

# MATH 4317: Analysis I

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# Lecture 1

## Aug. 22 – The Real Numbers

### 1.1 Number Systems

We start with the *natural numbers* <sup>1</sup>

$$\mathbb{N} = \{1, 2, 3, \dots\}.$$

These are perhaps the most natural in a way, since they are what we use to count things. They are closed under addition, but fail when it comes to subtraction. For example,  $1 - 2 = -1 \notin \mathbb{N}$ . So we must expand our number system to the *integers*

$$\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}.$$

We can now add, subtract, and multiply. But we run into problems when we start to consider quotients. For example,  $1 \div 2 = \frac{1}{2} \notin \mathbb{Z}$ . So we continue to the *rational numbers*

$$\mathbb{Q} = \left\{ \frac{p}{q} : p, q \in \mathbb{Z}, q \neq 0 \right\}.$$

We now have summation, subtraction, multiplication, and quotients. But there is still a problem. For example, consider the diagonal of a square with side length 1.

**Theorem 1.1.**  $\sqrt{2}$  is not a rational number. <sup>2</sup>

*Proof.* Argue by contradiction. Suppose  $\sqrt{2}$  is rational. Then we can write

$$\sqrt{2} = \frac{p}{q}$$

for some integers  $p, q$ . Further assume  $p$  and  $q$  have no common factors. Then

$$2 = \frac{p^2}{q^2} \implies p^2 = 2q^2.$$

So  $p$  is even and we can write  $p = 2r$  for some  $r \in \mathbb{Z}$ . Then

$$4r^2 = 2q^2 \implies 2r^2 = q^2.$$

So  $q$  is also even, and  $p, q$  share a common factor of 2. Contradiction. □

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<sup>1</sup>  $0 \notin \mathbb{N}$  for this class.

<sup>2</sup> In some sense, this shows that the notion of “rationals” is strictly weaker than the notion of “length.”

Another weakness of  $\mathbb{Q}$  is that we cannot take limits ( $\mathbb{Q}$  is not complete). For example, note that

$$(\sqrt{2} - 1)(\sqrt{2} + 1) = 2 - 1 = 1,$$

$$\sqrt{2} = 1 + \frac{1}{\sqrt{2} + 1} = 1 + \frac{1}{1 + 1 + \frac{1}{\sqrt{2} + 1}} = \dots$$

So if we define the rational sequence

$$a_1 = 1, \quad a_2 = 1 + \frac{1}{2}, \quad a_3 = 1 + \frac{1}{2 + \frac{1}{2}}, \quad a_4 = 1 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2}}}, \quad \dots,$$

then as  $n \rightarrow \infty$ ,  $a_n \rightarrow \sqrt{2} \notin \mathbb{Q}$ . Similarly,  $\sqrt[3]{2}, \sqrt{3}, \sqrt[5]{3}, \dots$  are all irrational.

## 1.2 Sets

Sets are any collections of objects. Given a set  $A$ , we write  $x \in A$  if  $x$  is an element of  $A$ . We write  $x \notin A$  otherwise. The **union** of two sets is

$$A \cup B = \{x : x \in A \text{ or } x \in B\},$$

and the **intersection** of two sets is

$$A \cap B = \{x : x \in A \text{ and } x \in B\}.$$

We use the notation

$$\bigcup_{k=1}^{\infty} A_k$$

to denote the countable union of a family of sets indexed by  $k$ .

## 1.3 Functions

**Definition 1.1.** Given two sets  $A$  and  $B$ , a **function** from  $A$  to  $B$  is a rule, relation, or mapping that takes each element  $x \in A$  and associates with it a single element in  $B$ . In this case, we write  $f : A \rightarrow B$ .

We call  $A$  the **domain** of  $f$  and  $B$  the **codomain** of  $f$ . The element in  $B$  associated with  $x \in A$  is  $f(x)$ , called the **image** of  $x$ . The **range** of  $f$  is

$$\text{range}(f) = \{y \in B : y = f(x) \text{ for some } x \in A\}.$$

We say  $f$  is:

1. **onto** or **surjective** if  $\text{range}(f) = B$ .
2. **one-to-one** or **injective** if  $x, x' \in A$  and  $x \neq x'$ , then  $f(x) \neq f(x')$ .
3. **bijective** if it is injective and surjective.

**Example 1.1.1.** First Dirichlet function:

$$g(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases} = \lim_{k \rightarrow \infty} \left( \lim_{j \rightarrow \infty} [\cos(k! \pi x)]^{2j} \right).$$

**Example 1.1.2.** Second Dirichlet function:

$$f(x) = \begin{cases} \frac{1}{q} & \text{if } x = \frac{p}{q} \in \mathbb{Q} \text{ in lowest terms} \\ 0 & \text{if } x \notin \mathbb{Q}. \end{cases}$$

**Example 1.1.3.** Absolute value:

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0. \end{cases}$$

Note that we have the following two properties:

- $|xy| = |x||y|$ .
- $|x + y| \leq |x| + |y|$ . This is called the *triangle inequality*.

## 1.4 Induction

If we have a set  $S \subseteq \mathbb{N}$  and

1.  $1 \in S$
2. if  $n \in S$ , then  $n + 1 \in S$

then  $S = \mathbb{N}$ .<sup>3</sup>

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<sup>3</sup>We always use induction in conjunction with  $\mathbb{N}$ .

# Lecture 2

## Aug. 24 – The Axiom of Completeness

The number system  $\mathbb{Q}$  is pretty good (it is a field), but recall that we are unable to take limits. For instance, take the sequence  $x_0 = 2$  and

$$x_{n+1} = \frac{1}{2} \left( x_n + \frac{2}{x_n} \right)$$

for  $n \geq 1$ . All the  $x_i$  are rational, but  $x_n \rightarrow \sqrt{2} \notin \mathbb{Q}$ . This shows that there are gaps in  $\mathbb{Q}$ . The real numbers  $\mathbb{R}$  will fill these gaps (completeness).

**Axiom 2.1** (Axiom of completeness). *Every nonempty set of real numbers that are bounded above has a least upper bound.*

Note that this least upper bound is *unique*.

### 2.1 Suprema and Infima

**Definition 2.1.** Let  $S \subseteq \mathbb{R}$ . The set  $S$  is **bounded above** if there exists  $u \in \mathbb{R}$  such that  $s \leq u$  for all  $s \in S$ . We say that  $u$  is an **upper bound** of  $S$ .

We define **bounded below** and **lower bound** similarly.

**Definition 2.2.**  $S$  is said to be **bounded** if it is both bounded above and below. Otherwise we say that  $S$  is **unbounded**.

**Example 2.2.1.**  $\mathbb{N} = \{1, 2, 3, \dots\}$  is bounded below but not above.

**Example 2.2.2.** The set

$$\left\{ \frac{1}{k} : k \in \mathbb{N} \right\} = \left\{ 1, \frac{1}{2}, \frac{1}{3}, \dots \right\}$$

is bounded.

**Example 2.2.3.**  $\emptyset$  is bounded.

**Definition 2.3.** We say  $u \in \mathbb{R}$  is the **least upper bound** or **supremum** of a nonempty set  $S \subseteq \mathbb{R}$  if

1.  $u$  is an upper bound of  $S$ .
2.  $u \leq v$  for any upper bound  $v$  of  $S$ .

We write  $u = \sup S$ .

The **greatest lower bound** or **infimum** of  $S$  is defined similarly, denoted  $\inf S$ .



**Example 2.3.1.**

$$S = \left\{ \frac{1}{k} : k \in \mathbb{N} \right\}.$$

$\sup S = 1, \inf S = 0.$

**Definition 2.4.** Let  $S \subseteq \mathbb{R}$ . We say a real number  $M \in S$  is a **maximal element** or **maximum** of  $S$  if  $s \leq M$  for all  $s \in S$ .

The **minimal element** or **minimum** is defined similarly.

**Example 2.4.1.**  $[0, 1)$  is bounded, but has no maximum. The minimum is 0.

**Example 2.4.2.** The set

$$\{2^{-n} : n \in \mathbb{N}\} = \left\{ \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots \right\}$$

is bounded, but has no minimum. The maximum is  $\frac{1}{2}$ .

**Example 2.4.3.**  $\emptyset$  is bounded but has no minimum or maximum.

**Exercise 2.1.** Let  $A \subseteq \mathbb{R}$  be bounded above. Let  $c \in \mathbb{R}$  and define

$$c + A := \{a + c : a \in A\}.$$

Then  $\sup(A + c) = c + \sup A$ .

*Proof.* Let  $s = \sup A$ . By definition, we know  $a \leq s$  for all  $a \in A$ , which implies  $a + c \leq s + c$ . So  $s + c$  is an upper bound for  $c + A$ . Now let  $b$  be an arbitrary upper bound for  $c + A$ . For all  $a \in A$ , we have  $a + c \leq b$ , which implies  $a \leq b - c$ . So  $b - c$  is an upper bound for  $A$ . By construction,  $s \leq b - c$ , so  $s + c \leq b$ . Therefore  $s + c = \sup(A + c)$ .  $\square$

**Lemma 2.1.** Assume  $s \in \mathbb{R}$  is an upper bound for a set  $A \subseteq \mathbb{R}$ . Then  $s = \sup A$  if and only if for every  $\epsilon > 0$ , there exists  $a \in A$  such that  $s - \epsilon < a$ .

*Proof.* ( $\Rightarrow$ ) Suppose  $\sup A = s$ . Then given any  $\epsilon > 0$ ,  $s - \epsilon$  cannot be an upper bound for  $A$ . So there exists  $a \in A$  such that  $a > s - \epsilon$ .

( $\Leftarrow$ ) Let  $b$  be an arbitrary upper bound for  $A$ . Suppose for contradiction that  $b < s$ . Set  $\epsilon = s - b > 0$ . Then by assumption we can find  $a \in A$  such that  $a > s - \epsilon = b$ . Contradiction. Therefore  $b \geq s$ , whence  $\sup A = s$ .  $\square$

## 2.2 Consequences of Completeness

### 2.2.1 1st Consequence: Nested Interval Properties

**Theorem 2.1** (Nested interval properties). For any  $n \in \mathbb{N}$ , assume that we are given a closed interval

$$I_n = [a_n, b_n] = \{x \in \mathbb{R} : a_n \leq x \leq b_n\}.$$

Assume  $I_n \supseteq I_{n+1}$ . Then the resulting nested sequence of closed intervals

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq \dots$$

has a nonempty intersection:

$$\bigcap_{n=1}^{\infty} I_n \neq \emptyset.$$

*Proof.* Define  $A = \{a_n\}$ . Note that  $A \neq \emptyset$ . For any  $n$ ,  $a_n \leq b_n \leq b_1$ . So  $x = \sup A$  exists. Furthermore, for any  $n$ ,  $b_n$  is an upper bound for  $A$ . So  $x \leq b_n$ . Since  $x = \sup A$ ,  $a_n \leq x$ . So  $x \in [a_n, b_n]$  for any  $n$ , whence

$$x \in \bigcap_{n=1}^{\infty} I_n.$$

Therefore, the intersection is nonempty. □

## 2.2.2 2nd Consequence: Archimedean Properties

**Theorem 2.2** (Archimedean properties).

1. Given any  $x \in \mathbb{R}$ , there is an  $n \in \mathbb{N}$  such that  $n > x$ .<sup>1</sup>
2. Given any real number  $y > 0$ , there is an  $\mathbb{N}$  such that  $\frac{1}{n} < y$ .

*Proof of (1).* Argue by contradiction. Suppose  $\mathbb{N}$  is bounded above. Then by the axiom of completeness,  $\alpha = \sup \mathbb{N}$  exists. By construction,  $\alpha - 1$  is not an upper bound for  $\mathbb{N}$ . So we can find  $n \in \mathbb{N}$  such that  $\alpha - 1 < n$ , which implies  $\alpha < n + 1 \in \mathbb{N}$ . Contradiction. □

*Proof of (2).* Follows from (1) by setting  $x = \frac{1}{y}$ . □

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<sup>1</sup>This is saying that  $\mathbb{N}$  is not bounded above.

# Lecture 3

## Aug. 29 – Completeness, Countability

### 3.1 Consequences of Completeness

#### 3.1.1 3rd Consequence: Density of $\mathbb{Q}$ in $\mathbb{R}$

**Theorem 3.1** (Density of  $\mathbb{Q}$  in  $\mathbb{R}$ ). *For all  $a, b \in \mathbb{R}$ ,  $a < b$ , there exists  $r \in \mathbb{Q}$  such that  $a < r < b$ .*

*Proof.* We want to find  $m \in \mathbb{Z}$ ,  $n \in \mathbb{N}$  such that

$$a < \frac{m}{n} < b.$$

By (2) of the Archimedean properties, we can find  $n \in \mathbb{N}$  such that

$$\frac{1}{n} < b - a.$$

Fix such an  $n$ . Then let  $m$  be the smallest integer such that  $m - 1 \leq na < m$ . By construction,

$$\begin{aligned} \frac{m}{n} - \frac{1}{n} &\leq a < \frac{m}{n}, \\ \frac{m}{n} &\leq a + \frac{1}{n} < b. \end{aligned}$$

Therefore,  $a < \frac{m}{n} < b$ . □

**Corollary 3.1.1.** *For all  $a, b \in \mathbb{Q}$ ,  $a < b$ , there exists  $t \in \mathbb{R} \setminus \mathbb{Q}$  such that  $a < t < b$ .*

#### 3.1.2 4th Consequence: Existence of $\sqrt{2}$

**Theorem 3.2** (Existence of  $\sqrt{2}$ ). *There exists  $s \in \mathbb{R}$ ,  $s > 0$  such that  $s^2 = 2$ .*

*Proof.* Define

$$S = \{x > 0 : x^2 < 2\} \subseteq \mathbb{R}.$$

$x = 1 \in S$ , so  $S \neq \emptyset$ . 2 is an upper bound for  $S$ , so  $S$  is bounded above. Then by the axiom of completeness,  $s = \sup S$  exists. We claim that  $s^2 = 2$ .

Suppose otherwise that  $s^2 < 2$ . Then we can find  $\epsilon > 0$  such that  $s + \epsilon \in S$ . Define  $\delta = 2 - s^2 > 0$ . Note that

$$(s + \epsilon)^2 - 2 = s^2 + 2s\epsilon + \epsilon^2 - 2 = -\delta + 2s\epsilon + \epsilon^2.$$

We know  $s \leq 2$  since 2 is an upper bound. Pick

$$\epsilon = \frac{\delta}{10000000000},$$

$$2s\epsilon + \epsilon \leq 4\epsilon + \epsilon^2 < \frac{\delta}{2}.$$

Then

$$(s + \epsilon)^2 - 2 < -\delta + \frac{\delta}{2} = -\frac{\delta}{2} < 0.$$

So  $s + \epsilon \in S$ , which contradicts with  $s = \sup S$ .

$s^2 > 2$  also leads to a contradiction (left as an exercise). Thus we must have  $s^2 = 2$ . □

## 3.2 Countability

**Definition 3.1.** We say two sets  $A$  and  $B$  have the same **cardinality** if there is a bijection  $f : A \rightarrow B$ . We write  $A \sim B$ .

**Definition 3.2.** We say that a set  $A$  is **finite** if  $A \sim \{1, 2, \dots, n\}$  for some integer  $n$ . We say that a set  $A$  is **countable** (or countably infinite) if  $A \sim \mathbb{N}$ . If a set  $A$  is not countable, then we say it is **uncountable**.

**Example 3.2.1.** Let  $E = \{2, 4, 6, 8, \dots\}$ .  $E$  is not finite but it is countable:  $E \sim \mathbb{N}$ . We can define  $f : \mathbb{N} \rightarrow E$  by  $f(n) = 2n$ .

**Example 3.2.2.**  $\mathbb{N} \sim \mathbb{Z}$ . The bijection  $f : \mathbb{N} \rightarrow \mathbb{Z}$  is given by

$$f(n) = \begin{cases} \frac{n-1}{2} & n \text{ is odd} \\ -\frac{n}{2} & n \text{ is even.} \end{cases}$$

**Example 3.2.3.**  $(-1, 1) \sim \mathbb{R}$ . The bijection  $f : (-1, 1) \rightarrow \mathbb{R}$  is given by

$$x \mapsto \frac{x}{x^2 - 1}.$$

**Theorem 3.3.**

1. The set  $\mathbb{Q}$  is countable.
2. The set  $\mathbb{R}$  is uncountable.

*Proof of (1).* Set  $A_1 = \{0\}$  and for  $n \geq 2$ ,

$$A_n = \left\{ \pm \frac{p}{q} : p, q \in \mathbb{N}, p, q \text{ in lowest terms, } p + q = n \right\}.$$

So the first few  $A_n$  are:

$$A_2 = \left\{ \frac{1}{1}, \frac{-1}{1} \right\},$$

$$A_3 = \left\{ \frac{1}{2}, \frac{2}{1}, \frac{-1}{2}, \frac{-2}{1} \right\},$$

etc. Note that  $A_n$  is finite and for all  $x \in \mathbb{Q}$ , there is an  $n \in \mathbb{N}$  such that  $x \in A_n$ . We can list elements in  $A_1, \dots, A_n$  and label them with integers in  $\mathbb{N}$ . Any element of  $A_n$  will be listed eventually. Then this pairing gives a bijection since the  $A_n$  are disjoint. So  $\mathbb{Q} \sim \mathbb{N}$ .  $\square$

*Proof of (2).* Argue by contradiction. Suppose  $f$  is one-to-one from  $\mathbb{N} \rightarrow \mathbb{R}$ . Set  $x_1 = f(1)$ ,  $x_2 = f(2)$ , etc. We can write

$$\mathbb{R} = \{x_1, x_2, \dots\}.$$

Let  $I_1$  be a closed interval such that  $x_1 \notin I_1$ . Pick  $I_2 \subseteq I_1$  such that  $x_2 \notin I_2$ . Continue this process such that  $I_{n+1} \subseteq I_n$  is a closed interval where  $x_{n+1} \notin I_{n+1}$ . By construction,

$$I_1 \supseteq I_2 \supseteq \dots \supseteq I_n \supseteq \dots$$

We know that

$$\bigcap_{n=1}^{\infty} I_n \neq \emptyset.$$

So we can find  $n_0$  such that

$$x_{n_0} \in \bigcap_{n=1}^{\infty} I_n.$$

This is a contradiction with  $x_{n_0} \notin I_{n_0}$ . Thus such an  $f$  cannot exist and  $\mathbb{R}$  is uncountable.  $\square$

### Theorem 3.4.

1. Let  $A \subseteq B$ . If  $B$  is countable, then  $A$  is either finite or countable.
2. If  $A_n$  is a countable set, then

$$\bigcup_{n=1}^{\infty} A_n$$

is also countable.

**Theorem 3.5** (Cantor's diagonal argument). *The open interval*

$$(0, 1) = \{x \in \mathbb{R} : 0 < x < 1\}$$

*is uncountable.*

*Proof.* Argue by contradiction. Assume  $f : \mathbb{N} \rightarrow (0, 1)$  is one-to-one and onto. Then for  $m \in \mathbb{N}$ , we can write (decimal expansion)

$$f(m) = 0.a_{m1}a_{m2}a_{m3}\dots \in (0, 1).$$

For every  $m, n \in \mathbb{N}$ ,  $a_{mn} \in \{0, \dots, 9\}$  is the  $n$ th digit in the decimal expansion of  $f(m)$ . We can write in a table

$$\begin{array}{cccccc} 1 & f(1) & a_{11} & a_{12} & a_{13} & \dots \\ 2 & f(2) & a_{21} & a_{22} & a_{23} & \dots \\ 3 & f(3) & a_{31} & a_{32} & a_{33} & \dots \\ & & & & & \vdots \end{array}$$

Take  $x = 0.b_1b_2b_3\dots$  where

$$b_n = \begin{cases} 2 & \text{if } a_{nn} \neq 2 \\ 3 & \text{if } a_{nn} = 2. \end{cases}$$

Then  $x \neq f(m)$  for any  $m \in \mathbb{N}$  (since  $b_m \neq a_{mm}$ ). This is a contradiction.

