

# MATH 4318: Analysis II

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# Contents

<b>1</b>	<b>Jan. 9 — The Derivative</b>	<b>2</b>
1.1	Defining the Derivative . . . . .	2
1.2	Immediate Properties . . . . .	2
1.3	Rules for Differentiation . . . . .	3
<b>2</b>	<b>Jan. 11 — The Mean Value Theorem</b>	<b>5</b>
2.1	The Mean Value Theorem . . . . .	5
2.2	Applications . . . . .	7
2.3	Cauchy's Mean Value Theorem . . . . .	8
<b>3</b>	<b>Jan. 16 — Taylor's Theorem</b>	<b>9</b>
3.1	Darboux's Lemma . . . . .	9
3.2	L'Hôpital's Rule . . . . .	9
3.3	Taylor's Theorem . . . . .	11

# Lecture 1

## Jan. 9 — The Derivative

### 1.1 Defining the Derivative

**Definition 1.1.** Let  $f$  be a real-valued function on an open interval  $U \subseteq \mathbb{R}$ . Let  $x_0 \in U$ , we say  $f$  is *differentiable* at  $x_0$  if

$$\lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

exists. If it does, then this limit, denoted by  $f'(x_0)$ , is called the *derivative* of  $f$  at  $x_0$ .

**Remark.** By definition, for any  $\epsilon > 0$ , there exists  $\delta > 0$  such that

$$\left| \frac{f(x) - f(x_0)}{x - x_0} - f'(x_0) \right| \leq \epsilon$$

if  $|x - x_0| < \delta$  and  $x \in U$ . Multiplying both sides by  $|x - x_0|$  yields

$$|f(x) - f(x_0) - f'(x_0)(x - x_0)| \leq \epsilon|x - x_0|.$$

In other words,

$$|f(x) - \varphi(x)| \leq \epsilon|x - x_0|$$

where  $\varphi(x) = f(x_0) + f'(x_0)(x - x_0)$ . In other words,  $\varphi(x)$  is a first-order approximation of  $f(x)$  near  $x_0$ . Geometrically, this is approximating the graph of  $y = f(x)$  by the tangent line  $y = \varphi(x)$ .

### 1.2 Immediate Properties

**Proposition 1.1.** Let  $U \subseteq \mathbb{R}$  be an open set and  $f : U \rightarrow \mathbb{R}$ . If  $f$  is differentiable at  $x_0 \in U$ , then  $f$  is continuous at  $x_0$ .

*Proof.* Pick any  $\epsilon_0 > 0$ . Then there exists  $\delta_0 > 0$  such that whenever  $|x - x_0| < \delta_0$  and  $x \in U$ ,

$$|f(x) - f(x_0) - f'(x_0)(x - x_0)| \leq \epsilon_0|x - x_0|.$$

By the triangle inequality,

$$|f(x) - f(x_0)| \leq \epsilon_0|x - x_0| + |f'(x_0)||x - x_0| = (\epsilon_0 + |f'(x_0)|)|x - x_0|.$$

Now for any  $\epsilon > 0$ , choose  $\delta = \min\{\delta_0, \epsilon/(\epsilon_0 + |f'(x_0)|)\}$ . Then

$$|f(x) - f(x_0)| \leq (\epsilon_0 + |f'(x_0)|)|x - x_0| < (\epsilon_0 + |f'(x_0)|)\delta \leq \epsilon$$

whenever  $|x - x_0| < \delta$  and  $x \in U$ . Thus  $f$  is continuous at  $x_0$ . □

**Example 1.1.1.** Take the function

$$f(x) = \begin{cases} x \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

Note that  $f$  is continuous on  $\mathbb{R}$ . For  $x \neq 0$ , continuity is clear since both  $x$  and  $\sin(1/x)$  are continuous. At  $x = 0$ , we have

$$\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} x \sin(1/x) = 0 = f(0)$$

since  $|x \sin(1/x)| \leq |x|$  for all  $x \in \mathbb{R}$ , so  $f$  is also continuous at  $x = 0$ . However,  $f$  is not differentiable at  $x = 0$ . Consider the limit

$$\lim_{x \rightarrow 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \rightarrow 0} \sin(1/x),$$

which does not exist since  $\sin(1/x)$  oscillates. So  $f$  is not differentiable at  $x = 0$ .

