

CS 2051: Project Title

Anthony Hong
Georgia Institute of Technology

Yongyu Qiang
Georgia Institute of Technology

Aravinth Venkatesh Natarajan
Georgia Institute of Technology

1 Background

- Definition of $\pi(x)$, the number of primes $\leq x$.
- Euclid's proof that $\lim_{x \rightarrow \infty} \pi(x) = \infty$
- Introduce the Riemann zeta function, $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$
- Use the $\zeta(2)$ case and $\frac{\sin x}{x}$ Taylor series to show $\zeta(2) = \frac{\pi^2}{6}$
- Use above idea to derive Euler product formula
- Use Euler product formula with $\zeta(1)$ to get that $\sum_p \frac{1}{p} = \infty$, also showing the infinitude of primes
- Notation for prime gaps: $g_n = p_{n+1} - p_n$
- A prime gap can be arbitrarily large: Consider $n! + 2, n! + 3, \dots, n! + n$
- Are there infinite small prime gaps? Introduce twin prime conjecture
- Move on to some other main topic?

2 Main result

A interesting problem is the rate of growth of prime gaps. $\pi(n)$ denotes the prime counting function, which counts the number of primes less than or equal to n . The Prime Number Theorem (PMT) states that $x/\ln(x)$ approximates $\pi(x)$. To be more precise, the PMT states:

$$\lim_{x \rightarrow \infty} \frac{\pi(x)}{\lfloor \frac{x}{\ln(x)} \rfloor} = 1$$

This means, as x gets larger, $x/\ln(x)$ will get better as an approximation for $\pi(x)$. This also implies that the average size of the gaps between consecutive primes up until x asymptotically approaches $\ln(x)$. So for a random number n in the interval $[x, x + kx]$ for large x and fixed k , the probability that n is prime is approximately $1/x \approx 1/n$.

Cramér's random model uses this idea as a naive approach to emulate the distribution of prime numbers. Consider a random subset of the natural numbers, where the independent probability that a number n is chosen is $1/\ln(n)$. Let's call this random set P' , where P is the set of actual prime numbers. Cramér conjectured that P' , which consists of our "fake primes", accurately models the distribution of P .

According to this heuristic, we have the resulting claim, which is known as Cramér's conjecture:

$$\limsup_{n \rightarrow \infty} \frac{p_{n+1} - p_n}{(\ln p_n)^2} = 1$$

where p_n denotes the n -th prime.

(Some directions to take these ideas):

- Problems with Cramér's naive model and ways we can improve it (with modern results)
- How Cramér's model fares depending on the size and location of the interval, calculating asymptotic statistics

3 Extension/application/generalisation

- Connections from Cramér's conjecture to the Riemann hypothesis
- Other ways to use Cramér's technique of random modeling

4 Preliminary Code and Illustrations

5 Reflection/Conclusion

6 References

Terence Tao's Blog