□

# Lecture 4

## Aug. 31 – Cantor’s Theorem, Sequences

### 4.1 Cantor’s Theorem

**Definition 4.1.** The **power set** of  $A$ , denoted  $\mathcal{P}(A)$ , is the collection of all subsets of  $A$ .

**Theorem 4.1** (Cantor’s theorem). *Given any set  $A$ , there does not exist a function  $f : A \rightarrow \mathcal{P}(A)$  which is surjective.*<sup>1</sup>

*Proof.* Argue by contradiction. Suppose  $f : A \rightarrow \mathcal{P}(A)$  is onto. Then for any  $a \in A$ ,  $f(a)$  is a subset of  $A$ . Since  $f$  is onto, for any subset  $B$  of  $A$ , we can find  $a \in A$  such that  $f(a) = B$ . Define

$$B = \{a \in A : a \notin f(a)\} \subseteq A.$$

We can find  $a' \in A$  such that  $f(a') = B$ . If  $a' \in B$ , then  $a' \notin f(a') = B$ , which is a contradiction.. If  $a' \notin B$ , this is a contradiction with the definition of  $B$ . Thus such  $f$  cannot exist.  $\square$

**Remark.** This means that the cardinality of  $\mathcal{P}(A)$  is strictly larger than that of  $A$ .

### 4.2 Sequences

**Definition 4.2.** A **sequence** is a function whose domain is  $\mathbb{N}$ .

We usually write  $\{a_n\}$ ,  $\{x_n\}$  or  $(a_n)$ ,  $(x_n)$  to denote sequences.

**Example 4.2.1.** The following

$$\left\{ \frac{1+n}{n} \right\}_{n=1}^{\infty} = \left\{ 2, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \dots \right\}$$

is a sequence.

**Example 4.2.2.**  $\{a_n\}$ , where  $a_n = 2^n$  for  $n \in \mathbb{N}$ , is a sequence.

**Example 4.2.3.** We can also define  $\{x_n\}$  recursively by  $x_1 = 2$  and

$$x_{n+1} = \frac{x_n + 1}{2}.$$

**Remark.** Sometimes a sequence is also labeled starting from  $n = 0$ .

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<sup>1</sup>Note that if  $\#(A) = n < \infty$ , this is true as  $\#(\mathcal{P}(A)) = 2^n \neq \#(A)$ .

### 4.2.1 Limits

**Definition 4.3.** A sequence  $\{a_n\}$  **converges** to a real number  $a$  if for every  $\epsilon > 0$ , we can find  $N \in \mathbb{N}$  such that for all  $n \geq N$ , one has  $|a_n - a| < \epsilon$ . We write  $\lim_{n \rightarrow \infty} a_n = a$ .

**Remark.** In analysis,  $\epsilon$  is always taken to be a positive number.

**Example 4.3.1.** The sequence  $\{1/n\}_{n=1}^{\infty}$  converges with

$$\lim_{n \rightarrow \infty} \frac{1}{n} = 0.$$

**Definition 4.4.** For  $\epsilon > 0$ , the  $\epsilon$ -**neighborhood** of  $a$  is defined to be

$$V_{\epsilon}(a) = \{x \in \mathbb{R} : |x - a| < \epsilon\}.$$

**Definition 4.5.** We say that  $a$  is the **limit** of a sequence  $\{a_n\}$  if for every  $\epsilon > 0$ ,  $V_{\epsilon}(a)$  contains all but finitely many elements of  $\{a_n\}$ .<sup>2</sup>

**Remark.** This definition of the limit is equivalent to the definition of convergence.

**Definition 4.6.** A sequence  $\{a_n\}$  that does not converge is said to be **divergent**.

**Theorem 4.2.** *The limit of a sequence, when it exists, must be unique.*

*Proof.* Homework problem. □

**Exercise 4.1.** Show

$$\lim_{n \rightarrow \infty} \frac{n+1}{n}$$

exists and

$$\lim_{n \rightarrow \infty} \frac{n+1}{n} = 1.$$

*Proof.* We show

$$\lim_{n \rightarrow \infty} \frac{n+1}{n} = 1.$$

For every  $\epsilon > 0$ , take  $N \in \mathbb{N}$  such that  $N > \frac{1}{\epsilon}$ . We have for all  $n \geq N$ ,

$$\left| \frac{n+1}{n} - 1 \right| = \left| \frac{1}{n} \right| \leq \frac{1}{N} < \epsilon.$$

Therefore,

$$\lim_{n \rightarrow \infty} \frac{n+1}{n} = 1$$

as desired. □

---

<sup>2</sup>This is the *topological* definition of the limit.



### 4.2.2 Tips for Showing Limits

To show the limit of a sequence, take the following steps:

1. Identify the limit  $a$ . This is always given by the problem or observation.
2.  $\forall \epsilon > 0$ .
3. Find  $N = N(\epsilon)$ . Do this in sketch paper (need computations and manipulations).
4. Set  $N$  as what is found in (3).
5. Check that  $N$  works.

# Lecture 5

## Sept. 5 – Limits and Limit Theorems

### 5.1 Review of Limits

**Example 5.0.1.** Find

$$\lim_{n \rightarrow \infty} \frac{1 + \sqrt{n}}{\sqrt{n}}.$$

*Proof.* We want to show that

$$\lim_{n \rightarrow \infty} \frac{1 + \sqrt{n}}{\sqrt{n}} = 1.$$

Fix  $\epsilon > 0$  and take  $N \in \mathbb{N}$  such that  $N > \frac{1}{\epsilon^2}$ . Then for any  $n > N$ ,

$$\left| \frac{1 + \sqrt{n}}{\sqrt{n}} - 1 \right| \leq \left| \frac{1}{\sqrt{n}} \right| \leq \frac{1}{\sqrt{N}} < \epsilon,$$

as desired. □

How can we understand this using the topological definition? For all  $\epsilon > 0$ , take  $V_\epsilon(1)$ . Pick  $N > \frac{1}{\epsilon^2}$ . Then we claim that  $V_\epsilon(1)$  contains all but at most  $N$  elements of  $\left\{ \frac{\sqrt{n}+1}{\sqrt{n}} \right\}$ . When  $n \geq N$ , we have

$$\left| \frac{\sqrt{n}+1}{\sqrt{n}} - 1 \right| < \epsilon,$$

i.e.  $\frac{\sqrt{n}+1}{\sqrt{n}} \in V_\epsilon(1)$ . So at most  $N$  elements might not be in  $V_\epsilon(1)$ .

### 5.2 Limit Theorems

#### 5.2.1 Algebraic Facts About Limits

**Definition 5.1.** A sequence  $\{x_n\}$  is said to be **bounded** if there exists  $M$  such that  $|x_n| \leq M$  for all  $n$ . Alternatively,  $\sup_n |x_n| \leq M$ .

**Theorem 5.1.** Every convergent sequence is bounded.

*Proof.* Suppose

$$\lim_{n \rightarrow \infty} x_n = l.$$

Take  $\epsilon = 1$ , we can find  $N$  such that for all  $n \geq N$ ,  $|x_n - l| < 1$ . By the triangle inequality,  $|x_n| < |l| + 1$  for  $n \geq N$ . Take

$$M = \max\{|x_1|, |x_2|, \dots, |x_{N-1}|, |l| + 1\}.$$

Then  $|x_n| \leq M$  for all  $n \in \mathbb{N}$ . □

**Theorem 5.2** (Algebraic limit theorem). *If*

$$\lim_{n \rightarrow \infty} a_n = a \quad \text{and} \quad \lim_{n \rightarrow \infty} b_n = b,$$

*then for all  $c \in \mathbb{R}$ ,*

$$(1) \lim_{n \rightarrow \infty} ca_n = ca, \quad (2) \lim_{n \rightarrow \infty} (a_n + b_n) = a + b, \quad \text{and} \quad (3) \lim_{n \rightarrow \infty} a_n b_n = ab.$$

*Furthermore, if  $b \neq 0$ , then*

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \frac{a}{b}. \quad (4)$$

*Proof.* (1) When  $c = 0$ , the result is trivial. When  $c \neq 0$ , for all  $\epsilon > 0$ , we set  $\epsilon' = \frac{\epsilon}{|c|}$ . Since  $\lim_{n \rightarrow \infty} a_n = a$ , we can find  $N_{\epsilon'}$  such that for all  $n \geq N_{\epsilon'}$ ,  $|a_n - a| < \epsilon'$ . When  $n > N_{\epsilon'}$ , we have

$$|ca_n - ca| = |c||a_n - a| < |c|\epsilon' = |c|\frac{\epsilon}{|c|} = \epsilon.$$

So  $\lim_{n \rightarrow \infty} ca_n = ca$ .

(2) For all  $\epsilon > 0$ , since  $a_n \rightarrow a$  and  $b_n \rightarrow b$ , we can find  $N_1$  and  $N_2$  such that when

$$\begin{aligned} n \geq N_1, \quad |a_n - a| &< \frac{\epsilon}{2}, \\ n \geq N_2, \quad |b_n - b| &< \frac{\epsilon}{2}. \end{aligned}$$

Take  $N = \max\{N_1, N_2\}$ . Then for all  $n \geq N$ ,

$$|a_n + b_n - (a + b)| = |a_n - a + b_n - b| \leq |a_n - a| + |b_n - b| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Therefore  $\lim_{n \rightarrow \infty} (a_n + b_n) = a + b$ . □

## 5.2.2 Order Limit Theorem

**Theorem 5.3** (Order limit theorem). *Let  $\{a_n\}$  and  $\{b_n\}$  be sequences such that*

$$\lim_{n \rightarrow \infty} a_n = a \quad \text{and} \quad \lim_{n \rightarrow \infty} b_n = b.$$

*(5) If  $a_n \geq 0$  for every  $n$ , then  $a \geq 0$ . (6) If  $a_n \leq b_n$ , then  $a \leq b$ . (7) If  $a_n \geq c$ , then  $a \geq c$ .*

*Proof.* (5) Argue by contradiction. Suppose  $a < 0$ . Take  $\epsilon = \frac{|a|}{2}$ . Since  $\lim_{n \rightarrow \infty} a_n = a$ , we can find  $N$  such that when  $n \geq N$ ,  $|a_n - a| < \epsilon$ . Note that this means

$$-\epsilon < a_n - a < \epsilon$$

Then we have

$$a_n < \epsilon + a = \frac{-a}{2} + a = \frac{a}{2} < 0.$$

Contradiction. □

### 5.2.3 Monotone Convergence Theorem

**Definition 5.2.** A sequence  $\{a_n\}$  is **increasing** if  $a_n \leq a_{n+1}$  for every  $n$  and **decreasing** if  $a_n \geq a_{n+1}$  for every  $n$ . A sequence is **monotone** if it is either increasing or decreasing.

**Theorem 5.4** (Monotone convergence theorem). *If a sequence is monotone and bounded, then it converges.*

*Proof.* Let  $\{a_n\}$  be increasing and bounded. Set  $A = \{a_n : n \in \mathbb{N}\}$ . Note that  $A \neq \emptyset$  and  $A$  is bounded. Therefore, by the axiom of completeness,  $s = \sup A \in \mathbb{R}$  exists. Then we claim that  $\lim_{n \rightarrow \infty} a_n = s$ . For every  $\epsilon > 0$ ,  $s - \epsilon$  is not an upper bound for  $A$ , so we can find  $N$  such that  $s - \epsilon < a_N \leq s$ . Since  $\{a_n\}$  is increasing, for all  $n \geq N$ , we know  $s - \epsilon < a_N \leq a_n \leq s$ , i.e.  $|a_n - s| < \epsilon$ . Therefore  $\lim_{n \rightarrow \infty} a_n = s$ .

For  $\{a_n\}$  decreasing and bounded, simply apply the previous result to  $\{-a_n\}$ . □

# Lecture 6

## Sept. 7 – Bolzano-Weierstrass Theorem

### 6.1 Review of Limits

**Theorem 6.1** (Squeeze theorem). Let  $\{x_n\}, \{y_n\}, \{z_n\}$  be sequences such that  $x_n \leq y_n \leq z_n$  for all  $n$ , and suppose that

$$\lim_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} z_n = l.$$

Then  $\lim_{n \rightarrow \infty} y_n = l$ .

*Proof.* Consider  $|y_n - l|$ . If

$$\begin{aligned} y_n - l \geq 0, & \text{ then } y_n - l \leq z_n - l, \\ y_n - l < 0, & \text{ then } |y_n - l| = l - y_n \leq l - x_n. \end{aligned}$$

So we have

$$|y_n - l| \leq |z_n - l| + |x_n - l|.$$

For all  $\epsilon > 0$ , there exist  $N_1, N_2$  such that for all  $n \geq N_1$ ,

$$|z_n - l| < \frac{\epsilon}{2},$$

and for all  $n \geq N_2$ ,

$$|x_n - l| < \frac{\epsilon}{2}.$$

Take  $N = \max\{N_1, N_2\}$ . If  $n \geq N$ , then

$$|y_n - l| \leq |z_n - l| + |x_n - l| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

So  $\lim_{n \rightarrow \infty} y_n = l$ . □

### 6.2 Subsequences and the Bolzano-Weierstrass Theorem

**Definition 6.1.** Let  $\{a_n\}$  be a sequence of real numbers. Let  $n_1 < n_2 < n_3 < \dots$  be an increasing sequence of natural numbers. Then  $\{a_{n_1}, a_{n_2}, \dots\}$  is a **subsequence** of  $\{a_n\}$ , and it is denoted by  $\{a_{n_k}\}$ .

**Example 6.1.1.** Let

$$\{a_n\} = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\right\}.$$

Then

$$\left\{\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \dots\right\}$$

is a subsequence of  $\{a_n\}$ . However, note that

$$\left\{\frac{1}{10}, \frac{1}{5}, \frac{1}{100}, \frac{1}{500}, \dots\right\}$$

is *not* a subsequence of  $\{a_n\}$  since the  $n_k$  are not strictly increasing. Similarly,

$$\left\{1, \frac{1}{3}, \frac{1}{3}, \frac{1}{5}, \frac{1}{5}, \dots\right\}$$

is also not a subsequence of  $\{a_n\}$ .

**Theorem 6.2.** *Subsequences of a convergent sequence converge to the same limit as the original sequence.*

*Proof.* Suppose  $\lim_{n \rightarrow \infty} a_n = a$ . So for every  $\epsilon > 0$ , there exists  $N$  such that  $|a_n - a| < \epsilon$  for all  $n \geq N$ . Consider an arbitrary subsequence  $\{a_{n_k}\}$ . Note that  $n_k \geq k$ . So when  $k \geq N$ ,

$$|a_{n_k} - a| < \epsilon.$$

Therefore  $\lim_{k \rightarrow \infty} a_{n_k} = a$ . □

**Example 6.1.2.** Let  $0 < b < 1$ . Clearly

$$1 > b > b^2 > b^3 > b^4 > \dots \geq 0.$$

The sequence  $\{b^n\}$  is decreasing and bounded below, so by the monotone convergence theorem,  $\lim_{n \rightarrow \infty} b^n = l \in \mathbb{R}$  exists. Note that  $\{b^{2n}\}$  is a subsequence of  $\{b^n\}$ , so by Theorem 6.2, we have  $\lim_{n \rightarrow \infty} b^{2n} = l$ . Note that  $b^{2n} = b^n b^n$ . By the algebraic limit theorem,

$$\lim_{n \rightarrow \infty} b^{2n} = \left(\lim_{n \rightarrow \infty} b^n\right) \left(\lim_{n \rightarrow \infty} b^n\right).$$

Therefore,  $l = l^2$ , so we have  $l = 0$  or  $l = 1$ . But the entire sequence is strictly less than 1 and decreasing, so  $l = 0$ .

**Example 6.1.3.** Consider the sequence

$$\{(-1)^n\} = \{-1, 1, -1, 1, \dots\}.$$

This sequence does not converge. But the subsequence

$$\{-1, -1, -1, \dots\}$$

does converge.

**Remark.** This shows that the converse of Theorem 6.2 is not true, i.e. a convergent subsequence does not imply that the original sequence converges.

**Example 6.1.4.** The sequence

$$a_n = \begin{cases} 1 & \text{if } n \text{ is prime,} \\ 0 & \text{otherwise} \end{cases}$$

does not converge.

**Exercise 6.1.** Show the limit of the sequence

$$\left\{1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \frac{1}{5}, \frac{1}{5}, -\frac{1}{5}, \frac{1}{5}, -\frac{1}{5}, \dots\right\}$$

*Proof.* The subsequence

$$\left\{\frac{1}{5}, \frac{1}{5}, \dots\right\}$$

converges to  $\frac{1}{5}$  while the subsequence

$$\left\{-\frac{1}{5}, -\frac{1}{5}, \dots\right\}$$

converges to  $-\frac{1}{5}$ . Thus the original sequence diverges.  $\square$

**Remark.** If we can find two subsequences that converge to different limits, then the original sequence diverges. This is the contrapositive of Theorem 6.2.

**Theorem 6.3** (Bolzano-Weierstrass theorem). *Every bounded sequence has a convergent subsequence.*<sup>1</sup>

*Proof.* Let  $\{a_n\}$  be a bounded sequence. So there exists  $M > 0$  such that  $\sup_n |a_n| < M$ . So  $a_n$  is contained in  $[-M, M]$ . Split  $[-M, M]$  into  $[-M, 0]$  and  $[0, M]$ . Pick one that contains infinitely many elements of  $\{a_n\}$  and call it  $I_1$ . Then pick  $a_{n_1} \in \{a_n\}$  such that  $a_{n_1} \in I_1$ . Split  $I_1$  again into two closed intervals of the same size. Take one of these two that contains infinitely many elements of  $\{a_n\}$  and call it  $I_2$ . Then take  $a_{n_2} \in \{a_n\}$  such that  $a_{n_2} \in I_2$ . Repeat this process to get  $I_{k+1} \subseteq I_k$  with  $|I_{k+1}| = \frac{1}{2}|I_k|$  such that  $I_{k+1}$  contains infinitely many elements of  $\{a_n\}$ .<sup>2</sup> Also pick  $a_{n_{k+1}} \in \{a_n\}$  such that  $a_{n_{k+1}} \in I_{k+1}$  with  $n_{k+1} > n_k$ .

By construction,  $\{a_{n_k}\}$  is a subsequence of  $\{a_n\}$  and  $a_{n_k} \in I_k$ . We have the  $I_k$  being closed intervals with

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq \dots$$

So there exists  $x \in \mathbb{R}$  such that  $x \in \bigcap_{k=1}^{\infty} I_k$ . Note that  $|I_k| = M \left(\frac{1}{2}\right)^{k-1}$ . Then we claim  $\lim_{k \rightarrow \infty} a_{n_k} = x$ .

Let  $\epsilon > 0$ . Take  $N$  such that

$$2^N > \frac{2M}{\epsilon}.$$

Then for every  $k \geq N$ , we have

$$|a_{n_k} - x| \leq M \left(\frac{1}{2}\right)^{k-1} < \epsilon$$

since  $a_{n_k}, x \in I_k$ . Thus  $\lim_{k \rightarrow \infty} a_{n_k} = x$ , and  $\{a_{n_k}\}$  is a convergent subsequence.  $\square$

<sup>1</sup>This demonstrates some kind of *compactness* of the real numbers.

<sup>2</sup>Here, by  $|I_k|$  we mean the length of the interval  $I_k$ .

# Lecture 7

## Sept. 12 – The Cauchy Criterion

### 7.1 Cauchy Sequences

**Definition 7.1.** A sequence  $\{a_n\}$  is called a **Cauchy** sequence if for every  $\epsilon > 0$ , there exists  $N \in \mathbb{N}$  such that for every  $m, n \geq N$ , one has  $|a_m - a_n| < \epsilon$ .<sup>1</sup>

**Theorem 7.1.** Every convergent sequence is a Cauchy sequence.

*Proof.* Assume  $\lim_{n \rightarrow \infty} a_n = a$ . Then for every  $\epsilon > 0$ , we can find  $N$  such that for every  $n \geq N$ , we have  $|a_n - a| < \epsilon/2$ . Then for every  $m, n \geq N$ , we have

$$|a_m - a_n| = |a_m - a + a - a_n| \leq |a_m - a| + |a - a_n| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

by the triangle inequality. □

**Lemma 7.1.** Every Cauchy sequence is bounded.

*Proof.* Suppose  $\{x_n\}$  is a Cauchy sequence. Pick  $\epsilon = 1$ . Then there exists  $N$  such that for all  $m, n \geq N$ , we have  $|x_m - x_n| < 1$ . Fixing  $m = N$ , we know that for all  $n \geq N$ ,  $|x_N - x_n| < 1$ . So  $|x_n| \leq |x_N| + 1$  for all  $n \geq N$ . Set

$$M = \max\{|x_1|, |x_2|, \dots, |x_{N-1}|, |x_N| + 1\}.$$

Then  $\sup |x_n| \leq M$  by construction. □

**Theorem 7.2** (Cauchy criterion). A sequence converges if and only if it is a Cauchy sequence.

*Proof.* ( $\Rightarrow$ ) This is Theorem 7.1.

