

# MATH 4318: Analysis II

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# Lecture 1

## Jan. 9 — The Derivative

### 1.1 Defining the Derivative

**Definition 1.1.** Let  $f$  be a real-valued function on an open interval  $U \subseteq \mathbb{R}$ . Let  $x_0 \in U$ , we say  $f$  is *differentiable* at  $x_0$  if

$$\lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

exists. If it does, then this limit, denoted by  $f'(x_0)$ , is called the *derivative* of  $f$  at  $x_0$ .

**Remark.** By definition, for any  $\epsilon > 0$ , there exists  $\delta > 0$  such that

$$\left| \frac{f(x) - f(x_0)}{x - x_0} - f'(x_0) \right| \leq \epsilon$$

if  $|x - x_0| < \delta$  and  $x \in U$ . Multiplying both sides by  $|x - x_0|$  yields

$$|f(x) - f(x_0) - f'(x_0)(x - x_0)| \leq \epsilon|x - x_0|.$$

In other words,

$$|f(x) - \varphi(x)| \leq \epsilon|x - x_0|$$

where  $\varphi(x) = f(x_0) + f'(x_0)(x - x_0)$ . In other words,  $\varphi(x)$  is a first-order approximation of  $f(x)$  near  $x_0$ . Geometrically, this is approximating the graph of  $y = f(x)$  by the tangent line  $y = \varphi(x)$ .

### 1.2 Immediate Properties

**Proposition 1.1.** Let  $U \subseteq \mathbb{R}$  be an open set and  $f : U \rightarrow \mathbb{R}$ . If  $f$  is differentiable at  $x_0 \in U$ , then  $f$  is continuous at  $x_0$ .

*Proof.* Pick any  $\epsilon_0 > 0$ . Then there exists  $\delta_0 > 0$  such that whenever  $|x - x_0| < \delta_0$  and  $x \in U$ ,

$$|f(x) - f(x_0) - f'(x_0)(x - x_0)| \leq \epsilon_0|x - x_0|.$$

By the triangle inequality,

$$|f(x) - f(x_0)| \leq \epsilon_0|x - x_0| + |f'(x_0)||x - x_0| = (\epsilon_0 + |f'(x_0)|)|x - x_0|.$$

Now for any  $\epsilon > 0$ , choose  $\delta = \min\{\delta_0, \epsilon/(\epsilon_0 + |f'(x_0)|)\}$ . Then

$$|f(x) - f(x_0)| \leq (\epsilon_0 + |f'(x_0)|)|x - x_0| < (\epsilon_0 + |f'(x_0)|)\delta \leq \epsilon$$

whenever  $|x - x_0| < \delta$  and  $x \in U$ . Thus  $f$  is continuous at  $x_0$ . □

**Example 1.1.1.** Take the function

$$f(x) = \begin{cases} x \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

Note that  $f$  is continuous on  $\mathbb{R}$ . For  $x \neq 0$ , continuity is clear since both  $x$  and  $\sin(1/x)$  are continuous. At  $x = 0$ , we have

$$\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} x \sin(1/x) = 0 = f(0)$$

since  $|x \sin(1/x)| \leq |x|$  for all  $x \in \mathbb{R}$ , so  $f$  is also continuous at  $x = 0$ . However,  $f$  is not differentiable at  $x = 0$ . Consider the limit

$$\lim_{x \rightarrow 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \rightarrow 0} \sin(1/x),$$

which does not exist since  $\sin(1/x)$  oscillates. So  $f$  is not differentiable at  $x = 0$ .

**Example 1.1.2.** Take the function  $f(x) = |x|$ , which is continuous everywhere on  $\mathbb{R}$ . However,  $f$  is not differentiable at  $x = 0$ , since

$$\lim_{x \rightarrow 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \rightarrow 0} \frac{|x|}{x}.$$

Note that

$$\frac{|x|}{x} = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0, \end{cases}$$

so the limit does not exist as  $x \rightarrow 0$ . Thus  $f$  is not differentiable at  $x = 0$ .

**Remark.** For the previous example, we can however define the *left (right) derivative* by

$$f'_-(x_0) = \lim_{x \rightarrow x_0^-} \frac{f(x) - f(x_0)}{x - x_0} \quad \text{and} \quad f'_+(x_0) = \lim_{x \rightarrow x_0^+} \frac{f(x) - f(x_0)}{x - x_0}.$$

If  $f$  is differentiable, then  $f'_-(x_0) = f'_+(x_0)$ . In the previous example,  $f'_-(0) = -1$  and  $f'_+(0) = 1$ . For the first example however, even  $f'_\pm(0)$  does not exist.

**Remark.** In one dimension, the existence of the derivative implies that the function is differentiable (the function is approximated by a linear function). However, in multiple dimensions, the existence of partial derivatives does not imply differentiability.

## 1.3 Rules for Differentiation

**Proposition 1.2.** Let  $U \subseteq \mathbb{R}$  be open and  $f, g : U \rightarrow \mathbb{R}$  be differentiable. Then

1.  $(f + g)'(x_0) = f'(x_0) + g'(x_0)$
2.  $(fg)'(x_0) = f'(x_0)g(x_0) + f(x_0)g'(x_0)$
3. if  $g(x_0) \neq 0$ , then  $(f/g)'(x_0) = (f'(x_0)g(x_0) - f(x_0)g'(x_0))/(g(x_0)^2)$ .

*Proof.* Find in textbook (Rosenlicht). □

**Proposition 1.3.** We have  $\frac{d}{dx}(c) = 0$ ,  $\frac{d}{dx}(x) = 1$ , and  $\frac{d}{dx}(x^n) = nx^{n-1}$  for all  $n \in \mathbb{N}$ .

*Proof.* We prove the last claim (the power rule) for  $n \geq 1$  by induction. The base case  $n = 1$  is the first claim which is true. Now suppose that the result holds for any  $n \leq k \in \mathbb{N}$ , and we show that it remains true for  $n = k + 1$ . By the product rule, we have

$$\frac{d}{dx}(x^{k+1}) = \frac{d}{dx}(x \cdot x^k) = \frac{d}{dx}(x) \cdot x^k + x \cdot \frac{d}{dx}(x^k) = x^k + x k x^{k-1} = (k+1)x^k.$$

Thus by induction this holds for all  $n \geq 1$ . We can do negative integers by the quotient rule.  $\square$

**Remark.** The power rule actually holds for any  $n \in \mathbb{R}$ .

**Proposition 1.4** (Chain rule). *Let  $U$  and  $V$  be open sets of  $\mathbb{R}$  and let  $f : U \rightarrow V, g : V \rightarrow \mathbb{R}$  be differentiable. Let  $x_0 \in U$  be such that  $f'(x_0)$  and  $g'(f(x_0))$  exist. Then  $(g \circ f)'(x_0)$  exists and*

$$(g \circ f)'(x_0) = g'(f(x_0))f'(x_0).$$

*Proof.* For any fixed  $y_0$  for which  $g'(y_0)$  exists, set

$$A(y, y_0) = \begin{cases} (g(y) - g(y_0))/(y - y_0) & \text{if } y \in V \text{ and } y \neq y_0 \\ g'(y_0) & \text{if } y = y_0. \end{cases}$$

Then  $A$  is continuous at  $y_0$ . To find  $(g \circ f)'(x_0)$ , observe that

$$\begin{aligned} \lim_{x \rightarrow x_0} \frac{g(f(x)) - g(f(x_0))}{x - x_0} &= \lim_{x \rightarrow x_0} \frac{A(f(x), f(x_0))(f(x) - f(x_0))}{x - x_0} \\ &= \lim_{x \rightarrow x_0} A(f(x), f(x_0)) \lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0} = g'(f(x_0))f'(x_0), \end{aligned}$$

by the continuity of  $A$  at  $f(x_0)$  and the differentiability of  $f$  at  $x_0$ .  $\square$

**Remark.** The rough idea of what we did here is

$$\begin{aligned} \lim_{x \rightarrow x_0} \frac{g(f(x)) - g(f(x_0))}{x - x_0} &= \lim_{x \rightarrow x_0} \frac{g(f(x)) - g(f(x_0))}{f(x) - f(x_0)} \frac{f(x) - f(x_0)}{x - x_0} \\ &= \lim_{x \rightarrow x_0} \frac{g(f(x)) - g(f(x_0))}{f(x) - f(x_0)} \lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0} = g'(f(x_0))f'(x_0). \end{aligned}$$

But does not quite work as stated since it might be that  $f(x) = f(x_0)$  even if  $x \neq x_0$ . We can fix this by introducing the function  $A$  as we did in the proof, though the overall idea is the same.

**Remark.** If  $f$  is monotone near  $x_0$ , then we can define the *inverse function*  $f^{-1}$  so that  $(f^{-1} \circ f)(x) = x$  near  $x_0$ . If  $f'(x_0)$  exists, then by the chain rule applied to  $x = (f^{-1} \circ f)(x)$  at  $x = x_0$  we have

$$1 = \frac{d}{dx}(f^{-1} \circ f)(x_0) = \frac{d}{dx}f^{-1}(f(x_0)) \cdot f'(x_0) \implies \frac{d}{dx}f^{-1}(f(x_0)) = \frac{1}{f'(x_0)}.$$

**Example 1.1.3.** Let  $f(x) = e^x$  with  $f^{-1}(x) = \ln(x)$ . Since  $f'(x) = f(x) = e^x$ , we have

$$\frac{d}{dx}f^{-1}(f(x_0)) = \frac{1}{f'(x_0)} \implies \frac{d}{dx}\ln(e^{x_0}) = \frac{1}{e^{x_0}}.$$

Letting  $e^{x_0} = h$ , we have  $\frac{d}{dx}\ln(x)|_{x=h} = 1/h$ , which recovers the familiar formula.

# Lecture 2

## Jan. 11 — The Mean Value Theorem

### 2.1 The Mean Value Theorem

**Lemma 2.1.** Let  $I \subseteq \mathbb{R}$  be open,  $f : I \rightarrow \mathbb{R}$  is differentiable at  $x_0 \in I$  and  $f'(x_0) \neq 0$ . Suppose  $f'(x_0) > 0$ , then there exists  $\delta > 0$  such that for any  $x \in (x_0 - \delta, x_0 + \delta)$ ,

1. if  $x > x_0$ , then  $f(x) > f(x_0)$ ,
2. if  $x < x_0$ , then  $f(x) < f(x_0)$ .

*Proof.* Take  $\epsilon = f'(x_0)/2$ . By the definition of the derivative, there exists  $\delta > 0$  such that for any  $|x - x_0| < \delta$ , we have

$$\left| \frac{f(x) - f(x_0)}{x - x_0} - f'(x_0) \right| < \epsilon = \frac{1}{2}f'(x_0).$$

By the triangle inequality,

$$\frac{f(x) - f(x_0)}{x - x_0} > \frac{1}{2}f'(x_0) > 0.$$

This quotient being positive immediately implies the desired results.  $\square$

**Theorem 2.1.** If  $f(x)$  is differentiable in an open interval  $I$  and  $f$  obtains its local maximum (or minimum) at  $x_0 \in I$ , then  $f'(x_0) = 0$ .