**Example 1.1.2.** Take the function  $f(x) = |x|$ , which is continuous everywhere on  $\mathbb{R}$ . However,  $f$  is not differentiable at  $x = 0$ , since

$$\lim_{x \rightarrow 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \rightarrow 0} \frac{|x|}{x}.$$

Note that

$$\frac{|x|}{x} = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0, \end{cases}$$

so the limit does not exist as  $x \rightarrow 0$ . Thus  $f$  is not differentiable at  $x = 0$ .

**Remark.** For the previous example, we can however define the *left (right) derivative* by

$$f'_-(x_0) = \lim_{x \rightarrow x_0^-} \frac{f(x) - f(x_0)}{x - x_0} \quad \text{and} \quad f'_+(x_0) = \lim_{x \rightarrow x_0^+} \frac{f(x) - f(x_0)}{x - x_0}.$$

If  $f$  is differentiable, then  $f'_-(x_0) = f'_+(x_0)$ . In the previous example,  $f'_-(0) = -1$  and  $f'_+(0) = 1$ . For the first example however, even  $f'_\pm(0)$  does not exist.

**Remark.** In one dimension, the existence of the derivative implies that the function is differentiable (the function is approximated by a linear function). However, in multiple dimensions, the existence of partial derivatives does not imply differentiability.

## 1.3 Rules for Differentiation

**Proposition 1.2.** Let  $U \subseteq \mathbb{R}$  be open and  $f, g : U \rightarrow \mathbb{R}$  be differentiable. Then

1.  $(f + g)'(x_0) = f'(x_0) + g'(x_0)$
2.  $(fg)'(x_0) = f'(x_0)g(x_0) + f(x_0)g'(x_0)$
3. if  $g(x_0) \neq 0$ , then  $(f/g)'(x_0) = (f'(x_0)g(x_0) - f(x_0)g'(x_0))/(g(x_0)^2)$ .

*Proof.* Find in textbook (Rosenlicht). □

**Proposition 1.3.** We have  $\frac{d}{dx}(c) = 0$ ,  $\frac{d}{dx}(x) = 1$ , and  $\frac{d}{dx}(x^n) = nx^{n-1}$  for all  $n \in \mathbb{N}$ .

*Proof.* We prove the last claim (the power rule) for  $n \geq 1$  by induction. The base case  $n = 1$  is the first claim which is true. Now suppose that the result holds for any  $n \leq k \in \mathbb{N}$ , and we show that it remains true for  $n = k + 1$ . By the product rule, we have

$$\frac{d}{dx}(x^{k+1}) = \frac{d}{dx}(x \cdot x^k) = \frac{d}{dx}(x) \cdot x^k + x \cdot \frac{d}{dx}(x^k) = x^k + xkx^{k-1} = (k+1)x^k.$$

Thus by induction this holds for all  $n \geq 1$ . We can do negative integers by the quotient rule.  $\square$

**Remark.** The power rule actually holds for any  $n \in \mathbb{R}$ .

**Proposition 1.4** (Chain rule). *Let  $U$  and  $V$  be open sets of  $\mathbb{R}$  and let  $f : U \rightarrow V, g : V \rightarrow \mathbb{R}$  be differentiable. Let  $x_0 \in U$  be such that  $f'(x_0)$  and  $g'(f(x_0))$  exist. Then  $(g \circ f)'(x_0)$  exists and*

$$(g \circ f)'(x_0) = g'(f(x_0))f'(x_0).$$

*Proof.* For any fixed  $y_0$  for which  $g'(y_0)$  exists, set

$$A(y, y_0) = \begin{cases} (g(y) - g(y_0))/(y - y_0) & \text{if } y \in V \text{ and } y \neq y_0 \\ g'(y_0) & \text{if } y = y_0. \end{cases}$$

