CS 2051: Project Title

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1 Background

- Definition of $\pi(x)$, the number of primes $\leq x$.
- Euclid's proof that $\lim_{x\to\infty} \pi(x) = \infty$
- Introduce the Riemann zeta function, $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$
- Use the $\zeta(2)$ case and $\frac{\sin x}{x}$ Taylor series to show $\zeta(2) = \frac{\pi^2}{6}$
- Use above idea to derive Euler product formula
- Use Euler product formula with $\zeta(1)$ to get that $\sum_p \frac{1}{p} = \infty$, also showing the infinitude of primes
- Notation for prime gaps: $g_n = p_{n+1} - p_n$
- A prime gap can be arbitrarily large: Consider $n! + 2, n! + 3, \ldots, n! + n$
- Are there infinite small prime gaps? Introduce twin prime conjecture
- Move on to some other main topic?

2 Main result

A interesting problem is the rate of growth of prime gaps. $\pi(n)$ denotes the prime counting function, which counts the number of primes less than or equal to n. The Prime Number Theorem (PMT) states that $x/\ln(x)$ approximates $\pi(n)$. To be more precise, the PMT states:

$$\lim_{x\to\infty} \frac{\pi(x)}{\lfloor \frac{x}{\ln(x)} \rfloor} = 1$$

This means, as x gets larger, $x/\ln(x)$ will get better as an approximation for $\pi(x)$. This also implies that the average size of the gaps between consecutive primes up until x asymptotically approaches $\ln(x)$. So for a random number n in the interval $[x, x + kx]$ for large x and fixed k, the probability that n is prime is approximately $1/x \approx 1/n$.

Cramér's random model uses this idea as a naive approach to emulate the distribution of prime numbers. Consider a random subset of the natural numbers, where the independent probability that a number n is chosen is $1/\ln(n)$. Let's call this random set P', where P is the set of actual prime numbers. Cramér conjectured that P', which consists of our "fake primes", accurately models the distribution of P.

According to this heuristic, we have the resulting claim, which is known as Cramér's conjecture:

$$\limsup_{n\to\infty} \frac{p_{n+1} - p_n}{(\ln p_n)^2} = 1$$

where p_n denotes the n-th prime.

(Some directions to take these ideas):

- Problems with Cramér's naive model and ways we can improve it (with modern results)
- How Cramér's model fares depending on the size and location of the interval, calculating asymptomatic statistics

3 Extension/application/generalisation

- Connections from Cramér's conjecture to the Riemann hypothesis
- Other ways to use Cramér's technique of random modeling

4 Preliminary Code and Illustrations

5 Reflection/Conclusion

6 References

Terence Tao's Blog