MATH 4317: Analysis I

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Lecture 1

Aug. 22 – The Real Numbers

1.1 Number Systems

We start with the natural numbers [1]

$$\mathbb{N} = \{1, 2, 3, \ldots\}.$$

These are perhaps the most natural in a way, since they are what we use to count things. They are closed under addition, but fail when it comes to subtraction. For example, $1 - 2 = -1 \notin \mathbb{N}$. So we must expand our number system to the integers

$$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}.$$

We can now add, subtract, and multiply. But we run into problems when we start to consider quotients. For example, $1 \div 2 = \frac{1}{2} \notin \mathbb{Z}$. So we continue to the rational numbers

$$\mathbb{Q} = \left\{ \frac{p}{q} : p, q \in \mathbb{Z}, \, q \neq 0 \right\}.$$

We now have summation, subtraction, multiplication, and quotients. But there is still a problem. For example, consider the diagonal of a square with side length 1.

Theorem 1.1. *$\sqrt{2}$ is not a rational number.* [2]

Proof. Argue by contradiction. Suppose $\sqrt{2}$ is rational. Then we can write

$$\sqrt{2} = \frac{p}{q}$$

for some integers p, q. Further assume p and q have no common factors. Then

$$2 = \frac{p^2}{q^2} \implies p^2 = 2q^2.$$

So p is even and we can write $p = 2r$ for some $r \in \mathbb{Z}$. Then

$$4r^2 = 2q^2 \implies 2r^2 = q^2.$$

So q is also even, and p, q share a common factor of 2. Contradiction. □

[1] $0 \notin \mathbb{N}$ for this class.

[2] In some sense, this shows that the notion of "rationals" is strictly weaker than the notion of "length."

Another weakness of $\mathbb{Q}$ is that we cannot take limits ($\mathbb{Q}$ is not complete). For example, note that

$$(\sqrt{2}-1)(\sqrt{2}+1) = 2-1 = 1,$$

$$\sqrt{2} = 1 + \frac{1}{\sqrt{2}+1} = 1 + \frac{1}{1+1+\frac{1}{\sqrt{2}+1}} = \dots.$$

So if we define the rational sequence

$$a_1 = 1, \quad a_2 = 1 + \frac{1}{2}, \quad a_3 = 1 + \frac{1}{2+\frac{1}{2}}, \quad a_4 = 1 + \frac{1}{2+\frac{1}{2+\frac{1}{2}}}, \quad \dots,$$

then as $n \to \infty$, $a_n \to \sqrt{2} \notin \mathbb{Q}$.

1.2 Sets

Sets are any collections of objects. Given a set A, we write $x \in A$ if x is an element of A. We write $x \notin A$ otherwise. The **union** of two sets is

$$A \cup B = \{x : x \in A \text{ or } x \in B\},$$

and the **intersection** of two sets is

$$A \cap B = \{x : x \in A \text{ and } x \in B\}.$$

We use the notation

$$\bigcup_{k=1}^{\infty} A_k$$

to denote the countable union of a family of sets indexed by k.

1.3 Functions

Definition 1.1. Given two sets A and B, a **function** from A to B is a rule, relation, or mapping that takes each element $x \in A$ and associates with it a single element in B. In this case, we write $f : A \to B$.

We call A the **domain** of f and B the **codomain** of f. The element in B associated with $x \in A$ is $f(x)$, called the **image** of x. The **range** of f is

$$\text{range}(f) = \{y \in B : y = f(x) \text{ for some } x \in A\}.$$

We say f is:

1. **onto** or **surjective** if $\text{range}(f) = B$.
2. **one-to-one** or **injective** if $x, x' \in A$ and $x \neq x'$, then $f(x) \neq f(x')$.
3. **bijective** if it is injective and surjective.

Example 1.1.1. First Dirichlet function:

$$g(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases} = \lim_{k\to\infty}\left(\lim_{j\to\infty}[\cos(k!\pi x)]^{2j}\right).$$

Example 1.1.2. Second Dirichlet function:

$$f(x) = \begin{cases} \frac{1}{q} & \text{if } x = \frac{p}{q} \in \mathbb{Q} \text{ in lowest terms} \\ 0 & \text{if } x \notin \mathbb{Q}. \end{cases}$$

Example 1.1.3. Absolute value:

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0. \end{cases}$$

Note that we have the following two properties:

- $|xy| = |x||y|$.
- $|x+y| \leq |x| + |y|$. This is called the *triangle inequality*.

1.4 Induction

If we have a set $S \subseteq \mathbb{N}$ and

1. $1 \in S$
2. if $n \in S$, then $n+1 \in S$

then $S = \mathbb{N}$. [3]

[3] We always use induction in conjunction with $\mathbb{N}$.

Lecture 2

Aug. 24 – The Axiom of Completeness

The number system $\mathbb{Q}$ is pretty good (it is a field), but recall that we are unable to take limits. For instance, take the sequence $x_0 = 2$ and

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right)$$

for $n \geq 1$. All the x_i are rational, but $x_n \to \sqrt{2} \notin \mathbb{Q}$. This shows that there are gaps in $\mathbb{Q}$. The real numbers $\mathbb{R}$ will fill these gaps (completeness).

Axiom 2.1 (Axiom of completeness)**.** *Every nonempty set of real numbers that are bounded above has a least upper bound.*

Note that this least upper bound is *unique*.

2.1 Suprema and Infima

Definition 2.1. Let $S \subseteq \mathbb{R}$. The set S is **bounded above** if there exists $u \in R$ such that $s \leq u$ for all $s \in S$. We say that u is an **upper bound** of S.

We define **bounded below** and **lower bound** similarly.

Definition 2.2. S is said to be **bounded** if it is both bounded above and below. Otherwise we say that S is **unbounded**.

Example 2.2.1. $\mathbb{N} = \{1, 2, 3, \dots\}$ is bounded below but not above.

Example 2.2.2. The set

$$\left\{\frac{1}{k} : k \in \mathbb{N}\right\} = \left\{1, \frac{1}{2}, \frac{1}{3}, \dots\right\}$$

is bounded.

Example 2.2.3. $\emptyset$ is bounded.

Definition 2.3. We say $u \in \mathbb{R}$ is the **least upper bound** or **supremum** of a nonempty set $S \subseteq \mathbb{R}$ if

1. u is an upper bound of S.
2. $u \leq v$ for any upper bound v of S.

We write $u = \sup S$.

The **greatest lower bound** or **infimum** of S is defined similarly, denoted $\inf S$.

Example 2.3.1.

$$S = \left\{\frac{1}{k} : k \in \mathbb{N}\right\}.$$

$\sup S = 1$, $\inf S = 0$.

Definition 2.4. Let $S \subseteq \mathbb{R}$. We say a real number $M \in S$ is a **maximal element** or **maximum** of S if $s \leq M$ for all $s \in S$.

The **minimal element** or **minimum** is defined similarly.

Example 2.4.1. $[0,1)$ is bounded, but has no maximum. The minimum is 0.