Then  $A$  is continuous at  $y_0$ . To find  $(g \circ f)'(x_0)$ , observe that

$$\begin{aligned} \lim_{x \rightarrow x_0} \frac{g(f(x)) - g(f(x_0))}{x - x_0} &= \lim_{x \rightarrow x_0} \frac{A(f(x), f(x_0))(f(x) - f(x_0))}{x - x_0} \\ &= \lim_{x \rightarrow x_0} A(f(x), f(x_0)) \lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0} = g'(f(x_0))f'(x_0), \end{aligned}$$

by the continuity of  $A$  at  $f(x_0)$  and the differentiability of  $f$  at  $x_0$ .  $\square$

**Remark.** The rough idea of what we did here is

$$\begin{aligned} \lim_{x \rightarrow x_0} \frac{g(f(x)) - g(f(x_0))}{x - x_0} &= \lim_{x \rightarrow x_0} \frac{g(f(x)) - g(f(x_0))}{f(x) - f(x_0)} \frac{f(x) - f(x_0)}{x - x_0} \\ &= \lim_{x \rightarrow x_0} \frac{g(f(x)) - g(f(x_0))}{f(x) - f(x_0)} \lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0} = g'(f(x_0))f'(x_0). \end{aligned}$$

But does not quite work as stated since it might be that  $f(x) = f(x_0)$  even if  $x \neq x_0$ . We can fix this by introducing the function  $A$  as we did in the proof, though the overall idea is the same.

**Remark.** If  $f$  is monotone near  $x_0$ , then we can define the *inverse function*  $f^{-1}$  so that  $(f^{-1} \circ f)(x) = x$  near  $x_0$ . If  $f'(x_0)$  exists, then by the chain rule applied to  $x = (f^{-1} \circ f)(x)$  at  $x = x_0$  we have

$$1 = \frac{d}{dx}(f^{-1} \circ f)(x_0) = \frac{d}{dx}f^{-1}(f(x_0)) \cdot f'(x_0) \implies \frac{d}{dx}f^{-1}(f(x_0)) = \frac{1}{f'(x_0)}.$$

**Example 1.1.3.** Let  $f(x) = e^x$  with  $f^{-1}(x) = \ln(x)$ . Since  $f'(x) = f(x) = e^x$ , we have

$$\frac{d}{dx}f^{-1}(f(x_0)) = \frac{1}{f'(x_0)} \implies \frac{d}{dx}\ln(e^{x_0}) = \frac{1}{e^{x_0}}.$$

Letting  $e^{x_0} = h$ , we have  $\frac{d}{dx}\ln(x)|_{x=h} = 1/h$ , which recovers the familiar formula.

# Lecture 2

## Jan. 11 — The Mean Value Theorem

### 2.1 The Mean Value Theorem

**Lemma 2.1.** Let  $I \subseteq \mathbb{R}$  be open,  $f : I \rightarrow \mathbb{R}$  is differentiable at  $x_0 \in I$  and  $f'(x_0) \neq 0$ . Suppose  $f'(x_0) > 0$ , then there exists  $\delta > 0$  such that for any  $x \in (x_0 - \delta, x_0 + \delta)$ ,

1. if  $x > x_0$ , then  $f(x) > f(x_0)$ ,
2. if  $x < x_0$ , then  $f(x) < f(x_0)$ .

*Proof.* Take  $\epsilon = f'(x_0)/2$ . By the definition of the derivative, there exists  $\delta > 0$  such that for any  $|x - x_0| < \delta$ , we have

$$\left| \frac{f(x) - f(x_0)}{x - x_0} - f'(x_0) \right| < \epsilon = \frac{1}{2}f'(x_0).$$

By the triangle inequality,

$$\frac{f(x) - f(x_0)}{x - x_0} > \frac{1}{2}f'(x_0) > 0.$$

This quotient being positive immediately implies the desired results.  $\square$

**Theorem 2.1.** If  $f(x)$  is differentiable in an open interval  $I$  and  $f$  obtains its local maximum (or minimum) at  $x_0 \in I$ , then  $f'(x_0) = 0$ .

