MATH 4318: Analysis II

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Contents

Lecture 1

Jan. 9 — The Derivative

1.1 Defining the Derivative

Definition 1.1. Let f be a real-valued function on an open interval $U \subseteq \mathbb{R}$. Let $x_0 \in U$, we say f is *differentiable* at x_0 if

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

exists. If it does, then this limit, denoted by $f'(x_0)$, is called the *derivative* of f at x_0.

Remark. By definition, for any $\epsilon > 0$, there exists $\delta > 0$ such that

$$\left| \frac{f(x) - f(x_0)}{x - x_0} - f'(x_0) \right| \leq \epsilon$$

if $|x - x_0| < \delta$ and $x \in U$. Multiplying both sides by $|x - x_0|$ yields

$$|f(x) - f(x_0) - f'(x_0)(x - x_0)| \leq \epsilon |x - x_0|.$$

In other words,

$$|f(x) - \varphi(x)| \leq \epsilon |x - x_0|$$

where $\varphi(x) = f(x_0) + f'(x_0)(x - x_0)$. In other words, $\varphi(x)$ is a first-order approximation of $f(x)$ near x_0. Geometrically, this is approximating the graph of $y = f(x)$ by the tangent line $y = \varphi(x)$.

1.2 Immediate Properties

Proposition 1.1. *Let $U \subseteq \mathbb{R}$ be an open set and $f : U \to \mathbb{R}$. If f is differentiable at $x_0 \in U$, then f is continuous at x_0.*

Proof. Pick any $\epsilon_0 > 0$. Then there exists $\delta_0 > 0$ such that whenever $|x - x_0| < \delta_0$ and $x \in U$,

$$|f(x) - f(x_0) - f'(x_0)(x - x_0)| \leq \epsilon_0 |x - x_0|.$$

By the triangle inequality,

$$|f(x) - f(x_0)| \leq \epsilon_0 |x - x_0| + |f'(x_0)||x - x_0| = (\epsilon_0 + |f'(x_0)|)|x - x_0|.$$

Now for any $\epsilon > 0$, choose $\delta = \min\{\delta_0, \epsilon/(\epsilon_0 + |f'(x_0)|)\}$. Then

$$|f(x) - f(x_0)| \leq (\epsilon_0 + |f'(x_0)|)|x - x_0| < (\epsilon_0 + |f'(x_0)|)\delta \leq \epsilon$$

whenever $|x - x_0| < \delta$ and $x \in U$. Thus f is continuous at x_0. $\square$

Example 1.1.1. Take the function

$$f(x) = \begin{cases} x\sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

Note that f is continuous on $\mathbb{R}$. For $x \neq 0$, continuity is clear since both x and $\sin(1/x)$ are continuous. At $x = 0$, we have

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} x\sin(1/x) = 0 = f(0)$$

since $|x\sin(1/x)| \leq |x|$ for all $x \in \mathbb{R}$, so f is also continuous at $x = 0$. However, f is not differentiable at $x = 0$. Consider the limit

$$\lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} \sin(1/x),$$

which does not exist since $\sin(1/x)$ oscillates. So f is not differentiable at $x = 0$.

Example 1.1.2. Take the function $f(x) = |x|$, which is continuous everywhere on $\mathbb{R}$. However, f is not differentiable at $x = 0$, since

$$\lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} \frac{|x|}{x}.$$

Note that

$$\frac{|x|}{x} = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0, \end{cases}$$

so the limit does not exist as $x \to 0$. Thus f is not differentiable at $x = 0$.

Remark. For the previous example, we can however define the *left (right) derivative* by

$$f'_-(x_0) = \lim_{x\to x_0^-} \frac{f(x) - f(x_0)}{x - x_0} \quad \text{and} \quad f'_+(x_0) = \lim_{x\to x_0^+} \frac{f(x) - f(x_0)}{x - x_0}.$$

If f is differentiable, then $f'_-(x_0) = f'_+(x_0)$. In the previous example, $f'_-(0) = -1$ and $f'_+(0) = 1$. For the first example however, even $f'_\pm(0)$ does not exist.

Remark. In one dimension, the existence of the derivative implies that the function is differentiable (the function is approximated by a linear function). However, in multiple dimensions, the existence of partial derivatives does not imply differentiability.

1.3 Rules for Differentiation

Proposition 1.2. *Let $U \subseteq \mathbb{R}$ be open and $f, g : U \to \mathbb{R}$ be differentiable. Then*

1. $(f + g)'(x_0) = f'(x_0) + g'(x_0)$
2. $(fg)'(x_0) = f'(x_0)g(x_0) + f(x_0)g'(x_0)$
3. *if* $g(x_0) \neq 0$, *then* $(f/g)'(x_0) = (f'(x_0)g(x_0) - f(x_0)g'(x_0))/(g(x_0)^2)$.

Proof. Find in textbook (Rosenlicht). □

Proposition 1.3. *We have* $\frac{d}{dx}(c) = 0$, $\frac{d}{dx}(x) = 1$, *and* $\frac{d}{dx}(x^n) = nx^{n-1}$ *for all* $n \in \mathbb{N}$.

Proof. We prove the last claim (the power rule) for $n \geq 1$ by induction. The base case $n = 1$ is the first claim which is true. Now suppose that the result holds for any $n \leq k \in \mathbb{N}$, and we show that it remains true for $n = k + 1$. By the product rule, we have

$$\frac{d}{dx}(x^{k+1}) = \frac{d}{dx}(x \cdot x^k) = \frac{d}{dx}(x) \cdot x^k + x \cdot \frac{d}{dx}(x^k) = x^k + xkx^{k-1} = (k+1)x^k.$$

Thus by induction this holds for all $n \geq 1$. We can do negative integers by the quotient rule. □

Remark. The power rule actually holds for any $n \in \mathbb{R}$.

Proposition 1.4 (Chain rule)**.** *Let U and V be open sets of $\mathbb{R}$ and let $f : U \to V, g : V \to \mathbb{R}$ be differentiable. Let $x_0 \in U$ be such that $f'(x_0)$ and $g'(f(x_0))$ exist. Then $(g \circ f)'(x_0)$ exists and*

$$(g \circ f)'(x_0) = g'(f(x_0))f'(x_0).$$

Proof. For any fixed y_0 for which $g'(y_0)$ exists, set

$$A(y, y_0) = \begin{cases} (g(y) - g(y_0))/(y - y_0) & \text{if } y \in V \text{ and } y \neq y_0 \\ g'(y_0) & \text{if } y = y_0. \end{cases}$$

Then A is continuous at y_0. To find $(g \circ f)'(x_0)$, observe that

$$\begin{aligned} \lim_{x \to x_0} \frac{g(f(x)) - g(f(x_0))}{x - x_0} &= \lim_{x \to x_0} \frac{A(f(x), f(x_0))(f(x) - f(x_0))}{x - x_0} \\ &= \lim_{x \to x_0} A(f(x), f(x_0)) \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = g'(f(x_0))f'(x_0), \end{aligned}$$

by the continuity of A at $f(x_0)$ and the differentiability of f at x_0. □

Remark. The rough idea of what we did here is

$$\begin{aligned} \lim_{x \to x_0} \frac{g(f(x)) - g(f(x_0))}{x - x_0} &= \lim_{x \to x_0} \frac{g(f(x)) - g(f(x_0))}{f(x) - f(x_0)} \frac{f(x) - f(x_0)}{x - x_0} \\ &= \lim_{x \to x_0} \frac{g(f(x)) - g(f(x_0))}{f(x) - f(x_0)} \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = g'(f(x_0))f'(x_0). \end{aligned}$$

But does not quite work as stated since it might be that $f(x) = f(x_0)$ even if $x \neq x_0$. We can fix this by introducing the function A as we did in the proof, though the overall idea is the same.

Remark. If f is monotone near x_0, then we can define the *inverse function* f^{-1} so that $(f^{-1} \circ f)(x) = x$ near x_0. If $f'(x_0)$ exists, then by the chain rule applied to $x = (f^{-1} \circ f)(x)$ at $x = x_0$ we have

$$1 = \frac{d}{dx}(f^{-1} \circ f)(x_0) = \frac{d}{dx} f^{-1}(f(x_0)) \cdot f'(x_0) \implies \frac{d}{dx} f^{-1}(f(x_0)) = \frac{1}{f'(x_0)}.$$

Example 1.1.3. Let $f(x) = e^x$ with $f^{-1}(x) = \ln(x)$. Since $f'(x) = f(x) = e^x$, we have

$$\frac{d}{dx} f^{-1}(f(x_0)) = \frac{1}{f'(x_0)} \implies \frac{d}{dx} \ln(e^{x_0}) = \frac{1}{e^{x_0}}.$$

Letting $e^{x_0} = h$, we have $\frac{d}{dx} \ln(x)\big|_{x=h} = 1/h$, which recovers the familiar formula.

Lecture 2

Jan. 11 — The Mean Value Theorem

2.1 The Mean Value Theorem

Lemma 2.1. *Let $I \subseteq \mathbb{R}$ be open, $f : I \to \mathbb{R}$ is differentiable at $x_0 \in I$ and $f'(x_0) \neq 0$. Suppose $f'(x_0) > 0$, then there exists $\delta > 0$ such that for any $x \in (x_0 - \delta, x_0 + \delta)$,*

1. *if $x > x_0$, then $f(x) > f(x_0)$,*
2. *if $x < x_0$, then $f(x) < f(x_0)$.*

Proof. Take $\epsilon = f'(x_0)/2$. By the definition of the derivative, there exists $\delta > 0$ such that for ay $|x - x_0| < \delta$, we have

$$\left| \frac{f(x) - f(x_0)}{x - x_0} - f'(x_0) \right| < \epsilon = \frac{1}{2} f'(x_0).$$

By the triangle inequality,

$$\frac{f(x) - f(x_0)}{x - x_0} > \frac{1}{2} f'(x_0) > 0.$$

This quotient being positive immediately implies the desired results. □

Theorem 2.1. *If $f(x)$ is differentiable in an open interval I and f obtains its local maximum (or minimum) at $x_0 \in I$, then $f'(x_0) = 0$.*

Proof. Suppose otherwise that $f'(x_0) \neq 0$. Assume without loss of generality that $f'(x_0) > 0$. Then by the previous lemma, there exists $\delta > 0$ such that for $x \in (x_0 - \delta, x_0 + \delta)$, if $x > x_0$ then $f(x) > f(x_0)$ and if $x < x_0$ then $f(x) < f(x_0)$. So x_0 cannot be a local maximum or minimum, which is a contradiction. □

Theorem 2.2 (Rolle's middle value theorem)**.** *Let $f(x)$ be continuous on $[a, b]$ and differentiable in (a, b). Suppose $f(a) = f(b)$, then there exists $x_0 \in (a, b)$ such that $f'(x_0) = 0$.*

Proof. Since f is continuous on a compact set, it obtains both a maximum and minimum on $[a, b]$. Let M be the maximum and m be the minimum. If $M = m$, then $f(x) \equiv M$ and $f'(x) = 0$ everywhere. If $M > m$, then at least one of the maximum or minimum must be obtained at an interior point $x_0 \in (a, b)$ since $f(a) = f(b)$. By the previous theorem, $f'(x_0) = 0$ at this point and we are done. □

Example 2.0.1. Show that the equation $4ax^3 + 3bx^2 + 2cx = a + b + c$ has at least one root in $(0, 1)$.