Example 2.4.2. The set

$$\{2^{-n} : n \in \mathbb{N}\} = \left\{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots\right\}$$

is bounded, but has no minimum. The maximum is $\frac{1}{2}$.

Example 2.4.3. $\emptyset$ is bounded but has no minimum or maximum.

Exercise 2.1. Let $A \subseteq \mathbb{R}$ be bounded above. Let $c \in \mathbb{R}$ and define

$$c + a := \{a + c : a \in A\}.$$

Then $\sup(A + c) = c + \sup A$.

Proof. Let $s = \sup A$. By definition, we know $a \leq s$ for all $a \in A$, which implies $a + c \leq s + c$. So $s + c$ is an upper bound for $c + A$. Now let b be an arbitrary upper bound for $c + A$. For all $a \in A$, we have $a + c \leq b$, which implies $a \leq b - c$. So $b - c$ is an upper bound for A. By construction, $s \leq b - c$, so $s + c \leq b$. Therefore $s + c = \sup(A + c)$. □

Lemma 2.1. *Assume $s \in \mathbb{R}$ is an upper bound for a set $A \subseteq \mathbb{R}$. Then $s = \sup A$ if and only if for every $\epsilon > 0$, there exists $a \in A$ such that $s - \epsilon < a$.*

Proof.
($\Longrightarrow$): Suppose $\sup A = s$. Then given any $\epsilon > 0$, $s - \epsilon$ cannot be an upper bound for A. So there exists $a \in A$ such that $a > s - \epsilon$.

($\Longleftarrow$): Let b be an arbitrary upper bound for A. Suppose for contradiction that $b < s$. Set $\epsilon = s - b > 0$. Then by assumption we can find $a \in A$ such that $a > s - \epsilon = b$. Contradiction. Therefore $b \geq s$, whence $\sup A = s$. □

2.2 Consequences of Completeness

2.2.1 1st Consequence: Nested Interval Properties

Theorem 2.1 (Nested interval properties)**.** *For any $n \in \mathbb{N}$, assume that we are given a closed interval*

$$I_n = [a_n, b_n] = \{x \in \mathbb{R} : a_n \leq x \leq b_n\}.$$

Assume $I_n \supseteq I_{n+1}$. Then the resulting nested sequence of closed intervals

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq \dots$$

has a nonempty intersection:

$$\bigcap_{n=1}^{\infty} I_n \neq \emptyset.$$

Proof. Define $A = \{a_n\}$. Note that $A \neq \emptyset$. For any n, $a_n \leq b_n \leq b_1$. So $x = \sup A$ exists. Furthermore, for any n, b_n is an upper bound for A. So $x \leq b_n$. Since $x = \sup A$, $a_n \leq x$. So $x \in [a_n, b_n]$ for any n, whence

$$x \in \bigcap_{n=1}^{\infty} I_n.$$

□

2.2.2 2nd Consequence: Archimedean Properties

Theorem 2.2 (Archimedean properties)**.**

1. *Given any $x \in \mathbb{R}$, there is an $n \in \mathbb{N}$ such that $n > x$.* [1]
2. *Given any real number $y > 0$, there is an $\mathbb{N}$ such that $\frac{1}{n} < y$.*

Proof of (1). Argue by contradiction. Suppose $\mathbb{N}$ is bounded above. Then by the axiom of completeness, $\alpha = \sup N$ exists. By construction, $\alpha - 1$ is not an upper bound for $\mathbb{N}$. So we can find $n \in N$ such that $\alpha - 1 < n$, which implies $\alpha < n + 1 \in \mathbb{N}$. Contradiction. □

Proof of (2). Follows from (1) by setting $x = \frac{1}{y}$. □

[1] This is saying that $\mathbb{N}$ is not bounded above.

Lecture 3

Aug. 29 – Completeness, Countability

3.1 Consequences of Completeness

3.1.1 3rd Consequence: Density of $\mathbb{Q}$ in $\mathbb{R}$

Theorem 3.1 (Density of $\mathbb{Q}$ in $\mathbb{R}$). *For all $a, b \in \mathbb{R}$, $a < b$, there exists $r \in \mathbb{Q}$ such that $a < r < b$.*

Proof. We want to find $m \in \mathbb{Z}$, $n \in \mathbb{N}$ such that

$$a < \frac{m}{n} < b.$$

By (2) of the Archimedean properties, we can find $n \in \mathbb{N}$ such that

$$\frac{1}{n} < b - a.$$

Fix such an n. Then let m be the smallest integer such that $m - 1 \leq na < m$. By construction,

$$\frac{m}{n} - \frac{1}{n} \leq a < \frac{m}{n},$$

$$\frac{m}{n} \leq a + \frac{1}{n} < b.$$

Therefore, $a < \frac{m}{n} < b$. □

Corollary 3.1.1. *For all $a, b \in \mathbb{Q}$, $a < b$, there exists $t \in \mathbb{R} \setminus \mathbb{Q}$ such that $a < t < b$.*

3.1.2 4th Consequence: Existence of $\sqrt{2}$

Theorem 3.2 (Existence of $\sqrt{2}$). *There exists $s \in \mathbb{R}$, $s > 0$ such that $s^2 = 2$.*

Proof. Define

$$S = \{x > 0 : x^2 < 2\} \subseteq \mathbb{R}.$$

$x = 1 \in S$, so $S \neq \emptyset$. 2 is an upper bound for S, so S is bounded above. Then by the axiom of completeness, $s = \sup S$ exists. We claim that $s^2 = 2$.

Suppose otherwise that $s^2 < 2$. Then we can find $\epsilon > 0$ such that $s + \epsilon \in S$. Define $\delta = 2 - s^2 > 0$. Note that

$$(s + \epsilon)^2 - 2 = s^2 + 2s\epsilon + \epsilon^2 - 2 = -\delta + 2s\epsilon + \epsilon^2.$$

We know $s \le 2$ since 2 is an upper bound. Pick

$$\epsilon = \frac{\delta}{100000000000},$$

$$2s\epsilon + \epsilon \le 4\epsilon + \epsilon^2 < \frac{\delta}{2}.$$

Then

$$(s+\epsilon)^2 - 2 < -\delta + \frac{\delta}{2} = -\frac{\delta}{2} < 0.$$

So $s+\epsilon \in S$, which contradicts with $s = \sup S$.

$s^2 > 2$ also leads to a contradiction (left as an exercise). Thus we must have $s^2 = 2$. □

3.2 Countability

Definition 3.1. We say two sets A and B have the same **cardinality** if there is a bijection $f : A \to B$. We write $A \sim B$.

Definition 3.2. We say that a set A is **finite** if $A \sim \{1, 2, \ldots, n\}$ for some integer n. We say that a set A is **countable** (or countably infinite) if $A \sim \mathbb{N}$. If a set A is not countable, then we say it is **uncountable**.

Example 3.2.1. Let $E = \{2, 4, 6, 8, \ldots\}$. E is not finite but it is countable: $E \sim \mathbb{N}$. We can define $f : \mathbb{N} \to E$ by $f(n) = 2n$.

