MATH 4431: Introduction to Topology

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Contents

Lecture 1

Aug. 20 — Review of Metric Spaces

1.1 Metric Spaces

Recall the definition of a *metric space*:

Definition 1.1. Given a set X, a function $d : X \times X \to \mathbb{R}$ is called a *metric* if

(i) (strong positivity) $d(x, y) \geq 0$ for all $x, y \in X$, and $d(x, y) = 0$ if and only if $x = y$,

(ii) (symmetry) $d(x, y) = d(y, x)$,

(iii) and (triangle inequality) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

Example 1.1.1. For any set X, we can define the *discrete metric* by

$$d(x, y) = \begin{cases} 1 & \text{if } x \neq y, \\ 0 & \text{otherwise.} \end{cases}$$

Verify as an exercise that this satisfies the triangle inequality.

Example 1.1.2. The Euclidean metric in $\mathbb{R}^n$ is

$$d(\overline{x}, \overline{y}) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$$

where $\overline{x} = (x_1, \ldots, x_n)$ and $\overline{y} = (y_1, \ldots, y_n)$.

1.2 Open Sets

Definition 1.2. The *open ball* of radius $R > 0$ around $x_0 \in X$ is

$$B_R(x_0) = \{y \in X \mid d(x_0, y) < R\}.$$

Given a set $S \subseteq X$, a point x_0 is called an interior point of S if there exists $r > 0$ such that $B_r(x_0) \subseteq S$. The set S is called *open* if all of its points are interior points.

Proposition 1.1. *The open ball $B_R(x)$ is open.*

Proof. Fix an arbitrary $y \in B_R(x)$, and observe that it suffices to show that y is an interior point. Take $r = R - d(x, y)$, and first note that $r > 0$ since $d(x, y) < R$. Now note that for all $z \in B_r(y)$, we have

$$d(x, z) \leq d(x, y) + d(y, z) < d(x, y) + (R - d(x, y)) = R,$$

so that $z \in B_R(x)$. Thus $B_r(y) \subseteq B_R(x)$, and so y is an interior point. $\square$

Corollary 1.0.1. $B_R(x) = \bigcup_{y \in B_R(x)} B_{r_y}(y)$*, where* $r_y = R - d(x, y)$*.*

Proof. We have $B_{r_y}(y) \subseteq B_R(x)$ for each $y \in B_R(x)$,[1] and so $\bigcup_{y \in B_R(x)} B_{r_y}(y) \subseteq B_R(x)$. For the reverse inclusion simply observe that $y \in B_{r_y}(y) \subseteq \bigcup_{y \in B_R(x)} B_{r_y}(y)$ for each $y \in B_R(x)$. $\square$

Proposition 1.2. *In a metric space* (X, d)*, the following are true:*

(i) $\varnothing, X$ *are open,*

(ii) if $\{S_i\}_{i \in I}$ *are open, then* $\bigcup_{i \in I} S_i$ *is open,*

(iii) and if $\{S_i\}_{i=1}^n$ *are open, then* $\bigcap_{i=1}^n S_i$ *is open.*

Proof. (i) The empty set is open vacuously. To see that X is open, simply take $R = 1$ for any $x \in X$.

(ii) Fix $x \in \bigcup_{i \in I} S_i$ arbitrary, so there exists $i_0 \in I$ with $x \in S_{i_0}$. Since S_{i_0} is open, x is an interior point and thus there exists $r > 0$ such that $B_r(x) \subseteq S_{i_0}$. But then $B_r(x) \subseteq S_{i_0} \subseteq \bigcup_{i \in I} S_i$, so x is an interior point of $\bigcup_{i \in I} S_i$ also and thus $\bigcup_{i \in I} S_i$ is open.

(iii) Now assume $x \in \bigcap_{i=1}^n S_i$. Then for each $1 \leq i \leq n$, there exists $r_i > 0$ such that $B_{r_i}(x) \subseteq S_i$. Then we can choose

$$r = \min\{r_1, \ldots, r_n\} > 0,$$

so that $B_r(x) \subseteq B_{r_i}(x) \subseteq S_i$ for each $1 \leq i \leq n$. Thus $B_r(x) \subseteq \bigcap_{i=1}^n S_i$ and $\bigcap_{i=1}^n S_i$ is open. $\square$

Remark. The above argument for the finite intersection property requires that there are only finitely many r_i. Otherwise it may very well be that $r = \inf\{r_i\} = 0$ and the argument fails.

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[1] Using the argument from the previous proposition.

Lecture 2

Aug. 22 — Topology, Basis, Continuity

2.1 Topological Spaces

Definition 2.1. A *topology* $\mathcal{T} \subseteq \mathcal{P}(X)$ is a collection of sets such that

(i) $\varnothing, X \in \mathcal{T}$,

(ii) for any index set I, if $\{s_i\}_{i \in I} \subseteq \mathcal{T}$, then $\bigcup_{i \in I} s_i \in \mathcal{T}$ (closure under arbitrary union),

(iii) and if $\{s_i\}_{i=1}^n \subseteq \mathcal{T}$, then $\bigcap_{i=1}^n s_i \in \mathcal{T}$ (closure under finite intersection).

A set with a topology, i.e. a pair $(X, \mathcal{T})$, is called a *topological space*. Elements of $\mathcal{T}$ are called *open sets*.

Example 2.1.1. The following are examples of topologies on a set X:

- The trivial topology: $\mathcal{T} = \{\varnothing, X\}$.
- The discrete topology: $\mathcal{T} = \mathcal{P}(X)$.[1]
- If (X, d) is a metric space, then $\mathcal{T} = \{\text{collection of metrically open sets}\}$ is a topology on X.

Remark. Not every topology is induced by a metric. For instance consider the trivial topology on $\mathbb{R}$.

2.2 Basis for a Topology

Definition 2.2. A collection $\mathcal{B} \subseteq \mathcal{P}(X)$ is called a *basis* if

(i) $\bigcup_{b \in \mathcal{B}} b = X$, i.e. $\mathcal{B}$ is a covering of X,

(ii) and if $x \in b_1 \cap b_2$ for any $b_1, b_2 \in B$, then there exists $b_3 \in \mathcal{B}$ such that $x \in b_3$ and $b_3 \subseteq b_1 \cap b_2$.

Theorem 2.1. *Given a set X and a basis $\mathcal{B}$, define*

$$\mathcal{T}_{\mathcal{B}} = \left\{ \bigcup_{i \in I} s_i \mid I \textit{ is any index set and } \{s_i\}_{i \in I} \subseteq \mathcal{B} \right\}.$$

Then $\mathcal{T}_{\mathcal{B}}$ is a topology on X.

Proof. First observe that $\varnothing, X \in \mathcal{T}_{\mathcal{B}}$: Picking $I = \varnothing$ gives $\bigcup_{i \in I} s_i = \varnothing \in \mathcal{T}_{\mathcal{B}}$ and picking $I = \mathcal{B}$ gives $\bigcup_{b \in \mathcal{B}} b = X \in \mathcal{T}_{\mathcal{B}}$ by the covering property of a basis.

[1] Note that the discrete topology is induced by the discrete metric.

Now assume $\{s_i\}_{i\in I} \subseteq \mathcal{T}_{\mathcal{B}}$. For each $i \in I$, we have $s_i \in \mathcal{T}_{\mathcal{B}}$ and so there exists an index set J_i such that $s_i = \bigcup_{j\in J_i} b_j$, where the $b_j \in \mathcal{B}$. Then

$$\bigcup_{i\in I} s_i = \bigcup_{i\in I}\bigcup_{j\in J_i} b_j,$$

which is a union of elements of $\mathcal{B}$ and hence is in $\mathcal{T}_{\mathcal{B}}$.

Finally assume $\{s_i\}_{i=1}^n \subseteq \mathcal{T}_{\mathcal{B}}$. Now as each $s_i \in \mathcal{T}_{\mathcal{B}}$, there exists J_i such that $s_i = \bigcup_{j\in J_i} b_j$. Then

$$\bigcap_{i=1}^n s_i = \bigcap_{i=1}^n \bigcup_{j\in J_i} b_j.$$

Now assume $x \in \bigcap_{i=1}^n s_i = \bigcap_{i=1}^n \bigcup_{j\in J_i} b_j$. For each $1 \le i \le n$, there exists $j_i \in J_i$ such that $x \in b_{j_i}$. Hence $x \in \bigcap_{i=1}^n b_{j_i}$. Now by induction on the intersection property of a basis, we can find $b_x \in \mathcal{B}$ with

$$x \in b_x \subseteq \bigcap_{i=1}^n b_{j_i}$$

Also observe that

$$\bigcap_{i=1}^n b_{j_i} \subseteq \bigcap_{i=1}^n \bigcup_{j\in J_i} b_j = \bigcap_{i=1}^n s_i$$

by construction, so we may write

$$\bigcap_{i=1}^n s_i = \bigcup_{x\in\bigcap_{i=1}^n s_i} b_x \in \mathcal{T}_{\mathcal{B}}$$

as a union of elements of $\mathcal{B}$. □

Definition 2.3. A *subbasis* $\mathcal{B} \subseteq \mathcal{P}(X)$ is a collection of sets such that $\bigcup_{b\in\mathcal{B}} b = X$.

