Solutions to *Fourier Analysis* by Stein and Shakarchi

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Contents

1 Chapter 1: The Genesis of Fourier Analysis

1.1 Exercises

1.1.1 Exercise 1.1

If $z = x + iy$ is a complex number with $x, y \in \mathbb{R}$, we define

$$|z| = (x^2 + y^2)^{1/2}$$

and call this quantity the **modulus** or **absolute value** of z.

(a) What is the geometric interpretation of $|z|$?

(b) Show that if $|z| = 0$, then $z = 0$.

(c) Show that if $\lambda \in \mathbb{R}$, then $|\lambda z| = |\lambda||z|$, where $|\lambda|$ denotes the standard absolute value of a real number.

(d) If z_1 and z_2 are two complex numbers, prove that

$$|z_1 z_2| = |z_1||z_2| \quad \text{and} \quad |z_1 + z_2| \leq |z_1| + |z_2|.$$

(e) Show that if $z \neq 0$, then $|1/z| = 1/|z|$.

Proof. (a) The geometric interpretation of $|z|$ is the distance from z as a vector in $\mathbb{R}^2$ to the origin.

(b) If $|z| = 0$, then $|z|^2 = x^2 + y^2 = 0$. But $x^2, y^2 \geq 0$ since $x, y \in \mathbb{R}$, so $x^2 = y^2 = 0$. Thus $x = y = 0$.

(c) If $\lambda \in \mathbb{R}$, then $\sqrt{\lambda^2} = |\lambda|$ and thus

$$|\lambda z| = |\lambda x + i\lambda y| = \sqrt{(\lambda x)^2 + (\lambda y)^2} = \sqrt{\lambda^2(x^2 + y^2)} = |\lambda|\sqrt{x^2 + y^2} = |\lambda||z|.$$

(d) Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. Then $z_1 z_2 = (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)$ and

$$|z_1z_2| = \sqrt{(x_1x_2 - y_1y_2)^2 + (x_1y_2 + x_2y_1)^2} = \sqrt{x_1^2x_2^2 + y_1^2y_2^2 + x_1^2y_2^2 + x_2^2y_1^2} = \sqrt{(x_1^2 + y_1^2)(x_2^2 + y_2^2)} = |z_1||z_2|,$$

where the mixed $x_1x_2y_1y_2$ terms cancel out. For the triangle inequality, we have

$$|z_1 + z_2|^2 = (x_1 + x_2)^2 + (y_1 + y_2)^2 = x_1^2 + y_1^2 + x_2^2 + y_2^2 + 2(x_1x_2 + y_1y_2) = |z_1|^2 + |z_2|^2 + 2(x_1x_2 + y_1y_2)$$

Now note that $x_1x_2 + y_1y_2 \leq \sqrt{x_1^2 + y_1^2}\sqrt{x_2^2 + y_2^2} = |z_1||z_2|$ by the Cauchy-Schwarz inequality on $\mathbb{R}^2$, so

$$|z_1 + z_2|^2 \leq |z_1|^2 + |z_2|^2 + 2|z_1||z_2| = (|z_1| + |z_2|)^2.$$

Taking square roots gives the desired result since the square root is monotonically increasing.

(e) If $z \neq 0$, then $1 = (1/z)z$ and thus

$$1 = \left|\frac{1}{z}z\right| = \left|\frac{1}{z}\right||z| \implies \left|\frac{1}{z}\right| = \frac{1}{|z|}$$

by (c). This is valid since $|z| \neq 0$ by (b). □

1.1.2 Exercise 1.2

If $z = x + iy$ is a complex number with $x, y \in \mathbb{R}$, we define the **complex conjugate** of z by

$$\overline{z} = x - iy.$$

(a) What is the geometric interpretation of $\overline{z}$?

(b) Show that $|z|^2 = z\overline{z}$.

(c) Prove that if z belongs to the unit circle, then $1/z = \overline{z}$.