( $\Leftarrow$ ) Suppose  $\{a_n\}$  is a Cauchy sequence. Since  $\{a_n\}$  is Cauchy, we know  $\sup |a_n| \leq M$  for some  $M \in \mathbb{R}$ . Then by the Bolzano-Weierstrass theorem, we can find a convergent subsequence  $\{a_{n_k}\}$  such that  $\lim_{k \rightarrow \infty} a_{n_k} = a$ . We show that we also have  $\lim_{n \rightarrow \infty} a_n = a$ .

For every  $\epsilon > 0$ , we can find  $N_1$  such that for all  $m, n \geq N_1$ , we have  $|a_m - a_n| < \epsilon/2$ . Since  $\lim_{k \rightarrow \infty} a_{n_k} = a$ , there is some  $K$  such that for all  $k \geq K$ , we have  $|a_{n_k} - a| < \epsilon/2$ . Take

$$N \geq \max\{N_1, n_K\}.$$

---

<sup>1</sup>The Cauchy condition controls the *oscillation* of the *tail* of a sequence.



We can find  $K_0$  such that  $n_{K_0} \geq N$ . Then for every  $n \geq N$ ,

$$|a_n - a| = |a_n - a_{n_{K_0}} + a_{n_{K_0}} - a| \leq |a_n - a_{n_{K_0}}| + |a_{n_{K_0}} - a| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

by the triangle inequality and the Cauchy condition.  $\square$

**Remark.** The Cauchy condition allows us to show that a sequence converges without explicitly providing its limit.

## 7.2 Revisiting Completeness

This is the way we have discussed completeness (ordered by implication):

- Axiom of Completeness
  - Nested intervals property
    - \* Bolzano-Weierstrass theorem
    - Cauchy criterion
  - Monotone convergence theorem.

But this is not the only way to do so: We have several ways of choosing axioms to define completeness. For example, we can also prove the nested intervals property using the monotone convergence theorem.

**Exercise 7.1.** The monotone convergence theorem implies the nested intervals property.

*Proof.* Let  $I_n = [a_n, b_n]$  with  $I_{n+1} \subseteq I_n$ . In particular,  $\{a_n\}$  is increasing and bounded ( $b_1$  is an upper bound). So by the monotone convergence theorem,  $\lim_{n \rightarrow \infty} a_n = a$  exists.

Left as an exercise to show that  $a \in I_n$  for all  $n$ .  $\square$

**Exercise 7.2.** Given the Archimedean property, the nested intervals property implies the Axiom of Completeness.

*Proof.* Note that  $\frac{1}{2^n} \rightarrow 0$  as  $n \rightarrow \infty$ . This is because for every  $\epsilon > 0$ , we can find  $N$  such that  $\frac{1}{N} < \epsilon$  by the Archimedean property. Then

$$\frac{1}{2^N} < \frac{1}{N}$$

for all  $N \in \mathbb{N}$ . So  $\lim_{n \rightarrow \infty} \frac{1}{2^n} = 0$ .

Now let  $S$  be a nonempty set which is bounded above. Let  $U$  be an upper bound for  $S$ . Take  $s \in S$ . Set  $a_1 = s$ ,  $b_1 = U$ . Consider

$$\frac{s+U}{2}.$$

If  $\frac{s+U}{2}$  is an upper bound for  $S$ , then we set  $a_2 = a_1 = s$ ,  $b_2 = \frac{s+U}{2}$ . If  $\frac{s+U}{2}$  is not an upper bound for  $S$ , then we set  $a_2 = \frac{s+U}{2}$ ,  $b_2 = b_1 = U$ . Note that  $[a_2, b_2] \subseteq [a_1, b_1]$ . Repeat the same process for  $a_n$  and  $b_n$  to obtain the closed intervals

$$[a_1, b_1] \supseteq [a_2, b_2] \supseteq [a_3, b_3] \supseteq \dots$$

By the nested interval properties, the intersection  $\bigcap_{n=1}^{\infty} [a_n, b_n]$  is nonempty. Note that

$$\begin{aligned} |[a_1, b_1]| &= |b_1 - a_1| = |U - s| \\ |[a_2, b_2]| &= |b_2 - a_2| = \left| \frac{U - s}{2} \right| \\ &\vdots \\ |[a_n, b_n]| &= |b_n - a_n| = \frac{2}{2^n} |U - s| \end{aligned}$$

So there is only one  $x \in \mathbb{R}$  such that  $x \in \bigcap_{n=1}^{\infty} [a_n, b_n]$ . We claim that  $\sup S = x$ .

Note that  $x \in [a_n, b_n]$  for all  $n$ . So  $a_n$  is not an upper bound and  $b_n$  is an upper bound. Suppose for contradiction that  $x$  is not an upper bound. Then there exists  $s_0 \in S$  such that  $s_0 > x$ . Since  $|[a_n, b_n]| \rightarrow 0$ , there exists an  $N$  such that whenever  $n \geq N$ ,

$$|[a_n, b_n]| < \frac{1}{2} |s_0 - x|.$$

Since  $x \in [a_n, b_n]$ , this implies that  $s_0 > b_n$ , which is a contradiction with  $b_n$  being an upper bound.

Use a similar idea to show that  $x$  is the *least* upper bound. □

**Remark.** These are all different ways to understand the same idea of completeness.

# Lecture 8

## Sept. 14 – Series

**Definition 8.1.** Let  $\{b_n\}$  be a sequence. An infinite **series** is formally given by

$$\sum_{n=1}^{\infty} b_n = b_1 + b_2 + \dots$$

**Definition 8.2.** We define the **partial sum** of a series by

$$s_m = \sum_{n=1}^m b_n.$$

### 8.1 Convergence of Series

**Definition 8.3.** The series  $\sum_{n=1}^{\infty} b_n$  **converges** to  $B$  if  $\lim_{m \rightarrow \infty} s_m = B$ . Otherwise we say that the series **diverges**.

**Example 8.3.1.** Consider the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots$$

We look at the partial sums for  $m > 1$ :

$$\begin{aligned} s_m &= \sum_{n=1}^m \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots + \frac{1}{m^2} \leq 1 + \frac{1}{2(1)} + \frac{1}{3(2)} + \frac{1}{4(3)} + \dots + \frac{1}{m(m-1)} \\ &= 1 + 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \dots + \frac{1}{m-1} - \frac{1}{m} \leq 2 - \frac{1}{m}. \end{aligned}$$

Note that  $\{s_m\}$  is a monotone sequence and it is bounded above by 2. Thus by the monotone convergence theorem,  $\{s_m\}$  converges and there is some  $B \in \mathbb{R}$  such that  $\lim_{m \rightarrow \infty} s_m = B$ .

**Remark.** Using some complex analysis, we can find  $B$  by way of residue calculations.

**Example 8.3.2.** Consider the harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \dots$$

We look at the partial sums

$$s_m = 1 + \frac{1}{2} + \cdots + \frac{1}{m}.$$

Note specifically that

$$\begin{aligned} s_4 &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} > 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} = 1 + \frac{1}{2} + 2\left(\frac{1}{4}\right) = 1 + \frac{1}{2} + \frac{1}{2} = 1 + 2\left(\frac{1}{2}\right) \\ s_8 &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} > 1 + \frac{1}{2} + 2\left(\frac{1}{4}\right) + 4\left(\frac{1}{8}\right) = 1 + 3\left(\frac{1}{2}\right) \\ &\vdots \\ s_{2^k} &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{2^{k-1}+1} + \frac{1}{2^{k-1}+2} + \cdots + \frac{1}{2^k} > 1 + \frac{k}{2} \end{aligned}$$

Thus  $\{s_{2^k}\}$  diverges, so  $\{s_m\}$  also diverges.

**Remark.** This type of trick (analyzing  $2^k$  terms) is called *dyadic analysis*, and it shows up frequently in analysis, particularly harmonic analysis.

**Theorem 8.1** (Cauchy condensation test). *Suppose  $\{b_n\}$  is decreasing and  $b_n \geq 0$  for all  $n$ . Then*

$$\sum_{n=1}^{\infty} b_n = b_1 + b_2 + b_3 + \dots$$

*converges if and only if*

$$\sum_{n=0}^{\infty} 2^n b_{2^n} = b_1 + 2b_2 + 4b_4 + \dots$$

*converges.*

*Proof.* First we show the backwards direction. Assume  $\sum_{n=0}^{\infty} 2^n b_{2^n}$  converges. Define

$$t_k = b_1 + \cdots + 2^k b_{2^k}.$$

By assumption,  $\{t_k\}$  converges. Note that  $t_k \geq 0$  and  $\sup_k t_k \leq M$  since convergent series are bounded. Set

$$s_m = \sum_{n=1}^m b_n.$$

Fix  $m$  and take  $k$  large such that  $m \leq 2^{k+1} - 1$ . Then  $s_m \leq s_{2^{k+1}-1}$  since  $b_n \geq 0$ . Observe that

$$\begin{aligned} s_{2^{k+1}-1} &= b_1 + (b_2 + b_3) + (b_4 + b_5 + b_6 + b_7) + \cdots + (b_{2^k} + \cdots + b_{2^{k+1}-1}) \\ &\leq b_1 + 2b_2 + 4b_4 + \cdots + 2^k b_{2^k}. \end{aligned}$$

So  $s_m \leq s_{2^{k+1}-1} \leq t_k \leq M$ . Thus  $\{s_m\}$  is increasing and bounded, so by the monotone convergence theorem,  $\lim_{m \rightarrow \infty} s_m = B \in \mathbb{R}$  exists.

Now we show the forwards direction. Argue by contraposition. Suppose  $\sum_{n=0}^{\infty} 2^n b_{2^n}$  diverges, then we show that  $\sum_{n=1}^{\infty} b_n$  also diverges. Just need to check that  $s_{2^k} \geq \frac{1}{2} + k$  (left as an exercise).  $\square$

**Corollary 8.1.1.** *The series*

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

*converges if and only if  $p > 1$ .*

*Proof.* Let  $b_n = \frac{1}{n^p}$  and  $b_{2^n} = \frac{1}{2^{np}}$ . Then we have

$$\sum_{n=0}^{\infty} 2^n b_{2^n} = \sum_{n=0}^{\infty} 2^{(1-p)n}.$$

The RHS is a geometric series, which converges if and only if  $p > 1$ . To see this, denote  $2^{1-p} = a$ . Then we have

$$\sum_{n=0}^{\infty} 2^{(1-p)n} = \sum_{n=0}^{\infty} a^n.$$

We can observe that the partial sums

$$t_k = \sum_{n=0}^k a^n = \frac{a^{k+1} - 1}{a - 1}$$

converges if and only if  $a^{k+1}$  converges. This happens if and only if  $a < 1$ , which happens if and only if  $p > 1$ .  $\square$

## 8.2 Properties of Series

**Theorem 8.2** (Algebraic limit theorem for series). *Let*

$$\sum_{n=1}^{\infty} a_n = A, \quad \sum_{n=1}^{\infty} b_n = B.$$

*Then for all  $c \in \mathbb{R}$ , we have*

$$\sum_{n=1}^{\infty} ca_n = cA, \quad \sum_{n=1}^{\infty} (a_n + b_n) = A + B.$$

*Proof.* Let  $\sum_{n=1}^{\infty} a_n = A$ . So  $s_m = \sum_{n=1}^m a_n$  converges. Set  $\lim_{m \rightarrow \infty} s_m = A$ . Define

$$t_m = \sum_{n=1}^m ca_n = c \sum_{n=1}^m a_n = cs_m.$$

Then by the algebraic limit theorem, we have  $\lim_{m \rightarrow \infty} t_m = c \lim_{m \rightarrow \infty} s_m = cA$ .  $\square$

**Theorem 8.3** (Cauchy criterion for series). *The series  $\sum_{n=1}^{\infty} a_n$  converges if and only if for all  $\epsilon > 0$ , there exist  $N$  such that whenever  $m, n \geq N$ , we have  $|a_{m+1} + \cdots + a_n| < \epsilon$ .*

*Proof.* The series  $\sum_{k=1}^{\infty} a_k$  converges if and only if  $s_m = \sum_{k=1}^m a_k$  converges. We show that  $\{s_m\}$  is a Cauchy sequence. For all  $\epsilon > 0$ , there exists  $N$  such that for all  $m, n \geq N$

$$|s_n - s_m| = |a_n + \cdots + a_{m+1}| < \epsilon.$$

The converse is the same inequality. □

**Corollary 8.3.1.** *If  $\sum_{n=1}^{\infty} a_n$  converges, then  $\lim_{n \rightarrow \infty} a_n = 0$ .*

*Proof.* Take  $m = n - 1$ . □

**Theorem 8.4.** *Assume  $\{a_n\}$  and  $\{b_n\}$  are sequences such that  $0 \leq a_n \leq b_n$  for all  $n$ . Then*

1.  $\sum_{n=1}^{\infty} b_n$  converges implies  $\sum_{n=1}^{\infty} a_n$  converges,
2. and  $\sum_{n=1}^{\infty} a_n$  diverges implies  $\sum_{n=1}^{\infty} b_n$  diverges.

*Proof.* For all  $m, n$ , we have

$$|a_{m+1} + \cdots + a_n| \leq |b_{m+1} + \cdots + b_n|.$$

Then apply the Cauchy criterion. □

**Definition 8.4.** A series is called **geometric** if it is of the form

$$\sum_{k=0}^{\infty} ar^k = a + ar + ar^2 + \dots$$

Note that the geometric series diverges when  $r = 1$  and  $a \neq 0$ . When  $r \neq 1$ , the partial sums

$$s_m = \sum_{k=0}^m ar^k = a \frac{1 - r^{m+1}}{1 - r}$$

converge if  $|r| < 1$ . In this case, as  $m \rightarrow \infty$ , we have

$$s_m \rightarrow \frac{a}{1 - r}.$$

# Lecture 9

## Sept. 19 – Absolute Convergence

### 9.1 Absolute Convergence

**Definition 9.1.** Consider a series  $\sum_{n=1}^{\infty} a_n$ . If

$$\sum_{n=1}^{\infty} |a_n|$$

converges, then we say  $\sum_{n=1}^{\infty} a_n$  **converges absolutely**.

**Theorem 9.1.** If  $\sum_{n=1}^{\infty} |a_n|$  converges, then  $\sum_{n=1}^{\infty} a_n$  converges.

*Proof.* For every  $\epsilon > 0$ , since  $\sum_{n=1}^{\infty} a_n$  converges, there is  $N$  such that for all  $m, k \geq N$ ,

$$\sum_{n=k+1}^m |a_n| < \epsilon.$$

This is by the Cauchy criterion for series. Then for all  $m, k \geq N$ , we have

$$\left| \sum_{n=k+1}^m a_n \right| \leq \sum_{n=k+1}^m |a_n| < \epsilon$$

by the triangle inequality. Apply the Cauchy criterion again to conclude that  $\sum_{n=1}^{\infty} a_n$  converges.  $\square$

**Theorem 9.2** (Alternating series test). If  $a_1 \geq a_2 \geq a_3 \geq \dots$  and  $\lim_{n \rightarrow \infty} a_n = 0$ , then the series

$$\sum_{n=1}^{\infty} (-1)^{n+1} a_n$$

converges.

*Proof.* Set  $s_m = \sum_{n=1}^m (-1)^{n+1} a_n$ . Check that

$$s_m - s_k = \sum_{n=k+1}^m (-1)^{n+1} a_n.$$

Suppose that  $m$  and  $k$  are odd, then

$$s_m - s_k = \underbrace{a_m - a_{m-1} + a_{m-2} - \cdots + a_{k+2} - a_{k+1}}_{\leq 0}.$$

So  $s_m - s_k \leq 0$ . We can also group the terms as as

$$s_m - s_k = a_m - \underbrace{a_{m-1} + a_{m-2} - \cdots - a_{k+3} - a_{k+2} - a_{k+1}}_{\geq 0} \geq a_m - a_{k+1}.$$

So  $|s_m - s_k| \leq |a_m| + |a_{k+1}|$  by the triangle inequality. Since  $\lim_{n \rightarrow \infty} a_n = 0$ , for all  $\epsilon > 0$ , there is  $N$  such that  $n \geq N$ , we have  $|a_n| < \epsilon$ . Then for all  $m, k \geq N$ ,

$$|s_m - s_k| \leq |a_m| + |a_{k+1}| < 2\epsilon.$$

Thus  $\{s_k\}$  converges. Left as exercise to check the other parities of  $m$  and  $k$  (group differently).  $\square$

**Example 9.1.1.** We saw previously that for  $a_n = \frac{1}{n}$ ,  $\sum_{n=1}^{\infty} a_n$  diverges. But  $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$  converges.

## 9.2 Rearrangements

**Definition 9.2.** Given a series  $\sum_{k=1}^{\infty} a_k$ , we say that a series  $\sum_{k=1}^{\infty} b_k$  is a **rearrangement** of  $\sum_{k=1}^{\infty} a_k$  if there is a bijection  $f : \mathbb{N} \rightarrow \mathbb{N}$  such that  $b_{f(k)} = a_k$  for all  $k \in \mathbb{N}$ .

**Example 9.2.1.** Let

$$\begin{aligned} S &= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \cdots, \\ \frac{1}{2}S &= \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \frac{1}{10} - \cdots, \\ S + \frac{1}{2}S &= 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} - \frac{1}{4} + \frac{1}{7} - \cdots. \end{aligned}$$

Notice that  $S + \frac{1}{2}S$  is a rearrangement of  $S$ . Supposing that  $S + \frac{1}{2}S$  converges to the same limit as  $S$ , we would have

$$S + \frac{1}{2}S = S,$$

or  $S = 0$ . This cannot be the case.

**Remark.** A rearrangement of a series might have different convergence properties from the original series.

**Theorem 9.3.** If a series converges absolutely to  $A$ , then any rearrangement of the series converges to the same limit  $A$ .

*Proof.* Let  $\sum_{k=1}^{\infty} a_k$  converge absolutely to  $A$ . Let  $\sum_{k=1}^{\infty} b_k$  be a rearrangement of  $\sum_{k=1}^{\infty} a_k$ . We set

$$s_n = \sum_{k=1}^n a_k, \quad t_m = \sum_{k=1}^m b_k.$$



We want to show that  $t_m$  converges to  $A$ . Since  $\lim_{n \rightarrow \infty} s_n = A$ , for every  $\epsilon > 0$ , there is  $N_1$  such that

$$|s_n - A| < \frac{\epsilon}{2}$$

for all  $n \in \mathbb{N}$ . Since  $\sum_{k=1}^{\infty} a_k$  converges absolutely, there is  $N_2$  such that for all  $n, m \geq N_2$ , we have

$$\sum_{k=m+1}^n |a_k| < \frac{\epsilon}{2}.$$

Since  $\sum_{k=1}^{\infty} b_k$  is a rearrangement of  $\sum_{k=1}^{\infty} a_k$ , we can write  $b_{f(k)} = a_k$  for some bijection  $f$ . Set

$$N = \max\{N_1, N_2\}, \quad M = \max\{f(k) : 1 \leq k \leq N\}.$$

Then for all  $m \geq M$ ,  $t_m - s_n$  will only consist of terms  $a_k$  for  $k > N$ . In particular,

$$|t_m - s_n| \leq \sum_{k=n}^{\infty} |a_k| < \frac{\epsilon}{2}.$$

Then we have

$$|t_m - A| = |t_m - s_n + s_n - A| \leq |t_m - s_n| + |s_n - A| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

So  $\lim_{m \rightarrow \infty} t_m = A$ . □

# Lecture 10

## Sept. 21 – Iterated Sums and Topology

### 10.1 Double Sums

Given a set of doubly indexed real numbers  $\{a_{ij} : i, j \in \mathbb{N}\}$ , consider the sums

$$\sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_{ij}, \quad \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij}.$$

Are these sums equal? The answer is no, in general.