Remark. One may define a basis $\widetilde{\mathcal{B}}$ from a subbasis $\mathcal{B}$ by adding all finite intersections of elements of $\mathcal{B}$. We get the covering property for free and adding the finite intersections gives us the intersection property of a basis.

Example 2.3.1. For $\mathbb{R}$ with the Euclidean metric, the following are bases for the standard topology:

- $\{B_R(x) \mid x \in \mathbb{R}, R > 0\}$.
- $\{B_R(x) \mid x \in \mathbb{R}, R > 0, R \in \mathbb{Q}\}$. For this use the fact that $\mathbb{Q}$ is dense in $\mathbb{R}$.

In particular this shows that a basis for a topology is not unique in general.

2.3 Continuous Functions

Definition 2.4. Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be two topological spaces. A function $f : X \to Y$ is called *continuous* if for any $O \in \mathcal{T}_Y$, we have $f^{-1}(O) \in \mathcal{T}_X$, i.e. the preimage of an open set is open.[2]

Example 2.4.1. Let X be equipped with the trivial topology $\{\varnothing, X\}$ and let $\mathbb{R}$ be equipped with the standard topology. Then the only continuous functions $f : X \to \mathbb{R}$ are the constant functions $f : x \mapsto c$ for fixed $c \in \mathbb{R}$. To see this, observe that

[2]Recall that $f^{-1}(O) = \{x \in X \mid f(x) \in O\}$.

- $x \mapsto c$ is continuous since any open set in $\mathbb{R}$ either contains c or does not, and so the preimage is either X or $\varnothing$.
- Suppose $f(x_1) = y_1$ and $f(x_2) = y_2$. Let $\epsilon = |y_1 - y_2|$ and observe that $x_1 \in f^{-1}(B_\epsilon(y_1))$ while $x_2 \notin f^{-1}(B_\epsilon(y_1))$, so $f^{-1}(B_\epsilon(y_1))$ is not open in X despite $B_\epsilon(y_1)$ being open in $\mathbb{R}$.

Example 2.4.2. Let X have the discrete topology $\mathcal{T} = \mathcal{P}(X)$ and let $\mathbb{R}$ have the standard topology. Then all functions $X \to \mathbb{R}$ are continuous since any preimage is a subset of X and thus in $\mathcal{P}(X)$.

Remark. In a way, the trivial topology has too few open sets while the discrete topology has too many.

Definition 2.5. Two topological spaces $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ are *topologically equivalent* or *homeomorphic* if there exists a bijection $f : X \to Y$ such that f and f^{-1} are continuous.

Remark. A bijective function f being continuous does not necessarily imply that its inverse f^{-1} is.

Example 2.5.1. Consider $(-\pi/2, \pi/2)$ equipped with the Euclidean metric. This is homeomorphic to $\mathbb{R}$ equipped with the Euclidean metric.[3] One homeomorphism is given by $\tan : (-\pi/2, \pi/2) \to \mathbb{R}$.

[3]Note that $(-\pi/2, \pi/2)$ is bounded while $\mathbb{R}$ is not.

Lecture 3

Aug. 27 — Closed Sets, Continuity, the Subspace Topology

3.1 Closed Sets

Definition 3.1. A set $S \subseteq X$ is called a *closed set* if $S^c = X \setminus S$ is open.

Example 3.1.1. In $\mathbb{R}$, observe that $[a, b]^c = (-\infty, a) \cup (b, \infty)$, which is a union of open sets and thus open. Thus the closed intervals $[a, b] \subseteq \mathbb{R}$ are closed.

Remark. This is not a dichotomy. Sets can be both open and closed (*clopen*), or even neither. Trivially, if X is any topological space, then $\varnothing$ and X are both open and closed.

Example 3.1.2. Let $X = \{0, 1\}$ and $\mathcal{T} = \mathcal{P}(X)$. Then $\{0\}$ is both open and closed.

Example 3.1.3. Let $X = \{1, 2, 3\}$ and $\mathcal{T} = \{\varnothing, X, \{1\}, \{1, 2\}\}$. Then $\{2\}$ is neither open nor closed.

Recall the following De Morgan's laws from set theory:

Proposition 3.1 (De Morgan's laws)**.** *Let I be an index set and $\{A_i\}_{i\in I}$ be sets. Then*

$$\left(\bigcup_{i\in I} A_i\right)^c = \bigcap_{i\in I} A_i^c \quad \textit{and} \quad \left(\bigcap_{i\in I} A_i\right)^c = \bigcup_{i\in I} A_i^c.$$

Corollary 3.0.1. *In a topological space $(X, \mathcal{T})$, we have:*

(i) $\varnothing, X$ are closed.

(ii) if $\{A_i\}_{i\in I}$ are closed, then $\bigcap_{i\in I} A_i$ is closed,

(iii) and if $\{A_i\}_{i=1}^n$ are closed, then so is $\bigcup_{i=1}^n A_i$.

This gives a dual characterization of a topology.

Proof. (i) We have $\varnothing^c = X \in \mathcal{T}$ and $X^c = \varnothing \in \mathcal{T}$.

(ii) Note that

$$\left(\bigcap_{i\in I} A_i\right)^c = \bigcup_{i\in I} A_i^c.$$

As each A_i is closed, we have $A_i^c \in \mathcal{T}$ is open, and hence $\bigcup_{i\in I} A_i^c \in \mathcal{T}$ is open. So $\bigcap_{i\in I} A_i$ is closed.

(*iii*) Observe that
$$\left(\bigcup_{i=1}^{n} A_i\right)^c = \bigcap_{i=1}^{n} A_i^c.$$
Each A_i is closed, so A_i^c is open. Thus $\bigcap_{i=1}^n A_i^c$ is open, and so $\bigcup_{i=1}^n A_i$ is closed. □

3.2 Properties of Continuity

Recall that a function $f : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ is continuous if for every $O \in \mathcal{T}_Y$, we have $f^{-1}(O) \in \mathcal{T}_X$.

Theorem 3.1. *A function $f : X \to Y$ is continuous if and only if for every C closed in Y, $f^{-1}(C)$ is closed in X.*

Proof. ($\Rightarrow$) Let $C \subseteq Y$ be closed. Note that
$$f^{-1}(C) = \{x \in X \mid f(x) \in C\},$$
so we have
$$(f^{-1}(C))^c = \{x \in X \mid f(x) \notin C\} = \{x \in X \mid f(x) \in C^c\} = f^{-1}(C^c).$$
Since C is closed, C^c is open and so $f^{-1}(C^c) = (f^{-1}(C))^c$ is open. Thus $f^{-1}(C)$ is closed.

($\Leftarrow$) Assume $S \subseteq Y$ is open. Note that
$$(f^{-1}(S))^c = \{x \in X \mid f(x) \in S\}^c = \{x \in X \mid f(x) \notin S\} = \{x \in X \mid f(x) \in S^c\} = f^{-1}(S^c).$$
Since S is open, S^c is closed and so $f^{-1}(S^c) = (f^{-1}(S))^c$ is closed by assumption. Thus $f^{-1}(S)$ is open, and so we see that f is continuous. □

Theorem 3.2 (Composition theorem)**.** *Let $(X, \mathcal{T}_X)$, $(Y, \mathcal{T}_Y)$, and $(Z, \mathcal{T}_Z)$ be topological spaces. Let*
$$f : X \to Y \quad \text{and} \quad g : Y \to Z$$
be continuous functions. Then $g \circ f : X \to Z$ is continuous.

