MATH 4317: Analysis I

Frank Qiang

Georgia Institute of Technology, Fall 2023

Contents

Lecture 1

Aug. 22 – The Real Numbers

1.1 Number Systems

We start with the natural numbers [1]

$$\mathbb{N} = \{1, 2, 3, \dots\}.$$

These are perhaps the most natural in a way, since they are what we use to count things. They are closed under addition, but fail when it comes to subtraction. For example, $1 - 2 = -1 \notin \mathbb{N}$. So we must expand our number system to the integers

$$\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}.$$

We can now add, subtract, and multiply. But we run into problems when we start to consider quotients. For example, $1 \div 2 = \frac{1}{2} \notin \mathbb{Z}$. So we continue to the rational numbers

$$\mathbb{Q} = \left\{ \frac{p}{q} : p, q \in \mathbb{Z}, \, q \neq 0 \right\}.$$

We now have summation, subtraction, multiplication, and quotients. But there is still a problem.

Consider the diagonal of a square with side length 1.

Theorem 1.1. *$\sqrt{2}$ is not a rational number.* [2]

Proof. Argue by contradiction. Suppose $\sqrt{2}$ is rational. Then we can write

$$\sqrt{2} = \frac{p}{q}$$

for some integers p, q. Further assume p and q have no common factors. Then

$$2 = \frac{p^2}{q^2} \implies p^2 = 2q^2.$$

[1] $0 \notin \mathbb{N}$ for this class.

[2] In some sense, this shows that the notion of “rationals” is strictly weaker than the notion of “length.”

So p is even and we can write $p = 2r$ for some $r \in \mathbb{Z}$. Then

$$4r^2 = 2q^2 \implies 2r^2 = q^2.$$

So q is also even, and p, q share a common factor of 2. Contradiction. $\square$

Another weakness of $\mathbb{Q}$ is that we cannot take limits ($\mathbb{Q}$ is not complete). For example, note that

$$(\sqrt{2}-1)(\sqrt{2}+1) = 2 - 1 = 1,$$
$$\sqrt{2} = 1 + \frac{1}{\sqrt{2}+1} = 1 + \frac{1}{1+1+\frac{1}{\sqrt{2}+1}} = \ldots.$$

So if we define the rational sequence

$$a_1 = 1, \quad a_2 = 1 + \frac{1}{2}, \quad a_3 = 1 + \frac{1}{2+\frac{1}{2}}, \quad a_4 = 1 + \frac{1}{2+\frac{1}{2+\frac{1}{2}}}, \quad \ldots,$$

then as $n \to \infty$, $a_n \to \sqrt{2} \notin \mathbb{Q}$.

1.2 Sets

Sets are any collections of objects. Given a set A, we write $x \in A$ if x is an element of A. We write $x \notin A$ otherwise. The **union** of two sets is

$$A \cup B = \{x : x \in A \text{ or } x \in B\},$$

and the **intersection** of two sets is

$$A \cap B = \{x : x \in A \text{ and } x \in B\}.$$

We use the notation

$$\bigcup_{k=1}^{\infty} A_k$$

to denote the countable union of a family of sets indexed by k.

1.3 Functions

Definition 1.1. Given two sets A and B, a **function** from A to B is a rule, relation, or mapping that takes each element $x \in A$ and associates with it a single element in B. In this case, we write $f : A \to B$.

We call A the **domain** of f and B the **codomain** of f. The element in B associated with $x \in A$ is $f(x)$, called the **image** of x. The **range** of f is

$$\text{range}(f) = \{y \in B : y = f(x) \text{ for some } x \in A\}.$$

We say f is:

1. **onto** or **surjective** if $\text{range}(f) = B$.
2. **one-to-one** or **injective** if $x, x' \in A$ and $x \neq x'$, then $f(x) \neq f(x')$.
3. **bijective** if it is injective and surjective.

