Patterns in Primes

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***Abstract*—**

I. Introduction

The distribution of the set of prime numbers is a topic long studied by mathematicians. We will explore some important results, with a focus on prime gaps. We will also cover Cramer's random model, a heuristic used to study distributions of seemingly random sets, such as the prime numbers.

II. Background

A common result many students learn early on in number theory is about the infinitude of prime numbers. This fact is also known as Euclid's theorem, and we include a short summary of Euclid's original proof.

Euclid's Theorem. *The set of all prime numbers is larger in cardinality than any finite collection of prime numbers.*

Proof. Consider

$$\{p_1, p_2, \dots, p_n\},$$

some arbitrary finite collection of prime numbers. Let

$$N = p_1 p_2 \dots p_n,$$

and consider $P = N + 1$. P is either prime or not prime.

First, let P be prime. Then, we have constructed a new prime number and we are done.

Now, let P not be prime. Let g be a prime factor of P. We propose that $g \notin \{p_1, p_2, \dots, p_n\}$. To show this, suppose for contradiction that $g \in \{p_1, p_2, \dots, p_n\}$. Then, since $p_1, p_2, \dots, p_n$ are all factors of N, we have $g|N$. $g|P$ and $g|N$, so we must also have $g|P - N$, i.e. $g|1$. But $g > 1$ (g is prime), so g cannot possibly divide 1. Therefore, $g \notin \{p_1, p_2, \dots, p_n\}$, and we have found a new prime, as required. □

A natural next step from here is to explore how prime numbers are distributed. For now, we'll focus particularly on prime gaps and how small or large they can be. A bit of thinking leads to the observation that there are certain restrictions on what prime gaps can look like. First, we can see that prime gaps can be odd only finitely many times.

Proposition. *There exist only finitely many odd prime gaps.*

Proof. Notice that all primes $p > 2$ are odd. Then $p_{n+1} - p_n$ is even for all $n > 1$, so there exists only finitely many n such that $p_{n+1} - p_n$ is odd. □

In fact, $n = 1$ yields the only odd prime gap, namely $(p_1, p_2) = (2, 3)$ with difference 1. On the other hand, we have a much more promising observation for large prime gaps, namely that we can make them arbitrarily large.

Proposition. *There exist prime gaps of arbitrarily large size.*

Proof. We'll show that given $n \in \mathbb{Z}^+$, we can construct an interval of size at least $n-1$ of only composite numbers. Then the first primes immediately before and after this interval will have gap of at least n.

Let $n \in \mathbb{Z}^+$. Now consider the interval

$$[n! + 2, n! + n].$$

By definition of the factorial, we have $i|n!$ for all $i \in [2, n]$. We also trivially have $i|i$. Therefore, we have $i|(n!+i)$, and so i is a divisor of $n!+i$ for all $i \in [2, n]$. Then all of $[n!+2, n!+n]$ is composite, and this interval has size $n - 1$, as desired. □

Although this result is nice, we soon realize that it does not give us a very strong bound, in the sense that it is rather wasteful. To find a prime gap of size n by this method, we must consider numbers of order $n!$. By Stirling's approximation, we have

$$n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

asymptotically, which is worse than exponential growth in n. Put in context, our current method suggests that finding a prime gap of size 10 requires us to find numbers of magnitude about 3 million. In reality, we can find such a gap of size 10 at $(p_{30}, p_{31}) = (113, 127)$, which is much smaller than 3 million, so certainly we can do better.

For that, we'll need better tools. Let us first define the prime counting function $\pi(n)$.

Definition. $\pi(n) :=$ number of primes $\leq n$.

Now we can introduce the prime number theorem, which characterizes the growth of $\pi(n)$ as n gets large. Note that we will use this result without proof in this paper, as even relatively simpler proofs rely on tools from analysis.