Proof. (a) The geometric interpretation of $\overline{z}$ is a reflection of z across the real axis.

(b) We have $z\overline{z} = (x + iy)(x - iy) = x^2 - i^2y^2 = x^2 + y^2 = |z|^2$ since $i^2 = -1$.

(c) If z belongs to the unit circle, then $|z| = 1$. Then by (b), $1 = |z|^2 = z\overline{z}$, so $1/z = \overline{z}$ since $z \neq 0$. □

2 Chapter 2: Basic Properties of Fourier Series

2.1 Exercises

2.1.1 Exercise 2.1

Suppose f is 2π-periodic and integrable on any finite interval. Prove that if $a, b \in \mathbb{R}$, then

$$\int_a^b f(x)\,dx = \int_{a+2\pi}^{b+2\pi} f(x)\,dx = \int_{a-2\pi}^{b-2\pi} f(x)\,dx.$$

Also prove that

$$\int_{-\pi}^{\pi} f(x+a)\,dx = \int_{-\pi}^{\pi} f(x)\,dx = \int_{-\pi+a}^{\pi+a} f(x)\,dx.$$

Proof. For the first chain of equalities, make the change of variables $u = x + 2\pi$ with $du = dx$ to get

$$\int_a^b f(x)\,dx = \int_{a+2\pi}^{b+2\pi} f(x-2\pi)\,dx = \int_{a+2\pi}^{b+2\pi} f(x)\,dx$$

since $f(x-2\pi) = f(x)$ by periodicity (taking $y = x - 2\pi$ in $f(y+2\pi) = f(y)$ yields this version). The second equality in the chain follows from using $u = x - 2\pi$. For the latter chain of equalities, first observe that the first and last integrals are equal by the change of variables $u = x + a$ with $du = dx$:

$$\int_{-\pi}^{\pi} f(x+a)\,dx = \int_{-\pi+a}^{\pi+a} f(u)\,du.$$

Now we show that the middle integral is equal to the last. Assume without loss of generality that $0 \le a \le 2\pi$ since f is 2π-periodic. Then we can write

$$\int_{-\pi}^{\pi} f(x)\,dx = \int_{-\pi}^{-\pi+a} f(x)\,dx + \int_{-\pi+a}^{\pi} f(x)\,dx$$

since $-\pi \le -\pi + a \le \pi$. Then apply the first inequality in the first chain to the first integral to get

$$\int_{-\pi}^{\pi} f(x)\,dx = \int_{\pi}^{\pi+a} f(x)\,dx + \int_{-\pi+a}^{\pi} f(x)\,dx = \int_{-\pi+a}^{\pi+a} f(x)\,dx,$$

as desired. □

2.1.2 Exercise 2.2

In this exercise we show how the symmetries of a function imply certain properties of its Fourier coefficients. Let f be a 2π-periodic Riemann integrable function defined on $\mathbb{R}$.

(a) Show that the Fourier series of the function f can be written as

$$f(\theta) \sim \hat{f}(0) + \sum_{n\ge 1} \left([\hat{f}(n) + \hat{f}(-n)] \cos n\theta + i[\hat{f}(n) - \hat{f}(-n)] \sin n\theta \right).$$

(b) Prove that if f is even, then $\hat{f}(n) = \hat{f}(-n)$, and we get a cosine series.

(c) Prove that if f is odd, then $\hat{f}(n) = -\hat{f}(-n)$, and we get a sine series.

(d) Suppose that $f(\theta + \pi) = f(\theta)$ for all $\theta \in \mathbb{R}$. Show that $\hat{f}(n) = 0$ for all odd n.

(e) Show that f is real-valued if and only if $\overline{\hat{f}(n)} = \hat{f}(-n)$ for all n.

