# MATH 4318: Analysis II

Frank Qiang
Instructor: Zhiwu Lin

Georgia Institute of Technology
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# Contents

# Lecture 1

# Jan. 9 — The Derivative

## 1.1 Defining the Derivative

**Definition 1.1.** Let $f$ be a real-valued function on an open interval $U \subseteq \mathbb{R}$. Let $x_0 \in U$, we say $f$ is *differentiable* at $x_0$ if

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

exists. If it does, then this limit, denoted by $f'(x_0)$, is called the *derivative* of $f$ at $x_0$.

**Remark.** By definition, for any $\epsilon > 0$, there exists $\delta > 0$ such that

$$\left| \frac{f(x) - f(x_0)}{x - x_0} - f'(x_0) \right| \leq \epsilon$$

if $|x - x_0| < \delta$ and $x \in U$. Multiplying both sides by $|x - x_0|$ yields

$$|f(x) - f(x_0) - f'(x_0)(x - x_0)| \leq \epsilon |x - x_0|.$$

In other words,

$$|f(x) - \varphi(x)| \leq \epsilon |x - x_0|$$

where $\varphi(x) = f(x_0) + f'(x_0)(x - x_0)$. In other words, $\varphi(x)$ is a first-order approximation of $f(x)$ near $x_0$. Geometrically, this is approximating the graph of $y = f(x)$ by the tangent line $y = \varphi(x)$.

## 1.2 Immediate Properties

**Proposition 1.1.** *Let $U \subseteq \mathbb{R}$ be an open set and $f : U \to \mathbb{R}$. If $f$ is differentiable at $x_0 \in U$, then $f$ is continuous at $x_0$.*

*Proof.* Pick any $\epsilon_0 > 0$. Then there exists $\delta_0 > 0$ such that whenever $|x - x_0| < \delta_0$ and $x \in U$,

$$|f(x) - f(x_0) - f'(x_0)(x - x_0)| \leq \epsilon_0 |x - x_0|.$$

By the triangle inequality,

$$|f(x) - f(x_0)| \leq \epsilon_0 |x - x_0| + |f'(x_0)||x - x_0| = (\epsilon_0 + |f'(x_0)|)|x - x_0|.$$

Now for any $\epsilon > 0$, choose $\delta = \min\{\delta_0, \epsilon/(\epsilon_0 + |f'(x_0)|)\}$. Then

$$|f(x) - f(x_0)| \leq (\epsilon_0 + |f'(x_0)|)|x - x_0| < (\epsilon_0 + |f'(x_0)|)\delta \leq \epsilon$$

whenever $|x - x_0| < \delta$ and $x \in U$. Thus $f$ is continuous at $x_0$. □

**Example 1.1.1.** Take the function

$$f(x) = \begin{cases} x\sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

Note that $f$ is continuous on $\mathbb{R}$. For $x \neq 0$, continuity is clear since both $x$ and $\sin(1/x)$ are continuous. At $x = 0$, we have

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} x\sin(1/x) = 0 = f(0)$$

since $|x\sin(1/x)| \leq |x|$ for all $x \in \mathbb{R}$, so $f$ is also continuous at $x = 0$. However, $f$ is not differentiable at $x = 0$. Consider the limit

$$\lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} \sin(1/x),$$

which does not exist since $\sin(1/x)$ oscillates. So $f$ is not differentiable at $x = 0$.

**Example 1.1.2.** Take the function $f(x) = |x|$, which is continuous everywhere on $\mathbb{R}$. However, $f$ is not differentiable at $x = 0$, since

$$\lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} \frac{|x|}{x}.$$

Note that

$$\frac{|x|}{x} = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0, \end{cases}$$

so the limit does not exist as $x \to 0$. Thus $f$ is not differentiable at $x = 0$.

