MATH 4318: Analysis II

Frank Qiang
Instructor: Zhiwu Lin

Georgia Institute of Technology
Spring 2024

Contents

Lecture 1

Jan. 9 — The Derivative

1.1 Defining the Derivative

Definition 1.1. Let f be a real-valued function on an open interval $U \subseteq \mathbb{R}$. Let $x_0 \in U$, we say f is *differentiable* at x_0 if

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

exists. If it does, then this limit, denoted by $f'(x_0)$, is called the *derivative* of f at x_0.

Remark. By definition, for any $\epsilon > 0$, there exists $\delta > 0$ such that

$$\left| \frac{f(x) - f(x_0)}{x - x_0} - f'(x_0) \right| \leq \epsilon$$

if $|x - x_0| < \delta$ and $x \in U$. Multiplying both sides by $|x - x_0|$ yields

$$|f(x) - f(x_0) - f'(x_0)(x - x_0)| \leq \epsilon |x - x_0|.$$

In other words,

$$|f(x) - \varphi(x)| \leq \epsilon |x - x_0|$$

where $\varphi(x) = f(x_0) + f'(x_0)(x - x_0)$. In other words, $\varphi(x)$ is a first-order approximation of $f(x)$ near x_0. Geometrically, this is approximating the graph of $y = f(x)$ by the tangent line $y = \varphi(x)$.

1.2 Immediate Properties

Proposition 1.1. *Let $U \subseteq \mathbb{R}$ be an open set and $f : U \to \mathbb{R}$. If f is differentiable at $x_0 \in U$, then f is continuous at x_0.*

Proof. Pick any $\epsilon_0 > 0$. Then there exists $\delta_0 > 0$ such that whenever $|x - x_0| < \delta_0$ and $x \in U$,

$$|f(x) - f(x_0) - f'(x_0)(x - x_0)| \leq \epsilon_0 |x - x_0|.$$

By the triangle inequality,

$$|f(x) - f(x_0)| \leq \epsilon_0 |x - x_0| + |f'(x_0)||x - x_0| = (\epsilon_0 + |f'(x_0)|)|x - x_0|.$$

Now for any $\epsilon > 0$, choose $\delta = \min\{\delta_0, \epsilon/(\epsilon_0 + |f'(x_0)|)\}$. Then

$$|f(x) - f(x_0)| \leq (\epsilon_0 + |f'(x_0)|)|x - x_0| < (\epsilon_0 + |f'(x_0)|)\delta \leq \epsilon$$

whenever $|x - x_0| < \delta$ and $x \in U$. Thus f is continuous at x_0. $\square$

Example 1.1.1. Take the function

$$f(x) = \begin{cases} x\sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

Note that f is continuous on $\mathbb{R}$. For $x \neq 0$, continuity is clear since both x and $\sin(1/x)$ are continuous. At $x = 0$, we have

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} x\sin(1/x) = 0 = f(0)$$

since $|x\sin(1/x)| \leq |x|$ for all $x \in \mathbb{R}$, so f is also continuous at $x = 0$. However, f is not differentiable at $x = 0$. Consider the limit

$$\lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} \sin(1/x),$$

which does not exist since $\sin(1/x)$ oscillates. So f is not differentiable at $x = 0$.

Example 1.1.2. Take the function $f(x) = |x|$, which is continuous everywhere on $\mathbb{R}$. However, f is not differentiable at $x = 0$, since

$$\lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} \frac{|x|}{x}.$$

Note that

$$\frac{|x|}{x} = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0, \end{cases}$$

so the limit does not exist as $x \to 0$. Thus f is not differentiable at $x = 0$.

Remark. For the previous example, we can however define the *left (right) derivative* by

$$f'_-(x_0) = \lim_{x\to x_0^-} \frac{f(x) - f(x_0)}{x - x_0} \quad \text{and} \quad f'_+(x_0) = \lim_{x\to x_0^+} \frac{f(x) - f(x_0)}{x - x_0}.$$

If f is differentiable, then $f'_-(x_0) = f'_+(x_0)$. In the previous example, $f'_-(0) = -1$ and $f'_+(0) = 1$. For the first example however, even $f'_\pm(0)$ does not exist.

