal set of transforms. Suppose without loss of generality that

$$\mathcal{T}_{\mathcal{F}} \supseteq \mathcal{T}_{\mathcal{G}}$$

That is, every transform allowed in $\mathcal{G}$ is also allowed in $\mathcal{F}$, but $\mathcal{F}$ may allow additional transforms not permitted in $\mathcal{G}$.

Now consider an evolution e. For every transform $\mathcal{T} \in \mathcal{T}_{\mathcal{G}}$, the entropy of e in both frameworks satisfies:

$$\mathrm{Ent}_{\mathcal{F}}(e) = 0 \quad \text{and} \quad \mathrm{Ent}_{\mathcal{G}}(e) = 0$$

However, for any transform $\mathcal{T} \in \mathcal{T}_{\mathcal{F}} \setminus \mathcal{T}_{\mathcal{G}}$, we still have:

$$\mathrm{Ent}_{\mathcal{F}}(e) = 0 \quad \text{but} \quad \mathrm{Ent}_{\mathcal{G}}(e) \geq 0 \quad \text{(possibly infinite)}$$

Thus, under the stricter framework $\mathcal{G}$, certain evolutions that are feasible in $\mathcal{F}$ may appear more disordered or even infeasible.

To measure this, we define the **relative entropy** of one framework with respect to another as:

$$\mathrm{Ent}_{\mathcal{G}}(\mathcal{F}) = \infty \quad \text{and} \quad \mathrm{Ent}_{\mathcal{F}}(\mathcal{G}) = 0$$

This quantifies how disordered framework $\mathcal{F}$ appears when viewed through the framework $\mathcal{G}$, and vice versa.

3.2. **Distinct Frameworks.** If none of the above conditions are true between the sets $\mathcal{T}_{\mathcal{F}}$ and $\mathcal{T}_{\mathcal{G}}$ then they are said to be *Distinct Frameworks.*

4. Comparing Entropies

Consider two evolutions,

$$j : a_1 \xrightarrow{c_1} b_1, \quad c_1 \in \mathcal{F}$$
$$k : a_2 \xrightarrow{c_2} b_2, \quad c_2 \in \mathcal{F}$$

We define the set

$$\mathcal{T}_{\mathcal{F}} = \{\dots\}$$

as the *fundamental transforms* allowed by $\mathcal{F}$. These fundamental transforms might themselves give rise to more complex transformations; however, under the specific framework $\mathcal{F}$, we treat them as irreducible.

Now, the entropy of an evolution is greater if its transform, in its most reduced form within the framework, utilizes *more fundamental transforms.* In other words, the evolution that uses more of the fundamental transforms permitted by $\mathcal{F}$ carries more disorder, and thus, more entropy.

Formally, we have:

$$\mathrm{Ent}_{\mathcal{F}}(j) > \mathrm{Ent}_{\mathcal{F}}(k)$$

if and only if

$$\mathrm{Ops}(c_1, \mathcal{F}) > \mathrm{Ops}(c_2, \mathcal{F})$$

where $\mathrm{Ops}(c, \mathcal{F})$ denotes the number of fundamental operations used by the transform c in its most reduced form, as allowed by framework $\mathcal{F}$.

4.1. **Relative Complexity in blocks.** Consider two evolutions,

$$j : a_1 \xrightarrow{c_1} b_1, c_1 \in \mathcal{F}$$
$$k : a_2 \xrightarrow{c_2} b_2, c_2 \in \mathcal{F}$$

Now, since they both *lie under the same framework*, then we can compare both evolutions, based on *which one carries more entropy*. This can be done by bringing, both transforms c_1 and c_2 to their most reducible forms, allowed within the framework $\mathcal{F}$ and then inspecting which one uses more amount of *fundamental transforms* from the set $\mathcal{T}_{\mathcal{F}}$, that is,for example,

$$\mathrm{Ent}_{\mathcal{F}}(j) > \mathrm{Ent}_{\mathcal{F}}(k), \quad \text{if and only if} \quad \mathrm{Ops}(c_1, \mathcal{F}) > \mathrm{Ops}(c_2, \mathcal{F})$$

Note that, it is possible that, initial block of some evolution is already a little more or less complex in comparison to other. But we are considering the *entropy of evolution* that is, how much disordered it made the final block, from the initial block.

Now, if we consider a case where a_1 and a_2 are identical, and b_1 and b_2 are identical too. Then does it implies that c_1 and c_2 in their most reducible forms within $\mathcal{F}$ are identical too?