*Proof.* Suppose otherwise that  $f'(x_0) \neq 0$ . Assume without loss of generality that  $f'(x_0) > 0$ . Then by the previous lemma, there exists  $\delta > 0$  such that for  $x \in (x_0 - \delta, x_0 + \delta)$ , if  $x > x_0$  then  $f(x) > f(x_0)$  and if  $x < x_0$  then  $f(x) < f(x_0)$ . So  $x_0$  cannot be a local maximum or minimum, which is a contradiction.  $\square$

**Theorem 2.2** (Rolle's middle value theorem). Let  $f(x)$  be continuous on  $[a, b]$  and differentiable in  $(a, b)$ . Suppose  $f(a) = f(b)$ , then there exists  $x_0 \in (a, b)$  such that  $f'(x_0) = 0$ .

*Proof.* Since  $f$  is continuous on a compact set, it obtains both a maximum and minimum on  $[a, b]$ . Let  $M$  be the maximum and  $m$  be the minimum. If  $M = m$ , then  $f(x) \equiv M$  and  $f'(x) = 0$  everywhere. If  $M > m$ , then at least one of the maximum or minimum must be obtained at an interior point  $x_0 \in (a, b)$  since  $f(a) = f(b)$ . By the previous theorem,  $f'(x_0) = 0$  at this point and we are done.  $\square$

**Example 2.0.1.** Show that the equation  $4ax^3 + 3bx^2 + 2cx = a + b + c$  has at least one root in  $(0, 1)$ .

*Proof.* Consider the equation

$$4ax^3 + 3bx^2 + 2cx - (a + b + c) = 0.$$

Notice that the left hand side is the derivative of the function

$$f(x) = ax^4 + bx^3 + cx^2 - (a + b + c)x.$$

So we just need to show that  $f'(x) = 0$  for some  $x$ . For this, we can check that  $f(0) = f(1) = 0$ , and thus by Rolle's theorem there exists  $x_0 \in (0, 1)$  such that  $f'(x_0) = 0$ . So  $x_0$  is a root.  $\square$

**Theorem 2.3** (Lagrange's middle value theorem). *Let  $f_9x$  be continuous on  $[a, b]$  and differentiable in  $(a, b)$ . Then there exists  $x_0 \in (a, b)$  such that*

$$f'(x_0) = \frac{f(b) - f(a)}{b - a}.$$

*Proof.* Subtract the secant line through  $(a, f(a))$  and  $(b, f(b))$  from  $f(x)$  to get

$$g(x) = f(x) - \frac{f(b) - f(a)}{b - a}(x - a).$$

Note that  $g(a) = g(b) = f(a)$ . So by Rolle's theorem, there exists  $x_0 \in (a, b)$  such that  $g'(x_0) = 0$ . But

$$0 = g'(x_0) = f'(x_0) - \frac{f(b) - f(a)}{b - a},$$

which is the desired result.  $\square$

**Corollary 2.3.1.** *Suppose  $f \in C([a, b])$ , i.e.  $f$  is continuous on  $[a, b]$ , and that  $f$  is differentiable in  $(a, b)$ . Then the following statements are equivalent:*

1.  $f'(x) \geq 0$  in  $(a, b)$ ,
2.  $f(x)$  is increasing, i.e. if  $x_1 > x_2$ , then  $f(x_1) \geq f(x_2)$ .

*In particular, if  $f'(x) > 0$  in  $(a, b)$ , then  $f(x)$  is strictly increasing, i.e. if  $x_1 > x_2$ , then  $f(x_1) > f(x_2)$ .*

*Proof.*  $(2 \Rightarrow 1)$  For any  $x_0 \in (a, b)$ ,

$$f'(x_0) = \lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h} \geq 0$$

since  $f(x_0 + h) - f(x_0) \geq 0$  for  $h > 0$  as  $f$  is increasing.

$(1 \Rightarrow 2)$  Take  $x_1 > x_2$ , then by Lagrange's theorem there exists  $\xi \in (x_2, x_1)$  such that

$$f(x_2) - f(x_1) = f'(\xi)(x_2 - x_1) \geq 0.$$

So  $f(x_1) \geq f(x_2)$ . The strict version follows from changing the above inequality to a strict one.  $\square$

## 2.2 Applications

**Example 2.0.2.** Show that

$$\frac{2}{2x+1} < \ln(1 + 1/x)$$

for any  $x > 0$ .