*Proof.* Suppose otherwise that  $f'(x_0) \neq 0$ . Assume without loss of generality that  $f'(x_0) > 0$ . Then by the previous lemma, there exists  $\delta > 0$  such that for  $x \in (x_0 - \delta, x_0 + \delta)$ , if  $x > x_0$  then  $f(x) > f(x_0)$  and if  $x < x_0$  then  $f(x) < f(x_0)$ . So  $x_0$  cannot be a local maximum or minimum, which is a contradiction.  $\square$

**Theorem 2.2** (Rolle's middle value theorem). Let  $f(x)$  be continuous on  $[a, b]$  and differentiable in  $(a, b)$ . Suppose  $f(a) = f(b)$ , then there exists  $x_0 \in (a, b)$  such that  $f'(x_0) = 0$ .

*Proof.* Since  $f$  is continuous on a compact set, it obtains both a maximum and minimum on  $[a, b]$ . Let  $M$  be the maximum and  $m$  be the minimum. If  $M = m$ , then  $f(x) \equiv M$  and  $f'(x) = 0$  everywhere. If  $M > m$ , then at least one of the maximum or minimum must be obtained at an interior point  $x_0 \in (a, b)$  since  $f(a) = f(b)$ . By the previous theorem,  $f'(x_0) = 0$  at this point and we are done.  $\square$

**Example 2.0.1.** Show that the equation  $4ax^3 + 3bx^2 + 2cx = a + b + c$  has at least one root in  $(0, 1)$ .

*Proof.* Consider the equation

$$4ax^3 + 3bx^2 + 2cx - (a + b + c) = 0.$$

Notice that the left hand side is the derivative of the function

$$f(x) = ax^4 + bx^3 + cx^2 - (a + b + c)x.$$

So we just need to show that  $f'(x) = 0$  for some  $x$ . For this, we can check that  $f(0) = f(1) = 0$ , and thus by Rolle's theorem there exists  $x_0 \in (0, 1)$  such that  $f'(x_0) = 0$ . So  $x_0$  is a root.  $\square$

**Theorem 2.3** (Lagrange's middle value theorem). *Let  $f(x)$  be continuous on  $[a, b]$  and differentiable in  $(a, b)$ . Then there exists  $x_0 \in (a, b)$  such that*

$$f'(x_0) = \frac{f(b) - f(a)}{b - a}.$$

*Proof.* Subtract the secant line through  $(a, f(a))$  and  $(b, f(b))$  from  $f(x)$  to get

$$g(x) = f(x) - \frac{f(b) - f(a)}{b - a}(x - a).$$

Note that  $g(a) = g(b) = f(a)$ . So by Rolle's theorem, there exists  $x_0 \in (a, b)$  such that  $g'(x_0) = 0$ . But

$$0 = g'(x_0) = f'(x_0) - \frac{f(b) - f(a)}{b - a},$$

which is the desired result.  $\square$

**Corollary 2.3.1.** *Suppose  $f \in C([a, b])$ , i.e.  $f$  is continuous on  $[a, b]$ , and that  $f$  is differentiable in  $(a, b)$ . Then the following statements are equivalent:*

1.  $f'(x) \geq 0$  in  $(a, b)$ ,
2.  $f(x)$  is increasing, i.e. if  $x_1 > x_2$ , then  $f(x_1) \geq f(x_2)$ .

*In particular, if  $f'(x) > 0$  in  $(a, b)$ , then  $f(x)$  is strictly increasing, i.e. if  $x_1 > x_2$ , then  $f(x_1) > f(x_2)$ .*

*Proof.*  $(2 \Rightarrow 1)$  For any  $x_0 \in (a, b)$ ,

$$f'(x_0) = \lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h} \geq 0$$

since  $f(x_0 + h) - f(x_0) \geq 0$  for  $h > 0$  as  $f$  is increasing.

$(1 \Rightarrow 2)$  Take  $x_1 > x_2$ , then by Lagrange's theorem there exists  $\xi \in (x_2, x_1)$  such that

$$f(x_2) - f(x_1) = f'(\xi)(x_2 - x_1) \geq 0.$$

So  $f(x_1) \geq f(x_2)$ . The strict version follows from changing the above inequality to a strict one.  $\square$

## 2.2 Applications

**Example 2.0.2.** Show that

$$\frac{2}{2x+1} < \ln(1 + 1/x)$$

for any  $x > 0$ .

*Proof.* Let  $f(x) = 2/(2x+1) - \ln(1 + 1/x)$ . Taking the derivative yields

$$f'(x) = \frac{1}{(2x+1)^2 x(x+1)} > 0,$$

so  $f$  is strictly increasing in  $(0, \infty)$ . Note that  $f \rightarrow 0$  as  $x \rightarrow \infty$ , so  $f(x) < 0$  for all  $x > 0$ . □

**Example 2.0.3.** Show that  $b/a > b^a/a^b$  when  $b > a > 1$ .

*Proof.* Take log on both sides to get  $\ln b - \ln a > a \ln b - b \ln a$ . This gives

$$(b-1) \ln a > (a-1) \ln b \iff \frac{\ln a}{a-1} > \frac{\ln b}{b-1}.$$

Note that this is a monotonicity property. So let  $f(x) = (\ln x)/(x-1)$  for  $x > 1$ . Then

$$f'(x) = \frac{x-1-x \ln x}{x(x-1)^2} < 0$$

when  $x > 1$  because  $x-1-x \ln x < 0$ . To see the last claim, define  $g(x) = x-1-x \ln x$  and note that  $g'(x) = -\ln x < 0$  for  $x > 1$ . But  $g(0) = 0$ , so  $g(x) < 0$  for  $x > 1$ . So  $f$  is strictly decreasing. □

**Example 2.0.4.** Show that

$$\lim_{x \rightarrow 0} \frac{e^x - e^{\sin x}}{x - \sin x} = 1.$$

*Proof.* Let  $f(x) = e^x$ . Then there exists  $\xi$  between  $x$  and  $\sin x$  such that

$$e^x - e^{\sin x} = (x - \sin x)e^{\xi(x)},$$

where the choice of  $\xi$  may vary for different  $x$ . Then

$$\lim_{x \rightarrow 0} \frac{e^x - e^{\sin x}}{x - \sin x} = \lim_{x \rightarrow 0} e^{\xi(x)}.$$

Now note that  $\xi(x)$  is always between  $x$  and  $\sin x$ , which both tend to 0 as  $x \rightarrow 0$ . So by the squeeze theorem we have  $\xi(x) \rightarrow 0$  as  $x \rightarrow 0$  and thus  $e^{\xi(x)} \rightarrow 1$  as  $x \rightarrow 0$ . □



## 2.3 Cauchy's Mean Value Theorem

**Theorem 2.4** (Cauchy's middle value theorem). *Let  $f, g \in C([a, b])$  and  $f, g$  be differentiable in  $(a, b)$ . Suppose  $g'(x) \neq 0$  for any  $x \in (a, b)$ . Then there exists  $x_0 \in (a, b)$  such that*

$$\frac{f'(x_0)}{g'(x_0)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

*Proof.* Use a similar construction as before and let

$$F(x) = f(x) - f(a) - \frac{f(b) - f(a)}{g(b) - g(a)}(g(x) - g(a)).$$

Note that  $F(b) = F(a) = 0$ , so by Rolle's theorem there exists  $x_0 \in (a, b)$  such that  $F'(x_0) = 0$ . Then

$$0 = F'(x_0) = f'(x_0) - \frac{f(b) - f(a)}{g(b) - g(a)}g'(x_0),$$

which implies the desired result. □

**Remark.** The  $g'(x) \neq 0$  condition guarantees that  $g$  is monotone, even if  $g'$  may fail to be continuous.

**Remark.** If  $g$  is a monotonically increasing function, we can view  $g$  as a mapping  $g : [a, b] \rightarrow [g(a), g(b)]$ , which we can view as a change of variables  $x \mapsto u$ . Since  $g$  is monotone, we have an inverse  $x = g^{-1}(u)$ . Then

$$f(x) = f(g^{-1}(u)) = (f \circ g^{-1})(u) = \tilde{f}(u).$$

By Lagrange's theorem,

$$\frac{\tilde{f}(g(b)) - \tilde{f}(g(a))}{g(b) - g(a)} = \tilde{f}'(u_0)$$

for some  $u_0 \in (g(a), g(b))$ . Now note that

$$\tilde{f}(g(b)) = (f \circ g^{-1})(g(b)) = f(b), \quad \tilde{f}(g(a)) = f(a).$$

So the left-hand side is precisely

$$\text{LHS} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

By the chain rule, we have

$$\text{RHS} = \tilde{f}'(u_0) = (f \circ g^{-1})'(u_0) = f'(g^{-1}(u_0))(g^{-1})'(u_0) = f'(x_0) \frac{1}{g'(x_0)}.$$

This recovers Cauchy's mean value theorem. So they are equivalent even if Cauchy's seems stronger.

# Lecture 3

## Jan. 16 — Taylor's Theorem

### 3.1 Darboux's Lemma

**Lemma 3.1** (Darboux's lemma). *If  $f$  is differentiable in  $(a, b)$ , continuous on  $[a, b]$  and  $f'(a) < f'(b)$ , then for any  $c \in (f'(a), f'(b))$ , there exists  $x_0 \in (a, b)$  such that  $f'(x_0) = c$ .*

*Proof.* See homework. □

**Remark.** There exists an example of a differentiable function  $f(x)$  but  $f'(x)$  is not continuous, e.g.

$$f(x) = \begin{cases} x^2 \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

We can compute that

$$f'(x) = \begin{cases} 2x \sin(1/x) - \cos(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0, \end{cases}$$

and we can verify as an exercise that  $f'(x)$  is not continuous at  $x = 0$ .

**Remark.** Darboux's lemma guarantees that  $g'(x) \neq 0$  implies either  $g'(x) > 0$  or  $g'(x) < 0$  everywhere in the conditions for Cauchy's mean value theorem.

