# MATH 4108: Abstract Algebra II

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# Contents

# Lecture 1

# Jan. 8 — Rings and Fields

## 1.1 Lots of Definitions

Recall the definitions of a ring and a field:

**Definition 1.1** (Ring)**.** A *ring* $R = (R, +, \cdot)$ is a non-empty set $R$ together with two binary operations $+$ and $\cdot$, called addition and multiplication respectively, which satisfy:

(R1) *Associative law for addition*: $(a + b) + c = a + (b + c)$ for all $a, b, c \in R$.

(R2) *Commutative law for addition*: $a + b = b + a$ for all $a, b \in R$.

(R3) *Existence of zero*: There exists $0 \in R$ such that $a + 0 = a$ for all $a \in R$.

(R4) *Existence of additive inverses*: For all $a \in R$, there exists $-a \in R$ such that $a + (-a) = 0$.[1]

(R5) *Associative law for multiplication*: $(ab)c = a(bc)$ for all $a, b, c \in R$.

(R6) *Distributive laws*: $a(b + c) = ab + ac$ and $(a + b)c = ac + bc$ for all $a, b, c \in R$.

**Definition 1.2** (Commutative ring)**.** In this class, we will mostly be interested in *commutative rings*, which satisfy the following additional property for multiplication:

(R7) *Commutative law for multiplication*: $ab = ba$ for all $a, b \in R$.

**Definition 1.3** (Ring with unity)**.** A ring *with unity* satisfies the additional property that

(R8) *Existence of unity*: There exists $1 \neq 0 \in R$ such that and $a1 = 1a = a$ for $a \in R$.

Note that a ring need not be commutative to have a unity.

**Definition 1.4** (Domain)**.** A commutative ring with unity is called a *(integral) domain* if it has the following cancellation property:

(R9) *Cancellation*: For all $a, b \in R$ and $c \neq 0$, $ca = cb$ implies $a = b$.

(R9') *No zero divisors*: For all $a, b \in R$, $ab = 0$ implies $a = 0$ or $b = 0$.

The conditions (R9) and (R9') are equivalent.

**Definition 1.5** (Field)**.** A commutative ring with unity is called a *field* if it has the following additional property for multiplicative inverses:

(R10) *Existence of multiplicative inverses*: For all $a \neq 0 \in R$, there exists $a^{-1} \in R$ such that $aa^{-1} = 1$.

[1] Note that we'll usually write $a - b$ in place of $a + (-b)$.

**Example 1.5.1.** Some examples of rings are $\mathbb{Z}/2\mathbb{Z}$, which also happens to be a field. The ring $\mathbb{Z}$ is a domain. The set $M_{2\times 2}(\mathbb{R})$ is a non-commutative ring with unity, and has zero divisors. The ring $\mathbb{Q}$ is a field.[2] The real polynomials in a single variable $\mathbb{R}[x]$ form a ring, which is a domain but not a field. The complex numbers $\mathbb{C}$ and the real numbers $\mathbb{R}$ both form a field. The even integers $2\mathbb{Z}$ form a commutative ring without unity. In general, $\mathbb{Z}/n\mathbb{Z}$ is a commutative ring with unity, and is a field if and only if $n$ is prime (and has zero divisors otherwise, if $n$ is composite).

**Remark.** If $(R, +, \cdot)$ is a ring, then $(R, +)$ is an abelian group. If $(K, +, \cdot)$ is a field, then $(K^*, \cdot)$ is an abelian group, where $K^* = K \setminus \{0\}$.

**Definition 1.6** (Group of units)**.** Let $R$ be a commutative ring with unity. The *group of units* of $R$ is

$$U = \{u \in R \mid \text{there exists } v \in R \text{ such that } uv = 1\}.$$

**Exercise 1.1.** Show that $U$ is in fact a group under multiplication.

**Definition 1.7** (Associate)**.** If $a, b \in R$ such that $a = ub$ for some $u \in U$, then $a$ and $b$ are called *associates*, denoted by $a \sim b$.

**Exercise 1.2.** Show that $\sim$ is in fact an equivalence relation.

**Example 1.7.1.** The group of units of $\mathbb{Z}$ is $\{1, -1\}$. The group of units of a field $K$ is $K^* = K \setminus \{0\}$.

**Exercise 1.3.** Let $R = \{a + b\sqrt{2} \mid a, b \in \mathbb{Z}\}$. Check the following:

1. $R$ is a commutative ring with unity.
2. The group of units of $R$ is $\{a + b\sqrt{2} \mid a, b \in \mathbb{Z}, |a^2 - 2b^2| = 1\}$.

**Definition 1.8** (Divisor)**.** Let $D$ be an integral domain, $a \in D \setminus \{0\}$, $b \in D$. Then $a$ divides $b$, or $a$ is a *divisor* or *factor* of $b$, denoted by $a|b$, if there exists $z \in D$ such that $az = b$. We write $a \nmid b$ if $a$ does not divide $b$. We say that $a$ is a *proper divisor* or that $a$ *properly divides* $b$ if $z$ is not a unit.

**Remark.** Equivalent, $a$ is a proper divisor of $b$ if and only if $a|b$ and $b \nmid a$.

**Definition 1.9** (Subring)**.** A *subring* $U$ of a ring $R$ is a non-empty subset of $R$ with the property that for all $a, b \in R$, $a, b \in U$ implies $a + b \in U$ and $ab \in U$, and $a \in U$ implies $-a \in U$.

**Remark.** Equivalently, $U$ is a subring of $R$ if and only if $a, b \in U$ implies $a - b \in U$ and $ab \in U$.

**Remark.** We automatically have $0 \in U$ since we can pick any $a \in U$, and then $0 = a - a \in U$.

**Definition 1.10** (Subfield)**.** A *subfield* of a field $K$ is a subset $E$ containing at least two elements such that $a, b \in E$ implies $a - b \in E$ and $a \in E, b \in E \setminus \{0\}$ implies $ab^{-1} \in E$. If $E$ is a subfield and $E \neq K$, then we say $E$ is a *proper* subfield.

**Remark.** As before, we can replace the last condition with the equivalent statement that $a, b \in E$ implies $ab \in E$ and $a \in E \setminus \{0\}$ implies $a^{-1} \in E$.

**Definition 1.11** (Ideal)**.** An *ideal* of $R$ is a non-empty subset $I$ of $R$ with the properties that $a, b \in I$ implies $a - b \in I$ and $a \in I, r \in R$ implies $ra \in I$.

**Remark.** All ideals are subrings, but the converse is not true in general.

**Example 1.11.1.** The integers $\mathbb{Z}$ form a subring of $\mathbb{R}$ but not an ideal.

[2] In fact, $\mathbb{Q}$ is somehow the smallest field containing $\mathbb{Z}$.

**Remark.** We trivially have that $\{0\}$ and $R$ are both ideals of $R$. An ideal $I$ is called *proper* if $\{0\} \subsetneq I \subsetneq R$.

**Theorem 1.1.** *Let $A = \{a_1, \ldots, a_n\}$ be a finite subset of a commutative ring $R$. Then the set*

$$Ra_1 + \cdots + Ra_n = \{x_1a_1 + \cdots + x_na_n \mid x_i \in R\}$$

*is the smallest ideal of $R$ containing $A$.*

*Proof.* See Howie. Check this is indeed an ideal and is contained in any other ideal containing $A$. □

**Definition 1.12** (Ideals generated by elements of a ring)**.** The set $Ra_1 + \cdots + Ra_n$ is the *ideal generated by* $a_1, \ldots, a_n$, denoted by $\langle a_1, \ldots, a_n \rangle$. If the ideal is generated by a single element $a \in R$, then we say that $Ra = \langle a \rangle$ is a *principal ideal*.

**Example 1.12.1.** In $\mathbb{Z}$, the ideal $\langle 2 \rangle = 2\mathbb{Z}$ are the even numbers. We have $\langle 2, 3 \rangle = \mathbb{Z}$, but $\langle 6, 8 \rangle = \langle 2 \rangle$.

**Theorem 1.2.** *Let $D$ be an integral domain with group of units $U$ and let $a, b \in D \setminus \{0\}$. Then*

1. *$\langle a \rangle \subseteq \langle b \rangle$ if and only if $b|a$,*
2. *$\langle a \rangle = \langle b \rangle$ if and only if $a \sim b$,*
3. *$\langle a \rangle = D$ if and only if $a \in U$.*

*Proof.* See Howie. □

**Definition 1.13** (Homomorphism of rings)**.** A *homomorphism* from a ring $R$ to a ring $S$ is a mapping $\varphi : R \to S$ such that $\varphi(a +_R b) = \varphi(a) +_S \varphi(b)$ and $\varphi(ab) = \varphi(a)\varphi(b)$ for all $a, b \in R$.

**Example 1.13.1.** The zero mapping $\varphi(a) = 0$ is always a homomorphism. The inclusion map $\iota : 2\mathbb{Z} \to \mathbb{Z}$ or $\iota : \mathbb{Z} \to \mathbb{Q}$ is a homomorphism.

**Theorem 1.3.** *Let $R, S$ be rings and $\varphi : R \to S$ a homomorphism. Then*

1. *$\varphi(0_R) = 0_S$,*
2. *$\varphi(-r) = -\varphi(r)$ for all $r \in R$,*
3. *the image $\varphi(R)$ is a subring of $S$.*

*Proof.* See Howie. □

**Definition 1.14** (Monomorphism)**.** Let $\varphi : R \to S$ be a homomorphism. If $\varphi$ is injective, we say that $\varphi$ is a *monomorphism* or an *embedding*.

**Example 1.14.1.** The inclusion map $\varphi : \mathbb{Z} \to \mathbb{R}$ given by $\varphi(n) = n$ is an embedding.

# Lecture 2

# Jan. 10 — Field of Fractions, Polynomials

## 2.1 Isomorphisms

**Definition 2.1** (Isomorphism)**.** If a homomorphism $\varphi : R \to S$ is both one-to-one and onto, then $\varphi$ is an *isomorphism* and we say $R$ and $S$ are *isomorphic*, denoted $R \cong S$.

**Definition 2.2** (Automorphism)**.** An isomorphism $\varphi : R \to R$ is called an *automorphism.*

**Example 2.2.1.** For any ring $R$, the identity map $\varphi : R \to R$ with $\varphi = \mathrm{id}$ is an automorphism.

**Exercise 2.1.** The complex conjugation $\varphi : \mathbb{C} \to \mathbb{C}$ with $\varphi(z) = \overline{z}$ is an automorphism.

**Definition 2.3** (Kernel)**.** Let $\varphi : R \to S$ be a homomorphism. The *kernel* of $\varphi$ is

$$\ker \varphi = \phi^{-1}(0_S) = \{a \in R : \varphi(a) = 0_S\}.$$

**Exercise 2.2.** For any homomorphism $\varphi$, $\ker \varphi$ is an ideal.

**Definition 2.4** (Residue class)**.** Let $I$ be an ideal of a ring $R$ and $a \in R$. The set

$$a + I = \{a + x \mid x \in I\}$$

is the *residue class* of $a$ modulo $I$.

**Exercise 2.3.** The set $R/I$ of residue classes modulo $I$ forms a ring with respect to the operations

$$(a + I) + (b + I) = (a + b) + I \quad \text{and} \quad (a + I)(b + I) = ab + I.$$

**Exercise 2.4.** The map $\theta_I : R \to R/I$ with $\theta_I(a) = a + I$ is a surjective homomorphism onto $R/I$ with kernel $I$. This map $\theta_I$ is called the *natural homomorphism* from $R$ to $R/I$.

**Example 2.4.1.** Consider $\mathbb{Z}$ and $I = \langle n \rangle = n\mathbb{Z}$. Then $\theta_I : \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ with $\theta_I(a) = a + \langle n \rangle$ is the natural homomorphism. There are $n$ residue classes, which are

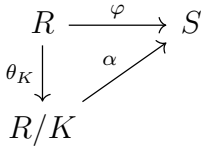
$$\langle n \rangle, \quad 1 + \langle n \rangle, \quad \ldots, \quad (n-1) + \langle n \rangle.$$

**Theorem 2.1.** *Let $n \in \mathbb{Z}_{>0}$. Then $\mathbb{Z}/n\mathbb{Z}$ is a field if and only if $n$ is prime.*

*Proof.* See Howie. □

**Remark.** If $n = 0$, then $\mathbb{Z}/0\mathbb{Z} \cong \mathbb{Z}$.

**Theorem 2.2.** *Let $\varphi : R \to S$ be a surjective homomorphism with kernel $K$. Then there is an isomorphism $\alpha : R/K \to S$ such that the following diagram commutes (i.e. $\varphi = \alpha \circ \theta_K$):*




*Proof.* See Howie. But the general idea is to define $\alpha : R/K \to S$ by $\alpha(a + K) = \varphi(a)$. Then need to check that $\alpha$ is well-defined and an isomorphism. □

## 2.2 Field of Fractions

The motivating question is: How do we get from $\mathbb{Z}$ to $\mathbb{Q}$? Recall that

$$\mathbb{Q} = \{a/b \mid a, b \in \mathbb{Z}, b \neq 0\},$$

where $a/c = b/d$ if $ad = bc$. We add and multiply fractions by

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd} \quad \text{and} \quad \frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}.$$

How do we do this more generally (construct a field out of an arbitrary integral domain)?

**Definition 2.5** (Field of fractions of a domain)**.** Let $D$ be an integral domain and

$$P = D \times (D \setminus \{0\}) = \{(a, b) \mid a, b \in D, b \neq 0.\}$$

Define an equivalence relation $\equiv$ on $P$ by $(a, b) \equiv (a', b')$ if $ab' = a'b$. Then the *field of fractions* of $D$ is

$$Q(D) = P/\equiv.$$

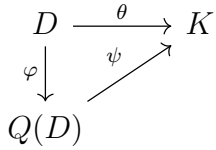
We denote the equivalence class $[a, b]$ by $a/b$, i.e. $a/b = c/d$ if $ad = bc$. We define addition and multiplication on $Q(D)$ by

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd} \quad \text{and} \quad \frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}.$$

**Exercise 2.5.** Do the following:

1. Check that $\equiv$ is an equivalence relation.
2. Check that these operations are well-defined.
3. Check that $Q(D)$ is a commutative ring with unity.
   - The zero element is $0/b$ for $b \neq 0$.
   - The unity element is $a/a$ for $a \neq 0$.
   - The negative of $a/b$ is $(-a)/b$ or equivalently $a/(-b)$.
   - The multiplicative inverse of $a/b$ is $b/a$ for $a, b \neq 0$.
4. Complete the previous exercise and check that $Q(D)$ is a field.

**Exercise 2.6.** The map $\varphi : D \to Q(D)$ defined by $\varphi(a) = a/1$ is a monomorphism. In particular, the field of fractions $Q(D)$ contains $D$ as a subring and $Q(D)$ is the smallest field containing $D$, in the sense that if $K$ is a field with the property that there exists a monomorphism $\theta : D \to K$, then there exists a monomorphism $\psi : Q(D) \to K$ such that the following diagram commutes:




## 2.3 The Characteristic of a Field

Note that for $a \in R$, we might write $a + a$ as $2a$ and $a + a + \cdots + a$ ($n$ times) as $na$. Furthermore, $0a = 0_R$ and $(-n)a = n(-a)$ for $n \in \mathbb{Z}_{>0}$. Thus $na$ has meaning for all $n \in \mathbb{Z}$.[1]

**Exercise 2.7.** For $a, b \in R$ and $m, n \in \mathbb{Z}$, we have $(ma)(nb) = (mn)(ab)$.

**Definition 2.6** (Characteristic of a ring)**.** For an arbitrary ring $R$, there are two possibilities:

1. $m1_R$ for $m \in \mathbb{Z}$ are all distinct. In this case, we say that $R$ has *characteristic* 0.
2. There exists $m, n \in \mathbb{N}$ such that $m1_R = (m + n)1_R$. In this case, we say that $R$ has *characteristic* $n$, where $n$ is the least positive $n$ for which this property holds.

We denote the characteristic of $R$ by $\operatorname{char} R$. If $\operatorname{char} R = n$, then $na = 0_R$ for all $a \in R$ since

$$na = (n1_R)a = 0a = 0.$$

**Example 2.6.1.** We have $\operatorname{char} \mathbb{Z}/n\mathbb{Z} = n$.

**Theorem 2.3.** *The characteristic of a field is either* 0 *or a prime.*

*Proof.* Let $K$ be a field and suppose $\operatorname{char} K = n \neq 0$ and $n$ is not prime. Then we can write $n = rs$ where $1 < r, s < n$. The minimal property of $n$ implies that $r1_K \neq 0$ and $s1_K \neq 0$. But then

$$r1_K \cdot s1_K = rs1_K = n1_K = 0,$$

which is impossible since $K$ is a field and thus has no zero divisors. □

**Remark.** Note the following:

1. If $K$ is a field with $\operatorname{char} K = 0$, then $K$ has a subring isomorphic to $\mathbb{Z}$, i.e. elements of the form $n1_K$ for $n \in \mathbb{Z}$, and $K$ has a subfield isomorphic to $\mathbb{Q}$, i.e.

   $$P(K) = \{m1_K/n1_K \mid m, n \in \mathbb{Z}, n \neq 0\}.$$

   This is the *prime subfield* of $K$, and any subfield of $K$ must contain $P(K)$.

2. If $K$ is a field with $\operatorname{char} K = p$, then the prime subfield of $K$ is

   $$P(K) = \{1_K, 2 \cdot 1_K, \ldots, (p-1) \cdot 1_K\},$$

   which is isomorphic to $\mathbb{Z}/p\mathbb{Z}$.

[1] This is saying that any abelian group is naturally a *module* over the integers $\mathbb{Z}$.

**Remark.** In other words, every field of characteristic 0 is an *extension* of $\mathbb{Q}$ (contains $\mathbb{Q}$ as a subfield), and every field of characteristic $p$ is an *extension* of $\mathbb{Z}/p\mathbb{Z}$ (contains $\mathbb{Z}/p\mathbb{Z}$ as a subfield).

**Remark.** If $\operatorname{char} K = 0$, then writing $a/n1_K$ as $a/n$ is fine. But if $\operatorname{char} K = p$, then $a/n$ does not make sense when $p|n$ (since $p \cdot 1_K = 0$).

**Theorem 2.4.** *If $K$ is a field with $\operatorname{char} K = p$, then for all $x, y \in K$, $(x+y)^p = x^p + y^p$.*

*Proof.* See Howie. Uses the binomial theorem. □

## 2.4 Polynomials

Let $R$ be a ring, then we have the polynomial ring over $R$

$$R[X] = \{a_0 + a_1X + \cdots + a_nX^n \mid a_i \in R, n \in \mathbb{N}\}.$$

If $f \in R[X]$, then it has *degree* $n$ if the last nonzero element in the sequence $\{a_0, a_1, \dots\}$ is $a_n$, denoted $\partial f = n$. By convention, the zero polynomial has degree $-\infty$. The coefficient $a_n$ is called the *leading coefficient*, and if $a_n = 1$, then $f$ is *monic*. Addition and multiplication work as expected:

$$(a_0 + a_1X + \cdots + a_mX^m) + (b_0 + b_1X + \cdots + b_nX^n) = (a_0 + b_0) + (a_1 + b_1)X + \dots$$

and

$$(a_0 + a_1X + \cdots + a_mX^m)(b_0 + b_1X + \cdots + b_nX^n) = c_0 + c_1X + \dots$$

where

$$c_k = \sum_{i+j=k}^{k} a_ib_j.$$

The ground ring $R$ sits inside of the polynomial ring $R[X]$. Take the monomorphism $\theta : R \to R[X]$ by $\theta(a) = a$, i.e. an element $a$ maps to the constant polynomial $a$.

**Theorem 2.5.** *Let $D$ be an integral domain. Then*

1. *$D[X]$ is an integral domain.*
2. *If $p, q \in D[X]$, then $\partial(p+q) \leq \max(\partial p, \partial q)$.*
3. *If $p, q \in D[X]$, then $\partial(pq) = \partial p + \partial q$.*
4. *The group of units of $D[X]$ coincides with the group of units of $D$.*

*Proof.* Statements (2) and (3) are left as exercises.

(1) We need to show that $D[X]$ has no zero divisors. For this, suppose that $p, q$ are nonzero polynomials with leading coefficients $a_m$ and $b_n$ respectively. Then the leading coefficient of $pq$ is $a_mb_n$, which is nonzero since $D$ is an integral domain and thus has no zero divisors. So $pq$ is nonzero.

(4) Let $p, q \in D[X]$ and suppose $pq = 1$. Since $\partial(pq) = \partial(1) = 0$, we must have $\partial p = \partial q = 0$. Thus $p, q \in D$ and $pq = 1$ if and only if $p$ and $q$ are in the group of units of $D$. □

Since $D[X]$ is a domain, we can consider polynomials in the variable $Y$ with coefficients in $D[X]$:

$$D[X, Y] = (D[X])[Y].$$

We can repeat this to get polynomials in $n$ variables: $D[X_1, X_2, \dots, X_n]$, which is an integral domain.