Example 1.1.1. First Dirichlet function:

$$g(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases} = \lim_{k\to\infty}\left(\lim_{j\to\infty}[\cos(k!\pi x)]^{2j}\right).$$

Example 1.1.2. Second Dirichlet function:

$$f(x) = \begin{cases} \frac{1}{q} & \text{if } x = \frac{p}{q} \in \mathbb{Q} \text{ in lowest terms} \\ 0 & \text{if } x \notin \mathbb{Q}. \end{cases}$$

Example 1.1.3. Absolute value:

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0. \end{cases}$$

Note that we have the following two properties:

- $|xy| = |x||y|$.
- $|x+y| \leq |x| + |y|$. This is called the *triangle inequality*.

1.4 Induction

If we have a set $S \subseteq \mathbb{N}$ and

1. $1 \in S$
2. if $n \in S$, then $n+1 \in S$

then $S = \mathbb{N}$. [3]

[3] We always use induction in conjunction with $\mathbb{N}$.

Lecture 2

Aug. 24 – The Axiom of Completeness

The number system $\mathbb{Q}$ is pretty good (it is a field), but recall that we are unable to take limits. For instance, take the sequence $x_0 = 2$ and

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right)$$

for $n \geq 1$. All the x_i are rational, but $x_n \to \sqrt{2} \notin \mathbb{Q}$. This shows that there are gaps in $\mathbb{Q}$. The real numbers $\mathbb{R}$ will fill these gaps (completeness).

Axiom 2.1 (Axiom of completeness)**.** *Every nonempty set of real numbers that are bounded above has a least upper bound.*

Note that this least upper bound is *unique*.

2.1 Suprema and Infima

Definition 2.1. Let $S \subseteq \mathbb{R}$. The set S is **bounded above** if there exists $u \in R$ such that $s \leq u$ for all $s \in S$. We say that u is an **upper bound** of S.

We define **bounded below** and **lower bound** similarly.

Definition 2.2. S is said to be **bounded** if it is both bounded above and below. Otherwise we say that S is **unbounded**.

Example 2.2.1. $\mathbb{N} = \{1, 2, 3, \dots\}$ is bounded below but not above.

Example 2.2.2. The set

$$\left\{\frac{1}{k} : k \in \mathbb{N}\right\} = \left\{1, \frac{1}{2}, \frac{1}{3}, \dots\right\}$$

is bounded.

Example 2.2.3. $\emptyset$ is bounded.

Definition 2.3. We say $u \in \mathbb{R}$ is the **least upper bound** or **supremum** of a nonempty set $S \subseteq \mathbb{R}$ if

1. u is an upper bound of S.

2. $u \le v$ for any upper bound v of S.

We write $u = \sup S$.

The **greatest lower bound** or **infimum** of S is defined similarly, denoted $\inf S$.

Example 2.3.1.

$$S = \left\{ \frac{1}{k} : k \in \mathbb{N} \right\}.$$

$\sup S = 1$, $\inf S = 0$.

Definition 2.4. Let $S \subseteq \mathbb{R}$. We say a real number $M \in S$ is a **maximal element** or **maximum** of S if $s \le M$ for all $s \in S$.

The **minimal element** or **minimum** is defined similarly.

Example 2.4.1. $[0, 1)$ is bounded, but has no maximum. The minimum is 0.

Example 2.4.2. The set

$$\{2^{-n} : n \in \mathbb{N}\} = \left\{ \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots \right\}$$

is bounded, but has no minimum. The maximum is $\frac{1}{2}$.

Example 2.4.3. $\emptyset$ is bounded but has no minimum or maximum.

Exercise 2.1. Let $A \subseteq \mathbb{R}$ be bounded above. Let $c \in \mathbb{R}$ and define

$$c + a := \{a + c : a \in A\}.$$

Then $\sup(A + c) = c + \sup A$.

Proof. Let $s = \sup A$. By definition, we know $a \le s$ for all $a \in A$, which implies $a + c \le s + c$. So $s + c$ is an upper bound for $c + A$. Now let b be an arbitrary upper bound for $c + A$. For all $a \in A$, we have $a + c \le b$, which implies $a \le b - c$. So $b - c$ is an upper bound for A. By construction, $s \le b - c$, so $s + c \le b$. Therefore $s + c = \sup(A + c)$. □

Lemma 2.1. *Assume $s \in \mathbb{R}$ is an upper bound for a set $A \subseteq \mathbb{R}$. Then $s = \sup A$ if and only if for every $\epsilon > 0$, there exists $a \in A$ such that $s - \epsilon < a$.*

Proof.
($\Longrightarrow$): Suppose $\sup A = s$. Then given any $\epsilon > 0$, $s - \epsilon$ cannot be an upper bound for A. So there exists $a \in A$ such that $a > s - \epsilon$.