**Example 10.0.1.** Define  $a_{ij}$  by

$$a_{ij} = \begin{cases} \left(\frac{1}{2}\right)^{j-i} & \text{if } j > i, \\ -1 & \text{if } i = j, \\ 0 & \text{if } j < i. \end{cases}$$

This looks like

	$a_{11}$	$a_{12}$	$a_{13}$	$a_{14}$	$\cdots$
$a_{11}$	-1	$\frac{1}{2}$	$\frac{1}{4}$	$\left(\frac{1}{2}\right)^3$	$\cdots$
$a_{21}$	0	-1	$\frac{1}{2}$	$\left(\frac{1}{2}\right)^2$	$\cdots$
$a_{31}$	0	0	-1	$\frac{1}{2}$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Then the first sum (over the columns) is

$$\sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_{ij} = \sum_{j=1}^{\infty} -\left(\frac{1}{2}\right)^{j-1} = -\frac{1}{1 - \frac{1}{2}} = -2.$$

Meanwhile, the second sum (over the rows) is

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij} = \sum_{i=1}^{\infty} \left( -1 + \sum_{k=1}^{\infty} \left(\frac{1}{2}\right)^k \right) = \sum_{i=1}^{\infty} (-1 + 1) = \sum_{i=1}^{\infty} 0 = 0.$$

Notice that these two sums are not the same.

**Remark.** We cannot always exchange the order of a double sum. <sup>1</sup>

<sup>1</sup>*Fubini's theorem* gives conditions under which we can do this for iterated *integrals* (when the integrand is *absolutely integrable*).

### 10.1.1 Convergence of Double Sums

**Definition 10.1.** We say that  $\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |a_{ij}|$  **converges** if for all  $i \in \mathbb{N}$ ,  $\sum_{j=1}^{\infty} |a_{ij}|$  converges to some real number  $b_i$  and  $\sum_{i=1}^{\infty} b_i$  converges.

**Theorem 10.1.** Consider  $\{a_{ij} : i, j \in \mathbb{N}\}$ . If

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |a_{ij}|$$

converges, then both

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij} \quad \text{and} \quad \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_{ij}$$

converge to the same limit, i.e.

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij} = \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_{ij} = \lim_{n \rightarrow \infty} S_{nn}$$

where  $S_{nn} = \sum_{i=1}^n \sum_{j=1}^n a_{ij}$ .

*Proof.* Go look this up. □

## 10.2 Basic Topology in $\mathbb{R}$

### 10.2.1 Open Sets

**Definition 10.2.** For all  $a \in \mathbb{R}$ ,  $\epsilon > 0$ , we define

$$V_{\epsilon}(a) = \{x \in \mathbb{R} : |x - a| < \epsilon\}$$

to be the  $\epsilon$ -neighborhood of  $a$ .

**Definition 10.3.** A set  $U \subseteq \mathbb{R}$  is **open** if for all  $a \in U$ , there exists  $\epsilon > 0$  such that  $V_{\epsilon}(a) \subseteq U$ .

**Example 10.3.1.** The set  $\mathbb{R}$  is open: Simply take  $\epsilon = 1$  for any choice of  $a \in \mathbb{R}$ .

**Example 10.3.2.** The open interval  $(c, d)$  is open: For any  $x \in (c, d)$ , take  $\epsilon = \min\{x - c, d - x\}$ .

**Theorem 10.2.**

1. The union of an arbitrary collection of open sets is open.
2. The intersection of a finite collection of open sets is open.

*Proof.* (1) Let  $\{U_{\lambda} : \lambda \in \Lambda\}$  be a collection of open sets and consider  $\bigcup_{\lambda \in \Lambda} U_{\lambda}$ . For every  $a \in \bigcup_{\lambda \in \Lambda} U_{\lambda}$ , there is  $\lambda'$  such that  $a \in U_{\lambda'}$ . Since  $U_{\lambda'}$  is open, there exists  $\epsilon > 0$  such that

$$V_{\epsilon}(a) \subseteq U_{\lambda'} \subseteq \bigcup_{\lambda \in \Lambda} U_{\lambda}.$$

So  $\bigcup_{\lambda \in \Lambda} U_{\lambda}$  is open.

(2) Let  $U_1, \dots, U_n$  be a collection of open sets and consider  $\bigcap_{j=1}^n U_j$ . For every  $a \in \bigcap_{j=1}^n U_j$ , note that  $a \in U_j$  for all  $j = 1, \dots, n$ . Since  $U_j$  is open, there exists  $\epsilon_j > 0$  such that  $V_{\epsilon_j}(a) \subseteq U_j$ . Then take

$$\epsilon = \min_{1 \leq j \leq n} \{\epsilon_j\},$$

which exists since the collection is finite. Since  $\epsilon_j > 0$ , we have  $\epsilon > 0$  as well. By construction,

$$V_\epsilon(a) \subseteq V_{\epsilon_j}(a) \subseteq U_j$$

for all  $j$ . So  $V_\epsilon(a) \subseteq \bigcap_{j=1}^n U_j$ , and thus  $\bigcap_{j=1}^n U_j$  is open.  $\square$

**Example 10.3.3.** Consider the family of open sets  $U_n = (-\frac{1}{n}, \frac{1}{n})$  for  $n \in \mathbb{N}$ . Notice that their intersection

$$\bigcap_{n=1}^{\infty} U_n = \{0\}$$

is not open.

## 10.2.2 Limit Points

**Definition 10.4.** A point  $x$  is a **limit point** of a set  $A$  if for all  $\epsilon > 0$ , we have

$$(V_\epsilon(x) \cap A) \setminus \{x\} \neq \emptyset,$$

i.e. there is some other point in the  $\epsilon$ -neighborhood of  $x$  that is also in  $A$ .

**Theorem 10.3.** A point  $x$  is a limit point of  $A$  if and only if  $x = \lim_{n \rightarrow \infty} a_n$  for some sequence  $\{a_n\}$  with  $a_n \neq x$  and  $a_n \in A$ .

*Proof.* ( $\Rightarrow$ ) Suppose  $x$  is a limit point of  $A$ . Take  $\epsilon = 1/n$  and pick  $a_n \in (V_{1/n}(x) \cap A) \setminus \{x\}$  such that  $a_n \neq x$ . For such a sequence  $\{a_n\}$ , for all  $\epsilon > 0$ , if  $N \geq 1/\epsilon$ , then for all  $n \geq N$ , we have

$$|a_n - x| \leq \frac{1}{N} < \epsilon.$$

So  $\lim_{n \rightarrow \infty} a_n = x$ .

( $\Leftarrow$ ) Assume such a sequence  $\{a_n\}$  exists. Then for any  $\epsilon > 0$ , there exists  $N$  such that  $|a_n - x| < \epsilon$  for all  $n \geq N$ . Note that  $a_N \in V_\epsilon(x)$ , and also  $a_N \in A$  and  $a_N \neq x$ . So  $a_N \in (V_\epsilon(x) \cap A) \setminus \{x\}$ , i.e. this set is not empty. Thus  $x$  is a limit point of  $A$ .  $\square$

# Lecture 11

## Sept. 26 – Closed Sets

### 11.1 Closed Sets

**Definition 11.1.** Let  $A \subseteq \mathbb{R}$ . An element  $x \in A$  is an **isolated point** of  $A$  if it is not a limit point of  $A$ , i.e. there exists  $\epsilon > 0$  such that  $V_\epsilon(x) \cap A = \{x\}$ .

**Definition 11.2.** A set  $A \subseteq \mathbb{R}$  is **closed** if it contains all of its limit points.

**Example 11.2.1.** The empty set and  $\mathbb{R}$  are closed. Moreover, any set without limit points is closed.

**Theorem 11.1.** A set  $A \subseteq \mathbb{R}$  is closed if and only if every Cauchy sequence in  $A$  converges to a limit in  $A$ .

*Proof.* ( $\Rightarrow$ ) Suppose  $A$  is closed and  $\{a_n\}$  is Cauchy with  $a_n \in A$  for all  $n$ . Since Cauchy sequences are convergent, let  $x = \lim_{n \rightarrow \infty} a_n$ . Now consider two cases. If there exists an  $n$  such that  $x = a_n$ , then we're done since  $x = a_n \in A$ . Otherwise, we have  $a_n \neq x$  for all  $n$ . By Theorem 10.3,  $x$  is a limit point of  $A$ . So  $x \in A$  as  $A$  is closed.

( $\Leftarrow$ ) Let  $x$  be a limit point of  $A$ . Then there exists a sequence  $\{a_n\}$  with  $a_n \in A$  and  $a_n \neq x$  for all  $n$  such that  $\lim_{n \rightarrow \infty} a_n = x$ . This means that  $\{a_n\}$  is Cauchy, so by assumption,  $x = \lim_{n \rightarrow \infty} a_n \in A$ . Thus every limit point of  $A$  is in  $A$ , so  $A$  is closed.  $\square$

**Example 11.2.2.** Consider the set

$$A = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\}.$$

First  $x \neq 0$ , we look at the following cases:

1. If  $x < 0$ , let  $\epsilon = |x|$ . Then  $V_\epsilon(x) = (2x, 0)$ , and  $V_\epsilon(x) \cap A = \emptyset$  since  $A \subseteq \mathbb{R}^+$ . So  $x < 0$  is not a limit point of  $A$ .
2. If  $x > 1$ , let  $\epsilon = x - 1$ . Then  $V_\epsilon(x) = (1, 2x - 1)$ , so  $V_\epsilon(x) \cap A = \emptyset$  as all  $y \in A$  satisfies  $y \in (0, 1]$ .
3. If  $x \in (0, 1]$ , then there exists  $n \in \mathbb{N}$  such that  $n > 1/x$ . Let  $n_0 = \min\{n \in \mathbb{N} : n > 1/x\}$ , which exists by the well-ordering principle. Noting that  $n_0 \geq 2$ , we have

$$\frac{1}{n_0} < x \leq \frac{1}{n_0 - 1}.$$

Now we look at two more cases:

(a) If  $x = \frac{1}{n_0 - 1}$ , let

$$\epsilon = x - \frac{1}{n_0} = \frac{1}{n_0 - 1} - \frac{1}{n_0}.$$

Then we have

$$V_\epsilon(x) = \left( \frac{1}{n_0}, \frac{2}{n_0 - 1} - \frac{1}{n_0} \right).$$

Note that  $(V_\epsilon \cap A) \setminus \{x\} = \emptyset$  if  $n_0 = 2$ . Otherwise,  $n_0 > 2$  and we have

$$\frac{2}{n_0 - 1} - \frac{1}{n_0} - \frac{1}{n_0 - 2} = \cdots = \frac{-2}{n_0(n_0 - 2)(n_0 - 1)} < 0.$$

So  $V_\epsilon(x) \subseteq \left( \frac{1}{n_0}, \frac{1}{n_0} - 2 \right)$ , which means that  $V_\epsilon(x) \cap A = \{x\}$ .

(b) Otherwise,  $x \in \left( \frac{1}{n_0}, \frac{1}{n_0 - 1} \right)$  and let

$$\epsilon = \min \left\{ x - \frac{1}{n_0}, \frac{1}{n_0 - 1} - x \right\}.$$

Then (left as exercise)  $V_\epsilon(x) \subseteq \left( \frac{1}{n_0}, \frac{1}{n_0 - 1} \right)$ , which implies  $V_\epsilon(x) \cap A = \emptyset$ .

So  $x \neq 0$  is not a limit point of  $A$ . However,  $x = 0 = \lim_{n \rightarrow \infty} \frac{1}{n}$ , so 0 is a limit point of  $A$ . But  $0 \notin A$ , so  $A$  is not closed.

**Example 11.2.3.** Let  $A = [a, b]$ . For any Cauchy sequence  $\{x_n\} \subseteq A$ , let  $x = \lim_{n \rightarrow \infty} x_n$ . Since  $x_n \geq a$ , we have  $x = \lim_{n \rightarrow \infty} x_n \geq a$ . Similarly,  $x_n \leq b$  implies that  $x \leq b$ . So  $x \in [a, b] = A$ , and thus  $A$  is closed.

**Example 11.2.4.** Consider  $\mathbb{Q}$ . For any  $x \in \mathbb{R}$ , for all  $n \in \mathbb{N}$  there exists  $a_n \in \mathbb{Q}$  with

$$\frac{1}{2n} < |a_n - x| < \frac{1}{n}.$$

Thus  $a_n \neq x$  and  $a_n \in \mathbb{Q}$  for all  $n$ , so  $\lim_{n \rightarrow \infty} a_n = x$  is a limit point of  $\mathbb{Q}$ . So  $\mathbb{Q}$  is not closed.

**Remark.** We can also define the real numbers as equivalence classes of Cauchy sequences.<sup>1</sup> Note that Cauchy sequences do not require an ordering (only a metric), so we can easily extend this definition to higher dimensions.

## 11.2 The Closure of a Set

**Definition 11.3.** Let  $A \subseteq \mathbb{R}$ . Define the **closure** of  $A$  as

$$\overline{A} = \{x : x \in A \text{ or } x \text{ is a limit point of } A\}.$$

**Theorem 11.2.** The closure of a set  $A$  is closed. Furthermore, if  $B$  is closed and  $A \subseteq B$ , then  $\overline{A} \subseteq B$ .<sup>2</sup>

*Proof.* Let  $x$  be a limit point of  $\overline{A}$ . We want to show that  $x$  is also a limit point of  $A$  (so we will have  $x \in \overline{A}$ ). If  $x \in A$ , then we're done. Otherwise,  $x \notin A$ , so for all  $\epsilon > 0$ , we have  $(V_{\epsilon/2}(x) \cap \overline{A}) \setminus \{x\} \neq \emptyset$ , so let  $y \in (V_{\epsilon/2}(x) \cap \overline{A}) \setminus \{x\}$ . If  $y \in A$ , then  $(V_\epsilon(x) \cap A) \setminus \{x\} \neq \emptyset$  and we're done. Otherwise,  $y \notin A$ , so  $y \in \overline{A}$  implies that  $y$  is a limit point of  $A$ . So there exists  $z \in (V_{\epsilon/2}(y) \cap A) \setminus \{y\}$ . Then

$$|x - z| \leq |x - y| + |y - z| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

<sup>1</sup>Using *Dedekind cuts* is another such way.

<sup>2</sup>So  $\overline{A}$  is the smallest closed set containing  $A$ .

So  $z \in (V_\epsilon(x) \cap A) \setminus \{x\}$ . Since this is true for all  $\epsilon > 0$ ,  $x$  is a limit point of  $A$ . Thus  $x \in \overline{A}$ , so  $\overline{A}$  is closed.

For the second part, let  $x \in \overline{A}$ . If  $x \in A$ , then  $A \subseteq B$  implies that  $x \in B$ . If  $x \notin A$ , then  $x$  is a limit point of  $A$ . So there exists a sequence  $\{a_n\}$  with  $a_n \in A$  for all  $n$  such that  $a_n \rightarrow x$ . Since  $a_n \in A \subseteq B$ , we have  $a_n \in B$  for all  $n$ . Since  $a_n \rightarrow x$ , we must have  $x \in B$  since  $B$  is closed. Thus  $A \subseteq B$ .  $\square$

**Corollary 11.2.1.** *If  $A \subseteq B$ , then  $\overline{A} \subseteq \overline{B}$ .*

*Proof.* Note that Cauchy sequences in  $A$  are also Cauchy sequences in  $B$ .  $\square$

# Lecture 12

## Sept. 28 – Compact Sets

### 12.1 Another Characterization of Closed Sets

**Definition 12.1.** Given a set  $A \subseteq \mathbb{R}$ , its **complement** is  $A^c = \{x \in \mathbb{R} : x \notin A\}$ .

**Theorem 12.1.** A set  $A \subseteq \mathbb{R}$  is closed if and only if  $A^c$  is open.

*Proof.* ( $\Rightarrow$ ) Suppose  $A$  is closed. Take any  $x \in A^c$ . Since  $A$  is closed and  $x \notin A$ , we know  $x$  is not a limit point of  $A$ . So there is  $\epsilon > 0$  such that  $(V_\epsilon(x) \cap A) \setminus \{x\} = \emptyset$ . Since  $x \notin A$ , this means that  $V_\epsilon(x) \cap A = \emptyset$ , which means that  $V_\epsilon(x) \subseteq A^c$ . So  $x$  is an interior point of  $A^c$ , and thus  $A^c$  is open.

( $\Leftarrow$ ) Suppose  $A^c$  is open. Let  $x$  be a limit point of  $A$ . Assume that  $x \notin A$ , i.e.  $x \in A^c$ . Since  $A^c$  is open, there is  $\epsilon > 0$  such that  $V_\epsilon(x) \subseteq A^c$ . But then  $V_\epsilon(x) \cap A = \emptyset$ , which is a contradiction with  $x$  being a limit point of  $A$ . So  $x \in A$ , and thus  $A$  is closed.  $\square$

**Corollary 12.1.1.** A set  $A \subseteq \mathbb{R}$  is open if and only if  $A^c$  is closed.

**Remark.** This is used in topology, where a collection of open sets that satisfies certain conditions<sup>1</sup> is called a *topology* of a space, and closed sets are defined as their complements. Furthermore, metrics are not necessary in this setting.

**Theorem 12.2.**

1. Let  $A_1, A_2, \dots, A_n \subseteq \mathbb{R}$  be closed. Then  $\bigcup_{i=1}^n A_i$  is closed.
2. Let  $A_\lambda \subseteq \mathbb{R}$ ,  $\lambda \in \Lambda$  be a family of closed subsets of  $\mathbb{R}$  indexed by  $\lambda \in \Lambda$ , where  $\Lambda$  is an index set. Then  $\bigcup_{\lambda \in \Lambda} A_\lambda$  is closed.

*Proof.* Left as an exercise.  $\square$

### 12.2 Compactness

**Definition 12.2.** A set  $A \subseteq \mathbb{R}$  is **compact** if any sequence  $\{a_n\}$  in  $A$  has a convergent subsequence  $\{a_{n_k}\}$  such that  $\lim_{k \rightarrow \infty} a_{n_k} \in A$ .<sup>2</sup>

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<sup>1</sup>The collection must be closed under finite intersection and arbitrary union.

<sup>2</sup>This definition is sometimes called *sequential compactness*.



**Remark.** Suppose we want to solve the differential equation

$$\begin{cases} x' = f(t, x) \\ x(t_0) = x_0 \end{cases}.$$

We can first transform this into the integral equation

$$x(t) = x_0 + \int_{t_0}^t f(s, x(s)) ds$$

with a test function  $f$ . Then we perform Picard iterations to continue. However, this method requires  $f \in C^1(\mathbb{R})$  (i.e.  $f$  is continuously differentiable), since that is what guarantees that the sequence of functions converges (closedness). But even without this condition (if  $f$  is only continuous), if we can show that the set of functions lies in a compact set, then we can find a subsequence of functions that do converge (though solutions may no longer be unique).

**Example 12.2.1.** The interval  $(0, 1]$  is not compact since it is not closed: it does not contain all of its limit points.

**Example 12.2.2.** The set  $\mathbb{R}$  is not compact since it is not bounded: an arbitrary sequence may not even have a convergent subsequence.

**Theorem 12.3.** *A set  $A \subseteq \mathbb{R}$  is compact if and only if  $A$  is bounded and closed.*<sup>3</sup>

*Proof.* ( $\Rightarrow$ ) Assume  $A$  is not bounded. Then for any  $n \in \mathbb{N}$ , there exists an  $x_n \in A$  such that  $|x_n| > M$ . Then  $\{x_n\}$  is a sequence in  $A$ . Since  $A$  is compact, there exists a subsequence

$$\{x_{n_k}\} = \{x_{n_1}, x_{n_2}, x_{n_3}, \dots\}$$

such that  $x = \lim_{k \rightarrow \infty} x_{n_k} \in A$ . But this implies that  $\{x_{n_k}\}$  is bounded, which contradicts the fact that  $|x_{n_k}| > n_k$  and  $n_k \rightarrow \infty$  and  $k \rightarrow \infty$ . Hence  $A$  must be bounded.