Example 3.2.2. $\mathbb{N} \sim \mathbb{Z}$. The bijection $f : \mathbb{N} \to \mathbb{Z}$ is given by

$$f(n) = \begin{cases} \frac{n-1}{2} & n \text{ is odd} \\ -\frac{n}{2} & n \text{ is even.} \end{cases}$$

Example 3.2.3. $(-1, 1) \sim \mathbb{R}$. The bijection $f : (-1, 1) \to \mathbb{R}$ is given by

$$x \mapsto \frac{x}{x^2 - 1}.$$

Theorem 3.3.

1. *$\mathbb{Q}$ is countable.*
2. *$\mathbb{R}$ is uncountable.*

Proof of (1). Set $A_1 = \{0\}$ and for $n \ge 2$,

$$A_n = \left\{ \pm\frac{p}{q} : p, q \in \mathbb{N}, p, q \text{ in lowest terms}, p + q = n \right\}.$$

So the first few A_n are:

$$A_2 = \left\{ \frac{1}{1}, \frac{-1}{1} \right\},$$

$$A_3 = \left\{ \frac{1}{2}, \frac{2}{1}, \frac{-1}{2}, \frac{-2}{1} \right\},$$

etc. Note that A_n is finite and for all $x \in \mathbb{Q}$, there is an $n \in \mathbb{N}$ such that $x \in A_n$. We can list elements in $A_1, \dots, A_n$ and label them with integers in $\mathbb{N}$. Any element of A_n will be listed eventually. Then this pairing gives a bijection since the A_n are disjoint. So $\mathbb{Q} \sim \mathbb{N}$. □

Proof of (2). Argue by contradiction. Suppose f is one-to-one from $\mathbb{N} \to \mathbb{R}$. Set $x_1 = f(1)$, $x_2 = f(2)$, etc. We can write

$$\mathbb{R} = \{x_1, x_2, \dots\}.$$

Let I_1 be a closed interval such that $x_1 \notin I_1$. Pick $I_2 \subseteq I_1$ such that $x_2 \notin I_2$. Continue this process such that $I_{n+1} \subseteq I_n$ is a closed interval where $x_{n+1} \notin I_{n+1}$. By construction,

$$I_1 \supseteq I_2 \supseteq \cdots \supseteq I_n \supseteq \dots.$$

We know that

$$\bigcap_{n=1}^{\infty} I_n \neq \emptyset.$$

So we can find n_0 such that

$$x_{n_0} \in \bigcap_{n=1}^{\infty} I_n.$$

This is a contradiction with $x_{n_0} \notin I_{n_0}$. Thus such an f cannot exist and $\mathbb{R}$ is uncountable. □

Theorem 3.4.

1. *Let $A \subseteq B$. If B is countable, then A is either finite or countable.*
2. *If A_n is a countable set, then*

$$\bigcup_{n=1}^{\infty} A_n$$

is also countable.

Theorem 3.5 (Cantor's diagonal argument)**.** *The open interval*

$$(0,1) = \{x \in \mathbb{R} : 0 < x < 1\}$$

is uncountable.

Proof. Argue by contradiction. Assume $f : \mathbb{N} \to (0,1)$ is one-to-one and onto. Then for $m \in \mathbb{N}$, we can write (decimal expansion)

$$f(m) = 0.a_{m1}a_{m2}a_{m3}\dots \in (0,1).$$

For every $m, n \in \mathbb{N}$, $a_{mn} \in \{0, \dots, 9\}$ is the nth digit in the decimal expansion of $f(m)$. We can write in a table

$$\begin{array}{cccccc}
1 & f(1) & a_{11} & a_{12} & a_{13} & \dots \\
2 & f(2) & a_{21} & a_{22} & a_{23} & \dots \\
3 & f(3) & a_{31} & a_{32} & a_{33} & \dots \\
 & & \vdots & & &
\end{array}$$

Take $x = 0.b_1 b_2 b_3 \ldots$ where
$$b_n = \begin{cases} 2 & \text{if } a_{nn} \neq 2 \\ 3 & \text{if } a_{nn} = 2. \end{cases}$$
Then $x \neq f(m)$ for any $m \in \mathbb{N}$ (since $b_m \neq a_{mm}$). This is a contradiction. $\square$

Lecture 4

Aug. 31 – Cantor's Theorem, Sequences

4.1 Cantor's Theorem

Definition 4.1. The **power set** of A, denoted $\mathcal{P}(A)$, is the collection of all subsets of A.

Theorem 4.1 (Cantor's theorem)**.** *Given any set A, there does not exist a function $f : A \to \mathcal{P}(A)$ which is surjective.* [1]

Proof. Argue by contradiction. Suppose $f : A \to \mathcal{P}(A)$ is onto. Then for any $a \in A$, $f(a)$ is a subset of A. Since f is onto, for any subset B of A, we can find $a \in A$ such that $f(a) = B$. Define

$$B = \{a \in A : a \notin f(a)\} \subseteq A.$$

We can find $a' \in A$ such that $f(a') = B$. If $a' \in B$, then $a' \notin f(a') = B$, which is a contradition.. If $a' \notin B$, this is a contradiction with the definition of B. Thus such f cannot exist. $\square$

Remark. This means that the cardinality of $\mathcal{P}(A)$ is strictly larger than that of A.

4.2 Sequences

Definition 4.2. A **sequence** is a function whose domain is $\mathbb{N}$.

We usually write $\{a_n\}$, $\{x_n\}$ or (a_n), (x_n) to denote sequences.

Example 4.2.1. The following

$$\left\{\frac{1+n}{n}\right\}_{n=1}^{\infty} = \left\{2, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \ldots\right\}$$

is a sequence.

Example 4.2.2. $\{a_n\}$, where $a_n = 2^n$ for $n \in \mathbb{N}$, is a sequence.

Example 4.2.3. We can also define $\{x_n\}$ recursively by $x_1 = 2$ and

$$x_{n+1} = \frac{x_n + 1}{2}.$$

Remark. Sometimes a sequence is also labeled starting from $n = 0$.

[1] Note that if $\#(A) = n < \infty$, this is true as $\#(\mathcal{P}(A)) = 2^n \neq \#(A)$.

4.2.1 Limits

Definition 4.3. A sequence $\{a_n\}$ **converges** to a real number a if for every $\epsilon > 0$, we can find $N \in \mathbb{N}$ such that for all $n \geq N$, one has $|a_n - a| < \epsilon$. We write $\lim_{n\to\infty} a_n = a$.

Remark. In analysis, ϵ is always taken to be a positive number.

Example 4.3.1. The sequence $\{1/n\}_{n=1}^{\infty}$ converges with

$$\lim_{n\to\infty} \frac{1}{n} = 0.$$

Definition 4.4. For $\epsilon > 0$, the ϵ-**neighborhood** of a is defined to be

$$V_\epsilon(a) = \{x \in \mathbb{R} : |x - a| < \epsilon\}.$$

Definition 4.5. We say that a is the **limit** of a sequence $\{a_n\}$ if for every $\epsilon > 0$, $V_\epsilon(a)$ contains all but finitely many elements of $\{a_n\}$. [2]

Remark. This definition of the limit is equivalent to the definition of convergence.