# Lecture 3

# Jan. 17 — Irreducible Polynomials

## 3.1 Principal Ideal Domains and Irreducibile Polynomials

**Definition 3.1.** The field of fractions of $D[X]$ consists of *rational forms*

$$\frac{a_0 + a_1X + \cdots + a_mX^m}{b_0 + b_1X + \cdots + b_nX^n}$$

where $b_0 + b_1X + \cdots + b_nX^n \neq 0$, denoted by $D(X)$.

**Definition 3.2.** A domain $D$ is a *principal ideal domain* (PID) if all of its ideals are principal.[1]

**Example 3.2.1.** The integers $\mathbb{Z}$ is a PID, since every ideal is of the form $\langle n \rangle$.

**Definition 3.3.** A non-zero, non-unit element $p$ in a domain $D$ is *irreducible* if it has no proper factors.

**Definition 3.4.** A domain $D$ is a *unique factorization domain* (UFD) if every non-unit $a \neq 0$ in $D$ has an essentially unique[2] factorization into irreducible elements.

**Example 3.4.1.** Again $\mathbb{Z}$ is a UFD, e.g. $12 = 2 \cdot 2 \cdot 3 = (-2) \cdot 2 \cdot (-3)$.

**Theorem 3.1.** *Every PID is a UFD.*

*Proof.* See Howie. □

**Theorem 3.2.** *If $K$ is a field, then $K[X]$ is a PID.*

*Proof.* See Howie. □

**Theorem 3.3.** *Let $p$ be an element in a PID $D$. Then the following are equivalent:*

1. *$p$ is irreducible.*
2. *$\langle p \rangle$ is maximal.*
3. *$D/\langle p \rangle$ is a field.*

*In particular if $f \in K[X]$, then $K[X]/\langle f \rangle$ is a field if and only if $f$ is irreducible.*

*Proof.* See Howie. □

[1] Recall that a principal ideal is one generated by a single element.
[2] As in, unique up to use of associates or adding in units.

**Definition 3.5.** Let $D$ be a domain and $\alpha \in D$. Let $\sigma_\alpha : D[X] \to D$ defined by

$$\sigma_\alpha(a_0 + a_1X + \cdots + a_nX^n) = a_0 + a_1\alpha + \cdots + a_n\alpha^n.$$

Note that we often write $\sigma_\alpha(f)$ as $f(\alpha)$. If $f(\alpha) = 0$, we say $\alpha$ is a *root* of $f$, or a *zero*.

**Exercise 3.1.** Check that $\sigma_\alpha$ is a homomorphism.

**Theorem 3.4.** *Let $K$ be a field, $\beta \in K$ and $f$ a non-zero polynomial in $K[X]$. Then $\beta$ is a root of $f$ if and only if $X - \beta | f$.*

*Proof.* See Howie. □

**Example 3.5.1.** We have $X^2 + 1$ in $\mathbb{R}[X]$ is irreducible, so $\mathbb{R}[X]/\langle X^2 + 1\rangle$ is a field. In fact this field is isomorphic to the complex numbers $\mathbb{C}$.

**Exercise 3.2.** Do the following:

1. Show that $\varphi : \mathbb{R}[X] \to \mathbb{C}$ given by

$$\varphi(a_0 + a_1X + \cdots + a_nX^n) = a_0 + a_1i + \cdots + a_ni^n$$

   is a surjective homomorphism.[3]

2. Show that $\ker\varphi = \langle X^2 + 1\rangle$.

So by the first isomorphism theorem we can conclude that $\mathbb{R}[X]/\langle X^2 + 1\rangle = \mathbb{R}/\ker\varphi \cong \varphi(\mathbb{R}[X]) = \mathbb{C}$.

**Theorem 3.5.** *Let $K$ be a field and $g \in K[X]$ an irreducible polynomial. Then $K[X]/\langle g\rangle$ is a field containing $K$ up to isomorphism.*

*Proof.* Since $g$ is irreducible, $K[X]/\langle g\rangle$ is a field. Now define $\varphi : K \to K[X]/\langle g\rangle$ by

$$\varphi(a) = a + \langle g\rangle.$$

(Left as an exercise to check that $\varphi$ is a homomorphism.) We need to show that $\varphi$ is injective. For this, take $a, b \in K$. If $a + \langle g\rangle = b + \langle g\rangle$, then $a - b \in \langle g\rangle$. But $K$ is a field, so this happens precisely when $a = b$. Thus $\varphi$ embeds $K$ into $K[X]/\langle g\rangle$, as desired. □

## 3.2 Irreducible Polynomials over $\mathbb{C}$, $\mathbb{R}$, $\mathbb{Q}$, and $\mathbb{Z}$

Our goal now is to study irreducible polynomials. Note that linear polynomials are irreducible, and recall that every polynomial in $\mathbb{C}$ factorizes, essentially uniquely, into linear factors. Furthermore, complex roots of real polynomials come in conjugate pairs, hence

$$g = a_0 + a_1X + \cdots + a_nX^n \in \mathbb{R}[X]$$

factors as

$$g = a_n(X - \beta_1)\ldots(X - \beta_r)(X - \gamma_1)(X - \overline{\gamma}_1)\ldots(X - \gamma_3)(X - \overline{\gamma}_s)$$

[3]Note that there's some technicality about this $\varphi$ not being a $\sigma_\alpha$ since we defined $\sigma_\alpha$ for $\alpha$ in the base domain, and $i$ is kind of somewhere else.

in $\mathbb{C}[X]$, where $\beta_1, \ldots, \beta_r \in \mathbb{R}$ and $\gamma_1, \ldots, \gamma_s \in \mathbb{C} \setminus \mathbb{R}$ and $r + 2s = n$. Thus over $\mathbb{R}[X]$, $g$ factors as

$$g = a_n(X - \beta_1) \ldots (X - \beta_r)(X^2 - (\gamma_1 + \overline{\gamma}_1)X + \gamma_1\overline{\gamma}_1) \ldots (X^2 - (\gamma_s + \overline{\gamma}_s)X + \gamma_s\overline{\gamma}_s)$$

in $\mathbb{R}[X]$, where the quadratic factors are irreducible in $\mathbb{R}[X]$.

**Exercise 3.3.** A quadratic $aX^2 + bX + c \in \mathbb{R}[X]$ is irreducible if and only if its discriminant $b^2 - 4ac < 0$.

Now we have pretty much characterized irreducible polynomials in $\mathbb{R}[X]$. But what about $\mathbb{Q}[X]$?

**Theorem 3.6.** *Let $g = a_0 + a_1X + a_2X^2 \in \mathbb{Q}[X]$. Then*

1. *If $g$ is irreducible over $\mathbb{R}$, then it is irreducible over $\mathbb{Q}$.*
2. *If $g = a_2(X - \beta_1)(X - \beta)$ with $\beta_1, \beta_2 \in \mathbb{R}$, then $g$ is irreducible in $\mathbb{Q}[X]$ if and only if $\beta_1$ and $\beta_2$ are irrational.*

*Proof.* (1) We show the contrapositive. If $g$ factors as

$$g = a_2(X - q_1)(X - q_2) \in \mathbb{Q}[X],$$

then $g$ also factors in $\mathbb{R}[X]$.

(2) If $\beta_1$ and $\beta_2$ are rational, then $g$ factors in $\mathbb{Q}[X]$ and is thus not irreducible. For the other direction, if $\beta_1$ and $\beta_2$ are irrational, then $g = a_2(X - \beta_1)(X - \beta_2)$ is the only factorization in $\mathbb{R}[X]$ since $\mathbb{R}[X]$ is a UFD, so there is no factorization in $\mathbb{Q}[X]$ into linear factors. □

**Example 3.5.2.** Are the following polynomials irreducible in $\mathbb{R}[X]$? In $\mathbb{Q}[X]$?

1. $X^2 + X + 1$ is irreducible over $\mathbb{R}$ and $\mathbb{Q}$ since $b^2 - 4ac = -3$.
2. $X^2 - X - 1$ has roots $(-1 \pm \sqrt{5})/2$, so it factors over $\mathbb{R}$ but is irreducible over $\mathbb{Q}$.
3. $X^2 + X - 2$ factors as $(X + 2)(X - 1)$ over $\mathbb{R}$ and $\mathbb{Q}$.

Now that we have studied irreducible polynomials in $\mathbb{R}[X]$ and $\mathbb{Q}[X]$, can a polynomial in $\mathbb{Z}[X]$ be irreducible over $\mathbb{Z}$ but not $\mathbb{Q}$? The answer is no!

**Theorem 3.7** (Gauss's lemma)**.** *Let $f$ be a polynomial in $\mathbb{Z}[X]$, irreducible over $\mathbb{Z}$. Then $f$ is irreducible over $\mathbb{Q}$.*

*Proof.* For sake of contradiction, suppose $f = gh$ with $g, h \in \mathbb{Q}[X]$ and $\partial g, \partial h < \partial f$. Then there exists $n \in \mathbb{Z}_{>0}$ such that $nf = g'h'$ where $g', h' \in \mathbb{Z}[X]$. Let $n$ be the smallest positive integer with this property. Let

$$g' = a_0 + a_1X + \cdots + a_kX^k$$
$$h' = b_0 + b_1X + \cdots + b_lX^l.$$

If $n = 1$, then $g' = g$ and $h' = h$, a contradiction. Now $n \geq 1$, so let $p$ be a prime factor of $n$.[4] Without loss of generality, assume $p$ divides $g'$, i.e. $g' = pg''$ where $g'' \in \mathbb{Z}[X]$. Then

$$\frac{n}{p}f = g''h',$$

contradicting the minimality of $n$. Hence $f$ cannot be factored over $\mathbb{Q}$. □

[4]Lemma: Either $p$ divides all the coefficients of $g'$ or $p$ divides all the coefficients of $h'$. Proof left as an exercise.

**Example 3.5.3.** Show that $g = X^3 + 2X^2 + 4X - 6$ is irreducible over $\mathbb{Q}$.

*Proof.* If $g$ factors over $\mathbb{Q}$, it factors over $\mathbb{Z}$ and at least one factor must be linear, i.e.

$$g = X^3 = 2X^2 + 4X - 6 = (X - a)(X^2 + bX + c)$$

where $a, b, c \in \mathbb{Z}$. We must have $ac = 6$, so $a \in \{\pm 1, \pm 2, \pm 3, \pm 6\}$ and $g(a) = 0$. We can check this:

| $a$ | $1$ | $-1$ | $2$ | $-2$ | $3$ | $-3$ | $-6$ | $6$ |
|---|---|---|---|---|---|---|---|---|
| $g(a)$ | $1$ | $-9$ | $1$ | $-10$ | $51$ | $-27$ | $306$ | $-174$ |

Hence $g$ is irreducible over $\mathbb{Z}$ and thus also irreducible over $\mathbb{Q}$. □

We could do this trick since the degree was 3, forcing a linear factor. What about degrees higher than 3?

**Theorem 3.8** (Eisenstein's criterion)**.** *Let $f = a_0 + a_1X + \cdots + a_nX^n \in \mathbb{Z}[X]$. Suppose there exists a prime $p$ such that*

1. *$p \nmid a_n$,*
2. *$p | a_i$ for $i = 0, \ldots, n-1$,*
3. *$p^2 \nmid a_0$.*

*Then $f$ is irreducible over $\mathbb{Q}$.*

*Proof.* By Gauss's lemma, it suffices to show that $f$ is irreducible over $\mathbb{Z}$. Suppose for sake of contradiction that $f = gh$ for

$$g = b_0 + b_1X + \cdots + b_rX^r \quad \text{and} \quad h = c_0 + c_1X + \cdots + c_sX^s,$$

$r, s < n$, and $r + s = n$. Note that $a_0 = b_0c_0$, so $p | a_0$ from (2) implies that $p | b_0$ or $p | c_0$. Since $p^2 \nmid a_0$, it cannot be both. Without loss of generality, assume $p | b_0$ and $p \nmid c_0$. Now suppose inductively that $p$ divides $b_0, \ldots, b_{k-1}$ where $1 \leq k \leq r$. Then

$$a_k = b_0c_k + b_1c_{k-1} + \cdots + b_{k-1}c_1 + b_kc_0$$

and since $p$ divides $a_k$, $b_0c_k$, $b_1c_{k-1}$, …, $b_{k-1}c_1$, it follows that $p | b_kc_0$. Since $p \nmid c_0$ by assumption, we must have $p | b_k$. Thus $p | b_r$ and since $a_n = b_rc_s$, we have $p | a_n$, contradicting (1). Hence is $f$ is irreducible. □

**Example 3.5.4.** The polynomial

$$X^5 + 2X^3 + \frac{8}{7}X^2 - \frac{4}{7}X + \frac{2}{7}$$

is irreducible over $\mathbb{Q}$.

*Proof.* Multiply by 7 and take the integer polynomial $7X^5 + 14X^3 + 8X^2 - 4X + 2$. Taking $p = 2$ satisfies Eisenstein's criterion, so this polynomial is irreducible over $\mathbb{Z}$ and thus also irreducible over $\mathbb{Q}$. □

**Example 3.5.5.** If $p > 2$ is prime, then show that

$$f = 1 + X + X^2 + \cdots + X^{p-1}$$

is irreducible over $\mathbb{Q}$.

*Proof.* First observe that

$$f = \frac{X^p - 1}{X - 1}.$$

Let $g(X) = f(X + 1)$. Then

$$\begin{aligned} g(X) &= \frac{(X+1)^p - 1}{(X+1) - 1} = \frac{1}{X}((X+1)^p - 1) = \frac{1}{X}\sum_{i=0}^{p} \binom{p}{i} X^{p-i} - 1 \\ &= \frac{1}{X}\sum_{i=0}^{p-1} \binom{p}{i} X^{p-i} = \sum_{i=0}^{p-1} \binom{p}{i} X^{p-i-1}. \end{aligned}$$

Note that $\binom{p}{1}, \binom{p}{2}, \ldots \binom{p}{p-1}$ are all divisible by $p$, so $g$ is irreducible by Eisenstein's criterion. Now if $f$ factors as $f = uv$, then $g(X) = u(X + 1)v(X + 1)$, which is a contradiction since $g$ is irreducible. □

# Lecture 4

# Jan. 22 — Field Extensions

## 4.1 More on Irreducibility

The following excerpt is from Howie:

> Another device for determining irreducibility over $\mathbb{Z}$ (and consequently over $\mathbb{Q}$) is to map the polynomial onto $\mathbb{Z}_p[X]$ for some suitably chosen prime $p$. Let $g = a_0 + a_1X + \cdots + a_nX^n \in \mathbb{Z}[X]$, and let $p$ be a prime not dividing $a_n$. For each $i$ in $\{0, 1, \ldots, n\}$, let $\overline{a}_i$ denote the residue class $a_i + \langle p \rangle$ in the field $\mathbb{Z}_p = \mathbb{Z}/\langle p \rangle$, and write the polynomial $\overline{a}_0 + \overline{a}_1X + \cdots + \overline{a}_nX^n$ as $\overline{g}$. Our choice of $p$ ensures that $\partial\overline{g} = n$. Suppose that $g = uv$, with $\partial u, \partial v < \partial f$ and $\partial u + \partial v = \partial g$. Then $\overline{g} = \overline{u}\,\overline{v}$. If we can show that $\overline{g}$ is irreducible in $\mathbb{Z}_p[X]$, then we have a contradiction, and we deduce that $g$ is irreducible. The advantage of transferring the problem from $\mathbb{Z}[X]$ to $\mathbb{Z}_p[X]$ is that $\mathbb{Z}_p$ is finite, and the verification of irreducibility is a matter of checking a finite number of cases.

**Example 4.0.1.** Show that

$$g = 7X^4 + 10X^3 - 2X^2 + 4X - 5$$

is irreducible over $\mathbb{Q}$.

*Proof.* Let $p = 3$ and

$$\overline{g} = X^4 + X^3 + X^2 + 1$$

This has no linear factors since

$$\overline{g}(0) = 1, \quad \overline{g}(1) = 2, \quad \overline{g}(-1) = 1.$$

So suppose

$$\overline{g} = X^4 + X^3 + X^2 + X + 1 = (X^2 + aX + b)(X^2 + cX + d)$$

in $\mathbb{Z}_3[x]$. Then for some $a, b, c, d \in \mathbb{Z}_3 = \{-1, 0, 1\}$, we have

$$\begin{cases} X^3 & a + c = 1 \\ X^2 & b + ac + d = 1 \\ X & ad + bc = 1 \\ 1 & bd = 1 \end{cases}$$

The first case is if $b = d = 1$, but this implies $ac = -1$, so $a = \pm 1$ and $c = \mp 1$. But $a + c = 1$, so this cannot happen. The second case is if $b = d = -1$. This implies that $ac = 0$ and $a + c = 1$. So if $a = 0$, then $c = 1$, so $1 = ad + bc = b$, which is a contradiction with $b = -1$. If $c = 0$, then $1 = ad + bc = d$,

which is a contradiction with $d = -1$. Thus $\overline{g}$ is irreducible in $\mathbb{Z}_3[x]$, so $g$ is irreducible in $\mathbb{Z}[x]$, and by Gauss's lemma, $g$ is irreducible in $\mathbb{Q}[x]$. □

**Remark.** If we had tried $p = 2$, then we have $\overline{g} = x^4 + 1 \in \mathbb{Z}_2[x]$, which is not in fact irreducible since

$$\overline{g} = x^4 + 1 = (x+1)^4 \in \mathbb{Z}_2[x].$$

## 4.2 Field Extensions

**Definition 4.1.** Let $K, L$ be fields and $\varphi : K \to L$ an injective homomorphism. Then $L$ is a *field extension* of $K$, denoted $L : K$.

**Example 4.1.1.** We have $\mathbb{C} : \mathbb{R}$ is a field extension.

**Definition 4.2.** Recall that $V$ is a *$K$-vector space* if

1. $V$ is an abelian group under $+$,
2. For $a, b \in K$ and $x, y \in V$, we have

$$\text{(i). } a(x+y) = ax + ay, \quad \text{(ii). } (a+b)x = ax + bx, \quad \text{(iii). } (ab)x = a(bx), \quad \text{(iv). } 1x = 1.$$

**Remark.** If $L : K$ is a field extension, then $L$ is a a vector space over $K$.

**Definition 4.3.** A *basis* for a vector space is a linearly independent spanning set.

**Example 4.3.1.** The complex numbers $\mathbb{C}$ is a $\mathbb{R}$-vector space with basis $\{1, i\}$. Bases are not unique, since $\{1+i, 1-i\}$ is another basis for $\mathbb{C}$.

**Example 4.3.2.** If there is a vector space that we know to be a field, then it is automatically a field extension of its ground field.

**Definition 4.4.** The *dimension* of $L$ is the cardinality of a basis for $L : K$.[1] The dimension is also called the *degree* of $L : K$, denoted $[L : K]$. We say that $L$ is a *finite extension* if $[L : K]$ is finite, and an *infinite extension* otherwise.

**Example 4.4.1.** We have $[\mathbb{C} : \mathbb{R}] = 2$, which is finite. On the other hand, $\mathbb{R} : \mathbb{Q}$ is an infinite extension.

**Theorem 4.1.** *Let $L : K$ be a field extension. Then $L = K$ if and only if $[L : K] = 1$.*

*Proof.* ($\Rightarrow$) If $L = K$, then $\{1\}$ is a basis for $L : K$, and thus $[L : K] = 1$.

($\Leftarrow$) If $[L : K] = 1$, then $\{x\}$ is a basis for $L : K$ for some $x \in L$. Then there exists some $a \in K$ such that $1 = ax$, so $x = a^{-1} \in K$. For every $y \in L$, there exists $b \in K$ such that $y = bx$. But then

$$y = bx = b(a^{-1}) \in K,$$

so $y \in K$ as well by closure. Thus $L = K$ as desired. □

**Remark.** Let $L : K$ and $M : L$ be field extensions with

$$K \xrightarrow{\alpha} L \xrightarrow{\beta} M$$

[1] Note that this is well-defined since any two bases of $L$ have the same length.

Then $M : K$ is also a field extension.