Now assume  $A$  is not closed. Then there exists a limit point  $x$  of  $A$  such that  $x \notin A$ . Since  $x$  is a limit point of  $A$ , there exists a sequence  $\{a_n\}$  such that  $a_n \in A$  and  $\lim_{n \rightarrow \infty} a_n = x$ . Now since  $A$  is compact, there exists a convergent subsequence  $\{a_{n_k}\}$  such that  $\lim_{k \rightarrow \infty} a_{n_k} \in A$ . But since  $\{a_n\}$  converges, we have

$$x = \lim_{n \rightarrow \infty} a_n = \lim_{k \rightarrow \infty} a_{n_k} \in A.$$

This is a contradiction with  $x \notin A$ . Hence  $A$  is closed.

( $\Leftarrow$ ) Suppose  $A$  is bounded and closed. Let  $\{a_n\}$  be a sequence in  $A$ . By the Bolzano-Weierstrass theorem, there exists a convergent subsequence  $\{a_{n_k}\}$ . Let  $x = \lim_{k \rightarrow \infty} a_{n_k}$ . But  $\{a_{n_k}\}$  is a convergent sequence in  $A$ , which is closed. So  $x \in A$ , and hence  $A$  is compact.  $\square$

**Example 12.2.3.** The union of intervals  $[1, 2] \cup [3, 4]$  is compact.

**Example 12.2.4.** The set

$$A = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\} \cup \{0\}$$

is compact (since we added the limit point 0).

<sup>3</sup>This applies to all Euclidean spaces (and pretty much only Euclidean spaces).

**Remark.** Why do we need the concept of compactness? Because Theorem 12.3 is no longer true in infinite dimensions.

**Theorem 12.4.** Suppose

$$k_1 \supseteq k_2 \supseteq k_3 \supseteq \dots$$

are non-empty compact sets. Then

$$\bigcap_{n=1}^{\infty} k_n \neq \emptyset.$$

*Proof.* Since  $k_n \neq \emptyset$  for all  $n$ , there exists  $a_n \in k_n$ . Then for any  $m \in \mathbb{N}$ ,  $\{a_n\}_{n=m}^{\infty}$  is a sequence in  $k_m$ , which is compact. For  $m = 1$ , there exists a convergent subsequence  $\{a_{n_k}\}$  such that  $x = \lim_{k \rightarrow \infty} a_{n_k} \in k_1$ . For any  $m \in \mathbb{N}$ , there is  $k \in \mathbb{N}$  such that for all  $k \geq k_m$ ,  $n_k \geq m$ . So

$$a_{n_k} \in k_{n_k} \subseteq k_m,$$

which means that  $\{a_{n_k}\}$  is a convergent sequence in  $k_m$ , which is closed. Thus  $x \in k_m$  for all  $m$ , which means that  $x \in \bigcap_{n=1}^{\infty} k_n$ . So  $\bigcap_{n=1}^{\infty} k_n$  is nonempty.  $\square$

**Example 12.2.5.** For  $U_n = (0, 1/n)$ . Then  $U_1 \supseteq U_2 \supseteq U_3 \supseteq \dots$ , but their intersection is empty.

However, this is not so surprising, since the measure of the sets  $U_n$  tends to 0.

**Example 12.2.6.** The sequence  $V_n = (n, \infty)$  also satisfies  $V_1 \supseteq V_2 \supseteq V_3 \supseteq \dots$ , but their intersection is empty, despite each  $V_n$  having infinite measure.

## 12.3 Another Definition of Compactness

**Theorem 12.5** (Heine-Borel theorem). A set  $A \subseteq \mathbb{R}$  is compact if and only if it satisfies the following:

(Covering property) For any family  $U_{\lambda}$ ,  $\lambda \in \Lambda$  of open subsets of  $\mathbb{R}$  such that  $A \subseteq \bigcup_{\lambda \in \Lambda} U_{\lambda}$ ,<sup>4</sup> there exists  $n \in \mathbb{N}$  and  $\lambda_1, \dots, \lambda_n \in \Lambda$  such that  $A \subseteq \bigcup_{k=1}^n U_{\lambda_k}$ .<sup>5</sup>

*Proof.* ( $\Leftarrow$ ) Assume that the covering property holds for  $A$ . For boundedness, let

$$A \subseteq \bigcup_{n=1}^{\infty} (-n, n) = \mathbb{R}.$$

By the covering property, there exist  $n_1 < n_2 < \dots < n_k$  such that

$$A \subseteq \bigcup_{i=1}^k (-n_i, n_i) = (-n_k, n_k).$$

So  $A$  is bounded. Now for closedness, suppose otherwise that  $A$  is not closed. So there exists a limit point  $x \notin A$ . Then

$$A \subseteq \mathbb{R} \setminus \{x\} = \bigcup_{n=1}^{\infty} \left\{ y : |y - x| > \frac{1}{n} \right\}.$$

<sup>4</sup>This is called an *open cover* of  $A$ .

<sup>5</sup>I.e. there exists a *finite subcover*.

By the covering property, there exist  $n_1 < n_2 < \dots < n_k$  such that

$$A \subseteq \bigcup_{i=1}^k \left\{ y : |y - x| > \frac{1}{n_i} \right\} = \left\{ y : |y - x| > \frac{1}{n_k} \right\} \subseteq \mathbb{R} \setminus V_{1/n_k}(x).$$

But  $x$  is a limit point of  $A$ , so there exists a  $z \in V_{1/n_k}(x) \cap A$ . This is a contradiction. Hence  $A$  is also closed, and thus  $A$  is compact.

( $\Rightarrow$ ) Assume  $A$  is compact and let  $U_\lambda$ ,  $\lambda \in \Lambda$  be an open cover of  $A$ . Suppose otherwise that there does not exist a finite subcover. Since  $A$  is compact,  $A$  is bounded. So there exists an  $M$  such that  $A \subseteq [-M, M]$ . Let  $A_1 = A$  and define a sequence  $A_n$  of sets inductively by  $A_n = A \cap [a_n, b_n] \neq \emptyset$  with

$$|b_n - a_n| = \frac{2M}{2^{n-1}},$$

and  $A_n$  can't be covered by a finite subcollection of  $U_\lambda$ . Suppose such  $A_n$  is defined. Then

$$A_n = \underbrace{\left( A \cap \left[ a_n, \frac{b_n + a_n}{2} \right] \right)}_{=k_1} \cup \underbrace{\left( A \cap \left[ \frac{a_n + b_n}{2}, b_n \right] \right)}_{=k_2}.$$

So either  $k_1$  or  $k_2$  can't be covered by a finite subcollection of  $U_\lambda$ . Let this one be  $A_{n+1}$ . Since  $A_n$  is closed and bounded,  $A_n$  is compact. Then we have a sequence of compact sets

$$A_1 \supseteq A_2 \supseteq A_3 \supseteq \dots,$$

so  $\bigcap_{n=1}^{\infty} A_n \neq \emptyset$ . Since  $|a_n - b_n| \rightarrow 0$  (prove this as exercise), there exists  $x \in \mathbb{R}$  such that  $\bigcap_{n=1}^{\infty} A_n = \{x\}$ . But there exists  $\lambda_0$  such that  $x \in U_{\lambda_0}$ , which is open. So there exists  $n$  such that  $V_{1/n}(x) \subseteq U_{\lambda_0}$ , which means that  $A_{n+1} \subseteq U_{\lambda_0}$ . This is a contradiction.  $\square$

**Example 12.2.7.** Let  $A = (0, 1)$ ,  $\Lambda = (0, 1)$ , and  $U_\lambda = (0, \lambda)$ . We have  $A \subseteq \bigcup_{\lambda \in (0, 1)} U_\lambda$ , but there does not exist a finite subcover.

# Lecture 13

## Oct. 3 – Perfect Sets

**Definition 13.1.** A set  $P \subseteq \mathbb{R}$  is **perfect** if it is closed and contains no isolated points.

**Example 13.1.1.** The closed intervals  $[a, b]$  are perfect.

### 13.1 The Cantor Set

**Example 13.1.2.** Define a sequence of sets inductively by  $C_0 = [0, 1]$  and removing its middle third to get  $C_1 = C_0 \setminus (\frac{1}{3}, \frac{2}{3})$ , i.e.

$$C_1 = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right].$$

Then we have

$$C_2 = \left(\left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{1}{3}\right]\right) \cup \left(\left[\frac{2}{3}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right]\right),$$

and so on, removing the middle third of each interval at each step. Note that  $C_n$  is a set consisting of  $2^n$  closed intervals each of length  $\frac{1}{3^n}$ . Then the (middle third) *Cantor set*  $C$  is defined as

$$C = \bigcap_{n=0}^{\infty} C_n.$$

**Remark.** If we consider the sum of the lengths of the intervals that we removed, we get

$$\frac{1}{3} + 2 \cdot \frac{1}{9} + 4 \cdot \frac{1}{27} + \cdots + 2^{n-1} \cdot \frac{1}{3^n} + \cdots = \frac{1}{3} \left( \frac{1}{1 - \frac{2}{3}} \right) = \frac{1}{3} \cdot 3 = 1.$$

So, in some sense, the “size”<sup>1</sup> of the Cantor set  $C$  is 0. However,  $C$  is uncountable. In particular, the cardinality of  $C$  is the same as the cardinality of  $\mathbb{R}$ . A lot of counterexamples in real analysis come from this Cantor set.

**Remark.** This means that the usual measure is not a good way to “catch” the Cantor set. Instead, we can consider *fractional* (or *fractal*) dimensions.

**Theorem 13.1.** *The Cantor set  $C$  is perfect.*

*Proof.* First note that  $C$  is a countable intersection of closed sets, so  $C$  is closed as well. To see that  $C$  has no isolated points, take an arbitrary  $x \in C$ . Since  $x \in C_n$  for all  $n$ , we can find  $x_n \in C_n$  such that  $x_n \neq x$  and  $|x_n - x| < \frac{1}{3^n}$ . Then  $\lim_{n \rightarrow \infty} x_n = x$  and  $x_n \neq x$ , so  $x$  is a limit point of  $C$ . Thus  $C$  is perfect.  $\square$

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<sup>1</sup>The (Lebesgue) *measure*.

## 13.2 Perfect Sets and Countability

**Theorem 13.2.** *A nonempty perfect set is uncountable.*

*Proof.* Note that if  $P$  is perfect, then  $P$  is infinite (if we only have finitely many points, then they must be isolated). Now suppose that  $P$  is only countably infinite. Then we can write

$$P = \{x_1, x_2, \dots\}.$$

Take  $I_1$  to be a closed interval such that  $x_1 \in I_1$  and  $x_1$  is not an endpoint of  $I_1$ . Since  $x_1$  is not isolated in  $P$ , we can find  $y_2 \in P$  with  $y_2 \neq x_1$  such that  $y_2 \in I_1$  and  $y_2$  is not an endpoint of  $I_1$ . Then let  $I_2 \subseteq I_1$  be a closed interval centered at  $y_2$  such that  $x_1 \notin I_2$ . For example, we can do this by setting

$$\epsilon = \frac{1}{2} \min\{y_2 - a, b - y_2, |x_1 - y_2|\}$$

and letting  $I_2 = [y_2 - \epsilon, y_2 + \epsilon]$ . Since  $y_2 \in P$ ,  $y_2$  is not isolated, so we can find  $y_3 \in P$  such that  $y_3$  is not an endpoint of  $I_2$  and  $y_3 \neq x_2$ . Pick  $I_3$  centered at  $y_3$  such that  $x_2 \notin I_3$  and  $I_3 \subseteq I_2$ . Note that  $I_3 \cap P \neq \emptyset$  since  $y_3 \in I_3 \cap P$ . From here, we continue by constructing  $I_{n+1} \subseteq I_n$  with  $x_n \notin I_{n+1}$  and  $I_{n+1} \cap P \neq \emptyset$ . Let  $k_n = I_n \cap P$ . Clearly  $k_n$  is closed (it is the intersection of two closed sets), and  $k_n$  is also bounded since  $k_n \subseteq I_n$ . So  $k_n$  is compact. By construction,  $k_{n+1} \subseteq k_n$ , so by the nested interval property of compact sets,  $\bigcap_{n=1}^{\infty} k_n \neq \emptyset$ . But  $k_n \subseteq I_n$  and  $x_n \notin I_{n+1}$  so  $x_n \notin k_{n+1}$ . Since  $k_n \subseteq P$ , we must have  $\bigcap_{n=1}^{\infty} k_n = \emptyset$ . This is a contradiction, so  $P$  must be uncountable.  $\square$

# Lecture 14

## Oct. 12 – Connected Sets

### 14.1 Review of Exam 1

**Exercise 14.1.** Investigate the convergence of the following series:

$$\sum_{n=1}^{\infty} \frac{\sin(\pi n/3)}{\sqrt{n}}.$$

Either state the test/reason used to show convergence (and verify the hypotheses required), or show why the sum diverges.

*Solution.* Observe that

$$\sin\left(\frac{\pi}{3}n\right) = \begin{cases} 0, & n = 3k \\ \frac{\sqrt{3}}{2}, & n = 6k + 1, 6k + 2 \\ -\frac{\sqrt{3}}{2}, & n = 6k + 4, 6k + 5. \end{cases}$$

But note that we cannot just regroup since the series is not absolutely convergent! Instead, look at the partial sums (here, the sum is finite so we can regroup) and show that

$$|s_m - s_n| \leq \frac{C}{\sqrt{n}}$$

for some constant  $C$ . Then argue by the Cauchy criterion that the series converges. □

### 14.2 Connected Sets

**Definition 14.1.** Two nonempty sets  $A, B \subseteq \mathbb{R}$  are **separated** if  $\overline{A} \cap B = \overline{B} \cap A = \emptyset$ .

**Definition 14.2.** A set  $E \subseteq \mathbb{R}$  is **disconnected** if  $E$  can be written as  $E = A \cup B$  such that  $A$  and  $B$  (both nonempty) are separated. A set which is not disconnected is **connected**.

**Example 14.2.1.** Let  $A = (1, 2)$  and  $B = (2, 5)$ . The set  $E = A \cup B = (1, 2) \cup (2, 5)$  is disconnected since

$$\overline{A} \cap B = [1, 2] \cap (2, 5) = \emptyset = (1, 2) \cap [2, 5] = A \cap \overline{B}.$$

**Example 14.2.2.** Let  $C = (1, 2)$  and  $D = [2, 5]$ . Then  $F = C \cup D$  is connected.

**Example 14.2.3.** The set of rational numbers  $\mathbb{Q}$  is disconnected. For example, we can take  $A = \mathbb{Q} \cap (-\infty, \sqrt{2})$  and  $B = \mathbb{Q} \cap (\sqrt{2}, \infty)$ . Clearly  $A \cup B = \mathbb{Q}$  but  $A$  and  $B$  are separated.

**Theorem 14.1.** A set  $E \subseteq \mathbb{R}$  is connected if and only if for every  $A, B \neq \emptyset$  such that  $E = A \cup B$ , there exists  $\{x_n\} \subseteq A$  such that  $\lim_{n \rightarrow \infty} x_n = x \in B$  or  $\{x_n\} \subseteq B$  such that  $\lim_{n \rightarrow \infty} x_n = x \in A$ .

*Proof.* ( $\Rightarrow$ ) Argue by contrapositive and suppose that the latter condition fails. Then we can find  $A^0, B^0 \neq \emptyset$  such that for all  $\{x_n\} \subseteq A^0$ , we have  $\lim_{n \rightarrow \infty} x_n = x \notin B^0$ . So  $\overline{A^0} \cap B^0 = \emptyset$ . By a similar argument,  $\overline{B^0} \cap A^0 = \emptyset$ . So  $E$  is disconnected.

( $\Leftarrow$ ) Use a similar argument by contrapositive. □

**Theorem 14.2.** A set  $E \subseteq \mathbb{R}$  is connected if and only if for any  $a, b \in E$ , we have  $a < c < b$  implies  $c \in E$ .

*Proof.* ( $\Rightarrow$ ) Let  $a, b \in E$  and pick  $a < c < b$ . Set  $A = (-\infty, c) \cap E$  and  $B = (c, \infty) \cap E$ . Note that  $A, B \neq \emptyset$  since  $a \in A$  and  $b \in B$ . Furthermore,

$$\overline{A} \cap B = A \cap \overline{B} = \emptyset.$$

If  $E = A \cup B$ , then  $E$  is disconnected, so  $A \cup B \neq E$ . But  $(A \cup B)^c = \{c\}$ , so we must have  $c \in E$ .

( $\Leftarrow$ ) Let  $E = A \cup B$  such that  $A, B \neq \emptyset$  and  $A \cap B = \emptyset$ . Take  $a_0 \in A, b_0 \in B$ . Without loss of generality, assume  $a_0 < b_0$ . Consider  $I_0 = [a_0, b_0]$ . Bisect  $I_0$  let  $x_0$  be the center of  $I_0$ . Since  $a_0 < x_0 < b_0$  and  $a_0, b_0 \in E$ , we must have  $x_0 \in E$ . But  $E = A \cup B$ , so  $x_0 \in A$  or  $x_0 \in B$ . If  $x_0 \in A$ , set  $a_1 = x_0$  and  $b_1 = b_0$ . Otherwise, set  $a_1 = a_0$  and  $b_1 = x_0$ . Then take  $I_1 = [a_1, b_1]$ . Repeat this process to construct a sequence of nested intervals with  $I_n = [a_n, b_n]$  such that  $a_n \in A$  and  $b_n \in B$ . By the nested interval property, we have  $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$ . But

$$|I_n| = |b_n - a_n| = \frac{1}{2^n}(b_0 - a_0) \rightarrow 0,$$

so  $\bigcap_{n=1}^{\infty} I_n = \{x\}$  for some  $x \in \mathbb{R}$ . So  $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} b_n = x$ . Since  $I_n \subseteq E$ , we have  $x \in E = A \cup B$ . So  $x \in A$  or  $x \in B$ . If  $x \in A$ , then  $A \cup \overline{B} \neq \emptyset$ , and if  $x \in B$ , then  $\overline{A} \cap B \neq \emptyset$ . Thus  $E$  is connected. □

## 14.3 Density in a Set

**Definition 14.3.** A set  $A \subseteq \mathbb{R}$  is called an  $F_\sigma$  **set** if it can be written as a countable union of closed sets. A set  $A \subseteq \mathbb{R}$  is called a  $G_\delta$  **set** if it can be written as a countable intersection of open sets.

**Remark.** Recall that the countable union of closed sets is not necessarily closed, and the countable intersection of open sets is not necessarily open.

**Definition 14.4.** A set  $G \subseteq \mathbb{R}$  is **dense** in  $\mathbb{R}$  if for every  $a, b \in \mathbb{R}$ , there exists  $x \in G$  such that  $a < x < b$ .

**Remark.** Recall that  $\mathbb{Q}$  is dense in  $\mathbb{R}$ .

**Theorem 14.3.** If  $\{G_1, G_2, \dots\}$  is a countable collection of dense open sets, then  $\bigcap_{n=1}^{\infty} G_n \neq \emptyset$ .

*Proof.* Take  $x_1 \in G_1$ . Since  $G_1$  is open, there exists  $\epsilon_1$  such that  $(x_1 - \epsilon_1, x_1 + \epsilon_1) \subseteq G_1$ . Take

$$I_1 = \left[ x_1 - \frac{\epsilon_1}{2}, x_1 + \frac{\epsilon_1}{2} \right] \subseteq G_1.$$

Since  $G_2$  is dense,  $I_1 \cap G_2 \neq \emptyset$ . Take  $x_2 \in I_1 \cap G_2$  such that  $x_2$  is not an endpoint of  $I_1$ . Since  $G_2$  is open, we can similarly find  $\epsilon_2$  such that  $(x_2 - \epsilon_2, x_2 + \epsilon_2) \subseteq G_2$ . Then take

$$I_2 = \left[ x_2 - \frac{\epsilon_2}{2}, x_2 + \frac{\epsilon_2}{2} \right] \cap I_1 \subseteq G_2.$$

Repeat this to get the sequence of nested intervals  $I_1 \supseteq I_2 \supseteq I_3 \supseteq \dots$ . Then  $\emptyset \neq \bigcap_{n=1}^{\infty} I_n \subseteq \bigcap_{n=1}^{\infty} G_n$ .  $\square$

**Corollary 14.3.1.** *The set of real numbers  $\mathbb{R}$  cannot be written as*

$$\mathbb{R} = \bigcup_{n=1}^{\infty} F_n$$

*such that  $F_n$  is closed and  $F_n$  contains no open nonempty interval.*

*Proof.* Suppose

$$\underbrace{\mathbb{R}^c}_{=\emptyset} = \left( \bigcup_{n=1}^{\infty} F_n \right)^c = \bigcap_{n=1}^{\infty} F_n^c$$

by de Morgan's laws. Then  $F_n^c$  is open and dense in  $\mathbb{R}$ . To see that  $F_n^c$  is dense in  $\mathbb{R}$ , let  $a, b \in \mathbb{R}$ . Note that  $(a, b) \not\subseteq F_n$ , so  $(a, b) \cap F_n^c \neq \emptyset$ . Hence there exists  $x \in F_n^c$  such that  $a < x < b$ . Then this is a contradiction with Theorem 14.3.  $\square$

**Definition 14.5.** A set  $E$  is **nowhere dense** if  $\overline{E}$  contains no nonempty open interval.

**Remark.** This is equivalent to saying that  $\overline{E}^c$  is dense.



# Lecture 15

## Oct. 17 – Functional Limits

### 15.1 The Limit of a Function

**Definition 15.1.** Let  $A \subseteq \mathbb{R}$  and  $f : A \rightarrow \mathbb{R}$ . Further let  $c$  be a limit point of  $A$  (not necessarily in  $A$ ). We say that

$$\lim_{x \rightarrow c} f(x) = L$$

provided that for all  $\epsilon > 0$ , there exists  $\delta > 0$  such that for all  $x \in A$ , if  $0 < |x - c| < \delta$ , then  $|f(x) - L| < \epsilon$ .