Proof. Let $S \subseteq Z$ be open. It suffices to show that $(g \circ f)^{-1}(S) \subseteq X$ is open. Note that
$$\begin{aligned}(g \circ f)^{-1}(S) = \{x \in X \mid (g \circ f)(x) \in S\} &= \{x \in X \mid f(x) \in g^{-1}(S)\} \\ &= \{x \in X \mid x \in f^{-1}(g^{-1}(S))\} = f^{-1}(g^{-1}(S)).\end{aligned}$$
Now as g is continuous, $g^{-1}(S)$ is open in Y. Finally as f is continuous, $f^{-1}(g^{-1}(S))$ is open in X. □

Theorem 3.3. *Assume $X = \bigcup_{\alpha \in I} U_\alpha$ for open sets U_α and let $f : X \to Y$. Assume that $f|_{U_\alpha} : U_\alpha \to Y$ is continuous for each $\alpha \in I$. Then f is continuous.*

Proof. Let $S \subseteq Y$ be open, and it suffices to show that $f^{-1}(S)$ is open. Observe that
$$f^{-1}(S) = f^{-1}(S) \cap X = f^{-1}(S) \cap \left(\bigcup_{\alpha \in I} U_\alpha\right) = \bigcup_{\alpha \in I} (f^{-1}(S) \cap U_\alpha) = \bigcup_{\alpha \in I} f|_{U_\alpha}^{-1}(S).$$
The $f|_{U_\alpha}$ are continuous, so each $f|_{U_\alpha}^{-1}(S)$ is open. Thus $f^{-1}(S)$ is open as a union of open sets. □

Theorem 3.4 (Pasting lemma)**.** *Assume X, Y are topological spaces and $A, B \subseteq X$ are open. Suppose $f_1 : A \to Y$ and $f_2 : B \to Y$ are continuous, and that $f_1 \equiv f_2$ on $A \cap B$. Then $f : A \cup B \to Y$ defined by*

$$f(x) = \begin{cases} f_1(x) & \text{if } x \in A, \\ f_2(x) & \text{if } x \in B \end{cases}$$

is continuous.

Proof. Let $S \subseteq Y$ be open, it suffices to show that $f^{-1}(S)$ is open. Observe that

$$f^{-1}(S) = f_1^{-1}(S) \cup f_2^{-1}(S).$$

Both $f_1^{-1}(S)$ and $f_2^{-1}(S)$ are open since f_1 and f_2 are continuous, so $f^{-1}(S)$ is open as their union. □

3.3 Subspace Topology

Definition 3.2. Let $(X, \mathcal{T}_X)$ be a topological space and $S \subseteq X$ a set. The *subspace topology* on S is defined as follows: $O \subseteq S$ is open if there exists $U \subseteq X$ open in X such that $U = O \cap S$.

Example 3.2.1. Let $\mathbb{R}$ be given the metric topology and $S = [0, 1]$.

- The set $[0, 1]$ is not open in $\mathbb{R}$, but it is open in the subspace topology on S since $[0, 1] = S \cap (-1, 2)$.
- The set $[0, 1/2)$ is neither open nor closed in $\mathbb{R}$, but $[0, 1/2) = S \cap (-1/2, 1/2)$, so it is open in S.

Theorem 3.5. *The subspace topology is indeed a topology.*

Proof. Let $(X, \mathcal{T}_X)$ be a topological space and $S \subseteq X$ be given the subspace topology.

(i) We have $S = S \cap X$ and $\varnothing = \varnothing \cap X$, so $S, \varnothing$ are open in S.

(ii) Let $\{U_\alpha\}_{\alpha \in I}$ be open in the subspace topology. Then for every $\alpha \in I$, there exists $O_\alpha \in \mathcal{T}$ such that $U_\alpha = S \cap O_\alpha$. Then

$$\bigcup_{\alpha \in I} U_\alpha = \bigcup_{\alpha \in I} (S \cap O_\alpha) = S \cap \left(\bigcup_{\alpha \in I} O_\alpha \right).$$

The $\{O_\alpha\}_{\alpha \in I}$ are open in X, so their union is open in X. Thus $\bigcup_{\alpha \in I} U_\alpha$ is open in the subspace topology.

(iii) Let $\{U_i\}_{i=1}^n$ be open in the subspace topology. Then there are O_i for $1 \leq i \leq n$ with $U_i = S \cap O_i$. Then we have

$$\bigcap_{i=1}^n U_i = \bigcap_{i=1}^n (S \cap O_i) = S \cap \left(\bigcap_{i=1}^n O_i \right).$$

As the $O_i \in \mathcal{T}$ are open, $\bigcap_{i=1}^n O_i$ is open in X. Thus $\bigcap_{i=1}^n U_i$ is open in the subspace topology. □

Theorem 3.6. *Assume $f : X \to Y$ is a continuous function and $S \subseteq X$ a subspace. Then $f|_S : S \to Y$ is continuous, where S is equipped with the subspace topology.*

Proof. Let $O \subseteq Y$ be an open set, it suffices to show that $f|_S^{-1}(O)$ is open in the subspace topology. But observing that $f|_S^{-1}(O) = f^{-1}(O) \cap S$ immediately shows that $f|_S^{-1}(O)$ is open in S since $f^{-1}(O)$ is open in X due to the continuity of f. □

Remark. The subspace topology is the smallest topology on S such that the inclusion map $i : S \to X$ given by $i(s) = s$ is a continuous function.

Remark. Let X be a topological space with subspaces $Y \subseteq X$ and $Z \subseteq Y$. Then the subspace topology on Z induced by the subspace Y is the same as the subspace topology on Z induced directly by X.

Remark. A topological space can have a subspace homeomorphic to itself. For instance, consider $\mathbb{R}$ and $(-\pi/2, \pi/2)$ with a homemorphism given by the tangent function.

Lecture 4

Aug. 29 — Connectedness

4.1 Connected Spaces

Definition 4.1. A *separation* of a topological space X is two open, nonempty sets $U, V \subseteq X$ such that $X = U \cup V$ and $U \cap V = \varnothing$. A space is called *connected* if there is no separation of the space.

Proposition 4.1. *If X is separated, i.e. $X = U \cup V$ with U, V open and disjoint, then U and V are both open and closed.*

Proof. Observe that U is open by assumption, and we have

$$U^c = X \setminus U = V,$$

which is also open by assumption. Hence U is closed. The case for V is identical. □

Example 4.1.1. Consider the following:

- The singleton space $\{x\}$ is connected. There are no two nonempty, disjoint open sets.
- Consider the space $X = \{0, 1\}$. This case depends on the choice of topology:
    1. With the trivial topology $\mathcal{T} = \{\varnothing, X\}$, the space is connected.
    2. With the discrete topology $\mathcal{T} = \{\varnothing, X, \{1\}, \{0\}\}$, X is disconnected since $X = \{0\} \cup \{1\}$.
    3. With the topology $\mathcal{T} = \{\varnothing, X, \{1\}\}$, the space is connected. The only nonempty sets $\{1\}, X$ are not disjoint and thus there can be no separation.

Theorem 4.1. *A space X is disconnected if and only if there exists a surjective map $f : X \to \{0, 1\}$ with the discrete topology.*

Proof. ($\Rightarrow$) If X is disconnected, then we may write $X = U \cup V$ with U, V open, disjoint, and nonempty. Then define

$$f(x) = \begin{cases} 0 & x \in U, \\ 1 & x \in V, \end{cases}$$

which is surjective as U, V are nonempty. To see that f is continuous, observe that

$$f^{-1}(\varnothing) = \varnothing, \quad f^{-1}(\{0,1\}) = X, \quad f^{-1}(\{0\}) = U, \quad f^{-1}(\{1\}) = V,$$

each of which are open. These are all of the open sets in the discrete topology, so f is continuous.

($\Leftarrow$) Assume there exists a surjective and continuous map $f : X \to \{0, 1\}$. Define

$$U = f^{-1}(\{0\}) \quad \text{and} \quad V = f^{-1}(\{1\}),$$

which are open since f is continuous. Observe that $U, V \neq \varnothing$ since f is surjective. Also $U \cap V = \varnothing$ since if there is any $x \in U \cap V$, then $f(x) = 0$ as $x \in U$ and $f(x) = 1$ as $x \in V$, a contradiction. Finally, $X = U \cup V$ since $f(x) = 0$ or $f(x) = 1$ for every $x \in X$, i.e. $x \in U$ or $x \in V$. So X is disconnected. □

4.2 Connected Sets

Definition 4.2. Let X be a topological space and $S \subseteq X$. Then S is called *connected* if it is connected in the subspace topology.

Theorem 4.2. *If A, B are connected sets and $A \cap B \neq \varnothing$, then $A \cup B$ is connected.*

Proof. Assume not. Then there exists a continuous, surjective map $f : A \cup B \to \{0, 1\}$ with the discrete topology. Consider $f|_A : A \to \{0, 1\}$, which is continuous in the subspace topology. Notice that $f(A)$ cannot be $\{0, 1\}$ since otherwise A is disconnected. Without loss of generality, assume $f(A) = \{0\}$ since A is nonempty. Now consider $f|_B : B \to \{0, 1\}$, which is also continuous. Similarly, notice that $f(B)$ cannot be $\{0, 1\}$. But there exists $p \in A \cap B$, and $f(p) = 0$ as $p \in A$. Then since $p \in B$, we must have $f(B) = \{0\}$. But then we get that $f(A \cup B) = \{0\} \neq \{0, 1\}$, a contradiction to surjectivity. □

Corollary 4.2.1. *A union of connected sets with "common points" is connected.*

Proof. Run induction (transfinite if the union is infinite) using the previous theorem. □

Theorem 4.3. *Closed intervals in $[a, b] \subseteq \mathbb{R}$ with the metric topology are connected.*

Proof. Assume otherwise that $[a, b] = U \cup V$ with U, V disjoint, open, and nonempty. Assume without loss of generality that $a \in U$. Since V is nonempty, there exists $c > a$ such that $c \in V$. Now consider $[a, c] \subseteq [a, b]$ with $U_1 = U \cap [a, c]$ and $V_1 = V \cap [a, c]$. By the least upper bound property of $\mathbb{R}$, since U_1 is nonempty and bounded from above, there exists $s = \sup U_1$ with $s \leq c$. Now either $s \in U_1$ or $s \notin U_1$.