**Remark.** For the previous example, we can however define the *left (right) derivative* by

$$f'_-(x_0) = \lim_{x\to x_0^-} \frac{f(x) - f(x_0)}{x - x_0} \quad \text{and} \quad f'_+(x_0) = \lim_{x\to x_0^+} \frac{f(x) - f(x_0)}{x - x_0}.$$

If $f$ is differentiable, then $f'_-(x_0) = f'_+(x_0)$. In the previous example, $f'_-(0) = -1$ and $f'_+(0) = 1$. For the first example however, even $f'_\pm(0)$ does not exist.

**Remark.** In one dimension, the existence of the derivative implies that the function is differentiable (the function is approximated by a linear function). However, in multiple dimensions, the existence of partial derivatives does not imply differentiability.

## 1.3 Rules for Differentiation

**Proposition 1.2.** *Let $U \subseteq \mathbb{R}$ be open and $f, g : U \to \mathbb{R}$ be differentiable. Then*

1. $(f + g)'(x_0) = f'(x_0) + g'(x_0)$
2. $(fg)'(x_0) = f'(x_0)g(x_0) + f(x_0)g'(x_0)$
3. *if* $g(x_0) \neq 0$, *then* $(f/g)'(x_0) = (f'(x_0)g(x_0) - f(x_0)g'(x_0))/(g(x_0)^2)$.

*Proof.* Find in textbook (Rosenlicht). □

**Proposition 1.3.** *We have* $\frac{d}{dx}(c) = 0$, $\frac{d}{dx}(x) = 1$, *and* $\frac{d}{dx}(x^n) = nx^{n-1}$ *for all* $n \in \mathbb{N}$.

*Proof.* We prove the last claim (the power rule) for $n \geq 1$ by induction. The base case $n = 1$ is the first claim which is true. Now suppose that the result holds for any $n \leq k \in \mathbb{N}$, and we show that it remains true for $n = k + 1$. By the product rule, we have

$$\frac{d}{dx}(x^{k+1}) = \frac{d}{dx}(x \cdot x^k) = \frac{d}{dx}(x) \cdot x^k + x \cdot \frac{d}{dx}(x^k) = x^k + xkx^{k-1} = (k+1)x^k.$$

Thus by induction this holds for all $n \geq 1$. We can do negative integers by the quotient rule. □

**Remark.** The power rule actually holds for any $n \in \mathbb{R}$.

**Proposition 1.4** (Chain rule)**.** *Let $U$ and $V$ be open sets of $\mathbb{R}$ and let $f : U \to V, g : V \to \mathbb{R}$ be differentiable. Let $x_0 \in U$ be such that $f'(x_0)$ and $g'(f(x_0))$ exist. Then $(g \circ f)'(x_0)$ exists and*

$$(g \circ f)'(x_0) = g'(f(x_0))f'(x_0).$$

*Proof.* For any fixed $y_0$ for which $g'(y_0)$ exists, set

$$A(y, y_0) = \begin{cases} (g(y) - g(y_0))/(y - y_0) & \text{if } y \in V \text{ and } y \neq y_0 \\ g'(y_0) & \text{if } y = y_0. \end{cases}$$

Then $A$ is continuous at $y_0$. To find $(g \circ f)'(x_0)$, observe that

$$\begin{aligned} \lim_{x \to x_0} \frac{g(f(x)) - g(f(x_0))}{x - x_0} &= \lim_{x \to x_0} \frac{A(f(x), f(x_0))(f(x) - f(x_0))}{x - x_0} \\ &= \lim_{x \to x_0} A(f(x), f(x_0)) \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = g'(f(x_0))f'(x_0), \end{aligned}$$

by the continuity of $A$ at $f(x_0)$ and the differentiability of $f$ at $x_0$. □

**Remark.** The rough idea of what we did here is

$$\begin{aligned} \lim_{x \to x_0} \frac{g(f(x)) - g(f(x_0))}{x - x_0} &= \lim_{x \to x_0} \frac{g(f(x)) - g(f(x_0))}{f(x) - f(x_0)} \frac{f(x) - f(x_0)}{x - x_0} \\ &= \lim_{x \to x_0} \frac{g(f(x)) - g(f(x_0))}{f(x) - f(x_0)} \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = g'(f(x_0))f'(x_0). \end{aligned}$$

But does not quite work as stated since it might be that $f(x) = f(x_0)$ even if $x \neq x_0$. We can fix this by introducing the function $A$ as we did in the proof, though the overall idea is the same.

