MATH 4318: Analysis II

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Spring 2024

Contents

Lecture 1

Jan. 9 — The Derivative

1.1 Defining the Derivative

Definition 1.1. Let f be a real-valued function on an open interval $U \subseteq \mathbb{R}$. Let $x_0 \in U$, we say f is *differentiable* at x_0 if

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

exists. If it does, then this limit, denoted by $f'(x_0)$, is called the *derivative* of f at x_0.

Remark. By definition, for any $\epsilon > 0$, there exists $\delta > 0$ such that

$$\left| \frac{f(x) - f(x_0)}{x - x_0} - f'(x_0) \right| \leq \epsilon$$

if $|x - x_0| < \delta$ and $x \in U$. Multiplying both sides by $|x - x_0|$ yields

$$|f(x) - f(x_0) - f'(x_0)(x - x_0)| \leq \epsilon |x - x_0|.$$

In other words,

$$|f(x) - \varphi(x)| \leq \epsilon |x - x_0|$$

where $\varphi(x) = f(x_0) + f'(x_0)(x - x_0)$. In other words, $\varphi(x)$ is a first-order approximation of $f(x)$ near x_0. Geometrically, this is approximating the graph of $y = f(x)$ by the tangent line $y = \varphi(x)$.

1.2 Immediate Properties

Proposition 1.1. *Let $U \subseteq \mathbb{R}$ be an open set and $f : U \to \mathbb{R}$. If f is differentiable at $x_0 \in U$, then f is continuous at x_0.*

Proof. Pick any $\epsilon_0 > 0$. Then there exists $\delta_0 > 0$ such that whenever $|x - x_0| < \delta_0$ and $x \in U$,

$$|f(x) - f(x_0) - f'(x_0)(x - x_0)| \leq \epsilon_0 |x - x_0|.$$

By the triangle inequality,

$$|f(x) - f(x_0)| \leq \epsilon_0 |x - x_0| + |f'(x_0)||x - x_0| = (\epsilon_0 + |f'(x_0)|)|x - x_0|.$$

Now for any $\epsilon > 0$, choose $\delta = \min\{\delta_0, \epsilon/(\epsilon_0 + |f'(x_0)|)\}$. Then

$$|f(x) - f(x_0)| \leq (\epsilon_0 + |f'(x_0)|)|x - x_0| = (\epsilon_0 + |f'(x_0)|)\delta < \epsilon$$

whenever $|x - x_0| < \delta$ and $x \in U$. Thus f is continuous at x_0. $\square$

Example 1.1.1. Take the function

$$f(x) = \begin{cases} x\sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

Note that f is continuous on $\mathbb{R}$. For $x \neq 0$, continuity is clear since both x and $\sin(1/x)$ are continuous. At $x = 0$, we have

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} x\sin(1/x) = 0 = f(0)$$

since $|x\sin(1/x)| \leq |x|$ for all $x \in \mathbb{R}$, so f is also continuous at $x = 0$. However, f is not differentiable at $x = 0$. Consider the limit

$$\lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} \sin(1/x),$$

which does not exist since $\sin(1/x)$ oscillates. So f is not differentiable at $x = 0$.

Example 1.1.2. Take the function $f(x) = |x|$, which is continuous everywhere on $\mathbb{R}$. However, f is not differentiable at $x = 0$, since

$$\lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} \frac{|x|}{x}.$$

Note that

$$\frac{|x|}{x} = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0, \end{cases}$$

so the limit does not exist as $x \to 0$. Thus f is not differentiable at $x = 0$.

Remark. For the previous example, we can however define the *left (right) derivative* by

$$f'_-(x_0) = \lim_{x\to x_0^-} \frac{f(x) - f(x_0)}{x - x_0} \quad \text{and} \quad f'_+(x_0) = \lim_{x\to x_0^+} \frac{f(x) - f(x_0)}{x - x_0}.$$

If f is differentiable, then $f'_-(x_0) = f'_+(x_0)$. In the previous example, $f'_-(0) = -1$ and $f'_+(0) = 1$. For the first example however, even $f'_\pm(0)$ does not exist.

Remark. In one dimension, the existence of the derivative implies that the function is differentiable (the function is approximated by a linear function). However, in multiple dimensions, the existence of partial derivatives does not imply differentiability.

