

MATH 3235: Probability Theory

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Chapter 1

Events and Probabilities

1.1 Probability Spaces

Definition 1.1. A *probability space* is a triple $(\Omega, \mathcal{F}, \mathbb{P})$ where

- Ω is called the *sample space* (the set of all possible outcomes of a random experiment);
- $\mathcal{F} \subseteq \mathcal{P}(\Omega)$, called the *event space*,¹ is nonempty and must satisfy:
 - (i) if $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$,
 - (ii) if $A_1, A_2, \dots \in \mathcal{F}$, then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$;
- \mathbb{P} is a probability measure on (Ω, \mathcal{F}) (to be defined later).

Remark. In general, when Ω is finite or countably infinite, one takes $\mathcal{F} = \mathcal{P}(\Omega)$.

Proposition 1.1. *We always have $\emptyset, \Omega \in \mathcal{F}$.*

Proof. Since $\mathcal{F} \neq \emptyset$, there exists some event $A \in \mathcal{F}$. Then we get $A^c \in \mathcal{F}$ and $\Omega = A \cup A^c \in \mathcal{F}$ by the complement and union properties of \mathcal{F} . Finally $\emptyset = \Omega^c \in \mathcal{F}$ by the complement property. \square

Definition 1.2. A *probability measure* on (Ω, \mathcal{F}) is a function $\mathbb{P} : \mathcal{F} \rightarrow [0, \infty)$ such that

- (i) $\mathbb{P}(\Omega) = 1$,
- (ii) and $\mathbb{P}(\bigcup_{k=1}^{\infty} A_k) = \sum_{k=1}^{\infty} \mathbb{P}(A_k)$ whenever $A_1, A_2, \dots \in \mathcal{F}$ are pairwise disjoint.²

Proposition 1.2. *The following properties hold for any probability measure \mathbb{P} on (Ω, \mathcal{F}) :*

- (1) *For any $A \in \mathcal{F}$, we have $\mathbb{P}(A^c) = 1 - \mathbb{P}(A)$.*
- (2) *Let $A, B \in \mathcal{F}$ with $A \subseteq B$. Then $\mathbb{P}(A) \leq \mathbb{P}(B)$.*
- (3) *Let $A, B, C \in \mathcal{F}$. Then*

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B).$$

This is the principle of inclusion-exclusion.

Proof. (1) Observe that $A \cup A^c = \Omega$ and $A \cap A^c = \emptyset$, so $1 = \mathbb{P}(\Omega) = \mathbb{P}(A \cup A^c) = \mathbb{P}(A) + \mathbb{P}(A^c)$.

¹The elements of \mathcal{F} are called *events*. Events with cardinality 1 are called *elementary*.

²i.e. $A_i \cap A_j = \emptyset$ whenever $i \neq j$.

(2) Write $B = A \cup (B \setminus A)$.³ Since $A \cap (B \setminus A) = \emptyset$, we have $\mathbb{P}(B) = \mathbb{P}(A) + \mathbb{P}(B \setminus A) \geq \mathbb{P}(A)$.⁴

(3) Left as an exercise. Follow similar ideas as in (2). \square

Remark. Observe that property (2) implies $\mathbb{P}(A) \leq \mathbb{P}(\Omega) = 1$ since any $A \subseteq \Omega$.

Example 1.2.1. Pick a point uniformly at random from the unit square $\Omega = [0, 1] \times [0, 1]$ and record its coordinates. Then the probability of the point being inside a fixed shape $S \subseteq \Omega$ is $|S|$, the area of S .

Remark. Note that \mathbb{P} only satisfies *countable* additivity. For instance let $\Omega = [0, 1]$ and \mathbb{P} be the uniform measure on Ω . Then $\Omega = \bigcup_{x \in [0, 1]} \{x\}$ and $\mathbb{P}(\{x\}) = 0$ for every $x \in [0, 1]$, but $\mathbb{P}(\Omega) = 1$. This is because the union $\bigcup_{x \in [0, 1]} \{x\}$ is uncountable.

Definition 1.3. Let Ω be finite and $\mathcal{F} = \mathcal{P}(\Omega)$. The uniform probability on (Ω, \mathcal{F}) is the one such that

$$\mathbb{P}(\{\omega\}) = \frac{1}{\text{card } \Omega} \quad \text{for all } \omega \in \Omega.$$

Proposition 1.3. Let \mathbb{P} be the uniform probability on a finite set Ω and let $A \in \mathcal{F}$. Then

$$\mathbb{P}(A) = \frac{\text{card } A}{\text{card } \Omega}.$$

Proof. Note that A is finite since Ω is and so we may enumerate its elements as $A = \{\omega_1, \omega_2, \dots, \omega_n\}$, where $n = \text{card } A$. Then the sets $\{\omega_i\}_{i=1}^n$ are pairwise disjoint and thus we have

$$\mathbb{P}(A) = \mathbb{P}\left(\bigcup_{i=1}^n \{\omega_i\}\right) = \sum_{i=1}^n \mathbb{P}(\{\omega_i\}) = \sum_{i=1}^n \frac{1}{\text{card } \Omega} = \frac{n}{\text{card } \Omega} = \frac{\text{card } A}{\text{card } \Omega},$$

which is the desired result. \square

1.2 Conditional Probability

Definition 1.4. Let $B \in \mathcal{F}$ such that $\mathbb{P}(B) > 0$. Then the *conditional probability* of A given B , written $\mathbb{P}(A|B)$, is given by

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}.$$

Remark. The intuition is that the extra information gained by knowing the occurrence of B should update our computation of the probability of A .

Remark. Another way to think about conditional probability is a restriction of the sample space to B .

Example 1.4.1. Suppose a family has two children, one of which is a girl. What is the probability the other is a girl? Define the sample space to be

$$\Omega = \{(B, G), (B, B), (G, G), (G, B)\}.$$

³Note that $B \setminus A \in \mathcal{F}$ since $B \setminus A = B \cap A^c = (B^c \cup A)^c$.

⁴Since $\mathbb{P} : \mathcal{F} \rightarrow [0, \infty)$, we have $\mathbb{P}(B \setminus A) \geq 0$.

Note that each elementary event is equally likely, i.e.

$$\mathbb{P}(\{(B, G)\}) = \mathbb{P}(\{(G, B)\}) = \mathbb{P}(\{(B, B)\}) = \mathbb{P}(\{(G, G)\}) = \frac{1}{4}.$$

Let $A = \{\text{both of them are } G\} = \{(G, G)\}$ and $B = \{\text{one is a girl}\} = \{(G, B), (B, G), (G, G)\}$. Then

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} = \frac{\mathbb{P}(\{(G, G)\})}{\mathbb{P}(\{(G, B), (B, G), (G, G)\})} = \frac{1/4}{3/4} = \frac{1}{3}.$$

Remark. If we instead condition on the event that one of them is a girl born on a Monday, then the probability changes! Carry out the calculation, and it should be $13/27$ for a 7-day week.

Example 1.4.2. A drug test is 98% accurate, i.e. a drug user tests positive 98% of the time and a non-drug user tests negative 98% of the time. Among a given population, it is known 2% of people use drugs. Suppose I pick a person at random in the population and this person tests positive. What is the probability that the person is a drug user? Define the events

$A = \text{the person is a drug user}$ and $B = \text{the person tests positive}$.

Then the goal is to compute $\mathbb{P}(A|B)$. The 98% accuracy assumption implies that $\mathbb{P}(B|A) = 0.98$. Now

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} = \frac{\mathbb{P}(B|A)\mathbb{P}(A)}{\mathbb{P}(B)}.$$

Note that $B = (B \cap A) \cup (B \cap A^c)$ and this is a disjoint union, so

$$\begin{aligned} \mathbb{P}(B) &= \mathbb{P}(B \cap A) + \mathbb{P}(B \cap A^c) = \mathbb{P}(B|A)\mathbb{P}(A) \\ &\quad + \mathbb{P}(B|A^c)\mathbb{P}(A^c) = 0.98(0.02) + 0.02(0.98) = 2(0.98)(0.02). \end{aligned}$$

Here we noted that the 98% accuracy of the test also implies that $\mathbb{P}(B|A^c) = 0.02$. Thus we get

$$\mathbb{P}(A|B) = \frac{0.98(0.02)}{2(0.98)(0.02)} = \frac{1}{2}.$$

Compute as an exercise that $\mathbb{P}(A^c|B^c) = 0.996$.

Remark. This test is designed clear non-drug users, not to identify drug users.

1.3 Bayes' Theorem

Definition 1.5. A *partition* of Ω is a collection or sequence of events $\{B_k\}_{k=1}^{\infty}$ such that

$$B_i \cap B_j = \emptyset \text{ for } i \neq j \quad \text{and} \quad \Omega = \bigcup_{k=1}^{\infty} B_k.$$

Remark. For any event A , observe that

$$A = A \cap \Omega = A \cap \left(\bigcup_{k=1}^{\infty} B_k \right) = \bigcup_{k=1}^{\infty} (A \cap B_k).$$

Then we get

$$\mathbb{P}(A) = \mathbb{P}\left(\bigcup_{k=1}^{\infty} (A \cap B_k)\right) = \sum_{k=1}^{\infty} \mathbb{P}(A \cap B_k) = \sum_{k=1}^{\infty} \mathbb{P}(A|B_k)\mathbb{P}(B_k).$$

This is the *partition theorem* in the book (Grimmett and Welsh). Now observe that

$$\mathbb{P}(B_i|A) = \frac{\mathbb{P}(B_i \cap A)}{\mathbb{P}(A)} = \frac{\mathbb{P}(B_i \cap A)}{\sum_{k=1}^{\infty} \mathbb{P}(A|B_k)\mathbb{P}(B_k)} = \frac{\mathbb{P}(A|B_i)\mathbb{P}(B_i)}{\sum_{k=1}^{\infty} \mathbb{P}(A|B_k)\mathbb{P}(B_k)}$$

for each $i = 1, 2, \dots$. This is *Bayes' theorem*, which relates posterior probabilities to prior probabilities.

1.4 Conditional Independence

Proposition 1.4. *Let B be such that $\mathbb{P}(B) > 0$. Then $Q : \mathcal{F} \rightarrow [0, 1]$ given by $A \mapsto Q(A) = \mathbb{P}(A|B)$ is a probability measure.*

Proof. (i) Observe that

$$Q(\Omega) = \mathbb{P}(\Omega|B) = \frac{\mathbb{P}(\Omega \cap B)}{\mathbb{P}(B)} = \frac{\mathbb{P}(B)}{\mathbb{P}(B)} = 1.$$

(ii) Let $\{A_k\}_{k=1}^{\infty} \subseteq \mathcal{F}$ be pairwise disjoint. Then observe that we have

$$\begin{aligned} Q\left(\bigcup_{k=1}^{\infty} A_k\right) &= \mathbb{P}\left(\bigcup_{k=1}^{\infty} A_k|B\right) = \frac{\mathbb{P}(\bigcup_{k=1}^{\infty} (A_k \cap B))}{\mathbb{P}(B)} \\ &= \frac{\sum_{k=1}^{\infty} \mathbb{P}(A_k \cap B)}{\mathbb{P}(B)} = \sum_{k=1}^{\infty} \mathbb{P}(A_k|B) = \sum_{k=1}^{\infty} Q(A_k). \end{aligned}$$

Thus Q is indeed a probability measure. □

Definition 1.6. Two events A and B are called *independent*, written $A \perp B$, if $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$.

Example 1.6.1. Let A, B be events with $\mathbb{P}(A) = 0.6$ and $\mathbb{P}(B) = 0.8$. Then $0.4 \leq \mathbb{P}(A \cap B) \leq 0.6$. This is because $A \cap B \subseteq A$ implies $\mathbb{P}(A \cap B) \leq \mathbb{P}(A) = 0.6$. Also noting that $A \cap B \subseteq \Omega$ and so $\mathbb{P}(A \cap B) \leq \mathbb{P}(\Omega) = 1$ implies that

$$\mathbb{P}(A \cap B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cup B) \geq 0.6 + 0.8 - 1 = 0.4.$$

Note that if A and B were independent, then we can immediately conclude $\mathbb{P}(A \cap B) = 0.6(0.8) = 0.48$.

Proposition 1.5. *Assume A and B are events with $0 < \mathbb{P}(A), \mathbb{P}(B) < 1$. The following are equivalent:*

1. A and B are independent,
2. $\mathbb{P}(A|B) = \mathbb{P}(A)$,
3. $\mathbb{P}(B|A) = \mathbb{P}(B)$,
4. $\mathbb{P}(A^c|B) = \mathbb{P}(A^c)$,
5. and $\mathbb{P}(B^c|A) = \mathbb{P}(B^c)$.

Proof. Note that $\mathbb{P}(A|B)\mathbb{P}(B) = \mathbb{P}(A \cap B)$ and $A \perp\!\!\!\perp B$ if and only if $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$. Then use cancellation since $\mathbb{P}(B) \neq 0$. Work out the rest as an exercise. \square

Definition 1.7. We say that three events A, B, C are *independent* if A, B, C are pairwise independent⁵ and $\mathbb{P}(A \cap B \cap C) = \mathbb{P}(A)\mathbb{P}(B)\mathbb{P}(C)$.

Remark. Pairwise independence does not imply independence. Consider flipping a fair coin twice. Let

$$A = \{\text{first flip is } T\}, \quad B = \{\text{second flip is } H\}, \quad C = \{\text{both flips are the same}\}.$$

Then A, B, C are pairwise independent but $\mathbb{P}(A \cap B \cap C) = 0 \neq \mathbb{P}(A)\mathbb{P}(B)\mathbb{P}(C) = 1/8$.

