A Random Model of the Primes

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Background

- Infinite number of primes (c. 300 BC, Euclid)
- $\pi(x) :=$ number of primes $\leq x$
- Prime number theorem: $\pi(x) \sim \frac{x}{\ln x}$ (1896, Hadamard and De la Vallée Poussin)

Prime Gaps

- Prime gap := difference between a pair of consecutive primes
- What does the sequence of prime gaps look like?

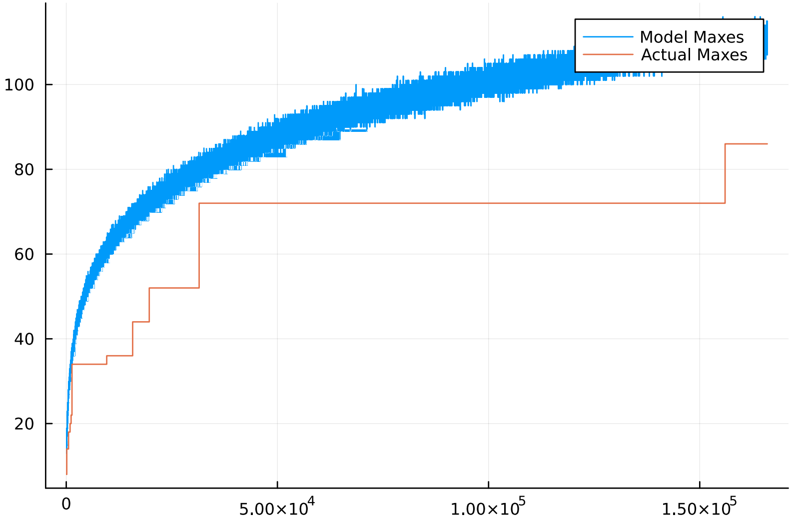
$$[1, 2, 2, 4, 2, 4, 2, 4, 6, 2]$$

  Notice that 1 is the only odd.
- How small/large can prime gaps get?
  - Small: twin prime conjecture; we have a constant bound by Yitang Zhang (2013), since improved to gap ≤ 246 infinitely often
  - Large: can find gaps of arbitary size (think $n!$); so slightly different goal, want to see how large of a gap we can expect in a given interval

Cramer's Random Model

- We know prime number distribution isn't exactly random, but it seems to behave that way: try to model as if it were
- In the interval $[2, n]$, we expect roughly $\frac{n}{\ln n}$ primes by prime number theorem. There are also roughly n total numbers $[2, n]$. So n has probability $\frac{1}{\ln n}$ of being prime?
- Naively assign each $x \in \mathbb{N}_{>2}$ a probability of $\frac{1}{\ln x}$ of being prime

First Iteration



Plot of estimated largest gap $\leq n$ vs. n, with 100 trials per n.

Why?

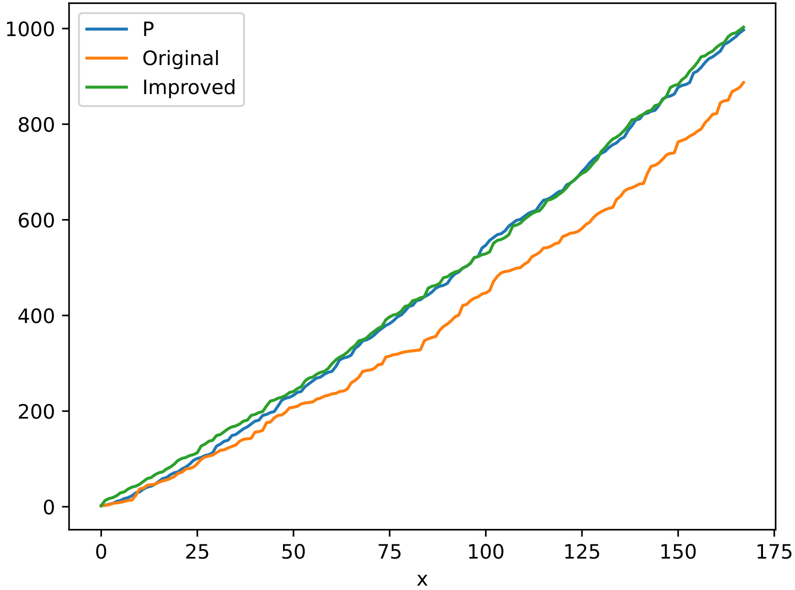
- We can study this random distribution using ideas from probability and statistics, can give extra insight
- Conducting many trials comes at relatively low cost (can be parallelized, will scale with current trend towards multithreaded computers)
- We can consider disjoint intervals completely independently (not true of traditional sieve methods)

Improvement to the Original Model

- Something obvious, the only even prime should be 2
- However, Cramer's original scheme gives all natural numbers $x > 2$ chance $\frac{1}{\ln x}$
- Idea: For every even number e, give its chance to $e + 1$
- Even number, $p(x) = 0$, odd number, $p(x) \approx \frac{2}{\ln x}$