Proof. Consider the equation

$$4ax^3 + 3bx^2 + 2cx - (a + b + c) = 0.$$

Notice that the left hand side is the derivative of the function

$$f(x) = ax^4 + bx^3 + cx^2 - (a + b + c)x.$$

So we just need to show that $f'(x) = 0$ for some x. For this, we can check that $f(0) = f(1) = 0$, and thus by Rolle's theorem there exists $x_0 \in (0, 1)$ such that $f'(x_0) = 0$. So x_0 is a root. □

Theorem 2.3 (Lagrange's middle value theorem)**.** *Let $f_9x)$ be continuous on $[a, b]$ and differentiable in (a, b). Then there exists $x_0 \in (a, b)$ such that*

$$f'(x_0) = \frac{f(b) - f(a)}{b - a}.$$

Proof. Subtract the secant line through $(a, f(a))$ and $(b, f(b))$ from $f(x)$ to get

$$g(x) = f(x) - \frac{f(b) - f(a)}{b - a}(x - a).$$

Note that $g(a) = g(b) = f(a)$. So by Rolle's theorem, there exists $x_0 \in (a, b)$ such that $g'(x_0) = 0$. But

$$0 = g'(x_0) = f'(x_0) - \frac{f(b) - f(a)}{b - a},$$

which is the desired result. □

Corollary 2.3.1. *Suppose $f \in C([a, b])$, i.e. f is continuous on $[a, b]$, and that f is differentiable in (a, b). Then the following statements are equivalent:*

1. *$f'(x) \geq 0$ in (a, b),*
2. *$f(x)$ is increasing, i.e. if $x_1 > x_2$, then $f(x_1) \geq f(x_2)$.*

In particular, if $f'(x) > 0$ in (a, b), then $f(x)$ is strictly increasing, i.e. if $x_1 > x_2$, then $f(x_1) > f(x_2)$.

Proof. $(2 \Rightarrow 1)$ For any $x_0 \in (a, b)$,

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} \geq 0$$

since $f(x_0 + h) - f(x_0) \geq 0$ for $h > 0$ as f is increasing.

$(1 \Rightarrow 2)$ Take $x_1 > x_2$, then by Lagrange's theorem there exists $\xi \in (x_2, x_1)$ such that

$$f(x_2) - f(x_1) = f'(\xi)(x_2 - x_1) \geq 0.$$

So $f(x_1) \geq f(x_2)$. The strict version follows from changing the above inequality to a strict one. □

2.2 Applications

Example 2.0.2. Show that
$$\frac{2}{2x+1} < \ln(1+1/x)$$
for any $x > 0$.

Proof. Let $f(x) = 2/(2x+1) - \ln(1+1/x)$. Taking the derivative yields
$$f'(x) = \frac{1}{(2x+1)^2 x(x+1)} > 0,$$
so f is strictly increasing in $(0, \infty)$. Note that $f \to 0$ as $x \to \infty$, so $f(x) < 0$ for all $x > 0$. □

Example 2.0.3. Show that $b/a > b^a/a^b$ when $b > a > 1$.

Proof. Take log on both sides to get $\ln b - \ln a > a \ln b - b \ln a$. This gives
$$(b-1)\ln a > (a-1)\ln b \quad\iff\quad \frac{\ln a}{a-1} > \frac{\ln b}{b-1}.$$
Note that this is a monotonicity property. So let $f(x) = (\ln x)/(x-1)$ for $x > 1$. Then
$$f'(x) = \frac{x - 1 - x\ln x}{x(x-1)^2} < 0$$
when $x > 1$ because $x - 1 - x\ln x < 0$. To see the last claim, define $g(x) = x - 1 - x\ln x$ and note that $g'(x) = -\ln x < 0$ for $x > 1$. But $g(0) = 0$, so $g(x) < 0$ for $x > 1$. So f is strictly decreasing. □

Example 2.0.4. Show that
$$\lim_{x\to 0} \frac{e^x - e^{\sin x}}{x - \sin x} = 1.$$

Proof. Let $f(x) = e^x$. Then there exists ξ between x and $\sin x$ such that
$$e^x - e^{\sin x} = (x - \sin x)e^{\xi(x)},$$
where the choice of ξ may vary for different x. Then
$$\lim_{x\to 0} \frac{e^x - e^{\sin x}}{x - \sin x} = \lim_{x\to 0} e^{\xi(x)}.$$
Now note that $\xi(x)$ is always between x and $\sin x$, which both tend to 0 as $x \to 0$. So by the squeeze theorem we have $\xi(x) \to 0$ as $x \to 0$ and thus $e^{\xi(x)} \to 1$ as $x \to 0$. □

2.3 Cauchy's Mean Value Theorem

Theorem 2.4 (Cauchy's middle value theorem)**.** *Let $f, g \in C([a,b])$ and f, g be differentiable in (a,b). Suppose $g'(x) \neq 0$ for any $x \in (a,b)$. Then there exists $x_0 \in (a,b)$ such that*

$$\frac{f'(x_0)}{g'(x_0)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

Proof. Use a similar construction as before and let

$$F(x) = f(x) - f(a) - \frac{f(b) - f(a)}{g(b) - g(a)}(g(x) - g(a)).$$

Note that $F(b) = F(a) = 0$, so by Rolle's theorem there exists $x_0 \in (a,b)$ such that $F'(x_0) = 0$. Then

$$0 = F'(x_0) = f'(x_0) - \frac{f(b) - f(a)}{g(b) - g(a)} g'(x_0),$$

which implies the desired result. $\square$

Remark. The $g'(x) \neq 0$ condition guarantees that g is monotone, even if g' may fail to be continuous.

Remark. If g is a monotonically increasing function, we can view g as a mapping $g : [a,b] \to [g(a), g(b)]$, which we can view as a change of variables $x \mapsto u$. Since g is monotone, we have an inverse $x = g^{-1}(u)$. Then

$$f(x) = f(g^{-1}(u)) = (f \circ g^{-1})(u) = \widetilde{f}(u).$$

By Lagrange's theorem,

$$\frac{\widetilde{f}(g(b)) - \widetilde{f}(g(a))}{g(b) - g(a)} = \widetilde{f}'(u_0)$$

for some $u_0 \in (g(a, g(b))$. Now note that

$$\widetilde{f}(g(b)) = (f \circ g^{-1})(g(b)) = f(b), \quad \widetilde{f}(g(a)) = f(a).$$

So the left-hand side is precisely

$$\text{LHS} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

By the chain rule, we have

$$\text{RHS} = \widetilde{f}'(u_0) = (f \circ g^{-1})'(u_0) = f'(g^{-1}(u_0))(g^{-1})'(u_0) = f'(x_0)\frac{1}{g'(x_0)}.$$

This recovers Cauchy's mean value theorem. So they are equivalent even if Cauchy's seems stronger.

Lecture 3

Jan. 16 — Taylor's Theorem

3.1 Darboux's Lemma

Lemma 3.1 (Darboux's lemma)**.** *If f is differentiable in (a, b), continuous on $[a, b]$ and $f'(a) < f'(b)$, then for any $c \in (f'(a), f'(b))$, there exists $x_0 \in (a, b)$ such that $f'(x_0) = c$.*

Proof. See homework. $\square$

Remark. There exists an example of a differentiable function $f(x)$ but $f'(x)$ is not continuous, e.g.

$$f(x) = \begin{cases} x^2 \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

We can compute that

$$f'(x) = \begin{cases} 2x \sin(1/x) - \cos(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0, \end{cases}$$

and we can verify as an exercise that $f'(x)$ is not continuous at $x = 0$.

Remark. Darboux's lemma guarantees that $g'(x) \neq 0$ implies either $g'(x) > 0$ or $g'(x) < 0$ everywhere in the conditions for Cauchy's mean value theorem.