**Remark.** If $f$ is monotone near $x_0$, then we can define the *inverse function* $f^{-1}$ so that $(f^{-1} \circ f)(x) = x$ near $x_0$. If $f'(x_0)$ exists, then by the chain rule applied to $x = (f^{-1} \circ f)(x)$ at $x = x_0$ we have

$$1 = \frac{d}{dx}(f^{-1} \circ f)(x_0) = \frac{d}{dx}f^{-1}(f(x_0)) \cdot f'(x_0) \implies \frac{d}{dx}f^{-1}(f(x_0)) = \frac{1}{f'(x_0)}.$$

**Example 1.1.3.** Let $f(x) = e^x$ with $f^{-1}(x) = \ln(x)$. Since $f'(x) = f(x) = e^x$, we have

$$\frac{d}{dx}f^{-1}(f(x_0)) = \frac{1}{f'(x_0)} \implies \frac{d}{dx}\ln(e^{x_0}) = \frac{1}{e^{x_0}}.$$

Letting $e^{x_0} = h$, we have $\frac{d}{dx}\ln(x)\big|_{x=h} = 1/h$, which recovers the familiar formula.

# Lecture 2

# Jan. 11 — The Mean Value Theorem

## 2.1 The Mean Value Theorem

**Lemma 2.1.** *Let $I \subseteq \mathbb{R}$ be open, $f : I \to \mathbb{R}$ is differentiable at $x_0 \in I$ and $f'(x_0) \neq 0$. Suppose $f'(x_0) > 0$, then there exists $\delta > 0$ such that for any $x \in (x_0 - \delta, x_0 + \delta)$,*

1. *if $x > x_0$, then $f(x) > f(x_0)$,*
2. *if $x < x_0$, then $f(x) < f(x_0)$.*

*Proof.* Take $\epsilon = f'(x_0)/2$. By the definition of the derivative, there exists $\delta > 0$ such that for ay $|x - x_0| < \delta$, we have

$$\left| \frac{f(x) - f(x_0)}{x - x_0} - f'(x_0) \right| < \epsilon = \frac{1}{2} f'(x_0).$$

By the triangle inequality,

$$\frac{f(x) - f(x_0)}{x - x_0} > \frac{1}{2} f'(x_0) > 0.$$

This quotient being positive immediately implies the desired results. $\square$

**Theorem 2.1.** *If $f(x)$ is differentiable in an open interval $I$ and $f$ obtains its local maximum (or minimum) at $x_0 \in I$, then $f'(x_0) = 0$.*

*Proof.* Suppose otherwise that $f'(x_0) \neq 0$. Assume without loss of generality that $f'(x_0) > 0$. Then by the previous lemma, there exists $\delta > 0$ such that for $x \in (x_0 - \delta, x_0 + \delta)$, if $x > x_0$ then $f(x) > f(x_0)$ and if $x < x_0$ then $f(x) < f(x_0)$. So $x_0$ cannot be a local maximum or minimum, which is a contradiction. $\square$

**Theorem 2.2** (Rolle's middle value theorem)**.** *Let $f(x)$ be continuous on $[a, b]$ and differentiable in $(a, b)$. Suppose $f(a) = f(b)$, then there exists $x_0 \in (a, b)$ such that $f'(x_0) = 0$.*