Remark. In one dimension, the existence of the derivative implies that the function is differentiable (the function is approximated by a linear function). However, in multiple dimensions, the existence of partial derivatives does not imply differentiability.

1.3 Rules for Differentiation

Proposition 1.2. *Let $U \subseteq \mathbb{R}$ be open and $f, g : U \to \mathbb{R}$ be differentiable. Then*

1. $(f + g)'(x_0) = f'(x_0) + g'(x_0)$
2. $(fg)'(x_0) = f'(x_0)g(x_0) + f(x_0)g'(x_0)$
3. *if* $g(x_0) \neq 0$, *then* $(f/g)'(x_0) = (f'(x_0)g(x_0) - f(x_0)g'(x_0))/(g(x_0)^2)$.

Proof. Find in textbook (Rosenlicht). □

Proposition 1.3. *We have* $\frac{d}{dx}(c) = 0$, $\frac{d}{dx}(x) = 1$, *and* $\frac{d}{dx}(x^n) = nx^{n-1}$ *for all* $n \in \mathbb{N}$.

Proof. We prove the last claim (the power rule) for $n \geq 1$ by induction. The base case $n = 1$ is the first claim which is true. Now suppose that the result holds for any $n \leq k \in \mathbb{N}$, and we show that it remains true for $n = k + 1$. By the product rule, we have

$$\frac{d}{dx}(x^{k+1}) = \frac{d}{dx}(x \cdot x^k) = \frac{d}{dx}(x) \cdot x^k + x \cdot \frac{d}{dx}(x^k) = x^k + xkx^{k-1} = (k+1)x^k.$$

Thus by induction this holds for all $n \geq 1$. We can do negative integers by the quotient rule. $\square$

Remark. The power rule actually holds for any $n \in \mathbb{R}$.

Proposition 1.4 (Chain rule)**.** *Let U and V be open sets of $\mathbb{R}$ and let $f : U \to V, g : V \to \mathbb{R}$ be differentiable. Let $x_0 \in U$ be such that $f'(x_0)$ and $g'(f(x_0))$ exist. Then $(g \circ f)'(x_0)$ exists and*

$$(g \circ f)'(x_0) = g'(f(x_0))f'(x_0).$$

Proof. For any fixed y_0 for which $g'(y_0)$ exists, set

$$A(y, y_0) = \begin{cases} (g(y) - g(y_0))/(y - y_0) & \text{if } y \in V \text{ and } y \neq y_0 \\ g'(y_0) & \text{if } y = y_0. \end{cases}$$

Then A is continuous at y_0. To find $(g \circ f)'(x_0)$, observe that

$$\begin{aligned} \lim_{x \to x_0} \frac{g(f(x)) - g(f(x_0))}{x - x_0} &= \lim_{x \to x_0} \frac{A(f(x), f(x_0))(f(x) - f(x_0))}{x - x_0} \\ &= \lim_{x \to x_0} A(f(x), f(x_0)) \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = g'(f(x_0))f'(x_0), \end{aligned}$$

by the continuity of A at $f(x_0)$ and the differentiability of f at x_0. $\square$

Remark. The rough idea of what we did here is

$$\begin{aligned} \lim_{x \to x_0} \frac{g(f(x)) - g(f(x_0))}{x - x_0} &= \lim_{x \to x_0} \frac{g(f(x)) - g(f(x_0))}{f(x) - f(x_0)} \frac{f(x) - f(x_0)}{x - x_0} \\ &= \lim_{x \to x_0} \frac{g(f(x)) - g(f(x_0))}{f(x) - f(x_0)} \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = g'(f(x_0))f'(x_0). \end{aligned}$$

But does not quite work as stated since it might be that $f(x) = f(x_0)$ even if $x \neq x_0$. We can fix this by introducing the function A as we did in the proof, though the overall idea is the same.