Prime Number Theorem. $\frac{n}{\ln(n)}$ asymptotically approximates $\pi(n)$. Put more formally,

$$\lim_{n\to\infty} \frac{\pi(n)}{\frac{n}{\ln(n)}} = 1.$$

From this result, we can quickly argue by the pigeonhole principle that there should exist a prime gap of size at least $\ln(n)$ in the interval $[2, n]$. If we take buckets to be the $\frac{n}{\ln(n)}$ prime numbers less than or equal to n and our pigeons to be the integers in $[2, n]$, then by the pigeonhole principle, at least one prime number will correspond to $\ln(n)$ or more integers, making a gap of size at least $\ln(n)$. Note that this pigeonhole argument already gives an asymptotically better bound for large gaps than our previous result, albeit just barely, as we only now only need to go to e^x for a gap of size x rather than $x!$.

We can also apply the same argument in reverse to get a small gap of size at most $\ln(n)$, but getting any better bounds generally requires the use of analysis. In an effort to avoid that, we can instead explore the Cramer random model, which considers randomness as a method to approximate $\pi(n)$.

III. Cramer's Random Model

Recall that the prime number theorem tells us for large n,

$$\pi(n) \approx \frac{n}{\ln(n)}.$$

But since we also have n total numbers less than or equal to n, we can divide by this size to get a rough prime density $\delta(n)$:

$$\delta(n) = \frac{\pi(n)}{n} = \frac{\frac{n}{\ln n}}{n} = \frac{1}{\ln(n)}.$$

So for a random number x in the interval $[n, n+kn]$ for large n and fixed k, the probability that x is prime is approximately $1/\ln(n) \approx 1/\ln(x)$. (As a side note, this density $\delta(n)$ is also the rationale behind an alternate approximation for $\pi(n)$ with the logarithmic integral $\mathrm{Li}(n)$ as

$$\pi(n) \approx \int_2^n \delta(x)\,dx = \int_2^n \frac{1}{\ln(x)}\,dx = \mathrm{Li}(n),$$

which actually turns out to be a much better approximation to $\pi(n)$ than the traditional prime number theorem. In fact, if we assume the Riemann hypothesis, we have that the error of $\mathrm{Li}(n)$ from $\pi(n)$ is bounded by $O(n^{1/2+\epsilon})$ for any $\epsilon > 0$, meaning that roughly the first half of the digits of $\mathrm{Li}(n)$ will be correct, but that is outside the scope of this paper.)

Equipped with this prime density, we arrive at Cramer's random model. In 1936, Harald Cramer, a Swedish mathematician proposed a probabilistic perspective for studying the distribution of primes. From first inspection, the primes don't seem to be distributed according to any clear pattern; they seem to be rather randomly scattered across the number line. So why don't we treat the primes as exactly that, a random distribution? In Cramer's original model, we assign each natural number $x > 2$ a probability

$$p(x) = \frac{1}{\ln(x)}$$

of being prime. The hope is that this random distribution will be within some reasonable margin of approximating the real prime number distribution. If we can achieve this, then not only do we have a computationally efficient way of estimating the prime distribution, but we can also bring in existing tools from probability theory to help. So let's give it a shot and see how reasonable this idea is.

We'll first explore large prime gaps and how accurate the random model predicts them to be compared to the real prime gaps. To do this, we computed the largest prime gap less than or equal to a natural number n for $100 \leq n \leq 1.66 \times 10^5$ for both distributions. In the random model, we assigned each natural number $x \leq n$ with its probability $x/\ln(x)$ of being prime. We then sorted these "primes" and calculated the largest gap between any consecutive two elements. For the actual distribution, we simply calculated the real largest prime gap. To mitigate the wild varation due to the random nature of the model, we also repeated each x in the random model for a total of 100 trials and plotted their average value.