*Proof.* Since $f$ is continuous on a compact set, it obtains both a maximum and minimum on $[a, b]$. Let $M$ be the maximum and $m$ be the minimum. If $M = m$, then $f(x) \equiv M$ and $f'(x) = 0$ everywhere. If $M > m$, then at least one of the maximum or minimum must be obtained at an interior point $x_0 \in (a, b)$ since $f(a) = f(b)$. By the previous theorem, $f'(x_0) = 0$ at this point and we are done. $\square$

**Example 2.0.1.** Show that the equation $4ax^3 + 3bx^2 + 2cx = a + b + c$ has at least one root in $(0, 1)$.

*Proof.* Consider the equation

$$4ax^3 + 3bx^2 + 2cx - (a+b+c) = 0.$$

Notice that the left hand side is the derivative of the function

$$f(x) = ax^4 + bx^3 + cx^2 - (a+b+c)x.$$

So we just need to show that $f'(x) = 0$ for some $x$. For this, we can check that $f(0) = f(1) = 0$, and thus by Rolle's theorem there exists $x_0 \in (0,1)$ such that $f'(x_0) = 0$. So $x_0$ is a root. □

**Theorem 2.3** (Lagrange's middle value theorem)**.** *Let $f_9x)$ be continuous on $[a,b]$ and differentiable in $(a,b)$. Then there exists $x_0 \in (a,b)$ such that*

$$f'(x_0) = \frac{f(b)-f(a)}{b-a}.$$

*Proof.* Subtract the secant line through $(a, f(a))$ and $(b, f(b))$ from $f(x)$ to get

$$g(x) = f(x) - \frac{f(b)-f(a)}{b-a}(x-a).$$

Note that $g(a) = g(b) = f(a)$. So by Rolle's theorem, there exists $x_0 \in (a,b)$ such that $g'(x_0) = 0$. But

$$0 = g'(x_0) = f'(x_0) - \frac{f(b)-f(a)}{b-a},$$

which is the desired result. □

**Corollary 2.3.1.** *Suppose $f \in C([a,b])$, i.e. $f$ is continuous on $[a,b]$, and that $f$ is differentiable in $(a,b)$. Then the following statements are equivalent:*

1. *$f'(x) \geq 0$ in $(a,b)$,*
2. *$f(x)$ is increasing, i.e. if $x_1 > x_2$, then $f(x_1) \geq f(x_2)$.*

*In particular, if $f'(x) > 0$ in $(a,b)$, then $f(x)$ is strictly increasing, i.e. if $x_1 > x_2$, then $f(x_1) > f(x_2)$.*

*Proof.* $(2 \Rightarrow 1)$ For any $x_0 \in (a,b)$,

$$f'(x_0) = \lim_{h\to 0} \frac{f(x_0+h) - f(x_0)}{h} \geq 0$$

since $f(x_0+h) - f(x_0) \geq 0$ for $h > 0$ as $f$ is increasing.

$(1 \Rightarrow 2)$ Take $x_1 > x_2$, then by Lagrange's theorem there exists $\xi \in (x_2, x_1)$ such that

$$f(x_2) - f(x_1) = f'(\xi)(x_2 - x_1) \geq 0.$$

So $f(x_1) \geq f(x_2)$. The strict version follows from changing the above inequality to a strict one. □

## 2.2 Applications

**Example 2.0.2.** Show that
$$\frac{2}{2x+1} < \ln(1+1/x)$$
for any $x > 0$.

*Proof.* Let $f(x) = 2/(2x+1) - \ln(1+1/x)$. Taking the derivative yields
$$f'(x) = \frac{1}{(2x+1)^2 x(x+1)} > 0,$$
so $f$ is strictly increasing in $(0, \infty)$. Note that $f \to 0$ as $x \to \infty$, so $f(x) < 0$ for all $x > 0$. □