Remark. If f is monotone near x_0, then we can define the *inverse function* f^{-1} so that $(f^{-1} \circ f)(x) = x$ near x_0. If $f'(x_0)$ exists, then by the chain rule applied to $x = (f^{-1} \circ f)(x)$ at $x = x_0$ we have

$$1 = \frac{d}{dx}(f^{-1} \circ f)(x_0) = \frac{d}{dx} f^{-1}(f(x_0)) \cdot f'(x_0) \implies \frac{d}{dx} f^{-1}(f(x_0)) = \frac{1}{f'(x_0)}.$$

Example 1.1.3. Let $f(x) = e^x$ with $f^{-1}(x) = \ln(x)$. Since $f'(x) = f(x) = e^x$, we have

$$\frac{d}{dx} f^{-1}(f(x_0)) = \frac{1}{f'(x_0)} \implies \frac{d}{dx} \ln(e^{x_0}) = \frac{1}{e^{x_0}}.$$

Letting $e^{x_0} = h$, we have $\frac{d}{dx} \ln(x)\big|_{x=h} = 1/h$, which recovers the familiar formula.

Lecture 2

Jan. 11 — The Mean Value Theorem

2.1 The Mean Value Theorem

Lemma 2.1. *Let $I \subseteq \mathbb{R}$ be open, $f : I \to \mathbb{R}$ is differentiable at $x_0 \in I$ and $f'(x_0) \neq 0$. Suppose $f'(x_0) > 0$, then there exists $\delta > 0$ such that for any $x \in (x_0 - \delta, x_0 + \delta)$,*

1. *if $x > x_0$, then $f(x) > f(x_0)$,*
2. *if $x < x_0$, then $f(x) < f(x_0)$.*

Proof. Take $\epsilon = f'(x_0)/2$. By the definition of the derivative, there exists $\delta > 0$ such that for ay $|x - x_0| < \delta$, we have

$$\left| \frac{f(x) - f(x_0)}{x - x_0} - f'(x_0) \right| < \epsilon = \frac{1}{2} f'(x_0).$$

By the triangle inequality,

$$\frac{f(x) - f(x_0)}{x - x_0} > \frac{1}{2} f'(x_0) > 0.$$

This quotient being positive immediately implies the desired results. □

Theorem 2.1. *If $f(x)$ is differentiable in an open interval I and f obtains its local maximum (or minimum) at $x_0 \in I$, then $f'(x_0) = 0$.*

Proof. Suppose otherwise that $f'(x_0) \neq 0$. Assume without loss of generality that $f'(x_0) > 0$. Then by the previous lemma, there exists $\delta > 0$ such that for $x \in (x_0 - \delta, x_0 + \delta)$, if $x > x_0$ then $f(x) > f(x_0)$ and if $x < x_0$ then $f(x) < f(x_0)$. So x_0 cannot be a local maximum or minimum, which is a contradiction. □

Theorem 2.2 (Rolle's middle value theorem)**.** *Let $f(x)$ be continuous on $[a, b]$ and differentiable in (a, b). Suppose $f(a) = f(b)$, then there exists $x_0 \in (a, b)$ such that $f'(x_0) = 0$.*

Proof. Since f is continuous on a compact set, it obtains both a maximum and minimum on $[a, b]$. Let M be the maximum and m be the minimum. If $M = m$, then $f(x) \equiv M$ and $f'(x) = 0$ everywhere. If $M > m$, then at least one of the maximum or minimum must be obtained at an interior point $x_0 \in (a, b)$ since $f(a) = f(b)$. By the previous theorem, $f'(x_0) = 0$ at this point and we are done. □

Example 2.0.1. Show that the equation $4ax^3 + 3bx^2 + 2cx = a + b + c$ has at least one root in $(0, 1)$.

Proof. Consider the equation

$$4ax^3 + 3bx^2 + 2cx - (a + b + c) = 0.$$

Notice that the left hand side is the derivative of the function

$$f(x) = ax^4 + bx^3 + cx^2 - (a + b + c)x.$$

So we just need to show that $f'(x) = 0$ for some x. For this, we can check that $f(0) = f(1) = 0$, and thus by Rolle's theorem there exists $x_0 \in (0, 1)$ such that $f'(x_0) = 0$. So x_0 is a root. □

