

# MATH 4317: Analysis I

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# Contents

<b>1</b>	<b>Aug. 22 – The Real Numbers</b>	<b>3</b>
1.1	Number Systems . . . . .	3
1.2	Sets . . . . .	4
1.3	Functions . . . . .	4
1.4	Induction . . . . .	5
<b>2</b>	<b>Aug. 24 – The Axiom of Completeness</b>	<b>6</b>
2.1	Suprema and Infima . . . . .	6
2.2	Consequences of Completeness . . . . .	7
2.2.1	1st Consequence: Nested Interval Properties . . . . .	7
2.2.2	2nd Consequence: Archimedean Properties . . . . .	8
<b>3</b>	<b>Aug. 29 – Completeness, Countability</b>	<b>9</b>
3.1	Consequences of Completeness . . . . .	9
3.1.1	3rd Consequence: Density of $\mathbb{Q}$ in $\mathbb{R}$ . . . . .	9
3.1.2	4th Consequence: Existence of $\sqrt{2}$ . . . . .	9
3.2	Countability . . . . .	10
<b>4</b>	<b>Aug. 31 – Cantor’s Theorem, Sequences</b>	<b>13</b>
4.1	Cantor’s Theorem . . . . .	13
4.2	Sequences . . . . .	13
4.2.1	Limits . . . . .	14
4.2.2	Tips for Showing Limits . . . . .	15
<b>5</b>	<b>Sept. 5 – Limits and Limit Theorems</b>	<b>16</b>
5.1	Review of Limits . . . . .	16
5.2	Limit Theorems . . . . .	16
5.2.1	Algebraic Facts About Limits . . . . .	16
5.2.2	Order Limit Theorem . . . . .	17
5.2.3	Monotone Convergence Theorem . . . . .	18
<b>6</b>	<b>Sept. 7 – Bolzano-Weierstrass Theorem</b>	<b>19</b>
6.1	Review of Limits . . . . .	19
6.2	Subsequences and the Bolzano-Weierstrass Theorem . . . . .	19
<b>7</b>	<b>Sept. 12 – The Cauchy Criterion</b>	<b>22</b>
7.1	Cauchy Sequences . . . . .	22
7.2	Revisiting Completeness . . . . .	23
<b>8</b>	<b>Sept. 14 – Series</b>	<b>25</b>

8.1	Convergence of Series . . . . .	25
8.2	Properties of Series . . . . .	27
<b>9</b>	<b>Sept. 19 – Absolute Convergence</b>	<b>29</b>
9.1	Absolute Convergence . . . . .	29
9.2	Rearrangements . . . . .	30
<b>10</b>	<b>Sept. 21 – Iterated Sums and Topology</b>	<b>32</b>
10.1	Double Sums . . . . .	32
10.1.1	Convergence of Double Sums . . . . .	33
10.2	Basic Topology in $\mathbb{R}$ . . . . .	33
10.2.1	Open Sets . . . . .	33
10.2.2	Limit Points . . . . .	34
<b>11</b>	<b>Sept. 26 – Closed Sets</b>	<b>35</b>
11.1	Closed Sets . . . . .	35
11.2	The Closure of a Set . . . . .	36
<b>12</b>	<b>Sept. 28 – Compact Sets</b>	<b>38</b>
12.1	Another Characterization of Closed Sets . . . . .	38
12.2	Compactness . . . . .	38
12.3	Another Definition of Compactness . . . . .	40
<b>13</b>	<b>Oct. 3 – Perfect Sets</b>	<b>42</b>
13.1	The Cantor Set . . . . .	42
13.2	Perfect Sets and Countability . . . . .	43

# Lecture 1

## Aug. 22 – The Real Numbers

### 1.1 Number Systems

We start with the natural numbers <sup>1</sup>

$$\mathbb{N} = \{1, 2, 3, \dots\}.$$

These are perhaps the most natural in a way, since they are what we use to count things. They are closed under addition, but fail when it comes to subtraction. For example,  $1 - 2 = -1 \notin \mathbb{N}$ . So we must expand our number system to the integers

$$\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}.$$

We can now add, subtract, and multiply. But we run into problems when we start to consider quotients. For example,  $1 \div 2 = \frac{1}{2} \notin \mathbb{Z}$ . So we continue to the rational numbers

$$\mathbb{Q} = \left\{ \frac{p}{q} : p, q \in \mathbb{Z}, q \neq 0 \right\}.$$

We now have summation, subtraction, multiplication, and quotients. But there is still a problem. For example, consider the diagonal of a square with side length 1.

**Theorem 1.1.**  $\sqrt{2}$  is not a rational number. <sup>2</sup>

*Proof.* Argue by contradiction. Suppose  $\sqrt{2}$  is rational. Then we can write

$$\sqrt{2} = \frac{p}{q}$$

for some integers  $p, q$ . Further assume  $p$  and  $q$  have no common factors. Then

$$2 = \frac{p^2}{q^2} \implies p^2 = 2q^2.$$

So  $p$  is even and we can write  $p = 2r$  for some  $r \in \mathbb{Z}$ . Then

$$4r^2 = 2q^2 \implies 2r^2 = q^2.$$

So  $q$  is also even, and  $p, q$  share a common factor of 2. Contradiction. □

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<sup>1</sup>  $0 \notin \mathbb{N}$  for this class.

<sup>2</sup> In some sense, this shows that the notion of “rationals” is strictly weaker than the notion of “length.”

Another weakness of  $\mathbb{Q}$  is that we cannot take limits ( $\mathbb{Q}$  is not complete). For example, note that

$$(\sqrt{2} - 1)(\sqrt{2} + 1) = 2 - 1 = 1,$$

$$\sqrt{2} = 1 + \frac{1}{\sqrt{2} + 1} = 1 + \frac{1}{1 + 1 + \frac{1}{\sqrt{2} + 1}} = \dots$$

So if we define the rational sequence

$$a_1 = 1, \quad a_2 = 1 + \frac{1}{2}, \quad a_3 = 1 + \frac{1}{2 + \frac{1}{2}}, \quad a_4 = 1 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2}}}, \quad \dots,$$

then as  $n \rightarrow \infty$ ,  $a_n \rightarrow \sqrt{2} \notin \mathbb{Q}$ .

## 1.2 Sets

Sets are any collections of objects. Given a set  $A$ , we write  $x \in A$  if  $x$  is an element of  $A$ . We write  $x \notin A$  otherwise. The **union** of two sets is

$$A \cup B = \{x : x \in A \text{ or } x \in B\},$$

and the **intersection** of two sets is

$$A \cap B = \{x : x \in A \text{ and } x \in B\}.$$

We use the notation

$$\bigcup_{k=1}^{\infty} A_k$$

to denote the countable union of a family of sets indexed by  $k$ .

## 1.3 Functions

**Definition 1.1.** Given two sets  $A$  and  $B$ , a **function** from  $A$  to  $B$  is a rule, relation, or mapping that takes each element  $x \in A$  and associates with it a single element in  $B$ . In this case, we write  $f : A \rightarrow B$ .

We call  $A$  the **domain** of  $f$  and  $B$  the **codomain** of  $f$ . The element in  $B$  associated with  $x \in A$  is  $f(x)$ , called the **image** of  $x$ . The **range** of  $f$  is

$$\text{range}(f) = \{y \in B : y = f(x) \text{ for some } x \in A\}.$$

We say  $f$  is:

1. **onto** or **surjective** if  $\text{range}(f) = B$ .
2. **one-to-one** or **injective** if  $x, x' \in A$  and  $x \neq x'$ , then  $f(x) \neq f(x')$ .
3. **bijective** if it is injective and surjective.

**Example 1.1.1.** First Dirichlet function:

$$g(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases} = \lim_{k \rightarrow \infty} \left( \lim_{j \rightarrow \infty} [\cos(k! \pi x)]^{2j} \right).$$

**Example 1.1.2.** Second Dirichlet function:

$$f(x) = \begin{cases} \frac{1}{q} & \text{if } x = \frac{p}{q} \in \mathbb{Q} \text{ in lowest terms} \\ 0 & \text{if } x \notin \mathbb{Q}. \end{cases}$$

**Example 1.1.3.** Absolute value:

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0. \end{cases}$$

Note that we have the following two properties:

- $|xy| = |x||y|$ .
- $|x + y| \leq |x| + |y|$ . This is called the *triangle inequality*.

## 1.4 Induction

If we have a set  $S \subseteq \mathbb{N}$  and

1.  $1 \in S$
2. if  $n \in S$ , then  $n + 1 \in S$

then  $S = \mathbb{N}$ .<sup>3</sup>

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<sup>3</sup>We always use induction in conjunction with  $\mathbb{N}$ .

# Lecture 2

## Aug. 24 – The Axiom of Completeness

The number system  $\mathbb{Q}$  is pretty good (it is a field), but recall that we are unable to take limits. For instance, take the sequence  $x_0 = 2$  and

$$x_{n+1} = \frac{1}{2} \left( x_n + \frac{2}{x_n} \right)$$

for  $n \geq 1$ . All the  $x_i$  are rational, but  $x_n \rightarrow \sqrt{2} \notin \mathbb{Q}$ . This shows that there are gaps in  $\mathbb{Q}$ . The real numbers  $\mathbb{R}$  will fill these gaps (completeness).

**Axiom 2.1** (Axiom of completeness). *Every nonempty set of real numbers that are bounded above has a least upper bound.*

Note that this least upper bound is *unique*.

### 2.1 Suprema and Infima

**Definition 2.1.** Let  $S \subseteq \mathbb{R}$ . The set  $S$  is **bounded above** if there exists  $u \in \mathbb{R}$  such that  $s \leq u$  for all  $s \in S$ . We say that  $u$  is an **upper bound** of  $S$ .

We define **bounded below** and **lower bound** similarly.

**Definition 2.2.**  $S$  is said to be **bounded** if it is both bounded above and below. Otherwise we say that  $S$  is **unbounded**.

**Example 2.2.1.**  $\mathbb{N} = \{1, 2, 3, \dots\}$  is bounded below but not above.

**Example 2.2.2.** The set

$$\left\{ \frac{1}{k} : k \in \mathbb{N} \right\} = \left\{ 1, \frac{1}{2}, \frac{1}{3}, \dots \right\}$$

is bounded.

**Example 2.2.3.**  $\emptyset$  is bounded.

**Definition 2.3.** We say  $u \in \mathbb{R}$  is the **least upper bound** or **supremum** of a nonempty set  $S \subseteq \mathbb{R}$  if

1.  $u$  is an upper bound of  $S$ .
2.  $u \leq v$  for any upper bound  $v$  of  $S$ .

We write  $u = \sup S$ .

The **greatest lower bound** or **infimum** of  $S$  is defined similarly, denoted  $\inf S$ .

**Example 2.3.1.**

$$S = \left\{ \frac{1}{k} : k \in \mathbb{N} \right\}.$$

$\sup S = 1, \inf S = 0.$

**Definition 2.4.** Let  $S \subseteq \mathbb{R}$ . We say a real number  $M \in S$  is a **maximal element** or **maximum** of  $S$  if  $s \leq M$  for all  $s \in S$ .

The **minimal element** or **minimum** is defined similarly.

**Example 2.4.1.**  $[0, 1)$  is bounded, but has no maximum. The minimum is 0.

**Example 2.4.2.** The set

$$\{2^{-n} : n \in \mathbb{N}\} = \left\{ \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots \right\}$$

is bounded, but has no minimum. The maximum is  $\frac{1}{2}$ .

**Example 2.4.3.**  $\emptyset$  is bounded but has no minimum or maximum.

**Exercise 2.1.** Let  $A \subseteq \mathbb{R}$  be bounded above. Let  $c \in \mathbb{R}$  and define

$$c + A := \{a + c : a \in A\}.$$

Then  $\sup(A + c) = c + \sup A$ .