### 3.2 L'Hôpital's Rule

**Theorem 3.1** (L'Hôpital's rule,  $0/0$ ). *Let  $f, g$  be differentiable in  $(a, b)$ ,  $\lim_{x \rightarrow a^+} f(x) = \lim_{x \rightarrow a^+} g(x) = 0$ , and  $g'(x) \neq 0$  for any  $x \in (a, b)$ . Then if  $\lim_{x \rightarrow a^+} f'(x)/g'(x)$  exists, we have*

$$\lim_{x \rightarrow a^+} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a^+} \frac{f'(x)}{g'(x)}.$$

*Proof.* By Cauchy's theorem, for any  $x \in (a, b)$ , there exists  $\xi(x) \in (a, x)$  such that

$$\frac{f(x)}{g(x)} = \frac{f(x) - f(a)}{g(x) - g(a)} = \frac{f'(\xi(x))}{g'(\xi(x))}.$$

If  $x \rightarrow a^+$ , then  $\xi(x) \rightarrow a^+$ , so

$$\lim_{x \rightarrow a^+} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a^+} \frac{f'(\xi(x))}{g'(\xi(x))} = \lim_{x \rightarrow a^+} \frac{f'(x)}{g'(x)},$$

as desired. □

**Corollary 3.1.1.** Let  $f, g$  be differentiable in  $(a, \infty)$ ,  $\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} g(x) = 0$ , and  $g'(x) \neq 0$  for any  $x \in (a, \infty)$ . Then if  $\lim_{x \rightarrow \infty} f'(x)/g'(x)$  exists, we have

$$\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = \lim_{x \rightarrow \infty} \frac{f'(x)}{g'(x)}.$$

*Proof.* Assume  $a > 0$ . Define  $\tilde{f}(y) = f(1/y)$  and  $\tilde{g}(y) = g(1/y)$  with  $y \in (0, 1/a)$ . By L'Hôpital's rule,

$$\lim_{y \rightarrow 0^+} \frac{\tilde{f}(y)}{\tilde{g}(y)} = \lim_{y \rightarrow 0^+} \frac{\tilde{f}'(y)}{\tilde{g}'(y)} = \lim_{y \rightarrow \infty} \frac{f'(1/y) \cdot (-1/y^2)}{g'(1/y) \cdot (-1/y^2)} = \lim_{x \rightarrow \infty} \frac{f'(x)}{g'(x)},$$

as desired. □

**Theorem 3.2** (L'Hôpital,  $\infty/\infty$ ). Let  $f, g$  be differentiable in  $(a, b)$ ,  $\lim_{x \rightarrow a^+} |f(x)| = \lim_{x \rightarrow a^+} |g(x)| = \infty$ , and  $g'(x) \neq 0$  for any  $x \in (a, b)$ . Then if  $\lim_{x \rightarrow a^+} f'(x)/g'(x)$  exists, we have

$$\lim_{x \rightarrow a^+} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a^+} \frac{f'(x)}{g'(x)}.$$

*Proof.* Left as an exercise. □

**Remark.** Saying that the absolute values of  $f$  and  $g$  go to infinity works, since the existence of the limit rules out oscillatory behavior.

**Remark.** These cases of  $\infty/\infty$  and  $0/0$  are called *indefinite types*. Other indefinite types include  $0 \cdot \infty$ ,  $0^0$ ,  $\infty^0$ ,  $1^\infty$ ,  $\infty - \infty$ , etc. But we can try to reduce them to the cases we know. For example, if  $f(x) \rightarrow 0^+$  and  $g(x) \rightarrow 0^+$  when  $x \rightarrow x_0$ , then  $\lim_{x \rightarrow x_0} f(x)^{g(x)}$  is  $0^0$ . Letting  $y(x) = f(x)^{g(x)}$ , we can take the log to get

$$\ln y(x) = g(x) \ln f(x) = \frac{\ln f(x)}{1/g(x)} = \frac{\infty}{\infty}.$$

**Example 3.0.1.** We can see that (this is a  $\infty - \infty$  case)

$$\lim_{x \rightarrow 0^+} \frac{1}{x^2} - \frac{\cot x}{x} = \lim_{x \rightarrow 0^+} \frac{1 + x \cot x}{x^2} = \lim_{x \rightarrow 0^+} \frac{-(\cot x - x \csc^2 x)}{2x \sin^2 x}.$$

Note that  $x \cot x = x \cos x / \sin x \rightarrow 1$  as  $x \rightarrow 0$ . Now note that  $\sin x / x \rightarrow 1$  as  $x \rightarrow 0$ , so we continue with

$$\lim_{x \rightarrow 0^+} \frac{-(\cot x - x \csc^2 x)}{2x \sin^2 x} = \lim_{x \rightarrow 0^+} \frac{x - \sin x \cos x}{2x^3} \frac{x^2}{\sin^2 x}$$

Since  $x^2 / \sin^2 x \rightarrow 1$  as  $x \rightarrow 0$ , we can look at the remaining part to get

$$\lim_{x \rightarrow 0^+} \frac{x - \sin x \cos x}{2x^3} = \lim_{x \rightarrow 0^+} \frac{1 - \cos 2x}{6x^2} = \lim_{x \rightarrow 0^+} \frac{2 \sin 2x}{12x} = \frac{1}{3}.$$

So  $\lim_{x \rightarrow 0^+} (1/x^2 - \cot x/x) = 1/3$ .

### 3.3 Taylor's Theorem

**Theorem 3.3** (Peano remainder term). *Let  $f : [a, b] \rightarrow \mathbb{R}$  be differentiable at  $x = a$  up to  $n$ th order of derivatives, i.e.  $f'(a), f''(a), \dots, f^{(n)}(a)$  exist. Then as  $x \rightarrow a^+$ , we have*

$$f(x) = \sum_{k=0}^n \frac{f^{(k)}(a)}{k!} (x-a)^k + o((x-a)^n).$$

Call the polynomial part of the above  $P_n(x)$ , which is also known as the Taylor polynomial of order  $n$ .

*Proof.* To show that the error term is  $o((x-a)^n)$ , we have

$$\lim_{x \rightarrow a^+} \frac{f(x) - P_n(x)}{(x-a)^n} = \lim_{x \rightarrow a^+} \frac{f'(x) - P'_n(x)}{n(x-a)^{n-1}} = \frac{1}{n!} \lim_{x \rightarrow a^+} \left[ \frac{f^{n-1}(x) - f^{n-1}(a)}{x-a} - f^{(n)}(a) \right] = 0$$

by L'Hôpital's rule, where we used the observation that  $f^{(k)}(a) = P_n^{(k)}(a)$  for  $1 \leq k \leq n$ . The final step is a result of the existence of  $f^{(n)}(a)$ .  $\square$

**Lemma 3.2** (Rolle's theorem for higher order derivatives). *Let  $f \in C^n([a, b])$  and differentiable to  $(n+1)$  order. If  $f'(a) = \dots = f^{(n)}(a) = 0$  and  $f(a) = f(b)$ , then there exists  $x_0 \in (a, b)$  such that  $f^{(n+1)}(x_0) = 0$ .*

*Proof.* Since  $f(a) = f(b)$ , by the usual Rolle's theorem there exists  $x_1 \in (a, b)$  such that  $f'(x_1) = 0$ . Then since  $f'(a) = f'(x_1) = 0$ , by Rolle's theorem again, there exists  $x_2 \in (a, x_1)$  such that  $f''(x_2) = 0$ . Repeat this to get  $x_{n+1} \in (a, x_n) \subseteq (a, b)$  such that  $f^{(n+1)}(x_{n+1}) = 0$ . Take  $x_0 = x_{n+1}$  to finish.  $\square$

**Theorem 3.4** (Lagrange remainder term). *Let  $f \in C^n([a, b])$ , in particular,  $f'(a), \dots, f^{(n)}(a)$  exist. Additionally, assume  $f$  is  $(n+1)$ -th differentiable in  $(a, b)$ .<sup>1</sup> Then*

$$f(x) = \sum_{k=0}^n \frac{f^{(k)}(a)}{k!} (x-a)^k + R_n(x) \quad \text{where} \quad R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} (x-a)^{n+1}$$

for some  $\xi \in [a, x]$ .

*Proof.* Define  $P(x) = P_n(x) + \lambda(x-a)^{n+1}$ , where we choose  $\lambda \in \mathbb{R}$  such that  $P(b) = f(b)$ , i.e.

$$\lambda = \frac{f(b) - P_n(b)}{(b-a)^{n+1}}.$$

Consider  $g(x) = f(x) - P(x)$ , which satisfies  $g(a) = g(b) = 0$  and  $g'(a) = \dots = g^{(n)}(a) = 0$ . Then by Rolle's theorem (higher order), there exists  $\xi \in (a, b)$  such that  $g^{(n+1)}(\xi) = 0$ . In other words,

$$f^{(n+1)}(\xi) - P^{(n+1)}(\xi) = 0 \implies f^{(n+1)}(\xi) - \underbrace{(n+1)! \frac{f(b) - P_n(b)}{(b-a)^{n+1}}}_{\lambda} = 0.$$

This implies that

$$f(b) = P_n(b) + \frac{1}{(n+1)!} f^{(n+1)}(\xi) (b-a)^{n+1},$$

and since we picked  $b$  arbitrarily (kind of), we can take  $b = x$  and we are done since  $\xi \in [a, b]$ .  $\square$

<sup>1</sup>Note that the  $(n+1)$ -th derivative need not be continuous here.

**Remark.** The choice of  $\xi$  in Lagrange's remainder term may (and likely does) vary for different  $x$ .

**Remark.** The Taylor polynomial is unique in the sense that if  $f : [a, b] \rightarrow \mathbb{R}$  and  $f'(a), \dots, f^{(n)}(a)$  exist, then if

$$f(x) = p(x) + o((x - a)^n)$$

as  $x \rightarrow a^+$  for some polynomial  $p(x)$  with  $\deg p \leq n$ , then  $p(x) = P_n(x) = \sum_{k=0}^n \frac{f^{(k)}(a)}{k!} (x - a)^k$ . This is because if  $Q(x) = p(x) - P_n(x)$ , then by Taylor's formula (Peano form), we get

$$\lim_{x \rightarrow a^+} \frac{Q(x)}{(x - a)^n} = \lim_{x \rightarrow a^+} \frac{p(x) - \sum_{k=0}^n \frac{f^{(k)}(a)}{k!} (x - a)^k}{(x - a)^n} = 0.$$

From here this implies that  $Q(x) = 0$  since  $\deg Q \leq n$ . Another way to see this is to plug in  $x = a$ , which deletes everything except the constant, and then ignore the constant and divide by  $(x - a)$  to repeat.

# Lecture 4

## Jan. 18 — Taylor Polynomials

### 4.1 Common Taylor Polynomials

We have

$$\begin{aligned}e^x &= 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + o(x^n), \\ \sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + \frac{(-1)^{n-1}x^{2n-1}}{(2n-1)!} + o(x^{2n}), \\ \cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + \frac{(-1)^n x^{2n}}{(2n)!} + o(x^{2n+1}), \\ (1+x)^\alpha &= 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \cdots + \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}x^n + o(x^n), \\ \ln(1+x) &= x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n-1}\frac{x^n}{n} + o(x^n).\end{aligned}$$

### 4.2 Combining Taylor Polynomials

**Remark.** If  $a = 0$  and  $f(x)$  is even in  $(-b, b)$ , then

$$f(x) = \sum_{k=0}^{n/2} a_k x^{2k} + o(x^n).$$

Similarly if  $f(x)$  is odd in  $(-b, b)$ , then

$$f(x) = \sum_{k=0}^{n/2} a_k x^{2k+1} + o(x^{n+1}).$$

**Remark.** To create new Taylor polynomials from known ones, we can observe that if  $f(x) = P_n(x) + o((x-a)^n)$  and  $g(x) = Q_n(x) + o((x-a)^n)$ , then

$$f(x) + g(x) = (P_n(x) + Q_n(x)) + o((x-a)^n) \quad \text{and} \quad f(x)g(x) = \underbrace{(P_n(x)Q_n(x))}_{\text{take first } n \text{ terms}} + o((x-a)^n).$$

If  $P_n(x) = \sum_{k=0}^n a_k (x-a)^k$  and  $Q_n(x) = \sum_{k=0}^n b_k (x-a)^k$ , then  $f(x)g(x)$  has Taylor polynomial  $\sum_{k=0}^n c_k (x-a)^k$  where

$$c_k = \sum_{i=0}^k a_i b_{k-i}.$$

If  $h(x) = f(x)/g(x)$  and  $g(x) \neq 0$  near  $x = a$ , then  $f(x) = h(x)g(x)$ . Let  $h(x) = \sum_{k=0}^n c_k(x-a)^k + o((x-a)^n)$ , then

$$a_k = \sum_{i=0}^k c_i b_{k-i}$$

for  $0 \leq k \leq n$ , after which we can solve for the  $c_k$ .

