# MATH 4318 Final Formula Sheet

## Differentiation

**Theorem 1** (Quotient rule)**.** *If $f, g : U \to \mathbb{R}$ are differentiable and $g(x_0) \neq 0$, then*

$$(f/g)'(x_0) = \frac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{g(x_0)^2}.$$

**Theorem 2** (Cauchy's mean value theorem)**.** *Let $f, g \in C([a,b])$ be differentiable in $(a,b)$. If $g'(x) \neq 0$ for any $x \in (a,b)$, then there exists $x_0 \in (a,b)$ such that*

$$\frac{f'(x_0)}{g'(x_0)} = \frac{f(b)-f(a)}{g(b)-g(a)}.$$

**Theorem 3.** *Let $R \in (0,\infty)$ and $f \in C^\infty(x_0 - R, x_0 + R)$. If there exists $M > 0$ such that for all $x \in (x_0 - R, x_0 + R)$, $|f^{(n)}(x)| \leq M$ for all $n \in \mathbb{N}$, then*

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

*for all $x \in (x_0 - R, x_0 + R)$.*

**Theorem 4** (Lagrange remainder)**.** *Let $f \in C^n([a,b])$ and assume that $f$ is $(n+1)$-times differentiable in $(a,b)$. Then*

$$f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1}$$

*for some $\xi \in [a,x]$.*

## Integration

**Definition 1** (Riemann integrability)**.** A function $f : [a,b] \to \mathbb{R}$ is *Riemann integrable* on $[a,b]$ if there exists $A \in \mathbb{R}$ such that for all $\epsilon > 0$, there exists $\delta > 0$ such that $|S - A| < \epsilon$ whenever $S$ is any Riemann sum on a partition of width $< \delta$. We call $A$ the *Riemann integral* of $f$ on $[a,b]$.

**Theorem 5** (Cauchy criterion for integrability)**.** *A function $f : [a,b] \to \mathbb{R}$ is integrable if and only if for any $\epsilon > 0$, there exists $\delta > 0$ such that $|S_1 - S_2| < \epsilon$ whenever $S_1$ and $S_2$ are Riemann sums for partitions of width $< \delta$.*

**Definition 2** (Step function)**.** We say $f : [a,b] \to \mathbb{R}$ is a *step function* if there exists a partition $a = x_0 < x_1 < \cdots < x_n = b$ such that $f$ is constant on each interval $(x_{i-1}, x_i)$.

**Theorem 6.** *A function $f : [a,b] \to \mathbb{R}$ is integrable if and only if for any $\epsilon > 0$, there exist step functions $f_1, f_2$ such that $f_1(x) \leq f(x) \leq f_2(x)$ for all $x \in [a,b]$ and*

$$\int_a^b (f_2 - f_1)\,dx < \epsilon.$$

**Definition 3** (Oscillation amplitude)**.** Let $f : [a,b] \to \mathbb{R}$ be bounded and $a = x_0 < x_1 < \cdots < x_n = b$ be a partition. Then

$$\omega_i(f) = \sup\{|f(x) - f(y)| : x, y \in [x_{i-1}, x_i)\}$$

is the *oscillation amplitude* of $f$ on $[x_{i-1}, x_i)$.

**Theorem 7.** *A function $f : [a,b] \to \mathbb{R}$ is integrable if and only if for any $\epsilon > 0$, there exists a partition such that*

$$\sum_{n=1}^{n} \omega_i(f)(x_i - x_{i-1}) < \epsilon.$$

**Theorem 8** (Du Bois-Reymond)**.** *Let $f$ be bounded on $[a,b]$. Then $f \in \mathcal{R}([a,b])$ if and only if for any $\epsilon, a > 0$, there exists a partition such that the total length of subintervals with $\omega_i(f) \geq \epsilon$ is $< a$.*

## Exchange of Limit Operations

**Theorem 9.** *Let $\{f_n\}$ be a sequence of functions on an open interval $U \subseteq \mathbb{R}$ such that each $f_n$ has a continuous derivative. Suppose $\{f_n'\}$ converges uniformly on $U$ and for some $a \in U$, $\{f_n'(a)\}$ converges. Then $\lim_{n\to\infty} f_n = f$ exists and $f$ is differentiable. Furthermore, we have $f' = \lim_{n\to\infty} f_n'$.*

**Theorem 10.** *Let $f_n : [a,b] \to \mathbb{R}$ be continuously differentiable. Suppose that $\sum_{n=1}^\infty f_n(x)$ converges pointwise on $[a,b]$ and $\sum_{n=1}^\infty f_n'(x)$ converges uniformly on $[a,b]$. Then $\sum_{n=1}^\infty f_n(x)$ converges uniformly on $[a,b]$ and*

$$\frac{d}{dx}\sum_{n=1}^{\infty} f_n(x) = \sum_{n=1}^{\infty} f_n'(x).$$

## Infinite Series

**Theorem 11.** *If $a_n \geq 0$, then $\sum_{n=1}^\infty a_n$ either converges or diverges to $\infty$.*

**Theorem 12** (Alternating series test)**.** *Let $\{a_n\}$ be a decreasing sequence with $a_n \to 0$ as $n \to \infty$. Then the series $\sum_{n=1}^\infty (-1)^{n+1} a_n$ converges (to $S$, say), and the partial sums $S_n$ have error $|S_n - S| \leq a_{n+1}$.*