*Proof.* Let  $s = \sup A$ . By definition, we know  $a \leq s$  for all  $a \in A$ , which implies  $a + c \leq s + c$ . So  $s + c$  is an upper bound for  $c + A$ . Now let  $b$  be an arbitrary upper bound for  $c + A$ . For all  $a \in A$ , we have  $a + c \leq b$ , which implies  $a \leq b - c$ . So  $b - c$  is an upper bound for  $A$ . By construction,  $s \leq b - c$ , so  $s + c \leq b$ . Therefore  $s + c = \sup(A + c)$ .  $\square$

**Lemma 2.1.** Assume  $s \in \mathbb{R}$  is an upper bound for a set  $A \subseteq \mathbb{R}$ . Then  $s = \sup A$  if and only if for every  $\epsilon > 0$ , there exists  $a \in A$  such that  $s - \epsilon < a$ .

*Proof.* ( $\Rightarrow$ ) Suppose  $\sup A = s$ . Then given any  $\epsilon > 0$ ,  $s - \epsilon$  cannot be an upper bound for  $A$ . So there exists  $a \in A$  such that  $a > s - \epsilon$ .

( $\Leftarrow$ ) Let  $b$  be an arbitrary upper bound for  $A$ . Suppose for contradiction that  $b < s$ . Set  $\epsilon = s - b > 0$ . Then by assumption we can find  $a \in A$  such that  $a > s - \epsilon = b$ . Contradiction. Therefore  $b \geq s$ , whence  $\sup A = s$ .  $\square$

## 2.2 Consequences of Completeness

### 2.2.1 1st Consequence: Nested Interval Properties

**Theorem 2.1** (Nested interval properties). For any  $n \in \mathbb{N}$ , assume that we are given a closed interval

$$I_n = [a_n, b_n] = \{x \in \mathbb{R} : a_n \leq x \leq b_n\}.$$

Assume  $I_n \supseteq I_{n+1}$ . Then the resulting nested sequence of closed intervals

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq \dots$$



has a nonempty intersection:

$$\bigcap_{n=1}^{\infty} I_n \neq \emptyset.$$

*Proof.* Define  $A = \{a_n\}$ . Note that  $A \neq \emptyset$ . For any  $n$ ,  $a_n \leq b_n \leq b_1$ . So  $x = \sup A$  exists. Furthermore, for any  $n$ ,  $b_n$  is an upper bound for  $A$ . So  $x \leq b_n$ . Since  $x = \sup A$ ,  $a_n \leq x$ . So  $x \in [a_n, b_n]$  for any  $n$ , whence

$$x \in \bigcap_{n=1}^{\infty} I_n.$$

□

## 2.2.2 2nd Consequence: Archimedean Properties

**Theorem 2.2** (Archimedean properties).

1. Given any  $x \in \mathbb{R}$ , there is an  $n \in \mathbb{N}$  such that  $n > x$ .<sup>1</sup>
2. Given any real number  $y > 0$ , there is an  $\mathbb{N}$  such that  $\frac{1}{n} < y$ .

*Proof of (1).* Argue by contradiction. Suppose  $\mathbb{N}$  is bounded above. Then by the axiom of completeness,  $\alpha = \sup \mathbb{N}$  exists. By construction,  $\alpha - 1$  is not an upper bound for  $\mathbb{N}$ . So we can find  $n \in \mathbb{N}$  such that  $\alpha - 1 < n$ , which implies  $\alpha < n + 1 \in \mathbb{N}$ . Contradiction. □

*Proof of (2).* Follows from (1) by setting  $x = \frac{1}{y}$ . □

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<sup>1</sup>This is saying that  $\mathbb{N}$  is not bounded above.

# Lecture 3

## Aug. 29 – Completeness, Countability

### 3.1 Consequences of Completeness

#### 3.1.1 3rd Consequence: Density of $\mathbb{Q}$ in $\mathbb{R}$

**Theorem 3.1** (Density of  $\mathbb{Q}$  in  $\mathbb{R}$ ). *For all  $a, b \in \mathbb{R}$ ,  $a < b$ , there exists  $r \in \mathbb{Q}$  such that  $a < r < b$ .*

*Proof.* We want to find  $m \in \mathbb{Z}$ ,  $n \in \mathbb{N}$  such that

$$a < \frac{m}{n} < b.$$

By (2) of the Archimedean properties, we can find  $n \in \mathbb{N}$  such that

$$\frac{1}{n} < b - a.$$

Fix such an  $n$ . Then let  $m$  be the smallest integer such that  $m - 1 \leq na < m$ . By construction,

$$\frac{m}{n} - \frac{1}{n} \leq a < \frac{m}{n},$$

$$\frac{m}{n} \leq a + \frac{1}{n} < b.$$

Therefore,  $a < \frac{m}{n} < b$ . □

**Corollary 3.1.1.** *For all  $a, b \in \mathbb{Q}$ ,  $a < b$ , there exists  $t \in \mathbb{R} \setminus \mathbb{Q}$  such that  $a < t < b$ .*

#### 3.1.2 4th Consequence: Existence of $\sqrt{2}$

**Theorem 3.2** (Existence of  $\sqrt{2}$ ). *There exists  $s \in \mathbb{R}$ ,  $s > 0$  such that  $s^2 = 2$ .*

*Proof.* Define

$$S = \{x > 0 : x^2 < 2\} \subseteq \mathbb{R}.$$

$x = 1 \in S$ , so  $S \neq \emptyset$ . 2 is an upper bound for  $S$ , so  $S$  is bounded above. Then by the axiom of completeness,  $s = \sup S$  exists. We claim that  $s^2 = 2$ .

Suppose otherwise that  $s^2 < 2$ . Then we can find  $\epsilon > 0$  such that  $s + \epsilon \in S$ . Define  $\delta = 2 - s^2 > 0$ . Note that

$$(s + \epsilon)^2 - 2 = s^2 + 2s\epsilon + \epsilon^2 - 2 = -\delta + 2s\epsilon + \epsilon^2.$$

We know  $s \leq 2$  since 2 is an upper bound. Pick

$$\epsilon = \frac{\delta}{10000000000},$$

$$2s\epsilon + \epsilon \leq 4\epsilon + \epsilon^2 < \frac{\delta}{2}.$$

Then

$$(s + \epsilon)^2 - 2 < -\delta + \frac{\delta}{2} = -\frac{\delta}{2} < 0.$$

So  $s + \epsilon \in S$ , which contradicts with  $s = \sup S$ .

$s^2 > 2$  also leads to a contradiction (left as an exercise). Thus we must have  $s^2 = 2$ . □

## 3.2 Countability

**Definition 3.1.** We say two sets  $A$  and  $B$  have the same **cardinality** if there is a bijection  $f : A \rightarrow B$ . We write  $A \sim B$ .

**Definition 3.2.** We say that a set  $A$  is **finite** if  $A \sim \{1, 2, \dots, n\}$  for some integer  $n$ . We say that a set  $A$  is **countable** (or countably infinite) if  $A \sim \mathbb{N}$ . If a set  $A$  is not countable, then we say it is **uncountable**.

**Example 3.2.1.** Let  $E = \{2, 4, 6, 8, \dots\}$ .  $E$  is not finite but it is countable:  $E \sim \mathbb{N}$ . We can define  $f : \mathbb{N} \rightarrow E$  by  $f(n) = 2n$ .

**Example 3.2.2.**  $\mathbb{N} \sim \mathbb{Z}$ . The bijection  $f : \mathbb{N} \rightarrow \mathbb{Z}$  is given by

$$f(n) = \begin{cases} \frac{n-1}{2} & n \text{ is odd} \\ -\frac{n}{2} & n \text{ is even.} \end{cases}$$

**Example 3.2.3.**  $(-1, 1) \sim \mathbb{R}$ . The bijection  $f : (-1, 1) \rightarrow \mathbb{R}$  is given by

$$x \mapsto \frac{x}{x^2 - 1}.$$

**Theorem 3.3.**

1. The set  $\mathbb{Q}$  is countable.
2. The set  $\mathbb{R}$  is uncountable.

*Proof of (1).* Set  $A_1 = \{0\}$  and for  $n \geq 2$ ,

$$A_n = \left\{ \pm \frac{p}{q} : p, q \in \mathbb{N}, p, q \text{ in lowest terms, } p + q = n \right\}.$$

So the first few  $A_n$  are:

$$A_2 = \left\{ \frac{1}{1}, \frac{-1}{1} \right\},$$

$$A_3 = \left\{ \frac{1}{2}, \frac{2}{1}, \frac{-1}{2}, \frac{-2}{1} \right\},$$

etc. Note that  $A_n$  is finite and for all  $x \in \mathbb{Q}$ , there is an  $n \in \mathbb{N}$  such that  $x \in A_n$ . We can list elements in  $A_1, \dots, A_n$  and label them with integers in  $\mathbb{N}$ . Any element of  $A_n$  will be listed eventually. Then this pairing gives a bijection since the  $A_n$  are disjoint. So  $\mathbb{Q} \sim \mathbb{N}$ .  $\square$

*Proof of (2).* Argue by contradiction. Suppose  $f$  is one-to-one from  $\mathbb{N} \rightarrow \mathbb{R}$ . Set  $x_1 = f(1)$ ,  $x_2 = f(2)$ , etc. We can write

$$\mathbb{R} = \{x_1, x_2, \dots\}.$$

Let  $I_1$  be a closed interval such that  $x_1 \notin I_1$ . Pick  $I_2 \subseteq I_1$  such that  $x_2 \notin I_2$ . Continue this process such that  $I_{n+1} \subseteq I_n$  is a closed interval where  $x_{n+1} \notin I_{n+1}$ . By construction,

$$I_1 \supseteq I_2 \supseteq \dots \supseteq I_n \supseteq \dots$$

We know that

$$\bigcap_{n=1}^{\infty} I_n \neq \emptyset.$$

So we can find  $n_0$  such that

$$x_{n_0} \in \bigcap_{n=1}^{\infty} I_n.$$

This is a contradiction with  $x_{n_0} \notin I_{n_0}$ . Thus such an  $f$  cannot exist and  $\mathbb{R}$  is uncountable.  $\square$

### Theorem 3.4.

1. Let  $A \subseteq B$ . If  $B$  is countable, then  $A$  is either finite or countable.
2. If  $A_n$  is a countable set, then

$$\bigcup_{n=1}^{\infty} A_n$$

is also countable.

**Theorem 3.5** (Cantor's diagonal argument). *The open interval*

$$(0, 1) = \{x \in \mathbb{R} : 0 < x < 1\}$$

*is uncountable.*

*Proof.* Argue by contradiction. Assume  $f : \mathbb{N} \rightarrow (0, 1)$  is one-to-one and onto. Then for  $m \in \mathbb{N}$ , we can write (decimal expansion)

$$f(m) = 0.a_{m1}a_{m2}a_{m3}\dots \in (0, 1).$$

For every  $m, n \in \mathbb{N}$ ,  $a_{mn} \in \{0, \dots, 9\}$  is the  $n$ th digit in the decimal expansion of  $f(m)$ . We can write in a table

1	$f(1)$	$a_{11}$	$a_{12}$	$a_{13}$	$\dots$
2	$f(2)$	$a_{21}$	$a_{22}$	$a_{23}$	$\dots$
3	$f(3)$	$a_{31}$	$a_{32}$	$a_{33}$	$\dots$
	$\vdots$				

Take  $x = 0.b_1b_2b_3\ldots$  where

$$b_n = \begin{cases} 2 & \text{if } a_{nn} \neq 2 \\ 3 & \text{if } a_{nn} = 2. \end{cases}$$

Then  $x \neq f(m)$  for any  $m \in \mathbb{N}$  (since  $b_m \neq a_{mm}$ ). This is a contradiction.

□

# Lecture 4

## Aug. 31 – Cantor’s Theorem, Sequences

### 4.1 Cantor’s Theorem

**Definition 4.1.** The **power set** of  $A$ , denoted  $\mathcal{P}(A)$ , is the collection of all subsets of  $A$ .

**Theorem 4.1** (Cantor’s theorem). *Given any set  $A$ , there does not exist a function  $f : A \rightarrow \mathcal{P}(A)$  which is surjective.*<sup>1</sup>

*Proof.* Argue by contradiction. Suppose  $f : A \rightarrow \mathcal{P}(A)$  is onto. Then for any  $a \in A$ ,  $f(a)$  is a subset of  $A$ . Since  $f$  is onto, for any subset  $B$  of  $A$ , we can find  $a \in A$  such that  $f(a) = B$ . Define

$$B = \{a \in A : a \notin f(a)\} \subseteq A.$$

We can find  $a' \in A$  such that  $f(a') = B$ . If  $a' \in B$ , then  $a' \notin f(a') = B$ , which is a contradiction.. If  $a' \notin B$ , this is a contradiction with the definition of  $B$ . Thus such  $f$  cannot exist.  $\square$

**Remark.** This means that the cardinality of  $\mathcal{P}(A)$  is strictly larger than that of  $A$ .

### 4.2 Sequences

**Definition 4.2.** A **sequence** is a function whose domain is  $\mathbb{N}$ .

We usually write  $\{a_n\}$ ,  $\{x_n\}$  or  $(a_n)$ ,  $(x_n)$  to denote sequences.