**Example 4.0.1.** Find the Taylor polynomial for  $\tan x$  up to  $n = 5$ .

*Proof.* Note that  $\tan x$  is odd, so we can write

$$\tan x = x + a_3 x^3 + a_5 x^5 + o(x^5).$$

Now since  $\tan x = \sin x / \cos x$ , we have  $\sin x = \tan x \cos x$ , so

$$x - \frac{x^3}{6} + \frac{x^5}{5!} + o(x^5) = (x + a_3 x^3 + a_5 x^5) \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!}\right)$$

We can solve to get

$$\begin{cases} -\frac{1}{6} = -\frac{1}{2} + a_3 \\ \frac{1}{5!} = \frac{1}{4!} - \frac{a_3}{2!} + a_5 \end{cases} \implies a_3 = -\frac{1}{3}, \quad a_5 = \frac{2}{15}$$

as the coefficients for the Taylor polynomial. □

**Remark.** If

$$f'(x) = \sum_{k=0}^n b_k (x-a)^k + o((x-a)^n),$$

then the anti-derivative of  $f(x)$  has

$$f(x) = f(x_0) + \sum_{k=0}^n a_{k+1} (x-a)^{k+1} + o((x-a)^{n+1}),$$

where  $a_{k+1} = b_k / (k+1)$  for  $0 \leq k \leq n$ . This is because

$$b_k = \frac{(f')^{(k)}(a)}{k!} = \frac{f^{(k+1)}(a)}{k!} \quad \text{and} \quad a_{k+1} = \frac{f^{(k+1)}(a)}{k+1} = \frac{1}{k+1} \frac{f^{(k+1)}(a)}{k!} = \frac{b_k}{k+1}.$$

**Example 4.0.2.** Find the Taylor polynomial for  $f(x) = \arctan x$ .

*Proof.* Recall that

$$f'(x) = \frac{1}{1+x^2} = \sum_{k=0}^n (-1)^k x^{2k}.$$

Using the above we get

$$f(x) = \arctan x = \sum_{k=0}^n (-1)^k \frac{x^{2k+1}}{2k+1} + o(x^{2n+2})$$

as the Taylor polynomial. □

## 4.3 Applications for Taylor Polynomials

### 4.3.1 Finding Limits

**Remark.** Let  $f(x) = ax^n + o(x^n)$  as  $x \rightarrow 0$  and  $g(x) = bx^n + o(x^n)$  where  $b \neq 0$ . Then

$$\lim_{x \rightarrow 0} \frac{f(x)}{g(x)} = \frac{a}{b}.$$

**Remark.** For the polynomial of  $f(g(x))$ , we can do

$$f(u) = \sum_{k=0}^n a_k(u - g(a))^k + o((u - g(a))^n), \quad \text{where } u = g(x) = \sum_{k=0}^n b_k(x - a)^k + o((x - a)^n).$$

Then we can substitute in  $u = g(x)$  to find the overall polynomial.

**Example 4.0.3.** Find

$$\lim_{x \rightarrow 0} \frac{\sqrt{1 + 2 \tan x} - e^x + x^2}{\arcsin x - \sin x}.$$

*Proof.* Note that

$$\begin{aligned} \sqrt{1 + 2 \tan x} - e^x + x^2 &= \frac{2x^3}{3} + o(x^3), \\ \arcsin x - \sin x &= \frac{x^3}{3} + o(x^3). \end{aligned}$$

So the desired limit is 2. □

**Remark.** If  $f(x) = ax^n + o(x^n)$  and  $g(x) = bx^m + o(x^m)$  for  $a, b \neq 0$ , then

$$\lim_{x \rightarrow 0} \frac{f(x)}{g(x)} = \begin{cases} a/b & \text{if } m = n, \\ 0 & \text{if } m < n, \\ \infty & \text{if } m > n. \end{cases}$$

**Example 4.0.4.** Assume  $f(x) = 1 + ax^n + o(x^n)$  where  $a \neq 0$  and

$$g(x) = \frac{1}{bx^n + o(x^n)}, \quad \text{i.e.} \quad \frac{1}{g(x)} = bx^n + o(x^n).$$

for  $b \neq 0$ . Then

$$\lim_{x \rightarrow 0} f(x)^{g(x)} = e^{a/b}.$$

Let  $y(x) = f(x)^{g(x)}$ , then  $\ln y(x) = g(x) \ln f(x)$ . Note that

$$\ln f(x) = \ln(1 + ax^n + o(x^n)) = ax^n + o(x^n),$$

so that

$$\frac{\ln f(x)}{1/g(x)} = \frac{ax^n + o(x^n)}{bx^n + o(x^n)} \rightarrow \frac{a}{b}$$

as  $x \rightarrow 0$ . Thus  $\ln y(x) \rightarrow a/b$  and  $y(x) \rightarrow e^{a/b}$  as  $x \rightarrow 0$ .



**Example 4.0.5.** Find

$$\lim_{x \rightarrow 0} [\cos(xe^x) - \ln(1-x) - x]^{\cot x^3}.$$

*Proof.* Here we have

$$f(x) = \cos(xe^x) - \ln(1-x) - x = 1 - \frac{2}{3}x^3 + o(x^3) \quad \text{and} \quad \frac{1}{g(x)} = \tan x^3 = x^3 + o(x^3).$$

Thus the limit is  $e^{-2/3}$ . □

### 4.3.2 Estimation

**Example 4.0.6.** Let  $f(x)$  be twice differentiable in  $[0, 1]$  and  $f(0) = f(1)$ . Further assume  $|f''(x)| \leq M$  for  $0 \leq x \leq 1$ . Prove that  $|f'(x)| \leq M/2$  for  $0 \leq x \leq 1$ .

*Proof.* Recall that Lagrange's form of Taylor's theorem says

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(\xi)}{2!}(x-a)^2$$

for some  $\xi$  between  $a$  and  $x$ . Thus for any  $x \in (0, 1)$ , we have

$$f(1) = f(x) + f'(x)(1-x) + \frac{f''(\xi_1)}{2}(1-x)^2.$$

Similarly, we have

$$f(0) = f(x) + f'(x)(-x) + \frac{f''(\xi_2)}{2}x^2.$$

Here  $x \leq \xi_1 \leq 1$  and  $0 \leq \xi_2 \leq x$ . Since  $f(1) = f(0)$ , we can solve for  $f'(x)$  to get

$$f'(x) = \frac{f''(\xi_2)x^2 - f''(\xi_1)(1-x)^2}{2}.$$

Then taking absolute values yields

$$|f'(x)| \leq M \left( \frac{x^2 + (1-x)^2}{2} \right) \leq \frac{M}{2} \max_{0 \leq x \leq 1} [x^2 + (1-x)^2] = \frac{M}{2},$$

as desired. □

**Example 4.0.7.** Let  $f(x)$  be twice differentiable in  $[0, 1]$  and  $f'(a) = f'(b) = 0$ . Then there exists  $\xi \in (a, b)$  such that

$$|f''(\xi)| \geq 4 \frac{|f(a) - f(b)|}{(b-a)^2}.$$

*Proof.* Note that this is equivalent to

$$|f(b) - f(a)| \leq f''(\xi) \left( \frac{b-a}{2} \right)^2.$$

Then we have

$$f\left(\frac{b+a}{2}\right) = f(a) + \frac{f''(\xi_1)}{2} \left(\frac{b-a}{2}\right)^2 = f(b) - \frac{f''(\xi_2)}{2} \left(\frac{b-a}{2}\right)^2,$$

so that

$$f(b) - f(a) = \frac{f''(\xi_2) + f''(\xi_1)}{2} \left( \frac{b-a}{2} \right)^2.$$

From here we have

$$|f(b) - f(a)| \leq \underbrace{\frac{|f''(\xi_1)| + |f''(\xi_2)|}{2}}_{=|f''(\xi)|} \left( \frac{b-a}{2} \right)^2$$

for some  $\xi \in (a, b)$  by Darboux's lemma, as desired.

□

# Lecture 5

## Jan. 23 — The Riemann Integral

### 5.1 The Anti-Derivative

Recall the *anti-derivative* from calculus:

**Definition 5.1.** Let  $f : U \rightarrow \mathbb{R}$  where  $U$  is an interval in  $\mathbb{R}$ . If there exists a differentiable function  $F : U \rightarrow \mathbb{R}$  such that  $F'(x) = f(x)$  for all  $x \in U$ , then  $F(x)$  is an *anti-derivative* of  $f$ , denoted

$$F(x) = \int f(x) dx.$$

This is also called the *indefinite integral* of  $f$ .

**Remark.** The anti-derivatives of a function can differ by a constant.

**Example 5.1.1.** Find an anti-derivative of  $f(x) = |x|$  for  $x \in \mathbb{R}$ .

*Proof.* If  $x > 0$ , we have  $f(x) = x$  and so  $F(x) = x^2/2$ . If  $x < 0$ , then  $f(x) = -x$  and so  $F(x) = -x^2/2$ . We can also write this as

$$F(x) = x \cdot \frac{|x|}{2}.$$

Clearly for  $x \neq 0$ , we have  $F'(x) = f(x)$ . At  $x = 0$ , we have

$$\lim_{x \rightarrow 0} \frac{F(x) - f(0)}{x} = \lim_{x \rightarrow 0} \frac{1}{2}|x| = 0,$$

so  $F'(0) = f(0)$  and  $F$  is an anti-derivative of  $f$ . □

**Remark.** The eventual goal is to show that any continuous function  $f : [a, b] \rightarrow \mathbb{R}$  has an anti-derivative.

**Example 5.1.2.** Find an anti-derivative for

$$f(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \\ -1 & \text{if } x < 0. \end{cases}$$

*Proof.* We can try to use  $F(x) = |x|$ , but recall that  $F$  is not differentiable at  $x = 0$ . More generally, suppose that  $f(x)$  has some anti-derivative  $F(x)$ , i.e.  $f(x) = F'(x)$ . By Darboux's theorem,  $f(x)$  must take all values in  $(-1, 1)$ , which is a contradiction with the definition of  $f$ . □

**Remark.** If  $f(x)$  has a jump discontinuity, then it has no anti-derivative.

## 5.2 The Riemann Integral

Recall from calculus that if  $f(x)$  is defined in  $[a, b]$  and  $F'(x) = f(x)$ , then we have<sup>1</sup>

$$\int_a^b f(x) dx = F(x) \Big|_a^b = F(b) - F(a).$$

We called this the *definite integral* of  $f$  in calculus, but we would like a more rigorous definition.