If $s \in U_1$ (note this implies $s \neq c$), then s is an interior point of U_1 since U_1 is open. So one may find a point t such that $t > s$ and $t \in U_1$. But then s is no longer an upper bound of U_1, a contradiction.

Otherwise $s \notin U_1$. Since U_1, V_1 cover $[a, c]$, we must then have $s \in V_1$ (note this implies $s \neq a$. Since V_1 is open, s is an interior point of V_1, and thus there exists $t < s$ such that $t \in V_1$ and t is an upper bound for U_1. This contradicts s being the least upper bound of U_1.

Since both cases lead to contradictions, we conclue that $[a, b]$ must be connected. □

Corollary 4.3.1. *Open intervals in $\mathbb{R}$ are connected, and $\mathbb{R}$ itself is connected.*

Proof. For some $N_0 \geq 1$ (for instance choose $N_0 \geq 2/(b - a)$) we can write

$$(a, b) = \bigcup_{n=N_0}^{\infty} \left[a + \frac{1}{n}, b - \frac{1}{n}\right],$$

Each of these closed intervals is connected by the previous theorem, and thus the union is connected by Corollary 4.2.1 since they overlap. Similarly writing $\mathbb{R} = \bigcup_{n=1}^{\infty} [-n, n]$ shows that $\mathbb{R}$ is connected. □

Corollary 4.3.2 (Intermediate value theorem)**.** *Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then for any $f(a) < t < f(b)$, there exists $c \in [a, b]$ such that $f(c) = t$.*

Proof. Assume not. We can consider the open sets $(-\infty, t)$ and (t, ∞) in $\mathbb{R}$. Then $f^{-1}((-\infty, t))$ and $f^{-1}((t, \infty))$ are open sets since f is continuous. They are clearly disjoint (since f must be well-defined), and also nonempty since $a \in f^{-1}((-\infty, t))$ and $b \in f^{-1}((t, \infty))$. Also since $f^{-1}(\{t\}) = \varnothing$ by assumption,

$$[a, b] = f^{-1}((-\infty, t)) \cup f^{-1}((t, \infty)).$$

But this is a separation of $[a, b]$, a contradiction since $[a, b]$ is connected. □

Proposition 4.2. *The open interval $(0, 1)$ is not homeomorphic to the closed interval $[0, 1]$.*

Proof. Removing any point from $(0, 1)$ disconnects it, but $[0, 1) = [0, 1] \setminus \{1\}$ remains connected.[1] □

Proposition 4.3. *The real line $\mathbb{R}$ is not homeomorphic to the plane $\mathbb{R}^n$ for any $n \geq 2$.*

Proof. Removing a point from $\mathbb{R}$ disconnects it but the same is not true for $\mathbb{R}^n$ when $n \geq 2$. □

[1] To see that $[0, 1)$ is connected, we can write $[0, 1) = \bigcup_{n=2}^{\infty} [0, 1 - 1/n]$.

Lecture 5

Sept. 3 — Path-Connectedness

5.1 More on Connectedness

Remark. The intervals $[a, b] \subseteq \mathbb{R}$ are homeomorphic to $[0, 1]$ for any $a < b$. We can take $f : [a, b] \to [0, 1]$ defined by

$$f(x) = \frac{1}{b-a}(x - a)$$

for instance as a homemorphism.

Lemma 5.1. *If X is connected and $f : X \to Y$ is continuous, then $f(X)$ is connected.*

Proof. This is part of Homework 2. □

Corollary 5.0.1. *The plane $\mathbb{R}^2$ is connected.*

Proof. Express $\mathbb{R}^2$ as the union of horizontal and vertical lines. Each line is the image of $\mathbb{R}$ and is thus connected by Lemma 5.1. Also any pair of horizontal and vertical lines must intersect, so we can use Corollary 4.2.1 to conclude that the union $\mathbb{R}^2$ is connected. □

Remark. We can extend this to $\mathbb{R}^3$ by embedding planes (copies of $\mathbb{R}^2$), and similarly for $\mathbb{R}^n$.

Proposition 5.1. *The unit circle $\mathbb{S}^1 \subseteq \mathbb{R}^2$ is connected.*

Proof. Define $\gamma : [0, 2\pi] \to \mathbb{R}^2$ by $\gamma(t) = (\cos t, \sin t)$. The image of γ is precisely $\mathbb{S}^1$. □

Proposition 5.2. *Define a relation $\sim$ on X by $x \sim y$ if there exists a connected subset $S \subseteq X$ such that $x, y \in S$. Then $\sim$ is an equivalence relation.*

Proof. For reflexivity, fix $x \in X$ and let S be the largest connected set containing x (this exists since we know at least $\{x\}$ must be connected). Then $x \in S$, so $x \sim x$.

For symmetry, fix $x, y \in X$. If $x \sim y$, then there exists a connected set S such that $x, y \in S$. But then $y, x \in S$, so we see that $y \sim x$.

For transitivity, assume that $x \sim y$ and $y \sim z$. Then there exists S_1 connected such that $x, y \in S_1$ and S_2 connected such that $y, z \in S_2$. Notice that $S_1 \cap S_2 \neq \varnothing$ since $y \in S_1 \cap S_2$. Then $S_1 \cup S_2$ is connected by Theorem 4.2 and $x, y, z \in S_1 \cap S_2$. In particular, $x, z \in S_1 \cap S_2$ and thus $x \sim z$.

So we see that $\sim$ is an equivalence relation. □

Definition 5.1. Let the equivalence relation $\sim$ be defined on X as in Proposition 5.2. Then we can write X as the disjoint union of the equivalence classes of $\sim$. These equivalence classes are called the *connected components* of X.

Remark. The connected components of a space are defined solely via topologies, so they must be invariant under homeomorphism.

Example 5.1.1. The letter S, sitting in $\mathbb{R}^2$, is not homeomorphic to the letter T. There is a point we can remove from T to give three connected components, but removing any point from S gives at most two such connected components.

5.2 Path-Connectedness

Remark. Connectedness is usually a very difficult property to verify. This motivates *path-connectedness*.

Definition 5.2. A set S is *path-connected* if for all $x, y \in S$, there exists a continuous map $\gamma : [0,1] \to S$ such that $\gamma(0) = x$ and $\gamma(1) = y$. Here $[0,1]$ is given the usual metric topology.

Lemma 5.2. *If S is path-connected, then S is connected.*

Proof. This is part of Homework 2. □

Remark. Unlike connectedness, it is immediately obvious that $\mathbb{R}^n$ is path-connected. Simply take the line segment between any two points. Then we can conclude connectedness by the previous lemma.

Example 5.2.1. There are spaces which are connected but not path-connected.

- Consider the *topologist's sine curve*, given by the union of the vertical segment $\{(0, y) \mid -1 \leq y \leq 1\}$ and the image of $(0, \infty)$ under $x \mapsto (x, \sin(1/x))$, is an example of such a space. See Homework 2 for more details.
- Consider the *cone* C in $\mathbb{R}^2$ defined by ($(0,1)$ denotes an open interval unless otherwise specified)

  $$C = ([0,1] \times \{0\}) \cup (K \times [0,1]) \cup (\{0\} \times [0,1]),$$

  where $K = \{1/n : n \in \mathbb{N}\}$. Note that C is clearly path-connected and hence also connected. Then define the space

  $$D = C \setminus (\{0\} \times (0,1)),$$

  which is now not path-connected (consider the point $(0,1) \in D$) but still connected.

Remark. Observe the following:

- One can define *path-connected components* in a similar manner as connected components.
- A continuous image of a path-connected space is path-connected. Simply compose the curve with the continuous map, which is now a path in the image.
- The union of path-connected spaces sharing a point is path-connected. Take two curves to the common point and concatenate them using the pasting lemma.
- In $\mathbb{R}^n$, connectedness is equivalent to path-connectedness. In general, this holds if you can get a basis of only connected sets.