For this, we state,

Lemma I. *Consider a group of evolutions,such that,*

$$e_0 : a \xrightarrow{c_0} b$$
$$e_1 : a \xrightarrow{c_1} b$$
$$e_2 : a \xrightarrow{c_2} b$$
$$\dots$$
$$e_n : a \xrightarrow{c_n} b$$

and,

$$\mathrm{Ent}_{\mathcal{F}}(e)_0 = \mathrm{Ent}_{\mathcal{F}}(e)_1 = \mathrm{Ent}_{\mathcal{F}}(e)_2 = \cdots = \mathrm{Ent}_{\mathcal{F}}(e)_n = 0$$

it implies, $c_0, c_1, c_2, \dots, c_n$ *at their most reduced form in* $\mathcal{F}$ *are identical.*

Proof. For the sake of contradiction we assume that, *none of the transforms at their most reduced forms are identical to each other.* *(end of proof)*

4.2. **Equal Frameworks.** Two frameworks $\mathcal{F}$ and $\mathcal{G}$ are said to be equal, if and only if, the sets, $\mathcal{T}_{\mathcal{F}}$ and $\mathcal{T}_{\mathcal{G}}$ follow,

$$\mathcal{T}_{\mathcal{F}} = \mathcal{T}_{\mathcal{G}}$$

4.3. **Decomposing Evolutions.** Given an evolution, the entropy it carries reflects the disorder present in the *final block*, quantified by how many *fundamental transforms*—as defined by a given framework—are required to realize that evolution.

Consider the evolution:

$$e : a \xrightarrow{c} b, \quad c \in \mathcal{F}$$

Suppose the transform c can be expressed as a composition of more transforms $\{c_0, c_1, c_2, \ldots, c_n\}$, each of which is *allowed* under the framework $\mathcal{F}$. Then, we may write the evolution as:

$$a \xrightarrow{c_0} a_0 \xrightarrow{c_1} a_1 \xrightarrow{c_2} a_2 \xrightarrow{c_3} \ldots \xrightarrow{c_n} b$$

For simplicity, when intermediate blocks are not of interest, this can be notated as:

$$e : a \xrightarrow{c_0}\xrightarrow{c_1}\xrightarrow{c_2}\xrightarrow{c_3} \ldots \xrightarrow{c_n} b$$

Here, the original transform c is *decomposed* into a sequence of individual transforms that are each valid within the framework $\mathcal{F}$.

Some of these transforms from $\{c_0, c_1, c_2, c_3, \ldots, c_n\}$ might even have an higher entropy than the transform c.

Now we can write this evolution e as,

$$\begin{aligned} e_0 &: a \xrightarrow{c_0} a_0 \\ e_1 &: a_0 \xrightarrow{c_1} a_1 \\ e_2 &: a_1 \xrightarrow{c_2} a_2 \\ e_3 &: a_2 \xrightarrow{c_3} a_3 \\ &\ldots \\ e_n &: a_{n-1} \xrightarrow{c_n} b \end{aligned}$$

And we can write this as,

$$e = e_0 \longrightarrow e_1 \longrightarrow e_2 \longrightarrow e_3 \longrightarrow \cdots \longrightarrow e_n$$

And now we are interested in inspecting the relation between, $\mathrm{Ent}_{\mathcal{F}}(e)_0, \mathrm{Ent}_{\mathcal{F}}(e)_1, \mathrm{Ent}_{\mathcal{F}}(e)_2, \mathrm{Ent}_{\mathcal{F}}(e)_3, \ldots, \mathrm{Ent}_{\mathcal{F}}(e)_n$ and $\mathrm{Ent}_{\mathcal{F}}(e)$.

An intuitive answer to this, is some sort of *conservation of total entropy* when the transform is decomposed in some other transforms. So we state the principle,

Principle I (Conservation of Entropy)**.** *Given an evolution e, which is decomposed into a sequence of intermediate evolutions:*

$$e = e_0 \longrightarrow e_1 \longrightarrow e_2 \longrightarrow \cdots \longrightarrow e_n$$

then the net entropy of the complete sequence equals the entropy of the original evolution e, under some framework $\mathcal{F}$ in which all corresponding transforms are allowed.

Proof. Consider the evolution defined under a framework $\mathcal{F}$:

$$e : a \xrightarrow{c} b \tag{1}$$

Suppose this evolution is decomposed as:

$$e = e_0 \longrightarrow e_1 \longrightarrow e_2 \longrightarrow \cdots \longrightarrow e_n$$

Assume, for the sake of contradiction, that the net entropy of this decomposition differs from the entropy of e. That is, entropy is not conserved across the sequence.

Now, if we compose the entire sequence back into a single evolution, we obtain:

$$e_0 : a \xrightarrow{c_0} b \tag{2}$$

But by *Lemma 1*, since both equations (1) and (2) represent the same overall evolution e, it must follow that $c = c_0$.

Since the transforms are identical, and they occur within the same framework $\mathcal{F}$, the entropy associated with them must also be identical. This contradicts our assumption that entropy was not conserved.