*Proof.* Let  $f(x) = 2/(2x+1) - \ln(1 + 1/x)$ . Taking the derivative yields

$$f'(x) = \frac{1}{(2x+1)^2 x(x+1)} > 0,$$

so  $f$  is strictly increasing in  $(0, \infty)$ . Note that  $f \rightarrow 0$  as  $x \rightarrow \infty$ , so  $f(x) < 0$  for all  $x > 0$ . □

**Example 2.0.3.** Show that  $b/a > b^a/a^b$  when  $b > a > 1$ .

*Proof.* Take log on both sides to get  $\ln b - \ln a > a \ln b - b \ln a$ . This gives

$$(b-1) \ln a > (a-1) \ln b \iff \frac{\ln a}{a-1} > \frac{\ln b}{b-1}.$$

Note that this is a monotonicity property. So let  $f(x) = (\ln x)/(x-1)$  for  $x > 1$ . Then

$$f'(x) = \frac{x-1-x \ln x}{x(x-1)^2} < 0$$

when  $x > 1$  because  $x-1-x \ln x < 0$ . To see the last claim, define  $g(x) = x-1-x \ln x$  and note that  $g'(x) = -\ln x < 0$  for  $x > 1$ . But  $g(0) = 0$ , so  $g(x) < 0$  for  $x > 1$ . So  $f$  is strictly decreasing. □

**Example 2.0.4.** Show that

$$\lim_{x \rightarrow 0} \frac{e^x - e^{\sin x}}{x - \sin x} = 1.$$

*Proof.* Let  $f(x) = e^x$ . Then there exists  $\xi$  between  $x$  and  $\sin x$  such that

$$e^x - e^{\sin x} = (x - \sin x)e^{\xi(x)},$$

where the choice of  $\xi$  may vary for different  $x$ . Then

$$\lim_{x \rightarrow 0} \frac{e^x - e^{\sin x}}{x - \sin x} = \lim_{x \rightarrow 0} e^{\xi(x)}.$$

Now note that  $\xi(x)$  is always between  $x$  and  $\sin x$ , which both tend to 0 as  $x \rightarrow 0$ . So by the squeeze theorem we have  $\xi(x) \rightarrow 0$  as  $x \rightarrow 0$  and thus  $e^{\xi(x)} \rightarrow 1$  as  $x \rightarrow 0$ . □



## 2.3 Cauchy's Mean Value Theorem

**Theorem 2.4** (Cauchy's middle value theorem). *Let  $f, g \in C([a, b])$  and  $f, g$  be differentiable in  $(a, b)$ . Suppose  $g'(x) \neq 0$  for any  $x \in (a, b)$ . Then there exists  $x_0 \in (a, b)$  such that*

$$\frac{f'(x_0)}{g'(x_0)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

*Proof.* Use a similar construction as before and let

$$F(x) = f(x) - f(a) - \frac{f(b) - f(a)}{g(b) - g(a)}(g(x) - g(a)).$$

Note that  $F(b) = F(a) = 0$ , so by Rolle's theorem there exists  $x_0 \in (a, b)$  such that  $F'(x_0) = 0$ . Then

$$0 = F'(x_0) = f'(x_0) - \frac{f(b) - f(a)}{g(b) - g(a)}g'(x_0),$$

which implies the desired result. □

**Remark.** The  $g'(x) \neq 0$  condition guarantees that  $g$  is monotone, even if  $g'$  may fail to be continuous.

**Remark.** If  $g$  is a monotonically increasing function, we can view  $g$  as a mapping  $g : [a, b] \rightarrow [g(a), g(b)]$ , which we can view as a change of variables  $x \mapsto u$ . Since  $g$  is monotone, we have an inverse  $x = g^{-1}(u)$ . Then

$$f(x) = f(g^{-1}(u)) = (f \circ g^{-1})(u) = \tilde{f}(u).$$

By Lagrange's theorem,

$$\frac{\tilde{f}(g(b)) - \tilde{f}(g(a))}{g(b) - g(a)} = \tilde{f}'(u_0)$$

for some  $u_0 \in (g(a), g(b))$ . Now note that

$$\tilde{f}(g(b)) = (f \circ g^{-1})(g(b)) = f(b), \quad \tilde{f}(g(a)) = f(a).$$

So the left-hand side is precisely

$$\text{LHS} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

By the chain rule, we have

$$\text{RHS} = \tilde{f}'(u_0) = (f \circ g^{-1})'(u_0) = f'(g^{-1}(u_0))(g^{-1})'(u_0) = f'(x_0) \frac{1}{g'(x_0)}.$$

This recovers Cauchy's mean value theorem. So they are equivalent even if Cauchy's seems stronger.

