MATH 4317: Analysis I

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Contents

Lecture 1

Aug. 22 – The Real Numbers

1.1 Number Systems

We start with the natural numbers [1]

$$\mathbb{N} = \{1, 2, 3, \dots\}.$$

These are perhaps the most natural in a way, since they are what we use to count things. They are closed under addition, but fail when it comes to subtraction. For example, $1 - 2 = -1 \notin \mathbb{N}$. So we must expand our number system to the integers

$$\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}.$$

We can now add, subtract, and multiply. But we run into problems when we start to consider quotients. For example, $1 \div 2 = \frac{1}{2} \notin \mathbb{Z}$. So we continue to the rational numbers

$$\mathbb{Q} = \left\{ \frac{p}{q} : p, q \in \mathbb{Z},\, q \neq 0 \right\}.$$

We now have summation, subtraction, multiplication, and quotients. But there is still a problem.

Consider the diagonal of a square with side length 1.

Theorem 1.1.1. *$\sqrt{2}$ is not a rational number.* [2]

Proof. Argue by contradiction. Suppose $\sqrt{2}$ is rational. Then we can write

$$\sqrt{2} = \frac{p}{q}$$

for some integers p, q. Further assume p and q have no common factors. Then

$$2 = \frac{p^2}{q^2} \implies p^2 = 2q^2.$$

[1] $0 \notin \mathbb{N}$ for this class.

[2] In some sense, this shows that the notion of "rationals" is strictly weaker than the notion of "length."

So p is even and we can write $p = 2r$ for some $r \in \mathbb{Z}$. Then

$$4r^2 = 2q^2 \implies 2r^2 = q^2.$$

So q is also even, and p, q share a common factor of 2. Contradiction. $\square$

Another weakness of $\mathbb{Q}$ is that we cannot take limits ($\mathbb{Q}$ is not complete). For example, note that

$$(\sqrt{2}-1)(\sqrt{2}+1) = 2 - 1 = 1,$$

$$\sqrt{2} = 1 + \frac{1}{\sqrt{2}+1} = 1 + \frac{1}{1+1+\frac{1}{\sqrt{2}+1}} = \dots.$$

So if we define the rational sequence

$$a_1 = 1, \quad a_2 = 1 + \frac{1}{2}, \quad a_3 = 1 + \frac{1}{2+\frac{1}{2}}, \quad a_4 = 1 + \frac{1}{2+\frac{1}{2+\frac{1}{2}}}, \quad \dots,$$

then as $n \to \infty$, $a_n \to \sqrt{2} \notin \mathbb{Q}$.

1.2 Sets

Sets are any collections of objects. Given a set A, we write $x \in A$ if x is an element of A. We write $x \notin A$ otherwise. The **union** of two sets is

$$A \cup B = \{x : x \in A \text{ or } x \in B\},$$

and the **intersection** of two sets is

$$A \cap B = \{x : x \in A \text{ and } x \in B\}.$$

We use the notation

$$\bigcup_{k=1}^{\infty} A_k$$

to denote the countable union of a family of sets indexed by k.

1.3 Functions

Definition 1.3.1. *Given two sets A and B, a* ***function*** *from A to B is a rule, relation, or mapping that takes each element $x \in A$ and associates with it a single element in B. In this case, we write $f : A \to B$.*

We call A the **domain** of f and B the **codomain** of f. The element in B associated with $x \in A$ is $f(x)$, called the **image** of x. The **range** of f is

$$\text{range}(f) = \{y \in B : y = f(x) \text{ for some } x \in A\}.$$

We say f is:

1. **onto** or **surjective** if range$(f) = B$.
2. **one-to-one** or **injective** if $x, x' \in A$ and $x \neq x'$, then $f(x) \neq f(x')$.
3. **bijective** if it is injective and surjective.

First Dirichlet function:

$$g(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases} = \lim_{k\to\infty}\left(\lim_{j\to\infty}[\cos(k!\pi x)]^{2j}\right).$$

Second Dirichlet function:

$$f(x) = \begin{cases} \frac{1}{q} & \text{if } x = \frac{p}{q} \in \mathbb{Q} \text{ in lowest terms} \\ 0 & \text{if } x \notin \mathbb{Q}. \end{cases}$$

Absolute value:

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0. \end{cases}$$

Note that we have the following two properties:

- $|xy| = |x||y|$.
- $|x+y| \leq |x| + |y|$. This is called the *triangle inequality*.

1.4 Induction

If we have a set $S \subseteq \mathbb{N}$ and

1. $1 \in S$
2. if $n \in S$, then $n+1 \in S$

then $S = \mathbb{N}$. [3]

[3] We always use induction in conjunction with $\mathbb{N}$.