Remark. Recall from homework that if $f : [0, 1] \to [0, 1]$ is continuous, then f has a fixed point, i.e. there exists $c \in [0, 1]$ with $f(c) = c$. This follows from a clever use of the intermediate value theorem. Now consider a more topological perspective. Consider the diagonal $\{(x, x) \mid x \in [0, 1]\}$ and look at the graph of f, which is contained in the closed unit square. This graph is path-connected as the image of a path-connected set and so there is a path between the points $(0, f(0))$ and $(1, f(1))$. But then this path must intersect the diagonal at some point, which gives a fixed point.

Theorem 5.1. *(Brouwer fixed point theorem) Let K be a closed, bounded, and convex set in $\mathbb{R}$. Then any continuous map $f : K \to K$ has a fixed point, i.e. there exists $c \in K$ such that $f(c) = c$.*

Remark. One can see the existence of the Nash equilibrium as a consequence of this theorem.

Remark. In $\mathbb{R}^2$, this theorem follows from the following claim. Let $X = \mathrm{maps}(\mathbb{S}^1, \mathbb{S}^1)$ be the set of all continuous maps from $\mathbb{S}^1$ to itself. Then Brouwer's fixed point theorem in $\mathbb{R}^2$ follows from the following:

Theorem 5.2. *The space* $\mathrm{maps}(\mathbb{S}^1, \mathbb{S}^1)$ *is not path connected.*

Lecture 6

Sept. 5 — Compactness

6.1 Note on the Subspace Topology

Remark. Let X be a topological space with topology $\mathcal{T}_X$, and let $Y \subseteq X$ be a subset endowed with a topology τ. Suppose that for any continuous $f : X \to Z$, there exists a continuous $\widetilde{f} : Y \to Z$ such that the following diagram commutes,

$$\begin{tikzcd} X \arrow[r, "f"] & Z \\ Y \arrow[u, "i"] \arrow[ur, dashed, "\widetilde{f}"'] & \end{tikzcd}$$

where $i : Y \to X$ is the inclusion map.[1] Then in Homework 2 we showed that $\mathcal{T}_Y \subseteq \tau$. We can see this as a *universal property* for the subspace topology.

6.2 Compactness

Definition 6.1. A set $C \subseteq X$ is called *compact* if for any *open cover*

$$C \subseteq \bigcup_{\alpha \in I} U_\alpha, \quad \text{each } U_\alpha \text{ is open,}$$

there exists a finite subcover $C \subseteq \bigcup_{i=1}^{n} U_{\alpha_i}$.

Example 6.1.1. Consider the following:

- In a finite topology, any set is compact. This is because any open cover is already finite.
- In a discrete space, i.e. $\mathcal{T} = \mathcal{P}(X)$, compact sets are precisely the finite sets. It is clear that finite sets are compact, for each x choose a single open set in the cover containing x. Conversely, if a set is compact, we can pick our open cover to contain only singletons, and the existence of a finite subcover means that the set has only finitely many elements.

Theorem 6.1 (Heine-Borel)**.** *Let $C \subseteq \mathbb{R}^n$ be a subset, where $\mathbb{R}^n$ is given the metric topology. Then C is compact if and only if C is closed and bounded.*

Proof. We postpone this proof until later. □

Lemma 6.1. *Let X be compact. If $Y \subseteq X$ is closed, then Y is compact.*

[1] Note that at least set-theoretically, this immediately defines $\widetilde{f} = f|_Y$. But a priori we do not know that $\widetilde{f}$ is continuous.

Proof. Let $Y \subseteq X$ closed be given, and assume that $Y \subseteq \bigcup_{\alpha \in I} U_\alpha$ an open cover. Since Y is closed, its complement Y^c is open. Then

$$Y^c \cup \bigcup_{\alpha \in I} U_\alpha$$

is an open cover of X since $X = Y \cup Y^c$. Since X is compact, there exists a finite subcover

$$X \subseteq Y^c \cup \bigcup_{i=1}^{n} U_{\alpha_i}.$$

Now observe that $Y \subseteq X$ and $Y \cap Y^c = \varnothing$, so actually $Y \subseteq \bigcup_{i=1}^n U_{\alpha_i}$, which is a finite subcover. □

Theorem 6.2. *Let X be compact and $f : X \to Y$ continuous. Then $f(X)$ is compact.*

Proof. Consider $f(X) \subseteq Y$ and let $f(X) \subseteq \bigcup_{\alpha \in I} V_\alpha$, an open cover in Y. Notice that

$$X = f^{-1}(f(X)) \subseteq f^{-1}\left(\bigcup_{\alpha \in I} V_\alpha\right) = \bigcup_{\alpha \in I} f^{-1}(V_\alpha).$$

Note that each $f^{-1}(V_\alpha)$ is open in X since f is continuous and V_α is open in Y, so this is in fact an open cover of X. Thus since X is compact, we may extract a finite subcover

$$X \subseteq \bigcup_{i=1}^{n} f^{-1}(V_{\alpha_i}).$$

Then we see that

$$f(X) \subseteq f\left(\bigcup_{i=1}^{n} f^{-1}(V_{\alpha_i})\right) \subseteq \bigcup_{i=1}^{n} V_{\alpha_i},$$

which is a finite subcover of $f(X)$. Therefore $f(X)$ is compact. □

Theorem 6.3. *Assume $\{C_j\}_{j=1}^m$ are compact subsets of X. Then $\bigcup_{j=1}^m C_j$ is compact.*

Proof. Assume $\bigcup_{j=1}^m C_j \subseteq \bigcup_{\alpha \in I} U_\alpha$, an open cover. Observe this is also an open cover of C_j for each $1 \le j \le m$, so we can extract a finite subcover, i.e. we can find $\alpha_{j,1}, \dots, \alpha_{j,n_j}$ with

$$C_j \subseteq \bigcup_{i=1}^{n_j} U_{\alpha_{j,i}}.$$

Then we see that

$$\bigcup_{j=1}^{m} C_j \subseteq \bigcup_{j=1}^{m} \bigcup_{i=1}^{n_j} U_{\alpha_{j,i}},$$

which is still a finite union. This is then a finite subcover of $\bigcup_{j=1}^m C_j$, so $\bigcup_{j=1}^m C_j$ is compact. □

Theorem 6.4 (Weierstrass)**.** *Let $f : [a, b] \to \mathbb{R}$ be continuous. Then $f([a, b])$ is bounded, and moreover there exist $x_{\max}, x_{\min} \in [a, b]$ such that $f(x_{\max}) \ge f(x) \ge f(x_{\min})$ for all $x \in [a, b]$.*

Proof. Since f is continuous and $[a, b]$ is compact (by Heine-Borel), $f([a, b]) \subseteq \mathbb{R}$ is compact. Thus by Heine-Borel, $f([a, b])$ is bounded. In particular, we can find M, m such that

$$m \leq f(x) \leq M \quad \text{for all } x \in [a, b].$$

For the second part, observe that $f([a, b])$ is bounded and nonempty, so $s = \sup f([a, b])$. Since this is the supremum, there must exist $y_i \in f([a, b])$ such that $y_i \to s$ as $i \to \infty$. Now observe that $f([a, b])$ is closed by Heine-Borel, and in particular it contains its limit points. Thus we obtain $s \in f([a, b])$. Then pick $x_{\max} \in f^{-1}(\{s\}) \subseteq [a, b]$, which will satisfy $f(x_{\max}) = s \geq f(x)$ for all $x \in [a, b]$. by construction.

The argument for finding $x_{\min} \in [a, b]$ is similar. □

Theorem 6.5. *Let X be a topological space, $K \subseteq X$ compact, and $f : K \to \mathbb{R}$ a continuous function. Then f is bounded over K and attains its minimum and maximum on K.*

Proof. The same argument goes through, replacing $[a, b]$ by the compact set K. □

Lecture 7

Sept. 10 — More Compactness

7.1 The Cantor Set

Define $I_0 = [0, 1]$ and remove the open middle-thirds interval to get

$$I_1 = [0, 1] \setminus (1/3, 2/3) = [0, 1/3] \cup [2/3, 1].$$

Continue by removing the middle thirds of each interval to get $I_2, I_3, \ldots$. Then the *Cantor set* is defined to be $K = \bigcap_{I \geq 0} I_n$. The Cantor set is compact and uncountable. See more on Homework 3.

7.2 The Heine-Borel Theorem

Theorem 7.1 (Heine-Borel)**.** *If $C \subseteq \mathbb{R}$, then C is compact if and only if C is closed and bounded.*

Proof. ($\Rightarrow$) This direction is easy, see Homework 3 for details.