**Example 4.2.1.** The following

$$\left\{ \frac{1+n}{n} \right\}_{n=1}^{\infty} = \left\{ 2, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \dots \right\}$$

is a sequence.

**Example 4.2.2.**  $\{a_n\}$ , where  $a_n = 2^n$  for  $n \in \mathbb{N}$ , is a sequence.

**Example 4.2.3.** We can also define  $\{x_n\}$  recursively by  $x_1 = 2$  and

$$x_{n+1} = \frac{x_n + 1}{2}.$$

**Remark.** Sometimes a sequence is also labeled starting from  $n = 0$ .

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<sup>1</sup>Note that if  $\#(A) = n < \infty$ , this is true as  $\#(\mathcal{P}(A)) = 2^n \neq \#(A)$ .

### 4.2.1 Limits

**Definition 4.3.** A sequence  $\{a_n\}$  **converges** to a real number  $a$  if for every  $\epsilon > 0$ , we can find  $N \in \mathbb{N}$  such that for all  $n \geq N$ , one has  $|a_n - a| < \epsilon$ . We write  $\lim_{n \rightarrow \infty} a_n = a$ .

**Remark.** In analysis,  $\epsilon$  is always taken to be a positive number.

**Example 4.3.1.** The sequence  $\{1/n\}_{n=1}^{\infty}$  converges with

$$\lim_{n \rightarrow \infty} \frac{1}{n} = 0.$$

**Definition 4.4.** For  $\epsilon > 0$ , the  $\epsilon$ -**neighborhood** of  $a$  is defined to be

$$V_{\epsilon}(a) = \{x \in \mathbb{R} : |x - a| < \epsilon\}.$$

**Definition 4.5.** We say that  $a$  is the **limit** of a sequence  $\{a_n\}$  if for every  $\epsilon > 0$ ,  $V_{\epsilon}(a)$  contains all but finitely many elements of  $\{a_n\}$ .<sup>2</sup>

**Remark.** This definition of the limit is equivalent to the definition of convergence.

**Definition 4.6.** A sequence  $\{a_n\}$  that does not converge is said to be **divergent**.

**Theorem 4.2.** *The limit of a sequence, when it exists, must be unique.*

*Proof.* Homework problem. □

**Exercise 4.1.** Show

$$\lim_{n \rightarrow \infty} \frac{n+1}{n}$$

exists and

$$\lim_{n \rightarrow \infty} \frac{n+1}{n} = 1.$$

*Proof.* We show

$$\lim_{n \rightarrow \infty} \frac{n+1}{n} = 1.$$

For every  $\epsilon > 0$ , take  $N \in \mathbb{N}$  such that  $N > \frac{1}{\epsilon}$ . We have for all  $n \geq N$ ,

$$\left| \frac{n+1}{n} - 1 \right| = \left| \frac{1}{n} \right| \leq \frac{1}{N} < \epsilon.$$

Therefore,

$$\lim_{n \rightarrow \infty} \frac{n+1}{n} = 1.$$

□

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<sup>2</sup>This is the *topological* definition of the limit.

### 4.2.2 Tips for Showing Limits

To show the limit of a sequence, take the following steps:

1. Identify the limit  $a$ . This is always given by the problem or observation.
2.  $\forall \epsilon > 0$ .
3. Find  $N = N(\epsilon)$ . Do this in sketch paper (need computations and manipulations).
4. Set  $N$  as what is found in (3).
5. Check that  $N$  works.



# Lecture 5

## Sept. 5 – Limits and Limit Theorems

### 5.1 Review of Limits

**Example 5.0.1.** Find

$$\lim_{n \rightarrow \infty} \frac{1 + \sqrt{n}}{\sqrt{n}}.$$

*Proof.* We want to show that

$$\lim_{n \rightarrow \infty} \frac{1 + \sqrt{n}}{\sqrt{n}} = 1.$$

Fix  $\epsilon > 0$  and take  $N \in \mathbb{N}$  such that  $N > \frac{1}{\epsilon^2}$ . Then for any  $n > N$ ,

$$\left| \frac{1 + \sqrt{n}}{\sqrt{n}} - 1 \right| \leq \left| \frac{1}{\sqrt{n}} \right| \leq \frac{1}{\sqrt{N}} < \epsilon,$$

as desired. □

How can we understand this using the topological definition? For all  $\epsilon > 0$ , take  $V_\epsilon(1)$ . Pick  $N > \frac{1}{\epsilon^2}$ . Then we claim that  $V_\epsilon(1)$  contains all but at most  $N$  elements of  $\left\{ \frac{\sqrt{n}+1}{\sqrt{n}} \right\}$ . When  $n \geq N$ , we have

$$\left| \frac{\sqrt{n}+1}{\sqrt{n}} - 1 \right| < \epsilon,$$

i.e.  $\frac{\sqrt{n}+1}{\sqrt{n}} \in V_\epsilon(1)$ . So at most  $N$  elements might not be in  $V_\epsilon(1)$ .

### 5.2 Limit Theorems

#### 5.2.1 Algebraic Facts About Limits

**Definition 5.1.** A sequence  $\{x_n\}$  is said to be **bounded** if there exists  $M$  such that  $|x_n| \leq M$  for all  $n$ . Alternatively,  $\sup_n |x_n| \leq M$ .

**Theorem 5.1.** Every convergent sequence is bounded.

*Proof.* Suppose

$$\lim_{n \rightarrow \infty} x_n = l.$$

Take  $\epsilon = 1$ , we can find  $N$  such that for all  $n \geq N$ ,  $|x_n - l| < 1$ . By the triangle inequality,  $|x_n| < |l| + 1$  for  $n \geq N$ . Take

$$M = \max\{|x_1|, |x_2|, \dots, |x_{N-1}|, |l| + 1\}.$$

Then  $|x_n| \leq M$  for all  $n \in \mathbb{N}$ . □

**Theorem 5.2** (Algebraic limit theorem). *If*

$$\lim_{n \rightarrow \infty} a_n = a \quad \text{and} \quad \lim_{n \rightarrow \infty} b_n = b,$$

*then for all  $c \in \mathbb{R}$ ,*

$$(1) \lim_{n \rightarrow \infty} ca_n = ca, \quad (2) \lim_{n \rightarrow \infty} (a_n + b_n) = a + b, \quad \text{and} \quad (3) \lim_{n \rightarrow \infty} a_n b_n = ab.$$

*Furthermore, if  $b \neq 0$ , then*

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \frac{a}{b}. \quad (4)$$

*Proof.* (1) When  $c = 0$ , the result is trivial. When  $c \neq 0$ , for all  $\epsilon > 0$ , we set  $\epsilon' = \frac{\epsilon}{|c|}$ . Since  $\lim_{n \rightarrow \infty} a_n = a$ , we can find  $N_{\epsilon'}$  such that for all  $n \geq N_{\epsilon'}$ ,  $|a_n - a| < \epsilon'$ . When  $n > N_{\epsilon'}$ , we have

$$|ca_n - ca| = |c||a_n - a| < |c|\epsilon' = |c|\frac{\epsilon}{|c|} = \epsilon.$$

So  $\lim_{n \rightarrow \infty} ca_n = ca$ .

(2) For all  $\epsilon > 0$ , since  $a_n \rightarrow a$  and  $b_n \rightarrow b$ , we can find  $N_1$  and  $N_2$  such that when

$$\begin{aligned} n \geq N_1, \quad |a_n - a| &< \frac{\epsilon}{2}, \\ n \geq N_2, \quad |b_n - b| &< \frac{\epsilon}{2}. \end{aligned}$$

Take  $N = \max\{N_1, N_2\}$ . Then for all  $n \geq N$ ,

$$|a_n + b_n - (a + b)| = |a_n - a + b_n - b| \leq |a_n - a| + |b_n - b| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Therefore  $\lim_{n \rightarrow \infty} (a_n + b_n) = a + b$ . □

### 5.2.2 Order Limit Theorem

**Theorem 5.3** (Order limit theorem). *Let  $\{a_n\}$  and  $\{b_n\}$  be sequences such that*

$$\lim_{n \rightarrow \infty} a_n = a \quad \text{and} \quad \lim_{n \rightarrow \infty} b_n = b.$$

*(5) If  $a_n \geq 0$  for every  $n$ , then  $a \geq 0$ . (6) If  $a_n \leq b_n$ , then  $a \leq b$ . (7) If  $a_n \geq c$ , then  $a \geq c$ .*

*Proof.* (5) Argue by contradiction. Suppose  $a < 0$ . Take  $\epsilon = \frac{|a|}{2}$ . Since  $\lim_{n \rightarrow \infty} a_n = a$ , we can find  $N$  such that when  $n \geq N$ ,  $|a_n - a| < \epsilon$ . Note that this means

$$-\epsilon < a_n - a < \epsilon$$

Then we have

$$a_n < \epsilon + a = \frac{-a}{2} + a = \frac{a}{2} < 0.$$

Contradiction. □

### 5.2.3 Monotone Convergence Theorem

**Definition 5.2.** A sequence  $\{a_n\}$  is **increasing** if  $a_n \leq a_{n+1}$  for every  $n$  and **decreasing** if  $a_n \geq a_{n+1}$  for every  $n$ . A sequence is **monotone** if it is either increasing or decreasing.

**Theorem 5.4** (Monotone convergence theorem). *If a sequence is monotone and bounded, then it converges.*

*Proof.* Let  $\{a_n\}$  be increasing and bounded. Set  $A = \{a_n : n \in \mathbb{N}\}$ . Note that  $A \neq \emptyset$  and  $A$  is bounded. Therefore, by the axiom of completeness,  $s = \sup A \in \mathbb{R}$  exists. Then we claim that  $\lim_{n \rightarrow \infty} a_n = s$ . For every  $\epsilon > 0$ ,  $s - \epsilon$  is not an upper bound for  $A$ , so we can find  $N$  such that  $s - \epsilon < a_N \leq s$ . Since  $\{a_n\}$  is increasing, for all  $n \geq N$ , we know  $s - \epsilon < a_N \leq a_n \leq s$ , i.e.  $|a_n - s| < \epsilon$ . Therefore  $\lim_{n \rightarrow \infty} a_n = s$ .

For  $\{a_n\}$  decreasing and bounded, simply let apply the previous result to  $\{-a_n\}$ . □

# Lecture 6

## Sept. 7 – Bolzano-Weierstrass Theorem

### 6.1 Review of Limits

**Theorem 6.1** (Squeeze theorem). Let  $\{x_n\}, \{y_n\}, \{z_n\}$  be sequences such that  $x_n \leq y_n \leq z_n$  for all  $n$ , and suppose that

$$\lim_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} z_n = l.$$

Then  $\lim_{n \rightarrow \infty} y_n = l$ .

*Proof.* Consider  $|y_n - l|$ . If

$$\begin{aligned} y_n - l \geq 0, & \text{ then } y_n - l \leq z_n - l, \\ y_n - l < 0, & \text{ then } |y_n - l| = l - y_n \leq l - x_n. \end{aligned}$$

So we have

$$|y_n - l| \leq |z_n - l| + |x_n - l|.$$

For all  $\epsilon > 0$ , there exist  $N_1, N_2$  such that for all  $n \geq N_1$ ,

$$|z_n - l| < \frac{\epsilon}{2},$$

and for all  $n \geq N_2$ ,

$$|x_n - l| < \frac{\epsilon}{2}.$$

Take  $N = \max\{N_1, N_2\}$ . If  $n \geq N$ , then

$$|y_n - l| \leq |z_n - l| + |x_n - l| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

So  $\lim_{n \rightarrow \infty} y_n = l$ . □

### 6.2 Subsequences and the Bolzano-Weierstrass Theorem

**Definition 6.1.** Let  $\{a_n\}$  be a sequence of real numbers. Let  $n_1 < n_2 < n_3 < \dots$  be an increasing sequence of natural numbers. Then  $\{a_{n_1}, a_{n_2}, \dots\}$  is a **subsequence** of  $\{a_n\}$ , and it is denoted by  $\{a_{n_k}\}$ .

