MATH 4318 Exam 2 Formula Sheet

Exchange of Limit Operations

Theorem 1. *Let $\{f_n\}$ be a uniformly convergent sequence of Riemann integrable functions on $[a,b]$. Then*

$$\int_a^b \lim_{n\to\infty} f_n(x)\,dx = \lim_{n\to\infty}\int_a^b f_n(x)\,dx.$$

Theorem 2. *Let $\{f_n\}$ be a sequence of functions on an open interval $U \subseteq \mathbb{R}$ such that each f_n has a continuous derivative. Suppose $\{f_n'\}$ converges uniformly on U and for some $a \in U$, $\{f_n'(a)\}$ converges. Then $\lim_{n\to\infty} f_n = f$ exists and f is differentiable. Furthermore, we have $f' = \lim_{n\to\infty} f_n'$.*

Theorem 3. *Let $f_n \in \mathcal{R}([a,b])$ for $n \in \mathbb{N}$. If $\sum_{n=1}^\infty f_n(x)$ converges uniformly on $[a,b]$, then $\sum_{n=1}^\infty f_n(x) \in \mathcal{R}([a,b])$ and*

$$\sum_{n=1}^\infty \int_a^b f_n(x)\,dx = \int_a^b \sum_{n=1}^\infty f_n(x)\,dx.$$

Theorem 4. *Let $f_n : [a,b] \to \mathbb{R}$ be continuously differentiable. Suppose that $\sum_{n=1}^\infty f_n(x)$ converges pointwise on $[a,b]$ and $\sum_{n=1}^\infty f_n'(x)$ converges uniformly on $[a,b]$. Then $\sum_{n=1}^\infty f_n(x)$ converges uniformly on $[a,b]$ and*

$$\frac{d}{dx}\sum_{n=1}^\infty f_n(x) = \sum_{n=1}^\infty f_n'(x).$$

Infinite Series Basics

Theorem 5. *A series $\sum_{n=1}^\infty a_n$ converges if and only if for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that if $n > m \geq N$, then $|a_{m+1} + \cdots + a_n| < \epsilon$.*

Theorem 6. *If $\sum_{n=1}^\infty a_n$ converges, then $a_n \to 0$ as $n \to \infty$.*

Theorem 7. *If $a_n \geq 0$, then $\sum_{n=1}^\infty a_n$ either converges or diverges to ∞.*

Theorem 8. *If $\sum_{n=1}^\infty a_n$ converges absolutely, then $\sum_{n=1}^\infty a_n$ converges and $\left|\sum_{n=1}^\infty a_n\right| \leq \sum_{n=1}^\infty |a_n|$.*

Convergence Tests

Theorem 9 (Comparison test)**.** *If $\sum_{n=1}^\infty a_n$ and $\sum_{n=1}^\infty b_n$ are two series such that $|a_n| \leq b_n$ and $\sum_{n=1}^\infty b_n$ converges, then $\sum_{n=1}^\infty a_n$ converges and $\left|\sum_{n=1}^\infty a_n\right| \leq \sum_{n=1}^\infty b_n$.*

Theorem 10 (Limit comparison)**.** *Let $\sum_{n=1}^\infty a_n$ and $\sum_{n=1}^\infty b_n$ be two positive series and suppose that $\lim_{n\to\infty} a_n/b_n = \ell > 0$. Then $\sum_{n=1}^\infty a_n$ converges if and only if $\sum_{n=1}^\infty b_n$ converges.*

Theorem 11 (Root test)**.** *Let $\sum_{n=1}^\infty a_n$ be a positive series and suppose that $\limsup_{n\to\infty} \sqrt[n]{a_n} = \ell$. Then*

1. *if $\ell < 1$, then $\sum_{n=1}^\infty a_n$ converges;*
2. *if $\ell > 1$, then $\sum_{n=1}^\infty a_n$ diverges.*

Theorem 12 (Integral test)**.** *Let $\{a_n\}$ be a positive decreasing sequence. If there exists a continuous decreasing f on $[1,\infty)$ such that $a_n = f(n)$, then $\sum_{n=1}^\infty a_n$ converges if and only if $\int_1^\infty f(x)\,dx$ converges.*

Theorem 13. *Let $\{a_n\}$ be a decreasing sequence with $a_n \to 0$ as $n \to \infty$. Then the series $\sum_{n=1}^\infty (-1)^{n+1} a_n$ converges (to S, say), and the partial sums S_n have error $|S_n - S| \leq a_{n+1}$.*

Series of Functions

Theorem 14 (Cauchy criterion)**.** *A series $\sum_{n=1}^\infty f_n(x)$ converges uniformly on $I \subseteq \mathbb{R}$ if and only if for every $\epsilon > 0$, there exists N such that whenever $n \geq N$, for any $x \in I$ and $p \in \mathbb{N}$ we have $|f_{n+1}(x) + \cdots + f_{n+p}(x)| < \epsilon$.*

Theorem 15. *If $\sum_{n=1}^\infty f_n(x)$ converges uniformly on I, then $f_n \to 0$ uniformly on I.*

Theorem 16. *Let $f_n \in C([a,b])$. If $\sum_{n=1}^\infty f_n(x)$ converges uniformly on (a,b), then it converges uniformly on $[a,b]$.*