3.2 L'Hôpital's Rule

Theorem 3.1 (L'Hôpital's rule, 0/0)**.** *Let f, g be differentiable in (a, b), $\lim_{x \to a^+} f(x) = \lim_{x \to a^+} g(x) = 0$, and $g'(x) \neq 0$ for any $x \in (a, b)$. Then if $\lim_{x \to a^+} f'(x)/g'(x)$ exists, we have*

$$\lim_{x \to a^+} \frac{f(x)}{g(x)} = \lim_{x \to a^+} \frac{f'(x)}{g'(x)}.$$

Proof. By Cauchy's theorem, for any $x \in (a, b)$, there exists $\xi(x) \in (a, x)$ such that

$$\frac{f(x)}{g(x)} = \frac{f(x) - f(a)}{g(x) - g(a)} = \frac{f'(\xi(x))}{g'(\xi(x))}.$$

If $x \to a^+$, then $\xi(x) \to a^+$, so

$$\lim_{x \to a^+} \frac{f(x)}{g(x)} = \lim_{x \to a^+} \frac{f'(\xi(x))}{g'(\xi(x))} = \lim_{x \to a^+} \frac{f'(x)}{g'(x)},$$

as desired. $\square$

Corollary 3.1.1. *Let f, g be differentiable in (a, ∞), $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = 0$, and $g'(x) \neq 0$ for any $x \in (a, \infty)$. Then if $\lim_{x\to\infty} f'(x)/g'(x)$ exists, we have*

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)}.$$

Proof. Assume $a > 0$. Define $\widetilde{f}(y) = f(1/y)$ and $\widetilde{g}(y) = g(1/y)$ with $y \in (0, 1/a)$. By L'Hôpital's rule,

$$\lim_{y\to0^+} \frac{\widetilde{f}(y)}{\widetilde{g}(y)} = \lim_{y\to0^+} \frac{\widetilde{f}'(y)}{\widetilde{g}'(y)} = \lim_{y\to\infty} \frac{f'(1/y)\cdot(-1/y^2)}{g'(1/y)\cdot(-1/y^2)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)},$$

as desired. $\square$

Theorem 3.2 (L'Hôpital, ∞/∞)**.** *Let f, g be differentiable in (a, b), $\lim_{x\to a^+} |f(x)| = \lim_{x\to a^+} |g(x)| = \infty$, and $g'(x) \neq 0$ for any $x \in (a, b)$. Then if $\lim_{x\to a^+} f'(x)/g'(x)$ exists, we have*

$$\lim_{x\to a^+} \frac{f(x)}{g(x)} = \lim_{x\to a^+} \frac{f'(x)}{g'(x)}.$$

Proof. Left as an exercise. $\square$

Remark. Saying that the absolute values of f and g go to infinity works, since the existence of the limit rules out oscillatory behavior.

Remark. These cases of ∞/∞ and $0/0$ are are called *indefinite types*. Other indefinite types include $0 \cdot \infty$, 0^0, ∞^0 1^∞, $\infty - \infty$, etc. But we can try to reduce them to the cases we know. For example, if $f(x) \to 0^+$ and $g(x) \to 0^+$ when $x \to x_0$, then $\lim_{x\to x_0} f(x)^{g(x)}$ is 0^0. Letting $y(x) = f(x)^{g(x)}$, we can take the log to get

$$\ln y(x) = g(x) \ln f(x) = \frac{\ln f(x)}{1/g(x)} = \frac{\infty}{\infty}.$$

Example 3.0.1. We can see that (this is a $\infty - \infty$ case)

$$\lim_{x\to0^+} \frac{1}{x^2} - \frac{\cot x}{x} = \lim_{x\to0^+} \frac{1 + x\cot x}{x^2} = \lim_{x\to0^+} \frac{-(\cot x - x\csc^2 x)}{2x\sin^2 x}.$$

Note that $x\cot x = x\cos x/\sin x \to 1$ as $x \to 0$. Now note that $\sin x/x \to 1$ as $x \to 0$, so we continue with

$$\lim_{x\to0^+} \frac{-(\cot x - x\csc^2 x)}{2x\sin^2 x} = \lim_{x\to0^+} \frac{x - \sin x\cos x}{2x^3} \frac{x^2}{\sin^2 x}$$

Since $x^2/\sin^2 x \to 1$ as $x \to 0$, we can look at the remaining part to get

$$\lim_{x\to0^+} \frac{x - \sin x\cos x}{2x^3} = \lim_{x\to0^+} \frac{1 - \cos 2x}{6x^2} = \lim_{x\to0^+} \frac{2\sin 2x}{12x} = \frac{1}{3}.$$

So $\lim_{x\to0^+}(1/x^2 - \cot x/x) = 1/3$.

3.3 Taylor's Theorem

Theorem 3.3 (Peano remainder term)**.** *Let $f : [a, b] \to \mathbb{R}$ be differentiable at $x = a$ up to nth order of derivatives, i.e. $f'(a), f''(a), \ldots, f^{(n)}(a)$ exist. Then as $x \to a^+$, we have*

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k + o((x-a)^n).$$

Call the polynomial part of the above $P_n(x)$, which is also known as the Taylor polynomial *of order n.*

Proof. To show that the error term is $o((x-a)^n)$, we have

$$\lim_{x \to a^+} \frac{f(x) - P_n(x)}{(x-a)^n} = \lim_{x \to a^+} \frac{f'(x) - P_n'(x)}{n(x-a)^{n-1}} = \frac{1}{n!} \lim_{x \to a^+} \left[\frac{f^{n-1}(x) - f^{n-1}(a)}{x-a} - f^{(n)}(a) \right] = 0$$

by L'Hôpital's rule, where we used the observation that $f^{(k)}(a) = P_n^{(k)}(a)$ for $1 \le k \le n$. The final step is a result of the existence of $f^{(n)}(a)$. $\square$

Lemma 3.2 (Rolle's theorem for higher order derivatives)**.** *Let $f \in C^n([a, b])$ and differentiable to $(n+1)$ order. If $f'(a) = \cdots = f^{(n)}(a) = 0$ and $f(a) = f(b)$, then there exists $x_0 \in (a, b)$ such that $f^{(n+1)}(x_0) = 0$.*

Proof. Since $f(a) = f(b)$, by the usual Rolle's theorem there exists $x_1 \in (a, b)$ such that $f'(x_1) = 0$. Then since $f'(a) = f'(x_1) = 0$, by Rolle's theorem again, there exists $x_2 \in (a, x_1)$ such that $f''(x_2) = 0$. Repeat this to get $x_{n+1} \in (a, x_n) \subseteq (a, b)$ such that $f^{(n+1)}(x_{n+1}) = 0$. Take $x_0 = x_{n+1}$ to finish. $\square$

Theorem 3.4 (Lagrange remainder term)**.** *Let $f \in C^n([a, b])$, in particular, $f'(a), \ldots, f^{(n)}(a)$ exist. Additionally, assume f is $(n+1)$-th differentiable in (a, b).*[1] *Then*

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k + R_n(x) \quad \textit{where} \quad R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1}$$

for some $\xi \in [a, x]$.

Proof. Define $P(x) = P_n(x) + \lambda(x-a)^{n+1}$, where we choose $\lambda \in \mathbb{R}$ such that $P(b) = f(b)$, i.e.

$$\lambda = \frac{f(b) - P_n(b)}{(b-a)^{n+1}}.$$

Consider $g(x) = f(x) - P(x)$, which satisfies $g(a) = g(b) = 0$ and $g'(a) = \cdots = g^{(n)}(a) = 0$. Then by Rolle's theorem (higher order), there exists $\xi \in (a, b)$ such that $g^{(n+1)}(\xi) = 0$. In other words,

$$f^{(n+1)}(\xi) - P^{(n+1)}(\xi) = 0 \implies f^{(n+1)}(\xi) - (n+1)! \underbrace{\frac{f(b) - P_n(b)}{(b-a)^{n+1}}}_{\lambda} = 0.$$

This implies that

$$f(b) = P_n(b) + \frac{1}{(n+1)!} f^{(n+1)}(\xi)(b-a)^{n+1},$$

and since we picked b arbitrarily (kind of), we can take $b = x$ and we are done since $\xi \in [a, b]$. $\square$

[1] Note that the $(n+1)$-th derivative need not be continuous here.

Remark. The choice of ξ in Lagrange's remainder term may (and likely does) vary for different x.

Remark. The Taylor polynomial is unique in the sense that if $f : [a, b] \to \mathbb{R}$ and $f'(a), \ldots, f^{(n)}(a)$ exist, then if

$$f(x) = p(x) + o((x-a)^n)$$

as $x \to a^+$ for some polynomial $p(x)$ with $\deg p \leq n$, then $p(x) = P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$. This is because if $Q(x) = p(x) - P_n(x)$, then by Taylor's formula (Peano form), we get

$$\lim_{x \to a^+} \frac{Q(x)}{(x-a)^n} = \lim_{x \to a^+} \frac{p(x) - \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k}{(x-a)^n} = 0.$$

From here this implies that $Q(x) = 0$ since $\deg Q \leq n$. Another way to see this is to plug in $x = a$, which deletes everything except the constant, and then ignore the constant and divide by $(x - a)$ to repeat.

Lecture 4

Jan. 18 — Taylor Polynomials

4.1 Common Taylor Polynomials

We have

$$
\begin{aligned}
e^x &= 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + o(x^n), \\
\sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + \frac{(-1)^{n-1}x^{2n-1}}{(2n-1)!} + o(x^{2n}), \\
\cos &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + \frac{(-1)^n x^{2n}}{(2n)!} + o(x^{2n+1}), \\
(1+x)^\alpha &= 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \cdots + \frac{\alpha(\alpha-1)\ldots(\alpha-n+1)}{n!}x^n + o(x^n), \\
\ln(1+x) &= x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots + (-1)^{n-1}\frac{x^n}{n} + o(x^n).
\end{aligned}
$$

4.2 Combining Taylor Polynomials

Remark. If $a = 0$ and $f(x)$ is even in $(-b, b)$, then

$$f(x) = \sum_{k=0}^{n/2} a_k x^{2k} + o(x^n).$$

Similarly if $f(x)$ is odd in $(-b, b)$, then

$$f(x) = \sum_{k=0}^{n/2} a_k x^{2k+1} + o(x^{n+1}).$$

Remark. To create new Taylor polynomials from known ones, we can observe that if $f(x) = P_n(x) + o((x-a)^n)$ and $g(x) = Q_n(x) + o((x-a)^n)$, then

$$f(x) + g(x) = (P_n(x) + Q_n(x)) + o((x-a)^n) \quad \text{and} \quad f(x)g(x) = \underbrace{(P_n(x)Q_n(x))}_{\text{take first } n \text{ terms}} + o((x-a)^n).$$

If $P_n(x) = \sum_{k=0}^n a_k(x-a)^k$ and $Q_n(x) = \sum_{k=0}^n b_k(x-a)^k$, then $f(x)g(x)$ has Taylor polynomial $\sum_{k=0}^n c_k(x-a)^k$ where

$$c_k = \sum_{i=0}^k a_i b_{k-i}.$$

If $h(x) = f(x)/g(x)$ and $g(x) \neq 0$ near $x = a$, then $f(x) = h(x)g(x)$. Let $h(x) = \sum_{k=0}^{n} c_k(x-a)^k + o((x-a)^n)$, then

$$a_k = \sum_{i=0}^{k} c_i b_{k-i}$$

for $0 \leq k \leq n$, after which we can solve for the c_k.