# Lecture 3

## Jan. 16 — Taylor's Theorem

### 3.1 Darboux's Lemma

**Lemma 3.1** (Darboux's lemma). *If  $f$  is differentiable in  $(a, b)$ , continuous on  $[a, b]$  and  $f'(a) < f'(b)$ , then for any  $c \in (f'(a), f'(b))$ , there exists  $x_0 \in (a, b)$  such that  $f'(x_0) = c$ .*

*Proof.* See homework. □

**Remark.** There exists an example of a differentiable function  $f(x)$  but  $f'(x)$  is not continuous, e.g.

$$f(x) = \begin{cases} x^2 \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

We can compute that

$$f'(x) = \begin{cases} 2x \sin(1/x) - \cos(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0, \end{cases}$$

and we can verify as an exercise that  $f'(x)$  is not continuous at  $x = 0$ .

**Remark.** Darboux's lemma guarantees that  $g'(x) \neq 0$  implies either  $g'(x) > 0$  or  $g'(x) < 0$  everywhere in the conditions for Cauchy's mean value theorem.

### 3.2 L'Hôpital's Rule

**Theorem 3.1** (L'Hôpital's rule,  $0/0$ ). *Let  $f, g$  be differentiable in  $(a, b)$ ,  $\lim_{x \rightarrow a^+} f(x) = \lim_{x \rightarrow a^+} g(x) = 0$ , and  $g'(x) \neq 0$  for any  $x \in (a, b)$ . Then if  $\lim_{x \rightarrow a^+} f'(x)/g'(x)$  exists, we have*

$$\lim_{x \rightarrow a^+} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a^+} \frac{f'(x)}{g'(x)}.$$

*Proof.* By Cauchy's theorem, for any  $x \in (a, b)$ , there exists  $\xi(x) \in (a, x)$  such that

$$\frac{f(x)}{g(x)} = \frac{f(x) - f(a)}{g(x) - g(a)} = \frac{f'(\xi(x))}{g'(\xi(x))}.$$

If  $x \rightarrow a^+$ , then  $\xi(x) \rightarrow a^+$ , so

$$\lim_{x \rightarrow a^+} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a^+} \frac{f'(\xi(x))}{g'(\xi(x))} = \lim_{x \rightarrow a^+} \frac{f'(x)}{g'(x)},$$

as desired. □

**Corollary 3.1.1.** *Let  $f, g$  be differentiable in  $(a, \infty)$ ,  $\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} g(x) = 0$ , and  $g'(x) \neq 0$  for any  $x \in (a, \infty)$ . Then if  $\lim_{x \rightarrow \infty} f'(x)/g'(x)$  exists, we have*

$$\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = \lim_{x \rightarrow \infty} \frac{f'(x)}{g'(x)}.$$

*Proof.* Assume  $a > 0$ . Define  $\tilde{f}(y) = f(1/y)$  and  $\tilde{g}(y) = g(1/y)$  with  $y \in (0, 1/a)$ . By L'Hôpital's rule,

$$\lim_{y \rightarrow 0^+} \frac{\tilde{f}(y)}{\tilde{g}(y)} = \lim_{y \rightarrow 0^+} \frac{\tilde{f}'(y)}{\tilde{g}'(y)} = \lim_{y \rightarrow \infty} \frac{f'(1/y) \cdot (-1/y^2)}{g'(1/y) \cdot (-1/y^2)} = \lim_{x \rightarrow \infty} \frac{f'(x)}{g'(x)},$$

as desired. □

**Theorem 3.2** (L'Hôpital,  $\infty/\infty$ ). *Let  $f, g$  be differentiable in  $(a, b)$ ,  $\lim_{x \rightarrow a^+} |f(x)| = \lim_{x \rightarrow a^+} |g(x)| = \infty$ , and  $g'(x) \neq 0$  for any  $x \in (a, b)$ . Then if  $\lim_{x \rightarrow a^+} f'(x)/g'(x)$  exists, we have*

$$\lim_{x \rightarrow a^+} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a^+} \frac{f'(x)}{g'(x)}.$$