Proof. (a) From the usual representation of the Fourier series of f, we have

$$f(\theta) \sim \sum_{n=-\infty}^{\infty} \hat{f}(n)e^{in\theta} = \hat{f}(0) + \sum_{n=1}^{\infty} \hat{f}(n)e^{in\theta} + \sum_{n=-\infty}^{1} \hat{f}(n)e^{in\theta} = \hat{f}(0) + \sum_{n=1}^{\infty} \left(\hat{f}(n)e^{in\theta} + \hat{f}(-n)e^{-in\theta}\right).$$

Writing $e^{in\theta} = \cos n\theta + i \sin n\theta$ and $e^{-in\theta} = \cos n\theta - i \sin n\theta$, we obtain

$$f(\theta) \sim \hat{f}(0) + \sum_{n=1}^{\infty} \left([\hat{f}(n) + \hat{f}(-n)] \cos n\theta + i[\hat{f}(n) - \hat{f}(-n)] \sin n\theta\right)$$

by grouping the real and imaginary parts.

(b) If f is even, then $f(\theta) = f(-\theta)$ and thus

$$\hat{f}(-n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta)e^{in\theta}\, d\theta = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(-\theta)e^{-in\theta}\, d\theta = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta)e^{-in\theta}\, d\theta = \hat{f}(n)$$

after making the change of variables $\theta \mapsto -\theta$ (note that we pick up a minus sign from the Jacobian but the bounds get reversed, so the two effects are canceled out). As a result, we have $\hat{f}(n) - \hat{f}(-n) = 0$ and the sine terms in the series vanish, so we are left with a cosine series.

(c) If f is odd, then $f(\theta) = -f(-\theta)$ and everything is the same as in (b) except we pick up an extra minus sign in the third equality, so we end up with $\hat{f}(-n) = -\hat{f}(n)$. As a result, we have $\hat{f}(n) + \hat{f}(-n) = 0$ and the cosine terms vanish instead, so we are left with a sine series.

(d) Let n be odd and look at

$$\hat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta)e^{-in\theta}\, d\theta = \frac{1}{2\pi} \int_{-\pi}^{0} f(\theta)e^{-in\theta}\, d\theta + \frac{1}{2\pi} \int_{0}^{\pi} f(\theta)e^{-in\theta}\, d\theta.$$

In the first integral, make the change of variables $\theta \mapsto \theta - \pi$ to get

$$\frac{1}{2\pi} \int_{-\pi}^{0} f(\theta)e^{-in\theta}\, d\theta = \frac{1}{2\pi} \int_{0}^{\pi} f(\theta - \pi)e^{-in(\theta-\pi)}\, d\theta = \frac{e^{\pi in}}{2\pi} \int_{0}^{\pi} f(\theta)e^{-in\theta}\, d\theta = -\frac{1}{2\pi} \int_{0}^{\pi} f(\theta)e^{-in\theta}\, d\theta$$

since $f(\theta - \pi) = f(\theta)$ by periodicity and $e^{\pi in} = (-1)^n = -1$ for odd n. Substituting this back in yields

$$\hat{f}(n) = -\frac{1}{2\pi} \int_{0}^{\pi} f(\theta)e^{-in\theta}\, d\theta. + \frac{1}{2\pi} \int_{0}^{\pi} f(\theta)e^{-in\theta}\, d\theta = 0,$$

as desired.

(e) ($\Longrightarrow$) If f is real-valued, then (we integrate over $[-\pi, \pi] \subseteq \mathbb{R}$, so the complex conjugate passes through)

$$\hat{f}(-n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta)e^{in\theta}\, d\theta = \frac{1}{2\pi} \int_{-\pi}^{\pi} \overline{f(\theta)}e^{in\theta}\, d\theta = \frac{1}{2\pi} \int_{-\pi}^{\pi} \overline{f(\theta)e^{-in\theta}}\, d\theta = \overline{\frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta)e^{-in\theta}\, d\theta} = \overline{\hat{f}(n)}$$

since $f(\theta) = \overline{f(\theta)}$ if and only if $f(\theta) \in \mathbb{R}$.