Hence, entropy must be conserved across the decomposition. *(end of proof)*

5. Evolution Graph

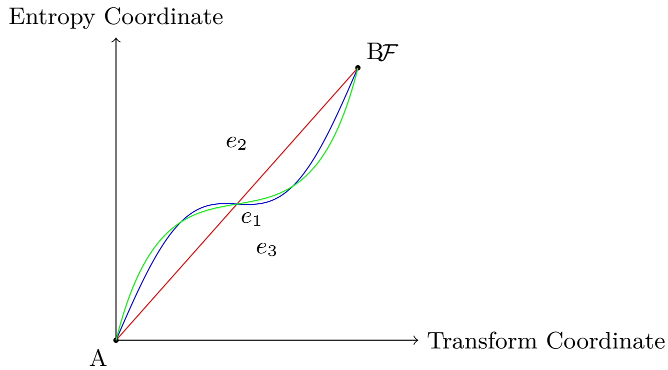
Consider an evolution:

$$e : A \longrightarrow B$$

This evolution can be accomplished through various transforms. For instance, consider three transforms c_1, c_2, and c_3, resulting in the evolutions:

$$e_1 : A \xrightarrow{c_1} B \quad , \quad e_2 : A \xrightarrow{c_2} B \quad , \quad e_3 : A \xrightarrow{c_3} B$$

Under a framework $\mathcal{F}$, we can represent these evolutions graphically as follows:



Each evolution is represented by a different *path*, showing different variations in entropy. However, the net change in entropy remains the same.

Any portion of the evolution graph can be regarded as a *sub-transform*.

Axes Interpretation. The x-axis, called the *Transform Coordinate*, represents the progression between blocks in the evolution.

The y-axis, called the *Entropy Coordinate*, represents the entropy of each infinitesimal sub-transform.

We conventionally begin from the origin $(0, 0)$, assuming the initial block is absolute (no prior transform), hence zero transform and entropy coordinates.

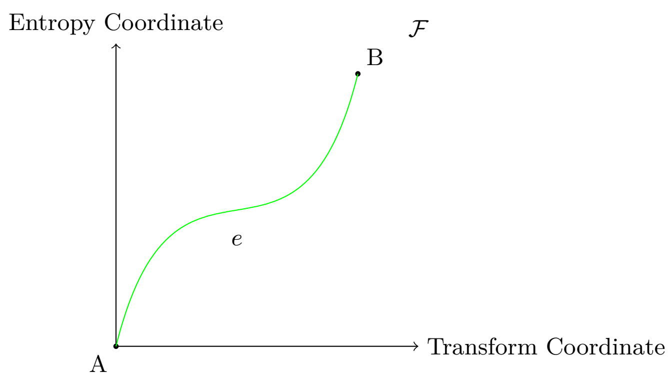
In summary: - **Transform coordinate** indicates the current sub-evolution step. - **Entropy coordinate** measures the entropy of that specific sub-transform.

5.1. Continuity in Evolution Graph.

Let us consider an evolution:

$$e : A \xrightarrow{c} B \quad , \quad c \in \mathcal{F}$$

and represent it as:



Now, assume for contradiction that the evolution e has a discontinuity at some point. This would imply that at that specific point, the entropy coordinate is undefined or non-finite.

This, in turn, suggests that the corresponding transform is *not feasible* under the framework $\mathcal{F}$.

Let this discontinuity occur at the c-th sub-evolution. Then, the decomposition:

$$e = e_0 \to e_1 \to \cdots \to e_c \to e_{c+1} \to \cdots \to e_k$$

is invalid, as e_c is non-permissible under $\mathcal{F}$.

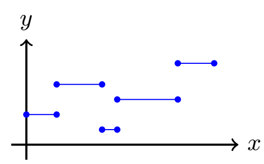
Hence, we conclude:

Lemma II. *If an evolution graph has a single discontinuity, then that evolution is not possible within the framework $\mathcal{F}$.*

However, special cases may arise. For example, two opposing discontinuities might occur, such that their infinite entropies cancel each other out—behaving like inverse transforms. In such cases, the evolution may still be feasible under $\mathcal{F}$.

Further analysis is required to fully explore such scenarios. For this, we investigate *how an actual evolution plot may look like.*

5.2. **Structure of an Evolution Graph.** The sample graphs shown in *subsection (5.1)* and above, were meant for introduction. However, almost all evolution aren't considered *infinitely smooth* under interpretation (they could be, but under our interpretation, we assume some fundamental transforms and build evolutions upon them), that is, given an evolution e, it cannot be neccesarily broken down in *infinitesimeal sub-transforms, within the framework*, hence an ideal evolution graph appears like,



Here each *step* denotes each irreducible transform within an evolution under a specified framework.

However, a graph can be transformed to contain discontinuities in such a way, that the net entropy still remain intact. That is, by introducing discontinuities in such a way that their infinite entropies get cancelled in total and we can replace that *part containing such discontinuities* with a well defined *sub-transform step.*

In this way, an evolution can still occur with net entropy being finite, still using some infinite entropy-transforms w.r.t. some framework. However, this always implies, *if the net entropy of an evolution is zero or non-infinite in general* it always has a pathway, with no discontinuities.

6. Principles, Theorems and Results