*Proof.* Left as an exercise. □

**Remark.** Saying that the absolute values of  $f$  and  $g$  go to infinity works, since the existence of the limit rules out oscillatory behavior.

**Remark.** These cases of  $\infty/\infty$  and  $0/0$  are called *indefinite types*. Other indefinite types include  $0 \cdot \infty$ ,  $0^0$ ,  $\infty^0$ ,  $1^\infty$ ,  $\infty - \infty$ , etc. But we can try to reduce them to the cases we know. For example, if  $f(x) \rightarrow 0^+$  and  $g(x) \rightarrow 0^+$  when  $x \rightarrow x_0$ , then  $\lim_{x \rightarrow x_0} f(x)^{g(x)}$  is  $0^0$ . Letting  $y(x) = f(x)^{g(x)}$ , we can take the log to get

$$\ln y(x) = g(x) \ln f(x) = \frac{\ln f(x)}{1/g(x)} = \frac{\infty}{\infty}.$$

**Example 3.0.1.** We can see that (this is a  $\infty - \infty$  case)

$$\lim_{x \rightarrow 0^+} \frac{1}{x^2} - \frac{\cot x}{x} = \lim_{x \rightarrow 0^+} \frac{1 + x \cot x}{x^2} = \lim_{x \rightarrow 0^+} \frac{-(\cot x - x \csc^2 x)}{2x \sin^2 x}.$$

Note that  $x \cot x = x \cos x / \sin x \rightarrow 1$  as  $x \rightarrow 0$ . Now note that  $\sin x / x \rightarrow 1$  as  $x \rightarrow 0$ , so we continue with

$$\lim_{x \rightarrow 0^+} \frac{-(\cot x - x \csc^2 x)}{2x \sin^2 x} = \lim_{x \rightarrow 0^+} \frac{x - \sin x \cos x}{2x^3} \frac{x^2}{\sin^2 x}$$

Since  $x^2 / \sin^2 x \rightarrow 1$  as  $x \rightarrow 0$ , we can look at the remaining part to get

$$\lim_{x \rightarrow 0^+} \frac{x - \sin x \cos x}{2x^3} = \lim_{x \rightarrow 0^+} \frac{1 - \cos 2x}{6x^2} = \lim_{x \rightarrow 0^+} \frac{2 \sin 2x}{12x} = \frac{1}{3}.$$

So  $\lim_{x \rightarrow 0^+} (1/x^2 - \cot x/x) = 1/3$ .

### 3.3 Taylor's Theorem

**Theorem 3.3** (Peano remainder term). *Let  $f : [a, b] \rightarrow \mathbb{R}$  be differentiable at  $x = a$  up to  $n$ th order of derivatives, i.e.  $f'(a), f''(a), \dots, f^{(n)}(a)$  exist. Then as  $x \rightarrow a^+$ , we have*

$$f(x) = \sum_{k=0}^n \frac{f^{(k)}(a)}{k!} (x-a)^k + o((x-a)^n).$$

Call the polynomial part of the above  $P_n(x)$ , which is also known as the Taylor polynomial of order  $n$ .

