MATH 4318 Final Formula Sheet

Differentiation

Theorem 1 (Quotient rule). *If $f, g : U \to \mathbb{R}$ are differentiable and $g(x_0) \neq 0$, then*

$$(f/g)'(x_0) = \frac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{g(x_0)^2}.$$

Theorem 2 (Cauchy's mean value theorem). *Let $f, g \in C([a,b])$ be differentiable in (a,b). If $g'(x) \neq 0$ for any $x \in (a,b)$, then there exists $x_0 \in (a,b)$ such that*

$$\frac{f'(x_0)}{g'(x_0)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

Theorem 3. *Let $R \in (0,\infty)$ and $f \in C^\infty(x_0 - R, x_0 + R)$. If there exists $M > 0$ such that for all $x \in (x_0 - R, x_0 + R)$, $|f^{(n)}(x)| \leq M$ for all $n \in \mathbb{N}$, then*

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

for all $x \in (x_0 - R, x_0 + R)$.

Theorem 4 (Lagrange remainder). *Let $f \in C^n([a,b])$ and assume that f is $(n+1)$-times differentiable in (a,b). Then*

$$f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1}$$

for some $\xi \in [a,x]$.

Integration

Definition 1 (Riemann integrability). A function $f : [a,b] \to \mathbb{R}$ is *Riemann integrable* on $[a,b]$ if there exists $A \in \mathbb{R}$ such that for all $\epsilon > 0$, there exists $\delta > 0$ such that $|S - A| < \epsilon$ whenever S is any Riemann sum on a partition of width $< \delta$. We call A the *Riemann integral* of f on $[a,b]$.

Theorem 5 (Cauchy criterion for integrability). *A function $f : [a,b] \to \mathbb{R}$ is integrable if and only if for any $\epsilon > 0$, there exists $\delta > 0$ such that $|S_1 - S_2| < \epsilon$ whenever S_1 and S_2 are Riemann sums for partitions of width $< \delta$.*

Definition 2 (Step function). We say $f : [a,b] \to \mathbb{R}$ is a *step function* if there exists a partition $a = x_0 < x_1 < \cdots < x_n = b$ such that f is constant on each interval (x_{i-1}, x_i).

Theorem 6. *A function $f : [a,b] \to \mathbb{R}$ is integrable if and only if for any $\epsilon > 0$, there exist step functions f_1, f_2 such that $f_1(x) \leq f(x) \leq f_2(x)$ for all $x \in [a,b]$ and*

$$\int_a^b (f_2 - f_1)\,dx < \epsilon.$$

Definition 3 (Oscillation amplitude). Let $f : [a,b] \to \mathbb{R}$ be bounded and $a = x_0 < x_1 < \cdots < x_n = b$ be a partition. Then

$$\omega_i(f) = \sup\{|f(x) - f(y)| : x, y \in [x_{i-1}, x_i)\}$$

is the *oscillation amplitude* of f on $[x_{i-1}, x_i)$.

Theorem 7. *A function $f : [a,b] \to \mathbb{R}$ is integrable if and only if for any $\epsilon > 0$, there exists a partition such that*

$$\sum_{n=1}^{n} \omega_i(f)(x_i - x_{i-1}) < \epsilon.$$

Theorem 8 (Du Bois-Reymond). *Let f be bounded on $[a,b]$. Then $f \in \mathcal{R}([a,b])$ if and only if for any $\epsilon, a > 0$, there exists a partition such that the total length of subintervals with $\omega_i(f) \geq \epsilon$ is $< a$.*

Exchange of Limit Operations

Theorem 9. *Let $\{f_n\}$ be a sequence of functions on an open interval $U \subseteq \mathbb{R}$ such that each f_n has a continuous derivative. Suppose $\{f_n'\}$ converges uniformly on U and for some $a \in U$, $\{f_n'(a)\}$ converges. Then $\lim_{n\to\infty} f_n = f$ exists and f is differentiable. Furthermore, we have $f' = \lim_{n\to\infty} f_n'$.*

Theorem 10. *Let $f_n : [a,b] \to \mathbb{R}$ be continuously differentiable. Suppose that $\sum_{n=1}^\infty f_n(x)$ converges pointwise on $[a,b]$ and $\sum_{n=1}^\infty f_n'(x)$ converges uniformly on $[a,b]$. Then $\sum_{n=1}^\infty f_n(x)$ converges uniformly on $[a,b]$ and*

$$\frac{d}{dx}\sum_{n=1}^{\infty} f_n(x) = \sum_{n=1}^{\infty} f_n'(x).$$

Infinite Series

Theorem 11. *If $a_n \geq 0$, then $\sum_{n=1}^\infty a_n$ either converges or diverges to ∞.*

Theorem 12 (Alternating series test). *Let $\{a_n\}$ be a decreasing sequence with $a_n \to 0$ as $n \to \infty$. Then the series $\sum_{n=1}^\infty (-1)^{n+1} a_n$ converges (to S, say), and the partial sums S_n have error $|S_n - S| \leq a_{n+1}$.*