**Theorem 4.2.** *For field extensions $L : K$ and $M : L$, we have $[M : L][L : K] = [M : K]$.*

*Proof.* Suppose $\{a_1, a_2, \dots a_r\}$ is a linearly independent subset of $M$ over $L$ and $\{b_1, b_2, \dots, b_s\}$ is a linearly independent subset of $L$ over $K$. Now we claim that

$$\{a_i b_j \mid 1 \leq i \leq r, 1 \leq j \leq s\}$$

is a linearly independent subset of $M$ over $K$. To see this, suppose

$$\sum_{i=1}^{r} \sum_{j=1}^{s} \lambda_{ij} a_i b_i = 0$$

for some $\lambda_{ij} \in K$. We can rewrite this as

$$\sum_{i=1}^{r} \left( \sum_{j=1}^{s} \lambda_{ij} b_j \right) a_i = 0.$$

Since the $a_i$ are linearly independent over $L$, it follows that

$$\sum_{j=1}^{s} \lambda_{ij} b_j = 0$$

for each $i = 1, \dots, r$. Since the $b_j$ are linearly independent over $K$, it follows that $\lambda_{ij} = 0$ for each $i, j$, which proves the claim. Returning to the main proof, if $[M : L]$ or $[L : K]$ is infinite, then $r$ or $s$ can be made arbitrarily large, so

$$\{a_i b_j \mid 1 \leq i \leq r, 1 \leq j \leq s\}$$

can also be made arbitrarily large, and hence $[M : K]$ is infinite. Now suppose $[M : L] = r < \infty$ and $[L : K] = s < \infty$. Let $\{a_1, a_2, \dots, a_r\}$ be a basis for $M : L$ and $\{b_1, b_2, \dots, b_s\}$ be a basis for $L : K$. We will show that

$$\{a_i b_j \mid 1 \leq i \leq r, 1 \leq j \leq s\}$$

is a basis for $M : K$. Since we already showed that $\{a_i b_j\}$ is linearly independent, it only remains to show that they span $M$ over $K$. For each $z \in M$, there exist $\lambda_1, \dots, \lambda_r \in L$ such that

$$z = \sum_{i=1}^{r} \lambda_i a_i.$$

Then for each $\lambda_i \in L$, there exist $\mu_{i1}, \dots, \mu_{is} \in K$ such that

$$\lambda_i = \sum_{j=1}^{s} \mu_{ij} b_j.$$

Combining this yields

$$z = \sum_{i=1}^{r} \sum_{j=1}^{s} \mu_{ij} a_i b_j$$

as desired, which finishes the proof. □

**Example 4.4.2.** Consider $\mathbb{Q}(\sqrt{2}) = \mathbb{Q}[\sqrt{2}] = \{a + b\sqrt{2} \mid a, b \in \mathbb{Q}\}$.

**Exercise 4.1.** Show that $\mathbb{Q}[\sqrt{2}]$ is a field. (Hint: $1/(a + b\sqrt{2}) = (a - b\sqrt{2})/(a^2 - 2b^2)$.)

**Definition 4.5.** Let $K$ be a subfield of $L$ and $S$ a subset of $L$. The *subfield of $L$ generated over $K$ by $S$*, denoted $K(S)$, is the intersection of all subfields of $L$ containing $K \cup S$. If $S = \{\alpha_1, \ldots, \alpha_n\}$ is finite, we write $K(\alpha_1, \ldots, \alpha_n)$.

**Theorem 4.3.** *Let $E$ be the elements in $L$ that can be expressed as quotients of finite $K$-linear combinations of finite products of elements in $S$. Then $K(S) = E$.*

*Proof.* To see that $K(S) \subseteq E$, simply check that $E$ is a subfield of $L$ containing $K \cup S$.

For $E \subseteq K(S)$, note that any subfield of $L$ containing $K$ and $S$ must contain all finite products of elements in $S$, all linear combinations of such products, and all quotients of such linear combinations. This is precisely what is means to have $E \subseteq K(S)$. □

**Definition 4.6.** A *simple extension* of $K$ is $K(\alpha)$, i.e. $S$ has a single element $\alpha \notin K$.

**Example 4.6.1.** The previous example $\mathbb{Q}(\sqrt{2})$ is a simple extension.

**Theorem 4.4.** *Let $L$ be a field, $K$ a subfield, and $\alpha \in L$. Then either*

1. *$K(\alpha)$ is isomorphic to $K(X)$, the field of rational forms with coefficients in $K$,*
2. *or there exists a unique monic polynomial $m \in K[X]$ with the property that for all $f \in K[X]$,*
   (a) *$f(\alpha) = 0$ if and only if $m|f$,*
   (b) *the field $K(\alpha)$ coincides with $K[\alpha]$, the ring of all polynomials in $\alpha$ with coefficients in $K$,*
   (c) *and $[K[\alpha] : K] = \partial m$.*

*Proof.* Suppose there does not exist nonzero $f \in K[X]$ such that $f(\alpha) = 0$. Then there exists a map $\varphi : K(X) \to K(\alpha)$ with $f/g \mapsto f(\alpha)/g(\alpha)$, which is defined since $g(\alpha) = 0$ only if $g$ is the zero polynomial. Note that $\varphi$ is a surjective homomorphism,[2] which one can check as an exercise. Now we show that $\varphi$ is also injective. To see this, suppose

$$\varphi(f/g) = \varphi(p/q),$$

which happens if and only if

$$f(\alpha)q(\alpha) - p(\alpha)g(\alpha) = 0.$$

in $L$. This happens if and only if $fq - pg = 0$ in $K[X]$, which happens if and only if $f/g = p/q$ in $K(X)$. This completes the first case of the theorem.

Now suppose there exists nonzero $g \in K[X]$ such that $g(\alpha) = 0$. Furthermore, suppose $g$ is a polynomial of least degree with this property. Let $a$ be the leading coefficient of $g$, and let $m = g/a$, so that $m$ is monic and $m(\alpha) = 0$ still. The reverse implication in (2a) is clear. For the forwards implication in (2a), note that by division with remainder for polynomials over a field, we can write

$$f = qm + r,$$

where $\partial r < \partial m$. By the minimality of $\partial m$, we must have $r = 0$, so $m|f$. For the uniqueness of $m$, suppose there exists $m'$ with the same properties. Then $m(\alpha) = m'(\alpha) = 0$, so $m|m'$ and $m'|m$, which

[2] Also check that $\varphi$ is well-defined.

implies that $m = m'$ since $m$ and $m'$ are monic. For the irreducibility of $m$, suppose for the sake of contradiction that $m = pq$ with $\partial p, \partial q < \partial m$. Then $m(\alpha) = p(\alpha)q(\alpha) = 0$, so either $p(\alpha) = 0$ or $q(\alpha) = 0$, which contradicts the minimality of $\partial m$.

Now we show (2b), which says that $K(\alpha) = K[\alpha]$. For this, consider $p(\alpha)/q(\alpha) \in K(\alpha)$ for $q(\alpha) \neq 0$. Then $m \nmid q$, and since $m$ is irreducible we have $\gcd(m, q) = 1$. Now by Theorem 2.15 of Howie (about gcd's in the Euclidean domain $K[X]$), there exist polynomials $a, b$ such that $aq + bm = 1$. Setting $X = \alpha$ yields $a(\alpha)q(\alpha) = 1$, so

$$\frac{p(\alpha)}{q(\alpha)} = p(\alpha)a(\alpha) \in K[\alpha].$$

Thus $K(\alpha) \subseteq K[\alpha]$. Since we already know that $K[\alpha] \subseteq K(\alpha)$, we conclude that $K(\alpha) = K[\alpha]$.

Finally we show (2c), which claims that $[K[\alpha] : K] = \partial m$. For this, suppose $\partial m = n$ and let

$$p(\alpha) \in K[\alpha] = K(\alpha).$$

Then $p = qm + r$ where $\partial r < \partial m = n$. We have $p(\alpha) = r(\alpha)$, so if

$$r = c_0 + c_1 X + \cdots + c_{n-1} X^{n-1}$$

for $c_i \in K$, then

$$p(\alpha) = c_0 + c_1\alpha + \cdots + c_{n-1}\alpha^{n-1}.$$

So $\{1, \alpha, \ldots, \alpha^{n-1}\}$ is a spanning set for $K[\alpha]$. To see that $\{1, \alpha, \ldots, \alpha^{n-1}\}$ is also linearly independent, suppose there exists $a_i \in K$ such that

$$a_0 + a_1\alpha + \cdots + a_{n-1}\alpha^{n-1} = 0.$$

Then $a_0 = \cdots = a_{n-1} = 0$ since otherwise we would have a polynomial

$$p = a_0 + a_1 X + \cdots + a_{n-1} X^{n-1}$$

with $\partial p \leq n - 1$ and $p(\alpha) = 0$, which is a contradiction with the minimality of $\partial m = n$. Thus $\{1, \alpha, \ldots, \alpha^{n-1}\}$ is a basis, and so $[K[\alpha] : K] = n = \partial m$. □

**Example 4.6.2.** Continuing the same example, note that

$$\mathbb{Q}[\sqrt{2}] = \{a + b\sqrt{2} \mid a, b \in \mathbb{Q}\} = \{a_0 + a_1\sqrt{2} + a_2\sqrt{2}^2 + a_3\sqrt{2}^3 + \cdots + a_n\sqrt{2}^n \mid a_i \in \mathbb{Q}\},$$

which falls in the second case of the previous theorem.

**Remark.** We also have $\mathbb{Q}[\sqrt{2}] = \mathbb{Q}[X]/\langle X^2 - 2\rangle$.

# Lecture 5

# Jan. 24 — Algebraic Extensions

## 5.1 Minimal Polynomials

**Remark.** The $m$ in the previous theorem from last class is called the *minimal polynomial* of $\alpha$.

**Example 5.0.1.** Let

$$\mathbb{Q}[i\sqrt{3}] = \{a + bi\sqrt{3} \mid a, b \in \mathbb{Q}\} \subseteq \mathbb{C}.$$

Here $m = X^2 + 3$, so this is a degree 2 extension.

**Exercise 5.1.** Write $1/(a + bi\sqrt{3})$ in the form $c + di\sqrt{3}$.

**Example 5.0.2.** Is $\mathbb{Q}(\sqrt{2}, \sqrt{3})$ a simple extension? In fact it is! Note that certainly

$$\mathbb{Q}(\sqrt{2} + \sqrt{3}) \subseteq \mathbb{Q}(\sqrt{2}, \sqrt{3}).$$

For the reverse inclusion, observe that $(\sqrt{3} + \sqrt{2})(\sqrt{3} - \sqrt{2}) = 1$, so

$$1/(\sqrt{3} + \sqrt{2}) = \sqrt{3} - \sqrt{2} \in \mathbb{Q}(\sqrt{2} + \sqrt{3}).$$

From this we have

$$(\sqrt{3} + \sqrt{2}) + (\sqrt{3} - \sqrt{2}) = 2\sqrt{3},$$

which implies that $\sqrt{3} \in \mathbb{Q}(\sqrt{2} + \sqrt{3})$. Similarly $\sqrt{2} \in \mathbb{Q}(\sqrt{2} + \sqrt{3})$, so that $\mathbb{Q}(\sqrt{2}, \sqrt{3}) \subseteq \mathbb{Q}(\sqrt{2} + \sqrt{3})$. Now we can consider

$$\mathbb{Q}(\sqrt{2}, \sqrt{3}) = \mathbb{Q}[\sqrt{2}, \sqrt{3}] = (\mathbb{Q}[\sqrt{2}])[\sqrt{3}].$$

First we have $[Q[\sqrt{2}] : \mathbb{Q}] = 2$. Note that $X^2 - 3$ is the minimal polynomial of $\sqrt{3}$ over $\mathbb{Q}[\sqrt{2}]$, so $[\mathbb{Q}[\sqrt{2}, \sqrt{3}] : \mathbb{Q}[\sqrt{2}]] = 2$. Hence $[\mathbb{Q}[\sqrt{2}, \sqrt{3}] : \mathbb{Q}] = 4$ with basis $\{1, \sqrt{2}, \sqrt{3}, \sqrt{6}\}$.[1] To find the minimal polynomial of $\sqrt{2} + \sqrt{3}$ over $\mathbb{Q}$, we can compute

$$(\sqrt{2} + \sqrt{3})^2 = 2 + 2\sqrt{6} + 3 = 5 + 2\sqrt{6}$$
$$(\sqrt{2} + \sqrt{3})^4 = 25 + 20\sqrt{6} + 24 = 49 + 20\sqrt{6}.$$

Thus $X^4 - 10X^2 + 1$ is the minimal polynomial, since $\alpha^4 - 10\alpha^2 + 1 = 0$ for $\alpha = \sqrt{2} + \sqrt{3}$.

[1] Since $\mathbb{Q}[\sqrt{2}, \sqrt{3}] = \mathbb{Q}[\alpha]$ where $\alpha = \sqrt{2} + \sqrt{3}$, we have $\{1, \alpha, \alpha^2, \alpha^3\}$ as another basis.

## 5.2 Algebraic Extensions

**Definition 5.1.** If $\alpha$ has a minimal polynomial over $K$, we say $\alpha$ is *algebraic* over $K$, and $K[\alpha] = K(\alpha)$ is an *algebraic extension* of $K$. A complex number that is algebraic over $\mathbb{Q}$ is called an *algebraic number*. Otherwise, if $K(\alpha) \cong K(X)$, then we say $\alpha$ is *transcendental* over $K$. A transcendental number $\alpha$ is a complex number that is transcendental over $\mathbb{Q}$.

**Example 5.1.1.** We have that $\mathbb{Q}(i\sqrt{3})$, $\mathbb{Q}(\sqrt{2})$, $\mathbb{Q}(\sqrt{3})$, and $\mathbb{Q}(\sqrt{2}, \sqrt{3})$ are all simple algebraic extensions of $\mathbb{Q}$, whereas $\mathbb{Q}(X)$ is a simple transcendental extension of $\mathbb{Q}$.

**Theorem 5.1.** *Let $K(\alpha)$ be a simple transcendental extension of $K$. Then $[K(\alpha) : K] = \infty$.*

*Proof.* Observe that $1, \alpha, \alpha^2, \ldots$ are linearly independent over $K$, since no minimal polynomial exists. □

**Definition 5.2.** An extension $L$ over $K$ is an *algebraic extension* if any element of $L$ is algebraic over $K$. Otherwise, $L$ is a *transcendental extension*.

**Theorem 5.2.** *Every finite extension is algebraic.*

*Proof.* Let $L : K$ be a finite extension and suppose for sake of contradiction that $\alpha \in L$ is transcendental over $K$. Then $1, \alpha, \alpha^2, \ldots$ are linearly independent, contradicting the fact that $L : K$ is finite. □

**Theorem 5.3.** *Let $L : K$ be a field extension and let $\mathcal{A}(L)$ be the set of elements in $L$ that are algebraic over $K$. Then $\mathcal{A}(L)$ is a subfield of $L$.*

*Proof.* See Howie. Just need to show the closure of algebraic elements under usual field operations. □

**Example 5.2.1.** For $L = \mathbb{C}$ and $K = \mathbb{Q}$, we have that $\mathcal{A}(\mathbb{C})$ is the field $\mathbb{A}$ of algebraic numbers.

**Theorem 5.4.** *The set of algebraic numbers $\mathbb{A}$ is countable.*

*Proof sketch.* Note that the set of monic polynomials of degree $n$ with coefficients in $\mathbb{Q}$ is countable, and each such polynomial has at most $n$ distinct roots in $\mathbb{C}$. Hence the number of roots of such polynomials is countable. Then $\mathbb{A}$ is the countable union of countable sets, so $\mathbb{A}$ is countable. □

**Theorem 5.5.** *Transcendental numbers exist.*

*Proof.* Since $|\mathbb{R}| = |\mathbb{C}| = 2^{\aleph_0} > \aleph_0$, we must have that $\mathbb{C} \setminus \mathbb{A}$ is nonempty. □

**Remark.** The above proof is very nonconstructive, what about actual examples of transcendental numbers? In 1844, Liouville constructed the following example:

$$\sum_{n=1}^{\infty} 10^{-n!},$$

which was shown to be transcendental. In 1873, Hermite showed that $e$ is transcendental, and in 1882, Lindemann showed that $\pi$ is transcendental.

**Theorem 5.6.** *Let $L : K$ be a field extension and $\alpha_1, \ldots, \alpha_n \in L$ have minimal polynomials $m_1, \ldots, m_n$, respectively. Then $[K(\alpha_1, \ldots, \alpha_n) : K] \leq \partial m_1 \partial m_2 \ldots \partial m_n$.*

*Proof.* See Howie. Uses induction and the fact that $[M : L][L : K] = [M : K]$. □

**Example 5.2.2.** Consider

$$[\mathbb{Q}[\sqrt{2}] : \mathbb{Q}] = [\mathbb{Q}[\sqrt{3}] : \mathbb{Q}] = [\mathbb{Q}[\sqrt{6}] : \mathbb{Q}] = 2,$$

but $[\mathbb{Q}[\sqrt{2}, \sqrt{3}, \sqrt{6}] : \mathbb{Q}] = 4$. So the bound in the previous theorem cannot be made into an equality.

**Proposition 5.1.** *A field extension $L : K$ is finite if and only if for some $n$, there exist $\alpha_1, \ldots, \alpha_n$ algebraic over $K$ such that $L = K(\alpha_1, \ldots, \alpha_n)$.*

*Proof.* ($\Leftarrow$) This is precisely the previous theorem.

($\Rightarrow$) Suppose $L : K$ is finite and $\{\alpha_1, \ldots, \alpha_n\}$ is a basis for $L$ over $K$. Since finite extensions are algebraic, the $\alpha_i$ must be algebraic. □

**Exercise 5.2.** Show that $\varphi : \mathbb{Q}[\sqrt{2}] \to \mathbb{Q}[X]/\langle X^2 - 2\rangle$ defined by

$$a + b\sqrt{2} \mapsto a + bX + \langle X^2 - 2\rangle$$

is an isomorphism.

**Theorem 5.7.** *Let $K$ be a field and $m$ a monic irreducible polynomial in $K[X]$. Then $L = K[X]/\langle m\rangle$ is a simple algebraic extension $K[\alpha]$ of $K$, and $\alpha = X + \langle m\rangle$ has minimal polynomial $m$ over $K$.*

*Proof.* First note that $L$ is indeed a field since $m$ is irreducible. Also $L : K$ is indeed a field extension since $\varphi : K \to L$ defined by $a \mapsto a + \langle m\rangle$ is an injective homomorphism. Now let $\alpha = X + \langle m\rangle$. For

$$f = a_0 + a_1X + \cdots + a_nX^n \in K[X],$$

we have

$$\begin{aligned} f(\alpha) &= a_0 + a_1\alpha + \cdots + a_n\alpha^n = a_0 + a_1(X + \langle m\rangle) + \cdots + a_n(X + \langle m\rangle)^n \\ &= a_0 + a_1X + \cdots + a_nX^n + \langle m\rangle = f + \langle m\rangle. \end{aligned}$$

So $f(\alpha) = 0$ if and only if $f \in \langle m\rangle$, i.e. $m|f$. Hence $m$ is the minimal polynomial of $\alpha$. □

# Lecture 6

# Jan. 29 — Geometric Constructions

## 6.1 $K$-Isomorphisms

Recall from last class that $L = K[X]/\langle m \rangle$ is a simple algebraic extension of $K$. In fact, we can show that the field $L$ is essentially unique, i.e. unique up to isomorphism.

**Theorem 6.1.** *Let $K$ be a field and and $f$ and an irreducible polynomial in $K[X]$. If $L$ and $L'$ are two extensions of $K$ containing roots $\alpha$ and $\alpha'$ respectively of $f$, then there exists an isomorphism $K[\alpha] \to K[\alpha']$ which fixes every element of $K$.*

*Proof sketch.* Suppose

$$f = a_0 + a_1 X + \cdots + a_n X^n.$$

Then $K[\alpha]$ consists of polynomials of the form

$$b_0 + b_1\alpha + \cdots + b_{n-1}\alpha^{n-1}.$$

This is because multiplication in $K[\alpha]$ relies on the observation that

$$\alpha^n = -\frac{1}{\alpha_n}(a_0 + a_1\alpha + \cdots + a_{n-1}\alpha^{n-1})$$

since $\alpha$ is a root of $f$. Define $\psi : K[\alpha] \to K[\alpha']$ by $\psi(g(a)) = g(\alpha')$ and show that $\psi$ is an isomorphism. □

**Exercise 6.1.** Check the following from the previous proof:

1. $\psi$ is one-to-one and onto,
2. $\psi$ fixes $K$,
3. and $\psi$ is a homomorphism.

For the last point, the addition is mostly straightforward but the multiplication is more involved since we need to reduce when we get $\alpha^n$ terms in the product.

**Definition 6.1.** A *$K$-isomorphism* is an isomorphism $\varphi : L \to L'$ such that $\varphi(x) = x$ for all $x \in K$.

**Example 6.1.1.** For $\mathbb{C} : \mathbb{R}$, the complex conjugation map $\varphi : \mathbb{C} \to \mathbb{C}$ given by $\varphi(a + bi) = a - bi$ is a $\mathbb{R}$-isomorphism.

**Example 6.1.2.** For $\mathbb{Q}[X]/\langle X^2+3\rangle : \mathbb{Q}$,[1] the map $\psi : \mathbb{Q}[X]/\langle X^2+3\rangle \to \mathbb{Q}[X]/\langle X^2+3\rangle$ given by

$$\psi(a+bX+\langle X^2+3\rangle) = a-bX+\langle X^2+3\rangle$$

is a $\mathbb{Q}$-isomorphism. The analogous map $\psi : \mathbb{Q}[i\sqrt{3}] \to \mathbb{Q}[i\sqrt{3}]$ given by $\psi(a+bi\sqrt{3}) = a-bi\sqrt{3}$ also works, which we can view as a restriction of the complex conjugation map to $\mathbb{Q}[i\sqrt{3}]$.

