A Random Model of the Primes

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Background

- Infinite number of primes (c. 300 BC, Euclid)
- $\pi(x) :=$ number of primes $\leq x$
- Prime number theorem: $\pi(x) \sim \frac{x}{\ln x}$ (1896, Hadamard and De la Vallée Poussin)

Prime Gaps

- Prime gap := difference between a pair of consecutive primes
- What does the sequence of prime gaps look like?

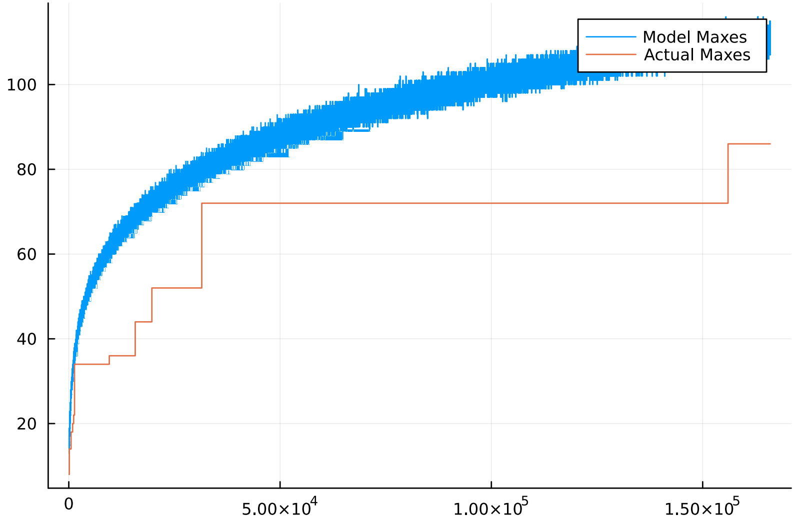
$$[1, 2, 2, 4, 2, 4, 2, 4, 6, 2]$$

  Notice that 1 is the only odd.
- How small/large can prime gaps get?
  - Small: twin prime conjecture; we have a constant bound by Yitang Zhang (2013), since improved to gap ≤ 246 infinitely often
  - Large: can find gaps of arbitary size (think $n!$); so slightly different goal, want to see how large of a gap we can expect in a given interval

Cramer's Random Model

- We know prime number distribution isn't exactly random, but it seems to behave that way: try to model as if it were
- In the interval $[2, n]$, we expect roughly $\frac{n}{\ln n}$ primes by prime number theorem. There are also roughly n total numbers $[2, n]$. So n has probability $\frac{1}{\ln n}$ of being prime?
- Naively assign each $x \in \mathbb{N}_{>2}$ a probability of $\frac{1}{\ln x}$ of being prime

First Iteration



Plot of estimated largest gap $\leq n$ vs. n, with 100 trials per n.

Why?

- We can study this random distribution using ideas from probability and statistics, can give extra insight
- Conducting many trials comes at relatively low cost (can be parallelized, will scale with current trend towards multithreaded computers)
- We can consider disjoint intervals completely independently (not true of traditional sieve methods)