($\Longleftarrow$): Let b be an arbitrary upper bound for A. Suppose for contradiction that $b < s$. Set $\epsilon = s - b > 0$. Then by assumption we can find $a \in A$ such that $a > s - \epsilon = b$. Contradiction. Therefore $b \ge s$, whence $\sup A = s$. □

2.2 Consequences of Completeness

2.2.1 1st Consequence: Nested Interval Properties

Theorem 2.1 (Nested interval properties)**.** *For any $n \in \mathbb{N}$, assume that we are given a closed interval*

$$I_n = [a_n, b_n] = \{x \in \mathbb{R} : a_n \leq x \leq b_n\}.$$

Assume $I_n \supseteq I_{n+1}$. Then the resulting nested sequence of closed intervals

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq \dots$$

has a nonempty intersection:

$$\bigcap_{n=1}^{\infty} I_n \neq \emptyset.$$

Proof. Define $A = \{a_n\}$. Note that $A \neq \emptyset$. For any n, $a_n \leq b_n \leq b_1$. So $x = \sup A$ exists. Furthermore, for any n, b_n is an upper bound for A. So $x \leq b_n$. Since $x = \sup A$, $a_n \leq x$. So $x \in [a_n, b_n]$ for any n, whence

$$x \in \bigcap_{n=1}^{\infty} I_n.$$

□

2.2.2 2nd Consequence: Archimedean Properties

Theorem 2.2 (Archimedean properties)**.**

1. *Given any $x \in \mathbb{R}$, there is an $n \in \mathbb{N}$ such that $n > x$.* [1]
2. *Given any real number $y > 0$, there is an $\mathbb{N}$ such that $\frac{1}{n} < y$.*

Proof of (1). Argue by contradiction. Suppose $\mathbb{N}$ is bounded above. Then by the axiom of completeness, $\alpha = \sup N$ exists. By construction, $\alpha - 1$ is not an upper bound for $\mathbb{N}$. So we can find $n \in N$ such that $\alpha - 1 < n$, which implies $\alpha < n + 1 \in \mathbb{N}$. Contradiction. □

Proof of (2). Follows from (1) by setting $x = \frac{1}{y}$. □

[1] This is saying that $\mathbb{N}$ is not bounded above.

Lecture 3

Aug. 29 – Completeness, Countability

3.1 Consequences of Completeness

3.1.1 3rd Consequence: Density of $\mathbb{Q}$ in $\mathbb{R}$

Theorem 3.1 (Density of $\mathbb{Q}$ in $\mathbb{R}$)**.** *For all $a, b \in \mathbb{R}$, $a < b$, there exists $r \in \mathbb{Q}$ such that $a < r < b$.*

Proof. We want to find $m \in \mathbb{Z}$, $n \in \mathbb{N}$ such that

$$a < \frac{m}{n} < b.$$

By (2) of the Archimedean properties, we can find $n \in \mathbb{N}$ such that

$$\frac{1}{n} < b - a.$$

Fix such an n. Then let m be the smallest integer such that $m - 1 \leq na < m$. By construction,

$$\frac{m}{n} - \frac{1}{n} \leq a < \frac{m}{n},$$

$$\frac{m}{n} \leq a + \frac{1}{n} < b.$$

Therefore, $a < \frac{m}{n} < b$. □

Corollary 3.1.1. *For all $a, b \in \mathbb{Q}$, $a < b$, there exists $t \in \mathbb{R} \setminus \mathbb{Q}$ such that $a < t < b$.*

3.1.2 4th Consequence: Existence of $\sqrt{2}$

Theorem 3.2 (Existence of $\sqrt{2}$)**.** *There exists $s \in \mathbb{R}$, $s > 0$ such that $s^2 = 2$.*

Proof. Define

$$S = \{x > 0 : x^2 < 2\} \subseteq \mathbb{R}.$$

$x = 1 \in S$, so $S \neq \emptyset$. 2 is an upper bound for S, so S is bounded above. Then by the axiom of completeness, $s = \sup S$ exists. We claim that $s^2 = 2$.