**Example 6.1.1.** Let

$$\{a_n\} = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\right\}.$$

Then

$$\left\{\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \dots\right\}$$

is a subsequence of  $\{a_n\}$ . However, note that

$$\left\{\frac{1}{10}, \frac{1}{5}, \frac{1}{100}, \frac{1}{500}, \dots\right\}$$

is *not* a subsequence of  $\{a_n\}$  since the  $n_k$  are not strictly increasing. Similarly,

$$\left\{1, \frac{1}{3}, \frac{1}{3}, \frac{1}{5}, \frac{1}{5}, \dots\right\}$$

is also not a subsequence of  $\{a_n\}$ .

**Theorem 6.2.** *Subsequences of a convergent sequence converge to the same limit as the original sequence.*

*Proof.* Suppose  $\lim_{n \rightarrow \infty} a_n = a$ . So for every  $\epsilon > 0$ , there exists  $N$  such that  $|a_n - a| < \epsilon$  for all  $n \geq N$ . Consider an arbitrary subsequence  $\{a_{n_k}\}$ . Note that  $n_k \geq k$ . So when  $k \geq N$ ,

$$|a_{n_k} - a| < \epsilon.$$

Therefore  $\lim_{k \rightarrow \infty} a_{n_k} = a$ . □

**Example 6.1.2.** Let  $0 < b < 1$ . Clearly

$$1 > b > b^2 > b^3 > b^4 > \dots \geq 0.$$

The sequence  $\{b^n\}$  is decreasing and bounded below, so by the monotone convergence theorem,  $\lim_{n \rightarrow \infty} b^n = l \in \mathbb{R}$  exists. Note that  $\{b^{2n}\}$  is a subsequence of  $\{b^n\}$ , so by Theorem 6.2, we have  $\lim_{n \rightarrow \infty} b^{2n} = l$ . Note that  $b^{2n} = b^n b^n$ . By the algebraic limit theorem,

$$\lim_{n \rightarrow \infty} b^{2n} = \left(\lim_{n \rightarrow \infty} b^n\right) \left(\lim_{n \rightarrow \infty} b^n\right).$$

Therefore,  $l = l^2$ , so we have  $l = 0$  or  $l = 1$ . But the entire sequence is strictly less than 1 and decreasing, so  $l = 0$ .

**Example 6.1.3.** Consider the sequence

$$\{(-1)^n\} = \{-1, 1, -1, 1, \dots\}.$$

This sequence does not converge. But the subsequence

$$\{-1, -1, -1, \dots\}$$

does converge.

**Remark.** This shows that the converse of Theorem 6.2 is not true, i.e. a convergent subsequence does not imply that the original sequence converges.

**Example 6.1.4.** The sequence

$$a_n = \begin{cases} 1 & \text{if } n \text{ is prime,} \\ 0 & \text{otherwise} \end{cases}$$

does not converge.

**Exercise 6.1.** Show the limit of the sequence

$$\left\{1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \frac{1}{5}, \frac{1}{5}, -\frac{1}{5}, \frac{1}{5}, -\frac{1}{5}, \dots\right\}$$

*Proof.* The subsequence

$$\left\{\frac{1}{5}, \frac{1}{5}, \dots\right\}$$

converges to  $\frac{1}{5}$  while the subsequence

$$\left\{-\frac{1}{5}, -\frac{1}{5}, \dots\right\}$$

converges to  $-\frac{1}{5}$ . Thus the original sequence diverges.  $\square$

**Remark.** If we can find two subsequences that converge to different limits, then the original sequence diverges. This is the contrapositive of Theorem 6.2.

**Theorem 6.3** (Bolzano-Weierstrass theorem). *Every bounded sequence has a convergent subsequence.*<sup>1</sup>

*Proof.* Let  $\{a_n\}$  be a bounded sequence. So there exists  $M > 0$  such that  $\sup_n |a_n| < M$ . So  $a_n$  is contained in  $[-M, M]$ . Split  $[-M, M]$  into  $[-M, 0]$  and  $[0, M]$ . Pick one that contains infinitely many elements of  $\{a_n\}$  and call it  $I_1$ . Then pick  $a_{n_1} \in \{a_n\}$  such that  $a_{n_1} \in I_1$ . Split  $I_1$  again into two closed intervals of the same size. Take one of these two that contains infinitely many elements of  $\{a_n\}$  and call it  $I_2$ . Then take  $a_{n_2} \in \{a_n\}$  such that  $a_{n_2} \in I_2$ . Repeat this process to get  $I_{k+1} \subseteq I_k$  with  $|I_{k+1}| = \frac{1}{2}|I_k|$  such that  $I_{k+1}$  contains infinitely many elements of  $\{a_n\}$ .<sup>2</sup> Also pick  $a_{n_{k+1}} \in \{a_n\}$  such that  $a_{n_{k+1}} \in I_{k+1}$  with  $n_{k+1} > n_k$ .

By construction,  $\{a_{n_k}\}$  is a subsequence of  $\{a_n\}$  and  $a_{n_k} \in I_k$ . We have the  $I_k$  being closed intervals with

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq \dots$$

So there exists  $x \in \mathbb{R}$  such that  $x \in \bigcap_{k=1}^{\infty} I_k$ . Note that  $|I_k| = M \left(\frac{1}{2}\right)^{k-1}$ . Then we claim  $\lim_{k \rightarrow \infty} a_{n_k} = x$ .

Let  $\epsilon > 0$ . Take  $N$  such that

$$2^N > \frac{2M}{\epsilon}.$$

Then for every  $k \geq N$ , we have

$$|a_{n_k} - x| \leq M \left(\frac{1}{2}\right)^{k-1} < \epsilon$$

since  $a_{n_k}, x \in I_k$ . Thus  $\lim_{k \rightarrow \infty} a_{n_k} = x$ , and  $\{a_{n_k}\}$  is a convergent subsequence.  $\square$

<sup>1</sup>This demonstrates some kind of *compactness* of the real numbers.

<sup>2</sup>Here, by  $|I_k|$  we mean the length of the interval  $I_k$ .

# Lecture 7

## Sept. 12 – The Cauchy Criterion

### 7.1 Cauchy Sequences

**Definition 7.1.** A sequence  $\{a_n\}$  is called a **Cauchy** sequence if for every  $\epsilon > 0$ , there exists  $N \in \mathbb{N}$  such that for every  $m, n \geq N$ , one has  $|a_m - a_n| < \epsilon$ .<sup>1</sup>

**Theorem 7.1.** Every convergent sequence is a Cauchy sequence.

*Proof.* Assume  $\lim_{n \rightarrow \infty} a_n = a$ . Then for every  $\epsilon > 0$ , we can find  $N$  such that for every  $n \geq N$ , we have  $|a_n - a| < \epsilon/2$ . Then for every  $m, n \geq N$ , we have

$$|a_m - a_n| = |a_m - a + a - a_n| \leq |a_m - a| + |a - a_n| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

by the triangle inequality. □

**Lemma 7.1.** Every Cauchy sequence is bounded.

*Proof.* Suppose  $\{x_n\}$  is a Cauchy sequence. Pick  $\epsilon = 1$ . Then there exists  $N$  such that for all  $m, n \geq N$ , we have  $|x_m - x_n| < 1$ . Fixing  $m = N$ , we know that for all  $n \geq N$ ,  $|x_N - x_n| < 1$ . So  $|x_n| \leq |x_N| + 1$  for all  $n \geq N$ . Set

$$M = \max\{|x_1|, |x_2|, \dots, |x_{N-1}|, |x_N| + 1\}.$$

Then  $\sup |x_n| \leq M$  by construction. □

**Theorem 7.2** (Cauchy criterion). A sequence converges if and only if it is a Cauchy sequence.

*Proof.* ( $\Rightarrow$ ) This is Theorem 7.1.

( $\Leftarrow$ ) Suppose  $\{a_n\}$  is a Cauchy sequence. Since  $\{a_n\}$  is Cauchy, we know  $\sup |a_n| \leq M$  for some  $M \in \mathbb{R}$ . Then by the Bolzano-Weierstrass theorem, we can find a convergent subsequence  $\{a_{n_k}\}$  such that  $\lim_{k \rightarrow \infty} a_{n_k} = a$ . We show that we also have  $\lim_{n \rightarrow \infty} a_n = a$ .

For every  $\epsilon > 0$ , we can find  $N_1$  such that for all  $m, n \geq N_1$ , we have  $|a_m - a_n| < \epsilon/2$ . Since  $\lim_{k \rightarrow \infty} a_{n_k} = a$ , there is some  $K$  such that for all  $k \geq K$ , we have  $|a_{n_k} - a| < \epsilon/2$ . Take

$$N \geq \max\{N_1, n_K\}.$$

---

<sup>1</sup>The Cauchy condition controls the *oscillation* of the *tail* of a sequence.

We can find  $K_0$  such that  $n_{K_0} \geq N$ . Then for every  $n \geq N$ ,

$$|a_n - a| = |a_n - a_{n_{K_0}} + a_{n_{K_0}} - a| \leq |a_n - a_{n_{K_0}}| + |a_{n_{K_0}} - a| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

by the triangle inequality and the Cauchy condition.  $\square$

**Remark.** The Cauchy condition allows us to show that a sequence converges without explicitly providing its limit.

## 7.2 Revisiting Completeness

This is the way we have discussed completeness (ordered by implication):

- Axiom of Completeness
  - Nested intervals property
    - \* Bolzano-Weierstrass theorem
    - Cauchy criterion
  - Monotone convergence theorem.

But this is not the only way to do so: We have several ways of choosing axioms to define completeness. For example, we can also prove the nested intervals property using the monotone convergence theorem.

**Exercise 7.1.** The monotone convergence theorem implies the nested intervals property.

*Proof.* Let  $I_n = [a_n, b_n]$  with  $I_{n+1} \subseteq I_n$ . In particular,  $\{a_n\}$  is increasing and bounded ( $b_1$  is an upper bound). So by the monotone convergence theorem,  $\lim_{n \rightarrow \infty} a_n = a$  exists.

Left as an exercise to show that  $a \in I_n$  for all  $n$ .  $\square$

**Exercise 7.2.** Given the Archimedean property, the nested intervals property implies the Axiom of Completeness.

*Proof.* Note that  $\frac{1}{2^n} \rightarrow 0$  as  $n \rightarrow \infty$ . This is because for every  $\epsilon > 0$ , we can find  $N$  such that  $\frac{1}{N} < \epsilon$  by the Archimedean property. Then

$$\frac{1}{2^N} < \frac{1}{N}$$

for all  $N \in \mathbb{N}$ . So  $\lim_{n \rightarrow \infty} \frac{1}{2^n} = 0$ .

Now let  $S$  be a nonempty set which is bounded above. Let  $U$  be an upper bound for  $S$ . Take  $s \in S$ . Set  $a_1 = s$ ,  $b_1 = U$ . Consider

$$\frac{s+U}{2}.$$

If  $\frac{s+U}{2}$  is an upper bound for  $S$ , then we set  $a_2 = a_1 = s$ ,  $b_2 = \frac{s+U}{2}$ . If  $\frac{s+U}{2}$  is not an upper bound for  $S$ , then we set  $a_2 = \frac{s+U}{2}$ ,  $b_2 = b_1 = U$ . Note that  $[a_2, b_2] \subseteq [a_1, b_1]$ . Repeat the same process for  $a_n$  and  $b_n$  to obtain the closed intervals

$$[a_1, b_1] \supseteq [a_2, b_2] \supseteq [a_3, b_3] \supseteq \dots$$



By the nested interval properties, the intersection  $\bigcap_{n=1}^{\infty} [a_n, b_n]$  is nonempty. Note that

$$\begin{aligned} |[a_1, b_1]| &= |b_1 - a_1| = |U - s| \\ |[a_2, b_2]| &= |b_2 - a_2| = \left| \frac{U - s}{2} \right| \\ &\vdots \\ |[a_n, b_n]| &= |b_n - a_n| = \frac{2}{2^n} |U - s| \end{aligned}$$

So there is only one  $x \in \mathbb{R}$  such that  $x \in \bigcap_{n=1}^{\infty} [a_n, b_n]$ . We claim that  $\sup S = x$ .