**Remark.** We do not require  $f$  to be defined at  $c$ .

**Example 15.1.1.** For the function  $f : \mathbb{R} \rightarrow \mathbb{R}$  given by

$$f = \begin{cases} 1 & \text{if } x \neq 0 \\ 5 & \text{if } x = 0, \end{cases}$$

we have  $\lim_{x \rightarrow 0} f(x) = 1$ .

**Example 15.1.2.** Take  $A = (-\infty, 0) \cup (0, \infty)$  and  $f = 1$  in  $A$ . Then  $\lim_{x \rightarrow 0} f(x) = 1$ .

We can alternatively use the following topological definition:

**Definition 15.2.** We say that

$$\lim_{x \rightarrow c} f(x) = L$$

if for all  $\epsilon > 0$ , there exists  $\delta > 0$  such that if  $x \in A$  with  $x \neq c$  and  $x \in V_\delta(c)$ , then  $f(x) \in V_\epsilon(L)$ .

### 15.2 Examples of Showing Limits

**Exercise 15.1.** Let  $f(x) = 3x + 1$ . Then  $\lim_{x \rightarrow 2} f(x) = 7$ .

*Proof.* Fix  $\epsilon > 0$  and let  $\delta = \frac{\epsilon}{3}$ . Then for all  $0 < |x - 2| < \delta$ , we have

$$|f(x) - 7| = |3x + 1 - 7| = |3x - 6| = 3|x - 2| < 3\delta = \epsilon.$$

Therefore  $\lim_{x \rightarrow 2} f(x) = 7$ . □

**Exercise 15.2.** Let  $f(x) = x^2$ . Then  $\lim_{x \rightarrow 2} f(x) = 4$ .

*Proof.* Note that

$$|f(x) - 4| = |x^2 - 4| = |x - 2||x + 2|.$$

Since we only care about  $f(x)$  when  $x$  is near 2, we take  $|x - 2| \leq 1$ . In this region,  $|x + 2| \leq 5$ .

Now fix  $\epsilon > 0$  and take  $\delta_1 = \frac{\epsilon}{5}$ . Then in the region  $|x - 2| \leq 1$ , for any  $0 < |x - 2| < \delta_1$ , we have

$$|f(x) - 4| = |x + 2||x - 2| < 5\delta_1 = 5 \cdot \frac{\epsilon}{5} = \epsilon.$$

Taking  $\delta = \min\{1, \delta_1\}$  does the job for any arbitrary  $0 < |x - 2| < \delta$ . So  $\lim_{x \rightarrow 2} f(x) = 4$ . □

**Exercise 15.3.** Let  $f(x) = x^3$ . Then  $\lim_{x \rightarrow 2} f(x) = 8$ .

*Proof.* Note that

$$|x^3 - 8| = |x - 2||x^2 + 2x + 4|.$$

First restrict  $|x - 2| < 1$ . Then

$$|x^2 + 2x + 4| \leq 9 + 6 + 4 = 19$$

in this region. So take  $\delta_1 = \frac{\epsilon}{19}$  and  $\delta = \min\{1, \delta_1\}$ . □

# Lecture 16

## Oct. 19 – Continuity

### 16.1 Algebraic Properties of Functional Limits

**Theorem 16.1.** Let  $f : A \rightarrow \mathbb{R}$  and  $c$  be a limit point of  $A$ . Then the following two statements are equivalent:

1.  $\lim_{x \rightarrow c} f(x) = L$ .
2. For any sequence  $\{x_n\} \subseteq A$  such that  $x_n \neq c$  and  $\lim_{n \rightarrow \infty} x_n = c$ , we have  $\lim_{n \rightarrow \infty} f(x_n) = L$ .

*Proof.* (1)  $\Rightarrow$  (2): By assumption, for all  $\epsilon > 0$ , there exists  $\delta > 0$  such that for any  $x \in A$ ,  $0 < |x - c| < \delta$  implies  $|f(x) - L| < \epsilon$ . Now let  $\{x_n\} \subseteq A$  be an arbitrary sequence satisfying  $x_n \neq c$  and  $\lim_{n \rightarrow \infty} x_n = c$ . Since  $x_n$  converges, for the  $\delta$  given above we can find  $N$  such that  $|x_n - c| < \delta$  for all  $n \geq N$ . Then by (1), we have  $|f(x_n) - L| < \epsilon$ . So for all  $n \geq N$ , we have  $|f(x_n) - L| < \epsilon$ , which means that  $\lim_{n \rightarrow \infty} f(x_n) = L$ .

(2)  $\Rightarrow$  (1): Argue by contradiction. Suppose that (1) fails. So there exists some  $\epsilon_0 > 0$  such that for all  $\delta > 0$  and  $|x - c| < \delta$ , we have  $|f(x) - L| > \epsilon_0$ . Fix  $\epsilon_0$  and take  $\delta_n = \frac{1}{n}$ . Pick  $x_n \neq c$  such that  $|x_n - c| < \frac{1}{n}$ . By construction,  $|f(x_n) - L| > \epsilon_0$ . Also note that clearly  $\lim_{n \rightarrow \infty} x_n = c$ . Then by (2), we have  $\lim_{n \rightarrow \infty} f(x_n) = L$ . Contradiction.  $\square$

**Corollary 16.1.1** (Algebraic limit theorem for functions). Given  $\lim_{x \rightarrow c} f(x) = L$  and  $\lim_{x \rightarrow c} g(x) = M$ ,

1.  $\lim_{x \rightarrow c} kf(x) = kL$ ,
2.  $\lim_{x \rightarrow c} (f(x) + g(x)) = L + M$ ,
3.  $\lim_{x \rightarrow c} f(x)g(x) = LM$ ,
4.  $\lim_{x \rightarrow c} \frac{f(x)}{g(x)} = \frac{L}{M}$  if  $M \neq 0$ .

*Proof.* Convert all the limits to sequences and apply the algebraic limit theorem.  $\square$

**Corollary 16.1.2.** Let  $f : A \rightarrow \mathbb{R}$  and  $c$  be a limit point of  $A$ . If there exist sequences  $\{x_n\}, \{y_n\} \subseteq A$  such that

$$\lim_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} y_n = c$$

but

$$\lim_{n \rightarrow \infty} f(x_n) \neq \lim_{n \rightarrow \infty} f(y_n),$$

then  $\lim_{x \rightarrow c} f(x)$  does not exist.

**Example 16.0.1.** Let  $f(x) = \sin(\frac{1}{x})$  and  $A = (-\infty, 0) \cup (0, \infty)$ . Note that  $c = 0$  is a limit point of  $A$ . Take

$$x_n = \frac{1}{2n\pi}, \quad y_n = \frac{1}{2n\pi + \frac{\pi}{2}}.$$

Then  $\lim_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} y_n = 0$ , but

$$f(x_n) = \sin(2n\pi) = 0, \quad f(y_n) = \sin(2n\pi + \frac{\pi}{2}) = 1.$$

So  $\lim_{x \rightarrow 0} f(x)$  does not exist because  $f$  oscillates too much at  $x = 0$ .

**Example 16.0.2.** Let

$$f(x) = \begin{cases} 1, & x < 0 \\ 2, & x \geq 0. \end{cases}$$

The limit does not exist at  $x = 0$  since there is a jump discontinuity.

**Example 16.0.3.** The function  $f(x) = \frac{1}{x}$  does not have a limit at  $x = 0$  since it diverges to infinity.

## 16.2 Continuous Functions

**Definition 16.1.** A function  $f : A \rightarrow \mathbb{R}$  is **continuous** at  $c \in A$  if for all  $\epsilon > 0$ , there exists  $\delta > 0$  such that whenever  $|x - c| < \delta$ , we have  $|f(x) - f(c)| < \epsilon$ .

**Remark.** Here we require  $f$  to be defined at  $c$ .

**Definition 16.2.** If  $f$  is continuous at every point of  $A$ , then we say  $f$  is **continuous on  $A$** .

**Theorem 16.2.** Let  $f : A \rightarrow \mathbb{R}$  and  $c$  be a limit point of  $A$ . Then  $f$  is continuous if one of the following hold:

1. For all  $\epsilon > 0$ , there exists  $\delta > 0$  such that whenever  $|x - c| < \delta$  for  $x \in A$ , then  $|f(x) - f(c)| < \epsilon$ .
2. For all  $\epsilon > 0$ , there exists  $\delta > 0$  such that if  $x \in V_\delta(c)$ , then  $f(x) \in V_\epsilon(f(c))$ .
3. For any sequence  $\{x_n\} \subseteq A$  such that  $\lim_{n \rightarrow \infty} x_n = c$ , we have  $\lim_{n \rightarrow \infty} f(x_n) = f(c)$ .

*Proof.* These follow from the topological and sequential definitions of limits. □

**Corollary 16.2.1.** Let  $f : A \rightarrow \mathbb{R}$  and  $c$  be a limit point of  $A$ . If there exists  $\{x_n\} \subseteq A$  with  $\lim_{n \rightarrow \infty} x_n = c$  but  $f(x_n) \not\rightarrow f(c)$ , then  $f$  is not continuous at  $c$ .

**Theorem 16.3.** Assume that  $f, g : A \rightarrow \mathbb{R}$  are continuous at  $c$ . Then for every  $k \in \mathbb{R}$ , each of  $kf, f + g, fg$  are also continuous at  $c$ . Also  $\frac{f}{g}$  is continuous at  $c$  if  $g(c) \neq 0$ .

*Proof.* Convert all of these to sequences. □

**Theorem 16.4.** Let  $f : A \rightarrow \mathbb{R}$  and  $g : B \rightarrow \mathbb{R}$  and assume that  $f(A) = \{f(x) : x \in A\} \subseteq B$ . Suppose that  $f(x)$  is continuous at  $c$  and  $g$  is continuous at  $f(c)$ . Then  $(g \circ f)(x) = g(f(x))$  is continuous at  $c$ .

*Proof.* For all  $\epsilon > 0$ , since  $g$  is continuous at  $f(c)$ , there exists  $\eta > 0$  such that if  $|y - f(c)| < \eta$ , one has  $|g(y) - g(f(c))| < \epsilon$ . Since  $f$  is continuous at  $c$ , for the  $\eta$  given above, we can find  $\delta > 0$  such that if  $|x - c| < \delta$ , one has  $|f(x) - f(c)| < \eta$ . Therefore, for every  $\epsilon > 0$ , we can find  $\delta$  such that if  $|x - c| < \delta$ , we have  $|g(f(x)) - g(f(c))| < \epsilon$  since  $|f(x) - f(c)| < \eta$ . So  $\lim_{x \rightarrow c} g(f(x)) = g(f(c))$ . □

## 16.3 Examples of Continuous Functions

**Example 16.2.1.** All polynomials are continuous on  $\mathbb{R}$ . To see this, first note that  $f(x) = x$  is continuous on  $\mathbb{R}$ . For all  $c > 0$ , we have  $f(c) = c$ . Now fix a  $c \in \mathbb{R}$ . Then for all  $\epsilon > 0$ , take  $\delta = \epsilon$ . Then for all  $x$  with  $|x - c| < \delta$ , we have

$$|f(x) - f(c)| = |x - c| < \delta = \epsilon.$$

So  $f$  is continuous at  $c$  and thus on all of  $\mathbb{R}$ . Now note that any polynomial  $p(x)$  can be written as

$$p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n.$$

Since constant functions are obviously continuous and we can write  $x^2 = x \cdot x$ , we conclude by the algebraic limit theorem that  $p(x)$  is also continuous on  $\mathbb{R}$ .

**Example 16.2.2.** Let

$$g(x) = \begin{cases} x \sin\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0. \end{cases}$$

Then  $g(x)$  is continuous at  $x = 0$ . This is because

$$|g(x) - g(0)| = \left| x \sin\left(\frac{1}{x}\right) - 0 \right| = \left| x \sin\left(\frac{1}{x}\right) \right| \leq |x|.$$

So taking  $\delta = \epsilon$  works for any  $\epsilon > 0$ .

**Example 16.2.3.** Let  $A = \{x \in \mathbb{R} : x \geq 0\}$  and define  $f : A \rightarrow \mathbb{R}$  by  $f(x) = \sqrt{x}$ . Then  $f$  is continuous on  $A$ . We can split the proof into two cases,  $c = 0$  and  $c > 0$ . First let  $c = 0$ . Then for all  $\epsilon > 0$ , take  $\delta = \epsilon^2$ . Then for any  $x \in A$  such that  $|x - 0| < \delta$ , we have

$$|f(x) - f(0)| = |\sqrt{x} - \sqrt{0}| = |\sqrt{x}| < \sqrt{\delta} = \epsilon.$$

So  $f$  is continuous at 0. Now consider  $c > 0$ . Note that

$$|f(x) - f(c)| = |\sqrt{x} - \sqrt{c}| = \left| \frac{(\sqrt{x} - \sqrt{c})(\sqrt{x} + \sqrt{c})}{\sqrt{x} + \sqrt{c}} \right| = \frac{|x - c|}{\sqrt{x} + \sqrt{c}} \leq \frac{|x - c|}{\sqrt{c}}.$$

Then for any  $\epsilon > 0$ , take  $\delta = \epsilon\sqrt{c}$ . Then for any  $x \in A$  such that  $|x - c| < \delta$ , we have

$$|f(x) - f(c)| \leq \frac{|x - c|}{\sqrt{c}} < \frac{\delta}{\sqrt{c}} = \epsilon.$$

So  $f$  is also continuous at  $c$ . Thus  $f$  is continuous on all of  $A$ .

**Example 16.2.4.** Let  $h(x) = \lfloor x \rfloor$ , where  $h(x)$  takes the largest integer  $n \leq x$ . Then we claim that  $h$  is not continuous on  $\mathbb{Z}$ , but it is continuous on  $\mathbb{R} \setminus \mathbb{Z}$ . For the first part, let  $m \in \mathbb{Z}$  and note that  $h(m) = m$ . Take  $x_n = m - \frac{1}{n}$ . Then  $\lim_{n \rightarrow \infty} x_n = m$  but  $h(x_n) = m - 1$ , so  $\lim_{n \rightarrow \infty} h(x_n) \neq h(m)$ . Thus  $h$  is not continuous at  $m$ . Now for the second part, let  $c \in \mathbb{R} \setminus \mathbb{Z}$ . Then there exists  $n \in \mathbb{Z}$  such that  $n < c < n + 1$ , so that  $h(c) = n$ . We can let  $\delta = \min\{c - n, n + 1 - c\}$ . Then for all  $|x - c| < \delta$ , we have  $h(x) = n = h(c)$ . So  $h$  is continuous at  $c$ .

# Lecture 17

## Oct. 24 – Uniform Continuity

### 17.1 Continuous Functions on Compact Sets

Recall that the notation for the image of a set  $B$  under a function  $f$  is  $f(B) = \{f(x) : x \in B\}$ .

**Theorem 17.1.** *Let  $f : A \rightarrow \mathbb{R}$  be continuous on  $A$  and let  $K \subseteq A$  be compact. Then  $f(K)$  is also compact.*

*Proof.* Take a sequence  $\{y_n\} \subseteq f(K)$ . So we can find  $x_n \in K$  such that  $f(x_n) = y_n$ . Note that  $\{x_n\} \subseteq K$ . Since  $K$  is compact, we can find  $\{x_{n_k}\}$  such that

$$\lim_{k \rightarrow \infty} x_{n_k} = x \in K.$$

Since  $f$  is continuous at  $x$  (since  $x \in K \subseteq A$ ), we have  $\lim_{k \rightarrow \infty} f(x_{n_k}) = f(x)$ . In other words,

$$\lim_{k \rightarrow \infty} y_{n_k} = f(x) \in f(K).$$

Thus  $f(K)$  is compact. □

**Remark.** The topological proof of this theorem goes as follows: Take an open cover of  $f(K)$ . Then the preimage of this open cover under  $f$  is an open cover of  $K$  (since the preimage of an open set under a continuous function is open). Since  $K$  is compact, we can find a finite subcover. Taking the image again of this subcover yields a finite subcover of  $f(K)$ .

**Theorem 17.2.** *If  $f : K \rightarrow \mathbb{R}$  is continuous on a compact set  $K \subseteq \mathbb{R}$ , then  $f$  attains both a maximum and minimum value on  $K$  (i.e. there exist  $x_0, x_1 \in K$  such that  $f(x_0) \leq f(x) \leq f(x_1)$  for all  $x \in K$ ).*

*Proof.* Note that  $K$  is compact, so  $f(K)$  is also compact. In particular,  $f(K)$  is bounded, so

$$\alpha = \sup_{x \in K} f(x)$$

exists. By construction, for every  $\epsilon > 0$ , we can always find  $x_\epsilon \in K$  such that  $f(x_\epsilon) \geq \alpha - \epsilon$ . Pick  $\epsilon = \frac{1}{n}$  and find  $x_{\epsilon_n}$  such that  $f(x_{\epsilon_n}) \geq \alpha - \frac{1}{n}$ . Then  $\{x_{\epsilon_n}\}$  is a sequence in  $K$ . Since  $K$  is compact, we can find

$$\lim_{k \rightarrow \infty} x_{\epsilon_{n_k}} = x \in K.$$

Note that

$$\alpha - \frac{1}{n_k} \leq f(x_{\epsilon_{n_k}}) \leq \alpha,$$

so letting  $k \rightarrow \infty$  and by the continuity of  $f$ , we have

$$\lim_{k \rightarrow \infty} \left( \alpha - \frac{1}{n_k} \right) \leq \lim_{k \rightarrow \infty} f(x_{\epsilon_{n_k}}) \leq \lim_{k \rightarrow \infty} \alpha.$$

This implies  $\alpha \leq f(x) \leq \alpha$ . So  $f(x) = \alpha$  is the maximum. The proof for a minimum is similar.  $\square$

**Remark.** In this proof, we find  $x_1$  and  $x_0$  in addition to simply showing existence.

## 17.2 Uniform Continuity

**Definition 17.1.** A function  $f : A \rightarrow \mathbb{R}$  is **uniformly continuous** on  $A$  if for every  $\epsilon > 0$ , there exists  $\delta > 0$  such that for all  $x, y \in A$  and  $|x - y| < \delta$ , one has  $|f(x) - f(y)| < \epsilon$ .

**Remark.** Here,  $\delta$  does not depend on a specific choice of  $x$ , as opposed to the usual continuity.

**Example 17.1.1.** The function  $f(x) = 3x + 1$  is uniformly continuous on  $\mathbb{R}$ . For any  $\epsilon > 0$ , take  $\delta = \epsilon/3$ . Then for all  $x, y \in \mathbb{R}$  with  $|x - y| < \delta$ , we have

$$|f(x) - f(y)| = 3|x - y| < 3\delta = \epsilon,$$

as required.

**Example 17.1.2.** Let  $g(x) = x^2$  on  $\mathbb{R}$ . If we let  $|x - y| < \delta$ , we have

$$|g(x) - g(y)| = |x + y||x - y| \leq |x + y|\delta.$$

Here, we clearly need  $\delta$  to depend on  $x$  and  $y$ , so  $g$  seems like it cannot be uniformly continuous.