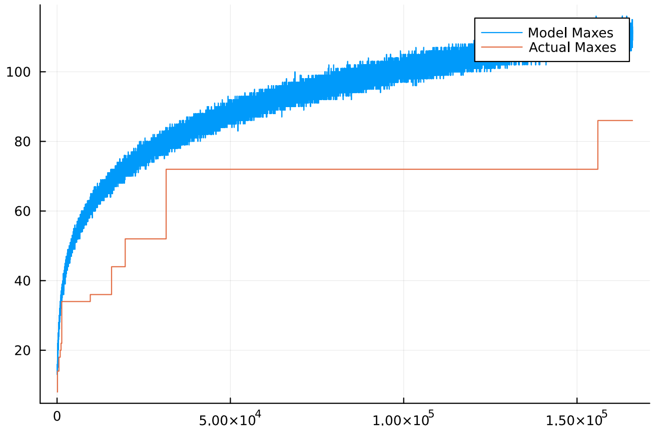


Fig. 1. Maximum prime gaps, size of largest gap $\leq n$ vs. n

Well. . . that's not quite exactly what we hoped it would be, but it's also not all that bad! We can see that for rather small values of x, the two graphs line up rather nicely, and the two graphs approach each other again around $n = 30{,}000$ (it also seems that the 100 trials did its job fairly well in reducing random variation). The two graphs do have their differences: the actual prime gap graph is a little too jagged for our (relatively) continuous-looking graph to model accurately. But it still seems that they behave similarly in a way; one could imagine that a curve of best fit through the actual graph would look rather similar to our model. It's not out of the world either to believe that the next jump in the prime gap will once again approach that of our model for larger n.

So not all hope is lost quite yet! We still have some more ideas. One first observation is that in the previous iteration of the model, we restricted our "fake primes" to being natural numbers. Of course, it perhaps makes sense to do so as primes are natural numbers, but why should we restrict ourselves in that way? What if we take $x \in \mathbb{R}$ instead of $x \in \mathbb{N}$? After all, differences, logarithms, and division are all just as well-defined on $\mathbb{R}$, so there's theoretically nothing preventing us from doing this.

Obviously, when we deal with $x \in \mathbb{R}$, we can no longer assign discrete probabilities to each element, as there's an infinite number of them! Instead, we need to assign an actual density function like the $\delta(x)$ mentioned earlier. But $\delta(x)$

poses some problems: for one, the integral

$$\lim_{n\to\infty} \int_2^n \delta(x)\, dx = \infty$$

doesn't converge (perhaps unsurprisingly, as there are indeed an infinite number of primes), so we can't hope to use it as a probability density without at least some modifications first. But for now, we can try something simpler.

As our first step into using a continuous model, we'll use the simplest possible density function: the uniform distribution. For a closed interval $[a, b]$, we can define the following uniform density function

$$f(x) = \begin{cases} \frac{1}{b-a}, & a \leq x \leq b \\ 0, & \text{otherwise.} \end{cases}$$

Then, since the prime number theorem predicts roughly $n/\ln(n)$ primes less than or equal to n, we can randomly sample $n/\ln(n)$ numbers uniformly from the interval $[2, n]$ to get an approximate distribution of primes. So we repeat the previous experiment, with the only change being the switch to this new method of generating our "fake primes."

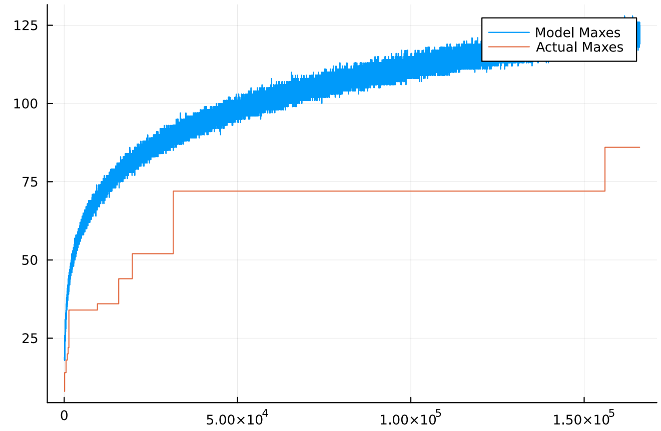


Fig. 2. Maximum prime gaps, size of largest gap $\leq n$ vs. n, (uniformly distributed model)

Wow, that's even worse than what we had before, which is actually rather surprising. We know that the prime number distribution is generally skewed to the right, meaning primes occur less frequently for larger n. This would suggest larger gaps for the than a uniform distribution would predict, but that's clearly not the case here. So, what went wrong? We have to recall that the $n/\ln(n)$ value from the prime number theorem is only an approximation. In fact, despite being asymptotically correct, it actually has a rather larger margin of error. We can immediately see this by plotting $\pi(n)$ and $n/\ln(n)$ on a graph.