($\Leftarrow$) First we show that $[a, b] \subseteq \mathbb{R}$ is compact. Let $\{U_\alpha\}_{\alpha \in I}$ be an open cover for $[a, b]$, i.e. $[a, b] \subseteq \bigcup_{\alpha \in I} U_\alpha$. Now define

$$R = \{x \in [a, b] \mid [a, x] \text{ has a finite subcover}\}$$

Clearly $a \in R$ since $a \in [a, b] \subseteq \bigcup_{\alpha \in I} U_\alpha$, so picking any single U_α with $a \in U_\alpha$ gives a finite subcover for $[a, a] = \{a\}$. The goal is now to show that $b \in R$. Observe that $a \in R$ implies $R = \varnothing$, and $R \subseteq [a, b]$, so

$$s = \sup R$$

exists by the completeness of $\mathbb{R}$. We proceed to show that $s \in R$ and then $s = b$, which will show that $b \in R$. As $s \in [a, b] \subseteq \bigcup_{\alpha \in I} U_\alpha$, we can find α_s such that $s \in U_{\alpha_s}$. Since U_{α_s} is open, we can find $\delta > 0$ such that $(s - \delta, s + \delta) \subseteq U_{\alpha_s}$. Then since s is a least upper bound of R, we can find $r \in R$ such that $s - \delta < r \leq s$. Now since $r \in R$, $[a, r]$ admits a finite subcover $\{U_{\alpha_i}\}_{i=1}^n$. Then

$$[a, s] = [a, r] \cup (s - \delta, s] \subseteq \left(\bigcup_{i=1}^n U_{\alpha_i} \right) \cup U_{\alpha_s}$$

is a finite subcover for $[a, s]$, so $s \in R$. Now observe that we actually covered

$$\left[a, s + \frac{\delta}{2}\right] = [a, r] \cup (s - \delta, s + \delta) \subseteq \left(\bigcup_{i=1}^n U_{\alpha_i} \right) \cup U_{\alpha_s}$$

in the previous construction. Then $s + \delta/2 \in R$, which contradicts the minimality of s unless $s = b$. Thus $b \in R$, so $[a, b]$ admits a finite subcover and thus $[a, b]$ is compact.

Now let $C \subseteq \mathbb{R}$ be an arbitrary closed and bounded set. Since C is bounded, there exists $I = (a, b)$ such that $C \subseteq I$. But then $C \subseteq I \subseteq \overline{I} = [a, b]$, so C is a closed subset of a compact set, hence compact. □

Remark. The Heine-Borel theorem also holds more generally in $\mathbb{R}^n$. A later theorem will say that the product of compact sets is compact in the product topology, and thus we can run the same argument as above but with boxes in $\mathbb{R}^n$ instead of intervals.

7.3 The Bolzano-Weierstrass Theorem

Definition 7.1. A point x is an *accumulation point* for a set S if for all open sets U containing x, we have $(U \setminus \{x\}) \cap S \neq \varnothing$.

Remark. We disallow constant sequences when talking about accumulation points.

Proposition 7.1. *Let* $\mathrm{Acc}(A)$ *be the set of accumulation points of a set* A*. Then* $\overline{A} = A \cup \mathrm{Acc}(A)$*.*

Proof. We show that $A \cup \mathrm{Acc}(A)$ is closed, which will imply $\overline{A} \subseteq A \cup \mathrm{Acc}(A)$ by the minimality of the closure. Write

$$(A \cup \mathrm{Acc}(A))^c = A^c \cap \mathrm{Acc}(A)^c.$$

Now assume $x \in A^c \cap \mathrm{Acc}(A)^c$. Since $x \notin \mathrm{Acc}(A)$, there exists U_x open such that $x \in U_x$ and

$$(A \setminus \{x\}) \cap U_x = \varnothing.$$

But also $x \notin A$, so $A \setminus \{x\} = A$ and $A \cap U_x = \varnothing$. Then we can write

$$(A \cup \mathrm{Acc}(A))^c = A^c \cap \mathrm{Acc}(A)^c = \bigcup_{x \in A^c \cap \mathrm{Acc}(A)^c} U_x.$$

This is a union of open sets, hence open, and so $A \cup \mathrm{Acc}(A)$ is closed.

For the other direction, assume $x \in A \cup \mathrm{Acc}(A)$. If $x \in A$, we are done, so assume $x \in \mathrm{Acc}(A) \setminus A$. Now assume otherwise that $x \notin \overline{A}$. Then $x \in (\overline{A})^c$, which is open. Set $U = (\overline{A})^c$, so that

$$U \cap (A \setminus \{x\}) = U \cap A = \varnothing.$$

But then this says that x is not an accumulation point, in contradiction. □

Definition 7.2. We say that a topological space X is *sequentially compact* if every bounded sequence has a convergent subsequence.

Theorem 7.2 (Bolzano-Weierstrass)**.** *Any bounded infinite set* $S \subseteq \mathbb{R}^n$ *has an accumulation point.*

Proof. Since S is bounded, find a compact set containing S. Then apply the later Theorem 7.4. □

Remark. In general, compactness is *not* equivalent to sequential compactness, but both imply the Bolzano-Weierstrass theorem. However, in many spaces (including metric spaces, in particular), the two notions coincide (and are also equivalent to the Bolzano-Weierstrass theorem).

Theorem 7.3. *A sequentially compact space has the Bolzano-Weierstrass property, namely that any bounded infinite set has an accumulation point.*

Proof. This is easy, pick a countable subset (i.e. a sequence) and apply sequential compactness. □

Theorem 7.4. *A compact space has the Bolzano-Weierstrass property, namely that any infinite set has an accumulation point.*

Proof. Let A be an infinite set in X, where X is compact. Assume otherwise that A has no accumulation points in X. Then there is no accumulation point for A outside of A, so $\mathrm{Acc}(A) \subseteq A$. This gives

$$\overline{A} = A \cup \mathrm{Acc}(A) = A,$$

so A is closed. Thus A is a closed subset of a compact space, hence compact. Now for any $a \in A$, pick an open set U_a such that $a \in U_a$ and $U_a \cap (A \setminus \{a\}) = \varnothing$. Write $A \subseteq \bigcup_{a \in A} U_a$, and by compactness we can find a finite subcover $A \subseteq \bigcup_{i=1}^n U_{a_i}$. Then observe that

$$A = A \cap \bigcup_{i=1}^{n} U_{a_i} = \bigcup_{i=1}^{n} (A \cap U_{a_i}) = \bigcup_{i=1}^{n} \{a_i\} = \{a_1, \ldots, a_n\},$$

This is in contradiction with A being infinite. □

Remark. Usually, this proof goes by showing that compactness implies sequential compactness, which then implies the Bolzano-Weierstrass property. But this proof avoids going through convergent sequences.

Lecture 8

Sept. 12 — Separation Axioms

8.1 Separation Axioms

Definition 8.1. A topological space is said to satisfy the T_0 *axiom* if the following holds: For every $a, b \in X$ with $a \neq b$, there exists U open such that either $a \in U$, $b \notin U$ or $b \in U$, $a \notin U$.

Remark. With the T_0 axiom, we cannot choose which point is in U and which is not. For instance take $X = \{a, b\}$ with topology $\mathcal{T} = \{\varnothing, \{a\}, X\}$. This space is T_0, but we can only choose U to contain a.

Definition 8.2. A space is said to satisfy the T_1 *axiom* if for every $a, b \in X$ with $a \neq b$, there exist U_a, U_b open such that $a \in U_a$, $b \notin U_a$ and $b \in U_b$, $a \notin U_b$.

Remark. With the T_1 axiom, U_a and U_b need not be disjoint.

Definition 8.3. A space is said to be T_2 or *Hausdorff* if the following holds: For every $a, b \in X$ with $a \neq b$, there exist U_a, U_b open such that $a \in U_a$, $b \in U_b$ and $U_a \cap U_b = \varnothing$.

Example 8.3.1. Metric spaces are Hausdorff. For any $a \neq b$, pick balls with radius $d(a, b)/2$ around a, b.

Theorem 8.1. *We have the proper inclusion $T_2 \subsetneq T_1 \subsetneq T_0$.*

Proof. The inclusions and $T_0 \neq T_1$ is clear (e.g. above). For $T_1 \neq T_2$ take the *line with two origins*[1]. $\square$

Theorem 8.2. *In a T_1 space, every singleton $\{x\}$ is closed.*

Proof. Fix $x \in X$. For every $y \neq x$, by the T_1 axiom we can find U_y open such that $y \in U_y$ and $x \notin U_y$. In particular, this means that $U_y \subseteq \{x\}^c$. Then we can write

$$\{x\}^c \subseteq \bigcup_{y \in \{x\}^c} U_y \subseteq \{x\}^c.$$

So $\{x\}^c = \bigcup_{y \in \{x\}^c} U_y$, which is open as a union of open sets. Thus $\{x\}$ is closed. $\square$

8.2 Properties of Hausdorff Spaces

Theorem 8.3. *In a Hausdorff space, a point x is an accumulation point of a set A if and only if every neighborhood of x contains infinitely many elements of A.*

[1] The *line with two origins* is $X = \mathbb{R} \cup \{p\}$ with topology generated by the open sets in $\mathbb{R}$ (with the metric topology), and adding $\widetilde{U} = (U \setminus \{0\}) \cup \{p\}$ for each open set $U \subseteq \mathbb{R}$ containing 0. One can separate 0 and p but not with disjoint sets.