Example 4.0.1. Find the Taylor polynomial for $\tan x$ up to $n = 5$.

Proof. Note that $\tan x$ is odd, so we can write

$$\tan x = x + a_3 x^3 + a_5 x^5 + o(x^5).$$

Now since $\tan x = \sin x / \cos x$, we have $\sin x = \tan x \cos x$, so

$$x - \frac{x^3}{6} + \frac{x^5}{5!} + o(x^5) = (x + a_3x^3 + a_5x^5)(1 - \frac{x^2}{2!} + \frac{x^4}{4!})$$

We can solve to get

$$\begin{cases} -\frac{1}{6} = -\frac{1}{2} + a_3 \\ \frac{1}{5!} = \frac{1}{4!} - \frac{a_3}{2!} + a_5 \end{cases} \implies \quad a_3 = -\frac{1}{3}, \quad a_5 = \frac{2}{15}$$

as the coefficients for the Taylor polynomial. $\square$

Remark. If

$$f'(x) = \sum_{k=0}^{n} b_k (x-a)^k + o((x-a)^n),$$

then the anti-derivative of $f(x)$ has

$$f(x) = f(x_0) + \sum_{k=0}^{n} a_{k+1}(x-a)k + 1 + o((x-a)^{n+1}),$$

where $a_{k+1} = b_k/(k+1)$ for $0 \leq k \leq n$. This is because

$$b_k = \frac{(f')^{(k)}(a)}{k!} = \frac{f^{(k+1)}(a)}{k!} \quad \text{and} \quad a_{k+1} = \frac{f^{(k+1)}(a)}{k+1} = \frac{1}{k+1}\frac{f^{(k+1)}(a)}{k!} = \frac{b_k}{k+1}.$$

Example 4.0.2. Find the Taylor polynomial for $f(x) = \arctan x$.

Proof. Recall that

$$f'(x) = \frac{1}{1+x^2} = \sum_{k=0}^{n} (-1)^k x^{2k}.$$

Using the above we get

$$f(x) = \arctan x = \sum_{k=0}^{n} (-1)^k \frac{x^{2k+1}}{2k+1} + o(x^{2n+2})$$

as the Taylor polynomial. $\square$

4.3 Applications for Taylor Polynomials

4.3.1 Finding Limits

Remark. Let $f(x) = ax^n + o(x^n)$ as $x \to 0$ and $g(x) = bx^n + o(x^n)$ where $b \neq 0$. Then

$$\lim_{x\to 0} \frac{f(x)}{g(x)} = \frac{a}{b}.$$

Remark. For the polynomial of $f(g(x))$, we can do

$$f(u) = \sum_{k=0}^{n} a_k(u - g(a))^k + o((u - g(a))^n), \quad \text{where} \quad u = g(x) = \sum_{k=0}^{n} b_k(x-a)^k + o((x-a)^n).$$

Then we can substitute in $u = g(x)$ to find the overall polynomial.

Example 4.0.3. Find

$$\lim_{x\to 0} \frac{\sqrt{1 + 2\tan x} - e^x + x^2}{\arcsin x - \sin x}.$$

Proof. Note that

$$\sqrt{1 + 2\tan x} - e^x + x^2 = \frac{2x^3}{3} + o(x^3),$$
$$\arcsin x - \sin x = \frac{x^3}{3} + o(x^3).$$

So the desired limit is 2. □

Remark. If $f(x) = ax^n + o(x^n)$ and $g(x) = bx^m + o(x^m)$ for $a, b \neq 0$, then

$$\lim_{x\to 0} \frac{f(x)}{g(x)} = \begin{cases} a/b & \text{if } m = n, \\ 0 & \text{if } m < n, \\ \infty & \text{if } m > n. \end{cases}$$

Example 4.0.4. Assume $f(x) = 1 + ax^n + o(x^n)$ where $a \neq 0$ and

$$g(x) = \frac{1}{bx^n + o(x^n)}, \quad \text{i.e.} \quad \frac{1}{g(x)} = bx^n + o(x^n).$$

for $b \neq 0$. Then

$$\lim_{x\to 0} f(x)^{g(x)} = e^{a/b}.$$

Let $y(x) = f(x)^{g(x)}$, then $\ln y(x) = g(x) \ln f(x)$. Note that

$$\ln f(x) = \ln(1 + ax^n + o(x^n)) = ax^n + o(x^n),$$

so that

$$\frac{\ln f(x)}{1/g(x)} = \frac{ax^n + o(x^n)}{bx^n + o(x^n)} \to \frac{a}{b}$$

as $x \to 0$. Thus $\ln y(x) \to a/b$ and $y(x) \to e^{a/b}$ as $x \to 0$.

Example 4.0.5. Find

$$\lim_{x\to 0}\left[\cos(xe^x) - \ln(1-x) - x\right]^{\cot x^3}.$$

Proof. Here we have

$$f(x) = \cos(xe^x) - \ln(1-x) - x = 1 - \frac{2}{3}x^3 + o(x^3) \quad \text{and} \quad \frac{1}{g(x)} = \tan x^3 = x^3 + o(x^3).$$

Thus the limit is $e^{-2/3}$. □

4.3.2 Estimation

Example 4.0.6. Let $f(x)$ be twice differentiable in $[0,1]$ and $f(0) = f(1)$. Further assume $|f''(x)| \le M$ for $0 \le x \le 1$. Prove that $|f'(x)| \le M/2$ for $0 \le x \le 1$.

Proof. Recall that Lagrange's form of Taylor's theorem says

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(\xi)}{2!}(x-a)^2$$

for some ξ between a and x. Thus for any $x \in (0,1)$, we have

$$f(1) = f(x) + f'(x)(1-x) + \frac{f''(\xi_1)}{2}(1-x)^2.$$

Similarly, we have

$$f(0) = f(x) + f'(x)(-x) + \frac{f''(\xi_2)}{2}x^2.$$

Here $x \le \xi_1 \le 1$ and $0 \le \xi_2 \le x$. Since $f(1) = f(2)$, we can solve for $f'(x)$ to get

$$f'(x) = \frac{f''(\xi_2)x^2 - f''(\xi_1)(1-x)^2}{2}.$$

Then taking absolute values yields

$$|f'(x)| \le M\left(\frac{x^2 + (1-x)^2}{2}\right) \le \frac{M}{2}\max_{0\le x\le 1}\left[x^2 + (1-x)^2\right] = \frac{M}{2},$$

as desired. □

Example 4.0.7. Let $f(x)$ be twice differentiable in $[0,1]$ and $f'(a) = f'(b) = 0$. Then there exists $\xi \in (a,b)$ such that

$$|f''(\xi)| \ge 4\frac{|f(a) - f(b)|}{(b-a)^2}.$$

Proof. Note that this is equivalent to

$$|f(b) - f(a)| \le f''(\xi)\left(\frac{b-a}{2}\right)^2.$$

Then we have

$$f\left(\frac{b+a}{2}\right) = f(a) + \frac{f''(\xi_1)}{2}\left(\frac{b-a}{2}\right)^2 = f(b) - \frac{f''(\xi_2)}{2}\left(\frac{b-a}{2}\right)^2,$$

so that

$$f(b) - f(a) = \frac{f''(\xi_2) + f''(\xi_1)}{2} \left(\frac{b-a}{2}\right)^2.$$

From here we have

$$|f(b) - f(a)| \leq \underbrace{\frac{|f''(\xi_1)| + |f''(\xi_2)|}{2}}_{=|f''(\xi)|} \left(\frac{b-a}{2}\right)^2$$

for some $\xi \in (a, b)$ by Darboux's lemma, as desired. $\square$

Lecture 5

Jan. 23 — The Riemann Integral

5.1 The Anti-Derivative

Recall the *anti-derivative* from calculus:

Definition 5.1. Let $f : U \to \mathbb{R}$ where U is an interval in $\mathbb{R}$. If there exists a differentiable function $F : U \to \mathbb{R}$ such that $F'(x) = f(x)$ for all $x \in U$, then $F(x)$ is an *anti-derivative* of f, denoted

$$F(x) = \int f(x)\, dx.$$

This is also called the *indefinite integral* of f.

Remark. The anti-derivatives of a function can differ by a constant.

Example 5.1.1. Find an anti-derivative of $f(x) = |x|$ for $x \in \mathbb{R}$.

Proof. If $x > 0$, we have $f(x) = x$ and so $F(x) = x^2/2$. If $x < 0$, then $f(x) = -x$ and so $F(x) = -x^2/2$. We can also write this as

$$F(x) = x \cdot \frac{|x|}{2}.$$

Clearly for $x \neq 0$, we have $F'(x) = f(x)$. At $x = 0$, we have

$$\lim_{x \to 0} \frac{F(x) - f(0)}{x} = \lim_{x \to 0} \frac{1}{2}|x| = 0,$$

so $F'(0) = f(0)$ and F is an anti-derivative of f. □

Remark. The eventual goal is to show that any continuous function $f : [a, b] \to \mathbb{R}$ has an anti-derivative.

Example 5.1.2. Find an anti-derivative for

$$f(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \\ -1 & \text{if } x < 0. \end{cases}$$

Proof. We can try to use $F(x) = |x|$, but recall that F is not differentiable at $x = 0$. More generally, suppose that $f(x)$ has some anti-derivative $F(x)$, i.e. $f(x) = F'(x)$. By Darboux's theorem, $f(x)$ must take all values in $(-1, 1)$, which is a contradiction with the definition of f. □

Remark. If $f(x)$ has a jump discontinuity, then it has no anti-derivative.

5.2 The Riemann Integral

Recall from calculus that if $f(x)$ is defined in $[a,b]$ and $F'(x) = f(x)$, then we have[1]

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a).$$

We called this the *definite integral* of f in calculus, but we would like a more rigorous definition.

Definition 5.2. Let $a, b \in \mathbb{R}$ and $a < b$. A *partition* of the interval $[a,b]$ is a finite sequence of numbers $x_0, x_1, \ldots, x_n$ such that $a = x_0 < x_1 < \cdots < x_n = b$.