Definition 4.6. A sequence $\{a_n\}$ that does not converge is said to be **divergent**.

Theorem 4.2. *The limit of a sequence, when it exists, must be unique.*

Proof. Homework problem. □

Exercise 4.1. Show

$$\lim_{n\to\infty} \frac{n+1}{n}$$

exists and

$$\lim_{n\to\infty} \frac{n+1}{n} = 1.$$

Proof. We show

$$\lim_{n\to\infty} \frac{n+1}{n} = 1.$$

For every $\epsilon > 0$, take $N \in \mathbb{N}$ such that $N > \frac{1}{\epsilon}$. We have for all $n \geq N$,

$$\left|\frac{n+1}{n} - 1\right| = \left|\frac{1}{n}\right| \leq \frac{1}{N} < \epsilon.$$

Therefore,

$$\lim_{n\to\infty} \frac{n+1}{n} = 1.$$

□

[2] This is the *topological* definition of the limit.

4.2.2 Tips for Showing Limits

To show the limit of a sequence, take the following steps:

1. Identify the limit a. This is always given by the problem or observation.
2. $\forall \epsilon > 0$.
3. Find $N = N(\epsilon)$. Do this in sketch paper (need computations and manipulations).
4. Set N as what is found in (3).
5. Check that N works.

Lecture 5

Sept. 5 – Limits and Limit Theorems

5.1 Review of Limits

Example 5.0.1. Find

$$\lim_{n\to\infty} \frac{1+\sqrt{n}}{\sqrt{n}}.$$

Proof. We want to show that

$$\lim_{n\to\infty} \frac{1+\sqrt{n}}{\sqrt{n}} = 1.$$

Fix $\epsilon > 0$ and take $N \in \mathbb{N}$ such that $N > \frac{1}{\epsilon^2}$. Then for any $n > N$,

$$\left|\frac{1+\sqrt{n}}{\sqrt{n}} - 1\right| \le \left|\frac{1}{\sqrt{n}}\right| \le \frac{1}{\sqrt{N}} < \epsilon,$$

as desired. □

How can we understand this using the topological definition? For all $\epsilon > 0$, take $V_\epsilon(1)$. Pick $N > \frac{1}{\epsilon^2}$. Then we claim that $V_\epsilon(1)$ contains all but at most N elements of $\left\{\frac{\sqrt{n}+1}{\sqrt{n}}\right\}$. When $n \ge N$, we have

$$\left|\frac{\sqrt{n}+1}{\sqrt{n}} - 1\right| < \epsilon,$$

i.e. $\frac{\sqrt{n}+1}{\sqrt{n}} \in V_\epsilon(1)$. So at most N elements might not be in $V_\epsilon(1)$.

5.2 Limit Theorems

5.2.1 Algebraic Facts About Limits

Definition 5.1. A sequence $\{x_n\}$ is said to be **bounded** if there exists M such that $|x_n| \le M$ for all n. Alternatively, $\sup_n |x_n| \le M$.

Theorem 5.1. *Every convergent sequence is bounded.*

Proof. Suppose
$$\lim_{n\to\infty} x_n = l.$$
Take $\epsilon = 1$, we can find N such that for all $n \geq N$, $|x_n - l| < 1$. By the triangle inequality, $|x_n| < |l| + 1$ for $n \geq N$. Take
$$M = \max\{|x_1|, |x_2|, \ldots, |x_{N-1}|, |l| + 1\}.$$
Then $|x_n| \leq M$ for all $n \in \mathbb{N}$. □

Theorem 5.2 (Algebraic limit theorem)**.** *If*
$$\lim_{n\to\infty} a_n = a \quad \textit{and} \quad \lim_{n\to\infty} b_n = b,$$
then for all $c \in \mathbb{R}$*,*
$$(1)\ \lim_{n\to\infty} ca_n = ca, \quad (2)\ \lim_{n\to\infty}(a_n + b_n) = a + b, \quad \textit{and} \quad (3)\ \lim_{n\to\infty} a_n b_n = ab.$$
Furthermore, if $b \neq 0$*, then*
$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{a}{b}. \tag{4}$$

Proof. (1) When $c = 0$, the result is trivial. When $c \neq 0$, for all $\epsilon > 0$, we set $\epsilon' = \frac{\epsilon}{|c|}$. Since $\lim_{n\to\infty} a_n = a$, we can find $N_{\epsilon'}$ such that for all $n \geq N_{\epsilon'}$, $|a_n - a| < \epsilon'$. When $n > N_{\epsilon'}$, we have
$$|ca_n - ca| = |c||a_n - a| < |c|e' = |c|\frac{\epsilon}{|c|} = \epsilon.$$
So $\lim_{n\to\infty} ca_n = ca$.

(2) For all $\epsilon > 0$, since $a_n \to a$ and $b_n \to b$, we can find N_1 and N_2 such that when
$$n \geq N_1, \quad |a_n - a| < \frac{\epsilon}{2},$$
$$n \geq N_2, \quad |b_n - b| < \frac{\epsilon}{2}.$$
Take $N = \max\{N_1, N_2\}$. Then for all $n \geq N$,
$$|a_n + b_n - (a + b)| = |a_n - a + b_n - b| \leq |a_n - a| + |b_n - b| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$
Therefore $\lim_{n\to\infty}(a_n + b_n) = a + b$. □

5.2.2 Order Theorem

Theorem 5.3 (Order theorem)**.** *Let* $\{a_n\}$ *and* $\{b_n\}$ *be sequences such that*
$$\lim_{n\to\infty} a_n = a \quad \textit{and} \quad \lim_{n\to\infty} b_n = b.$$
(5) If $a_n \geq 0$ *for every* n*, then* $a \geq 0$*. (6) If* $a_n \leq b_n$*, then* $a \leq b$*. (7) If* $a_n \geq c$*, then* $a \geq c$*.*

Proof. (5) Argue by contradiction. Suppose $a < 0$. Take $\epsilon = \frac{|a|}{2}$. Since $\lim_{n\to\infty} a_n = a$, we can find N such that when $n \geq N$, $|a_n - a| < \epsilon$. Note that this means
$$-\epsilon < a_n - a < \epsilon$$
Then we have
$$a_n < \epsilon + a = \frac{-a}{2} + a = \frac{a}{2} < 0.$$
Contradiction. □

5.2.3 Monotone Convergence Theorem

Definition 5.2. A sequence $\{a_n\}$ is **increasing** if $a_n \le a_{n+1}$ for every n and **decreasing** if $a_n \ge a_{n+1}$ for every n. A sequence is **monotone** if it is either increasing or decreasing.

