

# MATH 4317: Analysis I

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# Lecture 1

## Aug. 22 – The Real Numbers

### 1.1 Number Systems

We start with the natural numbers <sup>1</sup>

$$\mathbb{N} = \{1, 2, 3, \dots\}.$$

These are perhaps the most natural in a way, since they are what we use to count things. They are closed under addition, but fail when it comes to subtraction. For example,  $1 - 2 = -1 \notin \mathbb{N}$ . So we must expand our number system to the integers

$$\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}.$$

We can now add, subtract, and multiply. But we run into problems when we start to consider quotients. For example,  $1 \div 2 = \frac{1}{2} \notin \mathbb{Z}$ . So we continue to the rational numbers

$$\mathbb{Q} = \left\{ \frac{p}{q} : p, q \in \mathbb{Z}, q \neq 0 \right\}.$$

We now have summation, subtraction, multiplication, and quotients. But there is still a problem.

Consider the diagonal of a square with side length 1.

**Theorem 1.1.1.**  $\sqrt{2}$  is not a rational number. <sup>2</sup>

*Proof.* Argue by contradiction. Suppose  $\sqrt{2}$  is rational. Then we can write

$$\sqrt{2} = \frac{p}{q}$$

for some integers  $p, q$ . Further assume  $p$  and  $q$  have no common factors. Then

$$2 = \frac{p^2}{q^2} \implies p^2 = 2q^2.$$

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<sup>1</sup> $0 \notin \mathbb{N}$  for this class.

<sup>2</sup>In some sense, this shows that the notion of “rationals” is strictly weaker than the notion of “length.”

So  $p$  is even and we can write  $p = 2r$  for some  $r \in \mathbb{Z}$ . Then

$$4r^2 = 2q^2 \implies 2r^2 = q^2.$$

So  $q$  is also even, and  $p, q$  share a common factor of 2. Contradiction.  $\square$

Another weakness of  $\mathbb{Q}$  is that we cannot take limits ( $\mathbb{Q}$  is not complete). For example, note that

$$\begin{aligned} (\sqrt{2} - 1)(\sqrt{2} + 1) &= 2 - 1 = 1, \\ \sqrt{2} &= 1 + \frac{1}{\sqrt{2} + 1} = 1 + \frac{1}{1 + 1 + \frac{1}{\sqrt{2} + 1}} = \dots \end{aligned}$$

So if we define the rational sequence

$$a_1 = 1, \quad a_2 = 1 + \frac{1}{2}, \quad a_3 = 1 + \frac{1}{2 + \frac{1}{2}}, \quad a_4 = 1 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2}}}, \quad \dots,$$

then as  $n \rightarrow \infty$ ,  $a_n \rightarrow \sqrt{2} \notin \mathbb{Q}$ .

## 1.2 Sets

Sets are any collections of objects. Given a set  $A$ , we write  $x \in A$  if  $x$  is an element of  $A$ . We write  $x \notin A$  otherwise. The **union** of two sets is

$$A \cup B = \{x : x \in A \text{ or } x \in B\},$$

and the **intersection** of two sets is

$$A \cap B = \{x : x \in A \text{ and } x \in B\}.$$

We use the notation

$$\bigcup_{k=1}^{\infty} A_k$$

to denote the countable union of a family of sets indexed by  $k$ .

## 1.3 Functions

**Definition 1.3.1.** Given two sets  $A$  and  $B$ , a **function** from  $A$  to  $B$  is a rule, relation, or mapping that takes each element  $x \in A$  and associates with it a single element in  $B$ . In this case, we write  $f : A \rightarrow B$ .

We call  $A$  the **domain** of  $f$  and  $B$  the **codomain** of  $f$ . The element in  $B$  associated with  $x \in A$  is  $f(x)$ , called the **image** of  $x$ . The **range** of  $f$  is

$$\text{range}(f) = \{y \in B : y = f(x) \text{ for some } x \in A\}.$$

We say  $f$  is:

1. **onto** or **surjective** if  $\text{range}(f) = B$ .
2. **one-to-one** or **injective** if  $x, x' \in A$  and  $x \neq x'$ , then  $f(x) \neq f(x')$ .
3. **bijective** if it is injective and surjective.

