

MATH 4318 Exam 2 Formula Sheet

Exchange of Limit Operations

Theorem 1. Let $\{f_n\}$ be a uniformly convergent sequence of Riemann integrable functions on $[a, b]$. Then

$$\int_a^b \lim_{n \rightarrow \infty} f_n(x) dx = \lim_{n \rightarrow \infty} \int_a^b f_n(x) dx.$$

Theorem 2. Let $\{f_n\}$ be a sequence of functions on an open interval $U \subseteq \mathbb{R}$ such that each f_n has a continuous derivative. Suppose $\{f'_n\}$ converges uniformly on U and for some $a \in U$, $\{f'_n(a)\}$ converges. Then $\lim_{n \rightarrow \infty} f_n = f$ exists and f is differentiable. Furthermore, we have $f' = \lim_{n \rightarrow \infty} f'_n$.

Theorem 3. Let $f_n \in \mathcal{R}([a, b])$ for $n \in \mathbb{N}$. If $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly on $[a, b]$, then $\sum_{n=1}^{\infty} f_n(x) \in \mathcal{R}([a, b])$ and

$$\sum_{n=1}^{\infty} \int_a^b f_n(x) dx = \int_a^b \sum_{n=1}^{\infty} f_n(x) dx.$$

Theorem 4. Let $f_n : [a, b] \rightarrow \mathbb{R}$ be continuously differentiable. Suppose that $\sum_{n=1}^{\infty} f_n(x)$ converges pointwise on $[a, b]$ and $\sum_{n=1}^{\infty} f'_n(x)$ converges uniformly on $[a, b]$. Then $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly on $[a, b]$ and

$$\frac{d}{dx} \sum_{n=1}^{\infty} f_n(x) = \sum_{n=1}^{\infty} f'_n(x).$$

Infinite Series Basics

Theorem 5. A series $\sum_{n=1}^{\infty} a_n$ converges if and only if for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that if $n > m \geq N$, then $|a_{m+1} + \dots + a_n| < \epsilon$.

Theorem 6. If $\sum_{n=1}^{\infty} a_n$ converges, then $a_n \rightarrow 0$ as $n \rightarrow \infty$.

Theorem 7. If $a_n \geq 0$, then $\sum_{n=1}^{\infty} a_n$ either converges or diverges to ∞ .

Theorem 8. If $\sum_{n=1}^{\infty} a_n$ converges absolutely, then $\sum_{n=1}^{\infty} a_n$ converges and $|\sum_{n=1}^{\infty} a_n| \leq \sum_{n=1}^{\infty} |a_n|$.

Convergence Tests

Theorem 9 (Comparison test). If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are two series such that $|a_n| \leq b_n$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges and $|\sum_{n=1}^{\infty} a_n| \leq \sum_{n=1}^{\infty} b_n$.

Theorem 10 (Limit comparison). Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be two positive series and suppose that $\lim_{n \rightarrow \infty} a_n/b_n = \ell > 0$. Then $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{n=1}^{\infty} b_n$ converges.

Theorem 11 (Root test). Let $\sum_{n=1}^{\infty} a_n$ be a positive series and suppose that $\limsup_{n \rightarrow \infty} \sqrt[n]{a_n} = \ell$. Then

- if $\ell < 1$, then $\sum_{n=1}^{\infty} a_n$ converges;
- if $\ell > 1$, then $\sum_{n=1}^{\infty} a_n$ diverges.

Theorem 12 (Integral test). Let $\{a_n\}$ be a positive decreasing sequence. If there exists a continuous decreasing f on $[1, \infty)$ such that $a_n = f(n)$, then $\sum_{n=1}^{\infty} a_n$ converges if and only if $\int_1^{\infty} f(x) dx$ converges.

Theorem 13. Let $\{a_n\}$ be a decreasing sequence with $a_n \rightarrow 0$ as $n \rightarrow \infty$. Then the series $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ converges (to S , say), and the partial sums S_n have error $|S_n - S| \leq a_{n+1}$.

Series of Functions

Theorem 14 (Cauchy criterion). A series $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly on $I \subseteq \mathbb{R}$ if and only if for every $\epsilon > 0$, there exists N such that whenever $n \geq N$, for any $x \in I$ and $p \in \mathbb{N}$ we have $|f_{n+1}(x) + \dots + f_{n+p}(x)| < \epsilon$.

Theorem 15. If $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly on I , then $f_n \rightarrow 0$ uniformly on I .

Theorem 16. Let $f_n \in C([a, b])$. If $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly on (a, b) , then it converges uniformly on $[a, b]$.

Theorem 17 (Weierstrass M -test). If there exists a nonnegative and convergent series such that $|f_n(x)| \leq M_n$ for all $x \in I$, then $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly on I .