Suppose otherwise that $s^2 < 2$. Then we can find $\epsilon > 0$ such that $s + \epsilon \in S$. Define $\delta = 2 - s^2 > 0$. Note that

$$(s+\epsilon)^2 - 2 = s^2 + 2s\epsilon + \epsilon^2 - 2 = -\delta + 2s\epsilon + \epsilon^2.$$

We know $s \leq 2$ since 2 is an upper bound. Pick

$$\epsilon = \frac{\delta}{100000000000},$$

$$2s\epsilon + \epsilon \leq 4\epsilon + \epsilon^2 < \frac{\delta}{2}.$$

Then

$$(s+\epsilon)^2 - 2 < -\delta + \frac{\delta}{2} = -\frac{\delta}{2} < 0.$$

So $s + \epsilon \in S$, which contradicts with $s = \sup S$.

$s^2 > 2$ also leads to a contradiction (left as an exercise). Thus we must have $s^2 = 2$. $\square$

3.2 Countability

Definition 3.1. We say two sets A and B have the same **cardinality** if there is a bijection $f : A \to B$. We write $A \sim B$.

Definition 3.2. We say that a set A is **finite** if $A \sim \{1, 2, \dots, n\}$ for some integer n. We say that a set A is **countable** (or countably infinite) if $A \sim \mathbb{N}$. If a set A is not countable, then we say it is **uncountable**.

Example 3.2.1. Let $E = \{2, 4, 6, 8, \dots\}$. E is not finite but it is countable: $E \sim \mathbb{N}$. We can define $f : \mathbb{N} \to E$ by $f(n) = 2n$.

Example 3.2.2. $\mathbb{N} \sim \mathbb{Z}$. The bijection $f : \mathbb{N} \to \mathbb{Z}$ is given by

$$f(n) = \begin{cases} \frac{n-1}{2} & n \text{ is odd} \\ -\frac{n}{2} & n \text{ is even.} \end{cases}$$

Example 3.2.3. $(-1, 1) \sim \mathbb{R}$. The bijection $f : (-1, 1) \to \mathbb{R}$ is given by

$$x \mapsto \frac{x}{x^2 - 1}.$$

Theorem 3.3.

1. *$\mathbb{Q}$ is countable.*
2. *$\mathbb{R}$ is uncountable.*

Proof of (1). Set $A_1 = \{0\}$ and for $n \geq 2$,

$$A_n = \left\{ \pm \frac{p}{q} : p, q \in \mathbb{N}, p, q \text{ in lowest terms}, p + q = n \right\}.$$

So the first few A_n are:

$$A_2 = \left\{ \frac{1}{1}, \frac{-1}{1} \right\},$$

$$A_3 = \left\{ \frac{1}{2}, \frac{2}{1}, \frac{-1}{2}, \frac{-2}{1} \right\},$$

etc. Note that A_n is finite and for all $x \in \mathbb{Q}$, there is an $n \in \mathbb{N}$ such that $x \in A_n$. We can list elements in $A_1, \ldots, A_n$ and label them with integers in $\mathbb{N}$. Any element of A_n will be listed eventually. Then this pairing gives a bijection since the A_n are disjoint. So $\mathbb{Q} \sim \mathbb{N}$. □

Proof of (2). Argue by contradiction. Suppose f is one-to-one from $\mathbb{N} \to \mathbb{R}$. Set $x_1 = f(1)$, $x_2 = f(2)$, etc. We can write

$$\mathbb{R} = \{x_1, x_2, \ldots\}.$$

Let I_1 be a closed interval such that $x_1 \notin I_1$. Pick $I_2 \subseteq I_1$ such that $x_2 \notin I_2$. Continue this process such that $I_{n+1} \subseteq I_n$ is a closed interval where $x_{n+1} \notin I_{n+1}$. By construction,

$$I_1 \supseteq I_2 \supseteq \cdots \supseteq I_n \supseteq \ldots.$$

We know that

$$\bigcap_{n=1}^{\infty} I_n \neq \emptyset.$$

So we can find n_0 such that

$$x_{n_0} \in \bigcap_{n=1}^{\infty} I_n.$$

This is a contradiction with $x_{n_0} \notin I_{n_0}$. Thus such an f cannot exist and $\mathbb{R}$ is uncountable. □

Theorem 3.4.