1.5 Continuity of Probability Measures

Remark. We want a system of probability that can say if I flip a fair coin infinitely many times, then

$$\mathbb{P}(\text{never get heads}) = 0.$$

For this experiment, we can set the sample space to be

$$\Omega = \{\text{all sequences like } (H, T, T, \dots)\}.$$

The event space \mathcal{F} is a little complicated, but it includes events like $\{\text{heads on the } n\text{th throw}\}$ and their complements and countable unions. We would like to show that $\mathbb{P}(\{(T, T, T, \dots)\}) = 0$. We know that

$$\mathbb{P}(A_n) = \mathbb{P}(\text{no heads in first } n \text{ tosses}) = \frac{1}{2^n}.$$

As $n \rightarrow \infty$, we can see that $2^{-n} \rightarrow 0$. If we have $\mathbb{P}(A_n) \rightarrow \mathbb{P}(\{(T, T, T, \dots)\})$, then we can conclude that

$$\mathbb{P}(\{(T, T, T, \dots)\}) = \lim_{n \rightarrow \infty} \mathbb{P}(A_n) = \lim_{n \rightarrow \infty} \frac{1}{2^n} = 0.$$

Notice that $A_1 \supseteq A_2 \supseteq A_3 \supseteq \dots$ and $\bigcap_{n=1}^{\infty} A_n = \{(T, T, T, \dots)\}$. For sake of convenience, we will take complements and work with unions: We set

$$B_n = A_n^c = \{\text{at least one heads in first } n \text{ tosses}\},$$

then $B_1 \subseteq B_2 \subseteq B_3 \subseteq \dots$ and $\bigcup_{n=1}^{\infty} B_n = \Omega \setminus \{(T, T, T, \dots)\}$. But this union is not disjoint. To fix this, set $C_i = B_i \setminus B_{i-1}$, then

$$B_1 \cup \bigcup_{n=2}^{\infty} C_n = \Omega \setminus \{(T, T, T, \dots)\},$$

which is now a disjoint union. Taking probabilities, we can use countable additivity to get

$$\mathbb{P}(B_1 \cup C_2 \cup C_3 \cup \dots) = \mathbb{P}(B_1) + \mathbb{P}(C_2) + \mathbb{P}(C_3) + \dots \quad (*)$$

First, note that $B_1 \cup C_2 \cup C_3 \cup \dots = B_1 \cup B_2 \cup B_3 \cup \dots = \Omega \setminus \{(T, T, T, \dots)\}$. Thus in (*), the LHS is $1 - \mathbb{P}(\{(T, T, T, \dots)\})$. Now observe that in the RHS, we have $\mathbb{P}(C_i) = \mathbb{P}(B_i) - \mathbb{P}(B_{i-1})$. Then

$$\begin{aligned} \mathbb{P}(B_1) + \mathbb{P}(C_2) + \mathbb{P}(C_3) + \dots &= \lim_{n \rightarrow \infty} [\mathbb{P}(B_1) + \mathbb{P}(C_2) + \dots + \mathbb{P}(B_n)] \\ &= \lim_{n \rightarrow \infty} [\mathbb{P}(B_1) + (\mathbb{P}(B_2) - \mathbb{P}(B_1)) + \dots + (\mathbb{P}(B_n) - \mathbb{P}(B_{n-1}))] \\ &= \lim_{n \rightarrow \infty} \mathbb{P}(B_n) = \lim_{n \rightarrow \infty} [1 - \mathbb{P}(A_n)] = \lim_{n \rightarrow \infty} \left[1 - \frac{1}{2^n}\right] = 1. \end{aligned}$$

Thus matching the LHS and RHS, we see that $1 - \mathbb{P}(\{(T, T, T, \dots)\}) = 1$ and so $\mathbb{P}(\{(T, T, T, \dots)\}) = 0$.

⁵i.e. $\mathbb{P}(X \cap Y) = \mathbb{P}(X)\mathbb{P}(Y)$ for any $X, Y \in \{A, B, C\}$.

Theorem 1.1 (Continuity of probability measures). *If $\{B_i\}_{i=1}^\infty$ are nested events,⁶ then*

$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} B_i\right) = \lim_{n \rightarrow \infty} \mathbb{P}(B_n).$$

Proof. This was essentially the argument in the previous remark. □

1.6 Homework Problems

Problems #1, 2, 9, 10, 14, 16, 17, 19 from Grimmett and Welsh.

⁶i.e. $B_1 \subseteq B_2 \subseteq B_3 \subseteq \dots$

Chapter 2

Discrete Random Variables

2.1 Probability Mass Functions

Example 2.0.1. Consider the following game: Flip a fair coin 10 times and roll a fair die. I give you
(number of heads) \times (number on die) dollars.

This is a simple game, but it is kind of painful to write in terms of events (e.g. $\mathbb{P}(\text{win} \geq \$10)$). We would have to set

$$\Omega = \{\text{all sequences like } (H, T, H, H, T, T, T, T, T, H, 4)\}$$

and $\mathcal{F} = \mathcal{P}(\Omega)$. It is also not immediately obviously which sequences are in $\{\text{win} \geq \$10\}$. Instead, we would prefer something like

“Let H be the number of heads in 10 fair coin tosses and let R be the outcome of a roll of a fair die. Then you get HR dollars.”

How do we do this in our axiomatic framework? What are H, R ? Here are some observations:

- H, R are real numbers,
- and they are determined by the outcome of the experiment.

Thus we should think of H, R as functions from Ω to \mathbb{R} . These are examples of *discrete random variables*.

Remark. The name “random variable” is just historic. Really, H, R are non-random functions.

Remark. Can every function $X : \Omega \rightarrow \mathbb{R}$ be a discrete random variable? Note that we want to talk about probabilities like $\mathbb{P}(X = 17)$. This indicates that the event

$$\{X = 17\} = \{\omega \in \Omega : X(\omega) = 17\}$$

has to be in \mathcal{F} . So we require that X is *measurable*, i.e. for every $x \in \mathbb{R}$, we have $\{x \in \Omega : X(\omega) = x\} \in \mathcal{F}$. Also H, R must have special properties, for instance they can only take on finitely many values.

Definition 2.1. A function $X : \Omega \rightarrow \mathbb{R}$ is a *discrete random variable* if

- (i) for every $x \in \mathbb{R}$, we have $\{X = x\} \in \mathcal{F}$,
- (ii) and $X(\Omega) = \{x \in \mathbb{R} : x = X(\omega) \text{ for some } \omega\}$ is finite or countably infinite.

Remark. Often, we only care about what values X can take and with what probabilities. We store this data in a special function called the *probability mass function*.

Definition 2.2. Let X be a discrete random variable. Then its *probability mass function (pmf)* is

$$p_X : \mathbb{R} \rightarrow [0, 1] \quad \text{defined by} \quad p_X(s) = \mathbb{P}(X = s).$$

Example 2.2.1. Let X be the outcome of the roll of a fair die. Then

$$p_X(s) = \begin{cases} 1/6 & \text{if } s \in \{1, 2, 3, 4, 5, 6\}, \\ 0 & \text{otherwise.} \end{cases}$$

Remark. Another sentence we want to say is:

“A discrete random variable X takes values $\{1, 7, 9\}$ with probabilities $1/2, 1/3, 1/6$, respectively if and only if

$$p_X(s) = \begin{cases} 1/2 & \text{if } s = 1, \\ 1/3 & \text{if } s = 7, \\ 1/6 & \text{if } s = 9, \\ 0 & \text{otherwise.} \end{cases}$$

How do we know this exists? In other words, does there exist $(\Omega, \mathcal{F}, \mathbb{P})$ and $X : \Omega \rightarrow \mathbb{R}$ with this pmf?

Theorem 2.1. Let $S = \{s_i : i \in I\}$ be a countable subset of \mathbb{R} and let $\{\pi_i : i \in I\}$ be a collection of numbers such that $\pi_i \geq 0$ and

$$\sum_{i \in I} \pi_i = 1.$$

Then there exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a discrete random variable $X : \Omega \rightarrow \mathbb{R}$ such that

$$p_X(s) = \begin{cases} \pi_i & \text{if } s = s_i, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Take $\Omega = S$ and $\mathcal{F} = \mathcal{P}(S)$. Set

$$\mathbb{P}(A) = \sum_{i: s_i \in A} \pi_i$$

and define $X : \Omega \rightarrow \mathbb{R}$ given by $X(\omega) = \omega$. Then one can check that X has the desired pmf. \square

Remark. This allows us to just say

“Let X be a discrete random variable taking these values with these probabilities”

without worrying about the underlying $(\Omega, \mathcal{F}, \mathbb{P})$.

2.2 Common Discrete Random Variables

Example 2.2.2. Some common examples of discrete random variables are:

1. *Constant random variables:* Define $X : \Omega \rightarrow \mathbb{R}$ by $\omega \mapsto X(\omega) = C$.

2. *Bernoulli random variables*: For $0 < p < 1$, we say that $X \sim \text{Ber}(p)$ if

$$X = \begin{cases} 1 & \text{with probability } p, \\ 0 & \text{with probability } q = 1 - p. \end{cases}$$

This models a possibly unfair coin flip. The Bernoulli random variable X has pmf

$$p_X(s) = \begin{cases} p & \text{if } s = 1, \\ 1 - p & \text{if } s = 0, \\ 0 & \text{otherwise.} \end{cases}$$

3. *Binomial random variables*: For $n \in \mathbb{N}^* = \mathbb{N} \setminus \{0\}$ and $0 < p < 1$, we say that $X \sim \text{Bin}(n, p)$ if

$$\mathbb{P}(X = k) = \binom{n}{k} p^k (1 - p)^{n-k}$$

for $k = 0, 1, \dots, n$ and $\mathbb{P}(X = k) = 0$ otherwise. To that this is indeed a pmf, observe that

$$\sum_{k=0}^n \mathbb{P}(X = k) = \sum_{k=0}^n \binom{n}{k} p^k (1 - p)^{n-k} = (p + (1 - p))^n = 1^n = 1.$$

The $n = 1$ case reduces to a Bernoulli random variable.

4. *Geometric random variables*: For $0 < p < 1$, we say that $X \sim \text{Geo}(p)$ if

$$\mathbb{P}(X = k) = p(1 - p)^{k-1}$$

for $k = 1, 2, 3, \dots$ and $\mathbb{P}(X = k) = 0$ otherwise. The above function is clearly nonnegative and

$$\sum_{k=1}^{\infty} p(1 - p)^{k-1} = \frac{p}{1 - (1 - p)} = 1,$$

so this is indeed a pmf. The geometric random variable models the number of independent Bernoulli trials needed to obtain the first success.

Example 2.2.3. Consider the random variable X which counts the number of independent Bernoulli trials needed to get the 4th success. Note that the range of X is $\{4, 5, 6, \dots\}$. Then

$$\mathbb{P}(X = k) = \binom{k-1}{3} p^3 (1 - p)^{k-4} p = \binom{k-1}{3} p^4 (1 - p)^{k-4}$$

for $k = 4, 5, 6, \dots$ and $\mathbb{P}(X = k) = 0$ otherwise. This is because the last trial must be a success and the previous $k - 1$ trials need to contain 3 successes. Here $X = \text{NBin}(n = 4, p)$, the *negative binomial random variable*. In general, $X \sim \text{NBin}(n, p)$ takes on values $n, n + 1, n + 2, \dots$ and

$$\mathbb{P}(X = k) = \binom{k-1}{n-1} p^n (1 - p)^{k-n}$$

for $k = n, n + 1, n + 2, \dots$. Note that the $n = 1$ case reduces to a geometric random variable. The name comes from the binomial theorem with negative exponents.

Example 2.2.4. We say that X is a *Poisson random variable* with parameter $\lambda > 0$, written $X \sim \text{Poi}(\lambda)$, if X takes the values $k = 0, 1, 2, \dots$ with probability mass function

$$p_X(k) = \mathbb{P}(X = k) = \frac{e^{-\lambda} \lambda^k}{k!}.$$

Note that p_X is clearly nonnegative and

$$\sum_{k=0}^{\infty} p_X(k) = \sum_{k=0}^{\infty} \frac{e^{-\lambda} \lambda^k}{k!} = e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} = e^{-\lambda} e^{\lambda} = 1,$$

so p_X is indeed a pmf. One can view the Poisson random variable in the following manner: Suppose $X \sim \text{Bin}(n, p)$ with $n \gg 1$ and $p \ll 1$, e.g. $n = 10^5$ and $p = 10^{-4}$. Then

$$\mathbb{P}(X = 100) = \binom{10^5}{100} \left(\frac{1}{10^4}\right)^{100} \left(1 - \frac{1}{10^4}\right)^{10^5 - 100}.$$

This is very difficult to compute. Instead, we approximate this via the Poisson random variable.

Proposition 2.1. *Let $n \rightarrow \infty$ and $p = p(n) \rightarrow 0$ in such a way that $np(n) \rightarrow \lambda > 0$ as $n \rightarrow \infty$. Then*

$$\binom{n}{k} p^k (1-p)^{n-k} \xrightarrow{n \rightarrow \infty} \frac{e^{-\lambda} \lambda^k}{k!},$$

i.e. $p_X(k) \rightarrow p_Y(k)$ pointwise for $k = 0, 1, 2, \dots$, where $X \sim \text{Bin}(n, p)$ and $Y \sim \text{Poi}(\lambda)$.