Note that  $x \in [a_n, b_n]$  for all  $n$ . So  $a_n$  is not an upper bound and  $b_n$  is an upper bound. Suppose for contradiction that  $x$  is not an upper bound. Then there exists  $s_0 \in S$  such that  $s_0 > x$ . Since  $|[a_n, b_n]| \rightarrow 0$ , there exists an  $N$  such that whenever  $n \geq N$ ,

$$|[a_n, b_n]| < \frac{1}{2} |s_0 - x|.$$

Since  $x \in [a_n, b_n]$ , this implies that  $s_0 > b_n$ , which is a contradiction with  $b_n$  being an upper bound.

Use a similar idea to show that  $x$  is the *least* upper bound. □

**Remark.** These are all different ways to understand the same idea of completeness.

# Lecture 8

## Sept. 14 – Series

**Definition 8.1.** Let  $\{b_n\}$  be a sequence. An infinite **series** is formally given by

$$\sum_{n=1}^{\infty} b_n = b_1 + b_2 + \dots$$

**Definition 8.2.** We define the **partial sum** of a series by

$$s_m = \sum_{n=1}^m b_n.$$

### 8.1 Convergence of Series

**Definition 8.3.** The series  $\sum_{n=1}^{\infty} b_n$  **converges** to  $B$  if  $\lim_{m \rightarrow \infty} s_m = B$ . Otherwise we say that the series **diverges**.

**Example 8.3.1.** Consider the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots$$

We look at the partial sums for  $m > 1$ :

$$\begin{aligned} s_m &= \sum_{n=1}^m \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots + \frac{1}{m^2} \leq 1 + \frac{1}{2(1)} + \frac{1}{3(2)} + \frac{1}{4(3)} + \dots + \frac{1}{m(m-1)} \\ &= 1 + 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \dots + \frac{1}{m-1} - \frac{1}{m} \leq 2 - \frac{1}{m}. \end{aligned}$$

Note that  $\{s_m\}$  is a monotone sequence and it is bounded above by 2. Thus by the monotone convergence theorem,  $\{s_m\}$  converges and there is some  $B \in \mathbb{R}$  such that  $\lim_{m \rightarrow \infty} s_m = B$ .

**Remark.** Using some complex analysis, we can find  $B$  by way of residue calculations.

**Example 8.3.2.** Consider the harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \dots$$

We look at the partial sums

$$s_m = 1 + \frac{1}{2} + \cdots + \frac{1}{m}.$$

Note specifically that

$$\begin{aligned} s_4 &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} > 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} = 1 + \frac{1}{2} + 2\left(\frac{1}{4}\right) = 1 + \frac{1}{2} + \frac{1}{2} = 1 + 2\left(\frac{1}{2}\right) \\ s_8 &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} > 1 + \frac{1}{2} + 2\left(\frac{1}{4}\right) + 4\left(\frac{1}{8}\right) = 1 + 3\left(\frac{1}{2}\right) \\ &\vdots \\ s_{2^k} &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{2^{k-1}+1} + \frac{1}{2^{k-1}+2} + \cdots + \frac{1}{2^k} > 1 + \frac{k}{2} \end{aligned}$$

Thus  $\{s_{2^k}\}$  diverges, so  $\{s_m\}$  also diverges.

**Remark.** This type of trick (analyzing  $2^k$  terms) is called *dyadic analysis*, and it shows up frequently in analysis, particularly harmonic analysis.

**Theorem 8.1** (Cauchy condensation test). *Suppose  $\{b_n\}$  is decreasing and  $b_n \geq 0$  for all  $n$ . Then*

$$\sum_{n=1}^{\infty} b_n = b_1 + b_2 + b_3 + \dots$$

*converges if and only if*

$$\sum_{n=0}^{\infty} 2^n b_{2^n} = b_1 + 2b_2 + 4b_4 + \dots$$

*converges.*

*Proof.* First we show the backwards direction. Assume  $\sum_{n=0}^{\infty} 2^n b_{2^n}$  converges. Define

$$t_k = b_1 + \cdots + 2^k b_{2^k}.$$

By assumption,  $\{t_k\}$  converges. Note that  $t_k \geq 0$  and  $\sup_k t_k \leq M$  since convergent series are bounded. Set

$$s_m = \sum_{n=1}^m b_n.$$

Fix  $m$  and take  $k$  large such that  $m \leq 2^{k+1} - 1$ . Then  $s_m \leq s_{2^{k+1}-1}$  since  $b_n \geq 0$ . Observe that

$$\begin{aligned} s_{2^{k+1}-1} &= b_1 + (b_2 + b_3) + (b_4 + b_5 + b_6 + b_7) + \cdots + (b_{2^k} + \cdots + b_{2^{k+1}-1}) \\ &\leq b_1 + 2b_2 + 4b_4 + \cdots + 2^k b_{2^k}. \end{aligned}$$

So  $s_m \leq s_{2^{k+1}-1} \leq t_k \leq M$ . Thus  $\{s_m\}$  is increasing and bounded, so by the monotone convergence theorem,  $\lim_{m \rightarrow \infty} s_m = B \in \mathbb{R}$  exists.

Now we show the forwards direction. Argue by contraposition. Suppose  $\sum_{n=0}^{\infty} 2^n b_{2^n}$  diverges, then we show that  $\sum_{n=1}^{\infty} b_n$  also diverges. Just need to check that  $s_{2^k} \geq \frac{1}{2} + k$  (left as an exercise).  $\square$

**Corollary 8.1.1.** *The series*

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

*converges if and only if  $p > 1$ .*

*Proof.* Let  $b_n = \frac{1}{n^p}$  and  $b_{2^n} = \frac{1}{2^{np}}$ . Then we have

$$\sum_{n=0}^{\infty} 2^n b_{2^n} = \sum_{n=0}^{\infty} 2^{(1-p)n}.$$

The RHS is a geometric series, which converges if and only if  $p > 1$ . To see this, denote  $2^{1-p} = a$ . Then we have

$$\sum_{n=0}^{\infty} 2^{(1-p)n} = \sum_{n=0}^{\infty} a^n.$$

We can observe that the partial sums

$$t_k = \sum_{n=0}^k a^n = \frac{a^{k+1} - 1}{a - 1}$$

converges if and only if  $a^{k+1}$  converges. This happens if and only if  $a < 1$ , which happens if and only if  $p > 1$ .  $\square$

## 8.2 Properties of Series

**Theorem 8.2** (Algebraic limit theorem for series). *Let*

$$\sum_{n=1}^{\infty} a_n = A, \quad \sum_{n=1}^{\infty} b_n = B.$$

*Then for all  $c \in \mathbb{R}$ , we have*

$$\sum_{n=1}^{\infty} ca_n = cA, \quad \sum_{n=1}^{\infty} (a_n + b_n) = A + B.$$

*Proof.* Let  $\sum_{n=1}^{\infty} a_n = A$ . So  $s_m = \sum_{n=1}^m a_n$  converges. Set  $\lim_{m \rightarrow \infty} s_m = A$ . Define

$$t_m = \sum_{n=1}^m ca_n = c \sum_{n=1}^m a_n = cs_m.$$

Then by the algebraic limit theorem, we have  $\lim_{m \rightarrow \infty} t_m = c \lim_{m \rightarrow \infty} s_m = cA$ .  $\square$

**Theorem 8.3** (Cauchy criterion for series). *The series  $\sum_{n=1}^{\infty} a_n$  converges if and only if for all  $\epsilon > 0$ , there exist  $N$  such that whenever  $m, n \geq N$ , we have  $|a_{m+1} + \cdots + a_n| < \epsilon$ .*

*Proof.* The series  $\sum_{k=1}^{\infty} a_k$  converges if and only if  $s_m = \sum_{k=1}^m a_k$  converges. We show that  $\{s_m\}$  is a Cauchy sequence. For all  $\epsilon > 0$ , there exists  $N$  such that for all  $m, n \geq N$

$$|s_n - s_m| = |a_n + \cdots + a_{m+1}| < \epsilon.$$

The converse is the same inequality. □

**Corollary 8.3.1.** *If  $\sum_{n=1}^{\infty} a_n$  converges, then  $\lim_{n \rightarrow \infty} a_n = 0$ .*

*Proof.* Take  $m = n - 1$ . □

**Theorem 8.4.** *Assume  $\{a_n\}$  and  $\{b_n\}$  are sequences such that  $0 \leq a_n \leq b_n$  for all  $n$ . Then*

1.  $\sum_{n=1}^{\infty} b_n$  converges implies  $\sum_{n=1}^{\infty} a_n$  converges,
2. and  $\sum_{n=1}^{\infty} a_n$  diverges implies  $\sum_{n=1}^{\infty} b_n$  diverges.

*Proof.* For all  $m, n$ , we have

$$|a_{m+1} + \cdots + a_n| \leq |b_{m+1} + \cdots + b_n|.$$

Then apply the Cauchy criterion. □

**Definition 8.4.** A series is called **geometric** if it is of the form

$$\sum_{k=0}^{\infty} ar^k = a + ar + ar^2 + \dots$$

Note that the geometric series diverges when  $r = 1$  and  $a \neq 0$ . When  $r \neq 1$ , the partial sums

$$s_m = \sum_{k=0}^m ar^k = a \frac{1 - r^{m+1}}{1 - r}$$

converge if  $|r| < 1$ . In this case, as  $m \rightarrow \infty$ , we have

$$s_m \rightarrow \frac{a}{1 - r}.$$

# Lecture 9

## Sept. 19 – Absolute Convergence

### 9.1 Absolute Convergence

**Definition 9.1.** Consider a series  $\sum_{n=1}^{\infty} a_n$ . If

$$\sum_{n=1}^{\infty} |a_n|$$

converges, then we say  $\sum_{n=1}^{\infty} a_n$  **converges absolutely**.

**Theorem 9.1.** If  $\sum_{n=1}^{\infty} |a_n|$  converges, then  $\sum_{n=1}^{\infty} a_n$  converges.

*Proof.* For every  $\epsilon > 0$ , since  $\sum_{n=1}^{\infty} a_n$  converges, there is  $N$  such that for all  $m, k \geq N$ ,

$$\sum_{n=k+1}^m |a_n| < \epsilon.$$

This is by the Cauchy criterion for series. Then for all  $m, k \geq N$ , we have

$$\left| \sum_{n=k+1}^m a_n \right| \leq \sum_{n=k+1}^m |a_n| < \epsilon$$

by the triangle inequality. Apply the Cauchy criterion again to conclude that  $\sum_{n=1}^{\infty} a_n$  converges.  $\square$

**Theorem 9.2** (Alternating series test). If  $a_1 \geq a_2 \geq a_3 \geq \dots$  and  $\lim_{n \rightarrow \infty} a_n = 0$ , then the series

$$\sum_{n=1}^{\infty} (-1)^{n+1} a_n$$

converges.

*Proof.* Set  $s_m = \sum_{n=1}^m (-1)^{n+1} a_n$ . Check that

$$s_m - s_k = \sum_{n=k+1}^m (-1)^{n+1} a_n.$$

Suppose that  $m$  and  $k$  are odd, then

$$s_m - s_k = \underbrace{a_m - a_{m-1} + a_{m-2} - \cdots + a_{k+2} - a_{k+1}}_{\leq 0}.$$

So  $s_m - s_k \leq 0$ . We can also group the terms as as

$$s_m - s_k = a_m - \underbrace{a_{m-1} + a_{m-2} - \cdots - a_{k+3} - a_{k+2} - a_{k+1}}_{\geq 0} \geq a_m - a_{k+1}.$$

So  $|s_m - s_k| \leq |a_m| + |a_{k+1}|$  by the triangle inequality. Since  $\lim_{n \rightarrow \infty} a_n = 0$ , for all  $\epsilon > 0$ , there is  $N$  such that  $n \geq N$ , we have  $|a_n| < \epsilon$ . Then for all  $m, k \geq N$ ,

$$|s_m - s_k| \leq |a_m| + |a_{k+1}| < 2\epsilon.$$

Thus  $\{s_k\}$  converges. Left as exercise to check the other parities of  $m$  and  $k$  (group differently).  $\square$

**Example 9.1.1.** We saw previously that for  $a_n = \frac{1}{n}$ ,  $\sum_{n=1}^{\infty} a_n$  diverges. But  $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$  converges.