**Definition 5.2.** Let  $a, b \in \mathbb{R}$  and  $a < b$ . A *partition* of the interval  $[a, b]$  is a finite sequence of numbers  $x_0, x_1, \dots, x_n$  such that  $a = x_0 < x_1 < \dots < x_n = b$ .

**Definition 5.3.** The *width* of a partition  $x_0, x_1, \dots, x_n$  is  $\max\{x_i - x_{i-1} : i = 1, 2, \dots, n\}$ .

**Definition 5.4.** For any partition  $x_0, x_1, \dots, x_n$ , define the *Riemann sum* to be

$$S = \sum_{i=1}^n f(x'_i)(x_i - x_{i-1}),$$

where  $x'_i$  is any point between  $x_{i-1}$  and  $x_i$ , inclusive.<sup>2</sup>

**Definition 5.5.** Let  $a, b \in \mathbb{R}$  with  $a < b$  and  $f : [a, b] \rightarrow \mathbb{R}$ . We say  $f$  is *Riemann integrable* on  $[a, b]$  if there exists  $A \in \mathbb{R}$  such that for all  $\epsilon > 0$ , there exists  $\delta > 0$  such that  $|S - A| < \epsilon$  whenever  $S$  is any Riemann sum for a partition of  $[a, b]$  with width less than  $\delta$ . We call  $A$  the *Riemann integral* of  $f$  on  $[a, b]$  and denote it by

$$A = \int_a^b f(x) dx.$$

**Remark.** If  $f$  is Riemann integrable, then

$$A = \int_a^b f(x) dx$$

is unique. This is because if  $A$  and  $A'$  are two numbers for the Riemann integral, then for any  $\epsilon > 0$ , there exists  $\delta > 0$  such that

$$|A - S| < \epsilon \quad \text{and} \quad |A' - S| < \epsilon$$

for any Riemann sum  $S$  associated with a partition of width less than  $\delta$ . Then

$$|A - A'| \leq |A - S| + |A' - S| < 2\epsilon,$$

so  $A = A'$  and thus the Riemann integral is unique.

**Example 5.5.1.** Let  $f(x) = c$  on  $[a, b]$ , a constant function. Then for any partition  $x_0, x_1, \dots, x_n$ ,

$$S = \sum_{i=1}^n f(x'_i)(x_i - x_{i-1}) = \sum_{i=1}^n c(x_i - x_{i-1}) = c(b - a) \implies \int_a^b c dx = c(b - a).$$

<sup>1</sup>This is the *fundamental theorem of calculus*.

<sup>2</sup>The geometric intuition of the Riemann sum is an approximation for the *area* under the graph of  $f$  by rectangles.

**Example 5.5.2.** Fix  $\xi \in [a, b]$  and let  $f : [a, b] \rightarrow \mathbb{R}$  be defined by

$$f(x) = \begin{cases} 0 & \text{if } x \neq \xi \\ c & \text{if } x = \xi. \end{cases}$$

Check that

$$A = \int_a^b f(x) dx = 0.$$

*Proof.* For any partition  $a = x_0 < x_1 < \cdots < x_n = b$  with width  $\delta$ , we have

$$|S| = \left| \sum_{i=1}^n f(x'_i)(x_i - x_{i-1}) \right| \leq |c|2\delta$$

since  $\xi$  can be in at most two of the intervals of the partition. Then for any  $\epsilon > 0$ , choose  $\delta = \epsilon/(2|c|)$ , so that  $|S| < \epsilon$  for any partition of width less than  $\delta$ . From this we can conclude that  $A = 0$ .  $\square$

**Example 5.5.3.** Consider a step function. Let  $\alpha, \beta \in [a, b]$  with  $\alpha < \beta$ . Define  $f : [a, b] \rightarrow \mathbb{R}$  by

$$f(x) = \begin{cases} 1 & \text{if } x \in (\alpha, \beta) \\ 0 & \text{if } x \notin (\alpha, \beta) \text{ and } x \in [a, b]. \end{cases}$$

Note that  $f$  has no anti-derivative, but it is Riemann integrable. In fact,

$$\int_a^b f(x) dx = \beta - \alpha.$$

To see this, take any partition  $a = x_0 < x_1 < \cdots < x_n = b$  with width less than  $\delta$ . Then

$$S = \sum_{i=1}^n f(x'_i)(x_i - x_{i-1}) = \sum_{[x_{i-1}, x_i] \cap [\alpha, \beta] \neq \emptyset} f(x'_i)(x_i - x_{i-1}).$$

Each partition is in two classes: Either (1) it only partially intersects  $[\alpha, \beta]$  or (2) it is contained in  $[\alpha, \beta]$ . So

$$S = \underbrace{1(\text{total length of intervals of class 2})}_{I_1} + \underbrace{|f(x'_i)|(\text{total length of intervals of class 1})}_{I_2}.$$

We have  $|I_1 - (\beta - \alpha)| < 2\delta$  and  $|I_2| < 2\delta$  since there are at most two intervals of class 1. So

$$|S - (\beta - \alpha)| \leq |I_1| + |I_2| < 4\delta.$$

So  $f(x)$  is Riemann integrable and

$$\int_a^b f(x) dx = \beta - \alpha,$$

as desired.

**Example 5.5.4.** Define  $f : [a, b] \rightarrow \mathbb{R}$  by

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

Then  $f(x)$  is not Riemann integrable. For any partition  $a = x_0 < x_1 < \cdots < x_n = b$ ,

$$S = \sum_{i=1}^n f(x'_i)(x_i - x_{i-1}) = \begin{cases} b - a & \text{if } x'_i \text{ are all rational} \\ 0 & \text{if } x'_i \text{ are all irrational.} \end{cases}$$

We can always choose  $x'_i$  to be in either case since the rationals and irrationals are both dense in  $\mathbb{R}$ . So there is no  $A \in \mathbb{R}$  such that  $|A - S| < \epsilon$ , no matter how small we take  $\delta$  to be.

**Remark.** The function  $f$  from the previous example is not Riemann integrable, but it is Lebesgue integrable. In fact,

$$L = \int_a^b f(x) dx = 0$$

with respect to the Lebesgue measure. This is because the set of rational numbers  $\mathbb{Q}$  has measure zero.

## 5.3 Properties of the Riemann Integral

**Proposition 5.1.** *We have the following linearity properties of the Riemann integral:*

1. *If  $f, g : [a, b] \rightarrow \mathbb{R}$  are Riemann integrable, then  $f \pm g$  are also integrable and*

$$\int_a^b (f \pm g) dx = \int_a^b f(x) dx \pm \int_a^b g(x) dx$$

2. *For any  $c \in \mathbb{R}$ ,  $cf$  is integrable and*

$$\int_a^b cf dx = c \int_a^b f(x) dx.$$

*Proof.* See textbook, fairly straightforward. □

**Remark.** Since we only discuss Riemann integration in this class, we will sometimes simply say “integrable” instead of “Riemann integrable.”

**Proposition 5.2.** *If  $f : [a, b] \rightarrow \mathbb{R}$  is integrable and  $f(x) \geq 0$ , then*

$$\int_a^b f(x) dx \geq 0.$$

*Proof.* Let

$$A = \int_a^b f(x) dx.$$

Then for any  $\epsilon > 0$ , there exists  $\delta > 0$  such that for any partition of width  $< \delta$ , we have  $|A - S| < \epsilon$ . But

$$S = \sum_{i=1}^n f(x'_i)(x_i - x_{i-1}) \geq 0,$$

Then we have  $A > S - \epsilon \geq -\epsilon$ , so taking  $\epsilon \rightarrow 0$  gives  $A \geq 0$ . □

**Corollary 5.0.1.** *If  $f, g : [a, b] \rightarrow \mathbb{R}$  are integrable and  $f(x) \geq g(x)$  for all  $x \in [a, b]$ , then*

$$\int_a^b f(x) \, dx \geq \int_a^b g(x) \, dx.$$

*Proof.* By linearity,

$$\int_a^b f(x) \, dx - \int_a^b g(x) \, dx = \int_a^b (f(x) - g(x)) \, dx \geq 0$$

since  $f(x) - g(x) \geq 0$  by assumption. □

**Corollary 5.0.2.** *If  $f : [a, b] \rightarrow \mathbb{R}$  is integrable and  $m \leq f(x) \leq M$  for all  $x \in [a, b]$ , then*

$$m(b - a) \leq \int_a^b f(x) \, dx \leq M(b - a).$$

# Lecture 6

## Jan. 25 — Riemann Integrability

### 6.1 Conditions for Integrability

**Lemma 6.1.** *A function  $f : [a, b] \rightarrow \mathbb{R}$  is integrable if and only if for any  $\epsilon > 0$ , there exists  $\delta > 0$  such that  $|S_1 - S_2| < \epsilon$  whenever  $S_1$  and  $S_2$  are Riemann sums for partitions of width less than  $\delta$ .*

*Proof.* ( $\Rightarrow$ ) If  $f$  is integrable, then for any  $\epsilon > 0$ , there exists  $\delta > 0$  such that

$$\left| S - \int_a^b f(x) dx \right| < \frac{\epsilon}{2}$$

for any Riemann sum  $S$  of a partition with width less than  $\delta$ . Then

$$|S_1 - S_2| \leq \left| S_1 - \int_a^b f(x) dx \right| + \left| S_2 - \int_a^b f(x) dx \right| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

as desired.

( $\Leftarrow$ ) Take the special partition into intervals of equal length, with width  $(b-a)/n$ . Pick the middle point in each interval, and let

$$S_n = \sum_{i=1}^n f(x'_i)(x_i - x_{i-1})$$

be the corresponding Riemann sum. Now we check that  $\{S_n\}_{n=1}^\infty$  is a Cauchy sequence. This is because for any  $\epsilon > 0$ , if  $N$  is large enough, then for any  $n, m \geq N$ , we have  $|S_n - S_m| < \epsilon$  if  $1/N < \delta$ . Then  $\{S_n\}_{n=1}^\infty$  converges, so let  $\lim_{n \rightarrow \infty} S_n = A$ . Now for any  $\epsilon > 0$ , there exists  $\delta > 0$  such that for any Riemann sum  $S$  with width  $< \delta$ , if  $1/n < \delta$ , then  $|S_n - S| < \epsilon/2$ . So

$$|S - A| \leq |S_n - S| + |S_n - A| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

if  $n$  is large enough. Thus

$$A = \int_a^b f(x) dx$$

exists and is the Riemann integral of  $f$ . □

**Remark.** Recall the step function  $f : [a, b] \rightarrow \mathbb{R}$  given by

$$f(x) = \begin{cases} 1 & \text{if } x \in (\alpha, \beta) \subseteq [a, b] \\ 0 & \text{if } x \notin (\alpha, \beta). \end{cases}$$



Last time we saw that  $f$  is integrable and that

$$\int_a^b f(x) dx = \beta - \alpha.$$

Now let us consider a more general step function. We call  $f$  a *step function* on  $[a, b]$  if there exists a partition  $x_0 < x_1 < \cdots < x_n$  of  $[a, b]$  such that  $f(x)$  is constant on each subinterval  $(x_{i-1}, x_i)$ .