Theorem 2.3 (Lagrange's middle value theorem)**.** *Let $f_9x)$ be continuous on $[a, b]$ and differentiable in (a, b). Then there exists $x_0 \in (a, b)$ such that*

$$f'(x_0) = \frac{f(b) - f(a)}{b - a}.$$

Proof. Subtract the secant line through $(a, f(a))$ and $(b, f(b))$ from $f(x)$ to get

$$g(x) = f(x) - \frac{f(b) - f(a)}{b - a}(x - a).$$

Note that $g(a) = g(b) = f(a)$. So by Rolle's theorem, there exists $x_0 \in (a, b)$ such that $g'(x_0) = 0$. But

$$0 = g'(x_0) = f'(x_0) - \frac{f(b) - f(a)}{b - a},$$

which is the desired result. □

Corollary 2.3.1. *Suppose $f \in C([a, b])$, i.e. f is continuous on $[a, b]$, and that f is differentiable in (a, b). Then the following statements are equivalent:*

1. *$f'(x) \geq 0$ in (a, b),*
2. *$f(x)$ is increasing, i.e. if $x_1 > x_2$, then $f(x_1) \geq f(x_2)$.*

In particular, if $f'(x) > 0$ in (a, b), then $f(x)$ is strictly increasing, i.e. if $x_1 > x_2$, then $f(x_1) > f(x_2)$.

Proof. $(2 \Rightarrow 1)$ For any $x_0 \in (a, b)$,

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} \geq 0$$

since $f(x_0 + h) - f(x_0) \geq 0$ for $h > 0$ as f is increasing.

$(1 \Rightarrow 2)$ Take $x_1 > x_2$, then by Lagrange's theorem there exists $\xi \in (x_2, x_1)$ such that

$$f(x_2) - f(x_1) = f'(\xi)(x_2 - x_1) \geq 0.$$

So $f(x_1) \geq f(x_2)$. The strict version follows from changing the above inequality to a strict one. □

2.2 Applications

Example 2.0.2. Show that

$$\frac{2}{2x+1} < \ln(1+1/x)$$

for any $x > 0$.

Proof. Let $f(x) = 2/(2x+1) - \ln(1+1/x)$. Taking the derivative yields

$$f'(x) = \frac{1}{(2x+1)^2 x(x+1)} > 0,$$

so f is strictly increasing in $(0, \infty)$. Note that $f \to 0$ as $x \to \infty$, so $f(x) < 0$ for all $x > 0$. $\square$

Example 2.0.3. Show that $b/a > b^a/a^b$ when $b > a > 1$.

Proof. Take log on both sides to get $\ln b - \ln a > a \ln b - b \ln a$. This gives

$$(b-1)\ln a > (a-1)\ln b \quad \iff \quad \frac{\ln a}{a-1} > \frac{\ln b}{b-1}.$$

Note that this is a monotonicity property. So let $f(x) = (\ln x)/(x-1)$ for $x > 1$. Then

$$f'(x) = \frac{x - 1 - x\ln x}{x(x-1)^2} < 0$$

when $x > 1$ because $x - 1 - x\ln x < 0$. To see the last claim, define $g(x) = x - 1 - x\ln x$ and note that $g'(x) = -\ln x < 0$ for $x > 1$. But $g(0) = 0$, so $g(x) < 0$ for $x > 1$. So f is strictly decreasing. $\square$

Example 2.0.4. Show that

$$\lim_{x\to 0} \frac{e^x - e^{\sin x}}{x - \sin x} = 1.$$

Let $f(x) = e^x$. Then there exists ξ between x and $\sin x$ such that

$$e^x - e^{\sin x} = (x - \sin x)e^{\xi(x)},$$

where the choice of ξ may vary for different x. Then

$$\lim_{x\to 0} \frac{e^x - e^{\sin x}}{x - \sin x} = \lim_{x\to 0} e^{\xi(x)}.$$

Now note that $\xi(x)$ is always between x and $\sin x$, which both tend to 0 as $x \to 0$. So by the squeeze theorem we have $\xi(x) \to 0$ as $x \to 0$ and thus $e^{\xi(x)} \to 1$ as $x \to 0$.