Improvement to the Original Model

- Extrapolate idea: Instead of just 2, consider a small prefix of primes
- Only consider numbers coprime to those primes to be a part of the model
- Maintains randomness, but incorporates some structure from that prefix



P
Original
Improved
1000
800
600
400
200
0
0
25
50
75
100
125
150
175
x

Extension of Cramer's Model

- Cramer's model can be applied to other seemingly random sequences with known density
- e.g. Ulam numbers
- $U(1) = 1, U(2) = 2$
- $U(x)$ is least number which is a unique sum of two distinct earlier terms

Primes Themselves

- Fermat's Little Theorem, $a^{n-1} \equiv 1 \mod n \, \forall a$ coprime to n
- But, how accurate is this?
- Rewrite $n - 1 = 2^s d$
- $a^d \equiv 1 \pmod{n}$
- $a^{2^r d} \equiv -1 \pmod{n}, \quad 0 \leq r < s.$

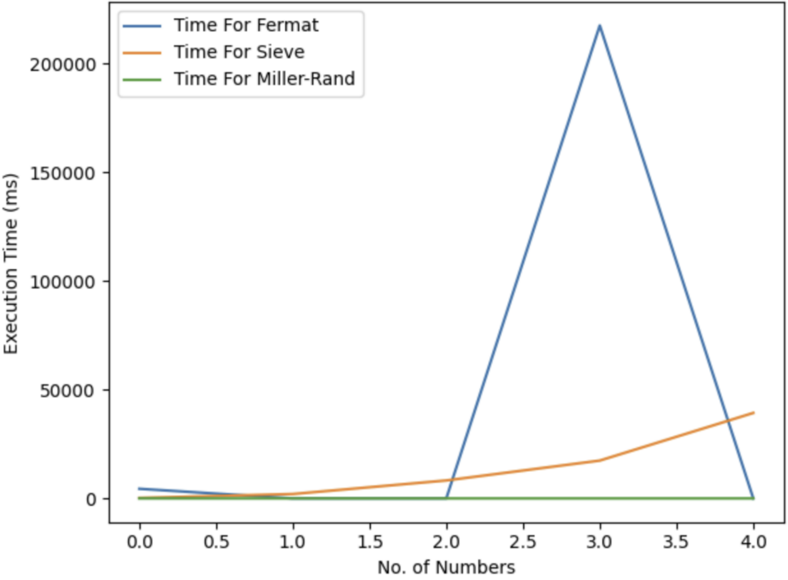
Miller Rabin

- No non-prime, odd, number exists that can pass for every a!!
- If the a chosen reveals that n is composite, it is called a witness. If it turns out it told us that a composite was prime, we call it a liar
- We don't know any way to deterministically find witnesses for a number
- So... Randomize!!
- It is shown that for $2 < a < n - 1$, there are at most $(1/4) * n$ liars

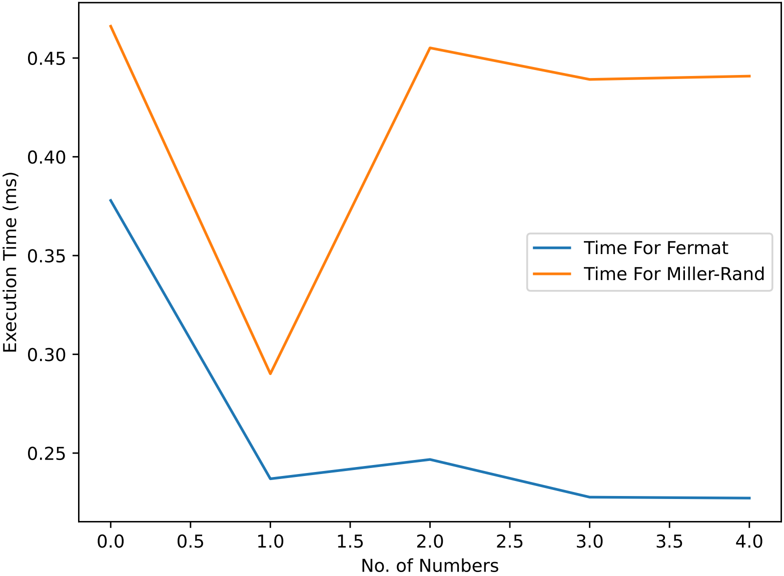
Miller-Rabin

- Doing k iterations of the algorithm, we have a confidence of 0.25^k that the number is prime
- In reality, the number of liars is much higher for most numbers
- This produces something known as an 'industry-grade' prime
- The running time is incredibly fast. The next slide will show Miller-Rabin in comparison with other algorithms

Running Time



Running Time



Other Primality tests

- The **The Baillie-PSW** primality test is one that is possibly deterministic
- It is a mix between a Miller-Rabin test with $a = 2$(chosen arbitrarily) and another class of numbers called Lucas primes
- There
  is no known crossover between the two sets of numbers upto 2^{64}