Power Series

Theorem 18. We have the following:

- If $\sum_{n=0}^{\infty} a_n x^n$ converges at $x = x_1 \neq 0$, then it converges absolutely for all x with $|x| < |x_1|$.
- If $\sum_{n=0}^{\infty} a_n x^n$ diverges at $x = x_2 \neq 0$, then it diverges for all x with $|x| > |x_2|$.

Theorem 19 (Hadamard's formula). For a power series $\sum_{n=0}^{\infty} a_n x^n$, let $L = \limsup_{n \rightarrow \infty} |a_n|^{1/n}$. Then its radius of convergence is $R = 1/L$.

Theorem 20. For a series $\sum_{n=0}^{\infty} a_n x^n$ with $a_n \neq 0$, if $\lim_{n \rightarrow \infty} |a_{n+1}/a_n| = L$, then its radius of convergence is $R = 1/L$.

Theorem 21. If $\sum_{n=0}^{\infty} a_n x^n$ has radius of convergence $R > 0$, then for any $0 < r < R$, the power series $\sum_{n=0}^{\infty} a_n r^n$ converges uniformly on $[-r, r]$. Moreover, if $\sum_{n=0}^{\infty} a_n x^n$ converges at $x = R < \infty$ (or $x = -R$), then $\sum_{n=0}^{\infty} a_n x^n$ converges uniformly on $[0, R]$ (or $[-R, 0]$).

Theorem 22. If $\sum_{n=0}^{\infty} a_n x^n$ has radius of convergence $R > 0$, then $f(x) = \sum_{n=0}^{\infty} a_n x^n \in C^{\infty}(-R, R)$.

Theorem 23. Suppose $f(x) = \sum_{n=0}^{\infty} a_n x^n$ has radius of convergence $R > 0$. Then for any $x \in (-R, R)$, $f \in \mathcal{R}([0, x])$ and

$$\int_0^x f(t) dt = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}.$$

Taylor Series

Theorem 24. Let $R \in (0, \infty)$ and $f \in C^{\infty}(x_0 - R, x_0 + R)$. If there exists $M > 0$ such that for all $x \in (x_0 - R, x_0 + R)$, $|f^{(n)}(x)| \leq M$ for all $n \in \mathbb{N}$, then

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n$$

for all $x \in (x_0 - R, x_0 + R)$.

Theorem 25 (Lagrange remainder). Let $f \in C^n([a, b])$ and assume that f is $(n+1)$ -times differentiable in (a, b) . Then

$$f(x) - \sum_{k=0}^n \frac{f^{(k)}(a)}{k!} (x - a)^k = \frac{f^{(n+1)}(\xi)}{(n+1)!} (x - a)^{n+1}$$

for some $\xi \in [a, x]$.

Contraction Mapping

Theorem 26 (Contraction mapping). Let (X, d) be a complete metric space and $T : X \rightarrow X$ a contraction mapping, i.e. there exists $0 < k < 1$ such that $d(Tx, Ty) \leq kd(x, y)$ for all $x, y \in X$. Then T admits a unique fixed point in X .

Theorem 27 (Newton's method). Let $f \in C^2([a, b])$ and $\hat{x} \in (a, b)$ such that $f(\hat{x}) = 0$ and $f'(\hat{x}) \neq 0$. Then there exists a neighborhood $U(\hat{x})$ of \hat{x} such that for all $x_0 \in U(\hat{x})$, the sequence

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

converges to \hat{x} as $n \rightarrow \infty$.

Theorem 28 (Picard-Lindelöf). Let $f(t, x) : [a, b] \times [c, d] \rightarrow \mathbb{R}$ be continuous in t and locally Lipschitz in x , i.e. it is Lipschitz in x for $|t| \leq h$ with h small enough, then the initial value problem

$$\begin{cases} x'(t) = f(t, x) \\ x(0) = \xi, \end{cases}$$

has a unique local solution. (Note: Consider

$$x(t) = \xi + \int_0^t f(\tau, x(\tau)) d\tau$$

by integrating. This is called a Picard iteration.)

Theorem 29 (Implicit function theorem). Let $f : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^m$ and $U \times V \subseteq \mathbb{R}^n \times \mathbb{R}^m$ be a neighborhood of (x_0, y_0) . Suppose f and $\partial f / \partial y$ are continuous on $U \times V$, and $f(x_0, y_0) = 0$,

$$\det \left(\frac{\partial f}{\partial y}(x_0, y_0) \right) \neq 0.$$

Then there exists a neighborhood $U_0 \times V_0 \subseteq U \times V$ of (x_0, y_0) and a unique continuous function $\varphi : U_0 \rightarrow V_0$ satisfying

$$\{f(x, \varphi(x)) = 0, \varphi(x_0) = y_0.$$