MATH 4317: Analysis I

Frank Qiang

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Contents

Lecture 1

Aug. 22 – The Real Numbers

1.1 Number Systems

We start with the natural numbers [1]

$$\mathbb{N} = \{1, 2, 3, \dots\}.$$

These are perhaps the most natural in a way, since they are what we use to count things. They are closed under addition, but fail when it comes to subtraction. For example, $1 - 2 = -1 \notin \mathbb{N}$. So we must expand our number system to the integers

$$\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}.$$

We can add, subtract, and multiply. But we run into problems when we start to consider quotients. For example, $1 \div 2 = \frac{1}{2} \notin \mathbb{Z}$. So we continue to the rational numbers

$$\mathbb{Q} = \left\{\frac{p}{q} : p, q \in \mathbb{Z}, q \neq 0\right\}.$$

We now have summation, subtraction, multiplication, and quotients. But there is still a problem.

Consider the diagonal of a square with side length 1.

Theorem 1.1.1. *$\sqrt{2}$ is not a rational number.* [2]

Proof. Argue by contradiction. Suppose $\sqrt{2}$ is rational. Then we can write

$$\sqrt{2} = \frac{p}{q}$$

for some integers p, q. Further assume p and q have no common factors. Then

$$2 = \frac{p^2}{q^2} \implies p^2 = 2q^2.$$

[1] $0 \notin \mathbb{N}$ for this class.

[2] In some sense, this shows that the notion of "rationals" is strictly weaker than the notion of "length."

So p is even and we can write $p = 2r$ for some $r \in \mathbb{Z}$. Then

$$4r^2 = 2q^2 \implies 2r^2 = q^2.$$

So q is also even, and p, q share a common factor of 2. Contradiction. □

Another weakness of $\mathbb{Q}$ is that we cannot take limits ($\mathbb{Q}$ is not complete). For example, note that

$$(\sqrt{2}-1)(\sqrt{2}+1) = 2 - 1 = 1,$$

$$\sqrt{2} = 1 + \frac{1}{\sqrt{2}+1} = 1 + \frac{1}{1 + 1 + \frac{1}{\sqrt{2}+1}} = \dots.$$

So if we define the rational sequence

$$a_1 = 1, \quad a_2 = 1 + \frac{1}{2}, \quad a_3 = 1 + \frac{1}{2+\frac{1}{2}}, \quad a_4 = 1 + \frac{1}{2+\frac{1}{2+\frac{1}{2}}}, \quad \dots,$$

then as $n \to \infty$, $a_n \to \sqrt{2} \notin \mathbb{Q}$.

1.2 Sets

Sets are any collections of objects. Given a set A, we write $x \in A$ if x is an element of A. We write $x \notin A$ otherwise. The **union** of two sets is

$$A \cup B = \{x : x \in A \text{ or } x \in B\},$$

and the **intersection** of two sets is

$$A \cap B = \{x : x \in A \text{ and } x \in B\}.$$

We use the notation

$$\bigcup_{k=1}^{\infty} A_k$$

to denote the countable union of a family of sets indexed by k.

1.3 Functions

Definition 1.3.1. *Given two sets A and B, a **function** from A to B is a rule, relation, or mapping that takes each element $x \in A$ and associates with it a single element in B. In this case, we write $f : A \to B$.*

We call A the **domain** of f and B the **codomain** of f. The element in B associated with $x \in A$ is $f(x)$, called the **image** of x. The **range** of f is

$$\text{range}(f) = \{y \in B : y = f(x) \text{ for some } x \in A\}.$$

We say f is:

1. **onto** or **surjective** if range$(f) = B$.
2. **one-to-one** or **injective** if $x, x' \in A$ and $x \neq x'$, then $f(x) \neq f(x')$.
3. **bijective** if it is injective and surjective.

First Dirichlet function:

$$g(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases} = \lim_{k\to\infty}\left(\lim_{j\to\infty}[\cos(k!\pi x)]^{2j}\right).$$

Second Dirichlet function:

$$f(x) = \begin{cases} \frac{1}{q} & \text{if } x = \frac{p}{q} \in \mathbb{Q} \text{ in lowest terms} \\ 0 & \text{if } x \notin \mathbb{Q}. \end{cases}$$

Absolute value:

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0. \end{cases}$$

Note that we have the following two properties:

- $|xy| = |x||y|$.
- $|x + y| \leq |x| + |y|$. This is called the *triangle inequality*.