**Theorem 17.3.** A function  $f : A \rightarrow \mathbb{R}$  fails to be uniformly continuous if and only if there exists  $\epsilon_0 > 0$  and two sequences  $\{x_n\}, \{y_n\} \subseteq A$  such that  $|x_n - y_n| \rightarrow 0$  but  $|f(x_n) - f(y_n)| \geq \epsilon_0$ .

*Proof.* ( $\Rightarrow$ ) This implies that there exists  $\epsilon_0 > 0$  such that for all  $\delta > 0$ , there exist  $x, y \in A$  with  $|x - y| < \delta$  but  $|f(x) - f(y)| \geq \epsilon_0$ . Take  $\delta = \frac{1}{n}$  and choose  $x_n, y_n \in A$  such that  $|x_n - y_n| < \frac{1}{n}$  and  $|f(x_n) - f(y_n)| \geq \epsilon_0$ .

( $\Leftarrow$ ) Take  $\epsilon = \epsilon_0$ . Then for all  $\delta > 0$ , since  $|x_n - y_n| \rightarrow 0$ , we can find  $|x_n - y_n| < \delta$ . But  $|f(x_n) - f(y_n)| \geq \epsilon_0$ . So  $f$  is not uniformly continuous.  $\square$

**Example 17.1.3.** Recall the example of  $g(x) = x^2$ . Take the sequences  $x_n = n$  and  $y_n = n + \frac{1}{n}$ . Then  $|x_n - y_n| = \frac{1}{n} \rightarrow 0$  but

$$|g(x_n) - g(y_n)| = 2 + \frac{1}{n^2} \geq 2,$$

so  $g$  is not uniformly continuous by the previous theorem. This is the formal proof.

**Example 17.1.4.** The function  $f(x) = \sin(\frac{1}{x})$  is not uniformly continuous. Let

$$x_n = \frac{1}{\frac{1}{2}\pi + 2n\pi}, \quad y_n = \frac{1}{\frac{3}{2}\pi + 2n\pi}.$$

Then  $f(x_n) = 1$  and  $f(y_n) = -1$ . So  $|x_n - y_n| \rightarrow 0$  but  $|f(x_n) - f(y_n)| = 2$ .

**Theorem 17.4.** If a function  $f$  is continuous over a compact set  $K$ , then  $f$  is uniformly continuous on  $K$ .

*Proof.* Argue by contradiction. Suppose  $f$  is not uniformly continuous. Then for some  $\epsilon_0 > 0$ , we can find sequences  $\{x_n\}, \{y_n\} \in K$  such that  $|x_n - y_n| \rightarrow 0$  and  $|f(x_n) - f(y_n)| \geq \epsilon_0$ . Since  $K$  is compact, we can find a subsequence  $\{x_{n_k}\}$  such that

$$\lim_{k \rightarrow \infty} x_{n_k} = x \in K.$$

Then we have

$$\lim_{k \rightarrow \infty} y_{n_k} = \lim_{k \rightarrow \infty} (y_{n_k} - x_{n_k}) + \lim_{k \rightarrow \infty} x_{n_k} = x$$

since  $|x_n - y_n| \rightarrow 0$ . Then by the continuity of  $f$  at  $x$ , we have

$$\lim_{k \rightarrow \infty} f(x_{n_k}) = f(x) = \lim_{k \rightarrow \infty} f(y_{n_k}),$$

so  $\lim_{k \rightarrow \infty} |f(x_{n_k}) - f(y_{n_k})| = 0$ . But by construction,  $|f(x_{n_k}) - f(y_{n_k})| \geq \epsilon_0$ . Contradiction.  $\square$

## 17.3 Intermediate Value Theorem

**Theorem 17.5** (Intermediate value theorem). *Let  $f : [a, b] \rightarrow \mathbb{R}$  be continuous. If there is  $L \in \mathbb{R}$  such that  $f(a) < L < f(b)$  or  $f(b) < L < f(a)$ , then there exists  $c \in (a, b)$  such that  $f(c) = L$ .*

*Proof.* It suffices to show that if  $f : [a, b] \rightarrow \mathbb{R}$  is continuous with  $f(a) < 0$  and  $f(b) > 0$ , then there exists  $c \in (a, b)$  such that  $f(c) = 0$ . This is because in the first case, we can define

$$g(x) = f(x) - L.$$

Then  $g(a) < 0$  and  $g(b) > 0$ . Similarly take  $h(x) = L - f(x)$  in the second case.

Now we prove the claim. Define

$$K = \{x \in (a, b) : f(x) \leq 0\}.$$

Note that  $f(a) < 0$  and  $a \in K$ , so  $K \neq \emptyset$ . Note that  $K$  is bounded above since  $b$  is an upper bound (since  $f(b) > 0$ ). Then by the axiom of completeness,  $c = \sup K$  exists. We show that  $f(c) > 0$  and  $f(c) < 0$  both lead to contradictions.

Suppose that  $f(c) > 0$ . Take  $\epsilon = \frac{f(c)}{2}$ . Then since  $f$  is continuous, we can find  $\delta$  such that

$$|f(x) - f(c)| < \frac{f(c)}{2}$$

whenever  $|x - c| < \delta$ . Then we have

$$-\frac{f(c)}{2} < f(x) - f(c) < \frac{f(c)}{2},$$

and rearranging yields

$$0 < \frac{f(c)}{2} < f(x) < \frac{3}{2}f(c).$$

Then any  $x$  such that  $|x - c| < \delta$  is an upper bound for  $K$ , so  $c$  is not the smallest upper bound for  $K$ . Contradiction.

The case where  $f(c) < 0$  leads to a similar contradiction. So we must have  $f(c) = 0$ .  $\square$

**Remark.** Alternatively, we can define  $I_0 = [a, b]$  and let  $z = \frac{a+b}{2}$ . Then if  $f(z) \leq 0$ , set  $a_1 = z$  and  $b_1 = b$ . Otherwise, set  $a_1 = a$  and  $b_1 = z$ . Then set  $I_1 = [a_1, b_1]$ , noting that  $I_1 \subseteq I_0$  and  $|I_1| = \frac{1}{2}|I_0|$ . Repeat this process and apply the nested interval property to find  $c \in \bigcap_{n=0}^{\infty} I_n$ . Then show that  $f(c) = 0$ .



# Lecture 18

## Oct. 26 – Discontinuity

### 18.1 Topological Proof of the Intermediate Value Theorem

**Theorem 18.1.** Let  $f : G \rightarrow \mathbb{R}$  be continuous. If  $E \subseteq G$  is connected, then  $f(E)$  is connected.

*Proof.* Suppose  $f(E) = A \cup B$  such that  $A \cap B = \emptyset$  and  $A, B \neq \emptyset$ . Let  $C = f^{-1}(A)$  and  $D = f^{-1}(B)$ . Note that  $C \cup D = E$  and  $C, D \neq \emptyset$  (since  $A, B \neq \emptyset$ ). Furthermore,  $C \cap D = \emptyset$  since  $A \cap B = \emptyset$ . Since  $E$  is connected, either  $\overline{C} \cap D \neq \emptyset$  or  $C \cap \overline{D} \neq \emptyset$ . So either we can find  $\{x_n\} \subseteq C$  such that  $\lim_{n \rightarrow \infty} x_n = x \in D$  or  $\{x_n\} \subseteq D$  such that  $\lim_{n \rightarrow \infty} x_n = x \in C$ . Without loss of generality, assume we are in the first case (just rename otherwise). So we have  $f(x_n) \in A$  and  $f(x) \in B$ . Since  $f$  is continuous,  $\lim_{n \rightarrow \infty} f(x_n) = f(x)$ , and thus  $f(x) \in \overline{A} \cap B \neq \emptyset$ . So  $f(E)$  is connected.  $\square$

Note that in  $\mathbb{R}$ , a connected set is precisely an interval, so we are now ready to give another proof of the intermediate value theorem:

**Theorem 18.2** (Intermediate value theorem). Let  $f : [a, b] \rightarrow \mathbb{R}$  be continuous. If there is  $L \in \mathbb{R}$  such that  $f(a) < L < f(b)$  or  $f(b) < L < f(a)$ , then there exists  $c \in (a, b)$  such that  $f(c) = L$ .

*Alternative proof.* Since  $[a, b]$  is a connected set,  $f([a, b])$  is also a connected set. Then  $f(a) \in f([a, b])$  and  $f(b) \in f([a, b])$ . Since  $f([a, b])$  is an interval,  $[f(a), f(b)] \subseteq f([a, b])$ . So for all  $L \in [f(a), f(b)]$ , there exists  $c \in [a, b]$  such that  $f(c) = L$ .  $\square$

### 18.2 Sets of Discontinuity

**Definition 18.1.** For  $f : \mathbb{R} \rightarrow \mathbb{R}$ , we define  $D_f \subseteq \mathbb{R}$  to be the set of points where  $f$  fails to be continuous.

Our goal will be to study the structure of  $D_f$ .

**Definition 18.2.** A function  $f : A \rightarrow \mathbb{R}$  is **increasing** if  $f(x) \leq f(y)$  for  $x < y$  with  $x, y \in A$ . Similarly, a function is **decreasing** if  $f(x) \geq f(y)$  for all  $x < y$  with  $x, y \in A$ .

**Definition 18.3.** A **monotone** function is a one that is either increasing or decreasing.

**Definition 18.4.** Let  $f : A \rightarrow \mathbb{R}$  and  $c$  be a limit point of  $A$ . Then

$$\lim_{x \rightarrow c^+} f(x) = L$$

if for all  $\epsilon > 0$ , there exists  $\delta > 0$  such that for all  $0 < x - c < \delta$ , one has  $|f(x) - f(c)| < \epsilon$ . Similarly, we define  $\lim_{x \rightarrow c^-} f(x) = L$  if the same is true for  $0 < c - x < \delta$ .

**Theorem 18.3.** Let  $f : A \rightarrow \mathbb{R}$  and  $c$  be a limit point of  $A$ . Then  $\lim_{x \rightarrow c} f(x) = L$  if and only if

$$\lim_{x \rightarrow c^+} f(x) = \lim_{x \rightarrow c^-} f(x) = L.$$

*Proof.* Homework problem. □

**Theorem 18.4.** Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a monotone function. Then  $D_f$  is either finite or countable.

*Proof.* Without loss of generality, assume that  $f$  is increasing. Then let  $c \in D_f$  and let  $\{x_n\}$  be an increasing sequence such that  $\lim_{n \rightarrow \infty} x_n = c$ . Then  $\{f(x_n)\}$  is a monotonically increasing sequence.

Now we check that  $\{f(x_n)\}$  is bounded. To do this, suppose otherwise that  $\{f(x_n)\}$  is not bounded. Then  $f(x_n) \rightarrow \infty$  monotonically as  $n \rightarrow \infty$ . Note that  $x_n \leq c$ . Since  $f(x_n) \rightarrow \infty$  as  $n \rightarrow \infty$ , we cannot define  $f(x)$  for  $x > c$ . This is a contradiction since  $f$  is defined on all of  $\mathbb{R}$ .

Thus  $f(x_n)$  is bounded and increasing, so  $\lim_{n \rightarrow \infty} f(x_n)$  exists by the monotone convergence theorem. Now define  $A_c = \lim_{n \rightarrow \infty} f(x_n)$ , and we claim that  $\lim_{x \rightarrow c^-} f(x) = A_c$ . To show this, take an arbitrary sequence  $\{y_n\}$  with  $y_n < c$  such that  $y_n \rightarrow c$ . Here we claim that  $\lim_{n \rightarrow \infty} f(y_n) = A_c$ . Since we have  $A_c = \lim_{n \rightarrow \infty} f(x_n)$ , for every  $\epsilon > 0$ , we can find  $N_0$  such that for all  $k \geq N_0$ , one has

$$A_c - \epsilon \leq f(x_k) \leq A_c + \epsilon.$$

Since  $y_n \rightarrow c$ , there exists  $N$  such that for any fixed  $n \geq N$ , we have  $x_{N_0} \leq y_n \leq x_{\tilde{N}}$  for some  $x_{\tilde{N}}$ . Then

$$A_c - \epsilon \leq f(x_{N_0}) \leq f(y_n) \leq f(x_{\tilde{N}}) \leq A_c + \epsilon,$$

so  $|f(y_n) - A_c| < \epsilon$  for all  $n \geq N$ . Thus we conclude that  $\lim_{n \rightarrow \infty} f(y_n) = A_c$ , and thus  $\lim_{x \rightarrow c^-} f(x) = A_c$ .

One can similarly show that  $\lim_{x \rightarrow c^+} f(x)$  exists, so let  $B_c = \lim_{x \rightarrow c^+} f(x)$ . Since  $f$  is increasing and  $f$  is not continuous at  $c$ , we have  $A_c < B_c$ . So by the density of  $\mathbb{Q}$  in  $\mathbb{R}$ , we can find  $q_c \in \mathbb{Q}$  such that  $q_c \in (A_c, B_c)$ . Then for any distinct  $c, c' \in D_f$ , we have  $q_c \neq q_{c'}$  since  $f$  is monotone. By construction, this is a bijection from  $D_f$  to  $\{q_c\}_{c \in D_f} \subseteq \mathbb{Q}$  given by  $c \mapsto q_c$ . Thus  $D_f$  is countable since  $\mathbb{Q}$  is countable. □

**Definition 18.5.** Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  and  $\alpha > 0$ . We say that  $f$  is  $\alpha$ -**continuous** at  $x$  if there exists  $\delta > 0$  such that for all  $y, z \in (x - \delta, x + \delta)$ , one has  $|f(y) - f(z)| < \alpha$ .

**Theorem 18.5.** Let  $f : \mathbb{R} \rightarrow \mathbb{R}$ . Then  $D_f$  is an  $F_\sigma$  set.

*Proof.* Let

$$D_f^\alpha = \{x \in \mathbb{R} : f \text{ is not } \alpha\text{-continuous at } x\}.$$

Note that  $D_f^\alpha$  is closed (show that  $(D_f^\alpha)^c$  is open). Also observe that  $D_f^{\alpha'} \subseteq D_f^\alpha$  for  $\alpha' > \alpha$ . Finally, if  $f$  is continuous at  $x$ , then  $f$  is  $\alpha$ -continuous for any fixed  $\alpha$ . So  $D_f^\alpha \subseteq D_f$  for all  $\alpha > 0$ . Then we can write

$$D_f = \bigcup_{n=1}^{\infty} D_f^{\alpha_n}$$

for  $\alpha_n = \frac{1}{n}$ . Note that we already have the reverse inclusion. To show the forward inclusion, we can observe that for any  $x \in D_f$ , one can find  $\alpha_0 > 0$  such that  $x \in D_f^{\alpha_0}$ . Then somehow finish. □

# Lecture 19

## Oct. 31 – Sequences of Functions

### 19.1 Pointwise Convergence

**Definition 19.1.** For all  $n \in \mathbb{N}$ , let  $f_n$  be a function defined on  $A \subseteq \mathbb{R}$ . The sequence of functions  $\{f_n\}$  **converges pointwise** on  $A$  to a function  $f$  if for all  $x \in A$ , we have  $\lim_{n \rightarrow \infty} f_n(x) = f(x)$ .

**Example 19.1.1.** Let  $f_n : \mathbb{R} \rightarrow \mathbb{R}$  be defined by

$$f_n(x) = \frac{x + nx}{n}.$$

For any fixed  $x \in \mathbb{R}$ , we have  $f_n(x) = x + \frac{x}{n} \rightarrow x$  as  $n \rightarrow \infty$ . So  $\{f_n\}$  converges pointwise on  $\mathbb{R}$  to  $f(x) = x$ .

**Example 19.1.2.** Let  $g_n : [0, 1] \rightarrow \mathbb{R}$  be defined by  $g_n(x) = x^n$ . First note that for  $x = 1$ ,

$$g_n(1) = 1^n = 1,$$

so  $\lim_{n \rightarrow \infty} g_n(1) = 1$ . For any fixed  $0 \leq x < 1$ ,

$$\lim_{n \rightarrow \infty} x^n = 0.$$

So  $\{g_n\}$  converges pointwise on  $[0, 1]$  to

$$g(x) = \begin{cases} 1 & \text{if } x = 1 \\ 0 & \text{if } 0 \leq x < 1. \end{cases}$$

**Example 19.1.3.** Let  $h_n : [-1, 1] \rightarrow \mathbb{R}$  be defined by

$$h_n(x) = x^{1 + \frac{1}{2n-1}}.$$

Note that  $2n - 1$  is odd for every  $n$ , so  $h_n(x)$  is well-defined even for  $x < 0$ . Observe that

$$h_n(x) = x \cdot x^{\frac{1}{2n-1}}.$$

For  $x = 0$ , we have  $h_n(0) = 0$ , so  $\lim_{n \rightarrow \infty} h_n(0) = 0$ . For  $0 < x \leq 1$ , we have

$$\lim_{n \rightarrow \infty} x^{\frac{1}{2n-1}} = 1.$$

Finally, if  $-1 \leq x < 0$ , then we have

$$\lim_{n \rightarrow \infty} x^{\frac{1}{2n-1}} = \lim_{n \rightarrow \infty} (-1)^{\frac{1}{2n-1}} (-x)^{\frac{1}{2n-1}} = -1 \lim_{n \rightarrow \infty} (-x)^{\frac{1}{2n-1}} = -1$$

since  $2n - 1$  is odd and  $0 < -x \leq 1$ . Then by the algebraic properties of limits, we have

$$\lim_{n \rightarrow \infty} h_n(x) = \begin{cases} 0 & \text{if } x = 0 \\ x & \text{if } 0 < x \leq 1 \\ -x & \text{if } -1 \leq x < 0. \end{cases}$$

So  $\{h_n\}$  converges pointwise on  $[-1, 1]$  to  $h(x) = |x|$ .

**Remark.** The pointwise limit of a sequence of continuous functions may fail to be continuous, and the pointwise limit of a sequence of differentiable functions may fail to be differentiable.

## 19.2 Uniform Convergence

**Definition 19.2.** Let  $f_n$  be functions on  $A \subseteq \mathbb{R}$ . We say that  $f_n \rightarrow f$  **uniformly** on  $A$  if for all  $\epsilon > 0$ , there exists  $N \in \mathbb{N}$  such that for all  $x \in A$  and  $n \geq N$ , one has  $|f_n(x) - f(x)| < \epsilon$ .

**Example 19.2.1.** Let  $g_n : \mathbb{R} \rightarrow \mathbb{R}$  be defined by

$$g_n = \frac{1}{n(1+x)^2}.$$

For any fixed  $x \in \mathbb{R}$ , we have

$$g_n(x) = \frac{1}{n(1+x)^2} \rightarrow 0,$$

so  $g_n(x) \rightarrow 0$  pointwise on  $\mathbb{R}$ . Further observe that

$$|g_n(x) - 0| = \frac{1}{n(1+x)^2} \leq \frac{1}{n}.$$

So for any  $\epsilon > 0$ , pick  $N \in \mathbb{N}$  such that  $N > \frac{1}{\epsilon}$ . Then for all  $x \in \mathbb{R}$  and  $n \geq N$ , we have

$$|g_n(x) - 0| \leq \frac{1}{n} < \epsilon.$$

So in fact  $g_n(x) \rightarrow 0$  uniformly.

**Example 19.2.2.** Let

$$g_n(x) = \frac{x^2 + nx}{n} = \frac{x^2}{n} + x.$$

For any fixed  $x \in \mathbb{R}$ , we have  $g_n(x) \rightarrow x$ , so  $g_n(x)$  converges pointwise to  $x$  on  $\mathbb{R}$ . Then we can note that

$$|g_n(x) - x| \leq \frac{x^2}{n},$$

so we need to take  $N > \frac{x^2}{\epsilon}$  to satisfy  $|g_n(x) - x| < \epsilon$  for  $n \geq N$ . So this convergence is not uniform.

**Theorem 19.1** (Cauchy criterion for uniform convergence). *Let  $\{f_n\}$  be a sequence of functions defined on  $A \subseteq \mathbb{R}$ . Then  $f_n$  converges uniformly on  $A$  if and only if for all  $\epsilon > 0$ , there exists  $N \in \mathbb{N}$  such that for all  $x \in A$  and  $m, n \geq N$ , we have  $|f_n(x) - f_m(x)| < \epsilon$ .*

*Proof.* Just apply the Cauchy criterion for sequences at every point.  $\square$

**Theorem 19.2.** Let  $\{f_n\}$  be a sequence of functions defined on  $A$  and suppose that  $f_n \rightarrow f$  uniformly on  $A$ . If each  $f_n$  is continuous at  $c \in A$ , then  $f$  is also continuous at  $c$ .