**Example 2.0.3.** Show that $b/a > b^a/a^b$ when $b > a > 1$.

*Proof.* Take log on both sides to get $\ln b - \ln a > a \ln b - b \ln a$. This gives
$$(b-1)\ln a > (a-1)\ln b \quad \iff \quad \frac{\ln a}{a-1} > \frac{\ln b}{b-1}.$$
Note that this is a monotonicity property. So let $f(x) = (\ln x)/(x-1)$ for $x > 1$. Then
$$f'(x) = \frac{x - 1 - x\ln x}{x(x-1)^2} < 0$$
when $x > 1$ because $x - 1 - x\ln x < 0$. To see the last claim, define $g(x) = x - 1 - x\ln x$ and note that $g'(x) = -\ln x < 0$ for $x > 1$. But $g(0) = 0$, so $g(x) < 0$ for $x > 1$. So $f$ is strictly decreasing. □

**Example 2.0.4.** Show that
$$\lim_{x\to 0} \frac{e^x - e^{\sin x}}{x - \sin x} = 1.$$
Let $f(x) = e^x$. Then there exists $\xi$ between $x$ and $\sin x$ such that
$$e^x - e^{\sin x} = (x - \sin x)e^{\xi(x)},$$
where the choice of $\xi$ may vary for different $x$. Then
$$\lim_{x\to 0} \frac{e^x - e^{\sin x}}{x - \sin x} = \lim_{x\to 0} e^{\xi(x)}.$$
Now note that $\xi(x)$ is always between $x$ and $\sin x$, which both tend to 0 as $x \to 0$. So by the squeeze theorem we have $\xi(x) \to 0$ as $x \to 0$ and thus $e^{\xi(x)} \to 1$ as $x \to 0$.

## 2.3 Cauchy's Mean Value Theorem

**Theorem 2.4** (Cauchy's middle value theorem)**.** *Let $f, g \in C([a, b])$ and $f, g$ be differentiable in $(a, b)$. Suppose $g'(x) \neq 0$ for any $x \in (a, b)$. Then there exists $x_0 \in (a, b)$ such that*

$$\frac{f'(x_0)}{g'(x_0)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

*Proof.* Use a similar construction as before and let

$$F(x) = f(x) - f(a) - \frac{f(b) - f(a)}{g(b) - g(a)}(g(x) - g(a)).$$

Note that $F(b) = F(a) = 0$, so by Rolle's theorem there exists $x_0 \in (a, b)$ such that $F'(x_0) = 0$. Then

$$0 = F'(x_0) = f'(x_0) - \frac{f(b) - f(a)}{g(b) - g(a)} g'(x_0),$$

which implies the desired result. □

**Remark.** The $g'(x) \neq 0$ condition guarantees that $g$ is monotone, even if $g'$ may fail to be continuous.

**Remark.** If $g$ is a monotonically increasing function, we can view $g$ as a mapping $g : [a, b] \to [g(a), g(b)]$, which we can view as a change of variables $x \mapsto u$. Since $g$ is monotone, we have an inverse $x = g^{-1}(u)$. Then

$$f(x) = f(g^{-1}(u)) = (f \circ g^{-1})(u) = \widetilde{f}(u).$$

By Lagrange's theorem,

$$\frac{\widetilde{f}(g(b)) - \widetilde{f}(g(a))}{g(b) - g(a)} = \widetilde{f}'(u_0)$$

for some $u_0 \in (g(a, g(b))$. Now note that

$$\widetilde{f}(g(b)) = (f \circ g^{-1})(g(b)) = f(b), \quad \widetilde{f}(g(a)) = f(a).$$

So the left-hand side is precisely

$$\text{LHS} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

By the chain rule, we have

$$\text{RHS} = \widetilde{f}'(u_0) = (f \circ g^{-1})'(u_0) = f'(g^{-1}(u_0))(g^{-1})'(u_0) = f'(x_0)\frac{1}{g'(x_0)}.$$

This recovers Cauchy's mean value theorem. So they are equivalent even if Cauchy's seems stronger.