Theorem 17 (Weierstrass M-test)**.** *If there exists a nonnegative and convergent series such that $|f_n(x)| \leq M_n$ for all $x \in I$, then $\sum_{n=1}^\infty f_n(x)$ converges uniformly on I.*

Power Series

Theorem 18. *We have the following:*

1. *If $\sum_{n=0}^\infty a_n x^n$ converges at $x = x_1 \neq 0$, then it converges absolutely for all x with $|x| < |x_1|$.*
2. *If $\sum_{n=0}^\infty a_n x^n$ diverges at $x = x_2 \neq 0$, then it diverges for all x with $|x| > |x_2|$.*

Theorem 19 (Hadamard's formula)**.** *For a power series $\sum_{n=0}^\infty a_n x^n$, let $L = \limsup_{n\to\infty} |a_n|^{1/n}$. Then its radius of convergence is $R = 1/L$.*

Theorem 20. *For a series $\sum_{n=0}^\infty a_n x^n$ with $a_n \neq 0$, if $\lim_{n\to\infty} |a_{n+1}/a_n| = L$, then its radius of convergence is $R = 1/L$.*

Theorem 21. *If $\sum_{n=0}^\infty a_n x^n$ has radius of convergence $R > 0$, then for any $0 < r < R$, the power series $\sum_{n=0}^\infty a_n r^n$ converges uniformly on $[-r,r]$. Moreover, if $\sum_{n=0}^\infty a_n x^n$ converges at $x = R < \infty$ (or $x = -R$), then $\sum_{n=0}^\infty a_n x^n$ converges uniformly on $[0,R]$ (or $[-R,0]$).*

Theorem 22. *If $\sum_{n=0}^\infty a_n x^n$ has radius of convergence $R > 0$, then $f(x) = \sum_{n=0}^\infty a_n x^n \in C^\infty(-R,R)$.*

Theorem 23. *Suppose $f(x) = \sum_{n=0}^\infty a_n x^n$ has radius of convergence $R > 0$. Then for any $x \in (-R,R)$, $f \in \mathcal{R}([0,x])$ and*

$$\int_0^x f(t)\,dt = \sum_{n=0}^\infty \frac{a_n}{n+1} x^{n+1}.$$

Taylor Series

Theorem 24. *Let $R \in (0,\infty)$ and $f \in C^\infty(x_0 - R, x_0 + R)$. If there exists $M > 0$ such that for all $x \in (x_0 - R, x_0 + R)$, $|f^{(n)}(x)| \leq M$ for all $n \in \mathbb{N}$, then*

$$f(x) = \sum_{n=0}^\infty \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

for all $x \in (x_0 - R, x_0 + R)$.

Theorem 25 (Lagrange remainder)**.** *Let $f \in C^n([a,b])$ and assume that f is $(n+1)$-times differentiable in (a,b). Then*

$$f(x) - \sum_{k=0}^n \frac{f^{(k)}(a)}{k!}(x-a)^k = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1}$$

for some $\xi \in [a,x]$.

Contraction Mapping

Theorem 26 (Contraction mapping)**.** *Let (X,d) be a complete metric space and $T : X \to X$ a contraction mapping, i.e. there exists $0 < k < 1$ such that $d(Tx,Ty) \leq kd(x,y)$ for all $x,y \in X$. Then T admits a unique fixed point in X.*

Theorem 27 (Newton's method)**.** *Let $f \in C^2([a,b])$ and $\hat{x} \in (a,b)$ such that $f(\hat{x}) = 0$ and $f'(\hat{x}) \neq 0$. Then there exists a neighborhood $U(\hat{x})$ of $\hat{x}$ such that for all $x_0 \in U(\hat{x})$, the sequence*

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

converges to $\hat{x}$ as $n \to \infty$.

Theorem 28 (Picard-Lindelöf)**.** *Let $f(t,x) : [a,b] \times [c,d] \to \mathbb{R}$ be continous in t and locally Lipschitz in x, i.e. it is Lipschitz in x for $|t| \leq h$ with h small enough, then the initial value problem*

$$\begin{cases} x'(t) = f(t,x) \\ x(0) = \xi, \end{cases}$$

has a unique local solution. (Note: Consider

$$x(t) = \xi + \int_0^t f(\tau, x(\tau))\,d\tau$$

by integrating. This is called a Picard iteration.*)*

Theorem 29 (Implicit function theorem)**.** *Let $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ and $U \times V \subseteq \mathbb{R}^n \times \mathbb{R}^m$ be a neighorhood of (x_0,y_0). Suppose f and $\partial f/\partial y$ are continuous on $U \times V$, and $f(x_0,y_0) = 0$,*

$$\det\left(\frac{\partial f}{\partial y}(x_0,y_0)\right) \neq 0.$$

Then there exists a neighborhood $U_0 \times V_0 \subseteq U \times V$ of (x_0,y_0) and a unique continuous function $\varphi : U_0 \to V_0$ satisfying

$$\begin{cases} f(x,\varphi(x)) = 0, \\ \varphi(x_0) = y_0. \end{cases}$$

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