**Lemma 6.2.** *If  $f : [a, b] \rightarrow \mathbb{R}$  is a step function for a partition  $x_0 < x_1 < \cdots < x_n$  and  $f(x) = c_i$  when  $x \in (x_{i-1}, x_i)$ , then  $f$  is integrable and*

$$\int_a^b f(x) dx = \sum_{i=1}^n c_i(x_i - x_{i-1}).$$

*Proof.* Define

$$\varphi_i(x) = \begin{cases} 1 & \text{if } x \in (x_{i-1}, x_i) \\ 0 & \text{otherwise.} \end{cases}$$

Now let

$$h = f - \sum_{i=1}^n c_i \varphi_i.$$

Then  $h(x)$  is nonzero only at  $\{x_i\}_{i=0}^n$ . Each  $\varphi_i$  is integrable and  $h$  is integrable with

$$\int_a^b h(x) dx = 0,$$

so  $f$  is also integrable and

$$\int_a^b f(x) dx = \sum_{i=1}^n c_i \int_a^b \varphi_i(x) dx = \sum_{i=1}^n c_i(x_i - x_{i-1})$$

by linearity and the integral of a simple step function that we calculated before. □

**Proposition 6.1.** *A function  $f : [a, b] \rightarrow \mathbb{R}$  is integrable if and only if for any  $\epsilon > 0$ , there exist step functions  $f_1, f_2$  such that  $f_1(x) \leq f(x) \leq f_2(x)$  for all  $x \in [a, b]$  and*

$$\int_a^b (f_2 - f_1) dx < \epsilon.$$

*Proof.* ( $\Leftarrow$ ) For any  $\epsilon > 0$ , choose step functions  $f_1, f_2$  such that

$$\int_a^b (f_2 - f_1) dx < \frac{\epsilon}{3}.$$

Then there exists  $\delta > 0$  such that for any partition with width  $< \delta$ , the Riemann sums  $S_1, S_2$  for  $f_1, f_2$  satisfy

$$|S_1 - \int_a^b f_1(x) dx| < \frac{\epsilon}{3} \quad \text{and} \quad |S_2 - \int_a^b f_2(x) dx| < \frac{\epsilon}{3}.$$

So for any partition width  $< \delta$ , the Riemann sum of  $f$  is

$$S = \sum_{i=1}^n f(x'_i)(x_i - x_{i-1}),$$

and  $S_1 \leq S \leq S_2$  since

$$S_1 = \sum_{i=1}^n f_1(x'_i)(x_i - x_{i-1}) \quad \text{and} \quad S_2 = \sum_{i=1}^n f_2(x'_i)(x_i - x_{i-1}).$$

So  $S$  is in the interval  $(S_1, S_2)$ , which has length  $< \epsilon$  by the triangle inequality on the previous results. For any two Riemann sums of  $f$  with partitions of width  $< \delta$ , we have  $|S' - S''| < \epsilon$ . Thus  $f$  is integrable.

( $\Rightarrow$ ) First we show that  $f$  is bounded in  $[a, b]$ . This is because for any  $\epsilon > 0$ , there exists  $\delta > 0$  such that any two Riemann sums  $S_1, S_2$  corresponding to partitions of width  $< \delta$  satisfy  $|S_1 - S_2| < \epsilon$ . Let

$$S_1 = \sum_{i=1}^n f(x'_i)(x_i - x_{i-1}),$$

and replace  $x'_{i_0} \in (x_{i_0-1}, x_{i_0})$  with  $x''_{i_0} \in (x_{i_0-1}, x_{i_0})$ . Keep  $x'_i$  for  $i \neq i_0$ . Define this new Riemann sum to be  $S_2$ . Then

$$|S_2 - S_1| \leq |f(x''_{i_0}) - f(x'_{i_0})|(x_{i_0} - x_{i_0-1}) < \epsilon,$$

so that

$$|f(x''_{i_0})| \leq |f(x'_{i_0})| + \frac{\epsilon}{x_{i_0} - x_{i_0-1}},$$

i.e.  $f$  is bounded in  $(x_{i_0-1}, x_{i_0})$  since  $x''_{i_0}$  was arbitrary. Since we also picked  $i_0$  arbitrarily, we can repeat this for any interval to conclude that  $f$  is bounded in  $[a, b]$ .

Now for any partition  $x_0 < x_1 < \dots < x_n$  with width  $< \delta$ , define

$$m_i = \inf\{f(x) : x \in (x_{i-1}, x_i)\} \quad \text{and} \quad M_i = \sup\{f(x) : x \in (x_{i-1}, x_i)\}.$$

Define the step function

$$f_1(x) = \begin{cases} m_i & \text{if } x \in (x_{i-1}, x_i) \\ \min\{m_1, \dots, m_n\} & \text{if } x = x_i \text{ for } i = 0, \dots, n. \end{cases}$$

Similarly define

$$f_2(x) = \begin{cases} M_i & \text{if } x \in (x_{i-1}, x_i) \\ \max\{M_1, \dots, M_n\} & \text{if } x = x_i \text{ for } i = 0, \dots, n. \end{cases}$$

Observe that  $f_1(x) \leq f(x) \leq f_2(x)$  for any  $x \in [a, b]$  by construction. Now we verify that

$$\int_a^b (f_2 - f_1) dx < \epsilon$$

if  $\delta > 0$  is small enough. This is because for any  $\eta > 0$ , there exists  $x'_i, x''_i \in [x_{i-1}, x_i]$  such that  $f(x'_i) < m_i + \eta$  and  $f(x''_i) > M_i - \eta$ . Then

$$\sum_{i=1}^n (f(x''_i) - f(x'_i))(x_i - x_{i-1}) > \sum_{i=1}^n (M_i - m_i - 2\eta)(x_i - x_{i-1}) = \int_a^b (f_2 - f_1) dx - 2\eta(b - a).$$

If  $\delta > 0$  is small enough, then

$$\sum_{i=1}^n (f(x'_i) - f(x''_i))(x_i - x_{i-1}) < \epsilon$$

since this is a difference of two Riemann sums with partitions of width  $< \delta$ . Thus

$$\int_a^b (f_2 - f_1) dx < \epsilon + 2\eta(b - a).$$

But  $\eta$  was arbitrary, so taking  $\eta \rightarrow 0$  gives the desired result.  $\square$

**Corollary 6.0.1.** *If  $f : [a, b] \rightarrow \mathbb{R}$  is integrable, then it is bounded.*

*Proof.* This was shown in the proof of the previous proposition.  $\square$

**Theorem 6.1.** *If  $f : [a, b] \rightarrow \mathbb{R}$  is continuous, then  $f$  is integrable.*

*Proof.* Since  $f$  is continuous on the compact set  $[a, b]$ , it is uniformly continuous. So for any  $\epsilon > 0$ , there exists  $\delta > 0$  such that for any  $x', x'' \in [a, b]$ , we have  $|f(x') - f(x'')| < \epsilon$  whenever  $|x' - x''| < \delta$ . Now let  $S_1, S_2$  be two Riemann sums with partitions of width  $< \delta$ . Assume without loss of generality that  $S_1, S_2$  are defined over the same partition (we can always combine two partitions to give a finer partition, if necessary). Let

$$S_1 = \sum_{i=1}^n f(x'_i)(x_i - x_{i-1}) \quad \text{and} \quad S_2 = \sum_{i=1}^n f(x''_i)(x_i - x_{i-1}).$$

Then

$$|S_1 - S_2| \leq \sum_{i=1}^n |f(x'_i) - f(x''_i)|(x_i - x_{i-1}) < \epsilon \sum_{i=1}^n (x_i - x_{i-1}) = \epsilon(b - a).$$

Since  $\epsilon > 0$  was arbitrary, we conclude that  $f$  is integrable by Lemma 6.1.  $\square$

## 6.2 The Fundamental Theorem of Calculus

**Theorem 6.2** (Fundamental theorem of calculus). *If  $f : [a, b] \rightarrow \mathbb{R}$  has anti-derivative  $F : [a, b] \rightarrow \mathbb{R}$  and  $f \in \mathcal{R}([a, b])$ ,<sup>1</sup> then*

$$\int_a^b f(x) dx = F(b) - F(a).$$

*Proof.* Since  $f$  is integrable, let

$$A = \int_a^b f(x) dx.$$

For any  $\epsilon > 0$ , there exists  $\delta > 0$  such that for any Riemann sum  $S$  with partition of width  $< \delta$ , we have  $|S - A| < \epsilon$ . Let  $x_0 < x_1 < \dots < x_n$  be a partition of width  $< \delta$ . Then by telescoping,

$$F(b) - F(a) = \sum_{i=1}^n (F(x_i) - F(x_{i-1})) = \sum_{i=1}^n f(x'_i)(x_i - x_{i-1})$$

---

<sup>1</sup>Here  $\mathcal{R}([a, b])$  is the class of Riemann integrable functions on  $[a, b]$ .

by Lagrange's mean value theorem, where  $x'_i \in (x_{i-1}, x_i)$ . Then

$$|F(b) - F(a) - A| = |S - A| < \epsilon,$$

so letting  $\epsilon \rightarrow 0$  gives  $F(b) - F(a) = A$ . □

**Remark.** The fundamental theorem of calculus requires both being Riemann integrable and having an anti-derivative, which do not always overlap. In fact, neither is a subset of the other.

**Example 6.0.1.** The step function

$$f(x) = \begin{cases} -1 & \text{if } 0 \leq x \leq 1 \\ 1 & \text{if } 1 < x \leq 2 \end{cases}$$

is integrable but has no anti-derivative.

**Example 6.0.2.** Define

$$F(x) = \begin{cases} 0 & \text{if } x = 0 \\ x^2 \sin(1/x) & \text{if } x \neq 0. \end{cases}$$

Then we have

$$F'(x) = f(x) = \begin{cases} 0 & \text{if } x = 0 \\ (-2/x)(\cos(1/x^2)) + 2x \sin(1/x^2) & \text{if } x \neq 0. \end{cases}$$

We can check that  $F'(0) = 0$  via the definition of the derivative. Note that  $f$  has an anti-derivative, namely  $F$ . However,  $f$  is not integrable since it is not bounded near  $x = 0$ .