Theorem 5.4 (Monotone convergence theorem)**.** *If a sequence is monotone and bounded, then it converges.*

Proof. Let $\{a_n\}$ be increasing and bounded. Set $A = \{a_n : n \in \mathbb{N}\}$. Note that $A \neq \emptyset$ and A is bounded. Therefore, by the axiom of completeness, $s = \sup A \in \mathbb{R}$ exists. Then we claim that $\lim_{n\to\infty} a_n = s$. For every $\epsilon > 0$, $s - \epsilon$ is not an upper bound for A, so we can find N such that $s - \epsilon < a_N \le s$. Since $\{a_n\}$ is increasing, for all $n \ge N$, we know $s - \epsilon < a_N \le a_n \le s$, i.e. $|a_n - s| < \epsilon$. Therefore $\lim_{n\to\infty} a_n = s$.

For $\{a_n\}$ decreasing and bounded, simply let apply the previous result to $\{-a_n\}$. $\square$

Lecture 6

Sept. 7 – Bolzano-Weierstrass Theorem

6.1 Review of Limits

Theorem 6.1 (Squeeze Theorem). *Let $\{x_n\}, \{y_n\}, \{z_n\}$ be sequences such that $x_n \le y_n \le z_n$ for all n, and suppose that*

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} z_n = l.$$

Then $\lim_{n\to\infty} y_n = l$.

Proof. Consider $|y_n - l|$. If

$$\begin{aligned} y_n - l \ge 0, &\quad \text{then} \quad y_n - l \le z_n - l, \\ y_n - l < 0, &\quad \text{then} \quad |y_n - l| = l - y_n \le l - x_n. \end{aligned}$$

So we have

$$|y_n - l| \le |z_n - l| + |x_n - l|.$$

For all $\epsilon > 0$, there exist N_1, N_2 such that for all $n \ge \mathbb{N}_1$,

$$|z_n - l| < \frac{\epsilon}{2},$$

and for all $n \ge N_2$,

$$|x_n - l| < \frac{\epsilon}{2}.$$

Take $N = \max\{N_1, N_2\}$. If $n \ge N$, then

$$|y_n - l| \le |z_n - l| + |x_n - l| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

So $\lim_{n\to\infty} y_n = l$. □

6.2 Subsequences and the Bolzano-Weierstrass Theorem

Definition 6.1. Let $\{a_n\}$ be a sequence of real numbers. Let $n_1 < n_2 < n_3 < \dots$ be an increasing sequence of natural numbers. Then $\{a_{n_1}, a_{n_2}, \dots,\}$ is a **subsequence** of $\{a_n\}$, and it is denoted by $\{a_{n_k}\}$.

Example 6.1.1. Let

$$\{a_n\} = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\right\}.$$

Then

$$\left\{\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \dots\right\}$$

is a subsequence of $\{a_n\}$. However, note that

$$\left\{\frac{1}{10}, \frac{1}{5}, \frac{1}{100}, \frac{1}{500}, \dots\right\}$$

is *not* a subsequence of $\{a_n\}$ since the the n_k are not strictly increasing. Similarly,

$$\left\{1, \frac{1}{3}, \frac{1}{3}, \frac{1}{5}, \frac{1}{5} \dots\right\}$$

is also not a subsequence of $\{a_n\}$.

Theorem 6.2. *Subsequences of a convergent sequence converge to the same limit as the original sequence.*

Proof. Suppose $\lim_{n\to\infty} a_n = a$. So for every $\epsilon > 0$, there exists N such that $|a_n - a| < \epsilon$ for all $n \geq N$. Consider an arbitrary subsequence $\{a_{n_k}\}$. Note that $n_k \geq k$. So when $k \geq N$,

$$|a_{n_k} - a| < \epsilon.$$

Therefore $\lim_{k\to\infty} a_{n_k} = a$. □

Example 6.1.2. Let $0 < b < 1$. Clearly

$$1 > b > b^2 > b^3 > b^4 > \cdots \geq 0.$$

The sequence $\{b^n\}$ is decreasing and bounded below, so by the monotone convergence theorem, $\lim_{n\to\infty} b^n = l \in \mathbb{R}$ exists. Note that $\{b^{2n}\}$ is a subsequence of $\{b^n\}$, so by Theorem 6.2, we have $\lim_{n\to\infty} b^{2n} = l$. Note that $b^{2n} = b^n b^n$. By the algebraic limit theorem,

$$\lim_{n\to\infty} b^{2n} = \left(\lim_{n\to\infty} b^n\right)\left(\lim_{n\to\infty} b^n\right).$$

Therefore, $l = l^2$, so we have $l = 0$ or $l = 1$. But the entire sequence is strictly less than 1 and decreasing, so $l = 0$.

Example 6.1.3. Consider the sequence

$$\{(-1)^n\} = \{-1, 1, -1, 1, \dots\}.$$

This sequence does not converge. But the subsequence

$$\{-1, -1, -1, \dots\}$$

does converge.

Remark. This shows that the converse of Theorem 6.2 is not true, i.e. a convergent subsequence does not imply that the original sequence converges.

Example 6.1.4. The sequence

$$a_n = \begin{cases} 1 & \text{if } n \text{ is prime,} \\ 0 & \text{otherwise} \end{cases}$$

does not converge.

Exercise 6.1. Show the limit of the sequence

$$\left\{1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \frac{1}{5}, \frac{1}{5}, -\frac{1}{5}, \frac{1}{5}, -\frac{1}{5}, \dots\right\}$$

Proof. The subsequence

$$\left\{\frac{1}{5}, \frac{1}{5}, \dots\right\}$$

converges to $\frac{1}{5}$ while the subsequence

$$\left\{-\frac{1}{5}, -\frac{1}{5}, \dots\right\}$$

converges to $-\frac{1}{5}$. Thus the original sequence diverges. □

Remark. If we can find two subsequences that converge to different limits, then the original sequence diverges. This is the contrapositive of Theorem 6.2.

Theorem 6.3 (Bolzano-Weierstrass theorem)**.** *Every bounded sequence has a convergent subsequence.* [1]

Proof. Let $\{a_n\}$ be a bounded be a bounded sequence. So there exists $M > 0$ such that $\sup_n |a_n| < M$. So a_n is contained in $[-M, M]$. Split $[-M, M]$ into $[-M, 0]$ and $[0, M]$. Pick one that contains infinitely many elements of $\{a_n\}$ and call it I_1. Then pick $a_{n_1} \in \{a_n\}$ such that $a_{n_1} \in I_1$. Split I_1 again into two closed intervals of the same size. Take one of these two that contains infinitely many elements of $\{a_n\}$ and call it I_2. Then take $a_{n_2} \in \{a_n\}$ such that $a_{n_2} \in I_2$. Repeat this process to to get $I_{k+1} \subseteq I_k$ with $|I_{k+1}| = \frac{1}{2}|I_k|$ such that I_{k+1} contains infinitely many elements of $\{a_n\}$. [2] Also pick $a_{n_{k+1}} \in \{a_n\}$ such that $a_{n_{k+1}} \in I_{k+1}$ with $n_{k+1} > n_k$.