Lecture 2

Aug. 24 – The Axiom of Completeness

The number system $\mathbb{Q}$ is pretty good (it is a field), but recall that we are unable to take limits. For instance, take the sequence $x_0 = 2$ and

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right)$$

for $n \geq 1$. All the x_i are rational, but $x_n \to \sqrt{2} \notin \mathbb{Q}$. This shows that there are gaps in $\mathbb{Q}$. The real numbers $\mathbb{R}$ will fill these gaps (completeness).

Axiom 2.0.1 (Axiom of completeness)**.** *Every nonempty set of real numbers that are bounded above has a least upper bound.*

Note that this least upper bound is *unique*.

2.1 Suprema and Infima

Definition 2.1.1. *Let $S \subseteq \mathbb{R}$. The set S is **bounded above** if there exists $u \in R$ such that $s \leq u$ for all $s \in S$. We say that u is an **upper bound** of S.*

We define **bounded below** and **lower bound** similarly.

Definition 2.1.2. *S is said to be **bounded** if it is both bounded above and below. Otherwise we say that S is **unbounded**.*

$\mathbb{N} = \{1, 2, 3, \dots\}$ is bounded below but not above.

$$\left\{\frac{1}{k} : k \in \mathbb{N}\right\} = \left\{1, \frac{1}{2}, \frac{1}{3}, \dots\right\}$$

is bounded.

$\emptyset$ is bounded.

Definition 2.1.3. *We say $u \in \mathbb{R}$ is the **least upper bound** or **supremum** of a nonempty set $S \subseteq \mathbb{R}$ if*

1. *u is an upper bound of S.*
2. *$u \leq v$ for any upper bound v of S.*

We write $u = \sup S$.

The **greatest lower bound** or **infimum** of S is defined similarly, denoted $\inf S$.

$$S = \left\{ \frac{1}{k} : k \in \mathbb{N} \right\}.$$

$\sup S = 1$, $\inf S = 0$.

Definition 2.1.4. *Let $S \subseteq \mathbb{R}$. We say a real number $M \in S$ is a* ***maximal element*** *or* ***maximum*** *of S if $s \leq M$ for all $s \in S$.*

The **minimal element** or **minimum** is defined similarly.

$[0,1)$ is bounded, but has no maximum. The minimum is 0.

$$\{2^{-n} : n \in \mathbb{N}\} = \{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots\}$$

is bounded, but has no minimum. The maximum is $\frac{1}{2}$.

$\emptyset$ is bounded but has no minimum or maximum.

Exercise 2.1.1. *Let $A \subseteq \mathbb{R}$ be bounded above. Let $c \in \mathbb{R}$ and define*

$$c + a := \{a + c : a \in A\}.$$

Then $\sup(A + c) = c + \sup A$.

Proof. Let $s = \sup A$. By definition, we know $a \leq s$ for all $a \in A$, which implies $a + c \leq s + c$. So $s + c$ is an upper bound for $c + A$. Now let b be an arbitrary upper bound for $c + A$. For all $a \in A$, we have $a + c \leq b$, which implies $a \leq b - c$. So $b - c$ is an upper bound for A. By construction, $s \leq b - c$, so $s + c \leq b$. Therefore $s + c = \sup(A + c)$. □

Lemma 2.1.0.1. *Assume $s \in \mathbb{R}$ is an upper bound for a set $A \subseteq \mathbb{R}$. Then $s = \sup A$ if and only if for every $\epsilon > 0$, there exists $a \in A$ such that $s - \epsilon < a$.*

Proof.
($\Longrightarrow$): Suppose $\sup A = s$. Then given any $\epsilon > 0$, $s - \epsilon$ cannot be an upper bound for A. So there exists $a \in A$ such that $a > s - \epsilon$.

($\Longleftarrow$): Let b be an arbitrary upper bound for A. Suppose for contradiction that $b < s$. Set $\epsilon = s - b > 0$. Then by assumption we can find $a \in A$ such that $a > s - \epsilon = b$. Contradiction. Therefore $b \geq s$, whence $\sup A = s$. □

2.2 Consequences of Completeness

2.2.1 1st Consequence: Nested Interval Properties

Theorem 2.2.1 (Nested interval properties)**.** *For any $n \in \mathbb{N}$, assume that we are given a closed interval*

$$I_n = [a_n, b_n] = \{x \in \mathbb{R} : a_n \leq x \leq b_n\}.$$

Assume $I_n \supseteq I_{n+1}$. Then the resulting nested sequence of closed intervals