First Dirichlet function:

$$g(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases} = \lim_{k \rightarrow \infty} \left( \lim_{j \rightarrow \infty} [\cos(k! \pi x)]^{2j} \right).$$

Second Dirichlet function:

$$f(x) = \begin{cases} \frac{1}{q} & \text{if } x = \frac{p}{q} \in \mathbb{Q} \text{ in lowest terms} \\ 0 & \text{if } x \notin \mathbb{Q}. \end{cases}$$

Absolute value:

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0. \end{cases}$$

Note that we have the following two properties:

- $|xy| = |x||y|$ .
- $|x + y| \leq |x| + |y|$ . This is called the *triangle inequality*.

## 1.4 Induction

If we have a set  $S \subseteq \mathbb{N}$  and

1.  $1 \in S$
2. if  $n \in S$ , then  $n + 1 \in S$

then  $S = \mathbb{N}$ .<sup>3</sup>

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<sup>3</sup>We always use induction in conjunction with  $\mathbb{N}$ .

## **Lecture 2**

### **Aug. 24 – The Axiom of Completeness**

# Lecture 3

## Aug. 29 – Completeness and Countability

### 3.1 Consequences of Completeness

#### 3.1.1 2nd Consequence: Density of $\mathbb{Q}$ in $\mathbb{R}$

**Theorem 3.1.1** (Density of  $\mathbb{Q}$  in  $\mathbb{R}$ ). *For all  $a, b \in \mathbb{R}$ ,  $a < b$ , there exists  $r \in \mathbb{Q}$  such that  $a < r < b$ .*

*Proof.* We want to find  $m \in \mathbb{Z}$ ,  $n \in \mathbb{N}$  such that

$$a < \frac{m}{n} < b.$$

By (ii) of the Archimedean property, we can find  $n \in \mathbb{N}$  such that

$$\frac{1}{n} < b - a.$$

Fix such an  $n$ . Then let  $m$  be the smallest integer such that  $m - 1 \leq na < m$ . By construction,

$$\frac{m}{n} - \frac{1}{n} \leq a < \frac{m}{n},$$

$$\frac{m}{n} \leq a + \frac{1}{n} < b.$$

Therefore,  $a < \frac{m}{n} < b$ . □

**Corollary 3.1.1.1.** *For all  $a, b \in \mathbb{Q}$ ,  $a < b$ , there exists  $t \in \mathbb{R} \setminus \mathbb{Q}$  such that  $a < t < b$ .*

#### 3.1.2 3rd Consequence: Existence of $\sqrt{2}$

**Theorem 3.1.2** (Existence of  $\sqrt{2}$ ). *There exists  $s \in \mathbb{R}$ ,  $s > 0$  such that  $s^2 = 2$ .*

*Proof.* Define

$$S = \{x > 0 : x^2 < 2\} \subseteq \mathbb{R}.$$

$x = 1 \in S$ , so  $S \neq \emptyset$ . 2 is an upper bound for  $S$ , so  $S$  is bounded above. Then by the axiom of completeness,  $s = \sup S$  exists. We claim that  $s^2 = 2$ .

Suppose otherwise that  $s^2 < 2$ . Then we can find  $\epsilon > 0$  such that  $s + \epsilon \in S$ . Define  $\delta = 2 - s^2 > 0$ . Note that

$$(s + \epsilon)^2 - 2 = s^2 + 2s\epsilon + \epsilon^2 - 2 = -\delta + 2s\epsilon + \epsilon^2.$$

We know  $s \leq 2$  since 2 is an upper bound. Pick

$$\epsilon = \frac{\delta}{100000000000},$$

$$2s\epsilon + \epsilon \leq 4\epsilon + \epsilon^2 < \frac{\delta}{2}.$$

Then

$$(s + \epsilon)^2 - 2 < -\delta + \frac{\delta}{2} = -\frac{\delta}{2} < 0.$$

So  $s + \epsilon \in S$ , which contradicts with  $s = \sup S$ .

$s^2 > 2$  also leads to a contradiction (left as an exercise). Thus we must have  $s^2 = 2$ . □

## 3.2 Countability

**Definition 3.2.1.** We say two sets  $A$  and  $B$  have the same **cardinality** if there is a bijection  $f : A \rightarrow B$ . We write  $A \sim B$ .