## 9.2 Rearrangements

**Definition 9.2.** Given a series  $\sum_{k=1}^{\infty} a_k$ , we say that a series  $\sum_{k=1}^{\infty} b_k$  is a **rearrangement** of  $\sum_{k=1}^{\infty} a_k$  if there is a bijection  $f : \mathbb{N} \rightarrow \mathbb{N}$  such that  $b_{f(k)} = a_k$  for all  $k \in \mathbb{N}$ .

**Example 9.2.1.** Let

$$\begin{aligned} S &= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \cdots, \\ \frac{1}{2}S &= \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \frac{1}{10} - \cdots, \\ S + \frac{1}{2}S &= 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} - \frac{1}{4} + \frac{1}{7} - \cdots. \end{aligned}$$

Notice that  $S + \frac{1}{2}S$  is a rearrangement of  $S$ . Supposing that  $S + \frac{1}{2}S$  converges to the same limit as  $S$ , we would have

$$S + \frac{1}{2}S = S,$$

or  $S = 0$ . This cannot be the case.

**Remark.** A rearrangement of a series might have different convergence properties from the original series.

**Theorem 9.3.** If a series converges absolutely to  $A$ , then any rearrangement of the series converges to the same limit  $A$ .

*Proof.* Let  $\sum_{k=1}^{\infty} a_k$  converge absolutely to  $A$ . Let  $\sum_{k=1}^{\infty} b_k$  be a rearrangement of  $\sum_{k=1}^{\infty} a_k$ . We set

$$s_n = \sum_{k=1}^n a_k, \quad t_m = \sum_{k=1}^m b_k.$$

We want to show that  $t_m$  converges to  $A$ . Since  $\lim_{n \rightarrow \infty} s_n = A$ , for every  $\epsilon > 0$ , there is  $N_1$  such that

$$|s_n - A| < \frac{\epsilon}{2}$$

for all  $n \in \mathbb{N}$ . Since  $\sum_{k=1}^{\infty} a_k$  converges absolutely, there is  $N_2$  such that for all  $n, m \geq N_2$ , we have

$$\sum_{k=m+1}^n |a_k| < \frac{\epsilon}{2}.$$

Since  $\sum_{k=1}^{\infty} b_k$  is a rearrangement of  $\sum_{k=1}^{\infty} a_k$ , we can write  $b_{f(k)} = a_k$  for some bijection  $f$ . Set

$$N = \max\{N_1, N_2\}, \quad M = \max\{f(k) : 1 \leq k \leq N\}.$$

Then for all  $m \geq M$ ,  $t_m - s_n$  will only consist of terms  $a_k$  for  $k > N$ . In particular,

$$|t_m - s_n| \leq \sum_{k=n}^{\infty} |a_k| < \frac{\epsilon}{2}.$$

Then we have

$$|t_m - A| = |t_m - s_n + s_n - A| \leq |t_m - s_n| + |s_n - A| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

So  $\lim_{m \rightarrow \infty} t_m = A$ . □



# Lecture 10

## Sept. 21 – Iterated Sums and Topology

### 10.1 Double Sums

Given a set of doubly indexed real numbers  $\{a_{ij} : i, j \in \mathbb{N}\}$ , consider the sums

$$\sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_{ij}, \quad \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij}.$$

Are these sums equal? The answer is no, in general.

**Example 10.0.1.** Define  $a_{ij}$  by

$$a_{ij} = \begin{cases} \left(\frac{1}{2}\right)^{j-i} & \text{if } j > i, \\ -1 & \text{if } i = j, \\ 0 & \text{if } j < i. \end{cases}$$

This looks like

	$a_{11}$	$a_{12}$	$a_{13}$	$a_{14}$	$\cdots$
$a_{11}$	-1	$\frac{1}{2}$	$\frac{1}{4}$	$\left(\frac{1}{2}\right)^3$	$\vdots$
$a_{21}$	0	-1	$\frac{1}{2}$	$\left(\frac{1}{2}\right)^2$	$\vdots$
$a_{31}$	0	0	-1	$\frac{1}{2}$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Then the first sum (over the columns) is

$$\sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_{ij} = \sum_{j=1}^{\infty} -\left(\frac{1}{2}\right)^{j-1} = -\frac{1}{1-\frac{1}{2}} = -2.$$

Meanwhile, the second sum (over the rows) is

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij} = \sum_{i=1}^{\infty} \left(-1 + \sum_{k=1}^{\infty} \left(\frac{1}{2}\right)^k\right) = \sum_{i=1}^{\infty} (-1 + 1) = \sum_{i=1}^{\infty} 0 = 0.$$

Notice that these two sums are not the same.

**Remark.** We cannot always exchange the order of a double sum. <sup>1</sup>

<sup>1</sup>*Fubini's theorem* gives conditions under which we can do this for iterated *integrals* (when the integrand is *absolutely integrable*).

### 10.1.1 Convergence of Double Sums

**Definition 10.1.** We say that  $\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |a_{ij}|$  **converges** if for all  $i \in \mathbb{N}$ ,  $\sum_{j=1}^{\infty} |a_{ij}|$  converges to some real number  $b_i$  and  $\sum_{i=1}^{\infty} b_i$  converges.

**Theorem 10.1.** Consider  $\{a_{ij} : i, j \in \mathbb{N}\}$ . If

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |a_{ij}|$$

converges, then both

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij} \quad \text{and} \quad \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_{ij}$$

converge to the same limit, i.e.

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij} = \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_{ij} = \lim_{n \rightarrow \infty} S_{nn}$$

where  $S_{nn} = \sum_{i=1}^n \sum_{j=1}^n a_{ij}$ .

*Proof.* Go look this up. □

## 10.2 Basic Topology in $\mathbb{R}$

### 10.2.1 Open Sets

**Definition 10.2.** For all  $a \in \mathbb{R}$ ,  $\epsilon > 0$ , we define

$$V_{\epsilon}(a) = \{x \in \mathbb{R} : |x - a| < \epsilon\}$$

to be the  $\epsilon$ -neighborhood of  $a$ .

**Definition 10.3.** A set  $U \subseteq \mathbb{R}$  is **open** if for all  $a \in U$ , there exists  $\epsilon > 0$  such that  $V_{\epsilon}(a) \subseteq U$ .

**Example 10.3.1.** The set  $\mathbb{R}$  is open: Simply take  $\epsilon = 1$  for any choice of  $a \in \mathbb{R}$ .

**Example 10.3.2.** The open interval  $(c, d)$  is open: For any  $x \in (c, d)$ , take  $\epsilon = \min\{x - c, d - x\}$ .

**Theorem 10.2.**

1. The union of an arbitrary collection of open sets is open.
2. The intersection of a finite collection of open sets is open.

*Proof.* (1) Let  $\{U_{\lambda} : \lambda \in \Lambda\}$  be a collection of open sets and consider  $\bigcup_{\lambda \in \Lambda} U_{\lambda}$ . For every  $a \in \bigcup_{\lambda \in \Lambda} U_{\lambda}$ , there is  $\lambda'$  such that  $a \in U_{\lambda'}$ . Since  $U_{\lambda'}$  is open, there exists  $\epsilon > 0$  such that

$$V_{\epsilon}(a) \subseteq U_{\lambda'} \subseteq \bigcup_{\lambda \in \Lambda} U_{\lambda}.$$

So  $\bigcup_{\lambda \in \Lambda} U_{\lambda}$  is open.

(2) Let  $U_1, \dots, U_n$  be a collection of open sets and consider  $\bigcap_{j=1}^n U_j$ . For every  $a \in \bigcap_{j=1}^n U_j$ , note that  $a \in U_j$  for all  $j = 1, \dots, n$ . Since  $U_j$  is open, there exists  $\epsilon_j > 0$  such that  $V_{\epsilon_j}(a) \subseteq U_j$ . Then take

$$\epsilon = \min_{1 \leq j \leq n} \{\epsilon_j\},$$

which exists since the collection is finite. Since  $\epsilon_j > 0$ , we have  $\epsilon > 0$  as well. By construction,

$$V_\epsilon(a) \subseteq V_{\epsilon_j}(a) \subseteq U_j$$

for all  $j$ . So  $V_\epsilon(a) \subseteq \bigcap_{j=1}^n U_j$ , and thus  $\bigcap_{j=1}^n U_j$  is open.  $\square$

**Example 10.3.3.** Consider the family of open sets  $U_n = (-\frac{1}{n}, \frac{1}{n})$  for  $n \in \mathbb{N}$ . Notice that their intersection

$$\bigcap_{n=1}^{\infty} U_n = \{0\}$$

is not open.

## 10.2.2 Limit Points

**Definition 10.4.** A point  $x$  is a **limit point** of a set  $A$  if for all  $\epsilon > 0$ , we have

$$(V_\epsilon(x) \cap A) \setminus \{x\} \neq \emptyset,$$

i.e. there is some other point in the  $\epsilon$ -neighborhood of  $x$  that is also in  $A$ .

**Theorem 10.3.** A point  $x$  is a limit point of  $A$  if and only if  $x = \lim_{n \rightarrow \infty} a_n$  for some sequence  $\{a_n\}$  with  $a_n \neq x$  and  $a_n \in A$ .

*Proof.* ( $\Rightarrow$ ) Suppose  $x$  is a limit point of  $A$ . Take  $\epsilon = 1/n$  and pick  $a_n \in (V_{1/n}(x) \cap A) \setminus \{x\}$  such that  $a_n \neq x$ . For such a sequence  $\{a_n\}$ , for all  $\epsilon > 0$ , if  $N \geq 1/\epsilon$ , then for all  $n \geq N$ , we have

$$|a_n - x| \leq \frac{1}{N} < \epsilon.$$

So  $\lim_{n \rightarrow \infty} a_n = x$ .

( $\Leftarrow$ ) Assume such a sequence  $\{a_n\}$  exists. Then for any  $\epsilon > 0$ , there exists  $N$  such that  $|a_n - x| < \epsilon$  for all  $n \geq N$ . Note that  $a_N \in V_\epsilon(x)$ , and also  $a_N \in A$  and  $a_N \neq x$ . So  $a_N \in (V_\epsilon(x) \cap A) \setminus \{x\}$ , i.e. this set is not empty. Thus  $x$  is a limit point of  $A$ .  $\square$

# Lecture 11

## Sept. 26 – Closed Sets

### 11.1 Closed Sets

**Definition 11.1.** Let  $A \subseteq \mathbb{R}$ . An element  $x \in A$  is an **isolated point** of  $A$  if it is not a limit point of  $A$ , i.e. there exists  $\epsilon > 0$  such that  $V_\epsilon(x) \cap A = \{x\}$ .

**Definition 11.2.** A set  $A \subseteq \mathbb{R}$  is **closed** if it contains all of its limit points.

**Example 11.2.1.** The empty set and  $\mathbb{R}$  are closed. Moreover, any set without limit points is closed.

**Theorem 11.1.** A set  $A \subseteq \mathbb{R}$  is closed if and only if every Cauchy sequence in  $A$  converges to a limit in  $A$ .

*Proof.* ( $\Rightarrow$ ) Suppose  $A$  is closed and  $\{a_n\}$  is Cauchy with  $a_n \in A$  for all  $n$ . Since Cauchy sequences are convergent, let  $x = \lim_{n \rightarrow \infty} a_n$ . Now consider two cases. If there exists an  $n$  such that  $x = a_n$ , then we're done since  $x = a_n \in A$ . Otherwise, we have  $a_n \neq x$  for all  $n$ . By Theorem 10.3,  $x$  is a limit point of  $A$ . So  $x \in A$  as  $A$  is closed.