*Proof.* To show that the error term is  $o((x-a)^n)$ , we have

$$\lim_{x \rightarrow a^+} \frac{f(x) - P_n(x)}{(x-a)^n} = \lim_{x \rightarrow a^+} \frac{f'(x) - P'_n(x)}{n(x-a)^{n-1}} = \frac{1}{n!} \lim_{x \rightarrow a^+} \left[ \frac{f^{n-1}(x) - f^{n-1}(a)}{x-a} - f^{(n)}(a) \right] = 0$$

by L'Hôpital's rule, where we used the observation that  $f^{(k)}(a) = P_n^{(k)}(a)$  for  $1 \leq k \leq n$ . The final step is a result of the existence of  $f^{(n)}(a)$ .  $\square$

**Lemma 3.2** (Rolle's theorem for higher order derivatives). *Let  $f \in C^n([a, b])$  and differentiable to  $(n+1)$  order. If  $f'(a) = \dots = f^{(n)}(a) = 0$  and  $f(a) = f(b)$ , then there exists  $x_0 \in (a, b)$  such that  $f^{(n+1)}(x_0) = 0$ .*

*Proof.* Since  $f(a) = f(b)$ , by the usual Rolle's theorem there exists  $x_1 \in (a, b)$  such that  $f'(x_1) = 0$ . Then since  $f'(a) = f'(x_1) = 0$ , by Rolle's theorem again, there exists  $x_2 \in (a, x_1)$  such that  $f''(x_2) = 0$ . Repeat this to get  $x_{n+1} \in (a, x_n) \subseteq (a, b)$  such that  $f^{(n+1)}(x_{n+1}) = 0$ . Take  $x_0 = x_{n+1}$  to finish.  $\square$

**Theorem 3.4** (Lagrange remainder term). *Let  $f \in C^n([a, b])$ , in particular,  $f'(a), \dots, f^{(n)}(a)$  exist. Additionally, assume  $f$  is  $(n+1)$ -th differentiable in  $(a, b)$ .<sup>1</sup> Then*

$$f(x) = \sum_{k=0}^n \frac{f^{(k)}(a)}{k!} (x-a)^k + R_n(x) \quad \text{where} \quad R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} (x-a)^{n+1}$$

for some  $\xi \in [a, x]$ .

*Proof.* Define  $P(x) = P_n(x) + \lambda(x-a)^{n+1}$ , where we choose  $\lambda \in \mathbb{R}$  such that  $P(b) = f(b)$ , i.e.

$$\lambda = \frac{f(b) - P_n(b)}{(b-a)^{n+1}}.$$

Consider  $g(x) = f(x) - P(x)$ , which satisfies  $g(a) = g(b) = 0$  and  $g'(a) = \dots = g^{(n)}(a) = 0$ . Then by Rolle's theorem (higher order), there exists  $\xi \in (a, b)$  such that  $g^{(n+1)}(\xi) = 0$ . In other words,

$$f^{(n+1)}(\xi) - P^{(n+1)}(\xi) = 0 \implies f^{(n+1)}(\xi) - \underbrace{(n+1)! \frac{f(b) - P_n(b)}{(b-a)^{n+1}}}_{\lambda} = 0.$$

This implies that

$$f(b) = P_n(b) + \frac{1}{(n+1)!} f^{(n+1)}(\xi) (b-a)^{n+1},$$

and since we picked  $b$  arbitrarily (kind of), we can take  $b = x$  and we are done since  $\xi \in [a, b]$ .  $\square$

<sup>1</sup>Note that the  $(n+1)$ -th derivative need not be continuous here.

**Remark.** The choice of  $\xi$  in Lagrange's remainder term may (and likely does) vary for different  $x$ .

**Remark.** The Taylor polynomial is unique in the sense that if  $f : [a, b] \rightarrow \mathbb{R}$  and  $f'(a), \dots, f^{(n)}(a)$  exist, then if

$$f(x) = p(x) + o((x - a)^n)$$

as  $x \rightarrow a^+$  for some polynomial  $p(x)$  with  $\deg p \leq n$ , then  $p(x) = P_n(x) = \sum_{k=0}^n \frac{f^{(k)}(a)}{k!} (x - a)^k$ . This is because if  $Q(x) = p(x) - P_n(x)$ , then by Taylor's formula (Peano form), we get

$$\lim_{x \rightarrow a^+} \frac{Q(x)}{(x - a)^n} = \lim_{x \rightarrow a^+} \frac{p(x) - \sum_{k=0}^n \frac{f^{(k)}(a)}{k!} (x - a)^k}{(x - a)^n} = 0.$$

From here this implies that  $Q(x) = 0$  since  $\deg Q \leq n$ .