Theorem 13. *Let $f_n \in C([a,b])$. If $\sum_{n=1}^\infty f_n(x)$ converges uniformly on (a,b), then it converges uniformly on $[a,b]$.*

Power Series

Theorem 14. *We have the following:*

1. *If $\sum_{n=0}^\infty a_n x^n$ converges at $x = x_1 \neq 0$, then it converges absolutely for all x with $|x| < |x_1|$.*
2. *If $\sum_{n=0}^\infty a_n x^n$ diverges at $x = x_2 \neq 0$, then it diverges for all x with $|x| > |x_2|$.*

Theorem 15 (Hadamard's formula). *For a power series $\sum_{n=0}^\infty a_n x^n$, let $L = \limsup_{n\to\infty} |a_n|^{1/n}$. Then its radius of convergence is $R = 1/L$.*

Theorem 16. *For a series $\sum_{n=0}^\infty a_n x^n$ with $a_n \neq 0$, if $\lim_{n\to\infty} |a_{n+1}/a_n| = L$, then its radius of convergence is $R = 1/L$.*

Theorem 17. *If $\sum_{n=0}^\infty a_n x^n$ has radius of convergence $R > 0$, then for any $0 < r < R$, the power series $\sum_{n=0}^\infty a_n r^n$ converges uniformly on $[-r, r]$. Moreover, if $\sum_{n=0}^\infty a_n x^n$ converges at $x = R < \infty$ (or $x = -R$), then $\sum_{n=0}^\infty a_n x^n$ converges uniformly on $[0, R]$ (or $[-R, 0]$).*

Theorem 18. *If $\sum_{n=0}^\infty a_n x^n$ has radius of convergence $R > 0$, then $f(x) = \sum_{n=0}^\infty a_n x^n \in C^\infty(-R, R)$.*

Theorem 19. *Suppose $f(x) = \sum_{n=0}^\infty a_n x^n$ has radius of convergence $R > 0$. Then for any $x \in (-R, R)$, $f \in \mathcal{R}([0, x])$ and*

$$\int_0^x f(t)\,dt = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}.$$

Differentiation in $\mathbb{R}^n$

Theorem 20. *Let $f : U \subseteq \mathbb{R}^n \to \mathbb{R}$ where U is open. Then f is differentiable at $x = a$ if and only if there exist functions $A_1, \ldots, A_n$ on U, continuous at $x = a$, such that*

$$f(x) - f(a) = A_1(x)(x_1 - a_1) + \cdots + A_n(x)(x_n - a_n)$$

for all $x \in U$. In this case, $\partial f/\partial x_i(a) = A_i(a)$.

Theorem 21. *Let U be an open set in $\mathbb{R}^n$ and suppose that $f : U \to \mathbb{R}$ has partial derivatives $f_1', \ldots, f_n'$ on U which are continuous on $x = a$. Then f is differentiable at $x = a$.*

Theorem 22 (Implicit function theorem). *Let $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ and $U \times V \subseteq \mathbb{R}^n \times \mathbb{R}^m$ be a neighborhood of (x_0, y_0). Suppose f and $\partial f/\partial y$ are continuous $U \times V$, and $f(x_0, y_0) = 0$ and*

$$\det\left(\frac{\partial f}{\partial y}(x_0, y_0)\right) \neq 0.$$

Then there exists a neighborhood $U_0 \times V_0 \subseteq U \times V$ of (x_0, y_0) and a unique continuous function $\varphi : U_0 \to V_0$ satisfying

$$\begin{cases} f(x, \varphi(x)) = 0, \\ \varphi(x_0) = y_0. \end{cases}$$

Theorem 23. *Let $f : [a,b] \times [c,d] \to \mathbb{R}$ be continuous and suppose that $\partial f/\partial y$ exists and is continuous on $[a,b] \times [c,d]$. Then*

$$\frac{d}{dy}\int_a^b f(x,y)\,dx = \int_a^b \frac{\partial f}{\partial y}(x,y)\,dx.$$

Integration in $\mathbb{R}^n$

Theorem 24 (Lebesgue's criterion for Riemann integrability). *Let $A \subseteq \mathbb{R}^n$ be a set with volume and let $f : A \to \mathbb{R}$ be a bounded function that is continuous except on a subset of A with zero volume. Then f is integrable on A.*

Theorem 25. *Suppose $f(x,y)$ is integrable on $D = [a,b] \times [c,d]$ and for each $x \in [a,b]$, $f(x,y)$ is integrable on $[c,d]$. Then*

$$\int_a^b dx \left[\int_c^d f(x,y)\,dy\right] = \iint_D f(x,y)\,dxdy$$

Reverse Triangle Inequality

Proposition 1. *For all $x, y \in \mathbb{R}$, we have $||x| - |y|| \leq |x - y|$.*

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