## 6.2 Applications to Geometric Constructions

Consider the straightedge and compass Constructions from geometry. Let $B_0$ be a set of points. Then we have the following operations:

1. (straightedge) Draw a straight line through any two points in $B_0$.
2. (compass) Draw a circle whose center is a point in $B_0$ passing through another point in $B_0$.

Let $C(B_0)$ be the set of points which are intersections of lines or circles obtained form $B_0$ by (1) and (2). Let $B_1 = B_0 \cup C(B_0)$, and proceed inductively to get $B_n = B_{n-1} \cup C(B_{n-1})$.

**Definition 6.2.** A point is *constructible from* $B_0$ if it belongs to $B_n$ for some $n$. A point is *constructible* if it is constructible from $\{O, I\}$ where $O = (0,0)$ and $I = (1,0)$.

**Example 6.2.1.** To find the midpoint of the line segment $OI$ from $B_0 = \{O, I\}$, we can do the following:

1. Draw a circle with center $O$ passing through $I$.
2. Draw a circle with center $I$ passing through $O$.
3. Mark points $P$ and $Q$ where these circles intersect. So $B_1 \supseteq \{O, I, P, Q\}$.
4. Draw a line connecting $P$ and $Q$.
5. Draw a line connecting $O$ and $I$.
6. Mark the point $M$ where $PQ$ and $OI$ meet. So $B_2 \supseteq \{O, I, P, Q, M\}$.

Thus $M$ is constructible from $\{O, I\}$.

The algebraic perspective is the following: Associate to $B_i$ the subfield of $\mathbb{R}$ generated by coordinates of points in $B_i$, i.e. view each coordinate of each point as an element and take the subfield generated.

**Example 6.2.2.** For $B_0 = \{(0,0),(1,0)\}$, we have $\{0,0,1,0\} \subseteq K_0 = \mathbb{Q}$ is the subfield of $\mathbb{R}$ generated by the coordinates of $B_0$. Next take[2]

$$B_1 = \{O, I, P, Q\} = \{(0,0),(1,0),(1/2, \pm\sqrt{3}/2)\},$$

so that $K_1 = \mathbb{Q}[\sqrt{3}]$ is the field generated by $B_1$. Then

$$B_2 = \{O, I, P, Q, M\} = \{(0,0),(1,0),(1/2, \pm\sqrt{3}/2),(1/2,0)\},$$

and the field generated by $B_2$ is still $K_2 = \mathbb{Q}[\sqrt{3}]$.

[1] Note that $\mathbb{Q}[X]/\langle X^2+3\rangle \cong \mathbb{Q}[i\sqrt{3}]$. The isomorphism is given by $a+bX+\langle X^3+3\rangle \mapsto a+bi\sqrt{3}$.

[2] There is some abuse of notation here since we take $B_i$ to be only some subset of all the actual possible points.

**Theorem 6.2.** *Let $P$ be a constructible point belonging to $B_n$, where $B_0 = \{(0,0),(1,0)\}$, and let $K_n$ be the field generated over $\mathbb{Q}$ by $B_n$. Then $[K_n : \mathbb{Q}]$ is a power of 2.*

*Proof sketch.* We proceed by induction. The base case is $K_0 = \mathbb{Q}$, so $[K_0 : \mathbb{Q}] = 1 = 2^0$. Now suppose $[K_{n-1} : \mathbb{Q}] = 2^k$ for some $k \geq 0$, and we want to show that $[K_n : K_{n-1}]$ is a power of 2. Observe that new points in $B_n$ can be obtained by

1. intersection of two lines,
2. intersection of a line and a circle,
3. or intersection of two circles.

In case (1), the intersection of two lines is given by solving a system of two linear equations, which only involves rational operations[3]. In other words, this case takes place entirely in $K_{n-1}$.

In case (2), the intersection of a line and a circle is given by solving of a system of one linear equation and one quadratic equation. Solving the linear equation for one of the variables and substituting into the quadratic equation reduces the system down to a single quadratic equation in a single variable. The solution involves $\sqrt{\Delta}$, where $\Delta$ is the discriminant. Then the new points are in $K_{n-1}[\sqrt{\Delta}]$.

In case (3), the intersection of two circles is given by solving a system of two quadratic equations. Subtracting the two quadratic equations yields a linear equation, which reduces back to case (2).

Thus the elements in $K_n$ are either in $K_{n-1}$ or $K_{n-1}[\sqrt{\Delta}]$ for some $\Delta \in K_{n-1}$.[4] Hence $[K_n : K_{n-1}]$ is either 1 or 2, so by induction $[K_n : \mathbb{Q}]$ is a power of 2. □

## 6.3 Classic Problems

### 6.3.1 Duplicating the Cube

Consider the problem of taking a cube of volume 1, and constructing a cube of volume 2. We need $\alpha$ such that $\alpha^3 = 2$. But $X^3 - 2$ is irreducible over $\mathbb{Q}$ by Eisenstein's criterion, so $[\mathbb{Q}[\alpha] : \mathbb{Q}] = 3$. This is not a power of 2, so $\alpha$ is not constructible and thus we cannot duplicate the cube.

### 6.3.2 Trisecting the Angle

Recall the triple angle formula:

$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta.$$

Suppose $\cos 3\theta = c$. So to find $\cos\theta$, we want a root of $4X^3 - 3X - c = 0$. This depends on $c$.

**Example 6.2.3.** If $3\theta = \pi/2$, then $c = 0$ and the polynomial factors into

$$4X^3 - 3X = 4X(4X^2 - 3),$$

so $[\mathbb{Q}[\alpha] : \mathbb{Q}] = [\mathbb{Q}[\sqrt{3}] : \mathbb{Q}] = 2$. So in fact we can trisect $\pi/2 = 90°$.

[3] By rational operations we mean addition, subtraction, multiplication, division.

[4] We can set it up so that we only gain one extra intersection, i.e. only one $\Delta$, at each step.

**Example 6.2.4.** If $3\theta = \pi/3$, then $c = 1/2$ and we have $4X^3 - 3X - 1/2$. Let

$$f(X) = 8X^3 - 6X - 1,$$

so that $g(X) = g(X/2) = X^3 - 3X - 1$. Note that $g$ does not factor over $\mathbb{Z}$ since that requires a linear factor of $X \pm 1$ but $g(\pm 1) \neq 0$. So $g$ is irreducible over $\mathbb{Z}$ and by Gauss's lemma, $g$ is irreducible over $\mathbb{Q}$. Thus $f$ is irreducible. Hence $[\mathbb{Q}[\alpha] : \mathbb{Q}] = 3$, so we cannot trisect $\pi/3$ with a straightedge and compass.

# Lecture 7

# Jan. 31 — Splitting Fields

## 7.1 Review of Notation

Recall that

$$\mathbb{Q}[X] = \{a_0 + a_1X + \cdots + a_nX^n : a_i \in \mathbb{Q}\}$$
$$\mathbb{Q}(X) = \{f/g : f, g \in \mathbb{Q}[X], g \neq 0\}/\sim,$$

where $\sim$ is the usual relation on fractions, e.g. $2f/2g = f/g$. Next, recall that

$$\mathbb{Q}[\sqrt{2}] = \{a_0 + a_1\sqrt{2} + \cdots + a_n\sqrt{2}^n : a_i \in \mathbb{Q}\} = \{a + b\sqrt{2} : a, b \in \mathbb{Q}\}$$

since $\sqrt{2}^2 = 2$. Also $\mathbb{Q}(\sqrt{2})$ is the smallest subfield of $\mathbb{R}$ containing $\mathbb{Q} \cup \{\sqrt{2}\}$. In this case, $\mathbb{Q}(\sqrt{2}) = \mathbb{Q}[\sqrt{2}]$ since

$$\frac{1}{a + b\sqrt{2}} = \frac{a - b\sqrt{2}}{a^2 - 2b^2}.$$

Next, we have

$$\begin{aligned}\mathbb{Q}[X]/\langle X^2 - 2\rangle &= \{a_0 + a_1X + \cdots + a_nX^n + \langle X^2 - 2\rangle : a_i \in \mathbb{Q}\} \\ &= \{a + bX + \langle X^2 - 2\rangle : a, b \in \mathbb{Q}\}\end{aligned}$$

since $X^2 + \langle X^2 - 2\rangle = 2 + \langle X^2 - 2\rangle$. In fact, $\mathbb{Q}[X]/\langle X^2 - 2\rangle \cong \mathbb{Q}[\sqrt{2}]$.[1]

## 7.2 Splitting Fields

The motivating question here is: When can we factor a polynomial into linear factors?

**Definition 7.1.** A polynomial *splits completely* over $K$ if it can be factored into linear factors over $K$.

**Example 7.1.1.** The polynomial $X^2+2$ splits completely over $\mathbb{Q}[i\sqrt{2}]$ since $X^2+2 = (X-i\sqrt{2})(X+i\sqrt{2})$.

**Example 7.1.2.** The polynomial $X^3 - 2$ is irreducible over $\mathbb{Q}$ by Eisenstein's criterion. However, it factors as

$$X^3 - 2 = (X - \alpha)(X^2 + \alpha X + \alpha^2)$$

in $\mathbb{Q}[\alpha]$, where $\alpha = \sqrt[3]{2}$. Also $X^2 + \alpha X + \alpha^2$ is irreducible over $\mathbb{Q}[\alpha]$, since its discriminant shows that it is irreducible even over $\mathbb{R}$. But in $\mathbb{C}$, we can factor it as

$$X^3 - 2 = (X - \alpha)(X - \alpha e^{2\pi i/3})(X - \alpha e^{4\pi i/3}).$$

A smaller field that $X^3 - 2$ splits completely over is $\mathbb{Q}[\sqrt[3]{2}, i\sqrt{3}]$.

[1] Here the isomorphism $\mathbb{Q}[X]/\langle X^2 - 2\rangle \to \mathbb{Q}[\sqrt{2}]$ is given by $a + bX + \langle X^2 - 2\rangle \mapsto a + b\sqrt{2}$.

**Definition 7.2.** Let $K$ be a field and $f \in K[X]$. An extension $L$ of $K$ is a *splitting field* for $f$ over $K$ if

1. $f$ splits completely over $L$,
2. and $f$ does not split completely over any subfield $E$ with $K < E < L$.

**Example 7.2.1.** From the last two examples, $\mathbb{Q}[i\sqrt{2}]$ is a splitting field over $\mathbb{Q}$ for $X^2+2$, and $\mathbb{Q}[\sqrt[3]{2}, i\sqrt{3}]$ is a splitting field for $X^3 - 2$ over $\mathbb{Q}$.

**Theorem 7.1.** *Let $K$ be a field and $f \in K[X]$ with $\partial f = n$. Then there exists a splitting field $L$ for $f$ over $K$ and $[L : K] \leq n!$.*

*Proof.* The proof is essentially the process we perform in the following example. At each step, construct an extension in which we can split off a linear factor from $f$. For more details, see Howie. □

**Example 7.2.2.** Let us find a splitting field for

$$f = X^5 + X^4 - X^3 - 3X^2 - 3X + 3$$

over $\mathbb{Q}$. Note that $\partial f = n$. Stare hard enough and we can see that

$$f = (X^3 - 3)(X^2 + X - 1),$$

where the first factor is irreducible by Eisenstein's criterion and the second factor is irreducible by checking the discriminant. Now add a root, say $\alpha = \sqrt[3]{3}$, and let $E_1 = \mathbb{Q}(\alpha)$. Then

$$f = (X - \alpha)(X^2 + \alpha X + \alpha^2)(X^2 + X - 1).$$

Note that $[E_1 : K] \leq n = \partial f$. Now let $E_2 = E_1(\alpha e^{2\pi i/3})$, so that

$$f = (X - \alpha)(X - \alpha e^{2\pi i/3})(X - \alpha e^{-2\pi i/3})(X^2 + X - 1).$$

Note that $[E_2 : \mathbb{Q}] \leq n(n-1)$. Next $E_3 = E_2(\alpha e^{-2\pi i/3})$ with

$$f = (X - \alpha)(X - \alpha e^{2\pi i/3})(X - \alpha e^{-2\pi i/3})(X^2 + X - 1).$$

Note that $[E_3 : K] \leq n(n-1)(n-2)$. Now let

$$\gamma = \frac{-1 + \sqrt{5}}{2}, \quad \delta = \frac{-1 - \sqrt{5}}{2}.$$

Let $E_4 = E_3(\gamma)$,

$$f = (X - \alpha)(X - \alpha e^{2\pi i/3})(X - \alpha e^{-2\pi i/3})(X - \gamma)(X - \delta).$$

Finally $E_5 = E_4(\delta)$ is the splitting field for $f$ over $\mathbb{Q}$. Note that we did much better than $n!$ here, since

$$[E_1 : \mathbb{Q}] = 3, \quad [E_2 : E_1] = 2, \quad [E_3 : E_2] = 1, \quad [E_4 : E_3] = 2, \quad [E_5 : E_4] = 1,$$

so $[E_5 : \mathbb{Q}] = 12 \leq 120$.

**Remark.** Splitting fields are unique (up to isomorphism).

**Theorem 7.2.** *Let $L$ and $L'$ be splitting fields of $f$ over $K$. Then there exists an isomorphism $\varphi : L \to L'$ fixing $K$.*

*Proof sketch.* Induct on the number of roots of $f$ that are not in $K$. The induction step uses Theorem 6.1 from last class giving an isomorphism $K[\alpha] \to K[\alpha']$ for $\alpha, \alpha'$ roots of an irreducible polynomial. □

**Example 7.2.3.** Let us find the splitting field of $f = X^4 - 2$ over $\mathbb{Q}$ and its degree. Note that $X^4 - 2$ is irreducible over $\mathbb{Q}$ by Eisenstein's criterion. Note that

$$X^4 - 2 = (X - \alpha)(X + \alpha)(X - i\alpha)(X + i\alpha)$$

where $\alpha = \sqrt[4]{2}$. So the splitting field is $\mathbb{Q}(\sqrt[4]{2}, i)$. For the degree, note that $[\mathbb{Q}(\sqrt[4]{2}) : \mathbb{Q}] = 4$ since the minimal polynomial of $\sqrt[4]{2}$ is $X^4 - 2$. A basis for this extension is $\{1, \sqrt[4]{2}, \sqrt[4]{2}^2, \sqrt[4]{2}^3\}$. Since $i \notin \mathbb{Q}(\sqrt[4]{2})$, we have $[\mathbb{Q}(\sqrt[4]{2}, i) : \mathbb{Q}(\sqrt[4]{2})] = 2$ since the minimal polynomial of $i$ over $\mathbb{Q}(\sqrt[4]{2})$ is $X^2 + 1$. Thus we see that the degree of the splitting field is $[\mathbb{Q}(\sqrt[4]{2}, i) : \mathbb{Q}] = 8$.

**Example 7.2.4.** Let us look at monic quadratic polynomials over $\mathbb{Z}_3 = \{-1, 0, 1\}$.[2] These are

$$\begin{array}{ccc} X^2 & X^2 + 1 & X^2 - 1 \\ X^2 + X & X^2 + X + 1 & X^2 + X - 1 \\ X^2 - X & X^2 - X + 1 & X^2 - X - 1. \end{array}$$

We have 0 is a root of the polynomials in the first column, 1 is a root of $X^2 - 1$ and $X^2 + X + 1$, and $-1$ is a root of $X^2 - X + 1$. So the irreducible polynomials over $\mathbb{Z}_3$ are

$$X^2 + 1, \quad X^2 + X - 1, \quad X^2 - X - 1.$$

Let $L = \mathbb{Z}_3[X]/\langle X^2 + 1\rangle$. Observe that $\alpha = X + \langle X^2 + 1\rangle$ satisfies

$$\alpha^2 = X^2 + \langle X^2 + 1\rangle = -1 + \langle X^2 + 1\rangle.$$

Hence $L$ is a splitting field for $X^2 + 1$ since $(X - \alpha)(X + \alpha) = X^2 + 1$. Similarly, $\mathbb{Z}_3[X]/\langle X^2 + X - 1\rangle$ is a splitting field for $X^2 + X - 1$ and $\mathbb{Z}_3[X]/\langle X^2 - X - 1\rangle$ is a splitting field for $X^2 - X - 1$. Note that each of these fields have $9 = 3^2$ elements since they are degree 2 extensions of $\mathbb{Z}_3$.

**Remark.** In $L$, we had $\alpha \in L$ such that $\alpha^2 = -1$ and addition is performed modulo 3. Now observe

$$(\alpha + 1)^2 + (\alpha + 1) - 1 = (\alpha^2 - \alpha + 1) + (\alpha + 1) - 1 = \alpha^2 - \alpha + \alpha + 1 + 1 - 1 = 0$$

since $\alpha^2 = -1$. So $\alpha + 1$ is a root of $X^2 + X - 1$ in $L$. By a similar computation, we see that $-\alpha + 1$ is a root of $X^2 + X - 1$, so $L$ is also a splitting field for $X^2 + X - 1$. Additionally, $\alpha - 1$ and $-\alpha - 1$ are roots of $X^2 - X - 1$, so $L$ is also a splitting field for $X^2 - X - 1$. So by uniqueness of splitting fields,

$$\mathbb{Z}_3[X]/\langle X^2 + 1\rangle \cong \mathbb{Z}_3[X]/\langle X^2 + X - 1\rangle \cong \mathbb{Z}_3[X]/\langle X^2 - X - 1\rangle.$$

**Exercise 7.1.** Find explicit isomorphisms between these fields.

## 7.3 Finite Fields

**Definition 7.3.** Let $f = a_0 + a_1X + \cdots + a_nX^n \in K[X]$. Then the *formal derivative* of $f$ is

$$Df = a_1 + 2a_2X + \cdots + na_nX^{n-1}.$$

**Exercise 7.2.** The usual formulas for derivatives

$$D(kf) = kDf, \quad D(f + g) = Df + Dg, \quad D(fg) = (Df)g + f(Dg)$$

all still hold for $f, g \in K[X]$ and $k \in K$.

[2]Note that as opposite to $\mathbb{Q}$, this field has finite characteristic.

# Lecture 8

# Feb. 5 — Finite Fields

## 8.1 Last Time

**Example 8.0.1.** The splitting field of $X^4 - 2$ over $\mathbb{Q}$ is $\mathbb{Q}(i, \sqrt[4]{2}])$ since

$$X^4 - 2 = (X - \sqrt[4]{2})(X + \sqrt[4]{2})(X - i\sqrt[4]{2})(X + i\sqrt[4]{2}).$$

**Example 8.0.2.** The splitting field of $Y^2 + 1$ over $\mathbb{Z}_3$ is $\mathbb{Z}_3[X]/\langle X^2 + 1\rangle$. If $\alpha = X + \langle X^2 + 1\rangle$, then

$$Y^2 + 1 = (Y - \alpha)(Y + \alpha).$$

Also the degree of this extension is $[Z_3[X]/\langle X^2 + 1\rangle : \mathbb{Z}_3] = 2$, and a basis for the extension is $\{1, X\}$.

## 8.2 Finite Fields

**Lemma 8.1.** *Let $f \in K[X]$, $K$ a field, and $L$ be a splitting field for $f$ over $K$. Then the roots of $f$ are distinct if and only if $f$ and $Df$ have no nonconstant common factor.*

*Proof.* ($\Leftarrow$) We show the contrapositive. Suppose $f$ has a repeated root $\alpha$ in $L$. Then

$$f = (X - \alpha)^r g$$

for some $r \geq 2$. Then

$$Df = (X - \alpha)^r Dg + r(X - \alpha)^{r-1} g,$$

so $Df$ and $f$ both have $X - \alpha$ as a factor.

($\Rightarrow$) Suppose the roots of $f$ are all distinct. Then for each root $\alpha$ of $f$ in $L$, we have

$$f = (X - \alpha)g,$$

where $g(\alpha) \neq 0$. Then

$$Df = (X - \alpha)Dg + g,$$

so that

$$(Df)(\alpha) = g(\alpha) \neq 0,$$

i.e. $X - \alpha \nmid Df$. This holds for factor of $f$ in $L[X]$, so $f$ and $Df$ have no common proper factors. □

**Theorem 8.1.** *Finite fields exist and are unique up to isomorphism. In particular,*

1. *Let $K$ be a finite field. Then $|K| = p^n$ for some prime $p$ and integer $n \geq 1$. Every element of $K$ is a root of $X^{p^n} - X$ and $K$ is a splitting field of $X^{p^n} - X$ over $\mathbb{Z}_p$.*
2. *Let $p$ be a prime and $n \in \mathbb{Z}$, $n \geq 1$. Then there exists a unique field of order $p^n$ up to isomorphism.*

*Proof.* (1) Let $\operatorname{char} K = p$. Then $K$ is a finite extension of $\mathbb{Z}_p$. Let $n = [K : \mathbb{Z}_p]$. If $\{\delta_1, \ldots, \delta_n\}$ is a basis for $K$ over $\mathbb{Z}_p$, then every element in $K$ can be uniquely written as

$$a_1\delta_1 + \cdots + a_n\delta_n$$

for some $a_i \in \mathbb{Z}_p$. There are $p^n$ such elements, so $|K| = p^n$. Then $|K^*| = p^n - 1$.[1] For any $\alpha \in K^*$, the order of $\alpha$ divides $p^n - 1$. So $\alpha^{p^n-1} = 1$, and hence $\alpha^{p^n} - \alpha = 0$. We also have $0^{p^n} - 0 = 0$ so every element in $K$ is a root of $X^{p^n} - X$. Hence $X^{p^n} - X$ splits completely over $K$. Since $X - \alpha$ is a factor of $X^{p^n} - X$ for each of the $p^n$ elements of $K$, $X^{p^n} - X$ does not split over any proper subfield of $K$. Thus we conclude that $K$ is a splitting field of $X^{p^n} - X$ over $\mathbb{Z}_p$.