(note to self, explore above idea of using real numbers, and then go into trying out other probability distributions like a Gaussian, maybe)

IV. Improvements to Cramer's Model

Although Cramer's original model makes a fair attempt at estimating prime number distributions, it is fundamentally flawed. The set of prime numbers is a non-random set. Primes have properties that make them deterministic. Although they exhibit pseudorandom behavior, they are not completely random. Hence, why mathematicans are obsessed with finding possible patterns. Cramer's model will generate a set of numbers that is completely random, considering only the independent probabilty $1 \;\ln(n)$. An improved version of the model would incorporate more characteristics of the actual set of prime numbers. Let's look at some of the more obvious improvements.

For any prime p, where $p > 2$, $p + 1$ is even and therefore cannot be prime. Currently in our model we do not consider parity, resulting in an arbitrary number of neighboring integer pairs. This never occurs in actual primes, with the exception of $(2, 3)$. Thus, let's update the probabilities of the model. To preserve the distribution in accordance to the Prime Number Theorem, lets "move" the probability of an even number e being selected, to $e + 1$. Thus, suppose n is an integer greater than 2. If n is even, then its updated probability of being selected is 0. If n is odd, then its probability is $1/\ln(n-1) + 1/\ln(n) \approx 2/\ln(n)$.

(Add models highlighting differences?)

V. Other Applications of Cramer's Model

The fundamental idea of Cramer's model is quite powerful. For any seemingly random set of numbers, we can emulate its density by using a random distribution. An interesting point of extension is, can we use this idea for similar sets of numbers other than the primes?

We introduce a lesser known sequence of numbers known as the Ulam numbers, named after Stanislaw Ulam, who first popularized the sequence in 1964. The Ulam sequence starts with $U_1 = 1$, $U_2 = 2$. Then for $n > 2$, U_n is the smallest integer greater than U_{n-1} that can be written as a unique sum of two distinct earlier terms. The first twenty Ulam numbers are:

$$1, 2, 3, 4, 6, 8, 11, 13, 16, 18, 26, 28, 36, 38, 47, 48, 53, 57, 62, 69$$

Similar to the prime numbers, the Ulam numbers are a deterministic set. They are fixed by a clear procedure. However, it is once again a seemingly random sequence. There are no equations to describe where Ulam numbers will show up on the integer number line. Let's run a program to see the distribution of Ulam numbers across a larger sample. Finding Ulam numbers effiently is a difficult task in itself, but we can do it naively.

(Insert Ulam Distribution here)

As can be seen by the distribution, Ulam numbers look quite linear. It has been shown that the first 3 million terms are close to the line $y = 13.51x$. Interestingly, Ulam himself conjectured that the natural density of Ulam numbers is 0. Meaning the distribution of Ulam numbers eventually converges to 0 as we go to infinity. But, as it turns out, calculations up to around a billion indicate that this is probably not the case. The observed density actually converges to around 0.074.

Using this density and the fact that Ulam numbers look to be quite linear, let's try to apply Cramer's model. We define a random variable such that

$$P(X = 1) = 0.074$$

(Show model vs actual Ulam numbers)

(Section on how the model trivializes analagous conjectures) In fact, we can apply Cramer's model to any sequence with significant natural density. This tool gives us a way to confidently guess the asymptotic statistics of seemingly random sets of numbers. Proving these results rigorously is very difficult, but by random modeling, we can get the next best thing, which is a statistics backed guess.

Cramér's random model uses this idea as a naive approach to emulate the distribution of prime numbers. Consider a random subset of the natural numbers, where the independent probability that a number n is chosen is $1/\ln(n)$. Let's call this random set P', where P is the set of actual prime numbers. Cramér conjectured that P', which consists of our "fake primes," accurately models the distribution of P.