Proof. Observe that

$$\begin{aligned} \binom{n}{k} p^k (1-p)^{n-k} &= \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k} = \frac{1}{k!} [n(n-1) \dots (n-k+1) p^k (1-p)^{-k} (1-p)^n] \\ &= \frac{1}{k!} \left[\frac{n(n-1) \dots (n-k+1)}{n^k} n^k p^k (1-p)^{-k} (1-p)^n \right]. \end{aligned}$$

Now notice that $n^k p^k = (np)^k \rightarrow \lambda^k$ since $np \rightarrow \lambda$, and

$$\lim_{n \rightarrow \infty} \frac{n(n-1) \dots (n-k+1)}{n^k} = 1 \quad \text{and} \quad \lim_{p \rightarrow 0} (1-p)^{-k} = 1.$$

Finally, setting $\lambda = np$,

$$(1-p)^n = \left(1 - \frac{\lambda}{n}\right)^n \rightarrow e^{-\lambda}.$$

Putting all of this together, we see that

$$\binom{n}{k} p^k (1-p)^{n-k} \xrightarrow{n \rightarrow \infty} \frac{e^{-\lambda} \lambda^k}{k!},$$

which is the desired result. □

2.3 Expectation of Random Variables

Remark. Suppose $X : \Omega \rightarrow \mathbb{R}$ is a discrete random variable and $h : \mathbb{R} \rightarrow \mathbb{R}$. Then we have:

$$\begin{array}{ccc} \Omega & \xrightarrow{X} & \mathbb{R} \\ & \searrow h(X) & \downarrow h \\ & & \mathbb{R} \end{array}$$

In particular, $h \circ X : \Omega \rightarrow \mathbb{R}$ is also a random variable.

Definition 2.3. Let X be a discrete random variable. The (*mathematical*) *expectation* of X is¹

$$\mathbb{E}[X] = \sum_{x \in \mathcal{R}(X)} xp_X(x)$$

if the above sum exists and converges absolutely,² where p_X is the probability mass function of X .

Remark. When X is discrete, the expectation coincides with the usual notion of a mean. In general, the expectation is some kind of weighted mean.

Remark. Observe that the sum in the definition of $\mathbb{E}[X]$ need not converge. Even worse, if it only converges conditionally, then by the Riemann rearrangement theorem we may get any real value we wish by reordering the sum. This is why we require absolute convergence.

Example 2.3.1. Set $Y = X^2$. Then we have

$$\mathbb{E}[X^2] = \mathbb{E}[Y] = \sum_{y \in \mathcal{R}(Y)} y\mathbb{P}(Y = y) = \sum_{y \in \mathcal{R}(X^2)} y\mathbb{P}(X^2 = y).$$

If we explicitly let

$$X = \begin{cases} 1 & \text{with probability } 1/2, \\ -1 & \text{with probability } 1/2, \end{cases}$$

we see that $\mathbb{E}[X] = 0$. We can also see that $\mathbb{E}[X^2] = 1$ since $X^2 = 1$ with probability 1. Equivalently, we can compute that

$$\mathbb{E}[X^2] = \sum_{y \in \mathcal{R}(X^2)} y\mathbb{P}(X^2 = y) = 1 \cdot \mathbb{P}(X^2 = 1) = 1.$$

Proposition 2.2 (Law of the unconscious statistician). *For any $h : \mathbb{R} \rightarrow \mathbb{R}$ and $X : \Omega \rightarrow \mathbb{R}$ discrete,*

$$\mathbb{E}[h(X)] = \sum_{x \in \mathcal{R}(X)} h(x)p_X(x)$$

where p_X is the pmf of X , provided these sums exist and converge absolutely.

Proof. Let $Y = h(X)$. Then we have

$$\mathbb{E}[h(X)] = \mathbb{E}[Y] = \sum_{y \in \mathcal{R}(Y)} yp_Y(y) = \sum_{y \in \mathcal{R}(Y)} y\mathbb{P}(h(X) = y) = \sum_{x \in \mathcal{R}(X)} h(x)\mathbb{P}(h(X) = y).$$

¹We write $\mathcal{R}(X)$ to denote the range of X .

²i.e. $\sum_{x \in \mathcal{R}(X)} |x|p_X(x) < \infty$.

Note that that y in the last term is $h(x)$, and thus $\mathbb{P}(h(X) = y) = \mathbb{P}(h(X) = h(x)) = p_X(x)$. Then

$$\mathbb{E}[h(X)] = \sum_{x \in \mathcal{R}(X)} h(x)p_X(x),$$

which is precisely the desired result. \square

Remark. In the discrete case, we do not require that h be measurable since $\mathcal{R}(X)$ is at most countable.

Proposition 2.3. *We have the following properties of expectation:*

- (i) If $X \geq 0$, then $\mathbb{E}[X] \geq 0$.
- (ii) If $X = C$ is constant, then $\mathbb{E}[X] = C$.
- (iii) If $\mathbb{E}[aX + bY] = a\mathbb{E}[X] + b\mathbb{E}[Y]$.

Proof. (i) Since $X \geq 0$, we have $\mathcal{R}(X) \subseteq [0, \infty)$ and thus

$$\mathbb{E}[X] = \sum_{x \in \mathcal{R}(X)} xp_X(x) \geq 0$$

since every term in the sum is nonnegative.

(ii) Since $\mathcal{R}(X) = \{C\}$, we have $\mathbb{P}(X = C) = 1$ and thus

$$\mathbb{E}[X] = \sum_{x \in \mathcal{R}(X)} xp_X(x) = C \cdot \mathbb{P}(X = C) = C.$$

This is the desired result.

(iii) We compute that

$$\begin{aligned} \mathbb{E}[aX + bY] &= \sum_{ax+by \in \mathcal{R}(aX+bY)} (ax+by)p_{aX+bY}(ax+by) = \sum_{x \in \mathcal{R}(X)} axp_X(x) + \sum_{y \in \mathcal{R}(Y)} byp_Y(y) \\ &= a \sum_{x \in \mathcal{R}(X)} xp_X(x) + b \sum_{y \in \mathcal{R}(Y)} yp_Y(y) = a\mathbb{E}[X] + b\mathbb{E}[Y], \end{aligned}$$

which is the desired equality. \square

Example 2.3.2. We compute the following:

1. Let $X \sim \text{Ber}(p)$. Then $\mathbb{E}[X] = 0(1-p) + 1p = p$.
2. Let $X \sim \text{Bin}(n, p)$. Then

$$\begin{aligned} \mathbb{E}[X] &= \sum_{k=0}^n k \binom{n}{k} p^k (1-p)^{n-k} = \sum_{k=1}^n k \binom{n}{k} p^k (1-p)^{n-k} = \sum_{k=1}^n k \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k} \\ &= \sum_{k=1}^n \frac{n!}{(k-1)!(n-k)!} p^k (1-p)^{n-k} = n \sum_{k=1}^n \frac{(n-1)!}{(k-1)!(n-k)!} p^k (1-p)^{n-k} \\ &= n \sum_{k=1}^n \binom{n-1}{k-1} p^k (1-p)^{n-k} = np \sum_{k=1}^n \binom{n-1}{k-1} p^{k-1} (1-p)^{n-1-(k-1)} = np. \end{aligned}$$

In the last step we re-index with $j = k-1$, and then recognize the terms as the pmf of a $\text{Bin}(n-1, p)$ random variable, which must sum to 1 over $0 \leq j \leq n-1$.

3. Let $X \sim \text{Geo}(p)$. Then

$$\begin{aligned}\mathbb{E}[X] &= \sum_{k=1}^{\infty} kp(1-p)^{k-1} = p \sum_{k=1}^{\infty} k(1-p)^{k-1} = p \sum_{k=1}^{\infty} \frac{d}{dx} (1-x)^k \Big|_{x=p} = p \frac{d}{dx} \sum_{k=1}^{\infty} (1-x)^k \Big|_{x=p} \\ &= p \frac{d}{dx} \frac{1-x}{1-(1-x)} \Big|_{x=p} = p \frac{d}{dx} \frac{1-x}{x} \Big|_{x=p} = p \frac{d}{dx} \left(1 - \frac{1}{x}\right) \Big|_{x=p} = p \cdot \frac{1}{p^2} = \frac{1}{p}.\end{aligned}$$

The exchange of the sum and derivative is justified since $0 < p < 1$, so we are in the region of uniform convergence of the power series.

2.4 Moments

Recall that by the law of the unconscious statistician, we have $\mathbb{E}[X^2] = \sum_{x \in \mathcal{R}(X)} x^2 p_X(x)$. More generally,

$$\mathbb{E}[X^k] = \sum_{x \in \mathcal{R}(X)} x^k p_X(x)$$

for $k \geq 1$, provided this series converges absolutely.

Definition 2.4. For a random variable X ,

- $\mathbb{E}[X^k]$ is called the *moment of order k* of X ,
- $\mathbb{E}[|X|^k]$ is called the *absolute moment of order k* of X ,
- $\mathbb{E}[(X - \mathbb{E}[X])^k]$ is called the *centered moment of order k* of X .
- and $\mathbb{E}[|X - \mathbb{E}[X]|^k]$ is called the *centered absolute moment of order k* of X .

2.5 Variance

Definition 2.5. Let X be a random variable with finite 2nd moment, i.e. we have $\mathbb{E}[X^2] < \infty$. Then the *variance* of X is given by

$$\text{Var}[X] = \mathbb{E}[(X - \mathbb{E}[X])^2].$$

Example 2.5.1. Define the random variables $X = 0$,

$$Y = \begin{cases} 1 & \text{with probability } 1/2, \\ -1 & \text{with probability } 1/2, \end{cases} \quad \text{and} \quad Z = \begin{cases} 10 & \text{with probability } 1/2, \\ -10 & \text{with probability } 1/2. \end{cases}$$

Then $\mathbb{E}[X] = \mathbb{E}[Y] = \mathbb{E}[Z] = 0$. But observe that we have

$$\begin{aligned}\text{Var}[X] &= \mathbb{E}[(X - \mathbb{E}[X])^2] = \mathbb{E}[X^2] = 0, \\ \text{Var}[Y] &= \mathbb{E}[(Y - \mathbb{E}[Y])^2] = \mathbb{E}[Y^2] = 1, \\ \text{Var}[Z] &= \mathbb{E}[(Z - \mathbb{E}[Z])^2] = \mathbb{E}[Z^2] = 100,\end{aligned}$$

which are not the same.

Remark. The variance is a measure of the spread of a random variable about its mean.

Definition 2.6. The positive square root of $\text{Var}[X]$ is called the *standard deviation* of X .

Proposition 2.4. We have the following properties of variance:

- (i) $\text{Var}[\alpha X] = \alpha^2 \text{Var}[X]$ for all $\alpha \in \mathbb{R}$,
- (ii) $\text{Var}[X] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$,
- (iii) $\text{Var}[X] = \mathbb{E}[X(X-1)] + \mathbb{E}[X] - (\mathbb{E}[X])^2$,
- (iv) and $\text{Var}[X] = 0$ if and only if X is constant.

Proof. (i) We can compute that

$$\text{Var}[\alpha X] = \mathbb{E}[(\alpha X - \mathbb{E}[\alpha X])^2] = \mathbb{E}[\alpha^2(X - \mathbb{E}[X])^2] = \alpha^2 \mathbb{E}[(X - \mathbb{E}[X])^2] = \alpha^2 \text{Var}[X],$$

by the linearity of expectation.

(ii) Again by the linearity of expectation, we have

$$\begin{aligned} \text{Var}[X] &= \mathbb{E}[(X - \mathbb{E}[X])^2] = \mathbb{E}[X^2 - 2X\mathbb{E}[X] + (\mathbb{E}[X])^2] \\ &= \mathbb{E}[X^2] - 2\mathbb{E}[X\mathbb{E}[X]] + \mathbb{E}[(\mathbb{E}[X])^2] \\ &= \mathbb{E}[X^2] - 2\mathbb{E}[X]\mathbb{E}[X] + (\mathbb{E}[X])^2 \end{aligned}$$

since $\mathbb{E}[X]$ and $(\mathbb{E}[X])^2$ are constants.

(iii) Simply write $\mathbb{E}[X(X-1)] = \mathbb{E}[X^2 - X] = \mathbb{E}[X^2] - \mathbb{E}[X]$ and apply (ii).

(iv) (\Leftarrow) If X is constant, then $X = \mathbb{E}[X]$, so

$$\text{Var}[X] = \mathbb{E}[(X - \mathbb{E}[X])^2] = \mathbb{E}[(\mathbb{E}[X] - \mathbb{E}[X])^2] = \mathbb{E}[0] = 0.$$

(\Rightarrow) Suppose $\text{Var}[X] = 0$. Then we find that

$$0 = \text{Var}[X] = \sum_{x \in \mathcal{R}(X)} (x - \mathbb{E}[X])^2 p_X(x).$$

This is a sum of nonnegative terms, so each term must be zero, i.e. $(x - \mathbb{E}[X])^2 p_X(x) = 0$. Since $x \in \mathcal{R}(X)$, we must have $p_X(x) > 0$, and thus $(x - \mathbb{E}[X])^2 = 0$. This gives $x = \mathbb{E}[X]$ for every $x \in \mathcal{R}(X)$, so we conclude that $X = \mathbb{E}[X]$ must be constant. \square

Exercise 2.1. Compute the variance for the following random variables:

- (i) $X \sim \text{Ber}(p)$. The answer should be $\text{Var}[X] = p(1-p)$.
- (ii) $X \sim \text{Bin}(n, p)$. The answer should be $\text{Var}[X] = np(1-p)$.
- (iii) $X \sim \text{Poi}(\lambda)$. The answer should be $\text{Var}[X] = \lambda$.
- (iv) $X \sim \text{Geo}(p)$. We know $\mathbb{E}[X] = 1/p$, what is $\text{Var}[X]$?
- (v) $X \sim \text{NBin}(r, p)$. We know $\mathbb{E}[X] = r/p$, what is $\text{Var}[X]$?

2.6 Conditional Expectation

Definition 2.7. Let X be a random variable and let B be an event such that $\mathbb{P}(B) > 0$. Then the *conditional expectation* of X given B is defined by

$$\mathbb{E}[X|B] = \sum_{x \in \mathcal{R}(X)} x \mathbb{P}(X = x|B) = \sum_{x \in \mathcal{R}(X)} \frac{x \mathbb{P}(\{X = x\} \cap B)}{\mathbb{P}(B)}.$$

Remark. Recall that $\{B_i\}_{i=1}^{\infty}$ is a partition of Ω if the B_i are pairwise disjoint events and $\Omega = \bigcup_{i=1}^{\infty} B_i$.