(2) Given a prime $p$ and an integer $n \geq 1$, let $L$ be the splitting field of $X^{p^n} - X$ over $\mathbb{Z}_p$. Note that

$$Df = p^n X^{p^n-1} - 1 = -1$$

since $\operatorname{char} \mathbb{Z}_p = p$. Then $Df$ and $f$ have no nonconstant common factors, so by Lemma 8.1, we see that $X^{p^n} - X$ has $p^n$ distinct roots in $L$. Let $K$ be the set of $p^n$ distinct roots, and we claim that $K$ is a subfield of $L$. To check this, let $a, b \in K$. Then by an extension of Theorem 2.4,

$$(a - b)^{p^n} = a^{p^n} - b^{p^n} = a - b$$

in $\mathbb{Z}_p$, $a - b \in K$. Also

$$(ab^{-1})^{p^n} = a^{p^n}(b^{p^n})^{-1} = ab^{-1},$$

so $ab^{-1} \in K$. Hence $K$ is a field of order $p^n$. In fact, $K = L$ since $K$ contains all the roots of $X^{p^n} - X$ and no proper subfield does. By uniqueness of splitting fields, $K$ is unique up to isomorphism. □

**Definition 8.1.** We call the field of order $p^n$ the *Galois field* of order $p^n$, denoted $\mathrm{GF}(p^n)$.

**Example 8.1.1.** We have $\mathrm{GF}(3^2) = \mathbb{Z}_3[X]/\langle X^2 + 1\rangle \cong \mathbb{Z}_3[X]/\langle X^2 + X - 1\rangle \cong \mathbb{Z}_3[X]/\langle X^2 - X - 1\rangle$.

**Remark.** Recall that for a finite group $G$ and $a \in G$, the *order* of $a$ is

$$\operatorname{ord}(a) = \min\{k \in \mathbb{N} : a^k = 1\}.$$

The *exponent* of $G$ is

$$\exp(G) = \min\{k \in \mathbb{N} : a^k = 1 \text{ for all } a \in G\}.$$

Also recall that $\operatorname{ord}(a)$ divides $|G|$ for all $a \in G$, and thus $\exp(G)$ divides $|G|$.

**Exercise 8.1.** Show that $\exp(G) = \operatorname{lcm}\{\operatorname{ord}(a) : a \in G\}$.

**Example 8.1.2.** For $S_3 = \{\mathrm{id}, (12), (23), (13), (123), (132)\}$, the order of the transpositions is 2 and the order of 3-cycles is 3. So we see that $\exp(S_3) = 6$.

**Proposition 8.1.** *If $G$ is a finite abelian group, then there exists $a \in G$ such that* $\operatorname{ord}(a) = \exp(G)$.

[1] Recall that $K^*$ is the set of nonzero elements of $K$, which forms a group under multiplication. We also call $K^*$ the group of units of $K$.

*Proof.* Suppose that

$$\exp(G) = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k},$$

where the $p_i$ are distinct primes and $\alpha_i \geq 1$ for all $i$. Since

$$\exp(G) = \operatorname{lcm}\{\operatorname{ord}(a) : a \in G\},$$

there exists $h_1 \in G$ such that $p_1^{\alpha_1} | \operatorname{ord}(h_1)$. So $\operatorname{ord}(h_1) = p_1^{\alpha_1} q_1$ where $q_1 | p_2^{\alpha_2} \dots p_k^{\alpha_k}$. Let $g_1 = h_1^{q_1}$. For each $m \geq 1$, we have $g_1^m = h_1^{mq_1}$, and

$$h_1^{mq_1} = 1 \iff p_1^{\alpha_1} q_1 | mq_1 \iff p_1^{\alpha_1} | m.$$

Hence $\operatorname{ord}(g_1) = p_1^{\alpha_1}$. Similarly for $i = 2, \dots, k$, we can find elements $g_i$ of order $p_i^{\alpha_i}$. Let

$$a = g_1 g_2 \dots g_k$$

and $n = \operatorname{ord}(a)$. Now check as an exercise that $\operatorname{ord}(a) = \exp(G)$. This relies on

$$a^n = g_1^n g_2^n \dots g_k^n = 1,$$

which uses the assumption that $G$ is abelian. □

**Remark.** The previous example shows that the abelian condition in this theorem is necessary.

**Corollary 8.1.1.** *If $G$ is a finite abelian group with $\exp(G) = |G|$, then $G$ is cyclic.*

**Theorem 8.2.** *The group of units $\mathrm{GF}(p^n)^*$ of a Galois field is cyclic.*

*Proof.* Let $e = \exp(\mathrm{GF}(p^n)^*)$. Then $a^e = 1$ for all $a \in \mathrm{GF}(p^n)^*$, so every element $a \in \mathrm{GF}(p^n)^*$ is a root of $X^e - 1$. Since $X^e - 1$ has at most $e$ roots, we see that $|\mathrm{GF}(p^n)^*| \leq e$. But $e \leq |\mathrm{GF}(p^n)^*|$ since $\exp(\mathrm{GF}(p^n)^*)$ divides $|\mathrm{GF}(p^n)^*|$. Hence $|\mathrm{GF}(p^n)^*| = e$, so by Corollary 8.1.1, $\mathrm{GF}(p^n)^*$ is cyclic. □

## 8.3 Automorphisms of Fields

**Example 8.1.3.** The complex conjugation $f : \mathbb{C} \to \mathbb{C}$ given by $f(a + bi) = a - bi$ is an automorphism of $\mathbb{C}$. Observe that $f(c) = c$ if and only if $c \in \mathbb{R}$.

**Theorem 8.3.** *Let $K$ be a field. The set $\operatorname{Aut} K$ of automorphisms of $K$ forms a group under composition.*

*Proof.* First observe that composition is associative. The identity element in $\operatorname{Aut} K$ is the identity map $\mathrm{id}_K$. For inverses, let $\alpha \in \operatorname{Aut} K$. Since $\alpha$ is a bijection, there exists an inverse map $\alpha^{-1} : K \to K$, where $\alpha^{-1}(x)$ is the unique element $s$ such that $\alpha(s) = x$. Now we check that $\alpha^{-1}$ is also a homomorphism. For this, let $x, y \in K$ and suppose that $\alpha^{-1}(x) = s$ and $\alpha^{-1}(y) = t$. Then $\alpha(s) = x$ and $\alpha(t) = y$, so

$$\alpha(s + t) = \alpha(s) + \alpha(t) = x + y$$

since $\alpha$ is a homomorphism. Then we see that

$$\alpha^{-1}(x + y) = s + t = \alpha^{-1}(x) + \alpha^{-1}(y).$$

Similarly, $\alpha(st) = xy$, so

$$\alpha^{-1}(xy) = st = \alpha^{-1}(x)\alpha^{-1}(y).$$

Hence $\alpha^{-1} \in \operatorname{Aut} K$ and $\alpha \circ \alpha^{-1} = \alpha^{-1} \circ \alpha = \mathrm{id}_K$, so $\operatorname{Aut} K$ is indeed a group. □

**Definition 8.2.** We call $\operatorname{Aut} K$ the *group of automorphisms* of $K$.

**Definition 8.3.** Let $L$ be a field extension of $K$. A *$K$-automorphism* is an automorphism $\alpha : L \to L$ such that $\alpha(x) = x$ for all $x \in K$. The *Galois group* of $L$ over $K$, denoted $\operatorname{Gal}(L : K)$, is the set of $K$-automorphisms of $L$. The *Galois group* $\operatorname{Gal}(f)$ of a polynomial $f \in K[X]$ is $\operatorname{Gal}(L : K)$ where $L$ is a splitting field of $f$ over $K$.

**Theorem 8.4.** *The Galois group* $\operatorname{Gal}(L : K)$ *is a subgroup of* $\operatorname{Aut} L$.

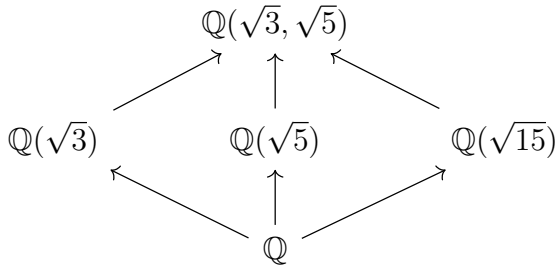
*Proof.* Clearly $\mathrm{id}_L \in \operatorname{Gal}(L : K)$ since it fixes all elements of $L$. Now let $\alpha, \beta \in \operatorname{Gal}(L : K)$. Then we have $\alpha(x) = x$ and $\beta(x) = x$ for all $x \in K$. Then $\beta^{-1}(x) = x$, which gives

$$\alpha\beta^{-1}(x) = \alpha(x) = x,$$

so $\alpha\beta^{-1} \in \operatorname{Gal}(L : K)$. Thus $\operatorname{Gal}(L : K)$ is a subgroup of $\operatorname{Aut} L$. □

**Remark.** The big idea here is that there is a correspondence between subfields $E$ with $K \subseteq E \subseteq L$ and subgroups $H$ of $\operatorname{Gal}(L : K)$.

**Exercise 8.2.** From a past homework, we identified the subfields of $\mathbb{Q}(\sqrt{3}, \sqrt{5})$ as:




Compare the subgroups of $\operatorname{Gal}(\mathbb{Q}(\sqrt{3}, \sqrt{5}) : \mathbb{Q})$ to the subfields of $\mathbb{Q}(\sqrt{3}, \sqrt{5})$ containing $\mathbb{Q}$.

# Lecture 9

# Feb. 7 — The Galois Correspondence

## 9.1 Automorphisms of Fields

**Example 9.0.1.** The complex conjugation $\beta : \mathbb{C} \to \mathbb{C}$ given by $\beta(a+bi) = a - bi$ is a nontrivial element of the Galois group of $\mathbb{C} : \mathbb{R}$. In fact, $\mathrm{Gal}(\mathbb{C} : \mathbb{R}) = \{\mathrm{id}, \beta\}$. Note that $\beta$ fixes $\mathbb{R}$, id fixes $\mathbb{C}$, and

$$\begin{array}{c} \mathbb{C} \\ \uparrow \\ \mathbb{R} \end{array}$$

## 9.2 The Galois Correspondence

**Definition 9.1.** Define

$$\Gamma(E) = \{\alpha \in \mathrm{Aut}\, L : \alpha(z) = z \text{ for all } z \in E\},$$
$$\Phi(H) = \{x \in L : \alpha(x) = x \text{ for all } \alpha \in H\},$$

where $E$ is a subfield of $L$ and $H$ is a subgroup of $\mathrm{Gal}(L : K)$. This is called the *Galois correspondence*.

**Example 9.1.1.** In the previous example of $\mathbb{C} : \mathbb{R}$, we have $\Gamma(\mathbb{C}) = \{\mathrm{id}\}$ and $\Gamma(\mathbb{R}) = \{\mathrm{id}, \beta\}$. We also have $\Phi(\{\mathrm{id}, \beta\}) = \mathbb{R}$ and $\Phi(\{\mathrm{id}\}) = \mathbb{C}$.

**Remark.** The goal is to determine: When are $\Gamma$ and $\Phi$ inverses of one another?

**Theorem 9.1.** *We have the following:*

1. *For every subfield $E$ of $L$ containing $K$, $\Gamma(E)$ is a subgroup of $\mathrm{Gal}(L : K)$.*
2. *Conversely, for every subgroup $H$ of $\mathrm{Gal}(L : K)$, $\Phi(H)$ is a subfield of $L$ containing $K$.*

*Proof.* See Howie. □

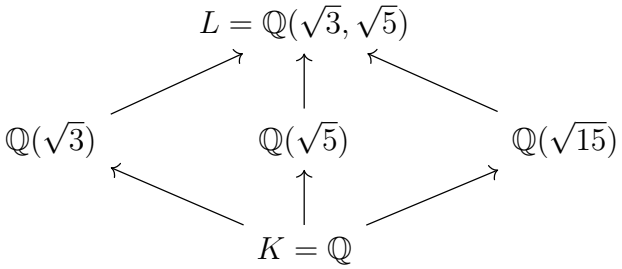
**Theorem 9.2.** *Let $z \in L \setminus K$. If $z$ is a root of $f \in K[X]$ and $\alpha \in \mathrm{Gal}(L : K)$, then $\alpha(z)$ is also a root of $f$.*

*Proof.* Let $f = a_0 + a_1 X + \cdots + a_n X^n$, where $a_i \in K$. Then since $\alpha$ fixes each $a_i \in K$, we have

$$\begin{aligned} f(\alpha(z)) = a_0 + a_1\alpha(z) + \cdots + a_n(\alpha(z))^n &= \alpha(a_0) + \alpha(a_1)\alpha(z) + \cdots + \alpha(a_n)(\alpha(z))^n \\ &= \alpha(a_0 + a_1 z + \cdots + a_n z^n) = \alpha(0) = 0, \end{aligned}$$

which completes the proof. □

**Example 9.1.2.** Recall this example from homework:




A basis for $L$ over $K$ is $\{1, \sqrt{3}, \sqrt{5}, \sqrt{15}\}$. Since $\sqrt{3}$ is a root of $X^2 - 3$, by the previous theorem, any element in $\mathrm{Gal}(L : K)$ must send $\sqrt{3} \mapsto \pm\sqrt{3}$. Similarly, any element must send $\sqrt{5} \mapsto \pm\sqrt{5}$. So the $\mathbb{Q}$-isomorphisms of $\mathbb{Q}(\sqrt{3}, \sqrt{5})$ are

$$\begin{aligned}
\alpha(a + b\sqrt{3} + c\sqrt{5} + d\sqrt{15}) &= a - b\sqrt{3} + c\sqrt{5} - d\sqrt{15}, \\
\beta(a + b\sqrt{3} + c\sqrt{5} + d\sqrt{15}) &= a + b\sqrt{3} - c\sqrt{5} - d\sqrt{15}, \\
\gamma(a + b\sqrt{3} + c\sqrt{5} + d\sqrt{15}) &= a - b\sqrt{3} - c\sqrt{5} + d\sqrt{15}, \\
\mathrm{id}(a + b\sqrt{3} + c\sqrt{5} + d\sqrt{15}) &= a + b\sqrt{3} + c\sqrt{5} + d\sqrt{15}.
\end{aligned}$$

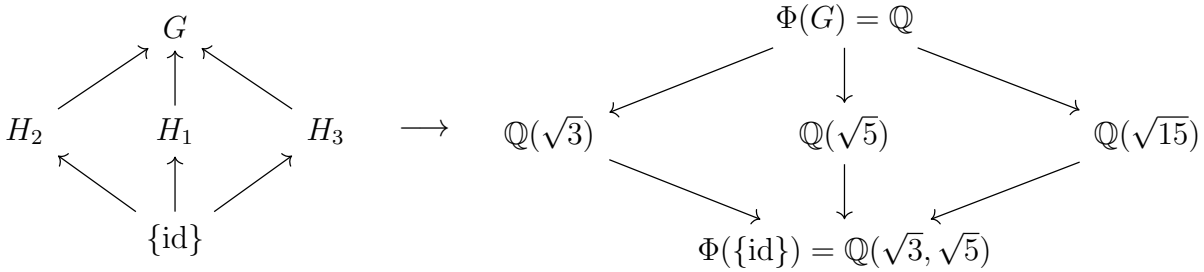
We can write the multiplication table for this group as:

| $\times$ | id | $\alpha$ | $\beta$ | $\gamma$ |
|---|---|---|---|---|
| id | id | $\alpha$ | $\beta$ | $\gamma$ |
| $\alpha$ | $\alpha$ | id | $\gamma$ | $\beta$ |
| $\beta$ | $\beta$ | $\gamma$ | id | $\alpha$ |
| $\gamma$ | $\gamma$ | $\beta$ | $\alpha$ | id |

The proper subgroups are $H_1 = \{\mathrm{id}, \alpha\}$, $H_2 = \{\mathrm{id}, \beta\}$, and $H_3 = \{\mathrm{id}, \gamma\}$. Also $\{\mathrm{id}\}$ and $G = \{\mathrm{id}, \alpha, \beta, \gamma\}$ are subgroups. Then

$$\begin{gathered}
\Phi(H_1) = \mathbb{Q}(\sqrt{5}), \quad \Phi(H_2) = \mathbb{Q}(\sqrt{3}), \quad \Phi(H_3) = \mathbb{Q}(\sqrt{15}), \\
\Phi(\{\mathrm{id}\}) = \mathbb{Q}(\sqrt{3}, \sqrt{5}), \quad \Phi(G) = \mathbb{Q}.
\end{gathered}$$

Under $\Phi$, this gives the diagram:




Also note that $\Gamma(\mathbb{Q}(\sqrt{3})) = \{\mathrm{id}, \alpha\}$ since

$$\alpha(a + b\sqrt{3} + c\sqrt{5} + d\sqrt{15}) = a - b\sqrt{3} + c\sqrt{5} - d\sqrt{15}.$$

**Exercise 9.1.** Show that $\Gamma$ is the inverse of $\Phi$ in the previous example.

**Theorem 9.3.** *Let $L : K$ be a field extension. Then*

1. *If $E_1, E_2$ are two subfields of $L$ containing $K$, then $E_1 \subseteq E_2$ implies $\Gamma(E_1) \supseteq \Gamma(E_2)$.*
2. *If $H_1, H_2$ are subgroups of $\mathrm{Gal}(L : K)$, then $H_1 \subseteq H_2$ implies $\Phi(H_1) \supseteq \Phi(H_2)$.*

*Proof.* (1) Suppose $E_1 \subseteq E_2$ and $\alpha \in \Gamma(E_2)$. Then $\alpha$ fixes every element in $E_2$, so since $E_1 \subseteq E_2$, $\alpha$ also fixes every element in $E_1$. Hence $\alpha \in \Gamma(E_1)$ by definition.

(2) Suppose $H_1 \subseteq H_2$ and let $z \in \Phi(H_2)$. Then $\alpha(z) = z$ for every $\alpha \in H_2$, and since $H_1 \subseteq H_2$, $\alpha(z) = z$ for every $\alpha \in H_1$ as well. Hence $z \in \Phi(H_1)$ by definition. □

**Remark.** Note that $\Gamma$ and $\Phi$ are not always inverses of one another.

**Example 9.1.3.** Consider the extension $\mathbb{Q}(\sqrt[3]{2}) : \mathbb{Q}$. If $\alpha \in \mathrm{Gal}(\mathbb{Q}(\sqrt[3]{2}) : \mathbb{Q})$, then

$$\alpha(\sqrt[3]{2})^3 = \alpha(2) = 2.$$

Since there is only one cube root of 2 in this field, we must have $\alpha(\sqrt[3]{2}) = \sqrt[3]{2}$. So $\mathrm{Gal}(\mathbb{Q}(\sqrt[3]{2}) : \mathbb{Q}) = \{\mathrm{id}\}$. So $\Gamma$ cannot be the inverse of $\Phi$ here since there are two subfields, namely $\mathbb{Q}(\sqrt[3]{2})$ and $\mathbb{Q}$. In particular,

$$\Gamma(\mathbb{Q}(\sqrt[3]{2})) = \Gamma(\mathbb{Q}) = \{\mathrm{id}\} \quad \text{and} \quad \Phi(\{\mathrm{id}\}) = \mathbb{Q}(\sqrt[3]{2}).$$

**Theorem 9.4.** *For any subfield $E$ of $L$ and subgroup $H$ of $\mathrm{Gal}(L : K)$, we have*

1. *$E \subseteq \Phi(\Gamma(E))$*
2. *and $H \subseteq \Gamma(\Phi(H))$.*

*Proof.* (1) Let $z \in E$. Then $\Gamma(E)$ is the set of all automorphisms fixing every element of $E$, and so $z$ is fixed by every element of $\Gamma(E)$. Hence $z \in \Phi(\Gamma(E))$.