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq \dots$$

has a nonempty intersection:

$$\bigcap_{n=1}^{\infty} I_n \neq \emptyset.$$

Proof. Define $A = \{a_n\}$. Note that $A \neq \emptyset$. For any n, $a_n \leq b_n \leq b_1$. So $x = \sup A$ exists. Furthermore, for any n, b_n is an upper bound for A. So $x \leq b_n$. Since $x = \sup A$, $a_n \leq x$. So $x \in [a_n, b_n]$ for any n, whence

$$x \in \bigcap_{n=1}^{\infty} I_n.$$

□

2.2.2 2nd Consequence: Archimedean Properties

Theorem 2.2.2 (Archimedean properties)**.**

1. *Given any $x \in \mathbb{R}$, there is an $n \in \mathbb{N}$ such that $n > x$.* [1]
2. *Given any real number $y > 0$, there is an $\mathbb{N}$ such that $\frac{1}{n} < y$.*

Proof of (1). Argue by contradiction. Suppose $\mathbb{N}$ is bounded above. Then by the axiom of completeness, $\alpha = \sup N$ exists. By construction, $\alpha - 1$ is not an upper bound for $\mathbb{N}$. So we can find $n \in N$ such that $\alpha - 1 < n$, which implies $\alpha < n + 1 \in \mathbb{N}$. Contradiction. □

Proof of (2). Follows from (1) by setting $x = \frac{1}{y}$. □

[1] This is saying that $\mathbb{N}$ is not bounded above.

Lecture 3

Aug. 29 – Completeness and Countability

3.1 Consequences of Completeness

3.1.1 3rd Consequence: Density of $\mathbb{Q}$ in $\mathbb{R}$

Theorem 3.1.1 (Density of $\mathbb{Q}$ in $\mathbb{R}$). *For all $a, b \in \mathbb{R}$, $a < b$, there exists $r \in \mathbb{Q}$ such that $a < r < b$.*

Proof. We want to find $m \in \mathbb{Z}$, $n \in \mathbb{N}$ such that

$$a < \frac{m}{n} < b.$$

By (ii) of the Archimedean property, we can find $n \in \mathbb{N}$ such that

$$\frac{1}{n} < b - a.$$

Fix such an n. Then let m be the smallest integer such that $m - 1 \leq na < m$. By construction,

$$\frac{m}{n} - \frac{1}{n} \leq a < \frac{m}{n},$$

$$\frac{m}{n} \leq a + \frac{1}{n} < b.$$

Therefore, $a < \frac{m}{n} < b$. $\square$

Corollary 3.1.1.1. *For all $a, b \in \mathbb{Q}$, $a < b$, there exists $t \in \mathbb{R} \setminus \mathbb{Q}$ such that $a < t < b$.*

3.1.2 4th Consequence: Existence of $\sqrt{2}$

Theorem 3.1.2 (Existence of $\sqrt{2}$). *There exists $s \in \mathbb{R}$, $s > 0$ such that $s^2 = 2$.*

Proof. Define

$$S = \{x > 0 : x^2 < 2\} \subseteq \mathbb{R}.$$

$x = 1 \in S$, so $S \neq \emptyset$. 2 is an upper bound for S, so S is bounded above. Then by the axiom of completeness, $s = \sup S$ exists. We claim that $s^2 = 2$.

Suppose otherwise that $s^2 < 2$. Then we can find $\epsilon > 0$ such that $s + \epsilon \in S$. Define $\delta = 2 - s^2 > 0$. Note that

$$(s+\epsilon)^2 - 2 = s^2 + 2s\epsilon + \epsilon^2 - 2 = -\delta + 2s\epsilon + \epsilon^2.$$

We know $s \leq 2$ since 2 is an upper bound. Pick

$$\epsilon = \frac{\delta}{100000000000},$$

$$2s\epsilon + \epsilon \leq 4\epsilon + \epsilon^2 < \frac{\delta}{2}.$$

Then

$$(s+\epsilon)^2 - 2 < -\delta + \frac{\delta}{2} = -\frac{\delta}{2} < 0.$$

So $s + \epsilon \in S$, which contradicts with $s = \sup S$.

$s^2 > 2$ also leads to a contradiction (left as an exercise). Thus we must have $s^2 = 2$. $\square$

3.2 Countability

Definition 3.2.1. *We say two sets A and B have the same* ***cardinality*** *if there is a bijection $f : A \to B$. We write $A \sim B$.*

Definition 3.2.2. *We say that a set A is* ***finite*** *if $A \sim \{1, 2, \ldots, n\}$ for some integer n. We say that a set A is* ***countable*** *(or countably infinite) if $A \sim \mathbb{N}$. If a set A is not countable, then we say it is* ***uncountable****.*

$E = \{2, 4, 6, 8, \ldots\}$.