Theorem 2.2 (Partition theorem in expectation). *Let X be a discrete random variable and $\{B_k\}_{k=1}^{\infty}$ be a partition of Ω with $\mathbb{P}(B_k) > 0$ for each $k \geq 1$. Then*

$$\mathbb{E}[X] = \sum_{k=1}^{\infty} \mathbb{E}[X|B_k] \mathbb{P}(B_k).$$

Proof. Use the definition of conditional expectation to write

$$\begin{aligned} \sum_{k=1}^{\infty} \mathbb{E}[X|B_k] \mathbb{P}(B_k) &= \sum_{k=1}^{\infty} \sum_{x \in \mathcal{R}(X)} x \mathbb{P}(X = x|B_k) \mathbb{P}(B_k) = \sum_{x \in \mathcal{R}(X)} x \sum_{k=1}^{\infty} \mathbb{P}(X = x|B_k) \mathbb{P}(B_k) \\ &= \sum_{x \in \mathcal{R}(X)} x \mathbb{P}(X = x) = \mathbb{E}[X] \end{aligned}$$

by the usual partition theorem. Exchanging the sums is permissible by absolute convergence of $\mathbb{E}[X]$. \square

Remark. One can see this as saying “the expectation of the conditional expectation is the expectation.”

Example 2.7.1. Suppose a coin flips heads with probability p and tails with probability $1 - p$. What is the expected length of the initial run (of consecutive heads if the first flip is heads, or of consecutive tails if the first flip is tails)? Let X be the length of the initial run and H be the event that the first flip is heads. Then

$$\mathbb{P}(X = k|H) = p^{k-1}(1 - p) \quad \text{for } k = 1, 2, \dots$$

Similarly we find

$$\mathbb{P}(X = k|H^c) = (1 - p)^{k-1}p \quad \text{for } k = 1, 2, \dots$$

Since $\{H, H^c\}$ is a partition of Ω , we can use the partition theorem in expectation to write

$$\mathbb{E}[X] = \mathbb{E}[X|H] \mathbb{P}(H) + \mathbb{E}[X|H^c] \mathbb{P}(H^c). \quad (*)$$

We can compute that

$$\mathbb{E}[X|H] = \sum_{x \in \mathcal{R}(X)} x \mathbb{P}(X = x|H) = \sum_{k=1}^{\infty} k p^{k-1} (1 - p) = (1 - p) \sum_{k=1}^{\infty} k p^{k-1} = \frac{1 - p}{(1 - p)^2} = \frac{1}{1 - p}.$$

Similarly we can find

$$\mathbb{E}[X|H^c] = \sum_{x \in \mathcal{R}(X)} x \mathbb{P}(X = x|H^c) = \sum_{k=1}^{\infty} k (1 - p)^{k-1} p = p \sum_{k=1}^{\infty} k (1 - p)^{k-1} = \frac{p}{p^2} = \frac{1}{p}.$$

Thus by substituting these values into (*) we obtain

$$\mathbb{E}[X] = \frac{1}{1-p} \cdot p + \frac{1}{p} \cdot (1-p) = \frac{1}{p(1-p)} - 2.$$

In particular, if $p = 1/2$, then we find that $\mathbb{E}[X] = 2$.

2.7 Homework Problems

Problems #1, 2, 4, 5, 6, 7, 9, 10 from Grimmett and Welsh.

Chapter 3

Multivariate Discrete Random Variables

3.1 Discrete Random Vectors

Definition 3.1. A *random vector* is a function from Ω to \mathbb{R}^d , where $d \geq 2$. We say that the random vector is *bivariate* if $d = 2$, *trivariate* if $d = 3$, etc.

Definition 3.2. A random vector is said to be *discrete* if its range is at most countably infinite.

Example 3.2.1. Let $d = 2$ and X be a 2-dimensional random vector. Then $X : \Omega \rightarrow \mathbb{R}^2$ is given by

$$\omega \mapsto X(\omega) = (X_1(\omega), X_2(\omega)).$$

In particular, each coordinate X_1 and X_2 is a function from Ω to \mathbb{R} and is thus itself a random variable.

Proposition 3.1. A function $X : \Omega \rightarrow \mathbb{R}^d$ with $d \geq 2$ is a random vector if and only if each of its coordinates X_1, X_2, \dots, X_d are random variables.

Proof. Most of this is immediate. More justification is necessary to ensure measurability (i.e. preimages of points are events), but that is the subject of a later course in probability. \square

Proposition 3.2. A random vector $X : \Omega \rightarrow \mathbb{R}^d$ is discrete if and only if each of its component random variables X_1, X_2, \dots, X_d are discrete.

Proof. (\Rightarrow) Observe that there is a surjection $\mathcal{R}(X) \rightarrow \mathcal{R}(X_i)$ by projecting onto the i th coordinate, so $\mathcal{R}(X_i)$ can only be at most countable since $\mathcal{R}(X)$ is.

(\Leftarrow) Notice that $\mathcal{R}(X) \subseteq \mathcal{R}(X_1) \times \mathcal{R}(X_2) \times \dots \times \mathcal{R}(X_d)$. Finite products of countable sets are countable, so $\mathcal{R}(X)$ is a subset of a countable set and thus countable. \square

Definition 3.3. Let $X : \Omega \rightarrow \mathbb{R}^d$ be a discrete random vector. The *probability mass function* of X is

$$p_X(x_1, x_2, \dots, x_d) = \mathbb{P}(X_1 = x_1, X_2 = x_2, \dots, X_d = x_d)$$

for all $(x_1, x_2, \dots, x_d) \in \mathcal{R}(X)$. This probability mass function must satisfy:

- (i) $p_X(x_1, \dots, x_d) \geq 0$,
- (ii) $\sum_{x_1 \in \mathcal{R}(X_1)} \sum_{x_2 \in \mathcal{R}(X_2)} \dots \sum_{x_d \in \mathcal{R}(X_d)} p_X(x_1, \dots, x_d) = 1$,

(iii) and for all $A \subseteq \mathbb{R}^d$,

$$\mathbb{P}(X \in A) = \sum_{x \in A \cap \mathcal{R}(X)} p_X(x_1, \dots, x_d).$$

Note that $A \cap \mathcal{R}(X)$ is countable even when A might not be.

3.2 Marginal Distributions

Definition 3.4. The *one-dimensional marginal pmf* p_{X_k} of a discrete random vector

$$X = (X_1, \dots, X_d) : \Omega \rightarrow \mathbb{R}^d$$

with joint pmf $p_X(x_1, \dots, x_d)$ is given by

$$p_{X_k}(x) = \sum_{x_1 \in \mathcal{R}(X_1)} \cdots \sum_{x_{k-1} \in \mathcal{R}(X_{k-1})} \sum_{x_{k+1} \in \mathcal{R}(X_{k+1})} \cdots \sum_{x_d \in \mathcal{R}(X_d)} p_X(x_1, \dots, x_{k-1}, x, x_{k+1}, \dots, x_d).$$

One can similarly define an *n-dimensional marginal pmf* by summing over all but n of the variables. Note that there are a total of $\binom{d}{n}$ *n-dimensional* marginal pmfs for X .

Remark. Note that we indeed have

$$\sum_{x \in \mathcal{R}(X_k)} p_{X_k}(x) = \sum_{x_1 \in \mathcal{R}(X_1)} \cdots \sum_{x_d \in \mathcal{R}(X_d)} p_X(x_1, \dots, x_d) = 1$$

since we can sum in any order by absolute convergence. A similar thing works for the other marginals.

3.3 Revisiting Expectation

Definition 3.5. Let $h : \mathbb{R}^d \rightarrow \mathbb{R}$ and X be a d -dimensional discrete random vector. Then the *expectation* of $h(X)$ is

$$\mathbb{E}[h(X)] = \sum_{x_1 \in \mathcal{R}(X_1)} \cdots \sum_{x_d \in \mathcal{R}(X_d)} h(x_1, \dots, x_d) p_X(x_1, \dots, x_d),$$

provided that this sum exists and converges absolutely. Here p_X is the joint pmf of X .

Remark. Observe that we have:

$$\begin{array}{ccc} \Omega & \xrightarrow{X} & \mathbb{R}^d \\ & \searrow h(X) & \downarrow h \\ & & \mathbb{R} \end{array}$$

In particular, we see that $h(X)$ is a random variable.

Proposition 3.3. Let X_1, X_2 be two discrete random variables and let $a, b \in \mathbb{R}$. Then

$$\mathbb{E}[aX_1 + bX_2] = a\mathbb{E}[X_1] + b\mathbb{E}[X_2],$$

provided the expectations exist.

Proof. Note that (X_1, X_2) is a random vector and thus has a joint pmf $p_{(X_1, X_2)}$. Define $h : \mathbb{R}^2 \rightarrow \mathbb{R}$ by

$$(x_1, x_2) \mapsto h(x_1, x_2) = ax_1 + bx_2.$$

Then we find that

$$\begin{aligned} \mathbb{E}[h(X_1, X_2)] &= \sum_{x_1 \in \mathcal{R}(X_1)} \sum_{x_2 \in \mathcal{R}(X_2)} h(x_1, x_2) p_{(X_1, X_2)}(x_1, x_2) \\ &= \sum_{x_1 \in \mathcal{R}(X_1)} \sum_{x_2 \in \mathcal{R}(X_2)} (ax_1 + bx_2) p_{(X_1, X_2)}(x_1, x_2) \\ &= \sum_{x_1 \in \mathcal{R}(X_1)} \sum_{x_2 \in \mathcal{R}(X_2)} ax_1 p_{(X_1, X_2)}(x_1, x_2) + \sum_{x_1 \in \mathcal{R}(X_1)} \sum_{x_2 \in \mathcal{R}(X_2)} bx_2 p_{(X_1, X_2)}(x_1, x_2) \\ &= a \sum_{x_1 \in \mathcal{R}(X_1)} x_1 \sum_{x_2 \in \mathcal{R}(X_2)} p_{(X_1, X_2)}(x_1, x_2) + b \sum_{x_1 \in \mathcal{R}(X_1)} x_2 \sum_{x_2 \in \mathcal{R}(X_2)} p_{(X_1, X_2)}(x_1, x_2) \\ &= a \sum_{x_1 \in \mathcal{R}(X_1)} x_1 p_{X_1}(x_1) + b \sum_{x_2 \in \mathcal{R}(X_2)} x_2 p_{X_2}(x_2) = a\mathbb{E}[X_1] + b\mathbb{E}[X_2], \end{aligned}$$

which is the desired result. Manipulating the sums above is justified by absolute convergence. \square

3.4 Independence of Random Variables

Definition 3.6. Two discrete random variables X and Y are said to be *independent*, written $X \perp\!\!\!\perp Y$, if for any $x \in \mathcal{R}(X)$ and $y \in \mathcal{R}(Y)$, the events $\{X = x\}$ and $\{Y = y\}$ are independent, i.e.

$$\mathbb{P}(\{X = x\} \cap \{Y = y\}) = \mathbb{P}(X = x)\mathbb{P}(Y = y).$$

Remark. We will often write $\mathbb{P}(X = x, Y = y)$ instead of $\mathbb{P}(\{X = x\} \cap \{Y = y\})$.

Theorem 3.1. *The discrete random variables X and Y are independent if and only if there exist functions $f, g : \mathbb{R} \rightarrow \mathbb{R}$ such that*

$$p_{(X, Y)}(x, y) = f(x)g(y)$$

for all $x, y \in \mathbb{R}$. Here $p_{(X, Y)}$ is the joint pmf of (X, Y) .

Proof. (\Leftarrow) Assume that $p_{(X, Y)}(x, y) = f(x)g(y)$ for all $x, y \in \mathbb{R}$. Then

$$p_X(x) = \sum_{y \in \mathcal{R}(Y)} p_{(X, Y)}(x, y) = \sum_{y \in \mathcal{R}(Y)} f(x)g(y) = f(x) \sum_{y \in \mathcal{R}(Y)} g(y),$$

and similarly by symmetry we find that

$$p_Y(y) = \sum_{x \in \mathcal{R}(X)} p_{(X, Y)}(x, y) = \sum_{x \in \mathcal{R}(X)} f(x)g(y) = g(y) \sum_{x \in \mathcal{R}(X)} f(x).$$

But $p_{(X, Y)}$ is the joint pmf, so we must have

$$1 = \sum_{x \in \mathcal{R}(X)} \sum_{y \in \mathcal{R}(Y)} f(x)g(y) = \sum_{x \in \mathcal{R}(X)} f(x) \sum_{y \in \mathcal{R}(Y)} g(y).$$

But then we can use this to write

$$\begin{aligned} p_{(X,Y)}(x, y) &= f(x)g(y) = f(x)g(y) \sum_{x \in \mathcal{R}(X)} f(x) \sum_{y \in \mathcal{R}(Y)} g(y) \\ &= \left(f(x) \sum_{y \in \mathcal{R}(Y)} g(y) \right) \left(g(y) \sum_{x \in \mathcal{R}(X)} f(x) \right) = p_X(x)p_Y(y), \end{aligned}$$

where the last line follows from the previous computations. This is the desired result.

(\Leftarrow) This is clear. By independence $p_{(X,Y)}(x, y) = p_X(x)p_Y(y)$, so we can set $f = p_X$ and $g = p_Y$. \square

Proposition 3.4. *Let X and Y be two independent discrete random variables. Then $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$.*

Proof. We can write

$$\begin{aligned} \mathbb{E}[XY] &= \sum_{x \in \mathcal{R}(X)} \sum_{y \in \mathcal{R}(Y)} xyp_{(X,Y)}(x, y) = \sum_{x \in \mathcal{R}(X)} \sum_{y \in \mathcal{R}(Y)} xyp_X(x)p_Y(y) \\ &= \left(\sum_{x \in \mathcal{R}(X)} xp_X(x) \right) \left(\sum_{y \in \mathcal{R}(Y)} yp_Y(y) \right) = \mathbb{E}[X]\mathbb{E}[Y] \end{aligned}$$

where the second step follows by independence. \square

Remark. More generally, the same argument shows that if X, Y are independent, then

$$\mathbb{E}[h_1(X)h_2(Y)] = \mathbb{E}[h_1(X)]\mathbb{E}[h_2(Y)]$$

for any functions $h_1, h_2 : \mathbb{R} \rightarrow \mathbb{R}$.