1.4 Induction

If we have a set $S \subseteq \mathbb{N}$ and

1. $1 \in S$
2. if $n \in S$, then $n + 1 \in S$

then $S = \mathbb{N}$. [3]

[3] We always use induction in conjunction with $\mathbb{N}$.

Lecture 2

Aug. 24 – The Axiom of Completeness

Lecture 3

Aug. 29 – Completeness and Countability

3.1 Consequences of Completeness

3.1.1 2nd Consequence: Density of $\mathbb{Q}$ in $\mathbb{R}$

Theorem 3.1.1 (Density of $\mathbb{Q}$ in $\mathbb{R}$)**.** *For all $a, b \in \mathbb{R}$, $a < b$, there exists $r \in \mathbb{Q}$ such that $a < r < b$.*

Proof. We want to find $m \in \mathbb{Z}$, $n \in \mathbb{N}$ such that

$$a < \frac{m}{n} < b.$$

By (ii) of the Archimedean property, we can find $n \in \mathbb{N}$ such that

$$\frac{1}{n} < b - a.$$

Fix such an n. Then let m be the smallest integer such that $m - 1 \leq na < m$. By construction,

$$\frac{m}{n} - \frac{1}{n} \leq a < \frac{m}{n},$$

$$\frac{m}{n} \leq a + \frac{1}{n} < b.$$

Therefore, $a < \frac{m}{n} < b$. □

Corollary 3.1.1.1. *For all $a, b \in \mathbb{Q}$, $a < b$, there exists $t \in \mathbb{R} \setminus \mathbb{Q}$ such that $a < t < b$.*

3.1.2 3rd Consequence: Existence of $\sqrt{2}$

Theorem 3.1.2 (Existence of $\sqrt{2}$)**.** *There exists $s \in \mathbb{R}$, $s > 0$ such that $s^2 = 2$.*

Proof. Define

$$S = \{x > 0 : x^2 < 2\} \subseteq \mathbb{R}.$$

$x = 1 \in S$, so $S \neq \emptyset$. 2 is an upper bound for S, so S is bounded above. Then by the axiom of completeness, $s = \sup S$ exists. We claim that $s^2 = 2$.

Suppose otherwise that $s^2 < 2$. Then we can find $\epsilon > 0$ such that $s+\epsilon \in S$. Define $\delta = 2 - s^2 > 0$. Note that

$$(s+\epsilon)^2 - 2 = s^2 + 2s\epsilon + \epsilon^2 - 2 = -\delta + 2s\epsilon + \epsilon^2.$$

We know $s \leq 2$ since 2 is an upper bound. Pick

$$\epsilon = \frac{\delta}{100000000000},$$

$$2s\epsilon + \epsilon \leq 4\epsilon + \epsilon^2 < \frac{\delta}{2}.$$

Then

$$(s+\epsilon)^2 - 2 < -\delta + \frac{\delta}{2} = -\frac{\delta}{2} < 0.$$

So $s+\epsilon \in S$, which contradicts with $s = \sup S$.

$s^2 > 2$ also leads to a contradiction (left as an exercise). Thus we must have $s^2 = 2$. □

3.2 Countability

Definition 3.2.1. *We say two sets A and B have the same **cardinality** if there is a bijection $f : A \to B$. We write $A \sim B$.*

Definition 3.2.2. *We say that a set A is **finite** if $A \sim \{1,2,\ldots,n\}$ for some integer n. We say that a set A is **countable** (or countably infinite) if $A \sim \mathbb{N}$. If a set A is not countable, then we say it is **uncountable**.*

$E = \{2,4,6,8,\ldots\}$.

E is not finite but it is countable: $E \sim \mathbb{N}$. We can define $f : \mathbb{N} \to E$ by $f(n) = 2n$.

$\mathbb{N} \sim \mathbb{Z}$.