E is not finite but it is countable: $E \sim \mathbb{N}$. We can define $f : \mathbb{N} \to E$ by $f(n) = 2n$.

$\mathbb{N} \sim \mathbb{Z}$.

The bijection $f : \mathbb{N} \to \mathbb{Z}$ is given by

$$f(n) = \begin{cases} \frac{n-1}{2} & n \text{ is odd} \\ -\frac{n}{2} & n \text{ is even.} \end{cases}$$

$(-1, 1) \sim \mathbb{R}$.

The bijection $f : (-1, 1) \to \mathbb{R}$ is given by

$$x \mapsto \frac{x}{x^2 - 1}.$$

Theorem 3.2.1.

1. *$\mathbb{Q}$ is countable.*
2. *$\mathbb{R}$ is uncountable.*

Proof of (1). Set $A_1 = \{0\}$ and for $n \geq 2$,

$$A_n = \left\{ \pm\frac{p}{q} : p, q \in \mathbb{N}, p, q \text{ in lowest terms}, p + q = n \right\}.$$

So the first few A_n are:

$$A_2 = \left\{ \frac{1}{1}, \frac{-1}{1} \right\},$$

$$A_3 = \left\{ \frac{1}{2}, \frac{2}{1}, \frac{-1}{2}, \frac{-2}{1} \right\},$$

etc. Note that A_n is finite and for all $x \in \mathbb{Q}$, there is an $n \in \mathbb{N}$ such that $x \in A_n$. We can list elements in $A_1, \ldots, A_n$ and label them with integers in $\mathbb{N}$. Any element of A_n will be listed eventually. Then this pairing gives a bijection since the A_n are disjoint. So $\mathbb{Q} \sim \mathbb{N}$. $\square$

Proof of (2). Argue by contradiction. Suppose f is one-to-one from $\mathbb{N} \to \mathbb{R}$. Set $x_1 = f(1)$, $x_2 = f(2)$, etc. We can write

$$\mathbb{R} = \{x_1, x_2, \ldots\}.$$

Let I_1 be a closed interval such that $x_1 \notin I_1$. Pick $I_2 \subseteq I_1$ such that $x_2 \notin I_2$. Continue this process such that $I_{n+1} \subseteq I_n$ is a closed interval where $x_{n+1} \notin I_{n+1}$. By construction,

$$I_1 \supseteq I_2 \supseteq \cdots \supseteq I_n \supseteq \ldots.$$

We know that

$$\bigcap_{n=1}^{\infty} I_n \neq \emptyset.$$

So we can find n_0 such that

$$x_{n_0} \in \bigcap_{n=1}^{\infty} I_n.$$

This is a contradiction with $x_{n_0} \notin I_{n_0}$. Thus such an f cannot exist and $\mathbb{R}$ is uncountable. $\square$

Theorem 3.2.2.

1. *Let $A \subseteq B$. If B is countable, then A is either finite or countable.*
2. *If A_n is a countable set, then*

$$\bigcup_{n=1}^{\infty} A_n$$

is also countable.

Theorem 3.2.3 (Cantor's theorem)**.** *The open interval*

$$(0,1) = \{x \in \mathbb{R} : 0 < x < 1\}$$

is uncountable.

Proof. Argue by contradiction. Assume $f : \mathbb{N} \to (0,1)$ is one-to-one and onto. Then for $m \in \mathbb{N}$, we can write (decimal expansion)

$$f(m) = 0.a_{m1}a_{m2}a_{m3}\ldots \in (0,1).$$

For every $m, n \in \mathbb{N}$, $a_{mn} \in \{0, \ldots, 9\}$ is the nth digit in the decimal expansion of $f(m)$. We can write in a table

$$\begin{array}{cccccc} 1 & f(1) & a_{11} & a_{12} & a_{13} & \ldots \\ 2 & f(2) & a_{21} & a_{22} & a_{23} & \ldots \\ 3 & f(3) & a_{31} & a_{32} & a_{33} & \ldots \\ & & & \vdots & & \end{array}$$

Take $x = 0.b_1b_2b_3\ldots$ where

$$b_n = \begin{cases} 2 & \text{if } a_{nn} \neq 2 \\ 3 & \text{if } a_{nn} = 2. \end{cases}$$

Then $x \neq f(m)$ for any $m \in \mathbb{N}$ (since $b_m \neq a_{mm}$). This is a contradiction. $\square$