1. *Let $A \subseteq B$. If B is countable, then A is either finite or countable.*
2. *If A_n is a countable set, then*

$$\bigcup_{n=1}^{\infty} A_n$$

is also countable.

Theorem 3.5 (Cantor's diagonal argument)**.** *The open interval*

$$(0, 1) = \{x \in \mathbb{R} : 0 < x < 1\}$$

is uncountable.

Proof. Argue by contradiction. Assume $f : \mathbb{N} \to (0,1)$ is one-to-one and onto. Then for $m \in \mathbb{N}$, we can write (decimal expansion)

$$f(m) = 0.a_{m1}a_{m2}a_{m3}\ldots \in (0,1).$$

For every $m, n \in \mathbb{N}$, $a_{mn} \in \{0, \ldots, 9\}$ is the nth digit in the decimal expansion of $f(m)$. We can write in a table

$$\begin{array}{cccccc} 1 & f(1) & a_{11} & a_{12} & a_{13} & \ldots \\ 2 & f(2) & a_{21} & a_{22} & a_{23} & \ldots \\ 3 & f(3) & a_{31} & a_{32} & a_{33} & \ldots \\ & & & \vdots & & \end{array}$$

Take $x = 0.b_1b_2b_3\ldots$ where

$$b_n = \begin{cases} 2 & \text{if } a_{nn} \neq 2 \\ 3 & \text{if } a_{nn} = 2. \end{cases}$$

Then $x \neq f(m)$ for any $m \in \mathbb{N}$ (since $b_m \neq a_{mm}$). This is a contradiction. $\square$

Lecture 4

Aug. 31 – Cantor's Theorem, Sequences

4.1 Cantor's Theorem

Definition 4.1. The **power set** of A, denoted $\mathcal{P}(A)$, is the collection of all subsets of A.

Theorem 4.1 (Cantor's theorem)**.** *Given any set A, there does not exist a function $f : A \to \mathcal{P}(A)$ which is surjective.* [1]

Proof. Argue by contradiction. Suppose $f : A \to \mathcal{P}(A)$ is onto. Then for any $a \in A$, $f(a)$ is a subset of A. Since f is onto, for any subset B of A, we can find $a \in A$ such that $f(a) = B$. Define

$$B = \{a \in A : a \notin f(a)\} \subseteq A.$$

We can find $a' \in A$ such that $f(a') = B$. If $a' \in B$, then $a' \notin f(a') = B$, which is a contradition.. If $a' \notin B$, this is a contradiction with the definition of B. Thus such f cannot exist. □

Remark. This means that the cardinality of $\mathcal{P}(A)$ is strictly larger than that of A.

4.2 Sequences

Definition 4.2. A **sequence** is a function whose domain is $\mathbb{N}$.

We usually write $\{a_n\}$, $\{x_n\}$ or (a_n), (x_n) to denote sequences.

Example 4.2.1. The following

$$\left\{\frac{1+n}{n}\right\}_{n=1}^{\infty} = \left\{2, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \dots\right\}$$

is a sequence.

Example 4.2.2. $\{a_n\}$, where $a_n = 2^n$ for $n \in \mathbb{N}$, is a sequence.

Example 4.2.3. We can also define $\{x_n\}$ recursively by $x_1 = 2$ and

$$x_{n+1} = \frac{x_n + 1}{2}.$$

[1] Note that if $\#(A) = n < \infty$, this is true as $\#(\mathcal{P}(A)) = 2^n \neq \#(A)$.

Remark. Sometimes a sequence is also labeled starting from $n = 0$.

4.2.1 Limits

Definition 4.3. A sequence $\{a_n\}$ **converges** to a real number a if for every $\epsilon > 0$, we can find $N \in \mathbb{N}$ such that for all $n \geq N$, one has $|a_n - a| < \epsilon$. We write $\lim_{n\to\infty} a_n = a$.