(2) Let $\alpha \in H$. Then $\Phi(H)$ is the set of elements of $L$ fixed by every element of $H$, and so $\alpha$ fixes every element of $\Phi(H)$. Hence $\alpha \in \Gamma(\Phi(H))$. □

**Remark.** Now the goal will be to find sufficient conditions for $\Gamma$ and $\Phi$ to be inverses of one another.

## 9.3 Normal Extensions

**Definition 9.2.** A field extension $L : K$ is *normal* if every irreducible polynomial in $K[X]$ having at least one root in $L$ splits completely over $L$.

**Example 9.2.1.** An nonexample is $\mathbb{Q}(\sqrt[3]{2}) : \mathbb{Q}$. This is not a normal extension since $X^3 - 2$ is irreducible and has a root in $\mathbb{Q}(\sqrt[3]{2})$, but does not split completely over $\mathbb{Q}(\sqrt[3]{2})$.

**Remark.** Is $\mathbb{Q}(\sqrt{2}) : \mathbb{Q}$ normal?

**Theorem 9.5.** *A finite extension $L : K$ is normal if and only if it is a splitting field for some polynomial in $K[X]$.*

*Proof.* ($\Rightarrow$) Let $L$ be a finite normal extension and $\{z_1, \ldots, z_n\}$ be a basis for $L : K$. let $m_i$ be the minimal polynomial for $z_i$, and let

$$m = m_1 m_2 \ldots m_n.$$

Each $m_i$ has at least one root $z_i$ in $L$, hence $m$ splits completely over $L$ since $L$ is normal. Since $L$ is generated by $z_1, \ldots, z_n$, it is not possible for $m$ to split over a proper subfield of $L$, hence $L$ is a splitting field for $m$ over $K$.

($\Leftarrow$) See Howie. Relies on the isomorphism $K(\alpha) \to K(\beta)$ for $\alpha, \beta$ roots of an irreducible polynomial $f$. We also need properties of degrees of field extensions. □

**Corollary 9.5.1.** *Let $L$ be a normal extension of $K$ and $E$ a subfield of $L$ containing $K$. Then every injective $K$-homomorphism $\varphi : E \to L$ can be extended to a $K$-automorphism $\varphi^*$ of $L$.*

$$\begin{array}{ccc} E & \xrightarrow{\varphi} & L \\ {\scriptstyle i}\big\downarrow & \nearrow_{\varphi^*} & \\ L & & \end{array}$$

*Proof.* By the theorem, there exists $f \in K[X]$ such that $L$ is a splitting field for $f$ over $K$. But $L$ is also a splitting field for $f$ over $E$ and $\varphi(E)$. From here, a slight generalization of the proof of uniqueness of splitting fields gives the desired $K$-automorphism of $L$ extending $\varphi$. □

**Example 9.2.2.** Let $L = \mathbb{Q}(\sqrt{3}, \sqrt{5})$, $K = \mathbb{Q}$, and $E = \mathbb{Q}(\sqrt{3})$. Define $\varphi : E \to L$ by

$$\varphi(a + b\sqrt{3}) = a - b\sqrt{3},$$

which is an injective $K$-homomorphism. We have the following diagram:

$$\begin{array}{ccc} \mathbb{Q}(\sqrt{3}) & \xrightarrow{\varphi} & \mathbb{Q}(\sqrt{3}, \sqrt{5}) \\ {\scriptstyle i}\big\downarrow & \nearrow_{\varphi^*} & \\ \mathbb{Q}(\sqrt{3}, \sqrt{5}) & & \end{array}$$

Then we can define

$$\varphi^*(a + b\sqrt{3} + c\sqrt{5} + d\sqrt{15}) = a - b\sqrt{3} + c\sqrt{5} - d\sqrt{15}$$

as an extension of $\varphi$. Note that we could have also defined

$$\varphi^*(a + b\sqrt{3} + c\sqrt{5} + d\sqrt{15}) = a - b\sqrt{3} - c\sqrt{5} + d\sqrt{15}.$$

**Remark.** From the previous example we see that $\varphi^*$ is not unique.

# Lecture 10

# Feb. 12 — Normal Closures

## 10.1 Normal Closures

Recall this theorem from last time:

> **Theorem 9.5.** A finite extension $L : K$ is normal if and only if it is a splitting field for some polynomial in $K[X]$.

A natural question to ask is: Can we always extend a finite extension to make it normal?

**Definition 10.1.** Let $L : K$ be a finite extension. A field $N$ containing $L$ is a *normal closure* of $L : K$ if

1. $N$ is a normal extension of $K$,
2. and if $E$ is a proper subfield of $N$ containing $L$, then $E$ is not a normal extension of $K$.

**Theorem 10.1.** *Let $L : K$ be a finite extension. Then*

1. *there exists a normal closure $N$ of $L$ over $K$,*
2. *and $N$ is unique up to isomorphism.*

*Proof.* Let $\{z_1, \ldots, z_n\}$ be a basis for $L : K$. Since $L : K$ is finite, each $z_i$ is algebraic over $K$, with say minimal polynomial $m_i \in K[X]$. Let

$$m = m_1 \ldots m_n,$$

and let $N$ be the splitting field of $m$ over $L$. Then $N$ is also a splitting field of $m$ over $K$, since $L$ is generated over $K$ by some of the roots of $m$ in $N$. Hence $N$ is a normal extension of $K$ containing $L$.

To see that $N$ is the smallest such field, suppose $E$ is a subfield of $N$ containing $L$, and suppose $E$ is normal. For each $m_i$, $E$ contains a root $z_i$, so the normality of $E$ implies that $E$ contains all the roots of $m$, so $E = N$. For uniqueness, see Howie. The proof relies on the uniqueness of splitting fields. □

**Definition 10.2.** Let $K_1, \ldots, K_n$ be subfields of $L$. The *join* of $K_1, \ldots, K_n$, denoted

$$K_1 \vee K_2 \vee \cdots \vee K_n,$$

is the smallest subfield of $L$ containing $K_1 \cup K_2 \cup \cdots \cup K_n$.

**Remark.** The smallest subfield of $L$ containing $K_1 \cup K_2$ is $K_1 \vee K_2 = K_1(K_2) = K_2(K_1)$, similar to how the smallest subfield of $\mathbb{R}$ containing $\mathbb{Q} \cup \{\sqrt{3}\}$ is $\mathbb{Q}(\sqrt{3})$.

**Example 10.2.1.** Let $\mathbb{Q}(\sqrt[3]{2}), \mathbb{Q}(e^{2\pi i/3} \cdot \sqrt[3]{2}) \subseteq \mathbb{C}$. Then $\mathbb{Q}(\sqrt[3]{2}) \vee \mathbb{Q}(e^{2\pi i/3} \cdot \sqrt[3]{2}) = \mathbb{Q}(\sqrt[3]{2}, i\sqrt{3})$, since

$$e^{2\pi i/3} \cdot \sqrt[3]{2} = -\frac{\sqrt[3]{2}}{2} + \frac{i\sqrt{3}}{2}\sqrt[3]{2}.$$

**Remark.** In the above example, we have $\mathbb{Q}(\sqrt[3]{2}) \cong \mathbb{Q}(e^{2\pi i/3} \cdot \sqrt[3]{2}) \cong \mathbb{Q}[X]/\langle X^3 - 2\rangle$.

**Corollary 10.1.1.** *Let $L : K$ be a finite extension, and $N$ the normal closure of $L : K$. Then*

$$N = L_1 \vee L_2 \vee \cdots \vee L_k,$$

*where $L_1, L_2, \ldots, L_k$ are subfields of $N$ containing $K$ isomorphic to $L$.*

*Proof.* As in the previous proof, suppose $\{z_1, \ldots, z_n\}$ is a basis for $L : K$, so $L = K(z_1, \ldots, z_n)$, and $m_i$ is a minimal polynomial for $z_i$, and $N$ a splitting field for $m = m_1 \ldots m_n$ over $K$. Let $z_i'$ be an arbitrary root of $m_i$. Since $z_i$ and $z_i'$ are both roots of $m_i$, there exists a $K$-isomorphism $\varphi : K(z_i) \to K(z_i')$, which by Corollary 9.5.1 implies there exists a $K$-automorphism $\varphi^* : N \to N$. We have that

$$z_i' \in \varphi^*(L) \cong L,$$

so every root of $m_i$ is contained in a subfield $L' = \varphi^*(L)$ of $N$ that contains $K$ and is isomorphic to $L$, since $\varphi^*$ is a $K$-automorphism. Since $N$ is generated over $K$ by the roots of $m$, it is generated by finitely many subfields containing $K$ and isomorphic to $L$. □

**Example 10.2.2.** Find the normal closure of $\mathbb{Q}(\sqrt[3]{2})$ over $\mathbb{Q}$. Following the proof of the theorem,

$$\{1, \sqrt[3]{2}, \sqrt[3]{2}^2\}$$

is a basis of $\mathbb{Q}(\sqrt[3]{2}) : \mathbb{Q}$. The minimal polynomials of $1, \sqrt[3]{2}, \sqrt[3]{2}^2$ are $X - 1, X^3 - 2, X^3 - 4$, respectively. The splitting field of

$$(X - 1)(X^3 - 2)(X^3 - 4)$$

over $\mathbb{Q}$ is $\mathbb{Q}(\sqrt[3]{2}, i\sqrt{3})$, since

$$X^3 - 2 = (X - \sqrt[3]{2})(X - e^{2\pi i/3}\sqrt[3]{2})(X - e^{-2\pi i/3}\sqrt[3]{2})$$

and

$$X^3 - 4 = (X - \sqrt[3]{2}^2)(X - e^{2\pi i/3}\sqrt[3]{2}^2)(X - e^{-2\pi i/3}\sqrt[3]{2}^2).$$

So $\mathbb{Q}(\sqrt[3]{2}, i\sqrt{3}) = L_1 \vee L_2 \vee L_3$, where $L_1 = \mathbb{Q}(\sqrt[3]{2})$, $L_2 = \mathbb{Q}(e^{2\pi i/3}\sqrt[3]{2})$, and $L_3 = \mathbb{Q}(e^{-2\pi i/3}\sqrt[3]{2})$, and

$$L_1 \cong L_2 \cong L_3 \cong \mathbb{Q}[X]/\langle X^3 - 2\rangle.$$

**Theorem 10.2.** *Let $L : K$ be a finite normal extension and $E$ a subfield of $L$ containing $K$. Then $E$ is a normal extension of $K$ if and only if every $K$-monomorphism of $E$ into $L$ is a $K$-automorphism of $E$.*

*Proof.* ($\Rightarrow$) Suppose $E : K$ is normal and let $\varphi : E \to L$ be a $K$-monomorphism. Now we would like to show that $\varphi(E) \subseteq E$. So let let $z \in E$ and suppose

$$m = a_0 + a_1 X + \cdots + a_n X^n$$

is the minimal polynomial of $z$ over $K$. Then

$$a_0 + a_1 z + \cdots + a_n z^n = 0,$$

so that

$$a_0 + a_1\varphi(z) + \cdots + a_n\varphi(z)^n = 0$$

since $\varphi$ is a homomorphism fixing $K$ pointwise. Hence $\varphi(z)$ is also a root of $m$ in $L$. Since $E : K$ is normal, the irreducible polynomial $m$ splits completely over $E$. Hence $\varphi(z) \in E$, so that $\varphi(E) \subseteq E$. Then[1]

$$[\varphi(E) : K] = [\varphi(E) : \varphi(K)] = [E : K] = [E : \varphi(E)][\varphi(E) : K],$$

so $[E : \varphi(E)] = 1$. Hence $\varphi(E) = E$, so $\varphi$ is a $K$-automorphism of $E$.

($\Leftarrow$) Suppose every $K$-monomorphism $E \to L$ is a $K$-automorphism of $E$. Let $f$ be an irreducible polynomial in $K[X]$ having a root $z \in E$. We need to show that $f$ splits completely over $E$. Since $L$ is normal, $f$ splits completely over $L$. Let $z'$ be another root of $f$ in $L$. Then there exists a $K$-automorphism $K(z) \to K(z')$ which sends $z \mapsto z'$, which by Corollary 9.5.1 extends to a $K$-automorphism $\psi$ of $L$. Let $\psi^* = \psi|_E$, i.e. the restriction of $\psi$ to $E$. By hypothesis, $\psi^*$ is a $K$-automorphism of $E$, so

$$z' = \psi(z) = \psi^*(z) \in E.$$

That is, $E$ is normal. □

**Example 10.2.3.** Consider $\mathbb{Q}(\sqrt[3]{2}) : \mathbb{Q}$, which is not normal. The $\mathbb{Q}$-monomorphism $\varphi : \mathbb{Q}(\sqrt[3]{2}) \to \mathbb{C}$ given by

$$\varphi(a + b\sqrt[3]{2} + c\sqrt[3]{2}^2) = a + be^{2\pi i/3}\sqrt[3]{2} + ce^{-2\pi i/3}\sqrt[3]{2}^2$$

is not an automorphism of $\mathbb{Q}(\sqrt[3]{2})$.

**Example 10.2.4.** Consider $\mathbb{Q}(\sqrt{2}) : \mathbb{Q}$, which is normal. The $\mathbb{Q}$-monomorphisms are id and

$$\varphi(a + b\sqrt{2}) = a - b\sqrt{2},$$

which are both $\mathbb{Q}$-automorphisms of $\mathbb{Q}(\sqrt{2})$.

## 10.2 Separable Extensions

**Definition 10.3.** An irreducible polynomial $f \in K[X]$ is *separable* over $K$ if it has no repeated roots over a splitting field. A polynomial $g \in K[X]$ is *separable* over $K$ if its irreducible factors are separable over $K$. An algebraic element in $L : K$ is *separable* over $K$ if its minimal polynomial is separable over $K$. An algebraic extension $L : K$ is *separable* if every $\alpha \in L$ is separable over $K$.

**Remark.** A polynomial like $(X - 2)^2$ actually *is* separable over $\mathbb{Q}$ since its irreducible factors are $X - 2$ and $X - 2$, which are each separable.

**Definition 10.4.** A field $K$ is *perfect* if every polynomial in $K[X]$ is separable over $K$.

**Theorem 10.3.** *We have the following:*

1. *Every field of characteristic* 0 *is perfect.*

[1] We need to make this argument since $E$ may be infinite, so injectivity does not imply bijectivity.

*2. Every finite field is perfect.*

*Proof.* (1) It suffices to show that if $\operatorname{char} K = 0$, then any irreducible polynomial $f$ is separable. Let

$$f = a_0 + a_1 X + \cdots + a_n X^n$$

for $n \geq 1$ and suppose $f$ is not separable. Then $f$ and $Df$ have a non-constant common factor $d$. Since $f$ is irreducible, $d$ must be a constant multiple of $f$, and thus $d$ cannot divide $Df$ unless

$$Df = a_1 + 2a_2 X + \cdots + n a_n X^{n-1}$$

is the zero polynomial, by comparing degrees. Then

$$a_1 = 2a_2 = \cdots = n a_n = 0.$$

Since $\operatorname{char} K = 0$, this implies

$$a_1 = a_2 = \cdots = a_n = 0,$$

and so $f = a_0$, a constant polynomial.[2] Contradiction. Hence $f$ is separable.

(2) The same argument as above implies the only possible inseparable irreducible polynomials are of the form[3]

$$f(X) = b_0 + b_1 X^p + b_2 X^{2p} \cdots + b_m X^{mp}.$$

Now Theorem 7.24 of Howie implies that if $K$ is finite, such a polynomial is reducible. Hence every irreducible polynomial is separable, so $K$ is perfect. See Howie for details. □

**Remark.** Recall that $\mathbb{Z}_p(X)$ is an example of an infinite field with characteristic $p$.

[2]Recall that an irreducible polynomial is by definition a non-unit.

[3]We can still conclude $k a_k = 0$ implies $a_k = 0$ when $k$ is not a multiple of $p$.

# Lecture 11

# Feb. 21 — Galois Extensions

## 11.1 Example of an Inseparable Extension

**Example 11.0.1.** The field $K = \mathbb{Z}_p(X)$ is not perfect. Consider the polynomial

$$f = Y^p - X \in \mathbb{Z}_p(X)[Y],$$

which is irreducible. Now let $L$ be a splitting field of $f$ over $K$ and $\alpha$ a root of $f$, i.e. $\alpha^p - X = 0$. Then

$$(Y - \alpha)^p = Y^p - \alpha^p = Y^p - X$$

by freshman exponentiation. In particular, $\alpha$ is a repeated root of $f$ in $L$.

## 11.2 Galois Extensions

**Definition 11.1.** A *Galois extension* of $K$ is a finite extension that is both normal and separable.

**Remark.** The main goal here is: For a Galois extension, $\Gamma$ and $\Phi$ are inverses of one another.

**Theorem 11.1.** *Let $L : K$ be a separable extension of degree $n$. Then there are exactly $n$ distinct $K$-monomorphisms of $L$ into a normal closure $N$ of $L$ over $K$.*

*Proof.* Use strong induction on the degree of $L : K$. See Howie for details. □

**Corollary 11.1.1.** *If $L : K$ is Galois, then $|\mathrm{Gal}(L : K)| = [L : K]$.*

*Proof.* If $L : K$ is Galois, then $L : K$ is normal and separable. So the previous theorem applies, where $L$ is its own normal closure. So we get exactly $[L : K]$ distinct $K$-monomorphisms of $L$ into $L$, which are precisely the $K$-automorphisms of $L$ and thus the elements of the Galois group. □

**Example 11.1.1.** The extension $\mathbb{Q}(\sqrt[3]{2}, i\sqrt{3}) : \mathbb{Q}$ is Galois with $[\mathbb{Q}(\sqrt[3]{2}, i\sqrt{3}) : \mathbb{Q}] = 6$. We could have

$$\sqrt[3]{2} \mapsto \sqrt[3]{2} \text{ or } e^{2\pi i/3}\sqrt[3]{2} \text{ or } e^{-2\pi i/3}\sqrt[3]{2} \quad \text{and} \quad i\sqrt{3} \mapsto i\sqrt{3} \text{ or } -i\sqrt{3}.$$

Combinining these options gives us 6 distinct maps, so these must in fact all be $\mathbb{Q}$-automorphisms of $\mathbb{Q}(\sqrt[3]{2}, i\sqrt{3})$, since we know the Galois group has size 6. In fact, $\mathrm{Gal}(\mathbb{Q}(\sqrt[3]{2}, i\sqrt{3}) : \mathbb{Q}) \cong S_3 \cong D_3$.

**Remark.** The proper nontrivial subfields of $\mathbb{Q}(\sqrt[3]{2}, i\sqrt{3})$ are $\mathbb{Q}(\sqrt[3]{2})$, $\mathbb{Q}(e^{2\pi i/3}\sqrt[3]{2})$, $\mathbb{Q}(e^{-2\pi i/3}\sqrt[3]{2})$, and $\mathbb{Q}(i\sqrt{3})$. Maybe draw a pretty diagram with this showing the Galois correspondence.

**Exercise 11.1.** Show that $\mathbb{Z}/6\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$.

**Exercise 11.2.** Show that $\mathbb{Z}/4\mathbb{Z} \ncong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

**Theorem 11.2.** *Let $L : K$ be a finite extension. Then $\Phi(\mathrm{Gal}(L : K)) = K$ if and only if $L : K$ is normal and separable.*

*Proof.* ($\Leftarrow$) Let $[L : K] = n$. By Corollary 11.1.1, we have $|\mathrm{Gal}(L : K)| = n$. Let $K' = \Phi(\mathrm{Gal}(L : K))$. By definition, $K \subseteq K'$. By Theorem 7.12 of Howie, we find that

$$[L : K'] = |\mathrm{Gal}(L : K)|.$$

Hence $[L : K'] = [L : K]$ and thus we conclude that $K = K'$.

($\Rightarrow$) See Howie. □

**Exercise 11.3.** Show that if $K \subseteq K'$ and $[L : K'] = [L : K]$, then $K = K'$.

**Theorem 11.3.** *Let $L : K$ be Galois and $E$ a subfield of $L$ containing $K$. If $\delta \in \mathrm{Gal}(L : K)$, then*

$$\Gamma(\delta(E)) = \delta\Gamma(E)\delta^{-1}.$$

*Proof.* We begin by showing $\delta\Gamma(E)\delta^{-1} \subseteq \Gamma(\delta(E))$. For this, let $\theta \in \Gamma(E)$ and $z' \in \delta(E)$. Then there exists a unique $z \in E$ such that $\delta(z) = z'$, since $\delta$ is an automorphism. Then

$$\delta\theta\delta^{-1}(z') = \delta\theta(z) = \delta(z) = z'$$

since $\delta(z) = z'$ and $\theta \in \Gamma(E)$. So we see that $\delta\theta\delta^{-1} \in \Gamma(\delta(E))$.