Proof. ($\Leftarrow$) This is clear.

($\Rightarrow$) Pick x an accumulation point of A, and assume otherwise that there exists a neighborhood U of x with only finitely many elements of A, i.e. $|(U \setminus \{x\}) \cap A| < \infty$. Then we can write

$$(U \setminus \{x\}) \cap A = \{a_1, \dots, a_n\} = \bigcup_{i=1}^{n} \{a_i\}.$$

Since our space is Hausdorff and thus also T_1, these singletons $\{a_i\}$ are closed. Then $(U \setminus \{x\}) \cap A$ is closed as a finite union of closed sets. Now since our space is Hausdorff, for every $1 \le i \le n$ we can separate a_i from x, i.e. there exists U_{x_i}, U_{a_i} open such that $x \in U_{x_i}$, $a_i \in U_{a_i}$ and $U_{x_i} \cap U_{a_i} = \varnothing$. Then

$$U' = U \cap \bigcap_{i=1}^{n} U_{x_i}$$

is open as a finite intersection of open sets. Also $x \in U'$ since $x \in U$ and $x \in U_{x_i}$ for each i. But

$$(U \setminus \{x\}) \cap A = \{a_1, \dots, a_n\} \subseteq \bigcup_{i=1}^{n} U_{a_i}$$

and $U_{a_j} \cap \bigcap_{i=1}^{n} U_{x_i} = \varnothing$ for all j, so $(U' \setminus \{x\}) \cap A = \varnothing$. Contradiction. $\square$

Remark. Maybe just the T_1 axiom is enough for this theorem. Think more about this.

Definition 8.4. A sequence $\{x_n\}_{n=1}^{\infty} \subseteq (X, \mathcal{T})$ *converges* to a point $x \in X$, written $x_n \to x$, if for any open set U containing x, there exists $N_0 \in \mathbb{N}$ such that $x_n \in U$ for every $n \ge N_0$.

Theorem 8.4. *In a Hausdorff space, a convergent sequence has a unique limit.*

Proof. Assume otherwise that $x_n \to L_1$ and $x_n \to L_2$ with $L_1 \ne L_2$. Then since our space is Hausdorff, we can find U_{L_1}, U_{L_2} open such that $L_1 \in U_{L_1}$, $L_2 \in U_{L_2}$ and $U_{L_1} \cap U_{L_2} = \varnothing$. Since $x_n \to L_1$, there exists $N_0 \in \mathbb{N}$ such that $x_n \in U_{L_1}$ for all $n \ge N_0$. Similarly we can find $N_0' \in \mathbb{N}$ with $x_n \in U_{L_2}$ for all $n \ge N_0'$ since $x_n \to L_2$. But then for $N = \max\{N_0, N_0'\}$, we have $x_N \in U_{L_1} \cap U_{L_2}$, a contradiction. $\square$

Theorem 8.5. *In a Hausdorff space, every compact set is closed.*

Proof. Let $C \subseteq X$ be compact, and we show that C^c is open. So fix $y \in C^c$. For any $x \in C$, since our space is Hausdorff, we can find U_x, U_y open such that $x \in U_x$, $y \in U_y$ and $U_x \cap U_y = \varnothing$. Now consider $\bigcup_{x \in C} U_x$. This is an open cover of C, so we can find a finite subcover $C \subseteq \bigcup_{i=1}^{n} U_{x_i}$ since C is compact. Then the finite intersection $\bigcap_{i=1}^{n} U_{y_i}$ is an open set contain y, and it is disjoint from C by construction since $U_{x_i} \cap U_{y_i} = \varnothing$ for each i. Now set $\widetilde{U}_y = \bigcap_{i=1}^{n} U_{y_i}$, so that

$$C^c \subseteq \bigcup_{y \in C^c} \widetilde{U}_y \subseteq C^c.$$

Thus $C^c = \bigcup_{y \in C^c} \widetilde{U}_y$, which is open as the union of open sets, so we conclude that C is closed. $\square$

Lecture 9

Sept. 17 — Compactification

9.1 Motivation for Compactification

Let $(X, \mathcal{T})$ be a topological space which is not compact. Usually we assume X is Hausdorff, and the goal is to find a compact space which looks like X, i.e. compactify X.

Remark. The naive idea is to take the trivial topology on X in place of $\mathcal{T}$, getting X_{trivial}. This is compact, the identity map $\text{id} : X \to X_{\text{trivial}}$ is continuous and bijective, but it is not a homeomorphism. This is bad because we forget all the topological structure on X, for instance every sequence converges to every point in X_{trivial}. We would like to compactify X while keeping as much structure as possible.

Example 9.0.1. Let $X = (0, 1)$ with the metric topology. Take $Y = [0, 1]$ with the metric topology, so X embeds into Y by the inclusion map.[1] Note that Y is compact by Heine-Borel.

Example 9.0.2. Let $X = (0, 1)$ with the metric topology. Take $Y = \mathbb{S}^1 \subseteq \mathbb{R}^2$ to be the unit circle, where $\mathbb{R}^2$ has the metric topology. Then X embeds into Y by the stereographic projection (technically $\mathbb{R}$ is embedded but $\mathbb{R}$ is homeomorphic to $(0, 1)$ by the arctangent) by adding one point at the north pole.

Example 9.0.3. For the open unit disk in $\mathbb{R}^2$, we can add its boundary to get the closed unit disk as a compactification (the closed unit disk is compact by Heine-Borel). This adds uncountably many points. An alternative is to add only a single point at infinity, and identify the boundary with this point.

9.2 One-Point Compactification

Definition 9.1. For a topological space X, the *one-point compactification* (or *Alexandroff compactification*) of X is the set $X^+ = X \cup \{\infty\}$ with topology generated by the basis

$$\{U \subseteq X \text{ open}\} \cup \{K^c \mid K \subseteq X \text{ is compact}\}.$$

Remark. One way to think about the one-point compactification is that we are forcing all unbounded sequences in X to converge to to the new point ∞.

Theorem 9.1. *For a Hausdorff topological space X, its one-point compactification X^+ is compact.*

Proof. Let $\{O_\alpha\}$ be an open cover of X^+, i.e. $X^+ = \bigcup_{\alpha \in I} O_\alpha$. Some open set must contain the point ∞, and each open set contains a basis element, so there exists $\alpha' \in I$ such that $O_{\alpha'}$ contains K^c, for some $K \subseteq X$ compact. Now since $K \subseteq X \subseteq X^+$, we see that $K \subseteq \bigcup_{\alpha \in I}(O_\alpha \cap X)$ is an open cover of K

[1] By X *embeds into* Y, we mean that there is a continuous injection from X to Y.

(note that the Hausdorff condition implies that every compact K is closed in X, and thus the K^c basis elements are open in X). So by compactness, there exists a finite subcover $K \subseteq \bigcup_{\alpha \in I_{\text{finite}}} (O_\alpha \cap X)$. Then

$$X^+ = O_{\alpha'} \cup K \subseteq O_{\alpha'} \cup \bigcup_{\alpha \in I_{\text{finite}}} (O_\alpha \cap X) \subseteq \bigcup_{\alpha \in I_{\text{finite}}} O_\alpha$$

since $K^c \subseteq O_{\alpha'}$. This is a finite subcover, so X is compact. $\square$

Remark. In analysis, we sometimes speak of a sequence diverging to ∞, i.e. the sequence eventually escapes any compact set. This is precisely convergence to ∞ in the one-point compactification.

Theorem 9.2. *Assume X is a Hausdorff, locally compact but not compact topological space.*[2] *Then the inclusion map* $\mathrm{id} : X \to X^+$ *is a dense, continuous embedding.*[3]

Proof. First clearly $\mathrm{id} : X \to \mathrm{id}(X) \subseteq X^+$ is injective. Now we show that id is continuous. It is enough to show that the preimage of basis elements of X^+ is open in X. If $U \subseteq X$ is open, then $\mathrm{id}^{-1}(U) = U \subseteq X$ is clearly open. Otherwise consider $K^c \cup \{\infty\}$ for $K \subseteq X$ compact. Then we have

$$\mathrm{id}^{-1}(K^c \cup \{\infty\}) = K^c \subseteq X.$$

Since X is Hausdorff, the compact set K is closed, and so K^c is open. Thus id is continuous.