Definition 5.3. The *width* of a partition $x_0, x_1, \ldots, x_n$ is $\max\{x_i - x_{i-1} : i = 1, 2, \ldots, n\}$.

Definition 5.4. For any partition $x_0, x_1, \ldots, x_n$, define the *Riemann sum* to be

$$S = \sum_{i=1}^{n} f(x_i')(x_i - x_{i-1}),$$

where x_i' is any point between x_{i-1} and x_i, inclusive.[2]

Definition 5.5. Let $a, b \in \mathbb{R}$ with $a < b$ and $f : [a,b] \to \mathbb{R}$. We say f is *Riemann integrable* on $[a,b]$ if there exists $A \in \mathbb{R}$ such that for all $\epsilon > 0$, there exists $\delta > 0$ such that $|S - A| < \epsilon$ whenever S is any Riemann sum for a partition of $[a,b]$ with width less than δ. We call A the *Riemann integral* of f on $[a,b]$ and denote it by

$$A = \int_a^b f(x)\,dx.$$

Remark. If f is Riemann integrable, then

$$A = \int_a^b f(x)\,dx$$

is unique. This is because if A and A' are two numbers for the Riemann integral, then for any $\epsilon > 0$, there exists $\delta > 0$ such that

$$|A - S| < \epsilon \quad \text{and} \quad |A' - S| < \epsilon$$

for any Riemann sum S associated with a partition of width less than δ. Then

$$|A - A'| \le |A - S| + |A' - S| < 2\epsilon,$$

so $A = A'$ and thus the Riemann integral is unique.

Example 5.5.1. Let $f(x) = c$ on $[a,b]$, a constant function. Then for any partition $x_0, x_1, \ldots, x_n$,

$$S = \sum_{i=1}^{n} f(x_i')(x_i - x_{i-1}) = \sum_{i=1}^{n} c(x_i - x_{i-1}) = c(b-a) \implies \int_a^b c\,dx = c(b-a).$$

[1] This is the *fundamental theorem of calculus.*

[2] The geometric intuition of the Riemann sum is an approximation for the *area* under the graph of f by rectangles.

Example 5.5.2. Fix $\xi \in [a, b]$ and let $f : [a, b] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 0 & \text{if } x \neq \xi \\ c & \text{if } x = \xi. \end{cases}$$

Check that

$$A = \int_a^b f(x)\, dx = 0.$$

Proof. For any partition $a = x_0 < x_1 < \cdots < x_n = b$ with width δ, we have

$$|S| = \left| \sum_{i=1}^n f(x_i')(x_i - x_{i-1}) \right| \leq |c| 2\delta$$

since ξ can be in at most two of the intervals of the partition. Then for any $\epsilon > 0$, choose $\delta = \epsilon/(2|c|)$, so that $|S| < \epsilon$ for any partition of width less than δ. From this we can conclude that $A = 0$. □

Example 5.5.3. Consider a step function. Let $\alpha, \beta \in [a, b]$ with $\alpha < \beta$. Define $f : [a, b] \to \mathbb{R}$ by

$$f(x) = \begin{cases} 1 & \text{if } x \in (\alpha, \beta) \\ 0 & \text{if } x \notin (\alpha, \beta) \text{ and } x \in [a, b]. \end{cases}$$

Note that f has no anti-derivative, but it is Riemann integrable. In fact,

$$\int_a^b f(x)\, dx = \beta - \alpha.$$

To see this, take any partition $a = x_0 < x_1 < \cdots < x_n = b$ with width less than δ. Then

$$S = \sum_{i=1}^n f(x_i')(x_i - x_{i-1}) = \sum_{[x_{i-1}, x_i] \cap [\alpha, \beta] \neq \varnothing} f(x_i')(x_i - x_{i-1}).$$

Each partition is in two classes: Either (1) it only partially intersects $[\alpha, \beta]$ or (2) it is contained in $[\alpha, \beta]$. So

$$S = \underbrace{1(\text{total length of intervals of class 2})}_{I_1} + \underbrace{|f(x_i')|(\text{total length of intervals of class 1})}_{I_2}.$$

We have $|I_1 - (\beta - \alpha)| < 2\delta$ and $|I_2| < 2\delta$ since there are at most two intervals of class 1. So

$$|S - (\beta - \alpha)| \leq |I_1| + |I_2| < 4\delta.$$

So $f(x)$ is Riemann integrable and

$$\int_a^b f(x)\, dx = \beta - \alpha,$$

as desired.

Example 5.5.4. Define $f : [a, b] \to \mathbb{R}$ by

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

Then $f(x)$ is not Riemann integrable. For any partition $a = x_0 < x_1 < \cdots < x_n = b$,

$$S = \sum_{i=1}^{n} f(x_i')(x_i - x_{i-1}) = \begin{cases} b-a & \text{if } x_i' \text{ are all rational} \\ 0 & \text{if } x_i' \text{ are all irrational.} \end{cases}$$

We can always choose x_i' to be in either case since the rationals and irrationals are both dense in $\mathbb{R}$. So there is no $A \in \mathbb{R}$ such that $|A - S| < \epsilon$, no matter how small we take δ to be.

Remark. The function f from the previous example is not Riemann integrable, but it is Lebesgue integrable. In fact,

$$L = \int_a^b f(x)\, dx = 0$$

with respect to the Lebesgue measure. This is because the set of rational numbers $\mathbb{Q}$ has measure zero.

5.3 Properties of the Riemann Integral

Proposition 5.1. *We have the following linearity properties of the Riemann integral:*

1. *If $f, g : [a, b] \to \mathbb{R}$ are Riemann integrable, then $f \pm g$ are also integrable and*

$$\int_a^b (f \pm g)\, dx = \int_a^b f(x)\, dx \pm \int_a^b g(x)\, dx$$

2. *For any $c \in \mathbb{R}$, cf is integrable and*

$$\int_a^b cf\, dx = c \int_a^b f(x)\, dx.$$

Proof. See textbook, fairly straightforward. □

Remark. Since we only discuss Riemann integration in this class, we will sometimes simply say "integrable" instead of "Riemann integrable."

Proposition 5.2. *If $f : [a, b] \to \mathbb{R}$ is integrable and $f(x) \geq 0$, then*

$$\int_a^b f(x)\, dx \geq 0.$$

Proof. Let

$$A = \int_a^b f(x)\, dx.$$

Then for any $\epsilon > 0$, there exists $\delta > 0$ such that for any partition of width $< \delta$, we have $|A - S| < \epsilon$. But

$$S = \sum_{i=1}^{n} f(x_i')(x_i - x_{i-1}) \geq 0,$$

Then we have $A > S - \epsilon \geq -\epsilon$, so taking $\epsilon \to 0$ gives $A \geq 0$. □

Corollary 5.0.1. *If $f, g : [a, b] \to \mathbb{R}$ are integrable and $f(x) \geq g(x)$ for all $x \in [a, b]$, then*

$$\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx.$$

Proof. By linearity,

$$\int_a^b f(x)\,dx - \int_a^b g(x)\,dx = \int_a^b (f(x) - g(x))\,dx \geq 0$$

since $f(x) - g(x) \geq 0$ by assumption. □

Corollary 5.0.2. *If $f : [a, b] \to \mathbb{R}$ is integrable and $m \leq f(x) \leq M$ for all $x \in [a, b]$, then*

$$m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a).$$

Lecture 6

Jan. 25 — Riemann Integrability

6.1 Conditions for Integrability

Lemma 6.1. *A function $f : [a, b] \to \mathbb{R}$ is integrable if and only if for any $\epsilon > 0$, there exists $\delta > 0$ such that $|S_1 - S_2| < \epsilon$ whenever S_1 and S_2 are Riemann sums for partitions of width less than δ.*

Proof. ($\Rightarrow$) If f is integrable, then for any $\epsilon > 0$, there exists $\delta > 0$ such that

$$\left| S - \int_a^b f(x)\,dx \right| < \frac{\epsilon}{2}$$

for any Riemann sum S of a partition with width less than δ. Then

$$|S_1 - S_2| \leq \left| S_1 - \int_a^b f(x)\,dx \right| + \left| S_2 - \int_a^b f(x)\,dx \right| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

as desired.

($\Leftarrow$) Take the special partition into intervals of equal length, with width $(a - b)/n$. Pick the middle point in each interval, and let

$$S_n = \sum_{i=1}^{n} f(x_i')(x_i - x_{i-1})$$

be the corresponding Riemann sum. Now we check that $\{S_n\}_{n=1}^{\infty}$ is a Cauchy sequence. This is because for any $\epsilon > 0$, if N is large enough, then for any $n, m \geq N$, we have $|S_n - S_m| < \epsilon$ if $1/N < \delta$. Then $\{S_n\}_{n=1}^{\infty}$ converges, so let $\lim_{n\to\infty} S_n = A$. Now for any $\epsilon > 0$, there exists $\delta > 0$ such that for any Riemann sum S with width $< \delta$, if $1/n < \delta$, then $|S_n - S| < \epsilon/2$. So

$$|S - A| \leq |S_n - S| + |S_n - A| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

if n is large enough. Thus

$$A = \int_a^b f(x)\,dx$$

exists and is the Riemann integral of f. $\square$

Remark. Recall the step function $f : [a, b] \to \mathbb{R}$ given by

$$f(x) = \begin{cases} 1 & \text{if } x \in (\alpha, \beta) \subseteq [a, b] \\ 0 & \text{if } x \notin (\alpha, \beta). \end{cases}$$

Last time we saw that f is integrable and that

$$\int_a^b f(x)\,dx = \beta - \alpha.$$

Now let us consider a more general step function. We call f a *step function* on $[a,b]$ if there exists a partition $x_0 < x_1 < \cdots < x_n$ of $[a,b]$ such that $f(x)$ is constant on each subinterval (x_{i-1}, x_i).