Remark. In analysis, ϵ is always taken to be a positive number.

Example 4.3.1. The sequence $\{1/n\}_{n=1}^{\infty}$ converges with

$$\lim_{n\to\infty} \frac{1}{n} = 0.$$

Definition 4.4. For $\epsilon > 0$, the ϵ-**neighborhood** of a is defined to be

$$V_\epsilon(a) = \{x \in \mathbb{R} : |x - a| < \epsilon\}.$$

Definition 4.5. We say that a is the **limit** of a sequence $\{a_n\}$ if for every $\epsilon > 0$, $V_\epsilon(a)$ contains all but finitely many elements of $\{a_n\}$. [2]

Remark. This definition of the limit is equivalent to the definition of convergence.

Definition 4.6. A sequence $\{a_n\}$ that does not converge is said to be **divergent**.

Theorem 4.2. *The limit of a sequence, when it exists, must be unique.*

Exercise 4.1. Show

$$\lim_{n\to\infty} \frac{n+1}{n}$$

exists and

$$\lim_{n\to\infty} \frac{n+1}{n} = 1.$$

Proof. We show

$$\lim_{n\to\infty} \frac{n+1}{n} = 1.$$

For every $\epsilon > 0$, take $N \in \mathbb{N}$ such that $N > \frac{1}{\epsilon}$. We have for all $n \geq N$,

$$\left|\frac{n+1}{n} - 1\right| = \left|\frac{1}{n}\right| \leq \frac{1}{N} < \epsilon.$$

Therefore,

$$\lim_{n\to\infty} \frac{n+1}{n} = 1.$$

□

[2] This is the *topological* definition of the limit.

4.2.2 Tips for Showing Limits

To show the limit of a sequence, take the following steps:

1. Identify the limit a. This is always given by the problem or observation.
2. $\forall \epsilon > 0$.
3. Find $N = N(\epsilon)$. Do this in sketch paper (need computations and manipulations).
4. Set N as what is found in (3).
5. Check that N works.

Lecture 5

Sept. 5 – Limits and Limit Theorems

5.1 Review of Limits

Example 5.0.1. Find

$$\lim_{n\to\infty} \frac{1+\sqrt{n}}{\sqrt{n}}.$$

Proof. We want to show that

$$\lim_{n\to\infty} \frac{1+\sqrt{n}}{\sqrt{n}} = 1.$$

Fix $\epsilon > 0$ and take $N \in \mathbb{N}$ such that $N > \frac{1}{\epsilon^2}$. Then for any $n > N$,

$$\left|\frac{1+\sqrt{n}}{\sqrt{n}} - 1\right| \le \left|\frac{1}{\sqrt{n}}\right| \le \frac{1}{\sqrt{N}} < \epsilon,$$

as desired. □

How can we understand this using the topological definition? For all $\epsilon > 0$, take $V_\epsilon(1)$. Pick $N > \frac{1}{\epsilon^2}$. Then we claim that $V_\epsilon(1)$ contains all but at most N elements of $\left\{\frac{\sqrt{n}+1}{\sqrt{n}}\right\}$. When $n \ge N$, we have

$$\left|\frac{\sqrt{n}+1}{\sqrt{n}} - 1\right| < \epsilon,$$

i.e. $\frac{\sqrt{n}+1}{\sqrt{n}} \in V_\epsilon(1)$. So at most N elements might not be in $V_\epsilon(1)$.

5.2 Limit Theorems

5.2.1 Algebraic Facts About Limits

Definition 5.1. A sequence $\{x_n\}$ is said to be **bounded** if there exists M such that $|x_n| \le M$ for all n. Alternatively, $\sup_n |x_n| \le M$.