Now for $\Gamma(\delta(E)) \subseteq \delta\Gamma(E)\delta^{-1}$, we will show that $\delta^{-1}\Gamma(\delta(E))\delta \subseteq \Gamma(E)$. Let $\theta' \in \Gamma(\delta(E))$ and $z \in E$. Then $\delta(z) \in \delta(E)$ and so $\theta'(\delta(z)) = \delta(z)$. Thus

$$(\delta^{-1}\theta'\delta)(z) = (\delta^{-1} \circ \delta)(z) = z,$$

so we get $\delta^{-1}\theta'\delta \in \Gamma(E)$, as desired. □

**Example 11.1.2.** Consider $\mathbb{Q}(\sqrt[3]{2}, i\sqrt{3}) : \mathbb{Q}$. Define the elements of $\mathrm{Gal}(\mathbb{Q}(\sqrt[3]{2}, i\sqrt{3}) : \mathbb{Q})$ by

$$\mu_1 : \sqrt[3]{2} \mapsto \sqrt[3]{2},\ i\sqrt{3} \mapsto -i\sqrt{3}, \quad \mu_2 : \sqrt[3]{2} \mapsto e^{2\pi i/3}\sqrt[3]{2},\ i\sqrt{3} \mapsto -i\sqrt{3},$$
$$\mu_3 : \sqrt[3]{2} \mapsto e^{-2\pi i/3}\sqrt[3]{2},\ i\sqrt{3} \mapsto -i\sqrt{3},$$
$$\rho_1 : \sqrt[3]{2} \mapsto e^{2\pi i/3}\sqrt[3]{2},\ i\sqrt{3} \mapsto i\sqrt{3}, \quad \rho_2 : \sqrt[3]{2} \mapsto e^{-2\pi i/3}\sqrt[3]{2},\ i\sqrt{3} \mapsto i\sqrt{3}.$$

Let $\delta = \mu_3$ and $E = \mathbb{Q}(\sqrt[3]{2})$. Then $\delta(E) = \mathbb{Q}(e^{-2\pi i/3}\sqrt[3]{2})$ since $\mu_3(\sqrt[3]{2}) = e^{-2\pi i/3}\sqrt[3]{2}$. Now

$$\begin{aligned}\mu_2(e^{-2\pi i/3}\sqrt[3]{2}) &= \mu_2(e^{-2\pi i/3})\mu_2(\sqrt[3]{2}) = \mu_2(-\frac{1}{2} - i\frac{\sqrt{3}}{2})\mu_2(\sqrt[3]{2}) \\ &= (-\frac{1}{2} + i\frac{\sqrt{3}}{2})(e^{2\pi i/3}\sqrt[3]{2}) = e^{2\pi i/3}e^{2\pi i/3}\sqrt[3]{2} = e^{-2\pi i/3}\sqrt[3]{2},\end{aligned}$$

so $\Gamma(\delta(E)) = \{\mathrm{id}, \mu_2\}$. Also $\Gamma(E) = \{\mathrm{id}, \mu_1\}$, and we find that

$$\delta\Gamma(E)\delta^{-1} = \{\delta\mathrm{id}\delta^{-1}, \delta\mu_1\delta^{-1}\} = \{\mathrm{id}, \mu_3\mu_1\mu_3^{-1}\} = \{\mathrm{id}, \mu_2\},$$

so indeed we have $\Gamma(\delta(E)) = \delta\Gamma(E)\delta^{-1}$ in this case.

# Lecture 12

# Feb. 26 — The Fundamental Theorem

## 12.1 Normal Subgroups

Recall the following:

**Definition 12.1.** A subgroup $H$ of $G$ is *normal* if

$$gHg^{-1} = H$$

for all $g \in G$ (equivalently, $gH = Hg$ for all $g \in G$). This is denoted $H \triangleleft G$.

**Remark.** If $G$ is abelian, then every subgroup of $G$ is normal.

**Exercise 12.1.** If $[G : H] = 2$, then $H$ is normal.

**Remark.** Normality is a necessary and sufficient condition for $G/H$ to be a well-defined group (with operation induced by the operation on $G$).

**Theorem 12.1.** *Let $\varphi : G \to G'$ be a surjective homomorphism with kernel $H$. Then there exists a unique isomorphism $\alpha : G/H \to G'$ such that the following diagram commutes:*

$$\begin{array}{ccc} G & \xrightarrow{\varphi} & G' \\ {\scriptstyle\pi}\downarrow & \nearrow_{\alpha} & \\ G/H & & \end{array}$$

*Here $\pi : G \to G/H$ is the canonical projection $g \mapsto gH$.*

## 12.2 The Fundamental Theorem of Galois Theory

**Theorem 12.2** (Fundamental theorem of Galois theory)**.** *Let $L : K$ be a separable, normal extension of finite degree $n$. Then*

1. *For all subfields $E$ of $L$ containing $K$ and for all subgroups $H$ of $\mathrm{Gal}(L : K)$,*
   - *(a) $\Phi(\Gamma(E)) = E$ and $|\Gamma(E)| = [L : E]$,*
   - *(b) $\Gamma(\Phi(H)) = H$ and $|\mathrm{Gal}(L : K)|/|\Gamma(E)| = [E : K]$.*
2. *A subfield $E$ is a normal extension of $K$ if and only if $\Gamma(E)$ is a normal subgroup of $\mathrm{Gal}(L : K)$. If $E : K$ is normal, then*

$$\mathrm{Gal}(E : K) \cong \mathrm{Gal}(L : K)/\Gamma(E).$$

*Proof.* (1) By a homework exercise, $L : K$ being normal implies that $L : E$ is normal. Also, by Howie's Theorem 7.26, $L : K$ being finite and separable implies that $L : E$ is separable. Hence $L : E$ is Galois, so $|\Gamma(E)| = [L : E]$. Then

$$[E : K] = \frac{[L : K]}{[L : E]} = \frac{|\mathrm{Gal}(L : K)|}{|\Gamma(E)|}.$$

Now $\Gamma(E) = \mathrm{Gal}(L : E)$, so $L : E$ being Galois and Howie's Theorem 7.30 imply that $\Phi(\Gamma(E)) = E$. Now let $H$ be a subgroup of $\mathrm{Gal}(L : K)$. We showed that $H \subseteq \Gamma(\Phi(H))$. Also $\Phi\Gamma\Phi = \Phi$, so

$$|H| = [L : \Phi(H)] = [L : \Phi\Gamma\Phi(H)] = |\Gamma\Phi(H)|$$

by Howie's Theorem 7.12. Now finiteness and $H \subseteq \Gamma(\Phi(H))$ imply that $H = \Gamma(\Phi(H))$.

(2) ($\Rightarrow$) Suppose $E : K$ is normal and let $\delta \in \mathrm{Gal}(L : K)$. Let $\delta' = \delta|_E$, the restriction of $\delta$ to $E$. Hence $\delta'$ is a monomorphism $E \to L$ and thus a $K$-automorphism of $E$, by Howie's Theorem 7.21. Hence

$$\delta(E) = \delta'(E) = E,$$

and so by Theorem 11.3,

$$\Gamma(E) = \Gamma(\delta(E)) = \delta\Gamma(E)\delta^{-1},$$

i.e. $\Gamma(E)$ is a normal subgroup of $\mathrm{Gal}(L : K)$.

($\Leftarrow$) Suppose $\Gamma(E)$ is a normal subgroup of $\mathrm{Gal}(L : K)$. Let $\delta_1$ be a $K$-monomorphism from $E$ to $L$. This extends (by Howie's Corollary 7.14) to a $K$-automorphism $\delta$ of $L$. Since $\Gamma(E)$ is normal, $\delta\Gamma(E)\delta^{-1} = \Gamma(E)$. Hence by Theorem 11.3, we get $\Gamma(\delta(E)) = \Gamma(E)$. Since $\Gamma$ is injective,

$$\delta_1(E) = \delta(E) = E,$$

so $\delta$ is a $K$-automorphism of $E$. By Howie's Theorem 7.21, this implies $E : K$ is normal.

Now suppose $E : K$ is normal, and we want to show that

$$\mathrm{Gal}(E : K) \cong \mathrm{Gal}(L : K)/\Gamma(E).$$

Let $\delta \in \mathrm{Gal}(L : K)$ and $\delta' = \delta|_E$. By Howie's Theorem 7.21, having $E : K$ be normal implies that $\delta'(E) = E$. Thus we can define $\theta : \mathrm{Gal}(L : K) \to \mathrm{Gal}(E : K)$ by $\delta \mapsto \delta'$, i.e. restricting $\delta$ to $E$. Clearly $\theta$ is surjective onto $\mathrm{Gal}(E : K)$. Also, we see that

$$\ker\theta = \{\delta \in \mathrm{Gal}(L : K) \mid \delta|_E = \mathrm{id}_E\} = \Gamma(E).$$

Hence by the first isomorphism theorem, $\mathrm{Gal}(E : K) \cong \mathrm{Gal}(L : K)/\ker\theta = \mathrm{Gal}(L : K)/\Gamma(E)$. □

**Exercise 12.2.** Show that $\Phi\Gamma\Phi = \Phi$.

**Exercise 12.3.** Check that $\theta$ is a homomorphism.

**Example 12.1.1.** Let $L = \mathbb{Q}(\sqrt[4]{2}, i)$ with $[L : \mathbb{Q}] = 8$. Any $\mathbb{Q}$-automorphism in $\mathrm{Gal}(L : \mathbb{Q})$ must map

$$i \mapsto \pm i, \quad \sqrt[4]{2} \mapsto \pm\sqrt[4]{2}, \pm i\sqrt[4]{2}.$$

So there are only 8 possible automorphisms, and thus each of these must in fact be automorphisms since $|\mathrm{Gal}(L : \mathbb{Q})| = [L : \mathbb{Q}] = 8$. We can enumerate these automorphisms via

$$\mathrm{id}, \quad \alpha : \sqrt[4]{2} \mapsto i\sqrt[4]{2}, i \mapsto i, \quad \beta : \sqrt[4]{2} \mapsto -\sqrt[4]{2}, i \mapsto i, \quad \gamma : \sqrt[4]{2} \mapsto -i\sqrt[4]{2}, i \mapsto i,$$
$$\lambda : \sqrt[4]{2} \mapsto \sqrt[4]{2}, i \mapsto -i, \quad \mu : \sqrt[4]{2} \mapsto i\sqrt[4]{2}, i \mapsto -i, \quad \nu : \sqrt[4]{2} \mapsto -\sqrt[4]{2}, i \mapsto -i,$$
$$\rho : \sqrt[4]{2} \mapsto -i\sqrt[4]{2}, i \mapsto -i.$$

Note that $\mathrm{Gal}(L:\mathbb{Q})$ is not abelian, as

$$\lambda\alpha(\sqrt[4]{2}) = \lambda(i\sqrt[4]{2}) = -i\sqrt[4]{2}, \quad \lambda\alpha(i) = \lambda(i) = i,$$

so $\lambda\alpha = \rho$. We can show as an exercise that $\alpha\lambda = \mu \neq \rho$, so $\lambda\alpha \neq \alpha\lambda$. The subgroups of $\mathrm{Gal}(L:\mathbb{Q})$ are

$$G = \mathrm{Gal}(L:\mathbb{Q}), \quad \{\mathrm{id}\}, \quad \{\mathrm{id},\beta\}, \quad \{\mathrm{id},\mu\}, \quad \{\mathrm{id},\nu\}, \quad \{\mathrm{id},\rho\},$$
$$\{\mathrm{id},\alpha,\beta,\gamma\}, \qquad \{\mathrm{id},\beta,\lambda,\nu\}, \quad \{\mathrm{id},\beta,\mu,\rho\}.$$

Now we could draw a nice subgroup lattice for this (identical to $D_4$, the dihedral group of order 8). The normal subgroups of $\mathrm{Gal}(L:\mathbb{Q})$ are

$$G, \quad \{\mathrm{id},\beta,\lambda,\nu\}, \quad \{\mathrm{id},\alpha,\beta,\gamma\}, \quad \{\mathrm{id},\beta,\mu,\rho\}, \quad \{\mathrm{id},\beta\}, \quad \{\mathrm{id}\}.$$

Let $H_1 = \{\mathrm{id},\alpha,\beta,\gamma\}$. Then $\Phi(H_1) = \mathbb{Q}(i)$. Also $\Phi(\{\mathrm{id},\lambda\}) = \mathbb{Q}(\sqrt[4]{2})$ and $\Phi(\{\mathrm{id},\nu\}) = \mathbb{Q}(i\sqrt[4]{2})$. We can also see that $\Phi(\{\mathrm{id},\mu\}) = \mathbb{Q}((1+i)\sqrt[4]{2})$ and $\Phi(\{\mathrm{id},\rho\}) = \mathbb{Q}((1-i)\sqrt[4]{2})$.

**Exercise 12.4.** Write out the multiplication table for $\mathrm{Gal}(L:\mathbb{Q})$.

# Lecture 13

# Feb. 28 — Join of Subgroups and Subfields

## 13.1 Join of Subgroups

Let $H_1, H_2$ be subgroups of $G$.

**Exercise 13.1.** Show that $H_1 \cap H_2$ is a subgroup of $G$.

**Remark.** In general, $H_1 \cup H_2$ is *not* a subgroup of $G$.

**Definition 13.1.** The *join* of $H_1$ and $H_2$, denoted $H_1 \vee H_2$, is the smallest subgroup of $G$ containing $H_1 \cup H_2$, i.e. $H_1 \vee H_2$ consists of all products of the form

$$a_1 b_1 \dots a_n b_n,$$

where $a_i \in H_1$ and $b_i \in H_2$ for all $n$.

**Remark.** Recall that if $E_1$ and $E_2$ are subfields of $L$, then $E_1 \cap E_2$ is also a subfield of $L$, as is the join

$$E_1 \vee E_2 = E_1(E_2) = E_2(E_1).$$

**Example 13.1.1.** In Example 12.1.1, we have $\{\mathrm{id}, \beta\} \vee \{\mathrm{id}, \lambda\} = \{\mathrm{id}, \beta, \lambda, \nu\}$. Now notice that

$$\Phi(\{\mathrm{id}, \beta\}) = \mathbb{Q}(i, \sqrt{2}), \quad \Phi(\{\mathrm{id}, \lambda\}) = \mathbb{Q}(\sqrt[4]{2}), \quad \Phi(\{\mathrm{id}, \beta, \lambda, \nu\}) = \mathbb{Q}(\sqrt{2}).$$

Notice that $\mathbb{Q}(i, \sqrt{2}) \cap \mathbb{Q}(\sqrt[4]{2}) = \mathbb{Q}(\sqrt{2})$.

**Theorem 13.1.** *Let $L : K$ be Galois and $E_1, E_2$ subfields of $L$ containing $K$. If*

$$\Gamma(E_1) = H_1, \quad \Gamma(E_2) = H_2,$$

*then $\Gamma(E_1 \cap E_2) = H_1 \vee H_2$ and $\Gamma(E_1 \vee E_2) = H_1 \cap H_2$.*

*Proof.* Certainly $E_1 \cap E_2 \subseteq E_1$, so $H_1 = \Gamma(E_1) \subseteq \Gamma(E_1 \cap E_2)$, since the Galois correspondence is order reversing. Similarly, $H_2 = \Gamma(E_2) \subseteq \Gamma(E_1 \cap E_2)$, so $H_1 \vee H_2 \subseteq \Gamma(E_1 \cap E_2)$. Now $H_1 \subseteq H_1 \vee H_2$, so we get $E_1 = \Phi(H_1) \supseteq \Phi(H_1 \vee H_2)$. Similarly, $E_2 = \Phi(H_2) \supseteq \Phi(H_1 \vee H_2)$, so $\Phi(H_1 \vee H_2) \subseteq E_1 \cap E_2$. Since $L : K$ is Galois, we get

$$H_1 \vee H_2 \supseteq \Gamma(E_1 \cap E_2)$$

by applying $\Gamma$ to both sides. So $\Gamma(E_1 \cap E_2) = H_1 \vee H_2$.

The proof for $\Gamma(E_1 \vee E_2) = H_1 \cap H_2$ is similar, see Howie for details. □

# Lecture 14

# Mar. 4 — Solvable Groups

## 14.1 Solvable Groups

**Definition 14.1.** A finite group $G$ is *solvable* if, for some $m \geq 0$, it has a finite series

$$\{\mathrm{id}\} = G_0 \subseteq G_1 \subseteq \cdots \subseteq G_m = G$$

of subgroups such that for $i = 0, \ldots, m-1$,

1. $G_i \triangleleft G_{i+1}$,
2. and $G_{i+1}/G_i$ is cyclic.

**Remark.** We require $G_i \triangleleft G_{i+1}$, but $G_i$ need not be normal in $G$.

**Example 14.1.1.** Let $G = \mathrm{Gal}(\mathbb{Q}(i, \sqrt[4]{2}), \mathbb{Q})$ from Example 12.1.1. We have

$$\{\mathrm{id}\} \subseteq \{\mathrm{id}, \lambda\} \subseteq \{\mathrm{id}, \beta, \lambda, \nu\} \subseteq G,$$

where $G_i \triangleleft G_{i+1}$ and $|G_{i+1}/G_i| = 2$, so it is cyclic. Observe that $\{\mathrm{id}, \lambda\}$ is not normal in G, since

$$\alpha\{\mathrm{id}, \lambda\} = \{\alpha, \mu\} \neq \{\alpha, \rho\} = \{\mathrm{id}, \lambda\}\alpha.$$

**Theorem 14.1.** *Every finite abelian group $G$ is solvable.*

*Proof.* Recall from the structure theorem for finitely generated abelian groups that every finite abelian group is a direct sum of cyclic groups. Then

$$G = U_1 \oplus U_2 \oplus \cdots \oplus U_k$$

where each $U_i$ is cyclic. Let

$$G_i = U_1 \oplus \cdots \oplus U_i.$$

Observe that $G_i \triangleleft G_{i+1}$ since $G$ is abelian, and $G_{i+1}/G_i \cong U_{i+1}$, which is cyclic. So $G$ is solvable. □

**Remark.** Recall that $S_n$ is the symmetric group on $n$ elements.

**Theorem 14.2.** *Every permutation can be expressed as a product of transpositions (i.e. 2-cycles).*

**Definition 14.2.** A permutation $\sigma$ is *even* (respectively *odd*) if $\sigma$ can be expressed as a product of an *even* (respectively *odd*) number of transpositions. This is well defined. The set

$$A_n = \text{subgroup of even permutations}$$

is called the *alternating group*.

**Example 14.2.1.** We have $S_3 = \{\mathrm{id}, (12), (23), (13), (123), (132)\}$. We can write

$$\{\mathrm{id}\} \subseteq \{\mathrm{id}, (123), (132)\} \subseteq S_3.$$

Call these $G_i$ for $i = 0, 1, 2$. Then $G_i \triangleleft G_{i+1}$, and $G_2/G_1 \cong \mathbb{Z}_2$ and $G_1/G_0 = G_1 \cong \mathbb{Z}_3$. So $S_3$ is solvable.

**Example 14.2.2.** The symmetric group $S_4$ is solvable. We can write

$$\{\mathrm{id}\} \subseteq \{\mathrm{id}, (12)(34)\} \subseteq \{\mathrm{id}, (12)(34), (13)(24), (14)(23)\} \subseteq A_4 \subseteq S_4.$$

Call the first three subgroups $G_i$ for $i = 0, 1, 2$. Then $G_i \triangleleft G_{i+1}$, and we have

$$S_4/A_4 \cong \mathbb{Z}_2, \quad A_4/G_2 \cong \mathbb{Z}_3, \quad G_2/G_1 \cong \mathbb{Z}_2, \quad G_1/G_0 \cong \mathbb{Z}_2.$$

**Exercise 14.1.** Show that $G_2 = \{\mathrm{id}, (12)(34), (13)(24), (14)(23)\} \triangleleft A_4$.

**Definition 14.3.** A group is *simple* if it has no proper normal subgroups.

**Remark.** A non-abelian simple group is not solvable.

**Theorem 14.3.** *For $n \geq 5$, the alternating group $A_n$ is simple.*

*Proof.* See Howie. □

**Theorem 14.4.** *We have the following:*

1. *If $G$ is solvable, then every subgroup of $G$ is solvable.*
2. *If $G$ is solvable and $N \triangleleft G$, then $G/N$ is solvable.*
3. *Let $N \triangleleft G$. Then $G$ is solvable if and only if $N$ and $G/N$ are solvable.*

*Proof.* (1) Since $G$ is solvable, there exists

$$\{\mathrm{id}\} = G_0 \subseteq G_1 \subseteq \cdots \subseteq G_m = G$$

where $G_i \triangleleft G_{i+1}$ and $G_{i+1}/G_i$ is cyclic. Now let $H$ be a subgroup of $G$. Let $K_i = G_i \cap H$. Now check as an exercise that

$$\{\mathrm{id}\} = K_0 \subseteq K_1 \subseteq \cdots \subseteq K_m = H$$

is the desired series of subgroups. In particular, check that $K_i \triangleleft K_{i+1}$ and $K_{i+1}/K_i$ is cyclic (show that it is a subgroup of the cyclic group $G_{i+1}/G_i$).