By construction, $\{a_{n_k}\}$ is a subsequence of $\{a_n\}$ and $a_{n_k} \in I_k$. We have the I_k being closed intervals with

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq \dots.$$

So there exists $x \in \mathbb{R}$ such that $x \in \bigcap_{k=1}^{\infty} I_k$. Note that $|I_k| = M\left(\frac{1}{2}\right)^{k-1}$. Then we claim that $\lim_{k\to\infty} a_{n_k} = x$.

Let $\epsilon > 0$. Take N such that

$$2^N > \frac{2M}{\epsilon}.$$

Then for every $k \geq N$, we have

$$|a_{n_k} - x| \leq M\left(\frac{1}{2}\right)^{k-1} < \epsilon$$

since $a_{n_k}, x \in I_k$. Thus $\lim_{k\to\infty} a_{n_k} = x$, and $\{a_{n_k}\}$ is a convergent subsequence. □

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[1] This demonstrates some kind of *compactness* of the real numbers.

[2] Here, by $|I_k|$ we mean the length of the interval I_k.

Lecture 7

Sept. 12 – The Cauchy Criterion

7.1 Cauchy Sequences

Definition 7.1. A sequence $\{a_n\}$ is called a **Cauchy** sequence if for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for every $m, n \geq N$, one has $|a_m - a_n| < \epsilon$. [1]

Theorem 7.1. *Every convergent sequence is a Cauchy sequence.*

Proof. Assume $\lim_{n\to\infty} a_n = a$. Then for every $\epsilon > 0$, we can find N such that for every $n \geq N$, we have $|a_n - a| < \epsilon/2$. Then for every $m, n \geq N$, we have

$$|a_m - a_n| = |a_m - a + a - a_n| \leq |a_m - a| + |a - a_n| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

by the triangle inequality. □

Lemma 7.1. *Every Cauchy sequence is bounded.*

Proof. Suppose $\{x_n\}$ is a Cauchy sequence. Pick $\epsilon = 1$. Then there exists N such that for all $m, n \geq N$, we have $|x_m - x_n| < 1$. Fixing $m = N$, we know that for all $n \geq N$, $|x_N - x_n| < 1$. So $|x_n| \leq |x_N| + 1$ for all $n \geq N$. Set

$$M = \max\{|x_1|, |x_2|, \ldots, |x_{N-1}|, |x_N| + 1\}.$$

Then $\sup |x_n| \leq M$ by construction. □

Theorem 7.2 (Cauchy criterion)**.** *A sequence converges if and only if it is a Cauchy sequence.*

Proof.
($\Longrightarrow$): This is Theorem 7.1.

($\Longleftarrow$): Suppose $\{a_n\}$ is a Cauchy sequence. Since $\{a_n\}$ is Cauchy, we know $\sup |a_n| \leq M$ for some $M \in \mathbb{R}$. Then by the Bolzano-Weierstrass theorem, we can find a convergent subsequence $\{a_{n_k}\}$ such that $\lim_{k\to\infty} a_{n_k} = a$. We show that we also have $\lim_{n\to\infty} a_n = a$.

For every $\epsilon > 0$, we can find N_1 such that for all $m, n \geq N_1$, we have $|a_m - a_n| < \epsilon/2$. Since $\lim_{k\to\infty} a_{n_k} = a$, there is some K such that for all $k \geq K$, we have $|a_{n_k} - a| < \epsilon/2$. Take

$$N \geq \max\{N_1, n_K\}.$$

[1] The Cauchy condition controls the *oscillation* of the *tail* of a sequence.

We can find K_0 such that $n_{K_0} \geq N$. Then for every $n \geq N$,

$$|a_n - a| = |a_n - a_{n_{K_0}} + a_{n_{K_0}} - a| \leq |a_n - a_{n_{K_0}}| + |a_{n_{K_0}} - a| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

by the triangle inequality and the Cauchy condition. □

Remark. The Cauchy condition allows us to show that a sequence converges without explicitly providing its limit.

7.2 Revisiting Completeness

This is the way we have discussed completeness (ordered by implication):

- Axiom of Completeness
  - Nested intervals property
    - Bolzano-Weierstrass theorem
      - Cauchy criterion
  - Monotone convergence theorem.

But this is not the only way to do so: We have several ways of choosing axioms to define completeness. For example, we can also prove the nested intervals property using the monotone convergence theorem.

Exercise 7.1. The monotone convergence theorem implies the nested intervals property.

Proof. Let $I_n = [a_n, b_n]$ with $I_{n+1} \subseteq I_n$. In particular, $\{a_n\}$ is increasing and bounded (b_1 is an upper bound). So by the monotone convergence theorem, $\lim_{n\to\infty} a_n = a$ exists.

Left as an exercise to show that $a \in I_n$ for all n. □

Exercise 7.2. Given the Archimedean property, the nested intervals property implies the Axiom of Completeness.

Proof. Note that $\frac{1}{2^n} \to 0$ as $n \to \infty$. This is because for every $\epsilon > 0$, we can find N such that $\frac{1}{N} < \epsilon$ by the Archimidean property. Then

$$\frac{1}{2^N} < \frac{1}{N}$$

for all $N \in \mathbb{N}$. So $\lim_{n\to\infty} \frac{1}{2^n} = 0$.

Now let S be a nonempty set which is bounded above. Let U be an upper bound for S. Take $s \in S$. Set $a_1 = s$, $b_1 = U$. Consider

$$\frac{s+U}{2}.$$

If $\frac{s+U}{2}$ is an upper bound for S, then we set $a_2 = a_1 = s$, $b_2 = \frac{s+U}{2}$. If $\frac{s+U}{2}$ is not an upper bound for S, then we set $a_2 = \frac{s+U}{2}$, $b_2 = b_1 = U$. Note that $[a_2, b_2] \subseteq [a_1, b_1]$. Repeat the same process for a_n and b_n to obtain the closed intervals

$$[a_1, b_1] \supseteq [a_2, a_2] \supseteq [a_3, b_3] \supseteq \dots.$$

By the nested interval properties, the intersection $\bigcap_{n=1}^{\infty}[a_n, b_n]$ is nonempty. Note that

$$\begin{aligned}
|[a_1, b_1]| &= |b_1 - a_1| = |U - s| \\
|[a_2, b_2]| &= |b_2 - a_2| = \left|\frac{U - s}{2}\right| \\
&\vdots \\
|[a_n, b_n]| &= |b_n - a_n| = \frac{2}{2^n}|U - s|
\end{aligned}$$

So there is only one $x \in \mathbb{R}$ such that $x \in \bigcap_{n=1}^{\infty}[a_n, b_n]$. We claim that $\sup S = x$.