Lemma 6.2. *If $f : [a,b] \to \mathbb{R}$ is a step function for a partition $x_0 < x_1 < \cdots < x_n$ and $f(x) = c_i$ when $x \in (x_{i-1}, x_i)$, then f is integrable and*

$$\int_a^b f(x)\,dx = \sum_{i=1}^n c_i(x_i - x_{i-1}).$$

Proof. Define

$$\varphi_i(x) = \begin{cases} 1 & \text{if } x \in (x_{i-1}, x_i) \\ 0 & \text{otherwise.} \end{cases}$$

Now let

$$h = f - \sum_{i=1}^n c_i \varphi_i.$$

Then $h(x)$ is nonzero only at $\{x_i\}_{i=0}^n$. Each φ_i is integrable and h is integrable with

$$\int_a^b h(x)\,dx = 0,$$

so f is also integrable and

$$\int_a^b f(x)\,dx = \sum_{i=1}^n c_i \int_a^b \varphi_i(x)\,dx = \sum_{i=1}^n c_i(x_i - x_{i-1})$$

by linearity and the integral of a simple step function that we calculated before. □

Proposition 6.1. *A function $f : [a,b] \to \mathbb{R}$ is integrable if and only if for any $\epsilon > 0$, there exist step functions f_1, f_2 such that $f_1(x) \le f(x) \le f_2(x)$ for all $x \in [a,b]$ and*

$$\int_a^b (f_2 - f_1)\,dx < \epsilon.$$

Proof. ($\Leftarrow$) For any $\epsilon > 0$, choose step functions f_1, f_2 such that

$$\int_a^b (f_2 - f_1)\,dx < \frac{\epsilon}{3}.$$

Then there exists $\delta > 0$ such that for any partition with width $< \delta$, the Riemann sums S_1, S_2 for f_1, f_2 satisfy

$$|S_1 - \int_a^b f_1(x)\,dx| < \frac{\epsilon}{3} \quad \text{and} \quad |S_2 - \int_a^b f_2(x)\,dx| < \frac{\epsilon}{3}.$$

So for any partition width $< \delta$, the Riemann sum of f is

$$S = \sum_{i=1}^{n} f(x_i')(x_i - x_{i-1}),$$

and $S_1 \le S \le S_2$ since

$$S_1 = \sum_{i=1}^{n} f_1(x_i')(x_i - x_{i-1}) \quad \text{and} \quad S_2 = \sum_{i=1}^{n} f_2(x_i')(x_i - x_{i-1}).$$

So S is in the interval (S_1, S_2), which has length $< \epsilon$ by the triangle inequality on the previous results. For any two Riemann sums of f with partitions of width $< \delta$, we have $|S' - S''| < \epsilon$. Thus f is integrable.

($\Rightarrow$) First we show that f is bounded in $[a, b]$. This is because for any $\epsilon > 0$, there exists $\delta > 0$ such that any two Riemann sums S_1, S_2 corresponding to partitions of width $< \delta$. satisfy $|S_1 - S_2| < \epsilon$. Let

$$S_1 = \sum_{i=1}^{n} f(x_i')(x_i - x_{i-1}),$$

and replace $x_{i_0}' \in (x_{i_0-1}, x_{i_0})$ with $x_{i_0}'' \in (x_{i_0-1}, x_{i_0})$. Keep x_i' for $i \neq i_0$. Define this new Riemann sum to be S_2. Then

$$|S_2 - S_1| \le |f(x_{i_0}'') - f(x_{i_0}')||x_{i_0} - x_{i_0-1}| < \epsilon,$$

so that

$$|f(x_{i_0}'')| \le |f(x_{i_0}')| + \frac{\epsilon}{x_{i_0} - x_{i_0-1}},$$

i.e. f is bounded in (x_{i_0-1}, x_{i_0}) since x_{i_0}'' was arbitrary. Since we also picked i_0 arbitrarily, we can repeat this for any interval to conclude that f is bounded in $[a, b]$.

Now for any partition $x_0 < x_1 < \cdots < x_n$ with width $< \delta$, define

$$m_i = \inf\{f(x) : x \in (x_{i-1}, x_i)\} \quad \text{and} \quad M_i = \sup\{f(x) : x \in (x_{i-1}, x_i)\}.$$

Define the step function

$$f_1(x) = \begin{cases} m_i & \text{if } x \in (x_{i-1}, x_i) \\ \min\{m_1, \ldots, m_n\} & \text{if } x = x_i \text{ for } i = 0, \ldots, n. \end{cases}$$

Similarly define

$$f_2(x) = \begin{cases} M_i & \text{if } x \in (x_{i-1}, x_i) \\ \max\{M_1, \ldots, M_n\} & \text{if } x = x_i \text{ for } i = 0, \ldots, n. \end{cases}$$

Observe that $f_1(x) \le f(x) \le f_2(x)$ for any $x \in [a, b]$ by construction. Now we verify that

$$\int_a^b (f_2 - f_1)\, dx < \epsilon$$

if $\delta > 0$ is small enough. This is because for any $\eta > 0$, there exists $x_i', x_i'' \in [x_{i-1}, x_i]$ such that $f(x_i') < m_i + \eta$ and $f(x_i'') > M_i - \eta$. Then

$$\sum_{i=1}^{n} (f(x_i'') - f(x_i'))(x_i - x_{i-1}) > \sum_{i=1}^{n} (M_i - m_i - 2\eta)(x_i - x_{i-1}) = \int_a^b (f_2 - f_1)\, dx - 2\eta(b - a).$$

If $\delta > 0$ is small enough, then

$$\sum_{i=1}^{n}(f(x_i'') - f(x_i'))(x_i - x_{i-1}) < \epsilon$$

since this a difference of two Riemann sums with partitions of width $< \delta$. Thus

$$\int_a^b (f_2 - f_1)\,dx < \epsilon + 2\eta(b-a).$$

But η was arbitrary, so taking $\eta \to 0$ gives the desired result. $\square$

Corollary 6.0.1. *If $f : [a,b] \to \mathbb{R}$ is integrable, then it is bounded.*

Proof. This was shown in the proof of the previous proposition. $\square$

Theorem 6.1. *If $f : [a,b] \to \mathbb{R}$ is continuous, then f is integrable.*

Proof. Since f is continuous on the compact set $[a,b]$, it is uniformly continuous. So for any $\epsilon > 0$, there exists $\delta > 0$ such that for any $x', x'' \in [a,b]$, we have $|f(x') - f(x'')| < \epsilon$ whenever $|x' - x''| < \delta$. Now let S_1, S_2 be two Riemann sums with partitions of width $< \delta$. Assume without loss of generality that S_1, S_2 are defined over the same partition (we can always combine two partitions to give a finer partition, if necessary). Let

$$S_1 = \sum_{i=1}^{n} f(x_i')(x_i - x_{i-1}) \quad \text{and} \quad S_2 = \sum_{i=1}^{n} f(x_i'')(x_i - x_{i-1}).$$

Then

$$|S_1 - S_2| \le \sum_{i=1}^{n} |f(x_i') - f(x_i'')|(x_i - x_{i-1}) < \epsilon \sum_{i=1}^{n}(x_i - x_{i-1}) = \epsilon(b-a).$$

Since $\epsilon > 0$ was arbitrary, we conclude that that f is integrable by Lemma 6.1. $\square$

6.2 The Fundamental Theorem of Calculus

Theorem 6.2 (Fundamental theorem of calculus)**.** *If $f : [a,b] \to \mathbb{R}$ has anti-derivative $F : [a,b] \to \mathbb{R}$ and $f \in \mathcal{R}([a,b])$,[1] then*

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Proof. Since f is integrable, let

$$A = \int_a^b f(x)\,dx.$$

For any $\epsilon > 0$, there exists $\delta > 0$ such that for any Riemann sum S with partition of width $< \delta$, we have $|S - A| < \epsilon$. Let $x_0 < x_1 < \cdots < x_n$ be a partition of width $< \delta$. Then by telescoping,

$$F(b) - F(a) = \sum_{i=1}^{n}(F(x_i) - F(x_{i-1})) = \sum_{i=1}^{n} f(x_i')(x_i - x_{i-1})$$

[1]Here $\mathcal{R}([a,b])$ is the class of Riemann integrable functions on $[a,b]$.

by Lagrange's mean value theorem, where $x_i' \in (x_{i-1}, x_i)$. Then

$$|F(b) - F(a) - A| = |S - A| < \epsilon,$$

so letting $\epsilon \to 0$ gives $F(b) - F(a) = A$. □

Remark. The fundamental theorem of calculus requires both being Riemann integrable and having an anti-derivative, which do not always overlap. In fact, neither is a subset of the other.

Example 6.0.1. The step function

$$f(x) = \begin{cases} -1 & \text{if } 0 \leq x \leq 1 \\ 1 & \text{if } 1 < x \leq 2 \end{cases}$$

is integrable but has no anti-derivative.

Example 6.0.2. Define

$$F(x) = \begin{cases} 0 & \text{if } x = 0 \\ x^2 \sin(1/x) & \text{if } x \neq 0. \end{cases}$$

Then we have

$$F'(x) = f(x) = \begin{cases} 0 & \text{if } x = 0 \\ (-2/x)(\cos(1/x^2)) + 2x \sin(1/x^2) & \text{if } x \neq 0. \end{cases}$$

We can check that $F'(0) = 0$ via the definition of the derivative. Note that f has an anti-derivative, namely F. However, f is not integrable since it is not bounded near $x = 0$.

Lecture 7

Jan. 30 — Riemann Integrability, Part 2

7.1 Conditions for an Anti-Derivative

Lemma 7.1. *Let $c \in (a, b)$. Then $f \in \mathcal{R}([a, b])$ if and only if $f \in \mathcal{R}([a, c])$ and $f \in \mathcal{R}([c, b])$. Moreover,*

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx. \tag{$*$}$$

Proof. ($\Rightarrow$) If $f \in \mathcal{R}([a, b])$, then for any $\epsilon > 0$, there exist two step functions f_1, f_2 such that $f_1 \le f \le f_2$ and

$$\int_a^b (f_2 - f_1)\,dx < \epsilon.$$

Let f_1, f_2 be the restrictions to $[a, c]$. Then still $f_1 \le f \le f_2$ on $[a, c]$ and

$$\int_a^c (f_2 - f_1) \le \int_a^b (f_2 - f_1) < \epsilon$$

since $f_2 - f_1$ is a nonnegative step function. (Note that the desired result is easy to verify for step functions.) So $f \in \mathcal{R}([a, c])$, and the same argument works to show that $f \in \mathcal{R}([c, b])$.