*Proof.* For all  $\epsilon > 0$ , since  $f_n \rightarrow f$  uniformly, we can find  $N \in \mathbb{N}$  such that whenever  $n \geq N$ , we have

$$|f_n(x) - f(x)| < \frac{\epsilon}{3}$$

for all  $x \in A$ . Take  $n = N$ . Since  $f_N$  is continuous at  $c$ , there exists  $\delta$  such that for all  $|x - c| < \delta$ , one has

$$|f_N(x) - f_N(c)| < \frac{\epsilon}{3}.$$

Then we can note that

$$|f(x) - f(c)| = |f(x) - f_N(x) + f_N(x) - f_N(c) + f_N(c) - f(c)|.$$

By the triangle inequality, we have

$$|f(x) - f(c)| \leq |f(x) - f_N(x)| + |f_N(x) - f_N(c)| + |f_N(c) - f(c)|.$$

So for all  $|x - c| < \delta$ , we have

$$|f(x) - f(c)| \leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon,$$

where the first and last terms are from uniform convergence and the middle term is from continuity. So for every  $\epsilon > 0$ , we can find  $\delta > 0$  such that for all  $|x - c| < \delta$ , we have  $|f(x) - f(c)| < \epsilon$ . Thus we can conclude that  $f$  is continuous at  $c$ .  $\square$

**Remark.** We can also pass differentiability and integrability through uniform limits.

## 19.3 Series of Functions

**Definition 19.3.** Let  $f$  and  $f_n$  be functions defined on  $A \subseteq \mathbb{R}$  for each  $n \in \mathbb{N}$ . We say that  $\sum_{n=1}^{\infty} f_n$  **converges pointwise** to  $f(x)$  if the partial sums

$$s_k(x) = \sum_{n=1}^k f_n(x)$$

converge pointwise to  $f(x)$  for every  $x \in A$ . Similarly, we say that  $\sum_{n=1}^{\infty} f_n$  **converges uniformly** to  $f(x)$  if the partial sums converge uniformly to  $f(x)$  for every  $x \in A$ .

**Theorem 19.3.** Let  $f_n(x)$  be a continuous function defined on a set  $A \subseteq \mathbb{R}$  and assume that  $\sum_{n=1}^{\infty} f_n(x)$  converges uniformly to  $f$ . Then  $f$  is continuous on  $A$ .

*Proof.* If  $f_n$  is continuous, then  $s_k(x) = \sum_{n=1}^k f_n(x)$  is continuous. Then we can pass continuity through the limit since the convergence of  $s_k(x)$  is uniform.  $\square$

**Theorem 19.4.** A series  $\sum_{n=1}^{\infty} f_n$  converges uniformly on  $A$  if and only if for all  $\epsilon > 0$ , there exists  $N \in \mathbb{N}$  such that for all  $n > m \geq N$  and  $x \in A$ , one has  $|f_{m+1}(x) + \cdots + f_n(x)| < \epsilon$ .

*Proof.* This is just the Cauchy criterion for series. □

**Corollary 19.4.1.** *For every  $n \in \mathbb{N}$ , let  $f_n$  be a function defined on  $A \subseteq \mathbb{R}$ . Assume*

$$|f_n(x)| \leq M_n$$

*for all  $x \in A$ . If  $\sum_{n=1}^{\infty} M_n$  converges, then  $\sum_{n=1}^{\infty} f_n$  converges uniformly on  $A$ .*

*Proof.* For every  $\epsilon > 0$ , since  $\sum_{i=1}^{\infty} M_n$  converges, there exists  $N$  such that for all  $m, n \geq N$ , we have

$$M_{m+1} + \cdots + M_n < \epsilon.$$

Note here that  $0 \leq |f_n(x)| \leq M_n$ , so we can drop the absolute values. Then for the same choice of  $N$ ,

$$|f_{m+1}(x) + \cdots + f_n(x)| \leq |f_{m+1}(x)| + \cdots + |f_n(x)| \leq M_{m+1} + \cdots + M_n < \epsilon.$$

Apply the Cauchy criterion again to finish. □

# Lecture 20

## Nov. 2 – Power Series

### 20.1 Power Series

**Definition 20.1.** A **power series** is a series of the form

$$\sum_{n=0}^{\infty} a_n x^n.$$

**Theorem 20.1.** *If*

$$\sum_{n=0}^{\infty} a_n x^n$$

*converges at  $x_0 \in \mathbb{R}$ , then it converges absolutely at any  $x$  such that  $|x| < |x_0|$ .*

*Proof.* Since  $\sum_{n=0}^{\infty} a_n x_0^n$  converges, there exists  $M > 0$  such that  $|a_n x_0^n| < M$  for all  $n$ . Then observe

$$|a_n x^n| = |a_n x_0^n| \left| \frac{x^n}{x_0^n} \right| = |a_n x_0^n| \left| \frac{x}{x_0} \right|^n.$$

Now let  $|x| < |x_0|$ , so that

$$\sum_{n=0}^{\infty} |a_n x^n| = \sum_{n=0}^{\infty} |a_n x_0^n| \left| \frac{x}{x_0} \right|^n \leq M \sum_{n=0}^{\infty} \left| \frac{x}{x_0} \right|^n.$$

Note that  $\left| \frac{x}{x_0} \right| < 1$ , so the right hand side converges by geometric series. Thus  $\sum_{n=0}^{\infty} |a_n x^n| < \infty$ .  $\square$

**Remark.** We need to make this argument since  $\sum_{n=0}^{\infty} a_n x_0^n$  need not converge absolutely. So we cannot simply do a direct comparison.

**Theorem 20.2.** *If*

$$\sum_{n=0}^{\infty} a_n x^n$$

*converges absolutely at  $x_0 \in \mathbb{R}$ , then it converges uniformly on  $[-c, c]$  where  $c = |x_0|$ .*

*Proof.* By assumption,  $\sum_{n=0}^{\infty} |a_n x_0^n| < \infty$ . Then for any  $x \in [-c, c]$ , note that  $|a_n x^n| \leq |a_n x_0^n|$ . Then

$$|a_{m+1} x^{m+1} + \cdots + a_n x^n| \leq |a_{m+1} x^{m+1}| + \cdots + |a_n x^n| \leq |a_{m+1} x_0^{m+1}| + \cdots + |a_n x_0^n|$$

for  $m < n$ . For any  $\epsilon$ , we can find  $N$  such that for any  $n > m \geq N$ , we have

$$|a_{m+1}x_0^{m+1}| + \cdots + |a_nx_0^n| < \epsilon.$$

Then by the Cauchy criterion for uniform convergence,  $\sum_{n=0}^{\infty} a_nx^n$  converges uniformly on  $[-c, c]$ .  $\square$

**Remark.** This result holds on a closed interval, whereas the previous holds on an open interval.

**Example 20.1.1.** Consider the sum

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} x^n,$$

which converges at  $x = 1$  by the alternating series test. Then by the previous theorem, this series converges absolutely on  $(-1, 1)$ . However, observe that at  $x = -1$ , we have

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} (-1)^n = \sum_{n=1}^{\infty} \frac{1}{n},$$

which diverges. This is an example of why the interval is open.

## 20.2 Abel's Theorem

**Lemma 20.1** (Abel's lemma). *Let  $\{b_n\}$  be a sequence satisfying  $b_1 \geq b_2 \geq b_3 \cdots \geq 0$  and  $\{a_n\}$  satisfy*

$$|s_k| = \left| \sum_{n=1}^k a_n \right| \leq A$$

*for some  $A > 0$ . Then  $|a_1b_1 + a_2b_2 + \cdots + a_nb_n| \leq Ab_1$ .*

*Proof.* See Homework 5. The key is summation by parts.  $\square$

**Theorem 20.3** (Abel's theorem). *If*

$$\sum_{n=0}^{\infty} a_nx^n$$

*converges at  $x = R > 0$ , then  $\sum_{n=0}^{\infty} a_nx^n$  converges uniformly on  $[0, R]$ . The same result holds for  $R < 0$ .*

*Proof.* Fix  $0 \leq x \leq R$ . Observe that

$$\sum_{n=0}^{\infty} a_nx^n = \sum_{n=0}^{\infty} a_n \frac{x^n}{R^n} R^n.$$

Note that  $\sum_{n=0}^{\infty} a_nR^n$  converges by assumption. By the Cauchy criterion, for every  $\epsilon > 0$ , there exists  $N$  such that for all  $n > m \geq N$ , we have

$$|a_{m+1}R^{m+1} + \cdots + a_nR^n| < \frac{\epsilon}{2}. \quad (+)$$

Now consider

$$|a_{m+1}x^{m+1} + \cdots + a_nx^n| = \left| a_{m+1}R^{m+1} \left( \frac{x}{R} \right)^{m+1} + \cdots + a_nR^n \left( \frac{x}{R} \right)^n \right|. \quad (*)$$



Then pick

$$b_1 = \left(\frac{x}{R}\right), \quad \dots, \quad b_{n-m} = \left(\frac{x}{R}\right)^n,$$

so that  $b_1 \geq b_2 \geq \dots \geq b_{n-m} \geq 0$  since  $1 \geq \frac{x}{R} \geq 0$ . Also pick

$$\alpha_1 = a_{m+1}R^{m+1}, \quad \dots, \quad \alpha_{n-m} = a_n R^n.$$

Then we have

$$(*) = |\alpha_1 b_1 + \alpha_2 b_2 + \dots + \alpha_{n-m} b_{n-m}|.$$

Note that  $\left|\sum_{j=1}^k \alpha_j\right| \leq \frac{\epsilon}{2}$  for  $k \leq n-m$  by (+). Then by Abel's lemma,

$$(*) = |\alpha_1 b_1 + \alpha_2 b_2 + \dots + \alpha_{n-m} b_{n-m}| \leq \frac{\epsilon}{2} \left(\frac{x}{R}\right)^{m+1} < \epsilon.$$

Therefore by the Cauchy criterion,  $\sum_{n=0}^{\infty} a_n x^n$  converges uniformly on  $[0, R]$ . □

**Remark.** Observe that for  $R < 0$  and  $x \in [R, 0]$ , we still have  $0 \leq \frac{x}{R} \leq 1$ , so the proof is the same.

**Theorem 20.4.** *If*

$$\sum_{n=0}^{\infty} a_n x^n$$

*converges pointwise on a set  $A \subseteq \mathbb{R}$ , then it converges uniformly on any compact set  $K \subseteq A$ .*

*Proof.* Since  $K$  is compact, it is closed and bounded. In particular,  $x_0 = \inf K$  exists and  $x_0 \in K$ . Similarly,  $x_1 = \sup K$  exists and  $x_1 \in K$ . Note that  $K \subseteq [x_0, x_1]$ . By assumption,  $\sum_{n=0}^{\infty} a_n x^n$  converges at  $x_0$  and  $x_1$ . Then by Abel's theorem,  $\sum_{n=0}^{\infty} a_n x^n$  converges uniformly on  $[x_0, x_1] \supseteq K$ .<sup>1</sup> □

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<sup>1</sup>To be more precise, we need some case work here depending on the signs of  $x_0$  and  $x_1$ .

# Lecture 21

## Nov. 7 – Weierstrass Approximation Theorem

### 21.1 Weierstrass Approximation Theorem

#### 21.1.1 Bernstein Polynomials

**Definition 21.1.** A **Bernstein polynomial** is a polynomial of the form

$$B_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$$

for  $k = 0, \dots, n$ .<sup>1</sup>

Clearly we have  $B_{n,k}(x) \geq 0$  for all  $x \in [0, 1]$ . We also have

$$B_{1,0} = 1 - x, \quad B_{1,1} = x, \quad B_{2,0} = (1-x)^2, \quad B_{2,1} = 2x(1-x), \quad B_{2,2} = x^2, \quad \dots$$

**Lemma 21.1.** *We have the following statements about Bernstein polynomials:*

1.  $\sum_{k=0}^n B_{n,k}(x) = 1$ ,
2.  $\sum_{k=0}^n k B_{n,k}(x) = nx$ ,
3. and  $\sum_{k=0}^n k(k-1) B_{n,k}(x) = n(n-1)x^2$ .

*Proof.* For (1), we note that

$$1 = (x + 1 - x)^n = \sum_{k=0}^n \binom{n}{k} x^k (1-x)^{n-k} = \sum_{k=0}^n B_{n,k}$$

by the binomial theorem. For (2), we can compute

$$\begin{aligned} \sum_{k=0}^n k B_{n,k}(x) &= \sum_{k=0}^n \frac{n!}{(n-k)!k!} x^k (1-x)^{n-k} k = nx \sum_{k=1}^n \frac{(n-1)!}{(n-k)!(k-1)!} x^{k-1} (1-x)^{n-k} \\ &= nx \sum_{j=0}^{n-1} \frac{(n-1)!}{(n-1-j)!j!} x^j (1-x)^{n-1-j} = nx \sum_{j=0}^{n-1} B_{n-1,j}(x) = nx \end{aligned}$$

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<sup>1</sup>Here  $\binom{n}{k} = \frac{n!}{(n-k)!k!}$  is the binomial coefficient.

by reindexing the sum and then using (1). Also we dropped  $k = 0$  in step two. Finally, for (3), we have

$$\begin{aligned} \sum_{k=0}^n k(k-1)B_{n,k}(x) &= \sum_{k=0}^n k(k-1) \frac{n!}{(n-k)!k!} x^k (1-x)^{n-k} = n(n-1)x^2 \sum_{k=2}^n \frac{(n-2)!}{(n-k)!(k-2)!} x^{k-2} (1-x)^{n-k} \\ &= n(n-1)x^2 \sum_{j=0}^{n-2} \frac{(n-2)!}{(n-2-j)!j!} x^j (1-x)^{n-2-j} = n(n-1)x^2 \sum_{j=0}^{n-2} B_{n-2,j}(x) = n(n-1)x^2, \end{aligned}$$

where we reindexed and used (1) again. □

**Remark.** By dividing by  $n$ , we can get  $(\widetilde{2})$ :

$$\sum_{k=0}^n \frac{k}{n} B_{n,k}(x) = x.$$

If we view  $\frac{k}{n}$  as an event and then  $B_{n,k}(x)$  as a probability, then this tells us  $x$  is the expectation. We can perform a similar process to get an analogous  $(\widetilde{3})$ :

$$\sum_{k=0}^n \left( \frac{k}{n} - x \right)^2 B_{n,k}(x) = \frac{x(1-x)}{n}.$$

Using the same interpretation, we can view this as a computation of the variance.

### 21.1.2 The Actual Theorem

**Theorem 21.1** (Weierstrass approximation theorem). *Let  $f : [a, b] \rightarrow \mathbb{R}$  be continuous. Given  $\epsilon > 0$ , there exists a polynomial  $p(x)$  such that*

$$|f(x) - p(x)| < \epsilon$$

for all  $x \in [a, b]$ .

*Proof.* First we can normalize the interval so that  $f : [0, 1] \rightarrow \mathbb{R}$ . Then define

$$B_n(f)(x) = \sum_{k=0}^n f\left(\frac{k}{n}\right) B_{n,k}(x)$$

It suffices to show that for all  $\epsilon > 0$ , there exists  $N$  such that for all  $n \geq N$  and  $x \in [0, 1]$ , we have  $|B_n(x) - f(x)| < \epsilon$ . Note that this somewhat resembles a Riemann sum.<sup>2</sup>

Now we proceed. Since  $f$  is continuous on the compact set  $[0, 1]$ , it is uniformly continuous. In other words, for all  $\epsilon > 0$ , there exists  $\delta > 0$  such that

$$|f(x) - f(y)| < \frac{\epsilon}{2}$$

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<sup>2</sup>While the usual Riemann sum gives the integral of  $f$  as  $n \rightarrow \infty$ , this construction recovers  $f$  itself instead.

for all  $|x - y| < \delta$ . Then we can bound the error by

$$\begin{aligned} B_n(x) - f(x) &= \sum_{k=0}^n f\left(\frac{k}{n}\right) B_{n,k}(x) - \sum_{k=0}^n f(x) B_{n,k}(x) = \sum_{k=0}^n \left[ f\left(\frac{k}{n}\right) - f(x) \right] B_{n,k}(x) \\ &= \sum_{\left|\frac{k}{n} - x\right| < \delta} \left[ f\left(\frac{k}{n}\right) - f(x) \right] B_{n,k}(x) + \sum_{\left|\frac{k}{n} - x\right| \geq \delta} \left[ f\left(\frac{k}{n}\right) - f(x) \right] B_{n,k}(x). \end{aligned}$$

Here we have used the result that  $\sum_{k=0}^n B_{n,k}(x) = 1$ . Then we continue with

$$|B_n(x) - f(x)| \leq \underbrace{\sum_{\left|\frac{k}{n} - x\right| < \delta} \left| f\left(\frac{k}{n}\right) - f(x) \right| B_{n,k}(x)}_{(\star)} + \underbrace{\sum_{\left|\frac{k}{n} - x\right| \geq \delta} \left| f\left(\frac{k}{n}\right) - f(x) \right| B_{n,k}(x)}_{(\star\star)}.$$

For  $(\star)$ , by uniform continuity we can bound

$$(\star) < \sum_{\left|\frac{k}{n} - x\right| < \delta} \frac{\epsilon}{2} B_{n,k}(x) = \frac{\epsilon}{2} \sum_{\left|\frac{k}{n} - x\right| < \delta} B_{n,k}(x) \leq \frac{\epsilon}{2} \sum_{k=0}^n B_{n,k}(x) = \frac{\epsilon}{2}.$$

Then for  $(\star\star)$ ,

$$(\star\star) \leq 2 \max_{x \in [0,1]} |f| \sum_{\left|\frac{k}{n} - x\right| \geq \delta} B_{n,k}(x)$$

since  $[0, 1]$  is compact. Now by  $(\widetilde{3})$ ,

$$\begin{aligned} \sum_{\left|\frac{k}{n} - x\right| \geq \delta} B_{n,k}(x) &= \sum_{\left|\frac{k}{n} - x\right| \geq \delta} \frac{\left(\frac{k}{n} - x\right)^2}{\left(\frac{k}{n} - x\right)^2} B_{n,k}(x) \leq \frac{1}{\delta^2} \sum_{\left|\frac{k}{n} - x\right| \geq \delta} \left(\frac{k}{n} - x\right)^2 B_{n,k}(x) \\ &\leq \frac{1}{\delta^2} \sum_{k=0}^n \left(\frac{k}{n} - x\right)^2 B_{n,k}(x) = \frac{1}{\delta^2} \frac{x(1-x)}{n}. \end{aligned}$$

Plugging this back in yields

$$(\star\star) \leq 2 \max_{x \in [0,1]} |f| \frac{1}{\delta^2 n}$$

since  $x(1-x) \leq 1$ . Now take

$$N > \frac{4 \max_{x \in [0,1]} |f|}{\epsilon \delta^2}.$$

So when  $n \geq N$ , we have  $(\star\star) < \epsilon/2$ . Then

$$|B_n(x) - f(x)| \leq (\star) + (\star\star) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

which finishes the proof. □

## 21.2 Arzela-Ascoli Theorem

**Definition 21.2.** We say that  $\{f_n(x)\}$  is **equicontinuous** on  $E \subseteq \mathbb{R}$  if for every  $\epsilon > 0$ , there exists  $\delta$  such that if  $|x - y| < \delta$  for  $x, y \in E$ , then  $|f_n(x) - f_n(y)| < \epsilon$  for all  $n$ .

**Remark.** The choice of  $\delta$  is independent of  $n$  here.

**Theorem 21.2** (Arzela-Ascoli theorem). *Let  $f_n$  be defined on  $[0, 1]$  for each  $n \in \mathbb{N}$ . Suppose that there is  $M > 0$  such that  $|f_n(x)| \leq M$  for every  $x \in [0, 1]$ , and that  $\{f_n(x)\}$  is equicontinuous on  $[0, 1]$ . Then there exists a subsequence of  $\{f_n\}$  which converges uniformly on  $[0, 1]$ .*

*Proof.* Next class. □