According to this heuristic, we have the resulting claim, which is known as Cramér's conjecture:

$$\limsup_{n\to\infty} \frac{p_{n+1} - p_n}{(\ln p_n)^2} = 1$$

where p_n denotes the n-th prime.

(Additional sections for these ideas (TBD)):

- Problems with Cramér's naive model and ways we can improve it (with modern results)
- How Cramér's model fares depending on the size and location of the interval, calculating asymptomatic statistics

VI. Bertrand's Postulate

Another problem in number theory is finding the bounds in which you would find a prime number. Of these, one of the paramount significance is **Bertrand's Postulate**:
Bertrand's postulate states that for an integer i > 1, there is at least one prime number p

$$i \leq n \leq 2i$$

The proof for it is as follows:
We'll start by proving Lemma 1:

$$\frac{4^n}{2n} \leq \binom{2n}{n}$$

$$4^n = (1+1)^{2n} = \sum_{k=0}^{2n} \binom{2n}{k}$$

Since, $\binom{2n}{0}$ is 1 and $\binom{2n}{2n}$ is 1, this is the same as

$$\equiv 2 + \sum_{k=1}^{2n-1} \binom{2n}{k}$$

Since the largest term in this summation is $\binom{2n}{n}$ (since for $\binom{n}{k}$, $k = n/2$ will give the largest term) and there are 2n terms,

$$(2 + \sum_{k=1}^{2n-1} \binom{2n}{k}) \leq (2n * \binom{2n}{n})$$

Therefore,

$$\frac{4^n}{2n} \leq \binom{2n}{n}$$

Let's now prove Lemma 2:
For a given prime p, let's define r as the greatest number for which $p^r | \binom{2n}{n}$. Lemma 2 is as follows, for such an r,

$$p^r \leq 2n$$

Firstly, we have to introduce Legendre's Formula:
Legendre's Formula states that for any prime number p, and any integer n, let's define the function $v_p(n)$ as the exponent of the largest power of p that divides n. Let L = $\lfloor \log_p n \rfloor$
Legendre's Formula is:

$$v_p(n!) = \sum_{i=1}^{L} \left\lfloor \frac{n}{p^i} \right\rfloor$$

$\binom{2n}{n}$ can also be written as $\frac{(2n)!}{n!*n!}$)
F Finding the largest exponent of p, r, that divides $\frac{(2n)!}{n!*n!}$ is the same as finding the largest exponent of p, r, that divides each component of $\frac{(2n)!}{n!*n!}$, i.e. $(2n)!$, $n!$

In this case, L = $\lfloor \log_p 2n \rfloor$
Writing this in terms Legendre's Formula, we get that:

$$v_p(\binom{2n}{n}) = \sum_{i=1}^{L} \left\lfloor \frac{2n}{p^i} \right\rfloor - 2\sum_{i=1}^{L} \left\lfloor \frac{n}{p^i} \right\rfloor$$

This is equivalent to:

$$v_p(\binom{2n}{n}) = \sum_{i=1}^{L} \left\lfloor \frac{2n}{p^i} \right\rfloor - 2\left\lfloor \frac{n}{p^i} \right\rfloor$$

Thinking intuitively, every term in $\sum_{i=1}^{L} \left\lfloor \frac{2n}{p^i} \right\rfloor - 2\left\lfloor \frac{n}{p^i} \right\rfloor$ must either be 0 or 1.
If ($\frac{2n}{p^i}$ mod 1) $\geq$ 0.5 then the term would be 1, otherwise the term is 0. Therefore, the maximum value of this function would be if all the terms were equal to 1. Since we are only dealing with positive numbers, and the exponent is a monotonic function if both numbers are positive numbers,

$$v_p(\binom{2n}{n}) = r \leq L$$

$$\equiv r <= \log_p 2n$$

Therefore, it follows that

$$p^r <= p^{\log_p 2n} = 2n$$

We are able to prove our initial Lemma 2,

$$p^r <= 2n$$

Now, we move on to proving Lemma 3: Let's place some bounds on p for the lemma we're going to prove. If

$$\frac{2n}{3} < p \leq n, thenR(n,p) = 0$$

$$\binom{2n}{n} = \frac{(2n)!}{n!^2}$$

Therefore, there are 2 factors of p in the numerator, p and 2p

In the denominator, there are 2 factors of p, each in n!