($\Leftarrow$) If $f \in \mathcal{R}([a, c])$ and $f \in \mathcal{R}([c, b])$, then for any $\epsilon > 0$, there exist step functions g_1, g_2, h_1, h_2 such that $g_1 \le f \le g_2$ on $[a, c]$, $h_1 \le f \le h_2$ on $[c, b]$, and

$$\int_a^c (g_2 - g_1)\,dx < \epsilon, \quad \int_c^b (h_2 - h_1)\,dx < \epsilon.$$

Now define

$$f_i = \begin{cases} g_i & \text{if } x \in [a, c) \\ h_i & \text{if } x \in [c, b] \end{cases}$$

for $i = 1, 2$. Then $f_1 \le f \le f_2$ on $[a, b]$, and

$$\int_a^b (f_2 - f_1)\,dx = \int_a^c (g_2 - g_1)\,dx + \int_c^b (h_2 - h_1)\,dx < 2\epsilon,$$

so $f \in \mathcal{R}([a, b])$. Now to prove ($*$), note that $f \in \mathcal{R}([a, c])$, so for any $\epsilon > 0$ there exist Riemann sums S_1 on $[a, c]$ and S_2 on $[c, b]$ such that

$$|S_1 - \int_a^c f(x)\,dx| < \frac{\epsilon}{3}, \quad |S_2 - \int_c^b f(x)\,dx| < \frac{\epsilon}{3}.$$

Now choose $\delta > 0$ such that if the Riemann sum S has partition with width $< \delta$, then

$$|S - \int_a^c f(x)\,dx| < \frac{\epsilon}{3}, \quad |S - \int_c^b f(x)\,dx| < \frac{\epsilon}{3}, \quad |S - \int_a^b f(x)\,dx| < \frac{\epsilon}{3}.$$

Now combine S_1, S_2 on $[a, b]$ to be a Riemann sum $S = S_1 + S_2$, so that

$$|S - \int_a^b f(x)\,dx| < \frac{\epsilon}{3}.$$

By the triangle inequality on the previous results,

$$\left| \int_a^b f(x)\,dx - \left(\int_a^c f(x)\,dx + \int_c^b f(x)\,dx \right) \right| < \epsilon.$$

Since ϵ is arbitrarily small, we conclude that

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

as desired. □

Remark. The formula $(*)$ is true for any three numbers a, b, c, as long as f is integrable. This is because by convention, if $a > b$, then

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx.$$

Theorem 7.1. *If $f : [a, b] \to \mathbb{R}$ is continuous, then*[1]

$$F(x) = \int_a^x f(\xi)\,d\xi$$

is an anti-derivative of f.

Proof. For any $x_0 \in (a, b)$, we check that $F'(x_0) = f(x_0)$. We can compute using Lemma 7.1 that

$$\begin{aligned} \left| \frac{F(x_0 + h) - F(x_0)}{h} - f(x_0) \right| &= \left| \frac{1}{h} \left(\int_a^{x_0+h} f(x)\,dx - \int_a^{x_0} f(x)\,dx \right) - f(x_0) \right| \\ &= \left| \frac{1}{h} \int_{x_0}^{x_0+h} f(x)\,dx - f(x_0) \right| = \left| \frac{1}{h} \int_{x_0}^{x_0+h} (f(x) - f(x_0))\,dx \right|. \end{aligned}$$

The last step is from observing

$$f(x_0) = \frac{1}{h} \int_{x_0}^{x_0+h} f(x_0)\,dx.$$

Since f is continuous, for any $\epsilon > 0$, there exists δ such that if $|x_0 - x| < \delta$, then $|f(x) - f(x_0)| < \epsilon$. This gives

$$\left| \frac{1}{h} \int_{x_0}^{x_0+h} (f(x) - f(x_0))\,dx \right| \le \frac{1}{h} \int_{x_0}^{x_0+h} |f(x) - f(x_0)|\,dx \le \frac{\epsilon h}{h} = \epsilon$$

if $|h| < \delta$. Thus,

$$\lim_{h \to 0} \frac{F(x_0 + h) - F(x_0)}{h} - f(x_0) = f(x_0),$$

so we indeed have $F'(x_0) = f(x_0)$. □

[1]Note that this integral is well-defined since any continuous function is integrable, and a continuous function restricted to a subset of its domain, i.e. $[a, x] \subseteq [a, b]$, remains continuous.

7.2 More Conditions for Integrability

Definition 7.1. Let $f : [a, b] \to \mathbb{R}$ be bounded and $x_0 < x_1 < \cdots < x_n$ be a partition of $[a, b]$. Define

$$\omega_i = \sup\{|f(x) - f(y)| : x, y \in [x_{i-1}, x_i)\}$$

for $i = 1, 2, \ldots, n$. We will call this the *oscillation amplitude* of f.

Theorem 7.2. *A function $f : [a, b] \to \mathbb{R}$ is integrable if and only if for any $\epsilon > 0$, there exists $\delta > 0$ such that for any partition with width $< \delta$, we have*

$$\sum_{i=1}^{n} \omega_i \Delta x_i < \epsilon,$$

where $\Delta x_i = x_i - x_{i-1}$.

Proof. ($\Leftarrow$) For any $\epsilon > 0$, choose any two Riemann sums S_1, S_2 over partitions with width $< \delta$. Assume without loss of generality that S_1 and S_2 are defined over the same (maybe refined) partition. Let

$$S_1 = \sum_{i=1}^{n} f(x_i')(x_i - x_{i-1}), \quad S_2 = \sum_{i=1}^{n} f(x_i'')(x_i - x_{i-1}).$$

Then we have

$$|S_1 - S_2| \le \sum_{i=1}^{n} |f(x_i') - f(x_i'')|\Delta x_i \le \sum_{i=1}^{n} \omega_i \Delta x_i < \epsilon.$$

Then by Lemma 6.1, we conclude that f is integrable.

($\Rightarrow$) Since f is integrable, by Lemma 6.1 we have that for any $\epsilon > 0$, there eixsts $\delta > 0$ such that for any two Riemann sums S_1, S_2 over partitions of with $< \delta$, we have $|S_1 - S_2| < \epsilon$. In the interval $[x_{i-1}, x_i]$, let

$$M_i = \sup_{x \in [x_{i-1}, x_i]} |f(x)|, \quad m_i = \inf_{x \in [x_{i-1}, x_i]} |f(x)|.$$

In particular note that $\omega_i = M_i - m_i$. Now for any $\eta > 0$, there exist $x_i', x_i'' \in [x_{i-1}, x_i]$ such that

$$f(x_i') > M_i - \eta, \quad f(x_i'') < m_i + \eta.$$

Let

$$S_1 = \sum_{i=1}^{n} f(x_i')\Delta x_i, \quad S_2 = \sum_{i=1}^{n} f(x_i'')\Delta x_i.$$

Then we have

$$|S_1 - S_2| \le \left| \sum_{i=1}^{n} (f(x_i') - f(x_i''))\Delta x_i \right|$$

Note that $f(x_i') - f(x_i'') \ge M_i - m_i - 2\eta$ for η sufficiently small. Thus

$$|S_1 - S_2| \ge \sum_{i=1}^{n} \omega_i \Delta x_i - 2\eta \sum_{i=1}^{n} \Delta x_i,$$

so that

$$\sum_{i=1}^{n} \omega_i \Delta x_i \le |S_1 - S_2| + 2\eta(b - a) < \epsilon + 2\eta(b - a).$$

From here letting $\eta \to 0$ gives the desired result. $\square$

Theorem 7.3. *A function $f : [a, b] \to \mathbb{R}$ is integrable if and only if for any $\epsilon > 0$, there exists a partition such that*

$$\sum_{i=1}^{n} \omega_i(f)\Delta x_i < \epsilon.$$

Proof. ($\Rightarrow$) This is immediate from the previous theorem.

($\Leftarrow$) We show that for any $\epsilon > 0$, there exists $\delta > 0$ such that for any partition with width $< \delta$, we have

$$\sum_{i=1}^{n} \omega_i(f)\Delta x_i < \epsilon.$$

This will imply that f is integrable by the previous theorem. Let $y_0 < y_1 < \cdots < y_m$ be the partition satisfying

$$\sum_{j=1}^{m} \omega_j(f)\Delta y_j < \epsilon,$$

and choose

$$\delta < \frac{1}{4} \min_{j=1,\ldots,m} \Delta y_j.$$

For any partition $x_0 < \cdots < x_n$ with width $< \delta$. We divide the intervals $[x_{i-1}, x_i]$ into two classes. The first case (1) is where $[x_{i-1}, x_i]$ is contained in one of the $[y_{j-1}, y_j]$, and the second case (2) is where $[x_{i-1}, x_i]$ contains an interior point y_j. In the first case, we have

$$\sum_{(1)} \omega_i(f)\Delta x_i \leq \sum_{j=1}^{m} \omega_j(f)\Delta y_j.$$

For the second case, since $[x_{i-1}, x_i]$ contains an interior point y_j but $\delta < \Delta y_j, \Delta y_{j+1}$, we must have

$$y_{j-1} < x_{i-1} < y_j < x_i < y_{j+1},$$

so that

$$\omega_i(f)\Delta x_i \leq \frac{1}{2}(\omega_j(f)\Delta y_j + \omega_{j+1}(f)\Delta y_{j+1}).$$

This implies

$$\sum_{(2)} \omega_i(f)\Delta x_i \leq \frac{1}{2}\sum_{j=1}^{m} \omega_j(f)\Delta y_j.$$

Thus

$$\sum_{i=1}^{n} \omega_i(f)\Delta x_i = \sum_{(1)} \omega_i(f)\Delta x_i + \sum_{(2)} \omega_i(f)\Delta x_i \leq 2\sum_{j=1}^{m} \omega_j(f)\Delta y_j < 2\epsilon,$$

so that f is integrable. $\square$

Lecture 8

Feb. 1 — Riemann Integrability, Part 3

8.1 Even More Conditions for Integrability

Example 8.0.1. If $f(x)$ is monotone on $[a, b]$, then $f \in \mathcal{R}([a, b])$.