# Lecture 7

## Jan. 30 — Riemann Integrability, Part 2

### 7.1 Conditions for an Anti-Derivative

**Lemma 7.1.** *Let  $c \in (a, b)$ . Then  $f \in \mathcal{R}([a, b])$  if and only if  $f \in \mathcal{R}([a, c])$  and  $f \in \mathcal{R}([c, b])$ . Moreover,*

$$\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx. \quad (*)$$

*Proof.* ( $\Rightarrow$ ) If  $f \in \mathcal{R}([a, b])$ , then for any  $\epsilon > 0$ , there exist two step functions  $f_1, f_2$  such that  $f_1 \leq f \leq f_2$  and

$$\int_a^b (f_2 - f_1) dx < \epsilon.$$

Let  $f_1, f_2$  be the restrictions to  $[a, c]$ . Then still  $f_1 \leq f \leq f_2$  on  $[a, c]$  and

$$\int_a^c (f_2 - f_1) \leq \int_a^b (f_2 - f_1) < \epsilon$$

since  $f_2 - f_1$  is a nonnegative step function. (Note that the desired result is easy to verify for step functions.) So  $f \in \mathcal{R}([a, c])$ , and the same argument works to show that  $f \in \mathcal{R}([c, b])$ .

( $\Leftarrow$ ) If  $f \in \mathcal{R}([a, c])$  and  $f \in \mathcal{R}([c, b])$ , then for any  $\epsilon > 0$ , there exist step functions  $g_1, g_2, h_1, h_2$  such that  $g_1 \leq f \leq g_2$  on  $[a, c]$ ,  $h_1 \leq f \leq h_2$  on  $[c, b]$ , and

$$\int_a^c (g_2 - g_1) dx < \epsilon, \quad \int_c^b (h_2 - h_1) dx < \epsilon.$$

Now define

$$f_i = \begin{cases} g_i & \text{if } x \in [a, c) \\ h_i & \text{if } x \in [c, b] \end{cases}$$

for  $i = 1, 2$ . Then  $f_1 \leq f \leq f_2$  on  $[a, b]$ , and

$$\int_a^b (f_2 - f_1) dx = \int_a^c (g_2 - g_1) dx + \int_c^b (h_2 - h_1) dx < 2\epsilon,$$

so  $f \in \mathcal{R}([a, b])$ . Now to prove (\*), note that  $f \in \mathcal{R}([a, c])$ , so for any  $\epsilon > 0$  there exist Riemann sums  $S_1$  on  $[a, c]$  and  $S_2$  on  $[c, b]$  such that

$$|S_1 - \int_a^c f(x) dx| < \frac{\epsilon}{3}, \quad |S_2 - \int_c^b f(x) dx| < \frac{\epsilon}{3}.$$

Now choose  $\delta > 0$  such that if the Riemann sum  $S$  has partition with width  $< \delta$ , then

$$|S - \int_a^c f(x) dx| < \frac{\epsilon}{3}, \quad |S - \int_c^b f(x) dx| < \frac{\epsilon}{3}, \quad |S - \int_a^b f(x) dx| < \frac{\epsilon}{3}.$$

Now combine  $S_1, S_2$  on  $[a, b]$  to be a Riemann sum  $S = S_1 + S_2$ , so that

$$|S - \int_a^b f(x) dx| < \frac{\epsilon}{3}.$$

By the triangle inequality on the previous results,

$$\left| \int_a^b f(x) dx - \left( \int_a^c f(x) dx + \int_c^b f(x) dx \right) \right| < \epsilon.$$

Since  $\epsilon$  is arbitrarily small, we conclude that

$$\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx$$

as desired. □

**Remark.** The formula  $(*)$  is true for any three numbers  $a, b, c$ , as long as  $f$  is integrable. This is because by convention, if  $a > b$ , then

$$\int_a^b f(x) dx = - \int_b^a f(x) dx.$$

**Theorem 7.1.** If  $f : [a, b] \rightarrow \mathbb{R}$  is continuous, then<sup>1</sup>

$$F(x) = \int_a^x f(\xi) d\xi$$

is an anti-derivative of  $f$ .

*Proof.* For any  $x_0 \in (a, b)$ , we check that  $F'(x_0) = f(x_0)$ . We can compute using Lemma 7.1 that

$$\begin{aligned} \left| \frac{F(x_0 + h) - F(x_0)}{h} - f(x_0) \right| &= \left| \frac{1}{h} \left( \int_a^{x_0+h} f(x) dx - \int_a^{x_0} f(x) dx \right) - f(x_0) \right| \\ &= \left| \frac{1}{h} \int_{x_0}^{x_0+h} f(x) dx - f(x_0) \right| = \left| \frac{1}{h} \int_{x_0}^{x_0+h} (f(x) - f(x_0)) dx \right|. \end{aligned}$$

The last step is from observing

$$f(x_0) = \frac{1}{h} \int_{x_0}^{x_0+h} f(x_0) dx.$$

Since  $f$  is continuous, for any  $\epsilon > 0$ , there exists  $\delta$  such that if  $|x_0 - x| < \delta$ , then  $|f(x) - f(x_0)| < \epsilon$ . This gives

$$\left| \frac{1}{h} \int_{x_0}^{x_0+h} (f(x) - f(x_0)) dx \right| \leq \frac{1}{h} \int_{x_0}^{x_0+h} |f(x) - f(x_0)| dx \leq \frac{\epsilon h}{h} = \epsilon$$

if  $|h| < \delta$ . Thus,

$$\lim_{h \rightarrow 0} \frac{F(x_0 + h) - F(x_0)}{h} - f(x_0) = f(x_0),$$

so we indeed have  $F'(x_0) = f(x_0)$ . □

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<sup>1</sup>Note that this integral is well-defined since any continuous function is integrable, and a continuous function restricted to a subset of its domain, i.e.  $[a, x] \subseteq [a, b]$ , remains continuous.

## 7.2 More Conditions for Integrability

**Definition 7.1.** Let  $f : [a, b] \rightarrow \mathbb{R}$  be bounded and  $x_0 < x_1 < \cdots < x_n$  be a partition of  $[a, b]$ . Define

$$\omega_i = \sup\{|f(x) - f(y)| : x, y \in [x_{i-1}, x_i]\}$$

for  $i = 1, 2, \dots, n$ . We will call this the *oscillation amplitude* of  $f$ .

**Theorem 7.2.** A function  $f : [a, b] \rightarrow \mathbb{R}$  is integrable if and only if for any  $\epsilon > 0$ , there exists  $\delta > 0$  such that for any partition with width  $< \delta$ , we have

$$\sum_{i=1}^n \omega_i \Delta x_i < \epsilon,$$

where  $\Delta x_i = x_i - x_{i-1}$ .

*Proof.* ( $\Leftarrow$ ) For any  $\epsilon > 0$ , choose any two Riemann sums  $S_1, S_2$  over partitions with width  $< \delta$ . Assume without loss of generality that  $S_1$  and  $S_2$  are defined over the same (maybe refined) partition. Let

$$S_1 = \sum_{i=1}^n f(x'_i)(x_i - x_{i-1}), \quad S_2 = \sum_{i=1}^n f(x''_i)(x_i - x_{i-1}).$$

Then we have

$$|S_1 - S_2| \leq \sum_{i=1}^n |f(x'_i) - f(x''_i)| \Delta x_i \leq \sum_{i=1}^n \omega_i \Delta x_i < \epsilon.$$

Then by Lemma 6.1, we conclude that  $f$  is integrable.

( $\Rightarrow$ ) Since  $f$  is integrable, by Lemma 6.1 we have that for any  $\epsilon > 0$ , there exists  $\delta > 0$  such that for any two Riemann sums  $S_1, S_2$  over partitions of width  $< \delta$ , we have  $|S_1 - S_2| < \epsilon$ . In the interval  $[x_{i-1}, x_i]$ , let

$$M_i = \sup_{x \in [x_{i-1}, x_i]} |f(x)|, \quad m_i = \inf_{x \in [x_{i-1}, x_i]} |f(x)|.$$

In particular note that  $\omega_i = M_i - m_i$ . Now for any  $\eta > 0$ , there exist  $x'_i, x''_i \in [x_{i-1}, x_i]$  such that

$$f(x'_i) > M_i - \eta, \quad f(x''_i) < m_i + \eta.$$

Let

$$S_1 = \sum_{i=1}^n f(x'_i) \Delta x_i, \quad S_2 = \sum_{i=1}^n f(x''_i) \Delta x_i.$$

Then we have

$$|S_1 - S_2| \leq \left| \sum_{i=1}^n (f(x'_i) - f(x''_i)) \Delta x_i \right|$$

Note that  $f(x'_i) - f(x''_i) \geq M_i - m_i - 2\eta$  for  $\eta$  sufficiently small. Thus

$$|S_1 - S_2| \geq \sum_{i=1}^n \omega_i \Delta x_i - 2\eta \sum_{i=1}^n \Delta x_i,$$

so that

$$\sum_{i=1}^n \omega_i \Delta x_i \leq |S_1 - S_2| + 2\eta(b - a) < \epsilon + 2\eta(b - a).$$

From here letting  $\eta \rightarrow 0$  gives the desired result. □

**Theorem 7.3.** *A function  $f : [a, b] \rightarrow \mathbb{R}$  is integrable if and only if for any  $\epsilon > 0$ , there exists a partition such that*

$$\sum_{i=1}^n \omega_i(f) \Delta x_i < \epsilon.$$

*Proof.* ( $\Rightarrow$ ) This is immediate from the previous theorem.

( $\Leftarrow$ ) We show that for any  $\epsilon > 0$ , there exists  $\delta > 0$  such that for any partition with width  $< \delta$ , we have

$$\sum_{i=1}^n \omega_i(f) \Delta x_i < \epsilon.$$

This will imply that  $f$  is integrable by the previous theorem. Let  $y_0 < y_1 < \dots < y_m$  be the partition satisfying

$$\sum_{j=1}^m \omega_j(f) \Delta y_j < \epsilon,$$

and choose

$$\delta < \frac{1}{4} \min_{j=1, \dots, m} \Delta y_j.$$

For any partition  $x_0 < \dots < x_n$  with width  $< \delta$ . We divide the intervals  $[x_{i-1}, x_i]$  into two classes. The first case (1) is where  $[x_{i-1}, x_i]$  is contained in one of the  $[y_{j-1}, y_j]$ , and the second case (2) is where  $[x_{i-1}, x_i]$  contains an interior point  $y_j$ . In the first case, we have

$$\sum_{(1)} \omega_i(f) \Delta x_i \leq \sum_{j=1}^m \omega_j(f) \Delta y_j.$$

For the second case, since  $[x_{i-1}, x_i]$  contains an interior point  $y_j$  but  $\delta < \Delta y_j, \Delta y_{j+1}$ , we must have

$$y_{j-1} < x_{i-1} < y_j < x_i < y_{j+1},$$

so that

$$\omega_i(f) \Delta x_i \leq \frac{1}{2} (\omega_j(f) \Delta y_j + \omega_{j+1}(f) \Delta y_{j+1}).$$

This implies

$$\sum_{(2)} \omega_i(f) \Delta x_i \leq \frac{1}{2} \sum_{j=1}^m \omega_j(f) \Delta y_j.$$

