MATH 3235: Probability Theory

Frank Qiang
Instructor: Christian Houdre

Georgia Institute of Technology
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Contents

Chapter 1

Events and Probabilities

1.1 Probability Spaces

Definition 1.1. A *probability space* is a triple $(\Omega, \mathcal{F}, \mathbb{P})$ where

- Ω is called the *sample space* (the set of all possible outcomes of a random experiment);
- $\mathcal{F} \subseteq \mathcal{P}(\Omega)$, called the *event space*,[1] is nonempty and must satisfy:
  (i) if $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$,
  (ii) if $A_1, A_2, \dots \in \mathcal{F}$, then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$;
- $\mathbb{P}$ is a probabilty measure on $(\Omega, \mathcal{F})$ (to be defined later).

Remark. In general, when Ω is finite or countably infinite, one takes $\mathcal{F} = \mathcal{P}(\Omega)$.

Proposition 1.1. *We always have* $\varnothing, \Omega \in \mathcal{F}$.

Proof. Since $\mathcal{F} \neq \varnothing$, there exists some event $A \in \mathcal{F}$. Then we get $A^c \in \mathcal{F}$ and $\Omega = A \cup A^c \in \mathcal{F}$ by the complement and union properties of $\mathcal{F}$. Finally $\varnothing = \Omega^c \in \mathcal{F}$ by the complement property. □

Definition 1.2. A *probability measure* on $(\Omega, \mathcal{F})$ is a function $\mathbb{P} : \mathcal{F} \to [0, \infty)$ such that

(i) $\mathbb{P}(\Omega) = 1$,

(ii) and $\mathbb{P}(\bigcup_{k=1}^{\infty} A_k) = \sum_{k=1}^{\infty} \mathbb{P}(A_k)$ whenever $A_1, A_2, \dots \in \mathcal{F}$ are pairwise disjoint.[2]

Proposition 1.2. *The following properties hold for any probability measure* $\mathbb{P}$ *on* $(\Omega, \mathcal{F})$*:*

(1) For any $A \in \mathcal{F}$*, we have* $\mathbb{P}(A^c) = 1 - \mathbb{P}(A)$.

(2) Let $A, B \in \mathcal{F}$ *with* $A \subseteq B$*. Then* $\mathbb{P}(A) \leq \mathbb{P}(B)$.

(3) Let $A, B, C \in \mathcal{F}$*. Then*
$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B).$$
This is the principle of inclusion-exclusion.

Proof. (1) Observe that $A \cup A^c = \Omega$ and $A \cap A^c = \varnothing$, so $1 = \mathbb{P}(\Omega) = \mathbb{P}(A \cup A^c) = \mathbb{P}(A) + \mathbb{P}(A^c)$.

[1] The elements of $\mathcal{F}$ are called *events*. Events with cardinality 1 are called *elementary*.
[2] i.e. $A_i \cap A_j = \varnothing$ whenever $i \neq j$.

(2) Write $B = A \cup (B \setminus A)$.[3] Since $A \cap (B \setminus A) = \varnothing$, we have $\mathbb{P}(B) = \mathbb{P}(A) + \mathbb{P}(B \setminus A) \geq \mathbb{P}(A)$.[4]

(3) Left as an exercise. Follow similar ideas as in (2). □

Remark. Observe that property (2) implies $\mathbb{P}(A) \leq \mathbb{P}(\Omega) = 1$ since any $A \subseteq \Omega$.

Example 1.2.1. Pick a point uniformly at random from the unit square $\Omega = [0,1] \times [0,1]$ and record its coordinates. Then the probability of the point being inside a fixed shape $S \subseteq \Omega$ is $|S|$, the area of S.

Remark. Note that $\mathbb{P}$ only satisfies *countable* additivity. For instance let $\Omega = [0,1]$ and $\mathbb{P}$ be the uniform measure on Ω. Then $\Omega = \bigcup_{x \in [0,1]} \{x\}$ and $\mathbb{P}(\{x\}) = 0$ for every $x \in [0,1]$, but $\mathbb{P}(\Omega) = 1$. This is because the union $\bigcup_{x \in [0,1]} \{x\}$ is uncountable.

Definition 1.3. Let Ω be finite and $\mathcal{F} = \mathcal{P}(\Omega)$. The uniform probability on $(\Omega, \mathcal{F})$ is the one such that

$$\mathbb{P}(\{\omega\}) = \frac{1}{\operatorname{card} \Omega} \quad \text{for all } \omega \in \Omega.$$

Proposition 1.3. *Let $\mathbb{P}$ be the uniform probability on a finite set Ω and let $A \in \mathcal{F}$. Then*

$$\mathbb{P}(A) = \frac{\operatorname{card} A}{\operatorname{card} \Omega}.$$

Proof. Note that A is finite since Ω is and so we may enumerate its elements as $A = \{\omega_1, \omega_2, \ldots, \omega_n\}$, where $n = \operatorname{card} A$. Then the sets $\{\omega_i\}_{i=1}^n$ are pairwise disjoint and thus we have

$$\mathbb{P}(A) = \mathbb{P}\left(\bigcup_{i=1}^n \{\omega_i\}\right) = \sum_{i=1}^n \mathbb{P}(\{\omega_i\}) = \sum_{i=1}^n \frac{1}{\operatorname{card} \Omega} = \frac{n}{\operatorname{card} \Omega} = \frac{\operatorname{card} A}{\operatorname{card} \Omega},$$

which is the desired result. □

1.2 Conditional Probability

Definition 1.4. Let $B \in \mathcal{F}$ such that $\mathbb{P}(B) > 0$. Then the *conditional probability* of A given B, written $\mathbb{P}(A|B)$, is given by

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}.$$

Remark. The intuition is that the extra information gained by knowing the occurrence of B should update our computation of the probability of A.

Remark. Another way to think about conditional probability is a restriction of the sample space to B.

1.3 Homework Problems

Problems #1, 2, 9, 10, 14 from Grimmett and Welsh.

[3]Note that $B \setminus A \in \mathcal{F}$ since $B \setminus A = B \cap A^c = (B^c \cup A)^c$.

[4]Since $\mathbb{P} : \mathcal{F} \to [0, \infty)$, we have $\mathbb{P}(B \setminus A) \geq 0$.