( $\Leftarrow$ ) Let  $x$  be a limit point of  $A$ . Then there exists a sequence  $\{a_n\}$  with  $a_n \in A$  and  $a_n \neq x$  for all  $n$  such that  $\lim_{n \rightarrow \infty} a_n = x$ . This means that  $\{a_n\}$  is Cauchy, so by assumption,  $x = \lim_{n \rightarrow \infty} a_n \in A$ . Thus every limit point of  $A$  is in  $A$ , so  $A$  is closed.  $\square$

**Example 11.2.2.** Consider the set

$$A = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\}.$$

First  $x \neq 0$ , we look at the following cases:

1. If  $x < 0$ , let  $\epsilon = |x|$ . Then  $V_\epsilon(x) = (2x, 0)$ , and  $V_\epsilon(x) \cap A = \emptyset$  since  $A \subseteq \mathbb{R}^+$ . So  $x < 0$  is not a limit point of  $A$ .
2. If  $x > 1$ , let  $\epsilon = x - 1$ . Then  $V_\epsilon(x) = (1, 2x - 1)$ , so  $V_\epsilon(x) \cap A = \emptyset$  as all  $y \in A$  satisfies  $y \in (0, 1]$ .
3. If  $x \in (0, 1]$ , then there exists  $n \in \mathbb{N}$  such that  $n > 1/x$ . Let  $n_0 = \min\{n \in \mathbb{N} : n > 1/x\}$ , which exists by the well-ordering principle. Noting that  $n_0 \geq 2$ , we have

$$\frac{1}{n_0} < x \leq \frac{1}{n_0 - 1}.$$

Now we look at two more cases:

(a) If  $x = \frac{1}{n_0 - 1}$ , let

$$\epsilon = x - \frac{1}{n_0} = \frac{1}{n_0 - 1} - \frac{1}{n_0}.$$

Then we have

$$V_\epsilon(x) = \left( \frac{1}{n_0}, \frac{2}{n_0 - 1} - \frac{1}{n_0} \right).$$

Note that  $(V_\epsilon \cap A) \setminus \{x\} = \emptyset$  if  $n_0 = 2$ . Otherwise,  $n_0 > 2$  and we have

$$\frac{2}{n_0 - 1} - \frac{1}{n_0} - \frac{1}{n_0 - 2} = \cdots = \frac{-2}{n_0(n_0 - 2)(n_0 - 1)} < 0.$$

So  $V_\epsilon(x) \subseteq \left( \frac{1}{n_0}, \frac{1}{n_0} - 2 \right)$ , which means that  $V_\epsilon(x) \cap A = \{x\}$ .

(b) Otherwise,  $x \in \left( \frac{1}{n_0}, \frac{1}{n_0 - 1} \right)$  and let

$$\epsilon = \min \left\{ x - \frac{1}{n_0}, \frac{1}{n_0 - 1} - x \right\}.$$

Then (left as exercise)  $V_\epsilon(x) \subseteq \left( \frac{1}{n_0}, \frac{1}{n_0 - 1} \right)$ , which implies  $V_\epsilon(x) \cap A = \emptyset$ .

So  $x \neq 0$  is not a limit point of  $A$ . However,  $x = 0 = \lim_{n \rightarrow \infty} \frac{1}{n}$ , so 0 is a limit point of  $A$ . But  $0 \notin A$ , so  $A$  is not closed.

**Example 11.2.3.** Let  $A = [a, b]$ . For any Cauchy sequence  $\{x_n\} \subseteq A$ , let  $x = \lim_{n \rightarrow \infty} x_n$ . Since  $x_n \geq a$ , we have  $x = \lim_{n \rightarrow \infty} x_n \geq a$ . Similarly,  $x_n \leq b$  implies that  $x \leq b$ . So  $x \in [a, b] = A$ , and thus  $A$  is closed.

**Example 11.2.4.** Consider  $\mathbb{Q}$ . For any  $x \in \mathbb{R}$ , for all  $n \in \mathbb{N}$  there exists  $a_n \in \mathbb{Q}$  with

$$\frac{1}{2n} < |a_n - x| < \frac{1}{n}.$$

Thus  $a_n \neq x$  and  $a_n \in \mathbb{Q}$  for all  $n$ , so  $\lim_{n \rightarrow \infty} a_n = x$  is a limit point of  $\mathbb{Q}$ . So  $\mathbb{Q}$  is not closed.

**Remark.** We can also define the real numbers as equivalence classes of Cauchy sequences.<sup>1</sup> Note that Cauchy sequences do not require an ordering (only a metric), so we can easily extend this definition to higher dimensions.

## 11.2 The Closure of a Set

**Definition 11.3.** Let  $A \subseteq \mathbb{R}$ . Define the **closure** of  $A$  as

$$\overline{A} = \{x : x \in A \text{ or } x \text{ is a limit point of } A\}.$$

**Theorem 11.2.** The closure of a set  $A$  is closed. Furthermore, if  $B$  is closed and  $A \subseteq B$ , then  $\overline{A} \subseteq B$ .<sup>2</sup>

*Proof.* Let  $x$  be a limit point of  $\overline{A}$ . We want to show that  $x$  is also a limit point of  $A$  (so we will have  $x \in \overline{A}$ ). If  $x \in A$ , then we're done. Otherwise,  $x \notin A$ , so for all  $\epsilon > 0$ , we have  $(V_{\epsilon/2}(x) \cap \overline{A}) \setminus \{x\} \neq \emptyset$ , so let  $y \in (V_{\epsilon/2}(x) \cap \overline{A}) \setminus \{x\}$ . If  $y \in A$ , then  $(V_\epsilon(x) \cap A) \setminus \{x\} \neq \emptyset$  and we're done. Otherwise,  $y \notin A$ , so  $y \in \overline{A}$  implies that  $y$  is a limit point of  $A$ . So there exists  $z \in (V_{\epsilon/2}(y) \cap A) \setminus \{y\}$ . Then

$$|x - z| \leq |x - y| + |y - z| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

<sup>1</sup>Using *Dedekind cuts* is another such way.

<sup>2</sup>So  $\overline{A}$  is the smallest closed set containing  $A$ .

So  $z \in (V_\epsilon(x) \cap A) \setminus \{x\}$ . Since this is true for all  $\epsilon > 0$ ,  $x$  is a limit point of  $A$ . Thus  $x \in \overline{A}$ , so  $\overline{A}$  is closed.

For the second part, let  $x \in \overline{A}$ . If  $x \in A$ , then  $A \subseteq B$  implies that  $x \in B$ . If  $x \notin A$ , then  $x$  is a limit point of  $A$ . So there exists a sequence  $\{a_n\}$  with  $a_n \in A$  for all  $n$  such that  $a_n \rightarrow x$ . Since  $a_n \in A \subseteq B$ , we have  $a_n \in B$  for all  $n$ . Since  $a_n \rightarrow x$ , we must have  $x \in B$  since  $B$  is closed. Thus  $A \subseteq B$ .  $\square$

**Corollary 11.2.1.** *If  $A \subseteq B$ , then  $\overline{A} \subseteq \overline{B}$ .*

*Proof.* Note that Cauchy sequences in  $A$  are also Cauchy sequences in  $B$ .  $\square$

# Lecture 12

## Sept. 28 – Compact Sets

### 12.1 Another Characterization of Closed Sets

**Definition 12.1.** Given a set  $A \subseteq \mathbb{R}$ , its **complement** is  $A^c = \{x \in \mathbb{R} : x \notin A\}$ .

**Theorem 12.1.** A set  $A \subseteq \mathbb{R}$  is closed if and only if  $A^c$  is open.

*Proof.* ( $\Rightarrow$ ) Suppose  $A$  is closed. Take any  $x \in A^c$ . Since  $A$  is closed and  $x \notin A$ , we know  $x$  is not a limit point of  $A$ . So there is  $\epsilon > 0$  such that  $(V_\epsilon(x) \cap A) \setminus \{x\} = \emptyset$ . Since  $x \notin A$ , this means that  $V_\epsilon(x) \cap A = \emptyset$ , which means that  $V_\epsilon(x) \subseteq A^c$ . So  $x$  is an interior point of  $A^c$ , and thus  $A^c$  is open.

( $\Leftarrow$ ) Suppose  $A^c$  is open. Let  $x$  be a limit point of  $A$ . Assume that  $x \notin A$ , i.e.  $x \in A^c$ . Since  $A^c$  is open, there is  $\epsilon > 0$  such that  $V_\epsilon(x) \subseteq A^c$ . But then  $V_\epsilon(x) \cap A = \emptyset$ , which is a contradiction with  $x$  being a limit point of  $A$ . So  $x \in A$ , and thus  $A$  is closed.  $\square$

**Corollary 12.1.1.** A set  $A \subseteq \mathbb{R}$  is open if and only if  $A^c$  is closed.

**Remark.** This is used in topology, where a collection of open sets that satisfies certain conditions<sup>1</sup> is called a *topology* of a space, and closed sets are defined as their complements. Furthermore, metrics are not necessary in this setting.

**Theorem 12.2.**

1. Let  $A_1, A_2, \dots, A_n \subseteq \mathbb{R}$  be closed. Then  $\bigcup_{i=1}^n A_i$  is closed.
2. Let  $A_\lambda \subseteq \mathbb{R}$ ,  $\lambda \in \Lambda$  be a family of closed subsets of  $\mathbb{R}$  indexed by  $\lambda \in \Lambda$ , where  $\Lambda$  is an index set. Then  $\bigcup_{\lambda \in \Lambda} A_\lambda$  is closed.

*Proof.* Left as an exercise.  $\square$

### 12.2 Compactness

**Definition 12.2.** A set  $A \subseteq \mathbb{R}$  is **compact** if any sequence  $\{a_n\}$  in  $A$  has a convergent subsequence  $\{a_{n_k}\}$  such that  $\lim_{k \rightarrow \infty} a_{n_k} \in A$ .<sup>2</sup>

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<sup>1</sup>The collection must be closed under finite intersection and arbitrary union.

<sup>2</sup>This definition is sometimes called *sequential compactness*.

**Remark.** Suppose we want to solve the differential equation

$$\begin{cases} x' = f(t, x) \\ x(t_0) = x_0 \end{cases}.$$

We can first transform this into the integral equation

$$x(t) = x_0 + \int_{t_0}^t f(s, x(s)) ds$$

with a test function  $f$ . Then we perform Picard iterations to continue. However, this method requires  $f \in C^1(\mathbb{R})$  (i.e.  $f$  is continuously differentiable), since that is what guarantees that the sequence of functions converges (closedness). But even without this condition (if  $f$  is only continuous), if we can show that the set of functions lies in a compact set, then we can find a subsequence of functions that do converge (though solutions may no longer be unique).

**Example 12.2.1.** The interval  $(0, 1]$  is not compact since it is not closed: it does not contain all of its limit points.

**Example 12.2.2.** The set  $\mathbb{R}$  is not compact since it is not bounded: an arbitrary sequence may not even have a convergent subsequence.

**Theorem 12.3.** A set  $A \subseteq \mathbb{R}$  is compact if and only if  $A$  is bounded and closed.<sup>3</sup>

*Proof.* ( $\Rightarrow$ ) Assume  $A$  is not bounded. Then for any  $n \in \mathbb{N}$ , there exists an  $x_n \in A$  such that  $|x_n| > n$ . Then  $\{x_n\}$  is a sequence in  $A$ . Since  $A$  is compact, there exists a subsequence

$$\{x_{n_k}\} = \{x_{n_1}, x_{n_2}, x_{n_3}, \dots\}$$

such that  $x = \lim_{k \rightarrow \infty} x_{n_k} \in A$ . But this implies that  $\{x_{n_k}\}$  is bounded, which contradicts the fact that  $|x_{n_k}| > n_k$  and  $n_k \rightarrow \infty$  and  $k \rightarrow \infty$ . Hence  $A$  must be bounded.