(2) Take

$$N/N = NG_0/N \subseteq NG_1/N \subseteq \ldots NG_m/N = G/N$$

as the desired series of subgroups. Here

$$NG = \{ng \mid n \in N, g \in G\}.$$

Check as an exercise that this construction works. For the cyclic part, verify that $(NG_{i+1}/N)/(NG_i/N)$ is a quotient of the cyclic group $G_{i+1}/G_i$. One of the isomorphism theorems may help here.

(3) ($\Rightarrow$) this follows from (1) and (2).

($\Leftarrow$) Suppose $N$ and $G/N$ are solvable. Then there exists a series

$$\{\mathrm{id}\} \subseteq N_0 \subseteq N_1 \subseteq \cdots \subseteq N_p = N$$

such that $N_i \triangleleft N_{i+1}$ and $N_{i+1}/N_i$ is cyclic, and a series

$$\{\mathrm{id}\} = N/N = G_0/N \subseteq G_1/N \subseteq \cdots \subseteq G_n/N = G/N$$

such that $G_i/N \triangleleft G_{i+1}/N$ and $(G_{i+1}/N)/(G_i/N) \cong G_{i+1}/G_i$ by one of the isomorphism theorems, so it is cyclic as well. Now check as an exercise that

$$\{\mathrm{id}\} = N_0 \subseteq N_1 \subseteq \cdots \subseteq N_p = N = G_0 \subseteq G_1 \subseteq \cdots \subseteq G_n = G$$

is the desired series (i.e. check the normal and cyclic conditions). □

**Corollary 14.4.1.** *For $n \geq 5$, $S_n$ is not solvable.*

*Proof.* For $n \geq 5$, $A_n$ is simple, hence it is not solvable. Now if $S_n$ were solvable, all of its subgroups would be solvable, which leads to a contradiction since $A_n \subseteq S_n$. □

## 14.2 Solvable Polynomials

**Definition 14.4.** A field extension $L : K$ is a *radical extension* if there exists a sequence

$$K = L_0 \subseteq L_1 \subseteq \cdots \subseteq L_m = L$$

such that $L_{j+1} = L_j(\alpha_j)$, where $\alpha_j$ is a root of a polynomial in $L_j[X]$ of the form $X^{n_j} - c_j$.

**Example 14.4.1.** For $L_0 = \mathbb{Q}$, we can take

$$\begin{aligned} &L_0 = \mathbb{Q}, \\ &L_1 = L_0(\alpha_0), \qquad \alpha_0^2 = 2, \\ &L_2 = L_1(\alpha_1), \qquad \alpha_1^5 = 3 + \sqrt{2} \in L_1, \\ &L_3 = L_2(\alpha_2), \qquad \alpha_2^2 = 2 + \sqrt[5]{3 + \sqrt{2}} \in L_2. \end{aligned}$$

This is a radical extension of $\mathbb{Q}$.

**Definition 14.5.** A polynomial $f \in K[X]$ is *solvable by radicals* if there is a splitting field for $f$ contained in a radical extension of $K$.

**Example 14.5.1.** Any quadratic $f = X^2 + bX + c \in \mathbb{Q}[X]$ is solvable by radicals, since its roots are

$$\frac{-b \pm \sqrt{b^2 - 4c}}{2}.$$

**Remark.** In the 16th and 17th centuries, mathematicians proved that cubics

$$X^3 + a_2X^2 + a_1X + a_0$$

and quartics

$$X^4 + a_3X^3 + a_2X^2 + a_1X + a_0$$

are solvable by radicals. For cubics, the idea is to *depress* the cubic, i.e. make a substitution to remove the quadratic term. Then we get

$$Y^3 + 3aY + b = 0.$$

By a lengthy algebra argument, the roots are

$$q + r, \quad q\omega + r\omega^2, \quad q\omega^2 + r\omega,$$

where

$$q = \left(\frac{1}{2}(-b + \sqrt{b^2 + 4a^3})\right)^{1/3}$$

and we have similar expressions for $r$ and $\omega$. A similar but longer algebraic manipulations can be made for quartics. In particular, the expressions for the roots of cubics and quartics only involve radicals.

**Theorem 14.5.** *Let $L : K$ be a radical extension and $N$ the normal closure of $L$ over $K$. Then $N$ is also a radical extension of $K$.*

*Proof.* By Corollary 10.1.1, we have

$$N = L_1 \vee \cdots \vee L_k,$$

where each $L_i \cong L$, hence they are all radical. Now it suffices to show that the join of two radical extensions is radical. For this, let

$$L_1 = K(\alpha_1, \ldots, \alpha_m), \quad L_2 = K(\beta_1, \ldots, \beta_n),$$

where $\alpha_i^{k_i} \in K(\alpha_1, \ldots, \alpha_{i-1})$ and $\beta_j^{l_j} \in K(\beta_1, \ldots, \beta_{j-1})$. Then

$$L_1 \vee L_2 = K(\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_n),$$

where $\alpha_i^{k_i} \in K(\alpha_1, \ldots, \alpha_{i-1})$ and $\beta_j^{l_j} \in K(\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_{j-1})$, so $L_1 \vee L_2$ is radical. □

**Remark.** Radical extensions involve polynomials of the form $X^m - c$. Let us look more closely at $X^m - 1$. We focus on fields $K$ of characteristic 0, so that the splitting field $L$ of $X^m - 1$ over $K$ is normal and separable.

**Lemma 14.1.** *The set $R$ of roots of $X^m - 1$ is a cyclic group under multiplication.*

*Proof.* Check as an exercise that $R$ is indeed a subgroup of $L$. To see that it is cyclic, recall that

$$\exp(R) = \text{smallest positive integer } e \text{ such that } a^e = 1 \text{ for all } a \in R.$$

Clearly we have $\exp(R) \leq |R|$. Now observe that $x^e - 1$ has at most $e$ roots, so $|R| \leq e$. Hence $e = |R| = m$, so $(R, \cdot)$ is cyclic. □

**Definition 14.6.** A *primitive $m$th root of unity* $\omega$ is a generator for $(R, \cdot)$.

**Remark.** We have

$$R = \{1, \omega, \omega^2, \ldots, \omega^{m-1}\},$$

and $\omega^i$ is a primitive $m$th root of unity if $\gcd(m, i) = 1$.

**Definition 14.7.** Let $P_m = \{\text{primitive } m\text{th roots of unity}\}$. The *cyclotomic polynomial* $\Phi_m$ is

$$\Phi_m = \prod_{\varepsilon \in P_m} (X - \varepsilon).$$

# Lecture 15

# Mar. 6 — Cyclotomic Polynomials

## 15.1 Cyclotomic Polynomials

**Example 15.0.1.** For $X^p - 1$ with $p$ prime, all roots except 1 are primitive:

$$X^p - 1 = (X-1)(X^{p-1} + X^{p-2} + \cdots + 1).$$

So we get $\Phi_p = X^{p-1} + X^{p-2} + \cdots + 1$.

**Example 15.0.2.** Consider $f = X^{12} - 1$, where $L \subseteq \mathbb{C}$ is the splitting field of $f$ over $\mathbb{Q}$. We have

$$P_{12} = \{\omega, \omega^5, \omega^7, \omega^{11}\},$$

the powers of $\omega = e^{2\pi i/12}$ relatively prime to 12. This gives

$$\begin{aligned}\Phi_{12} &= (X-\omega)(X-\omega^{11})(X-\omega^5)(X-\omega^7) = (X^2 - (\omega+\omega^{11})X + 1)(X^2 - (\omega^5+\omega^7)X + 1) \\ &= (X^2 - \sqrt{3}X + 1)(X^2 + \sqrt{3}X + 1) = X^4 - X^2 + 1,\end{aligned}$$

since $\omega^{11} = \overline{\omega}$ with $\mathrm{Re}(\omega) = \sqrt{3}/2$ (a similar analysis works for $\omega^5$ and $\omega^7$). We have $P_6 = \{\omega^2, \omega^{10}\}$, so

$$(X-\omega^2)(X-\omega^{10}) = X^2 - (\omega^2+\omega^{10})X + 1 = X^2 - X + 1$$

Next $P_3 = \{\omega^3, \omega^9\} = \{\pm i\}$, so

$$\Phi_4 = (X-i)(X+i) = X^2 + 1.$$

Now $P_3 = \{\omega^4, \omega^8\}$, so

$$\Phi_2 = (X-\omega^4)(X-\omega^8) = X^2 + X + 1.$$

Finally $P_2 = \{\omega^6\}$, so $\Phi_2 = X + 1$, and $P_1 = \{1\} = \{\omega^{12}\}$, so $\Phi_1 = X - 1$.

**Remark.** Observe that

$$X^{12} - 1 = \prod_{d|12} \Phi_d = (X-1)(X+1)(X^2+X+1)(X^2+1)(X^2-X+1)(X^4-X^2+1).$$

This works in general, i.e.

$$X^m - 1 = \prod_{d|m} \Phi_d.$$

Note that we need $1|m$ and $m|m$ here.

**Remark.** The question here is: Does $\Phi_d$ always have coefficients in $K$?

**Lemma 15.1.** *Let $K, L$ be fields and $K \subseteq L$. If $f, g \in L[X]$ such that $f, fg \in K[X]$, then $g \in K[X]$.*

*Proof.* Let

$$f = a_0 + a_1X + \cdots + a_mX^m$$

for $a_i \in K$, $a_m \neq 0$, and

$$g = b_0 + b_1X + \cdots + b_nX^n$$

for $b_i \in L$, $b_n \neq 0$. Then

$$fg = c_0 + c_1X + \cdots + c_{m+n}X^{m+n}$$

for $c_i \in K$, so $b_n = c_{m+n}/a_m \in K$. Now suppose inductively that $b_j \in K$ for all $j > r$. Then

$$c_{m+r} = a_mb_r + a_{m-1}b_{r+1} + \cdots + a_{m-n+r}b_n$$

where $a_i = 0$ if $i < 0$. Then we get that

$$b_r = \frac{c_{m+r} - a_{m-1}b_{r+1} - \cdots - a_{m-n+r}b_n}{a_m}.$$

Since each $a_i \in K$, $c_{m+r} \in K$, and $b_j \in K$ for $j > r$, we get that $b_r \in K$. So in fact $b_j \in K$ for all $j$ by induction, and thus $g \in K[X]$. □

**Theorem 15.1.** *Let* $\operatorname{char} K = 0$ *(so the prime subfield $K_0 \cong \mathbb{Q}$). Suppose $K$ contains the $m$th roots of unity, where $m \geq 2$. Then for every divisor $d$ of $m$, $\Phi_d \in K_0[X]$.*

*Proof.* Note that $\Phi_1 = X - 1 \in K_0[X]$. Let $d|m$, $d \neq 1$, and suppose inductively that $\Phi_r \in K_0[X]$ for all proper divisors $r$ of $d$. Now

$$X^d - 1 = \left( \prod_{r|d, r \neq d} \Phi_r \right) \Phi_d,$$

so Lemma 15.1 gives $\Phi_d \in K_0[X]$. □

**Remark.** In fact, $\Phi_m \in \mathbb{Z}_[X]$.

**Theorem 15.2.** *The cyclotomic polynomials $\Phi_m$ are irreducible over $\mathbb{Q}$.*

*Proof.* See Howie. □

## 15.2 The Galois Groups of Cyclotomic Polynomials

**Remark.** When we talk about the *Galois group of a polynomial*, we mean the Galois group of the splitting field of that polynomial.

**Theorem 15.3.** *Let $L$ be a splitting field over $\mathbb{Q}$ of $X^m - 1$. Then* $\operatorname{Gal}(L : \mathbb{Q}) \cong \mathbb{Z}_m^*$.

*Proof.* Let $\omega$ be a primitive $m$th root of unity and $\sigma \in \operatorname{Gal}(L : \mathbb{Q})$. Since $L = \mathbb{Q}(\omega)$, $\sigma(\omega)$ must be another primitive $m$th root of unity, so $\sigma \in \operatorname{Gal}(L : \mathbb{Q})$ if and only if $\sigma(\omega) = \omega^{k_\sigma}$ where $\gcd(k_\sigma, m) = 1$. Then $\sigma \mapsto k_\sigma$ is an isomorphism $\operatorname{Gal}(L : \mathbb{Q}) \to \mathbb{Z}_m^*$, so $\operatorname{Gal}(L : \mathbb{Q}) \cong \mathbb{Z}_m^*$. □

**Exercise 15.1.** Show that the map $\sigma \mapsto k_\sigma$ is an isomorphism $\operatorname{Gal}(L : \mathbb{Q}) \to \mathbb{Z}_m^*$.

**Corollary 15.3.1.** *If $L$ is a splitting field of $X^p - 1$ over $\mathbb{Q}$ with $p$ prime, then $\mathrm{Gal}(L : \mathbb{Q})$ is cyclic.*

*Proof.* By Theorem 15.3, $\mathrm{Gal}(L : \mathbb{Q}) \cong \mathbb{Z}_p^*$, which we have previously shown is cyclic. □

**Example 15.0.3.** Consider the splitting field $\mathbb{Q}(\omega)$ of $X^8 - 1$ over $\mathbb{Q}$, where $\omega = e^{2\pi i/8} = e^{\pi i/4}$. Then

$$\mathrm{Gal}(\mathbb{Q}(\omega) : \mathbb{Q}) = \{\omega \mapsto \omega, \omega \mapsto \omega^3, \omega \mapsto \omega^5, \omega \mapsto \omega^7\} \cong \mathbb{Z}_8^*.$$

In particular, $\mathrm{Gal}(\mathbb{Q}(\omega) : \mathbb{Q})$ is not cyclic since every element has order 2.

**Example 15.0.4.** Consider the splitting field $\mathbb{Q}(\omega)$ of $X^5 - 1$ over $\mathbb{Q}$, where $\omega = e^{2\pi i/5}$. Then

$$\mathrm{Gal}(\mathbb{Q}(\omega) : \mathbb{Q}) = \{\omega \mapsto \omega, \omega \mapsto \omega^2, \omega \mapsto \omega^3, \omega \mapsto \omega^4\} \cong \mathbb{Z}_5^*.$$

**Theorem 15.4.** *Let $f = X^m - a \in K[X]$, where $\mathrm{char}\, K = 0$. Let $L$ be a splitting for $f$ over $K$. Then*

1. *$L$ contains a primitive $m$th root of unity $\omega$,*
2. *$\mathrm{Gal}(L : K(\omega))$ is cyclic, with order dividing $m$,*
3. *and $|\mathrm{Gal}(L : K)| = m$ if and only if $f$ is irreducible over $K(\omega)$.*

*Proof.* If $\alpha$ is a root of $f$, then over $L$ we have

$$f = (X - \alpha)(X - \omega\alpha)(X - \omega^2\alpha)\dots(X - \omega^{m-1}\alpha)$$

where $\omega$ is a primitive $m$th root of unity. Since $\alpha, \omega\alpha \in L$, this proves (1). Thus $L = K(\omega, \alpha)$, and an element $\sigma \in \mathrm{Gal}(L : K(\omega))$ is determined by $\sigma(\alpha)$, which must be another root of $f$. Hence $\sigma(\alpha) = \omega^{k_\sigma}\alpha$ for some $k_\sigma \in \{0, 1, \dots, m-1\}$. Now for $\sigma, \tau \in \mathrm{Gal}(L : K(\omega))$,

$$\sigma \circ \tau(\alpha) = \sigma(\omega^{k_\tau}\alpha) = \omega^{k_\tau}\sigma(\alpha) = \omega^{k_\tau}\omega^{k_\sigma}\alpha = \omega^{k_\sigma + k_\tau}\alpha,$$

so $\sigma \mapsto k_\sigma$ is a homomorphism $\mathrm{Gal}(L : K(\omega)) \to \mathbb{Z}_m$. This homomorphism is injective since

$$k_\sigma \equiv 0 \pmod{m}$$

if and only if $m | k_\sigma$, if and only if $\sigma(\alpha = \alpha)$. Hence $\mathrm{Gal}(L : K(\omega))$ is isomorphic to a subgroup of the cyclic group $\mathbb{Z}_m$, so $\mathrm{Gal}(L : K(\omega))$ is cyclic (subgroups of cyclic groups are cyclic). This proves (2).

(3) ($\Leftarrow$) Suppose $f$ is irreducible over $K(\omega)$. Then by the Galois correspondence

$$|\mathrm{Gal}(L : K(\omega))| = [L : K(\omega)] = \partial f = m,$$

where the second equality follows from the characterization of simple algebraic extensions. So we get $\mathrm{Gal}(L : K(\omega)) \cong \mathbb{Z}_m$, since we already showed that $\mathrm{Gal}(L : K(\omega))$ is isomorphic to a subgroup of $\mathbb{Z}_m$.

(3) $\Rightarrow$) We show the contrapositive. Suppose $f$ is not irreducible over $K(\omega)$, so $f$ has a monic proper factor $g$ with $\partial g < m$. Let $\beta$ be a root of $g$. Then

$$X^m - a = (X - \beta)(X - \omega\beta)\dots(X - \omega^{m-1}\beta),$$

so $L = K(\omega, \beta)$ is a splitting field for $f$ over $K(\omega)$. Hence

$$|\mathrm{Gal}(L : K(\omega))| = [L : K(\omega)] = \partial g < m,$$

so the Galois group is a proper subgroup of $\mathbb{Z}_m$. □

**Theorem 15.5** (Abel's theorem)**.** *Let* $\operatorname{char} K = 0$*, $p$ prime, and $a \in K$. If $X^p - a$ is reducible over $K$, then it has a linear factor $X - c$ in $K[X]$.*

*Proof.* Suppose $f = X^p - a$ is reducible over $K$. Let $g \in K[X]$ be a monic irreducible factor of $f$ of egree $d$. If $d = 1$, then we are done, so suppose $1 < d < p$. Let $L$ be a splitting field of $f$ over $K$, and $\beta$ a root of $f$ in $L$. Then in $L[X]$,

$$g = (X - \omega^{n_1}\beta)(X - \omega^{n_2}\beta)\dots(X - \omega^{n_d}\beta)$$

where $\omega$ is a primitive $p$th root of unity and $0 \le n_1 < n_2 < \cdots < n_d < p$. Suppose

$$g = X^d - b_{d-1}X^{d-1} + \cdots + (-1)^d b_0.$$

Then we have

$$b_0 = \omega^{n_1+n_2+\cdots+n_d}\beta^d = \omega^n\beta^d$$

where $n = n_1 + n_2 + \cdots + n_d$. So

$$b_0^p = \omega^{pn}\beta^{pd} = (\beta^p)^d = a^d$$

since $\omega^p = 1$ and $\beta$ is a $p$th root of $a$. We have $\gcd(d, p) = 1$ since $p$ is prime, so there exist $s, t \in \mathbb{Z}$ such that $sd + tp = 1$. Then since $a^d = b_0^p$, we get that

$$a = a^{sd+tp} = a^{sd}a^{tp} = b_0^{sp}a^{tp} = (b_0^s a^t)^p.$$

Now $X - b_0^s a^t$ is the desired linear factor of $f$ in $K[X]$. □

**Example 15.0.5.** Let $L$ be the splitting field of $X^5 - 8$ over $\mathbb{Q}$. We have $L = \mathbb{Q}(\sqrt[5]{7}, \omega)$, where $\omega = e^{2\pi i/5}$. Note that the minimum polynomial of $\omega$ is $X^4 + X^3 + X^2 + X + 1$. What is $\operatorname{Gal}(L : \mathbb{Q})$? First we show that $X^5 - 7$ is irreducible over $\mathbb{Q}(\omega)$. To do this, suppose not. Then by Abel's theorem, $X^5 - 7$ has a linear factor $X - c$ in $\mathbb{Q}(\omega)[X]$, i.e. $c = \sqrt[5]{7} \in \mathbb{Q}(\omega)$ and $[\mathbb{Q}(c) : \mathbb{Q}] = 5$. But if $c \in \mathbb{Q}(\omega)$, then

$$[\mathbb{Q}(c) : \mathbb{Q}] \le [\mathbb{Q}(\omega) : \mathbb{Q}] = 4,$$

a contradiction. Now notice that the roots of $X^5 - 7$ in $\mathbb{C}$ are

$$\alpha, \omega\alpha, \omega^2\alpha, \omega^3\alpha, \omega^4\alpha,$$

where $\alpha = \sqrt[5]{7}$. Since $|\operatorname{Gal}(L : \mathbb{Q})| = 20$, define the maps

$$\sigma_{p,q} : \alpha \mapsto \omega^p\alpha, \quad \omega \mapsto \omega^q$$

for $0 \le p \le 4$ and $1 \le q \le 4$. Then we can write

$$\operatorname{Gal}(L : \mathbb{Q}) = \{\sigma_{p,q} \mid 0 \le p \le 4, 1 \le q \le 4\},$$

where the identity element is $\mathrm{id} = \sigma_{0,1}$.

**Exercise 15.2.** Check that

$$\sigma_{p,q}\sigma_{r,s} = \sigma_{rq+p,qs}$$

in the above example, where the subscripts are taken modulo 5 (i.e. compute $\sigma_{p,q}\sigma_{r,s}(\alpha)$ and $\sigma_{p,q}\sigma_{r,s}(\omega)$).