Note that $x \in [a_n, b_n]$ for all n. So a_n is not an upper bound and b_n is an upper bound. Suppose for contradiction that x is not an upper bound. Then there exists $s_0 \in S$ such that $s_0 > x$. Since $|[a_n, b_n]| \to 0$, there exists an N such that whenever $n \geq N$,

$$|[a_n, b_n]| < \frac{1}{2}|s_0 - x|.$$

Since $x \in [a_n, b_n]$, this implies that $s_0 > b_n$, which is a contradiction with b_n being an upper bound.

Use a similar idea to show that x is the *least* upper bound. □

Remark. These are all different ways to understand the same idea of completeness.

Lecture 8

Sept. 14 – Series

Definition 8.1. Let $\{b_n\}$ be a sequence. An infinite **series** is formally given by

$$\sum_{n=1}^{\infty} b_n = b_1 + b_2 + \dots.$$

Definition 8.2. We define the **partial sum** of a series by

$$s_m = \sum_{n=1}^{m} b_n.$$

8.1 Convergence of Series

Definition 8.3. The series $\sum_{n=1}^{\infty} b_n$ **converges** to B if $\lim_{m\to\infty} s_m = B$. Otherwise we say that the series **diverges**.

Example 8.3.1. Consider the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3}^2 + \dots.$$

We look at the partial sums for $m > 1$:

$$\begin{aligned} s_m = \sum_{n=1}^{m} \frac{1}{n^2} &= 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots + \frac{1}{m^2} \leq 1 + \frac{1}{2(1)} + \frac{1}{3(2)} + \frac{1}{4(3)} + \dots + \frac{1}{m(m-1)} \\ &= 1 + 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \dots + \frac{1}{m-1} - \frac{1}{m} \leq 2 - \frac{1}{m}. \end{aligned}$$

Note that $\{s_m\}$ is a monotone sequence and it is bounded above by 2. Thus by the monotone convergence theorem, $\{s_m\}$ converges and there is some $B \in \mathbb{R}$ such that $\lim_{m\to\infty} s_m = B$.

Remark. Using some complex analysis, we can find B by way of residue calculations.

Example 8.3.2. Consider the harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \dots.$$

We look at the partial sums

$$s_m = 1 + \frac{1}{2} + \cdots + \frac{1}{m}.$$

Note specifically that

$$\begin{aligned}
s_4 &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} > 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} = 1 + \frac{1}{2} + 2\left(\frac{1}{4}\right) = 1 + \frac{1}{2} + \frac{1}{2} = 1 + 2\left(\frac{1}{2}\right) \\
s_8 &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} > 1 + \frac{1}{2} + 2\left(\frac{1}{4}\right) + 4\left(\frac{1}{8}\right) = 1 + 3\left(\frac{1}{2}\right) \\
&\vdots \\
s_{2^k} &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{2^{k-1}+1} + \frac{1}{2^{k-1}+2} + \cdots + \frac{1}{2^k} > 1 + \frac{k}{2}
\end{aligned}$$

Thus $\{s_{2^k}\}$ diverges, so $\{s_m\}$ also diverges.

Remark. This type of trick (analyzing 2^k terms) is called *dyadic analysis*, and it shows up frequently in analysis, particularly harmonic analysis.

Theorem 8.1 (Cauchy condensation test)**.** *Suppose $\{b_n\}$ is decreasing and $b_n \geq 0$ for all n. Then*

$$\sum_{n=1}^{\infty} b_n = b_1 + b_2 + b_3 + \ldots$$

converges if and only if

$$\sum_{n=0}^{\infty} 2^n b_{2^n} = b_1 + 2b_2 + 4b_4 + \ldots$$

converges.

Proof. First we show the backwards direction. Assume $\sum_{n=0}^{\infty} 2^n b_{2^n}$ converges. Define

$$t_k = b_1 + \cdots + 2^k b_{2^k}.$$

By assumption, $\{t_k\}$ converges. Note that $t_k \geq 0$ and $\sup_k t_k \leq M$ since convergent series are bounded. Set

$$s_m = \sum_{n=1}^{m} b_n.$$

Fix m and take k large such that $m \leq 2^{k+1} - 1$. Then $s_m \leq s_{2^{k+1}-1}$ since $b_n \geq 0$. Observe that

$$\begin{aligned}
s_{2^{k+1}-1} &= b_1 + (b_2 + b_3) + (b_4 + b_5 + b_6 + b_7) + \cdots + (b_{2^k} + \cdots + b_{2^{k+1}-1}) \\
&\leq b_1 + 2b_2 + 4b_4 + \cdots + 2^k b_{2^k}.
\end{aligned}$$

So $s_m \leq 2^{k+1} - 1 \leq t_k \leq M$. Thus $\{s_m\}$ is increasing and bounded, so by monotone convergence, $\lim_{m\to\infty} s_m = B \in \mathbb{R}$ exists.

Now we show the forwards direction. Argue by contraposition. Suppose $\sum_{n=0}^{\infty} 2^n b_{2^n}$ diverges, then we show that $\sum_{n=1}^{\infty} b_n$ also diverges. Just need to check that $s_{2^k} \geq \frac{1}{2} + k$ (left as an exercise). $\square$

Corollary 8.1.1. *The series*

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

converges if and only if $p > 1$.

Proof. Let $b_n = \frac{1}{n^p}$ and $b_{2^n} = \frac{1}{2^{np}}$. Then we have

$$\sum_{n=0}^{\infty} 2^n b_{2^n} = \sum_{n=0}^{\infty} 2^{(1-p)n}.$$

The RHS is a geometric series, which converges if and only if $p > 1$. To see this, denote $2^{1-p} = a$. Then we have

$$\sum_{n=0}^{\infty} 2^{(1-p)n} = \sum_{n=0}^{\infty} a^n.$$

We can observe that the partial sums

$$t_k = \sum_{n=0}^{k} a^n = \frac{a^{k+1} - 1}{a - 1}$$

converges if and only if a^{k+1} converges. This happens if and only if $a < 1$, which happens if and only if $p > 1$. $\square$

8.2 Properties of Series

Theorem 8.2 (Algebraic limit theorem for series). *Let*

$$\sum_{n=1}^{\infty} a_n = A, \quad \sum_{n=1}^{\infty} b_n = B.$$

Then for all $c \in \mathbb{R}$, we have

$$\sum_{n=1}^{\infty} ca_n = cA, \quad \sum_{n=1}^{\infty} (a_n + b_n) = A + B.$$

Proof. Let $\sum_{n=1}^{\infty} a_n = A$. So $s_m = \sum_{n=1}^{m} a_n$ converges. Set $\lim_{n\to\infty} s_m = A$. Define

$$t_m = \sum_{n=1}^{m} ca_n = c\sum_{n=1}^{m} a_n = cs_m.$$

Then by the algebraic limit theorem, we have $\lim_{m\to\infty} t_m = c\lim_{m\to\infty} s_m = cA$. $\square$

Theorem 8.3 (Cauchy criterion for series). *The series $\sum_{n=1}^{\infty} a_n$ converges if and only if for all $\epsilon > 0$, there exist N such that whenever $m, n \geq N$, we have $|a_{m+1} + \cdots + a_n| < \epsilon$.*