**Definition 3.2.2.** We say that a set  $A$  is **finite** if  $A \sim \{1, 2, \dots, n\}$  for some integer  $n$ . We say that a set  $A$  is **countable** (or countably infinite) if  $A \sim \mathbb{N}$ . If a set  $A$  is not countable, then we say it is **uncountable**.

$$E = \{2, 4, 6, 8, \dots\}.$$

$E$  is not finite but it is countable:  $E \sim \mathbb{N}$ . We can define  $f : \mathbb{N} \rightarrow E$  by  $f(n) = 2n$ .

$$\mathbb{N} \sim \mathbb{Z}.$$

The bijection  $f : \mathbb{N} \rightarrow \mathbb{Z}$  is given by

$$f(n) = \begin{cases} \frac{n-1}{2} & n \text{ is odd} \\ -\frac{n}{2} & n \text{ is even.} \end{cases}$$

$$(-1, 1) \sim \mathbb{R}.$$

The bijection  $f : (-1, 1) \rightarrow \mathbb{R}$  is given by

$$x \mapsto \frac{x}{x^2 - 1}.$$



**Theorem 3.2.1.**

1.  $\mathbb{Q}$  is countable.
2.  $\mathbb{R}$  is uncountable.

*Proof of (1).* Set  $A_1 = \{0\}$  and for  $n \geq 2$ ,

$$A_n = \left\{ \pm \frac{p}{q} : p, q \in \mathbb{N}, p, q \text{ in lowest terms}, p + q = n \right\}.$$

So the first few  $A_n$  are:

$$A_2 = \left\{ \frac{1}{1}, \frac{-1}{1} \right\},$$

$$A_3 = \left\{ \frac{1}{2}, \frac{2}{1}, \frac{-1}{2}, \frac{-2}{1} \right\},$$

etc. Note that  $A_n$  is finite and for all  $x \in \mathbb{Q}$ , there is an  $n \in \mathbb{N}$  such that  $x \in A_n$ . We can list elements in  $A_1, \dots, A_n$  and label them with integers in  $\mathbb{N}$ . Any element of  $A_n$  will be listed eventually. Then this pairing gives a bijection since the  $A_n$  are disjoint. So  $\mathbb{Q} \sim \mathbb{N}$ .  $\square$

*Proof of (2).* Argue by contradiction. Suppose  $f$  is one-to-one from  $\mathbb{N} \rightarrow \mathbb{R}$ . Set  $x_1 = f(1)$ ,  $x_2 = f(2)$ , etc. We can write

$$\mathbb{R} = \{x_1, x_2, \dots\}.$$

Let  $I_1$  be a closed interval such that  $x_1 \notin I_1$ . Pick  $I_2 \subseteq I_1$  such that  $x_2 \notin I_2$ . Continue this process such that  $I_{n+1} \subseteq I_n$  is a closed interval where  $x_{n+1} \notin I_{n+1}$ . By construction,

$$I_1 \supseteq I_2 \supseteq \dots \supseteq I_n \supseteq \dots$$

We know that

$$\bigcap_{n=1}^{\infty} I_n \neq \emptyset.$$

So we can find  $n_0$  such that

$$x_{n_0} \in \bigcap_{n=1}^{\infty} I_n.$$

This is a contradiction with  $x_{n_0} \notin I_{n_0}$ . Thus such an  $f$  cannot exist and  $\mathbb{R}$  is uncountable.  $\square$

**Theorem 3.2.2.**

1. Let  $A \subseteq B$ . If  $B$  is countable, then  $A$  is either finite or countable.
2. If  $A_n$  is a countable set, then

$$\bigcup_{n=1}^{\infty} A_n$$

is also countable.

**Theorem 3.2.3** (Cantor's theorem). *The open interval*

$$(0, 1) = \{x \in \mathbb{R} : 0 < x < 1\}$$

*is uncountable.*

*Proof.* Argue by contradiction. Assume  $f : \mathbb{N} \rightarrow (0, 1)$  is one-to-one and onto. Then for  $m \in \mathbb{N}$ , we can write (decimal expansion)

$$f(m) = 0.a_{m1}a_{m2}a_{m3}\dots \in (0, 1)$$

For every  $m, n \in \mathbb{N}$ ,  $a_{mn} \in \{0, \dots, 9\}$  is the  $n$ th digit in the decimal expansion of  $f(m)$ . We can write in a table

$$\begin{array}{cccccc} 1 & f(1) & a_{11} & a_{12} & a_{13} & \dots \\ 2 & f(2) & a_{21} & a_{22} & a_{23} & \dots \\ 3 & f(3) & a_{31} & a_{32} & a_{33} & \dots \\ & & & \vdots & & \end{array}$$

Take  $x = 0.b_1b_2b_3\dots$  where

$$b_n = \begin{cases} 2 & \text{if } a_{nn} \neq 2 \\ 3 & \text{if } a_{nn} = 2. \end{cases}$$

Then  $x \neq f(m)$  for any  $m \in \mathbb{N}$  (since  $b_m \neq a_{mm}$ ). This is a contradiction. □