The conditions mean that 3p is too large to be in the numerator

Due to this, the factors of p in the numerator and denominator cancel out so the greatest power of p that divides this number is 1, i.e. p^0, giving R = 0

With this, we prove Lemma 3

Moving on to Lemma 4:

Let's define the primorial function:

$$x\# = \Pi_{p \leq x} x$$

I.e., x# is the product of all primes less than or equal to x

Lemma 4 states that for x $\geq$ 3, x# $\leq$ 4^x

We proceed by induction:

This claim is easily verifiable for small odd numbers. Try one yourself!

For even numbers, x# = (x - 1)# as for x $\geq$ 3, there are no even prime numbers

Therefore, it is also bounded by x# $\leq$ 4^x

Let's take a larger odd number, n = 2m + 1

$$\Pi_{p \leq n} p = \Pi_{p \leq m+1} * \Pi_{m+2 \leq p \leq 2m+1}$$

Since the Inductive Hypothesis is true for smaller odd numbers,

$$\Pi_{p \leq m+1} = 4^{m+1}$$

Now, let's take a detour, let's look at a binomial expansion

$$(1+1)^{2m+1} = \sum_{k=0}^{2m+1} \binom{2m+1}{k} 1^{2m+1-k} 1^k$$

Necessarily

$$\Pi_{m+2 \leq p \leq 2m+1} \leq \binom{2m+1}{m}$$

Since you can notice that all the primes on the left will divide $\binom{2m+1}{m}$

$$\binom{2m+1}{m} \leq \sum_{k=0}^{2m+1} \binom{2m+1}{k} 1^{2m+1-k} 1^k$$

$$\sum_{k=0}^{2m+1} \binom{2m+1}{k} 1^{2m+1-k} 1^k = 2^{2m+1}$$

Also, since

$$\binom{2m+1}{m} = \binom{2m+1}{m+1}$$

We can divide our bound by 2 since those numbers will be repeated

Therefore,

$$\Pi_{m+2 \leq p \leq 2m+1} \leq 2^{2m}$$

Consequently,

$$\Pi_{p \leq n} p = 4^{m+1} * 2^{2m}$$

Which is equivalent to

$$\Pi_{p \leq n} p \leq 4^n$$

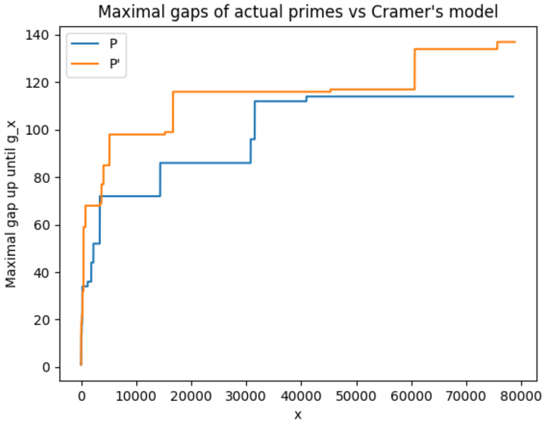
Now, we are finally ready to prove Bertrand's Postulate:

VII. Extension/application/generalisation

- Connections from Cramér's conjecture to the Riemann hypothesis
- Other ways to use Cramér's technique of random modeling

VIII. Preliminary Code and Illustrations

Cramér's random model allows us to heuristically test properties of primes. In this example, we graphically compare the maximal prime gap of the model and the actual primes.



IX. Accuracy of the Model

Under heuristic testing with variations of Cramér's model, we should be able to support strong statements such as Bertrand's postulate and Legendre's conjecture. However, comparing with the actual primes, it should be clear that Cramér's model is inaccurate. Further work would involve the creation and tuning of other random models, to more closely emulate prime distributions.

References

[1] Terence Tao's Blog

[2] Wikipedia Article on Prime Gaps