Proof. Suppose $f(x)$ is monotone increasing on $[a, b]$ and $f(x)$ is not constant (since the result is trivial if f is constant). Then $f(a) \le f(x) \le f(b)$. For any $\epsilon > 0$, for any partition $x_0 < \cdots < x_n$ with width

$$\delta < \frac{\epsilon}{f(b) - f(a)},$$

we have on $[x_{i-1}, x_i]$ that $M_i = f(x_i)$ and $f(x_{i-1}) = m_i$ since f is monotone. Then

$$\omega_i(f) = f(x_i) - f(x_{i-1}) = M_i - m_i.$$

Thus

$$\sum_{i=1}^{n} \omega_i(f)\Delta x_i = \sum_{i=1}^{n} (f(x_i) - f(x_{i-1}))\Delta x_i < \frac{\epsilon}{f(b) - f(a)} \sum_{i=1}^{n} (f(x_i) - f(x_{i-1})) = \epsilon$$

since the sum telescopes and comes out to $f(b) - f(a)$. Thus f is integrable. □

Theorem 8.1 (Du Bois-Reymond)**.** *Let f be bounded on $[a, b]$. Then $f \in \mathcal{R}([a, b])$ if and only if for any $\epsilon, a > 0$, there exists a partition such that the total length of subintervals with $\{\omega_i(f) \le \epsilon\}$ is $< a$.*

Proof. For any partition $x_0 < \cdots < x_n$, split

$$\sum_{i=1}^{n} \omega_i(f)\Delta x_i = \sum_{(A)} \omega_i(f)\Delta x_i + \sum_{(B)} \omega_i(f)\Delta x_i$$

where (A) is over subintervals with width $\omega_i(f) < \epsilon$ and (B) is over subintervals with width $\omega_i(f) \ge \epsilon$.

($\Rightarrow$) Let

$$\Omega = \sup_{x,y \in [a,b]} |f(x) - f(y)|.$$

For any $\epsilon > 0$, for

$$\epsilon_1 = \frac{\epsilon}{2(b-a)} \quad \text{and} \quad a = \frac{\epsilon}{2\Omega},$$

by assumption there exists a partition $x_0 < \cdots < x_n$ such that

$$\sum_{i=1}^{n} \omega_i(f)\Delta x_i = \sum_{(A)} \omega_i(f)\Delta x_i + \sum_{(B)} \omega_i(f)\Delta x_i$$
$$< \frac{\epsilon}{2(b-a)} \sum_{(a)} \Delta x_i + \Omega \sum_{(B)} \Delta x_i < \frac{\epsilon}{2(b-a)}(b-a) + \Omega\frac{\epsilon}{2\Omega} = \epsilon.$$

So we see that $f \in \mathcal{R}([a,b])$ as desired.

($\Rightarrow$) If $f \in \mathcal{R}([a,b])$, then for any $\epsilon, a > 0$, there exists a partition $x_0 < \cdots < x_n$ such that

$$\sum_{i=1}^{n} \omega_i(f)\Delta x_i < a\epsilon.$$

Then we have

$$\epsilon \sum_{(B)} \Delta x_i \le \sum_{(B)} \omega_i(f)\Delta x_i < a\epsilon \implies \sum_{(B)} \Delta x_i < a,$$

which shows the desired result. $\square$

Corollary 8.1.1. *If $f : [a,b] \to \mathbb{R}$ is bounded and has only finitely many discontinuity points, then $f \in \mathcal{R}([a,b])$.*

Proof. Suppose $f(x)$ has p discontinuity points on $[a,b]$ and $m \le f(x) \le M$ for all $x \in [a,b]$. Then for any $\epsilon > 0$, first (1) we construct p small open intervals on $[a,b]$ containing the p discontinuity points with

$$\text{total length} < \frac{\epsilon}{2(M-m)}.$$

Next (2) for any subintervals in $[a,b]$ excluding the above p subintervals, f is continuous on them, so there exists a partition such that

$$\sum_{(2)} \omega_i(f)\Delta x_i < \frac{\epsilon}{2}.$$

Now combine (1) and (2) to get

$$\sum_{i=1}^{n} \omega_i(f)\Delta = \sum_{(1)} \omega_i(f)\Delta x_i + \sum_{(2)} \omega_i(f)\Delta x_i < (M-m)\frac{\epsilon}{2(M-m)} + \frac{\epsilon}{2} = \epsilon.$$

Thus $f \in \mathcal{R}([a,b])$, as desired. $\square$

Example 8.0.2. Consider

$$f(x) = \begin{cases} \sin(1/x) & \text{if } x \ne 0 \\ A & \text{if } x = 0 \end{cases}$$

for any constant $A \in \mathbb{R}$. Then by the previous corollary, $f \in \mathcal{R}([0,1])$.

Theorem 8.2. *If $f, g \in \mathcal{R}([a,b])$, then $fg \in \mathcal{R}([a,b])$.*

Proof. Since f, g are integrable, they are bounded. So assume $|f|, |g| \le M$. Then for any $\epsilon > 0$, there exists $\delta > 0$ such that for any partition of width $< \delta$, we have

$$\sum_{i=1}^{n} \omega_i(f)\Delta x_i < \frac{\epsilon}{2M}, \quad \sum_{i=1}^{n} \omega_i(g)\Delta x_i < \frac{\epsilon}{2M}.$$

Notice

$$\omega_i(fg) \le M(\omega_i(f) + \omega_i(g))$$

because

$$\begin{aligned} |f(x)g(x) - f(y)g(y) &\le |g(x)||f(x) - f(y)| + |f(y)||g(x) - g(y)| \\ &\le M(|f(x) - f(y)| + |g(x) - g(y)|). \end{aligned}$$

Taking suprememes over $x, y \in [x_{i-1}, x_i]$ from here gives $\omega_i(fg) \le M(\omega_i(f) + \omega_i(g))$. Then

$$\sum_{i=1}^{n} \omega_i(fg)\Delta x_i \le M\left(\sum_{i=1}^{n} \omega_i(f)\Delta x_i + \sum_{i=1}^{n} \omega_i(g)\Delta x_i\right) < M\left(\frac{\epsilon}{2M} + \frac{\epsilon}{2M}\right) = \epsilon.$$

Thus $fg \in \mathcal{R}([a,b])$ as desired. □

Theorem 8.3. *If $f \in \mathcal{R}([a,b])$, then $|f| \in \mathcal{R}([a,b])$ and*

$$\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx.$$

Proof. Since $f \in \mathcal{R}([a,b])$, for any $\epsilon > 0$ there exists a partition $x_0 < \cdots < x_n$ such that

$$\sum_{i=1}^{n} \omega_i(f)\Delta x_i < \epsilon.$$

Since

$$\big||f(x)| - |f(y)|\big| \le |f(x) - f(y)|,$$

taking supremums over $x, y \in [x_{i-1}, x_i]$ gives $\omega_i(|f|) \le \omega_i(f)$. Then

$$\sum_{i=1}^{n} \omega_i(|f|)\Delta x_i \le \sum_{i=1}^{n} \omega_i(f)\Delta x_i < \epsilon.$$

So we indeed have $|f| \in \mathcal{R}([a,b])$. Now observe that $-|f| \le f \le |f|$. After integrating, we get

$$-\int_a^b |f(x)|\,dx \le \int_a^b f(x)\,dx \le \int_a^b |f(x)|\,dx.$$

This immediately implies the desired result. □

Example 8.0.3 (Cauchy-Schwarz)**.** If $f, g \in \mathcal{R}([a,b])$, then

$$\left|\int_a^b f(x)g(x)\,dx\right| \le \left(\int_a^b f(x)^2\,dx\right)^{1/2} \left(\int_a^b g(x)^2\,dx\right)^{1/2}. \qquad (*)$$

Proof. Let

$$A = \int_a^b f^2\, dx, \quad B = \int_a^b |fg|\, dx, \quad C = \int_a^b g^2\, dx.$$

Note that it suffices to show that $B^2 \leq AC$, which will imply ($*$) by the previous theorem. Then

$$0 \leq \int_a^b (t|f| - |g|)^2\, dx = At^2 - 2Bt + C$$

for any $t \in \mathbb{R}$. So the discriminant must satisfy $(2B)^2 - 4AC \leq 0$, which gives $B^2 \leq AC$ as desired. □

Example 8.0.4 (Riemann-Lebesgue lemma)**.** If $f \in \mathcal{R}([a, b])$, then

$$\lim_{\lambda\to\infty} \int_a^b f(x)\sin(\lambda x)\, dx = 0.$$

Proof. Since $f \in \mathcal{R}([a, b])$, for any $\epsilon > 0$ there exists a partition $x_0 < \cdots < x_n$ of $[a, b]$ such that

$$\sum_{i=1}^n \omega_i(f)\Delta x_i < \frac{\epsilon}{2}.$$

Also assume $|f| \leq M$ on $[a, b]$ since f is integrable. Then we choose

$$\lambda > \frac{4nM}{\epsilon}.$$

We can estimate

$$\begin{aligned}
\left|\int_a^b f(x)\sin(\lambda x)\, dx\right| &= \left|\sum_{i=1}^n \int_{x_{i-1}}^{x_i} (f(x) - f(x_i) + f(x_i))\sin(\lambda x)\, dx\right| \\
&\leq \sum_{i=1}^n |f(x_i)| \left|\int_{x_{i-1}}^{x_i} \sin(\lambda x)\, dx\right| + \sum_{i=1}^n \int_{x_{i-1}}^{x_i} \underbrace{|f(x) - f(x_i)|}_{\leq \omega_i(f)} \underbrace{|\sin(\lambda x)|}_{\leq 1}\, dx \\
&\leq M \sum_{i=1}^n \frac{\overbrace{|\cos(\lambda x_i) - \cos(\lambda x_{i-1})|}^{\leq 2}}{\lambda} + \sum_{i=1}^n \int_{x_{i-1}}^{x_i} \omega_i(f)\, dx \\
&\leq M\frac{2n}{\lambda} + \sum_{i=1}^n \omega_i(f)\Delta x_i < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\end{aligned}$$

So as $\lambda \to \infty$, the integral goes to 0. □

Remark. Recall that

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is irrational} \\ 1 & \text{if } x \text{ is rational} \end{cases}$$

is not Riemann integrable, but we might expect that this should integrate to 0. The Lebesgue integral will fix this, which was discovered much later.