2.3 Cauchy's Mean Value Theorem

Theorem 2.4 (Cauchy's middle value theorem)**.** *Let $f, g \in C([a,b])$ and f, g be differentiable in (a,b). Suppose $g'(x) \neq 0$ for any $x \in (a,b)$. Then there exists $x_0 \in (a,b)$ such that*

$$\frac{f'(x_0)}{g'(x_0)} = \frac{f(b)-f(a)}{g(b)-g(a)}.$$

Proof. Use a similar construction as before and let

$$F(x) = f(x) - f(a) - \frac{f(b)-f(a)}{g(b)-g(a)}(g(x)-g(a)).$$

Note that $F(b) = F(a) = 0$, so by Rolle's theorem there exists $x_0 \in (a,b)$ such that $F'(x_0) = 0$. Then

$$0 = F'(x_0) = f'(x_0) - \frac{f(b)-f(a)}{g(b)-g(a)} g'(x_0),$$

which implies the desired result. □

Remark. The $g'(x) \neq 0$ condition guarantees that g is monotone, even if g' may fail to be continuous.

Remark. If g is a monotonically increasing function, we can view g as a mapping $g : [a,b] \to [g(a), g(b)]$, which we can view as a change of variables $x \mapsto u$. Since g is monotone, we have an inverse $x = g^{-1}(u)$. Then

$$f(x) = f(g^{-1}(u)) = (f \circ g^{-1})(u) = \widetilde{f}(u).$$

By Lagrange's theorem,

$$\frac{\widetilde{f}(g(b)) - \widetilde{f}(g(a))}{g(b)-g(a)} = \widetilde{f}'(u_0)$$

for some $u_0 \in (g(a, g(b))$. Now note that

$$\widetilde{f}(g(b)) = (f \circ g^{-1})(g(b)) = f(b), \quad \widetilde{f}(g(a)) = f(a).$$

So the left-hand side is precisely

$$\text{LHS} = \frac{f(b)-f(a)}{g(b)-g(a)}.$$

By the chain rule, we have

$$\text{RHS} = \widetilde{f}'(u_0) = (f \circ g^{-1})'(u_0) = f'(g^{-1}(u_0))(g^{-1})'(u_0) = f'(x_0)\frac{1}{g'(x_0)}.$$

This recovers Cauchy's mean value theorem. So they are equivalent even if Cauchy's seems stronger.

Lecture 3

Jan. 16 — Taylor's Theorem

3.1 Darboux's Lemma

Lemma 3.1 (Darboux's lemma). *If f is differentiable in (a, b), continuous on $[a, b]$ and $f'(a) < f'(b)$, then for any $c \in (f'(a), f'(b))$, there exists $x_0 \in (a, b)$ such that $f'(x_0) = c$.*

Proof. See homework. $\square$

Remark. There exists an example of a differentiable function $f(x)$ but $f'(x)$ is not continuous, e.g.

$$f(x) = \begin{cases} x^2 \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

We can compute that

$$f'(x) = \begin{cases} 2x \sin(1/x) - \cos(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0, \end{cases}$$

and we can verify as an exercise that $f'(x)$ is not continuous at $x = 0$.

Remark. Darboux's lemma guarantees that $g'(x) \neq 0$ implies either $g'(x) > 0$ or $g'(x) < 0$ everywhere in the conditions for Cauchy's mean value theorem.

3.2 L'Hôpital's Rule

Theorem 3.1 (L'Hôpital's rule, 0/0). *Let f, g be differentiable in (a, b), $\lim_{x \to a^+} f(x) = \lim_{x \to a^+} g(x) = 0$, and $g'(x) \neq 0$ for any $x \in (a, b)$. Then if $\lim_{x \to a^+} f'(x)/g'(x)$ exists, we have*

$$\lim_{x \to a^+} \frac{f(x)}{g(x)} = \lim_{x \to a^+} \frac{f'(x)}{g'(x)}.$$

Proof. By Cauchy's theorem, for any $x \in (a, b)$, there exists $\xi(x) \in (a, x)$ such that

$$\frac{f(x)}{g(x)} = \frac{f(x) - f(a)}{g(x) - g(a)} = \frac{f'(\xi(x))}{g'(\xi(x))}.$$