Definition 3.7. Let X and Y be two random variables. Then

- X, Y are *uncorrelated* if $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$, i.e. $\mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] = 0$,
- X, Y are *positively correlated* if $\mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] > 0$,
- and X, Y are *negatively correlated* if $\mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] < 0$.

Remark. The previous result shows that if X, Y are independent, then they are uncorrelated.

Example 3.7.1. However, the converse is not true in general. Let X take the values $-1, 0, 1$ with probability $1/3$. Clearly $\mathbb{E}[X] = 0$. Now set

$$Y = \begin{cases} 0 & \text{if } X = 0, \\ 1 & \text{if } X \neq 0. \end{cases}$$

First observe that X and Y are dependent since

$$\mathbb{P}(X = 0, Y = 1) = 0 \neq \frac{1}{3} \cdot \frac{2}{3} = \mathbb{P}(X = 0)\mathbb{P}(Y = 1).$$

Now since $\mathbb{E}[X] = 0$, we clearly have $\mathbb{E}[X]\mathbb{E}[Y] = 0$. Also we can compute that

$$\begin{aligned}\mathbb{E}[XY] &= \sum_{x \in \mathcal{R}(X)} \sum_{y \in \mathcal{R}(Y)} xy \mathbb{P}(X = x, Y = y) \\ &= -1 \cdot 1 p_{(X,Y)}(-1, 1) + 1 \cdot 1 p_{(X,Y)}(1, 1) = -\frac{1}{3} + \frac{1}{3} = 0.\end{aligned}$$

In particular, this means that $\mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] = 0 - 0 = 0$, so X and Y are uncorrelated. However, as we previously computed, X and Y are not independent.

Remark. If we instead demand that $\mathbb{E}[h_1(X)h_2(Y)] = \mathbb{E}[h_1(X)]\mathbb{E}[h_2(Y)]$ for all functions $h_1, h_2 : \mathbb{R} \rightarrow \mathbb{R}$, then we are indeed able to conclude that X and Y are independent.

Proposition 3.5. *If $\mathbb{E}[h_1(X)h_2(Y)] = \mathbb{E}[h_1(X)]\mathbb{E}[h_2(Y)]$ for all functions $h_1, h_2 : \mathbb{R} \rightarrow \mathbb{R}$, then X and Y are independent.*

Proof. For each $x_1 \in \mathcal{R}(X)$ and $y_1 \in \mathcal{R}(Y)$, choose $h_1(x) = \mathbb{1}_{\{x_1\}}$ and $h_2(y) = \mathbb{1}_{\{y_1\}}$. Then

$$\mathbb{E}[h_1(X)h_2(Y)] = \mathbb{E}[\mathbb{1}_{\{x_1\}}(X)\mathbb{1}_{\{y_1\}}(Y)] = \mathbb{P}(X = x_1, Y = y_1)$$

and also

$$\mathbb{E}[h_1(X)]\mathbb{E}[h_2(Y)] = \mathbb{E}[\mathbb{1}_{\{x_1\}}(X)]\mathbb{E}[\mathbb{1}_{\{y_1\}}(Y)] = \mathbb{P}(X = x_1)\mathbb{P}(Y = y_1),$$

which implies the desired result since $\mathbb{E}[h_1(X)h_2(Y)] = \mathbb{E}[h_1(X)]\mathbb{E}[h_2(Y)]$. \square

3.5 Convolution and Random Variables

Definition 3.8. The *convolution* of two pmfs p_X, p_Y is given by

$$(p_X * p_Y)(z) = \sum_{x \in \mathcal{R}(X)} p_X(x)p_Y(z - x) = \sum_{y \in \mathcal{R}(Y)} p_X(z - y)p_Y(y).$$

Proposition 3.6. *Let X and Y be two independent discrete random variables with pmfs p_X and p_Y , respectively. Then $X + Y$ is discrete and has pmf given by the convolution $p_X * p_Y$.*

Proof. It is clear that $X + Y$ is discrete, so it suffices to find its pmf. Let X take the values $x \in \mathcal{R}(X)$ and Y take the values $y \in \mathcal{R}(Y)$. Let $Z = X + Y$ take the values $z \in \mathcal{R}(X + Y)$. Then we find that

$$p_Z(z) = \mathbb{P}(Z = z) = \mathbb{P}(X + Y = z) = \mathbb{P}\left(\bigcup_{x \in \mathcal{R}(X)} (\{X = x\} \cap \{Y = z - x\})\right).$$

These events are pairwise disjoint, so we have

$$\begin{aligned}p_Z(z) &= \sum_{x \in \mathcal{R}(X)} \mathbb{P}(\{X = x\} \cap \{Y = z - x\}) \\ &= \sum_{x \in \mathcal{R}(X)} \mathbb{P}(X = x)\mathbb{P}(Y = z - x) = \sum_{x \in \mathcal{R}(X)} p_X(x)p_Y(z - x)\end{aligned}$$

by independence. This is precisely the convolution of p_X and p_Y . \square

Proposition 3.7. Let X_1, \dots, X_n be n independent discrete random variables with pmfs p_{X_1}, \dots, p_{X_n} . Then $X_1 + \dots + X_n$ is discrete and has pmf $p_{X_1} * p_{X_2} * \dots * p_{X_{n-1}} * p_{X_n}$.

Proof. This follows from induction using the previous result and convolution being associative.¹ \square

Example 3.8.1. Let $X \sim \text{Ber}(p)$ and $Y \sim \text{Ber}(p)$, and let X and Y be independent. The pmf of $X + Y$ is the convolution of the pmfs of X and Y . Now $X + Y$ takes the values $z = 0, 1, 2$, and

$$p_{X+Y}(z) = \sum_{x \in \mathcal{R}(X)} p_X(x)p_Y(z-x).$$

For $z = 0$, we have

$$p_{X+Y}(0) = \sum_{x=0}^1 p_X(x)p_Y(-x) = p_X(0)p_Y(0) + p_X(1)p_Y(-1) = (1-p)(1-p) + 0 = (1-p)^2$$

since $p_Y(-1) = 0$. For $z = 1$, we have

$$p_{X+Y}(1) = \sum_{x=0}^1 p_X(x)p_Y(1-x) = p_X(0)p_Y(1) + p_X(1)p_Y(0) = (1-p)p + p(1-p) = 2p(1-p).$$

Finally, when $z = 2$, we see that

$$p_{X+Y}(2) = \sum_{x=0}^1 p_X(x)p_Y(2-x) = p_X(0)p_Y(2) + p_X(1)p_Y(1) = 0 + p(p) = p^2$$

since $p_Y(2) = 0$. In particular, this shows us that $X + Y \sim \text{Bin}(2, p)$. Induct to get the general case.

3.6 Indicator Functions

Definition 3.9. The *indicator function* of a set A is

$$\mathbb{1}_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases}$$

Remark. We have the following properties of indicator functions:

(i) $\mathbb{1}_{A \cap B} = \mathbb{1}_A \mathbb{1}_B$. To see this, we can write

$$(\mathbb{1}_A \mathbb{1}_B)(x) = \mathbb{1}_A(x) \mathbb{1}_B(x) = \begin{cases} 1 & \text{if } x \in A \text{ and } x \in B, \\ 0 & \text{otherwise.} \end{cases}$$

(ii) $\mathbb{1}_A + \mathbb{1}_{A^c} = 1$.

(iii) $\mathbb{1}_{A \cup B} = 1 - \mathbb{1}_{A^c \cap B^c}$.

¹One way to see that convolution is associative is to note that the addition of random variables is associative.

(iv) $\mathbb{1}_{A\Delta B} = \mathbb{1}_A + \mathbb{1}_B \pmod{2}$. Here $A\Delta B = (A \setminus B) \cup (B \setminus A)$ is the *symmetric difference* of A and B .

Remark. Observe that $\mathbb{1}_A$ is a function $\mathbb{1}_A : \Omega \rightarrow \mathbb{R}$. In particular, if A is an event, then $\mathbb{1}_A$ is a random variable and

$$\mathbb{E}[\mathbb{1}_A] = 0 \cdot \mathbb{P}(\mathbb{1}_A = 0) + 1 \cdot \mathbb{P}(\mathbb{1}_A = 1) = \mathbb{P}(\mathbb{1}_A = 1) = \mathbb{P}(A).$$

Using this, we can take expectations on property (ii) above to get

$$\mathbb{E}[\mathbb{1}_A + \mathbb{1}_{A^c}] = \mathbb{E}[\mathbb{1}_A] + \mathbb{E}[\mathbb{1}_{A^c}] = \mathbb{P}(A) + \mathbb{P}(A^c) = 1.$$

Now observe that property (iii) says that

$$\begin{aligned} \mathbb{1}_{A \cup B} &= 1 - \mathbb{1}_{A^c \cap B^c} = 1 - \mathbb{1}_{A^c} \mathbb{1}_{B^c} = 1 - (1 - \mathbb{1}_A)(1 - \mathbb{1}_B) \\ &= 1 - (1 - \mathbb{1}_A - \mathbb{1}_B + \mathbb{1}_A \mathbb{1}_B) = \mathbb{1}_A + \mathbb{1}_B - \mathbb{1}_A \mathbb{1}_B. \end{aligned}$$

From here taking expectations gives

$$\begin{aligned} \mathbb{P}(A \cup B) &= \mathbb{E}[\mathbb{1}_{A \cup B}] = \mathbb{E}[\mathbb{1}_A + \mathbb{1}_B - \mathbb{1}_A \mathbb{1}_B] \\ &= \mathbb{E}[\mathbb{1}_A] + \mathbb{E}[\mathbb{1}_B] - \mathbb{E}[\mathbb{1}_A \mathbb{1}_B] = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B). \end{aligned}$$

More generally, let A_1, \dots, A_n be arbitrary events with indicators $\mathbb{1}_{A_1}, \dots, \mathbb{1}_{A_n}$. Let $A = \bigcup_{i=1}^n A_i$. Then

$$\begin{aligned} \mathbb{1}_A &= \mathbb{1}_{\bigcup_{i=1}^n A_i} = 1 - \mathbb{1}_{\bigcap_{i=1}^n A_i^c} = 1 - \prod_{i=1}^n \mathbb{1}_{A_i^c} = 1 - \prod_{i=1}^n (1 - \mathbb{1}_{A_i}) \\ &= \sum_{i=1}^n \mathbb{1}_{A_i} - \sum_{1 \leq i < j \leq n} \mathbb{1}_{A_i} \mathbb{1}_{A_j} + \sum_{1 \leq i < j < k \leq n} \mathbb{1}_{A_i} \mathbb{1}_{A_j} \mathbb{1}_{A_k} - \dots + (-1)^{n+1} \mathbb{1}_{A_1} \dots \mathbb{1}_{A_n}. \end{aligned}$$

At this point, taking expectations precisely recovers the general inclusion-exclusion formula.

3.7 Homework Problems

Problems #2, 3, 5, 7, 12, 13, 14 from Grimmett and Welsh.

Chapter 4

Probability Generating Functions

4.1 Probability Generating Functions

The probability generating function is a concept for discrete random variables, and more precisely for those X taking values in $\{0, 1, 2, \dots\}$. Such random variables have a pmfs $p_X(k) = p_k = \mathbb{P}(X = k)$ for $k = 0, 1, 2, \dots$. Then $\{p_k\}_{k \geq 0} \subseteq \mathbb{R}$ is a sequence such that $p_k \geq 0$ for each k and $\sum_{k=0}^{\infty} p_k = 1$.

Definition 4.1. Let X be a discrete random variable taking values in $\{0, 1, 2, \dots\}$ with pmf given by $\mathbb{P}(X = k) = p_k$ for $k = 0, 1, 2, \dots$. Then the *probability generating function (pgf)* of X is given by

$$G_X(s) = p_0 + p_1s + p_2s^2 + \dots = \mathbb{E}[s^X],$$

for the values of s for which $\mathbb{E}[|s^X|] = \sum_{k=0}^{\infty} p_k |s|^k < \infty$.

Proposition 4.1. The series $G_X(s) = \sum_{k=0}^{\infty} p_k s^k$ converges absolutely for $s \in \mathbb{R}$ where $|s| \leq 1$.

Proof. Observe that $G_X(s) = \mathbb{E}[s^X]$ and for $|s| \leq 1$,

$$\mathbb{E}[|s^X|] = \sum_{k=0}^{\infty} p_k |s|^k \leq \sum_{k=0}^{\infty} p_k = 1$$

since $p_k = p_X(k)$ is a pmf. Thus the interval of convergence is guaranteed to contain $[-1, 1]$. \square

Example 4.1.1. Consider the following probability generating functions:

- Let $X \sim \text{Ber}(p)$. Then

$$G_X(s) = s^0 \mathbb{P}(X = 0) + s^1 \mathbb{P}(X = 1) = 1(1 - p) + ps = 1 - p + ps.$$

This exists for all $s \in \mathbb{R}$.

- Let $X \sim \text{Bin}(n, p)$. Then

$$G_X(s) = \sum_{k=0}^n s^k \binom{n}{k} p^k (1 - p)^{n-k} = \sum_{k=0}^n \binom{n}{k} (ps)^k (1 - p)^{n-k} = (1 - p + ps)^n.$$

This again exists for all $s \in \mathbb{R}$.