Now assume  $A$  is not closed. Then there exists a limit point  $x$  of  $A$  such that  $x \notin A$ . Since  $x$  is a limit point of  $A$ , there exists a sequence  $\{a_n\}$  such that  $a_n \in A$  and  $\lim_{n \rightarrow \infty} a_n = x$ . Now since  $A$  is compact, there exists a convergent subsequence  $\{a_{n_k}\}$  such that  $\lim_{k \rightarrow \infty} a_{n_k} \in A$ . But since  $\{a_n\}$  converges, we have

$$x = \lim_{n \rightarrow \infty} a_n = \lim_{k \rightarrow \infty} a_{n_k} \in A.$$

This is a contradiction with  $x \notin A$ . Hence  $A$  is closed.

( $\Leftarrow$ ) Suppose  $A$  is bounded and closed. Let  $\{a_n\}$  be a sequence in  $A$ . By the Bolzano-Weierstrass theorem, there exists a convergent subsequence  $\{a_{n_k}\}$ . Let  $x = \lim_{k \rightarrow \infty} a_{n_k}$ . But  $\{a_n\}$  is a convergent sequence in  $A$ , which is closed. So  $x \in A$ , and hence  $A$  is compact.  $\square$

**Example 12.2.3.** The union of intervals  $[1, 2] \cup [3, 4]$  is compact.

**Example 12.2.4.** The set

$$A = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\} \cup \{0\}$$

is compact (since we added the limit point 0).

<sup>3</sup>This applies to all Euclidean spaces (and pretty much only Euclidean spaces).



**Remark.** Why do we need the concept of compactness? Because Theorem 12.3 is no longer true in infinite dimensions.

**Theorem 12.4.** Suppose

$$k_1 \supseteq k_2 \supseteq k_3 \supseteq \dots$$

are non-empty compact sets. Then

$$\bigcap_{n=1}^{\infty} k_n \neq \emptyset.$$

*Proof.* Since  $k_n \neq \emptyset$  for all  $n$ , there exists  $a_n \in k_n$ . Then for any  $m \in \mathbb{N}$ ,  $\{a_n\}_{n=m}^{\infty}$  is a sequence in  $k_m$ , which is compact. For  $m = 1$ , there exists a convergent subsequence  $\{a_{n_k}\}$  such that  $x = \lim_{k \rightarrow \infty} a_{n_k} \in k_1$ . For any  $m \in \mathbb{N}$ , there is  $k \in \mathbb{N}$  such that for all  $k \geq k_m$ ,  $n_k \geq m$ . So

$$a_{n_k} \in k_{n_k} \subseteq k_m,$$

which means that  $\{a_{n_k}\}$  is a convergent sequence in  $k_m$ , which is closed. Thus  $x \in k_m$  for all  $m$ , which means that  $x \in \bigcap_{n=1}^{\infty} k_n$ . So  $\bigcap_{n=1}^{\infty} k_n$  is nonempty.  $\square$

**Example 12.2.5.** For  $U_n = (0, 1/n)$ . Then  $U_1 \supseteq U_2 \supseteq U_3 \supseteq \dots$ , but their intersection is empty.

However, this is not so surprising, since the measure of the sets  $U_n$  tends to 0.

**Example 12.2.6.** The sequence  $V_n = (n, \infty)$  also satisfies  $V_1 \supseteq V_2 \supseteq V_3 \supseteq \dots$ , but their intersection is empty, despite each  $V_n$  having infinite measure.

## 12.3 Another Definition of Compactness

**Theorem 12.5.** A set  $A \subseteq \mathbb{R}$  is compact if and only if it satisfies the following property:

(Covering property) For any family  $U_{\lambda}$ ,  $\lambda \in \Lambda$  of open subsets of  $\mathbb{R}$  such that  $A \subseteq \bigcup_{\lambda \in \Lambda} U_{\lambda}$ ,<sup>4</sup> there exists  $n \in \mathbb{N}$  and  $\lambda_1, \dots, \lambda_n \in \Lambda$  such that  $A \subseteq \bigcup_{k=1}^n U_{\lambda_k}$ .<sup>5</sup>

*Proof.* ( $\Leftarrow$ ) Assume that the covering property holds for  $A$ . For boundedness, let

$$A \subseteq \bigcup_{n=1}^{\infty} (-n, n) = \mathbb{R}.$$

By the covering property, there exist  $n_1 < n_2 < \dots < n_k$  such that

$$A \subseteq \bigcup_{i=1}^k (-n_i, n_i) = (-n_k, n_k).$$

So  $A$  is bounded. Now for closedness, suppose otherwise that  $A$  is not closed. So there exists a limit point  $x \notin A$ . Then

$$A \subseteq \mathbb{R} \setminus \{x\} = \bigcup_{n=1}^{\infty} \left\{ y : |y - x| > \frac{1}{n} \right\}.$$

<sup>4</sup>This is called an *open cover* of  $A$ .

<sup>5</sup>I.e. there exists a finite subcover.

By the covering property, there exist  $n_1 < n_2 < \dots < n_k$  such that

$$A \subseteq \bigcup_{i=1}^k \left\{ y : |y - x| > \frac{1}{n_i} \right\} = \left\{ y : |y - x| > \frac{1}{n} \right\}.$$

But  $x$  is a limit point of  $A$ , so there exists a  $z \in V_{1/n_k}(x) \cap A$ . This is a contradiction. Hence  $A$  is also closed, and thus  $A$  is compact.

( $\Rightarrow$ ) Assume  $A$  is compact and let  $U_\lambda$ ,  $\lambda \in \Lambda$  be an open cover of  $A$ . Suppose otherwise that there does not exist a finite subcover. Since  $A$  is compact,  $A$  is bounded. So there exists an  $M$  such that  $A \subseteq [-M, M]$ . Let  $A_1 = A$  and define a sequence  $A_n$  of sets inductively by  $A_n = A \cap [a_n, b_n] \neq \emptyset$  with

$$|b_n - a_n| = \frac{2M}{2^{n-1}},$$

and  $A_n$  can't be covered by a finite subcollection of  $U_\lambda$ . Suppose such  $A_n$  is defined. Then

$$A_n = \underbrace{\left( A \cap \left[ a_n, \frac{b_n + a_n}{2} \right] \right)}_{=k_1} \cup \underbrace{\left( A \cap \left[ \frac{a_n + b_n}{2}, b_n \right] \right)}_{=k_2}.$$

So either  $k_1$  or  $k_2$  can't be covered by a finite subcollection of  $U_\lambda$ . Let this one be  $A_{n+1}$ . Since  $A_n$  is closed and bounded,  $A_n$  is compact. Then we have a sequence of compact sets

$$A_1 \supseteq A_2 \supseteq A_3 \supseteq \dots,$$

so  $\bigcap_{n=1}^{\infty} A_n \neq \emptyset$ . Since  $|a_n - b_n| \rightarrow 0$  (prove this as exercise), there exists  $x \in \mathbb{R}$  such that  $\bigcap_{n=1}^{\infty} A_n = \{x\}$ . But there exists  $\lambda_0$  such that  $x \in U_{\lambda_0}$ , which is open. So there exists  $n$  such that  $V_{1/n}(x) \subseteq U_{\lambda_0}$ , which means that  $A_{n+1} \subseteq U_{\lambda_0}$ . This is a contradiction.  $\square$

**Example 12.2.7.** Let  $A = (0, 1)$ ,  $\Lambda = (0, 1)$ , and  $U_\lambda = (0, \lambda)$ . We have  $A \subseteq \bigcup_{\lambda \in (0, 1)} U_\lambda$ , but there does not exist a finite subcover.

# Lecture 13

## Oct. 3 – Perfect Sets

**Definition 13.1.** A set  $P \subseteq \mathbb{R}$  is **perfect** if it is closed and contains no isolated points.

**Example 13.1.1.** The closed intervals  $[a, b]$  are perfect.

### 13.1 The Cantor Set

**Example 13.1.2.** Define a sequence of sets inductively by  $C_0 = [0, 1]$  and removing its middle third to get  $C_1 = C_0 \setminus (\frac{1}{3}, \frac{2}{3})$ , i.e.

$$C_1 = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right].$$

Then we have

$$C_2 = \left(\left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{1}{3}\right]\right) \cup \left(\left[\frac{2}{3}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right]\right),$$

and so on, removing the middle third of each interval at each step. Note that  $C_n$  is a set consisting of  $2^n$  closed intervals each of length  $\frac{1}{3^n}$ . Then the (middle third) *Cantor set*  $C$  is defined as

$$C = \bigcap_{n=0}^{\infty} C_n.$$

**Remark.** If we consider the sum of the lengths of the intervals that we removed, we get

$$\frac{1}{3} + 2 \cdot \frac{1}{9} + 4 \cdot \frac{1}{27} + \cdots + 2^{n-1} \cdot \frac{1}{3^n} + \cdots = \frac{1}{3} \left( \frac{1}{1 - \frac{2}{3}} \right) = \frac{1}{3} \cdot 3 = 1.$$

So, in some sense, the “size”<sup>1</sup> of the Cantor set  $C$  is 0. However,  $C$  is uncountable. In particular, the cardinality of  $C$  is the same as the cardinality of  $\mathbb{R}$ . A lot of counterexamples in real analysis come from this Cantor set.

**Remark.** This means that the usual measure is not a good way to “catch” the Cantor set. Instead, we can consider *fractional* (or *fractal*) dimensions.

**Theorem 13.1.** *The Cantor set  $C$  is perfect.*

*Proof.* First note that  $C$  is a countable intersection of closed sets, so  $C$  is closed as well. To see that  $C$  has no isolated points, take an arbitrary  $x \in C$ . Since  $x \in C_n$  for all  $n$ , we can find  $x_n \in C_n$  such that  $x_n \neq x$  and  $|x_n - x| < \frac{1}{3^n}$ . Then  $\lim_{n \rightarrow \infty} x_n = x$  and  $x_n \neq x$ , so  $x$  is a limit point of  $C$ . Thus  $C$  is perfect.  $\square$

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<sup>1</sup>The (Lebesgue) *measure*.

## 13.2 Perfect Sets and Countability

**Theorem 13.2.** *A nonempty perfect set is uncountable.*

*Proof.* Note that if  $P$  is perfect, then  $P$  is infinite (if we only have finitely many points, then they must be isolated). Now suppose that  $P$  is only countably infinite. Then we can write

$$P = \{x_1, x_2, \dots\}.$$

Take  $I_1$  to be a closed interval such that  $x_1 \in I_1$  and  $x_1$  is not an endpoint of  $I_1$ . Since  $x_1$  is not isolated in  $P$ , we can find  $y_2 \in P$  with  $y_2 \neq x_1$  such that  $y_2 \in I_1$  and  $y_2$  is not an endpoint of  $I_1$ . Then let  $I_2 \subseteq I_1$  be a closed interval centered at  $y_2$  such that  $x_1 \notin I_2$ . For example, we can do this by setting

$$\epsilon = \frac{1}{2} \min\{y_2 - a, b - y_2, |x_1 - y_2|\}$$

and letting  $I_2 = [y_2 - \epsilon, y_2 + \epsilon]$ . Since  $y_2 \in P$ ,  $y_2$  is not isolated, so we can find  $y_3 \in P$  such that  $y_3$  is not an endpoint of  $I_2$  and  $y_3 \neq x_2$ . Pick  $I_3$  centered at  $y_3$  such that  $x_2 \notin I_3$  and  $I_3 \subseteq I_2$ . Note that  $I_3 \cap P \neq \emptyset$  since  $y_3 \in I_3 \cap P$ . From here, we continue by constructing  $I_{n+1} \subseteq I_n$  with  $x_n \notin I_{n+1}$  and  $I_{n+1} \cap P \neq \emptyset$ . Let  $k_n = I_n \cap P$ . Clearly  $k_n$  is closed, and  $k_n$  is also bounded since  $k_n \subseteq I_n$ . So  $k_n$  is compact. By construction,  $k_{n+1} \subseteq k_n$ , so by the nested interval property of compact sets,  $\bigcap_{n=1}^{\infty} k_n \neq \emptyset$ . But  $k_n \subseteq I_n$  and  $x_n \notin I_{n+1}$  so  $x_n \notin k_{n+1}$ . Since  $k_n \subseteq P$ , we must have  $\bigcap_{n=1}^{\infty} k_n = \emptyset$ . This is a contradiction, so  $P$  must be uncountable.  $\square$