Proof. The series $\sum_{k=1}^{\infty} a_k$ converges if and only if $s_m = \sum_{k=1}^{m} a_k$ converges. We show that $\{s_m\}$ is a Cauchy sequence. For all $\epsilon > 0$, there exists N such that for all $m, n \geq N$

$$|s_n - s_m| = |a_n + \cdots + a_{m+1}| < \epsilon.$$

The converse is the same inequality. □

Corollary 8.3.1. *If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.*

Proof. Take $m = n - 1$. □

Theorem 8.4. *Assume $\{a_n\}$ and $\{b_n\}$ are sequences such that $0 \leq a_n \leq b_n$ for all n. Then*

1. *$\sum_{n=1}^{\infty} b_n$ converges implies $\sum_{n=1}^{\infty} a_n$ converges,*
2. *and $\sum_{n=1}^{\infty} a_n$ diverges implies $\sum_{n=1}^{\infty} b_n$ diverges.*

Proof. For all m, n, we have

$$|a_{m+1} + \cdots + a_n| \leq |b_{m+1} + \cdots + b_n|.$$

Then apply the Cauchy criterion. □

Definition 8.4. A series is called **geometric** if it is of the form

$$\sum_{k=0}^{\infty} ar^k = a + ar + ar^2 + \ldots.$$

Note that the geometric series diverges when $r = 1$ and $a \neq 0$. When $r \neq 1$, the partial sums

$$s_m = \sum_{k=0}^{m} ar^k = a\frac{1 - r^{m+1}}{1 - r}$$

converge if $|r| < 1$. In this case, as $m \to \infty$, we have

$$s_m \to \frac{a}{1 - r}.$$

Lecture 9

Sept. 19 – Absolute Convergence

9.1 Absolute Convergence

Definition 9.1. Consider a series $\sum_{n=1}^{\infty} a_n$. If

$$\sum_{n=1}^{\infty} |a_n|$$

converges, then we say $\sum_{n=1}^{\infty} a_n$ **converges absolutely**.

Theorem 9.1. *If $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.*

Proof. For every $\epsilon > 0$, since $\sum_{n=1}^{\infty} a_n$ converges, there is N such that for all $m, k \geq N$,

$$\sum_{n=k+1}^{m} |a_n| < \epsilon.$$

This is by the Cauchy criterion for series. Then for all $m, k \geq N$, we have

$$\left| \sum_{n=k+1}^{m} a_n \right| \leq \sum_{n=k+1}^{m} |a_n| < \epsilon$$

by the triangle inequality. Applying the Cauchy criterion again, we conclude that $\sum_{n=1}^{\infty} a_n$ converges. $\square$

Theorem 9.2 (Alternating series test)**.** *If $a_1 \geq a_2 \geq a_3 \geq \dots$ and $\lim_{n\to\infty} a_n = 0$, then the series*

$$\sum_{n=1}^{\infty} (-1)^{n+1} a_n$$

converges.

Proof. Set $s_m = \sum_{n=1}^{m} (-1)^{n+1} a_n$. Check that

$$s_m - s_k = \sum_{n=k+1}^{m} (-1)^{n+1} a_n.$$

Suppose that m and k are odd, then

$$s_m - s_k = \underbrace{a_m - a_{m-1}}_{\leq 0} + a_{m-2} - \cdots + a_{k+2} - a_{k+1}.$$

So $s_m - s_k \leq 0$. We can also group the terms as as

$$s_m - s_k = a_m \underbrace{-a_{m-1} + a_{m-2}}_{\geq 0} - \cdots - a_{k+3} - a_{k+2} - a_{k+1} \geq a_m - a_{k+1}.$$

So $|s_m - s_k| \leq |a_m| + |a_{k+1}|$ by the triangle inequality. Since $\lim_{n\to\infty} a_n = 0$, for all $\epsilon > 0$, there is N such that $n \geq N$, we have $|a_n| < \epsilon$. Then for all $m, k \geq N$,

$$|s_m - s_k| \leq |a_m| + |a_{k+1}| < 2\epsilon.$$

Thus $\{s_k\}$ converges. Left as exercise to check the other parities of m and k (group differently). □

Example 9.1.1. We saw previously that for $a_n = \frac{1}{n}$, $\sum_{n=1}^{\infty} a_n$ diverges. But $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ converges.

9.2 Rearrangements

Definition 9.2. Given a series $\sum_{k=1}^{\infty} a_k$, we say that a series $\sum_{k=1}^{\infty} b_k$ is a **rearrangement** of $\sum_{k=1}^{\infty} a_k$ if there is a bijection $f : \mathbb{N} \to \mathbb{N}$ such that $b_{f(k)} = a_k$ for all $k \in \mathbb{N}$.

Example 9.2.1. Let

$$\begin{aligned} S &= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \cdots, \\ \frac{1}{2}S &= \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \frac{1}{10} - \cdots, \\ S + \frac{1}{2}S &= 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} - \frac{1}{4} + \frac{1}{7} - \cdots. \end{aligned}$$

Notice that $S + \frac{1}{2}S$ is a rearrangment of S. Supposing that $S + \frac{1}{2}S$ converges to the same limit as S, we would have

$$S + \frac{1}{2}S = S,$$

or $S = 0$. This cannot be the case.

Remark. A rearrangement of a series might have different convergence properties from the original series.

Theorem 9.3. *If a series converges absolutely to A, then any rearrangement of the series converges to the same limit A.*

Proof. Let $\sum_{k=1}^{\infty} a_k$ converge absolutely to A. Let $\sum_{k=1}^{\infty} b_k$ be a rearrangement of $\sum_{k=1}^{\infty} a_k$. We set

$$s_n = \sum_{k=1}^{n} a_k, \quad t_m = \sum_{k=1}^{m} b_k.$$

We want to show that t_m converges to A. Since $\lim_{n\to\infty} s_n = A$, for every $\epsilon > 0$, there is N_1 such that

$$|s_n - A| < \frac{\epsilon}{2}$$

for all $n \in \mathbb{N}$. Since $\sum_{k=1}^{\infty} a_k$ converges absolutely, there is N_2 such that for all $n, m \geq N_2$, we have

$$\sum_{k=m+1}^{n} |a_k| < \frac{\epsilon}{2}.$$

Since $\sum_{k=1}^{\infty} b_k$ is a rearrangement of $\sum_{k=1}^{\infty} a_k$, we can write $b_{f(k)} = a_k$ for some bijection f. Set

$$N = \max\{N_1, N_2\}, \quad M = \max\{f(k) : 1 \leq k \leq N\}.$$

Then for all $m \geq M$, $t_m - s_n$ will only consist of terms a_k for $k > N$. In particular,

$$|t_m - s_n| \leq \sum_{k=n}^{\infty} |a_k| < \frac{\epsilon}{2}.$$

Then we have

$$|t_m - A| = |t_m - s_n + s_n - A| \leq |t_m - s_n| + |s_n - A| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

So $\lim_{m\to\infty} t_m = A$. □