- Let $X \sim \text{Geo}(p)$. Then

$$G_X(s) = \sum_{k=1}^{\infty} p(1-p)^{k-1} s^k = ps \sum_{k=1}^{\infty} (1-p)^{k-1} s^{k-1} = ps \sum_{k=0}^{\infty} ((1-p)s)^k.$$

When $|(1-p)s| < 1$, we get

$$G_X(s) = \frac{ps}{1 - (1-p)s}$$

Thus G_X exists for $|s| < 1/(1-p)$.

- Let $X \sim \text{Poi}(\lambda)$. Then

$$G_X(s) = \sum_{k=0}^{\infty} \frac{e^{-\lambda} \lambda^k}{k!} s^k = e^{-\lambda} \sum_{k=0}^{\infty} \frac{(\lambda s)^k}{k!} = e^{-\lambda} e^{\lambda s} = e^{\lambda(s-1)}.$$

This series converges for all $s \in \mathbb{R}$.

4.2 Properties of PGFs

Theorem 4.1. *Let X and Y have probability generating functions G_X and G_Y , respectively. Then*

$$G_X(s) = G_Y(s) \quad \text{for all } s \text{ in their interval of convergence}$$

if and only if $\mathbb{P}(X = k) = \mathbb{P}(Y = k)$ for all $k = 0, 1, 2, \dots$.

Proof. (\Leftarrow) If $\mathbb{P}(X = k) = \mathbb{P}(Y = k)$ for each k , then

$$G_X(s) = \sum_{k=0}^{\infty} \mathbb{P}(X = k) s^k = \sum_{k=0}^{\infty} \mathbb{P}(Y = k) s^k = G_Y(s).$$

This holds for every s for which the above series converge.

(\Rightarrow) Assume that $G_X(s) = G_Y(s)$ for all $|s| \leq 1$. First observe that plugging in $s = 0$ in $G_X(s)$ and $G_Y(s)$ immediately gives $\mathbb{P}(X = 0) = \mathbb{P}(Y = 0)$. Then because we have absolute convergence for $|s| \leq 1$ and thus uniform convergence on $|s| \leq 1$, we can differentiate term by term to get

$$\frac{d}{ds} G_X(s) = \frac{d}{ds} \sum_{k=0}^{\infty} \mathbb{P}(X = k) s^k = \sum_{k=0}^{\infty} \frac{d}{ds} [\mathbb{P}(X = k) s^k] = \sum_{k=0}^{\infty} \mathbb{P}(X = k) k s^{k-1}$$

and

$$\frac{d}{ds} G_Y(s) = \frac{d}{ds} \sum_{k=0}^{\infty} \mathbb{P}(Y = k) s^k = \sum_{k=0}^{\infty} \frac{d}{ds} [\mathbb{P}(Y = k) s^k] = \sum_{k=0}^{\infty} \mathbb{P}(Y = k) k s^{k-1}.$$

These are equal since $G_X(s) = G_Y(s)$, so we can again evaluate at $s = 0$ to get

$$1 \cdot \mathbb{P}(X = 1) = 1 \cdot \mathbb{P}(Y = 1),$$

i.e. $\mathbb{P}(X = 1) = \mathbb{P}(Y = 1)$. Continue by induction to get $k! \cdot \mathbb{P}(X = k) = k! \cdot \mathbb{P}(Y = k)$ for every $k \geq 1$, so $\mathbb{P}(X = k) = \mathbb{P}(Y = k)$ for each k . This is the desired result. \square

Theorem 4.2. *Let X be an integer-valued random variable with pgf G_X . Then for every $\ell \geq 1$,*

$$\left. \frac{d^\ell}{ds^\ell} G_X(s) \right|_{s=1} = \mathbb{E}[X(X-1)\dots(X-\ell+1)]$$

Proof. As before,

$$\frac{d^\ell}{ds^\ell} G_X(s) = \sum_{k=0}^{\infty} k(k-1)\dots(k-\ell+1)\mathbb{P}(X=k)s^{k-\ell}$$

for all $|s| \leq 1$. Evaluating at $s = 1$ gives

$$\frac{d^\ell}{ds^\ell} G_X(s) = \sum_{k=\ell}^{\infty} k(k-1)\dots(k-\ell+1)\mathbb{P}(X=k) = \mathbb{E}[X(X-1)\dots(X-\ell+1)],$$

as desired. In particular, this expectation is finite if and only if the series on the left converges. \square

Remark. Recall from before that $\text{Var}[X] = \mathbb{E}[X(X-1)] + \mathbb{E}[X] - (\mathbb{E}[X])^2$. We can compute this via

$$\text{Var}[X] = G_X''(1) + G_X'(1) - (G_X'(1))^2$$

using the probability generating function if X is integer-valued.

Remark. The value $\mathbb{E}[X(X-1)\dots(X-\ell+1)]$ is called the *factorial moment of order ℓ* of X .

Theorem 4.3. *Let X, Y be independent random variables. Then $G_{X+Y}(s) = G_X(s)G_Y(s)$ for all $|s| \leq 1$.*

Proof. We have $G_{X+Y}(s) = \mathbb{E}[s^{X+Y}] = \mathbb{E}[s^X s^Y] = \mathbb{E}[s^X]\mathbb{E}[s^Y] = G_X(s)G_Y(s)$ by independence. \square

Remark. Another way to show this is to use the convolution formula for the pmf of $X + Y$.

Corollary 4.3.1. *For independent random variables X_1, \dots, X_n , we have*

$$G_{X_1+\dots+X_n}(s) = \prod_{k=1}^n G_{X_k}(s)$$

Remark. We actually also have the converse: If $G_{X+Y}(s) = G_X(s)G_Y(s)$ for all $|s| \leq 1$, then X and Y are independent. One can show this using the convolution formula.

4.3 The Random Sum Formula

Let $S = X_1 + \dots + X_N$, where X_1, X_2, \dots are independent random variables and N is an integer-valued random variable independent of the X_i . If $N = 0$, then the sum is empty and by convention we say that $X_1 + \dots + X_N = 0$. If $N \geq 1$, then we define

$$(X_1 + \dots + X_N)(\omega) = X_1(\omega) + X_2(\omega) + \dots + X_{N(\omega)}(\omega).$$

Theorem 4.4 (Random sum formula). *For integer-valued independent random variables N, X_1, X_2, \dots ,*

$$G_{X_1+\dots+X_N}(s) = G_N(G_X(s)),$$

given that X_1, X_2, \dots are identically distributed with $p_X = p_{X_1} = p_{X_2} = \dots$.

Proof. Since the events $\{N = n\}$ partition the sample space Ω , by the partition theorem we can write

$$\begin{aligned} G_{X_1+\dots+X_N}(s) &= \mathbb{E}[s^{X_1+\dots+X_N}] = \sum_{n=0}^{\infty} \mathbb{E}[s^{X_1+\dots+X_N} | N = n] \mathbb{P}(N = n) \\ &= \sum_{n=0}^{\infty} \mathbb{E}[s^{X_1+\dots+X_n}] \mathbb{P}(N = n). \end{aligned}$$

Now by independence of the X_i , we can split the expectation and get

$$G_{X_1+\dots+X_N}(s) = \sum_{n=0}^{\infty} G_{X_1}(s) \dots G_{X_n}(s) \mathbb{P}(N = n) = \sum_{n=0}^{\infty} (G_X(s))^n \mathbb{P}(N = n)$$

since the X_i are identically distributed. Now the value on the right is precisely $G_N(G_X(s))$. \square

Remark. We will use “i.i.d.” to denote “independent and identically distributed”.

Remark. Set $S = X_1 + \dots + X_N$, so that $G_S(s) = G_N(G_X(s))$. Now differentiating $G_S(s)$ gives

$$G'_S(s) = G'_N(G_X(s))G'_X(s).$$

At $s = 1$, we get $G'_S(1) = G'_N(G_X(1))G'_X(1) = G'_N(1)G'_X(1)$. But $G'_Y(1) = \mathbb{E}[Y]$, so this says that

$$\mathbb{E}[S] = \mathbb{E}[N]\mathbb{E}[X]$$

One can continue this process. For instance, differentiating again yields

$$G''_S(s) = \frac{d}{ds} [G'_N(G_X(s))G'_X(s)] = G''_N(G_X(s))G'_X(s)G'_X(s) + G'_N(G_X(s))G''_X(s).$$

Evaluating at $s = 1$, we obtain

$$\begin{aligned} \mathbb{E}[S(S-1)] &= G''_S(1) = G''_N(1)(G'_X(1))^2 + G'_N(1)G''_X(1) \\ &= \mathbb{E}[N(N-1)](\mathbb{E}[X])^2 + \mathbb{E}[N]\mathbb{E}[X(X-1)]. \end{aligned}$$

This allows us to find $\text{Var}[S]$ with some more work.

Example 4.1.2. Suppose that we have 20 rabbits, and each has probability $1/2$ of escaping. The next morning, each remaining rabbit gives birth to a number of offspring following a Poisson distribution with parameter $\lambda = 3$, and we want to determine the pgf of the number of offspring. To do this, let N be the number of rabbits not escaping, and observe that $N \sim \text{Bin}(20, 1/2)$. Then

$$G_S(s) = G_{X_1+\dots+X_N}(s) = G_N(G_X(s))$$

by the random sum formula, where $X \sim \text{Poi}(3)$. Then

$$G_N(s) = \left(\frac{1}{2} + \frac{1}{2}s\right)^{20} \quad \text{and} \quad G_X(s) = e^{3(s-1)},$$

so we find that

$$G_S(s) = G_N(G_X(s)) = \left(\frac{1}{2} + \frac{1}{2}e^{3(s-1)}\right)^{20}.$$

In particular, $\mathbb{E}[S] = \mathbb{E}[N]\mathbb{E}[X] = 10(3) = 30$.

4.4 Homework Problems

Problems #2, 5, 6, 8, 9, 10, 11 from Grimmett and Welsh.

Chapter 5

Continuous Random Variables

5.1 Distribution Functions

Remark. When X is discrete, we have the pmf $p_X(x) = \mathbb{P}(X = x)$ for $x \in \mathbb{R}$. Note that $p_X(x) > 0$ for only countably many x , and outside of $\mathcal{R}(X)$ we “set $p_X(x) = 0$ ” (really p_X is only defined on $\mathcal{R}(X)$). We will now study the situation where we have $p_X(x) = 0$ everywhere.

Definition 5.1. Let X be a random variable. Then the (*cumulative*) *distribution function (cdf)* of X is

$$F_X(x) = \mathbb{P}(X \leq x) \quad \text{for } x \in \mathbb{R}.$$

Remark. Note that we can write

$$\mathbb{P}(X \leq x) = \mathbb{P}(\{\omega \in \Omega : X(\omega) \leq x\}) = \mathbb{P}(\{\omega \in \Omega : X(\omega) \in (-\infty, x]\}) = \mathbb{P}(X^{-1}((-\infty, x])).$$

This hints that $X^{-1}((-\infty, x])$ must be in the event space \mathcal{F} .

Example 5.1.1. Suppose that X is discrete. Then

$$F_X(x) = \sum_{y \in \mathcal{R}(X), y \leq x} \mathbb{P}(X = y).$$

- If $X \sim \text{Ber}(p)$, then

$$F_X(x) = \mathbb{P}(X \leq x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 - p & \text{if } 0 \leq x < 1 \\ 1 & \text{if } x \geq 1. \end{cases}$$

- If $X \sim \text{Bin}(n, p)$, then

$$F_X(x) = \begin{cases} 0 & \text{if } x < 0 \\ \sum_{\ell=0}^{k-1} \binom{n}{\ell} p^\ell (1-p)^{n-\ell} & \text{if } k-1 \leq x < k, \text{ for } k = 1, 2, \dots, n, \\ 1 & \text{if } x \geq n. \end{cases}$$

- If $X \sim \text{Geo}(p)$, then

$$F_X(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 - (1-p)^k & \text{if } k \leq x < k+1, \text{ for } k = 0, 1, 2, \dots \end{cases}$$

This is because for $k \leq x < k+1$, we have

$$\mathbb{P}(X \leq x) = \sum_{\ell=1}^k p(1-p)^{\ell-1} = p \sum_{\ell=0}^{k-1} p(1-p)^{\ell} = p \frac{1 - (1-p)^k}{1 - (1-p)} = 1 - (1-p)^k$$

- If $X \sim \text{Poi}(\lambda)$, then

$$F_X(x) = \begin{cases} 0 & \text{if } x < 0 \\ \sum_{k=0}^n e^{-\lambda} \lambda^k / k! & \text{if } n \leq x < n+1, \text{ for } n = 0, 1, 2, \dots \end{cases}$$

Proposition 5.1. *Let X be a random variable with distribution function F_X . Then*

- (i) F_X is a function $F_X : \mathbb{R} \rightarrow [0, 1]$,
- (ii) F_X is non-decreasing, i.e. if $x \leq y$, then $F_X(x) \leq F_X(y)$,
- (iii) $\lim_{x \rightarrow \infty} F_X(x) = 1$ and $\lim_{x \rightarrow -\infty} F_X(x) = 0$,
- (iv) F_X is continuous from the right, i.e. $\lim_{y \rightarrow x^+} F_X(y) = F_X(x)$ for every $x \in \mathbb{R}$,
- (v) F_X has a left-hand limit at every $x \in \mathbb{R}$, i.e. $F_X(x^-) = \lim_{y \rightarrow x^-} F_X(y)$ exists.
- (vi) $\mathbb{P}(X = x) = F_X(x) - F_X(x^-)$,
- (vii) and $\mathbb{P}(X \in [a, b]) = F_X(b) - F_X(a^-)$.

Proof. (i) Under F , we have $x \mapsto F(x) = \mathbb{P}(X \leq x) \in [0, 1]$.

(ii) Observe that $\{X \leq x\} \subseteq \{X \leq y\}$ if $x \leq y$, so $F_X(x) = \mathbb{P}(X \leq x) \leq \mathbb{P}(X \leq y) = F_X(y)$.