**Theorem 13.** *Let $f_n \in C([a,b])$. If $\sum_{n=1}^\infty f_n(x)$ converges uniformly on $(a,b)$, then it converges uniformly on $[a,b]$.*

## Power Series

**Theorem 14.** *We have the following:*

1. *If $\sum_{n=0}^\infty a_n x^n$ converges at $x = x_1 \neq 0$, then it converges absolutely for all $x$ with $|x| < |x_1|$.*
2. *If $\sum_{n=0}^\infty a_n x^n$ diverges at $x = x_2 \neq 0$, then it diverges for all $x$ with $|x| > |x_2|$.*

**Theorem 15** (Hadamard's formula)**.** *For a power series $\sum_{n=0}^\infty a_n x^n$, let $L = \limsup_{n\to\infty} |a_n|^{1/n}$. Then its radius of convergence is $R = 1/L$.*

**Theorem 16.** *For a series $\sum_{n=0}^\infty a_n x^n$ with $a_n \neq 0$, if $\lim_{n\to\infty} |a_{n+1}/a_n| = L$, then its radius of convergence is $R = 1/L$.*

**Theorem 17.** *If $\sum_{n=0}^\infty a_n x^n$ has radius of convergence $R > 0$, then for any $0 < r < R$, the power series $\sum_{n=0}^\infty a_n r^n$ converges uniformly on $[-r, r]$. Moreover, if $\sum_{n=0}^\infty a_n x^n$ converges at $x = R < \infty$ (or $x = -R$), then $\sum_{n=0}^\infty a_n x^n$ converges uniformly on $[0, R]$ (or $[-R, 0]$).*

**Theorem 18.** *If $\sum_{n=0}^\infty a_n x^n$ has radius of convergence $R > 0$, then $f(x) = \sum_{n=0}^\infty a_n x^n \in C^\infty(-R, R)$.*

**Theorem 19.** *Suppose $f(x) = \sum_{n=0}^\infty a_n x^n$ has radius of convergence $R > 0$. Then for any $x \in (-R, R)$, $f \in \mathcal{R}([0,x])$ and*

$$\int_0^x f(t)\,dt = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}.$$

## Differentiation in $\mathbb{R}^n$

**Theorem 20.** *Let $f : U \subseteq \mathbb{R}^n \to \mathbb{R}$ where $U$ is open. Then $f$ is differentiable at $x = a$ if and only if there exist functions $A_1, \ldots, A_n$ on $U$, continuous at $x = a$, such that*

$$f(x) - f(a) = A_1(x)(x_1 - a_1) + \cdots + A_n(x)(x_n - a_n)$$

*for all $x \in U$. In this case, $\partial f/\partial x_i(a) = A_i(a)$.*

**Theorem 21.** *Let $U$ be an open set in $\mathbb{R}^n$ and suppose that $f : U \to \mathbb{R}$ has partial derivatives $f_1', \ldots, f_n'$ on $U$ which are continuous on $x = a$. Then $f$ is differentiable at $x = a$.*

**Theorem 22** (Implicit function theorem)**.** *Let $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ and $U \times V \subseteq \mathbb{R}^n \times \mathbb{R}^m$ be a neighborhood of $(x_0, y_0)$. Suppose $f$ and $\partial f/\partial y$ are continuous $U \times V$, and $f(x_0, y_0) = 0$ and*

$$\det\left(\frac{\partial f}{\partial y}(x_0, y_0)\right) \neq 0.$$

*Then there exists a neighborhood $U_0 \times V_0 \subseteq U \times V$ of $(x_0, y_0)$ and a unique continuous function $\varphi : U_0 \to V_0$ satisfying*

$$\begin{cases} f(x, \varphi(x)) = 0, \\ \varphi(x_0) = y_0. \end{cases}$$

**Theorem 23.** *Let $f : [a,b] \times [c,d] \to \mathbb{R}$ be continuous and suppose that $\partial f/\partial y$ exists and is continuous on $[a,b] \times [c,d]$. Then*

$$\frac{d}{dy}\int_a^b f(x,y)\,dx = \int_a^b \frac{\partial f}{\partial y}(x,y)\,dx.$$

## Integration in $\mathbb{R}^n$

**Theorem 24** (Lebesgue's criterion for Riemann integrability)**.** *Let $A \subseteq \mathbb{R}^n$ be a set with volume and let $f : A \to \mathbb{R}$ be a bounded function that is continuous except on a subset of $A$ with zero volume. Then $f$ is integrable on $A$.*

**Theorem 25.** *Suppose $f(x,y)$ is integrable on $D = [a,b] \times [c,d]$ and for each $x \in [a,b]$, $f(x,y)$ is integrable on $[c,d]$. Then*

$$\int_a^b dx\left[\int_c^d f(x,y)\,dy\right] = \iint_D f(x,y)\,dxdy$$

## Reverse Triangle Inequality

**Proposition 1.** *For all $x, y \in \mathbb{R}$, we have $||x| - |y|| \leq |x - y|$.*

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