The bijection $f : \mathbb{N} \to \mathbb{Z}$ is given by

$$f(n) = \begin{cases} \frac{n-1}{2} & n \text{ is odd} \\ -\frac{n}{2} & n \text{ is even.} \end{cases}$$

$(-1,1) \sim \mathbb{R}$.

The bijection $f : (-1,1) \to \mathbb{R}$ is given by

$$x \mapsto \frac{x}{x^2-1}.$$

Theorem 3.2.1.

1. *$\mathbb{Q}$ is countable.*
2. *$\mathbb{R}$ is uncountable.*

Proof of (1). Set $A_1 = \{0\}$ and for $n \geq 2$,

$$A_n = \left\{ \pm\frac{p}{q} : p, q \in \mathbb{N},\, p, q \text{ in lowest terms},\, p + q = n \right\}.$$

So the first few A_n are:

$$A_2 = \left\{ \frac{1}{1}, \frac{-1}{1} \right\},$$

$$A_3 = \left\{ \frac{1}{2}, \frac{2}{1}, \frac{-1}{2}, \frac{-2}{1} \right\},$$

etc. Note that A_n is finite and for all $x \in \mathbb{Q}$, there is an $n \in \mathbb{N}$ such that $x \in A_n$. We can list elements in $A_1, \ldots, A_n$ and label them with integers in $\mathbb{N}$. Any element of A_n will be listed eventually. Then this pairing gives a bijection since the A_n are disjoint. So $\mathbb{Q} \sim \mathbb{N}$. □

Proof of (2). Argue by contradiction. Suppose f is one-to-one from $\mathbb{N} \to \mathbb{R}$. Set $x_1 = f(1)$, $x_2 = f(2)$, etc. We can write

$$\mathbb{R} = \{x_1, x_2, \ldots\}.$$

Let I_1 be a closed interval such that $x_1 \notin I_1$. Pick $I_2 \subseteq I_1$ such that $x_2 \notin I_2$. Continue this process such that $I_{n+1} \subseteq I_n$ is a closed interval where $x_{n+1} \notin I_{n+1}$. By construction,

$$I_1 \supseteq I_2 \supseteq \cdots \supseteq I_n \supseteq \ldots.$$

We know that

$$\bigcap_{n=1}^{\infty} I_n \neq \emptyset.$$

So we can find n_0 such that

$$x_{n_0} \in \bigcap_{n=1}^{\infty} I_n.$$

This is a contradiction with $x_{n_0} \notin I_{n_0}$. Thus such an f cannot exist and $\mathbb{R}$ is uncountable. □

Theorem 3.2.2.

1. *Let $A \subseteq B$. If B is countable, then A is either finite or countable.*
2. *If A_n is a countable set, then*

$$\bigcup_{n=1}^{\infty} A_n$$

is also countable.

Theorem 3.2.3 (Cantor's theorem). *The open interval*

$$(0,1) = \{x \in \mathbb{R} : 0 < x < 1\}$$

is uncountable.

Proof. Argue by contradiction. Assume $f : \mathbb{N} \to (0,1)$ is one-to-one and onto. Then for $m \in \mathbb{N}$, we can write (decimal expansion)

$$f(m) = 0.a_{m1}a_{m2}a_{m3}\ldots \in (0,1)$$

For every $m, n \in \mathbb{N}$, $a_{mn} \in \{0, \ldots, 9\}$ is the nth digit in the decimal expansion of $f(m)$. We can write in a table

$$\begin{array}{cccccc} 1 & f(1) & a_{11} & a_{12} & a_{13} & \ldots \\ 2 & f(2) & a_{21} & a_{22} & a_{23} & \ldots \\ 3 & f(3) & a_{31} & a_{32} & a_{33} & \ldots \\ & & & \vdots & & \end{array}$$

Take $x = 0.b_1 b_2 b_3 \ldots$ where

$$b_n = \begin{cases} 2 & \text{if } a_{nn} \neq 2 \\ 3 & \text{if } a_{nn} = 2. \end{cases}$$

Then $x \neq f(m)$ for any $m \in \mathbb{N}$ (since $b_m \neq a_{mm}$). This is a contradiction. □