(iii) Fix any increasing sequence $\{a_n\}$ with $a_n \rightarrow \infty$. Then $\{X \leq a_n\}$ is an increasing sequence of events and $\bigcup_{n=1}^{\infty} \{X \leq a_n\} = \Omega$ since $a_n \rightarrow \infty$. Thus by the continuity from below of probability measures,

$$\lim_{n \rightarrow \infty} F_X(a_n) = \lim_{n \rightarrow \infty} \mathbb{P}(X \leq a_n) = \mathbb{P}\left(\bigcup_{n=1}^{\infty} \{X \leq a_n\}\right) = \mathbb{P}(\Omega) = 1$$

Since this holds for any increasing sequence $\{a_n\}$, we have $\lim_{x \rightarrow \infty} F_X(x) = 1$. We immediately obtain $\lim_{x \rightarrow -\infty} F_X(x) = 0$ as well by taking complements in the above argument.

(iv) Take sequences again. Note that $\bigcap_{n=1}^{\infty} \{X \leq a_n\} = \{X \leq y\}$ if $\{a_n\}$ is decreasing and $a_n \rightarrow y$.

(v) We informally write

$$\begin{aligned} F_X(x^-) &= \lim_{y \rightarrow x^-} F_X(y) = \lim_{y \rightarrow x^-} \mathbb{P}(X \leq y) = \lim_{y \rightarrow x^-} \mathbb{P}(X \in (-\infty, y]) \\ &= \mathbb{P}(X \in (-\infty, x)) = \mathbb{P}(X < x). \end{aligned}$$

To justify this, one can take sequences and argue as before.

(vi) This is because $F_X(x) - F_X(x^-) = \mathbb{P}(X \leq x) - \mathbb{P}(X < x) = \mathbb{P}(X = x)$.

(vii) Write $\mathbb{P}(X \in [a, b]) = \mathbb{P}(X \leq b) - \mathbb{P}(X < a) = F_X(b) - F_X(a^-)$. □

Remark. Observe that we also have

$$\mathbb{P}(a < X \leq b) = \mathbb{P}(X \in (a, b]) = \mathbb{P}(X \leq b) - \mathbb{P}(X \leq a) = F_X(b) - F_X(a).$$

We get analogous results for the probability of X being in any interval.

5.2 Continuous Random Variables

Definition 5.2. A random variable X is *continuous* if its distribution function F_X is continuous.

Definition 5.3. A random variable X is *absolutely continuous* if F_X is absolutely continuous, i.e.

$$F_X(x) = \int_{-\infty}^x f_X(t) dt$$

for some function $f_X : \mathbb{R} \rightarrow \mathbb{R}$ such that $f_X(x) \geq 0$ for all $x \in \mathbb{R}$ and

$$\int_{-\infty}^{\infty} f_X(x) dx = 1$$

In this case, f_X is called a *(probability) density function (pdf)* of X .

Remark. Note that density functions of X may differ on a set of measure zero, e.g. at a point.

Remark. If X is discrete, then F_X is *not* continuous, and thus certainly not absolutely continuous.

Remark. If X is absolutely continuous with pdf f_X , then

$$\mathbb{P}(X \in (a, b]) = \int_a^b f_X(x) dx.$$

In particular, this says that for all $x \in \mathbb{R}$, we must have

$$\int_{\{x\}} f_X(t) dt = 0.$$

More generally, if X is continuous, then $\mathbb{P}(X = x) = 0$ for every $x \in \mathbb{R}$.

5.3 Common Continuous Random Variables

Example 5.3.1. The following are common continuous random variables:

- *Uniform random variables:* We say that $X \sim \text{Unif}([0, 1])$ if

$$F_X(x) = \begin{cases} 0 & \text{if } x < 0 \\ x & \text{if } 0 \leq x < 1 \\ 1 & \text{if } x \geq 1. \end{cases}$$

In general, for $-\infty < a < b < \infty$, we say that $X \sim \text{Unif}([a, b])$ if $(X - a)/(b - a) \sim \text{Unif}([0, 1])$. In this case, X has pdf

$$f_X(x) = \begin{cases} 1/(b - a) & \text{if } a \leq x \leq b \\ 0 & \text{otherwise.} \end{cases}$$

Note that f_X is continuous and also differentiable everywhere except at $x = a$ and $x = b$.

- *Exponential random variables:* For $\lambda > 0$, we say that $X \sim \text{Exp}(\lambda)$ if X has density function

$$f_X(t) = \begin{cases} \lambda e^{-\lambda t} & \text{if } t > 0 \\ 0 & \text{otherwise.} \end{cases}$$

The cdf of X is

$$F_X(x) = \int_{-\infty}^x f_X(t) dt = \begin{cases} 0 & \text{if } x \leq 0 \\ 1 - e^{-\lambda x} & \text{if } x > 0. \end{cases}$$

Note that F_X is continuous and also differentiable everywhere except $x = 0$.

Remark. The *fundamental theorem of calculus* says that for a fixed $c \in \mathbb{R}$,

$$\frac{d}{dx} \int_c^x f(t) dt = f(x).$$

When applied to distribution functions, this says that

$$\frac{d}{dx} F_X(x) = \frac{d}{dx} \int_{-\infty}^x f_X(t) dt = f_X(x),$$

for almost all $x \in \mathbb{R}$ (whenever F_X is differentiable at x).

Example 5.3.2. A random variable X is said to be *normal* (or *Gaussian*) with parameters $\mu \in \mathbb{R}$ and σ^2 such that $\sigma > 0$, written $X \sim \mathcal{N}(\mu, \sigma^2)$, if its density is given by

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/(2\sigma^2)}$$

for $-\infty < x < \infty$. When $\mu = 0$, $\sigma^2 = 1$, then X is said to be the *standard normal* distribution, i.e.

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}.$$

To verify that this is a density, first note that clearly $f_X(x) \geq 0$. Now set $I = \int_{-\infty}^{\infty} e^{-x^2/2} dx$. Then

$$I^2 = \int_{-\infty}^{\infty} e^{-x^2/2} dx \int_{-\infty}^{\infty} e^{-y^2/2} dy = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)/2} dx dy.$$

This step is justified by the Fubini-Tonelli theorem since $e^{-(x^2+y^2)/2} \geq 0$. Then change to polar coordinates (with $(x, y) = (r \cos \theta, r \sin \theta)$ and $dx dy = r dr d\theta$) to get

$$I^2 = \int_0^{\infty} \int_0^{2\pi} r e^{-r^2/2} dr d\theta = 2\pi \int_0^{\infty} r e^{-r^2/2} dr = 2\pi \left[e^{-r^2/2} \right]_{r=0}^{r=\infty} = 2\pi.$$

Thus $I = \sqrt{2\pi}$ (since $I \geq 0$), so $\int_{-\infty}^{\infty} f_X(x) dx = I/\sqrt{2\pi} = 1$. Thus f_X is a density. The fact that the general pdf f_X is a density follows from a change of coordinates $y = (x - \mu)/\sigma$ with $dx = \sigma dy$, so that

$$\frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{-(x-\mu)^2/(2\sigma^2)} dx = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{-y^2/2} \sigma dy = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-y^2/2} dy = 1,$$

where the latter integral is what we computed previously.

Example 5.3.3. A random variable X has the *standard Cauchy* distribution if its pdf is given by

$$f_X(x) = \frac{1}{\pi(1+x^2)}, \quad -\infty < x < \infty.$$

Clearly we have $f_X(x) \geq 0$, and also we can compute that

$$\int_{-\infty}^{\infty} \frac{1}{\pi(1+x^2)} dx = \frac{1}{\pi} \lim_{N, M \rightarrow \infty} \int_{-M}^N \frac{1}{1+x^2} dx = \frac{1}{\pi} \lim_{N, M \rightarrow \infty} [\arctan x]_{-M}^N = \frac{1}{\pi} \left(\frac{\pi}{2} + \frac{\pi}{2} \right) = 1.$$

5.4 Expectation of Continuous Random Variables

Theorem 5.1 (Law of the subconscious statistician). *Let X be a random variable with pdf f_X , and let $h : \mathbb{R} \rightarrow \mathbb{R}$ be a sufficiently nice function.¹ Let $Y = h(X)$. Then*

$$\mathbb{E}[Y] = \mathbb{E}[h(X)] = \int_{-\infty}^{\infty} h(x)f_X(x) dx,$$

provided that $\int_{-\infty}^{\infty} h(x)f_X(x) dx$ converges absolutely, i.e. $\int_{-\infty}^{\infty} |h(x)|f_X(x) dx < \infty$.²

Proof. Use the Riemann sum approximation. First restrict to a compact interval, e.g. $[0, 1]$. Then

$$\int_0^1 h(x)f_X(x) dx \approx \sum_{i=1}^n h(x_i)f_X(x_i)\Delta x$$

for large n . Now appeal to the discrete version of this result. □

Example 5.3.4. Let $X \sim \mathcal{N}(0, 1)$. Then we would like to find

$$\mathbb{E}[X] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} xe^{-x^2/2} dx.$$

First we must check that this integral converges absolutely. We write

$$\mathbb{E}[|X|] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |x|e^{-x^2/2} dx = \lim_{M_1, M_2 \rightarrow \infty} \frac{1}{\sqrt{2\pi}} \int_{-M_1}^{M_2} |x|e^{-x^2/2} dx.$$

In the finite setting, we can split the integral to get

$$\begin{aligned} \mathbb{E}[|X|] &= \lim_{M_1, M_2 \rightarrow \infty} \left[\frac{1}{\sqrt{2\pi}} \int_0^{M_2} xe^{-x^2/2} dx - \frac{1}{\sqrt{2\pi}} \int_{-M_1}^0 xe^{-x^2/2} dx \right] \\ &= \lim_{M_2 \rightarrow \infty} \left[\frac{1}{\sqrt{2\pi}} (-e^{-x^2/2}) \right]_{x=0}^{x=M_2} + \lim_{M_1 \rightarrow \infty} \left[\frac{1}{\sqrt{2\pi}} e^{-x^2/2} \right]_{x=-M_1}^{x=0} \\ &= \lim_{M_2 \rightarrow \infty} \frac{1}{\sqrt{2\pi}} (1 - e^{-M_2^2/2}) + \lim_{M_1 \rightarrow \infty} \frac{1}{\sqrt{2\pi}} (1 - e^{-M_1^2/2}) = \sqrt{2/\pi}. \end{aligned}$$

¹Here h must be *measurable*.

²The integral of $h(x)f_X(x)$ must be absolutely convergent for its *Lebesgue integral* to exist.

This establishes that $\mathbb{E}[X]$ converges absolutely. Thus we can argue that

$$\mathbb{E}[X] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x e^{-x^2/2} dx = 0$$

since $x e^{-x^2/2}$ is an odd function on \mathbb{R} (we can restrict to $[-M, M]$ and then take $M \rightarrow \infty$).

Example 5.3.5. Now let $X \sim \mathcal{N}(\mu, \sigma^2)$, and we find $\mathbb{E}[X]$. We have

$$\mathbb{E}[X] = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} x e^{-(x-\mu)^2/(2\sigma^2)} dx.$$

Make the change of variables $y = (x - \mu)/\sigma$ with $dx = \sigma dy$ to get

$$\begin{aligned} \mathbb{E}[X] &= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} (\sigma y + \mu) e^{-y^2/2} \sigma dy = \frac{\sigma}{\sqrt{2\pi}} \int_{-\infty}^{\infty} y e^{-y^2/2} dy + \frac{\mu}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-y^2/2} dy \\ &= 0 + \mu = \mu \end{aligned}$$

Example 5.3.6. Let $X \sim \mathcal{N}(0, 1)$, and we find $\mathbb{E}[X^2]$. We have

$$\mathbb{E}[X^2] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^2 e^{-x^2/2} dx.$$

This integrand is nonnegative, so no need to justify absolute convergence. Now integrate by parts with $u(x) = x$ and $v'(x) = x e^{-x^2/2}$ (with $u'(x) = 1$ and $v(x) = -e^{-x^2/2}$) to get

$$\mathbb{E}[X^2] = \frac{1}{\sqrt{2\pi}} \left[-x e^{-x^2/2} \right]_{x=-\infty}^{x=\infty} + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2} dx = 0 + 1 = 1.$$

In particular, this gives $\text{Var}[X] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = 1 - 0 = 1$.

Example 5.3.7. In general, for $X \sim \mathcal{N}(\mu, \sigma^2)$, we can compute

$$\text{Var}[X] = \mathbb{E}[(X - \mathbb{E}[X])^2] = \mathbb{E}[(X - \mu)^2] = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} (x - \mu)^2 e^{-(x-\mu)^2/(2\sigma^2)} dx.$$

Now change variables with $y = (x - \mu)/\sigma$ and $dx = \sigma dy$ to get

$$\text{Var}[X] = \frac{1}{2\pi\sigma^2} \int_{-\infty}^{\infty} \sigma^2 y^2 e^{-y^2/2} \sigma dy = \frac{\sigma^2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} y^2 e^{-y^2/2} dy = \sigma^2.$$

Remark. The previous computations show that for $X \sim \mathcal{N}(\mu, \sigma^2)$, we have $\mathbb{E}[X] = \mu$ and $\text{Var}[X] = \sigma^2$.

Example 5.3.8. Let X be a standard Cauchy random variable. We compute

$$\begin{aligned} \mathbb{E}[|X|] &= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{|x|}{1+x^2} dx \geq \frac{1}{\pi} \int_1^{\infty} \frac{x}{1+x^2} dx = \frac{1}{\pi} \lim_{N \rightarrow \infty} \int_1^N \frac{x}{1+x^2} dx \\ &= \frac{1}{\pi} \lim_{N \rightarrow \infty} \left[\frac{\ln(1+x^2)}{2} \right]_{x=1}^{x=N} = \frac{1}{\pi} \lim_{N \rightarrow \infty} \left[\frac{\ln(1+N^2)}{2} - \frac{\ln 2}{2} \right] = \infty. \end{aligned}$$

In particular, this means that $\mathbb{E}[X]$ does not exist. This is an example of a *heavy-tailed* random variable.