Thus

$$\sum_{i=1}^n \omega_i(f) \Delta x_i = \sum_{(1)} \omega_i(f) \Delta x_i + \sum_{(2)} \omega_i(f) \Delta x_i \leq 2 \sum_{j=1}^m \omega_j(f) \Delta y_j < 2\epsilon,$$

so that  $f$  is integrable. □



# Lecture 8

## Feb. 1 — Riemann Integrability, Part 3

### 8.1 Even More Conditions for Integrability

**Example 8.0.1.** If  $f(x)$  is monotone on  $[a, b]$ , then  $f \in \mathcal{R}([a, b])$ .

*Proof.* Suppose  $f(x)$  is monotone increasing on  $[a, b]$  and  $f(x)$  is not constant (since the result is trivial if  $f$  is constant). Then  $f(a) \leq f(x) \leq f(b)$ . For any  $\epsilon > 0$ , for any partition  $x_0 < \cdots < x_n$  with width

$$\delta < \frac{\epsilon}{f(b) - f(a)},$$

we have on  $[x_{i-1}, x_i]$  that  $M_i = f(x_i)$  and  $f(x_{i-1}) = m_i$  since  $f$  is monotone. Then

$$\omega_i(f) = f(x_i) - f(x_{i-1}) = M_i - m_i.$$

Thus

$$\sum_{i=1}^n \omega_i(f) \Delta x_i = \sum_{i=1}^n (f(x_i) - f(x_{i-1})) \Delta x_i < \frac{\epsilon}{f(b) - f(a)} \sum_{i=1}^n (f(x_i) - f(x_{i-1})) = \epsilon$$

since the sum telescopes and comes out to  $f(b) - f(a)$ . Thus  $f$  is integrable.  $\square$

**Theorem 8.1** (Du Bois-Reymond). *Let  $f$  be bounded on  $[a, b]$ . Then  $f \in \mathcal{R}([a, b])$  if and only if for any  $\epsilon, a > 0$ , there exists a partition such that the total length of subintervals with  $\omega_i(f) \geq \epsilon$  is  $< a$ .*

*Proof.* For any partition  $x_0 < \cdots < x_n$ , split

$$\sum_{i=1}^n \omega_i(f) \Delta x_i = \sum_{(A)} \omega_i(f) \Delta x_i + \sum_{(B)} \omega_i(f) \Delta x_i$$

where (A) is over subintervals with width  $\omega_i(f) < \epsilon$  and (B) is over subintervals with width  $\omega_i(f) \geq \epsilon$ .

( $\Rightarrow$ ) Let

$$\Omega = \sup_{x, y \in [a, b]} |f(x) - f(y)|.$$

For any  $\epsilon > 0$ , for

$$\epsilon_1 = \frac{\epsilon}{2(b-a)} \quad \text{and} \quad a = \frac{\epsilon}{2\Omega},$$

by assumption there exists a partition  $x_0 < \cdots < x_n$  such that

$$\begin{aligned} \sum_{i=1}^n \omega_i(f) \Delta x_i &= \sum_{(A)} \omega_i(f) \Delta x_i + \sum_{(B)} \omega_i(f) \Delta x_i \\ &< \frac{\epsilon}{2(b-a)} \sum_{(a)} \Delta x_i + \Omega \sum_{(B)} \Delta x_i < \frac{\epsilon}{2(b-a)} (b-a) + \Omega \frac{\epsilon}{2\Omega} = \epsilon. \end{aligned}$$

So we see that  $f \in \mathcal{R}([a, b])$  as desired.

( $\Rightarrow$ ) If  $f \in \mathcal{R}([a, b])$ , then for any  $\epsilon, a > 0$ , there exists a partition  $x_0 < \cdots < x_n$  such that

$$\sum_{i=1}^n \omega_i(f) \Delta x_i < a\epsilon.$$

Then we have

$$\epsilon \sum_{(B)} \Delta x_i \leq \sum_{(B)} \omega_i(f) \Delta x_i < a\epsilon \implies \sum_{(B)} \Delta x_i < a,$$

which shows the desired result.  $\square$

**Corollary 8.1.1.** *If  $f : [a, b] \rightarrow \mathbb{R}$  is bounded and has only finitely many discontinuity points, then  $f \in \mathcal{R}([a, b])$ .*

*Proof.* Suppose  $f(x)$  has  $p$  discontinuity points on  $[a, b]$  and  $m \leq f(x) \leq M$  for all  $x \in [a, b]$ . Then for any  $\epsilon > 0$ , first (1) we construct  $p$  small open intervals on  $[a, b]$  containing the  $p$  discontinuity points with

$$\text{total length} < \frac{\epsilon}{2(M-m)}.$$

Next (2) for any subintervals in  $[a, b]$  excluding the above  $p$  subintervals,  $f$  is continuous on them, so there exists a partition such that

$$\sum_{(2)} \omega_i(f) \Delta x_i < \frac{\epsilon}{2}.$$

Now combine (1) and (2) to get

$$\sum_{i=1}^n \omega_i(f) \Delta x_i = \sum_{(1)} \omega_i(f) \Delta x_i + \sum_{(2)} \omega_i(f) \Delta x_i < (M-m) \frac{\epsilon}{2(M-m)} + \frac{\epsilon}{2} = \epsilon.$$

Thus  $f \in \mathcal{R}([a, b])$ , as desired.  $\square$

**Example 8.0.2.** Consider

$$f(x) = \begin{cases} \sin(1/x) & \text{if } x \neq 0 \\ A & \text{if } x = 0 \end{cases}$$

for any constant  $A \in \mathbb{R}$ . Then by the previous corollary,  $f \in \mathcal{R}([0, 1])$ .

**Theorem 8.2.** *If  $f, g \in \mathcal{R}([a, b])$ , then  $fg \in \mathcal{R}([a, b])$ .*

*Proof.* Since  $f, g$  are integrable, they are bounded. So assume  $|f|, |g| \leq M$ . Then for any  $\epsilon > 0$ , there exists  $\delta > 0$  such that for any partition of width  $< \delta$ , we have

$$\sum_{i=1}^n \omega_i(f) \Delta x_i < \frac{\epsilon}{2M}, \quad \sum_{i=1}^n \omega_i(g) \Delta x_i < \frac{\epsilon}{2M}.$$

Notice

$$\omega_i(fg) \leq M(\omega_i(f) + \omega_i(g))$$

because

$$\begin{aligned} |f(x)g(x) - f(y)g(y)| &\leq |g(x)||f(x) - f(y)| + |f(y)||g(x) - g(y)| \\ &\leq M(|f(x) - f(y)| + |g(x) - g(y)|). \end{aligned}$$

Taking suprememes over  $x, y \in [x_{i-1}, x_i]$  from here gives  $\omega_i(fg) \leq M(\omega_i(f) + \omega_i(g))$ . Then

$$\sum_{i=1}^n \omega_i(fg) \Delta x_i \leq M \left( \sum_{i=1}^n \omega_i(f) \Delta x_i + \sum_{i=1}^n \omega_i(g) \Delta x_i \right) < M \left( \frac{\epsilon}{2M} + \frac{\epsilon}{2M} \right) = \epsilon.$$

Thus  $fg \in \mathcal{R}([a, b])$  as desired. □

**Theorem 8.3.** *If  $f \in \mathcal{R}([a, b])$ , then  $|f| \in \mathcal{R}([a, b])$  and*

$$\left| \int_a^b f(x) dx \right| \leq \int_a^b |f(x)| dx.$$

*Proof.* Since  $f \in \mathcal{R}([a, b])$ , for any  $\epsilon > 0$  there exists a partition  $x_0 < \cdots < x_n$  such that

$$\sum_{i=1}^n \omega_i(f) \Delta x_i < \epsilon.$$

Since

$$||f(x)| - |f(y)|| \leq |f(x) - f(y)|,$$

taking supremums over  $x, y \in [x_{i-1}, x_i]$  gives  $\omega_i(|f|) \leq \omega_i(f)$ . Then

$$\sum_{i=1}^n \omega_i(|f|) \Delta x_i \leq \sum_{i=1}^n \omega_i(f) \Delta x_i < \epsilon.$$

So we indeed have  $|f| \in \mathcal{R}([a, b])$ . Now observe that  $-|f| \leq f \leq |f|$ . After integrating, we get

$$-\int_a^b |f(x)| dx \leq \int_a^b f(x) dx \leq \int_a^b |f(x)| dx.$$

This immediately implies the desired result. □

**Example 8.0.3** (Cauchy-Schwarz). If  $f, g \in \mathcal{R}([a, b])$ , then

$$\left| \int_a^b f(x)g(x) dx \right| \leq \left( \int_a^b f(x)^2 dx \right)^{1/2} \left( \int_a^b g(x)^2 dx \right)^{1/2}. \quad (*)$$

*Proof.* Let

$$A = \int_a^b f^2 dx, \quad B = \int_a^b |fg| dx, \quad C = \int_a^b g^2 dx.$$

Note that it suffices to show that  $B^2 \leq AC$ , which will imply  $(*)$  by the previous theorem. Then

$$0 \leq \int_a^b (t|f| - |g|)^2 dx = At^2 - 2Bt + C$$

for any  $t \in \mathbb{R}$ . So the discriminant must satisfy  $(2B)^2 - 4AC \leq 0$ , which gives  $B^2 \leq AC$  as desired.  $\square$

**Example 8.0.4** (Riemann-Lebesgue lemma). If  $f \in \mathcal{R}([a, b])$ , then

$$\lim_{\lambda \rightarrow \infty} \int_a^b f(x) \sin(\lambda x) dx = 0.$$

*Proof.* Since  $f \in \mathcal{R}([a, b])$ , for any  $\epsilon > 0$  there exists a partition  $x_0 < \cdots < x_n$  of  $[a, b]$  such that

$$\sum_{i=1}^n \omega_i(f) \Delta x_i < \frac{\epsilon}{2}.$$

Also assume  $|f| \leq M$  on  $[a, b]$  since  $f$  is integrable. Then we choose

$$\lambda > \frac{4nM}{\epsilon}.$$

We can estimate

$$\begin{aligned} \left| \int_a^b f(x) \sin(\lambda x) dx \right| &= \left| \sum_{i=1}^n \int_{x_{i-1}}^{x_i} (f(x) - f(x_i) + f(x_i)) \sin(\lambda x) dx \right| \\ &\leq \sum_{i=1}^n |f(x_i)| \left| \int_{x_{i-1}}^{x_i} \sin(\lambda x) dx \right| + \sum_{i=1}^n \int_{x_{i-1}}^{x_i} \underbrace{|f(x) - f(x_i)|}_{\leq \omega_i(f)} \underbrace{|\sin(\lambda x)|}_{\leq 1} dx \\ &\leq M \sum_{i=1}^n \frac{\overbrace{|\cos(\lambda x_i) - \cos(\lambda x_{i-1})|}^{\leq 2}}{\lambda} + \sum_{i=1}^n \int_{x_{i-1}}^{x_i} \omega_i(f) dx \\ &\leq M \frac{2n}{\lambda} + \sum_{i=1}^n \omega_i(f) \Delta x_i < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

So as  $\lambda \rightarrow \infty$ , the integral goes to 0.  $\square$

**Remark.** Recall that

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is irrational} \\ 1 & \text{if } x \text{ is rational} \end{cases}$$

is not Riemann integrable, but we might expect that this should integrate to 0. The Lebesgue integral will fix this, which was discovered much later.