1.3 Rules for Differentiation

Proposition 1.2. *Let $U \subseteq \mathbb{R}$ be open and $f, g : U \to \mathbb{R}$ be differentiable. Then*

1. $(f + g)'(x_0) = f'(x_0) + g'(x_0)$
2. $(fg)'(x_0) = f'(x_0)g(x_0) + f(x_0)g'(x_0)$
3. $(f/g)'(x_0) = (f'(x_0)g(x_0) - f(x_0)g'(x_0))/(g(x_0)^2)$.

Proof. Find in textbook (Rosenlicht). □

Proposition 1.3. *We have $\frac{d}{dx}(c) = 0$, $\frac{d}{dx}(x) = 1$, and $\frac{d}{dx}(x^n) = nx^{n-1}$ for all $n \in \mathbb{N}$.*

Proof. We prove the last claim (the power rule) for $n \geq 1$ by induction. The base case $n = 1$ is the first claim which is true. Now suppose that the result holds for any $n \leq k \in \mathbb{N}$, and we show that it remains true for $n = k + 1$. By the product rule, we have

$$\frac{d}{dx}(x^{k+1}) = \frac{d}{dx}(x \cdot x^k) = \frac{d}{dx}(x) \cdot x^k + x \cdot \frac{d}{dx}(x^k) = x^k + xkx^{k-1} = (k+1)x^k.$$

Thus by induction this holds for all $n \geq 1$. We can do negative integers by the quotient rule. □

Remark. The power rule actually holds for any $n \in \mathbb{R}$.

Proposition 1.4 (Chain rule)**.** *Let U and V be open sets of $\mathbb{R}$ and let $f : U \to V, g : V \to \mathbb{R}$ be differentiable. Let $x_0 \in U$ be such that $f'(x_0)$ and $g'(f(x_0))$ exist. Then $(g \circ f)'(x_0)$ exists and*

$$(g \circ f)'(x_0) = g'(f(x_0))f'(x_0).$$

Proof. For any fixed y_0 for which $g'(y_0)$ exists, set

$$A(y, y_0) = \begin{cases} (g(y) - g(y_0))/(y - y_0) & \text{if } y \in V \text{ and } y \neq y_0 \\ g'(y_0) & \text{if } y = y_0. \end{cases}$$

Then A is continuous at y_0. To find $(g \circ f)'(x_0)$, observe that

$$\begin{aligned} \lim_{x \to x_0} \frac{g(f(x)) - g(f(x_0))}{x - x_0} &= \lim_{x \to x_0} \frac{A(f(x), f(x_0))(f(x) - f(x_0))}{x - x_0} \\ &= \lim_{x \to x_0} A(f(x), f(x_0)) \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = g'(f(x_0))f'(x_0), \end{aligned}$$

by the continuity of A at $f(x_0)$ and the differentiability of f at x_0. □

Remark. The rough idea of what we did here is

$$\begin{aligned} \lim_{x \to x_0} \frac{g(f(x)) - g(f(x_0))}{x - x_0} &= \lim_{x \to x_0} \frac{g(f(x)) - g(f(x_0))}{f(x) - f(x_0)} \frac{f(x) - f(x_0)}{x - x_0} \\ &= \lim_{x \to x_0} \frac{g(f(x)) - g(f(x_0))}{f(x) - f(x_0)} \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = g'(f(x_0))f'(x_0). \end{aligned}$$

But does not quite work as stated since it might be that $f(x) = f(x_0)$ even if $x \neq x_0$. We can fix this by introducing the function A as we did in the proof, though the overall idea is the same.

Remark. If f is monotone near x_0, then we can define the *inverse function* f^{-1} so that $(f^{-1} \circ f)(x) = x$ near x_0. If $f'(x_0)$ exists, then by the chain rule applied to $x = (f^{-1} \circ f)(x)$ at $x = x_0$ we have

$$1 = \frac{d}{dx}(f^{-1} \circ f)(x_0) = \frac{d}{dx} f^{-1}(f(x_0)) \cdot f'(x_0) \implies \frac{d}{dx} f^{-1}(f(x_0)) = \frac{1}{f'(x_0)}.$$

Example 1.1.3. Let $f(x) = e^x$ with $f^{-1}(x) = \ln(x)$. Since $f'(x) = f(x) = e^x$, we have

$$\frac{d}{dx} f^{-1}(f(x_0)) = \frac{1}{f'(x_0)} \implies \frac{d}{dx} \ln(e^{x_0}) = \frac{1}{e^{x_0}}.$$

Letting $e^{x_0} = h$, we have $\frac{d}{dx} \ln(x)\big|_{x=h} = 1/h$, which recovers the familiar formula.