Remark. For any random variable X , if $\mathbb{E}[|X|] = \infty$ then $\mathbb{E}[X^2] = \infty$ also. This is because

$$\mathbb{E}[X^2] = \int_{-\infty}^{\infty} x^2 f_X(x) dx \geq \int_{|x| \geq 1} x^2 f_X(x) dx \geq \int_{|x| \geq 1} |x| f_X(x) dx = \infty,$$

since $x^2 \geq |x|$ for $|x| \geq 1$. Note that $|x| f_X(x)$ is bounded on $|x| \leq 1$, so its integral on $[-1, 1]$ is finite.

5.5 The Gamma and Beta Random Variables

Definition 5.4. Define the *gamma function* Γ for $\alpha > 0$ by

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx.$$

Proposition 5.2. *The gamma function Γ has the following properties:*

- (i) $\Gamma(1) = 1$,
- (ii) $\Gamma(\alpha + 1) = \alpha\Gamma(\alpha)$, and in particular $\Gamma(n) = (n-1)!$ for positive integers n ,
- (iii) and $\Gamma(1/2) = \sqrt{\pi}$.

Proof. (i) We have

$$\Gamma(1) = \int_0^\infty e^{-x} dx = 1.$$

(ii) We can integrate by parts with $u(x) = x^\alpha$ and $v'(x) = e^{-x}$ to get

$$\Gamma(\alpha + 1) = \int_0^\infty x^\alpha e^{-x} dx = [x^\alpha(-e^{-x})]_0^\infty + \alpha \int_0^\infty x^{\alpha-1} e^{-x} dx = \alpha\Gamma(\alpha).$$

The second part follows by induction on n .

(iii) Change variables with $u = \sqrt{x}$ and $dx/\sqrt{x} = du$ to get

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty \frac{e^{-x}}{\sqrt{x}} dx = \int_0^\infty 2e^{-u^2} du = \sqrt{\pi}.$$

Note that one can extend this by induction to compute $\Gamma(\alpha)$ at positive half integers. □

Example 5.4.1. A random variable X has the *standard gamma* distribution if its pdf is given by

$$f_X(x) = \frac{1}{\Gamma(\alpha)} x^{\alpha-1} e^{-x}, \quad x \in (0, \infty)$$

for a constant $\alpha > 0$.³ In general, X is a *gamma random variable* with parameters $\alpha > 0$ and $\lambda > 0$ if

$$f_X(x; \alpha, \lambda) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}, \quad x > 0.$$

Clearly $f_X(x) \geq 0$. For the integral, make the change of variables $y = \lambda x$ with $dx = dy/\lambda$ to get

$$\int_0^\infty \frac{\lambda^\alpha x^{\alpha-1} e^{-\lambda x}}{\Gamma(\alpha)} dx = \frac{1}{\Gamma(\alpha)} \int_0^\infty \lambda^\alpha \left(\frac{y}{\lambda}\right)^{\alpha-1} e^{-y} \frac{dy}{\lambda} = \frac{\lambda^\alpha}{\Gamma(\alpha)\lambda^\alpha} \int_0^\infty y^{\alpha-1} e^{-y} dy = 1.$$

Remark. There are relationships between the gamma, Poisson, and geometric random variables, which are useful in industrial engineering.

³If $\alpha = 1$, then this reduces to the exponential distribution.

Example 5.4.2. A random variable X is said to be a *beta random variable* with parameters $s, t > 0$ if

$$f_X(x; s, t) = \frac{1}{\beta(s, t)} x^{s-1} (1-x)^{t-1}, \quad 0 < x < 1,$$

where β is the *beta function* given by

$$\beta(s, t) = \int_0^1 x^{s-1} (1-x)^{t-1} dx = \frac{\Gamma(s)\Gamma(t)}{\Gamma(s+t)}.$$

Definition 5.5. Taking a gamma random variable with parameters $\alpha = n/2$ and $\lambda = 1/2$, one gets

$$f_X\left(x; \frac{n}{2}, \frac{1}{2}\right) = \frac{1}{2\Gamma(n/2)} \left(\frac{x}{2}\right)^{n/2-1} e^{-x/2}, \quad x > 0,$$

which is called a χ^2 (*chi-squared*) *random variable* with n degrees of freedom.

Remark. The reason we care about χ^2 random variables is that if $Z \sim \mathcal{N}(0, 1)$, then $Z^2 = \chi^2(1)$. In general, if $Z_1, \dots, Z_n \sim \mathcal{N}(0, 1)$ are independent, then $Z_1^2 + \dots + Z_n^2 = \chi^2(n)$.

5.6 Functions of Continuous Random Variables

Remark. Suppose X has pdf f_X and $h : \mathbb{R} \rightarrow \mathbb{R}$, so $Y = h(X)$ is a random variable. If h is “nice,” then Y has a density f_Y . What is the relationship between f_X and f_Y ?

Remark. Note that f_Y need not have a density in general. Let $X \sim \text{Unif}([0, 1])$ and

$$h(x) = \begin{cases} \pi & \text{if } 0 \leq x < 1/2 \\ e & \text{if } 1/2 \leq x \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

Then $Y = h(X)$ only takes two values with positive probability, so it cannot have a density (it is discrete).

Example 5.5.1. Suppose first that $h(x) = ax + b$ with $a > 0$. Then $Y = h(X) = aX + b$, and

$$F_Y(y) = \mathbb{P}(Y \leq y) = \mathbb{P}(aX + b \leq y) = \mathbb{P}\left(X \leq \frac{y-b}{a}\right) = F_X\left(\frac{y-b}{a}\right).$$

Then we can differentiate to get

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy} F_X\left(\frac{y-b}{a}\right) = \frac{1}{a} f_X\left(\frac{y-b}{a}\right).$$

If $a < 0$, then the inequality flips and we get

$$F_Y(y) = \mathbb{P}\left(X \geq \frac{y-b}{a}\right) = 1 - \mathbb{P}\left(X < \frac{y-b}{a}\right) = 1 - F_X\left(\frac{y-b}{a}\right),$$

so differentiating yields

$$f_Y(y) = \frac{d}{dy} F_Y(y) = -\frac{1}{a} f_X\left(\frac{y-b}{a}\right).$$

Note that $a < 0$, so $-a > 0$ and thus $f_Y(y) \geq 0$.

Theorem 5.2. *Let X be an absolutely continuous random variable with density function f_X , and let $h : \mathbb{R} \rightarrow \mathbb{R}$ be strictly increasing and differentiable. Then $Y = h(X)$ has a density function given by*

$$f_Y(y) = f_X(h^{-1}(y)) \frac{d}{dy} h^{-1}(y) \quad \text{for all } y \in \mathbb{R}.$$

Proof. Since h is strictly increasing, applying its inverse preserves the inequality, so that

$$F_Y(y) = \mathbb{P}(h(X) \leq y) = \mathbb{P}(X \leq h^{-1}(y)).$$

Differentiating this via the chain rule gives the desired formula for f_Y . □

Remark. Note that

$$\frac{d}{dy} h^{-1}(y) = \frac{1}{h'(h^{-1}(y))}.$$

This is because $h^{-1}(h(x)) = x$, so differentiating via the chain rule gives

$$1 = \frac{d}{dx} x = \frac{d}{dx} h^{-1}(h(x)) = (h^{-1})'(h(x)) h'(x).$$

Dividing by $h'(x)$ and setting $x = h^{-1}(y)$ gives the desired formula.

Remark. Using the previous theorem, we can create random number generators following a specified distribution, given a random number generator following a uniform distribution on $[0, 1]$.

5.7 Geometric Probability

Example 5.5.2 (Bertrand's paradox). Consider a unit disk U , and choose a chord C of the disk at random. Let T be the equilateral triangle with one side C (there are two, choose the one that points towards the center of the disk). Then what is $\mathbb{P}(T \subseteq U)$? The issue comes from how we choose C :

1. Suppose we choose angles $X, Y \in [0, 2\pi]$ uniformly at random, and then choose the chord C to go through X, Y . This amounts to choose the difference uniformly at random in $[0, 2\pi]$, since the problem is rotationally invariant. We need $|X - Y| \leq 2\pi/3$, so

$$\mathbb{P}(T \subseteq U) = \mathbb{P}(-2\pi/3 \leq X - Y \leq 2\pi/3) = \frac{2}{3}$$

2. Alternatively we could choose a point X at random and then choose a random angle $\theta \in [0, \pi/2]$ (from the tangent line to U at X). Clearly X does not matter, and we need $\theta \leq \pi/3$. This gives

$$\mathbb{P}(T \subseteq U) = \mathbb{P}(\theta \leq \pi/3) = 1/3 = \frac{\pi/3}{\pi/2} = \frac{2}{3}.$$

3. We could also choose a direction θ , which determines a line through the circle. Then choose a point $M \in [-1, 1]$ uniformly at random on this line, which determines a chord if we let M be the midpoint and say that the chord is perpendicular to the line. Then we need $-1/2 \leq M \leq 1/2$, so

$$\mathbb{P}(T \subseteq U) = \mathbb{P}(-1/2 \leq M \leq 1/2) = \frac{1}{2}.$$

4. Finally, we could simply choose the midpoint $M \in U$ at random. We need M to be within the circle of radius $1/2$ with the same center as M . Thus in this case

$$\mathbb{P}(T \subseteq U) = \frac{\pi - \pi/4}{\pi} = \frac{3}{4}.$$

We see that, depending on the method in which we choose the chord, the answer may be different.

Example 5.5.3 (Buffon's needle). Consider horizontal lines in the plane, each spaced distance 1 apart. Drop a needle of length 1 at random. What is the probability that the needle intersects a line?

- One way to proceed is to choose (X, Y) first as the position of the center of the needle. Then choose $\theta \in [0, \pi]$ to be the angle of the needle. Note the actual position of the needle does not matter, so we can simply choose $Y \in [0, 1]$. Now there are two cases:

1. If the needle intersects with the upper line, then we need

$$Y - \frac{1}{2} \sin \theta \leq 1 \leq Y + \frac{1}{2} \sin \theta \iff 1 - \frac{1}{2} \sin \theta \leq Y \leq 1 + \frac{1}{2} \sin \theta.$$

Note that the second inequality is always true, so we only need $Y \geq 1 - \frac{1}{2} \sin \theta$.

2. If the needle intersects with the lower line, then we need

$$Y - \frac{1}{2} \sin \theta \leq 0 \leq Y + \frac{1}{2} \sin \theta \iff -\frac{1}{2} \sin \theta \leq Y \leq \frac{1}{2} \sin \theta.$$

This time the first inequality is always true, so we just need $Y \leq \frac{1}{2} \sin \theta$.

So for the needle to intersect the lines, we need $Y \leq \frac{1}{2} \sin \theta$ or $Y \geq 1 - \frac{1}{2} \sin \theta$. Consider the subset of $[0, \pi] \times [0, 1]$ which satisfies the above condition. There are two symmetric regions of area

$$\int_0^\pi \frac{1}{2} \sin \theta d\theta = \frac{1}{2} [-\cos \theta]_{\theta=0}^{\theta=\pi} = \frac{1}{2} [1 - (-1)] = 1.$$

Thus the area is 2, and the probability is $2/\pi$.

This gives one way to estimate the value of π . Let I_n be the number of times the needle intersects with a line in n trials. Then $I_n/n \rightarrow 2/\pi$ as $n \rightarrow \infty$, so

$$\hat{\pi}_n = \frac{2n}{I_n} \rightarrow \pi.$$

This is called the *Monte Carlo method*, which can also be used to estimate integrals.⁴

Example 5.5.4 (Stick breaking). Choose $X, Y \in [0, 1]$ uniformly at random. This determines two cuts of $[0, 1]$, which breaks the interval into three segments. What is the probability that these segments form the sides of a triangle?

- The pieces are of length $X, Y - X, 1 - Y$ (assume that $X \leq Y$, which we can do since we are choosing uniformly at random). By the triangle inequality, we need

$$\begin{cases} X + (Y - X) \geq 1 - Y \\ (Y - X) + (1 - Y) > X \\ X + (1 - Y) > Y - X \end{cases} \implies \begin{cases} Y > 1/2 \\ X < 1/2 \\ Y < 1/2 + X. \end{cases}$$

Draw the region in $[0, 1] \times [0, 1]$ that satisfies the above, and we find that the probability is $1/4$.

⁴Note that there are much more efficient methods than Monte Carlo for estimating π .

Example 5.5.5. Choose n points on a circle uniformly at random. What is the probability that all n points lie in some semicircle?

- Let A be the event {all n points lie in some semicircle}. The goal is to work with easier events. Number the points by P_1, P_2, \dots, P_n . Then let A_i be the event that P_1, P_2, \dots, P_n lie in the semicircle which begins at P_i . Then we easily find that $\mathbb{P}(A_i) = 1/2^{n-1}$ for each $1 \leq i \leq n$. Also

$$A = A_1 \cup A_2 \cup \dots \cup A_n,$$

where the union is disjoint, since we cannot have two different points both be the first point in a semicircle (note that $\mathbb{P}(P_i = P_j) = 0$ for $i \neq j$). Thus

$$\mathbb{P}(A) = \mathbb{P}(A_1 \cup \dots \cup A_n) = \mathbb{P}(A_1) + \dots + \mathbb{P}(A_n) = n\mathbb{P}(A_1) = \frac{n}{2^{n-1}}$$

since the union is disjoint.

If the problem is more difficult (the union is not disjoint), one can try to apply inclusion-exclusion.

5.8 Homework Problems

Problems #1, 2, 3, 4, 7, 9, 10, 14 from Grimmett and Welsh.