If $x \to a^+$, then $\xi(x) \to a^+$, so

$$\lim_{x \to a^+} \frac{f(x)}{g(x)} = \lim_{x \to a^+} \frac{f'(\xi(x))}{g'(\xi(x))} = \lim_{x \to a^+} \frac{f'(x)}{g'(x)},$$

as desired. $\square$

Corollary 3.1.1. *Let f, g be differentiable in (a, ∞), $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = 0$, and $g'(x) \neq 0$ for any $x \in (a, \infty)$. Then if $\lim_{x\to\infty} f'(x)/g'(x)$ exists, we have*

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)}.$$

Proof. Assume $a > 0$. Define $\widetilde{f}(y) = f(1/y)$ and $\widetilde{g}(y) = g(1/y)$ with $y \in (0, 1/a)$. By L'Hôpital's rule,

$$\lim_{y\to 0^+} \frac{\widetilde{f}(y)}{\widetilde{g}(y)} = \lim_{y\to 0^+} \frac{\widetilde{f}'(y)}{\widetilde{g}'(y)} = \lim_{y\to\infty} \frac{f'(1/y)\cdot(-1/y^2)}{g'(1/y)\cdot(-1/y^2)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)},$$

as desired. □

Theorem 3.2 (L'Hôpital, ∞/∞)**.** *Let f, g be differentiable in (a, b), $\lim_{x\to a^+} |f(x)| = \lim_{x\to a^+} |g(x)| = \infty$, and $g'(x) \neq 0$ for any $x \in (a, b)$. Then if $\lim_{x\to a^+} f'(x)/g'(x)$ exists, we have*

$$\lim_{x\to a^+} \frac{f(x)}{g(x)} = \lim_{x\to a^+} \frac{f'(x)}{g'(x)}.$$

Proof. Left as an exercise. □

Remark. Saying that the absolute values of f and g go to infinity works, since the existence of the limit rules out oscillatory behavior.

Remark. These cases of ∞/∞ and $0/0$ are are called *indefinite types*. Other indefinite types include $0\cdot\infty$, 0^0, ∞^0 1^∞, $\infty - \infty$, etc. But we can try to reduce them to the cases we know. For example, if $f(x) \to 0^+$ and $g(x) \to 0^+$ when $x \to x_0$, then $\lim_{x\to x_0} f(x)^{g(x)}$ is 0^0. Letting $y(x) = f(x)^{g(x)}$, we can take the log to get

$$\ln y(x) = g(x) \ln f(x) = \frac{\ln f(x)}{1/g(x)} = \frac{\infty}{\infty}.$$

Example 3.0.1. We can see that (this is a $\infty - \infty$ case)

$$\lim_{x\to 0^+} \frac{1}{x^2} - \frac{\cot x}{x} = \lim_{x\to 0^+} \frac{1 + x\cot x}{x^2} = \lim_{x\to 0^+} \frac{-(\cot x - x\csc^2 x)}{2x\sin^2 x}.$$

Note that $x\cot x = x\cos x/\sin x \to 1$ as $x \to 0$. Now note that $\sin x/x \to 1$ as $x \to 0$, so we continue with

$$\lim_{x\to 0^+} \frac{-(\cot x - x\csc^2 x)}{2x\sin^2 x} = \lim_{x\to 0^+} \frac{x - \sin x\cos x}{2x^3} \frac{x^2}{\sin^2 x}$$

Since $x^2/\sin^2 x \to 1$ as $x \to 0$, we can look at the remaining part to get

$$\lim_{x\to 0^+} \frac{x - \sin x\cos x}{2x^3} = \lim_{x\to 0^+} \frac{1 - \cos 2x}{6x^2} = \lim_{x\to 0^+} \frac{2\sin 2x}{12x} = \frac{1}{3}.$$

So $\lim_{x\to 0^+}(1/x^2 - \cot x/x) = 1/3$.