Theorem 5.1. *Every convergent sequence is bounded.*

Proof. Suppose
$$\lim_{n\to\infty} x_n = l.$$
Take $\epsilon = 1$, we can find N such that for all $n \geq N$, $|x_n - l| < 1$. By the triangle inequality, $|x_n| < |l| + 1$ for $n \geq N$. Take
$$M = \max\{|x_1|, |x_2|, \ldots, |x_{N-1}|, |l| + 1\}.$$
Then $|x_n| \leq M$ for all $n \in \mathbb{N}$. □

Theorem 5.2 (Algebraic limit theorem)**.** *If*
$$\lim_{n\to\infty} a_n = a \quad \text{and} \quad \lim_{n\to\infty} b_n = b,$$
then for all $c \in \mathbb{R}$,
$$(1)\ \lim_{n\to\infty} ca_n = ca, \quad (2)\ \lim_{n\to\infty}(a_n + b_n) = a + b, \quad \text{and} \quad (3)\ \lim_{n\to\infty} a_n b_n = ab.$$
Furthermore, if $b \neq 0$, then
$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{a}{b}. \tag{4}$$

Proof. (1) When $c = 0$, the result is trivial. When $c \neq 0$, for all $\epsilon > 0$, we set $\epsilon' = \frac{\epsilon}{|c|}$. Since $\lim_{n\to\infty} a_n = a$, we can find $N_{\epsilon'}$ such that for all $n \geq N_{\epsilon'}$, $|a_n - a| < \epsilon'$. When $n > N_{\epsilon'}$, we have
$$|ca_n - ca| = |c||a_n - a| < |c|e' = |c|\frac{\epsilon}{|c|} = \epsilon.$$
So $\lim_{n\to\infty} ca_n = ca$.

(2) For all $\epsilon > 0$, since $a_n \to a$ and $b_n \to b$, we can find N_1 and N_2 such that when
$$n \geq N_1, \quad |a_n - a| < \frac{\epsilon}{2},$$
$$n \geq N_2, \quad |b_n - b| < \frac{\epsilon}{2}.$$
Take $N = \max\{N_1, N_2\}$. Then for all $n \geq N$,
$$|a_n + b_n - (a + b)| = |a_n - a + b_n - b| \leq |a_n - a| + |b_n - b| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$
Therefore $\lim_{n\to\infty}(a_n + b_n) = a + b$. □

5.2.2 Order Theorem

Theorem 5.3 (Order theorem)**.** *Let $\{a_n\}$ and $\{b_n\}$ be sequences such that*
$$\lim_{n\to\infty} a_n = a \quad \text{and} \quad \lim_{n\to\infty} b_n = b.$$
(5) If $a_n \geq 0$ for every n, then $a \geq 0$. (6) If $a_n \leq b_n$, then $a \leq b$. (7) If $a_n \geq c$, then $a \geq c$.

Proof. (5) Argue by contradiction. Suppose $a < 0$. Take $\epsilon = \frac{|a|}{2}$. Since $\lim_{n\to\infty} a_n = a$, we can find N such that when $n \geq N$, $|a_n - a| < \epsilon$. Note that this means

$$-\epsilon < a_n - a < \epsilon$$

Then we have

$$a_n < \epsilon + a = \frac{-a}{2} + a = \frac{a}{2} < 0.$$

Contradiction. □

5.2.3 Monotone Convergence Theorem

Definition 5.2. A sequence $\{a_n\}$ is **increasing** if $a_n \leq a_{n+1}$ for every n and **decreasing** if $a_n \geq a_{n+1}$ for every n. A sequence is **monotone** if it is either increasing or decreasing.

Theorem 5.4 (Monotone convergence theorem)**.** *If a sequence is monotone and bounded, then it converges.*

Proof. Let $\{a_n\}$ be increasing and bounded. Set $A = \{a_n : n \in \mathbb{N}\}$. Note that $A \neq \emptyset$ and A is bounded. Therefore, by the axiom of completeness, $s = \sup A \in \mathbb{R}$ exists. Then we claim that $\lim_{n\to\infty} a_n = s$. For every $\epsilon > 0$, $s - \epsilon$ is not an upper bound for A, so we can find N such that $s - \epsilon < a_N \leq s$. Since $\{a_n\}$ is increasing, for all $n \geq N$, we know $s - \epsilon < a_N \leq a_n \leq s$, i.e. $|a_n - s| < \epsilon$. Therefore $\lim_{n\to\infty} a_n = s$.

For $\{a_n\}$ decreasing and bounded, simply let apply the previous result to $\{-a_n\}$. □