(e) ($\Longleftarrow$) Now assume that $\overline{\hat{f}(n)} = \hat{f}(-n)$ for all n. By the second equality of the previous calculation this implies that

$$\int_{-\pi}^{\pi} f(\theta)e^{in\theta}\,d\theta = \int_{-\pi}^{\pi} \overline{f(\theta)}e^{in\theta}\,d\theta \implies \int_{-\pi}^{\pi} (f(\theta) - \overline{f(\theta)})e^{in\theta}\,d\theta = 0$$

for every n. Letting $g(\theta) = f(\theta) - \overline{f(\theta)}$, we can rewrite this as $\hat{g}(n) = 0$ for all n. From here we can conclude

$$0 = g(\xi) = f(\xi) - \overline{f(\xi)}$$

whenever g (and thus f) is continuous at ξ. Since f is Riemann integrable, it is continuous almost everywhere, so $f \equiv \overline{f}$ almost everywhere and thus f is real-valued almost everywhere. $\square$

2.1.3 Exercise 2.10

Suppose f is a periodic function of period 2π which belongs to the class C^k. Show that

$$\hat{f}(n) = O(1/|n|^k) \quad \text{as } |n| \to \infty.$$

This notation means that there exists a constant C such that $|\hat{f}(n)| \leq C/|n|^k$. We could also write this as $|n|^k \hat{f}(n) = O(1)$, where $O(1)$ means bounded.

[Hint: Integrate by parts.]

Proof. We proceed by induction on k. When $k = 0$, note that f is continuous on a compact set and thus uniformly continuous, so f is Riemann integrable. Then (noting that $|e^{in\theta}| = 1$)

$$|\hat{f}(n)| = \left|\frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)e^{-in\theta}\,d\theta\right| \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(\theta)|\,d\theta \leq \frac{1}{2\pi}2\pi \max_{-\pi\leq\theta\leq\pi} |f(\theta)| = \max_{-\pi\leq\theta\leq\pi} |f(\theta)|,$$

where we know that the maximum exists since f is continuous on a compact set. Thus $\hat{f}$ is bounded, i.e. $\hat{f}(n) = O(1)$. Now assume for induction that the result holds for some fixed $k \in \mathbb{N}$, i.e. that $\hat{g}(n) = O(1/|n|^k)$ for any 2π-periodic function $g \in C^k$, and we show that it remains true for any 2π-periodic function $f \in C^{k+1}$. Note that $k + 1 \geq 1$ for $k \in \mathbb{N}$, so that f is at least once continuously differentiable. Thus we can integrate by parts (differentiating f and integrating $e^{-in\theta}$) to see that

$$\hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)e^{-in\theta}\,d\theta = \frac{1}{2\pi}\left[f(\theta)\frac{e^{-in\theta}}{-in}\right]_{-\pi}^{\pi} + \frac{1}{2\pi in}\int_{-\pi}^{\pi} f'(\theta)e^{-in\theta}\,d\theta = \frac{1}{in}\widehat{f'}(n),$$

where the first term vanishes since f and $e^{-in\theta}$ are both 2π-periodic. Now since f is 2π-periodic and $(k+1)$-times continuously differentiable, f' is also 2π-periodic but only k-times continuously differentiable. So we can apply the induction hypothesis to conclude that $\widehat{f'}(n) = O(1/|n|^k)$. From here it follows that

$$|\hat{f}(n)| = \left|\frac{1}{in}\widehat{f'}(n)\right| = \frac{1}{|n|}\left|\widehat{f'}(n)\right| \leq \frac{1}{|n|}\frac{C}{|n|^k} = \frac{C}{|n|^{k+1}},$$

for some constant C since $\widehat{f'}(n) = O(1/|n|^k)$. Thus $\hat{f}(n) = O(1/|n|^{k+1})$, and we finish by induction. $\square$