3.3 Taylor's Theorem

Theorem 3.3 (Peano remainder term)**.** *Let $f : [a, b] \to \mathbb{R}$ be differentiable at $x = a$ up to nth order of derivatives, i.e. $f'(a), f''(a), \ldots, f^{(n)}(a)$ exist. Then as $x \to a^+$, we have*

$$f(x) = \sum_{k=0}^{n} \frac{f^(k)(a)}{k!}(x-a)^k + o((x-a)^n).$$

Call the polynomial part of the above $P_n(x)$, which is also known as the Taylor polynomial *of order n.*

Proof. To show that the error term is $o((x-a)^n)$, we have

$$\lim_{x\to a^+} \frac{f(x) - P_n(x)}{(x-a)^n} = \lim_{x\to a^+} \frac{f'(x) - P_n'(x)}{n(x-a)^{n-1}} = \frac{1}{n!} \lim_{x\to a^+} \left[\frac{f^{n-1}(x) - f^{n-1}(a)}{x-a} - f^{(n)}(a) \right] = 0$$

by L'Hôpital's rule, where we used the observation that $f^{(k)}(a) = P_n^{(k)}(a)$ for $1 \le k \le n$. The final step is a result of the existence of $f^{(n)}(a)$. □

Lemma 3.2 (Rolle's theorem for higher order derivatives)**.** *Let $f \in C^([a, b])$ and differentiable to $(n+1)$ order. If $f'(a) = \cdots = f^{(n)}(a) = 0$ and $f(a) = f(b)$, then there exists $x_0 \in (a, b)$ such that $f^{(n+1)}(x_0) = 0$.*

Proof. Since $f(a) = f(b)$, by the usual Rolle's theorem there exists $x_1 \in (a, b)$ such that $f'(x_1) = 0$. Then since $f'(a) = f'(x_1) = 0$, by Rolle's theorem again, there exists $x_2 \in (a, x_1)$ such that $f''(x_2) = 0$. Repeat this to get $x_{n+1} \in (a, x_n) \subseteq (a, b)$ such that $f^{(n+1)}(x_{n+1}) = 0$. Take $x_0 = x_{n+1}$ to finish. □

Theorem 3.4 (Lagrange remainder term)**.** *Let $f \in C^n([a, b])$, in particular, $f'(a), \ldots, f^{(n)}(a)$ exist. Additionally, assume f is $(n+1)$-th differentiable in (a, b).*[1] *Then*

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k + R_n(x) \quad \textit{where} \quad R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1}$$

for some $\xi \in [a, x]$.

Proof. Define $P(x) = P_n(x) + \lambda(x-a)^{n+1}$, where we choose $\lambda \in \mathbb{R}$ such that $P(b) = f(b)$, i.e.

$$\lambda = \frac{f(b) - P_n(b)}{(b-a)^{n+1}}.$$

Consider $g(x) = f(x) - P(x)$, which satisfies $g(a) = g(b) = 0$ and $g'(a) = \cdots = g^{(n)}(a) = 0$. Then by Rolle's theorem (higher order), there exists $\xi \in (a, b)$ such that $g^{(n+1)}(\xi) = 0$. In other words,

$$f^{(n+1)} - P^{(n+1)}(\xi) = 0 \implies f^{(n+1)}(\xi) - (n+1)! \underbrace{\frac{f(b) - P_n(b)}{(b-a)^{n+1}}}_{\lambda} = 0.$$

This implies that

$$f(b) = P_n(b) + \frac{1}{(n+1)!} f^{(n+1)}(\xi)(b-a)^{n+1},$$

and since we picked b arbitrarily (kind of), we can take $b = x$ and we are done since $\xi \in [a, x]$. □

[1] Note that the $(n+1)$-th derivative need not be continuous here.

Remark. The choice of ξ in Lagrange's remainder term may (and likely does) vary for different x.

Remark. The Taylor polynomial is unique in the sense that if $f : [a, b] \to \mathbb{R}$ and $f'(a), \ldots, f^{(n)}(a)$ exist, then if

$$f(x) = p(x) + o((x - a)^n) \text{as} x \to a^+$$

as $x \to a^+$ for some polynomial $p(x)$ with $\deg p \leq n$, then $p(x) = P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x - a)^k$. This is because if $Q(x) = p(x) - P_n(x)$, then by Taylor's formula (Peano form), we get

$$\lim_{x \to a^+} \frac{Q(x)}{(x - a)^n} = \lim_{x \to a^+} \frac{p(x) - \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x - a)^k}{(x - a)^n} = 0.$$

From here this implies that $Q(x) = 0$ since $\deg Q \leq n$.