Finally we show density, i.e. $\overline{X} = X^+$, where the closure is taken in X^+. To do this, suppose otherwise that $\overline{X} \neq X^+$. Clearly $X \subseteq \overline{X}$, so if $\overline{X} \neq X^+$, we must have $\overline{X} = X$ (since $X^+ = X \cup \{\infty\}$). But then X is closed in X^+, which is compact, so X is also compact in X^+ since X and thus X^+ is Hausdorff (since X is locally compact). This implies (exercise) that X itself is compact. Contradiction. $\square$

Remark. If X is already compact, then $X^c = \{\infty\}$ is open in X^+. Obviously X^+ is compact since X is and every sequence converging to ∞ must eventually be constant. In particular, X^+ must be disconnected, and the extra point ∞ just sits there to the side.

Remark. Due to the above, we must assume that X is not compact in order to get a dense embedding.

Example 9.1.1. Consider the space $X = [0, 1)$. Then the one-point compactification of X is $[0, 1]$.

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[2]A space X is *locally compact* if for every $x \in X$, there exists U open with $x \in U$ such that $\overline{U}$ is compact.

[3]The embedding is *dense* if $\overline{X} = X^+$.

Lecture 10

Sept. 19 — Miscellaneous Topics

10.1 Local Compactness and One-Point Compactification

Definition 10.1. A map $f : X \to Y$ is called *open* if for every $U \subseteq X$ open, $f(U)$ is open. A *topological embedding* is an injective, continuous, and open map.

Remark. This ensures that f with codomain restricted to $f(U)$ is a homeomorphism.

Example 10.1.1. Demanding only that f is injective and continuous is not enough. Let $(X, \mathcal{T})$ be any topological space and let X_{trivial} be X with the trivial topology. Then the identity map $\mathrm{id} : X \to X_{\text{trivial}}$ is injective, continuous, but not open is general if $\mathcal{T}$ is not trivial. So this is *not* a topological embedding.

Theorem 10.1. *If X is Hausdorff and locally compact, then $X^+ = X \cup \{\infty\}$ is Hausdorff.*

Proof. Pick $y, y' \in X^+$ with $y \neq y'$. If $y, y' \in X$, then since X is Hausdorff, there are disjoint open sets $U, U' \subseteq X$ with $y \in U$ and $y' \in U$. But then $U, U' \subseteq X^+$ are still open and disjoint in X^+ since the identity map embeds X into X^+. Thus U, U' are disjoint open sets separating y, y'.

Now assume without loss of generality that $y' = \infty$. By local compactness, there is a open set $U \subseteq X$ containing y with compact closure. Then observe that $(\overline{U})^c \cup \{\infty\}$ is open in X^+ since $\overline{U}$ is compact. Since U is also open in X^+ and clearly disjoint from $(\overline{U})^c \cup \{\infty\}$, we have separated y, ∞. □

10.2 Distance to a Closed Set

Definition 10.2. Let (X, d) be a metric space. For $x \in X$ and $A \subseteq X$, the *distance from x to A* is

$$d(x, A) = \inf\{d(x, a) : a \in A\}.$$

Remark. This infimum is achieved in $\mathbb{R}^n$, but not necessarily in a more general metric space.

Proposition 10.1. *For any $x \in \mathbb{R}^n$ and closed set $A \subseteq \mathbb{R}^n$, there exists $a \in A$ with $d(x, A) = d(x, a)$.*

Proof. Pick $a_0 \in A$ and note that $d(x, a_0) < \infty$. Then set $R = 2d(x, a_0) > 0$. Consider $\overline{B_R(x)} \cap A$, which is a closed and bounded set containing a_0, hence compact by Heine-Borel. Now apply the compact case to $\overline{B_R(x)} \cap A$ to get $a_{\min}$ with $d(x, a_{\min}) = d(x, \overline{B_R(x)} \cap A)$. Now any $a \in A$ with $a \notin \overline{B_R(x)}$ satisfies

$$d(x, a) \geq R > d(x, a_0) \geq d(x, a_{\min}),$$

and hence it cannot be the minimum. Thus we must have $d(x, a_{\min}) = d(x, A)$. □

Remark. Define

$$\ell^2(\mathbb{N}) = \left\{ \{a_n\}_{n=1}^\infty \subseteq \mathbb{R} : \sum_{n=1}^\infty a_n^2 < \infty \right\}.$$

This is an inner product space over $\mathbb{R}$, and in particular we can induce a metric

$$d(\{a_n\}, \{b_n\}) = \sqrt{\sum_{n=1}^\infty |a_n - b_n|^2}$$

to turn (X, d) into a complete metric space. Also notice that $\{e_i\}_{i=1}^\infty$, where e_i is the sequence with 1 in the ith position and 0 everywhere else, is an orthonormal basis for $\ell^2(\mathbb{N})$.

Lemma 10.1. *Define the set*

$$A = \left\{ \left(1 + \frac{1}{i}\right) e_i \right\}_{i=1}^\infty .$$

Then $d(\{0\}, A) = 1$. In particular, this is an example of a metric space where the infimum of the distance from a point to a closed set is not achieved.

Proof. Observe that

$$d\left(\{0\}, (1+1/i)e_i\right) = 1 + \frac{1}{i},$$

and so

$$d(\{0\}, A) = \inf\{d(\{0\}, (1+1/i)e_i : i \in \mathbb{N}\} = \inf\left\{1 + \frac{1}{i} : i \in \mathbb{N}\right\} = 1.$$

In particular, this infimum is clearly not achieved since $d(\{0\}, (1+1/i)e_i) = 1 + 1/i > 1$ for each $i \in \mathbb{N}$. Now we show that A is closed by showing that it contains its limit points. For this, first observe that $d(a, a') \geq 1$ for any $a, a' \in A$ with $a \neq a'$. To see this, we can compute that

$$d\left((1+1/i)e_i, (1+1/j)e_j\right) = \sqrt{\left(1 + \frac{1}{i}\right)^2 + \left(1 + \frac{1}{j}\right)^2} \geq \sqrt{2} \geq 1$$

whenever $i \neq j$. Now assume we have $\{a_n\} \subseteq A$ with $a_n \to x$. In particular, $\{a_n\}$ must be a Cauchy sequence, and so for $\epsilon = 1/2$, there exists $N_0 \in \mathbb{N}$ such that for all $n, m \geq N_0$, we have $d(a_n, a_m) < 1/2$. But $d(a, a') \geq 1$, so the sequence must stabilize after N_0, and hence $x = a_n$ for $n \geq N_0$. In particular, $x \in A$, so we conclude that A is closed. This finishes the example. $\square$

10.3 Nested Intersections in Compact Hausdorff Spaces

Proposition 10.2. *Let X be a compact Hausdorff space, and $Y_i \subseteq X$ be closed and connected for $i \in I$. Assume the $\{Y_i\}$ are totally ordered, i.e. $Y_i \subseteq Y_j$ or $Y_j \subseteq Y_i$ for all $i, j \in I$. Then $\bigcap_{i \in I} Y_i$ is connected.*

Proof. Let U, V be a separation of $Y = \bigcap_{i \in I} Y_i$. Then $Y = U \cup V$ and U, V are open in Y, disjoint, and nonempty. In particular, we can find U', V' open in X such that $U = U' \cap Y$ and $V = V' \cap Y$. However, U', V' may no longer separate Y. This is why we need the Hausdorff condition. Use the next lemma to fix the proof from here, see more details in Homework 4. $\square$

Lemma 10.2. *In a compact Hausdorff space, if C_1, C_2 are two compact disjoint sets, then there exist U_1, U_2 open and disjoint such that $C_1 \subseteq U_1, C_2 \subseteq U_2$ and $U_1 \cap U_2 = \varnothing$.*

Proof. First we show this in the case where $C_1 = \{x\}$ is a singleton and $C_2 = C$. For all $y \in C$, consider the pair x and y. Then there exists $U_{x,y}, V_{x,y}$ open such that $x \in U_{x,y}$, $y \in V_{x,y}$, and $U_{x,y} \cap V_{x,y} = \varnothing$. Observe that $\bigcup_{y \in C} V_{x,y}$ is an open cover of C, so by compactness there exists a finite subcover $C \subseteq V = \bigcup_{i=1}^{n} V_{x,y_i}$. Then $x \in U = \bigcap_{i=1}^{n} U_{x,y_i}$, which is open as a finite intersection of open sets. Also each U_{x,y_i} is disjoint from V_{x,y_i}, so U is disjoint from V. Then $U_1 = U$ and $U_2 = V$ are the desired open sets.

Now let C_1, C_2 be any two disjoint compact sets. For all $x \in C_1$, there are open sets U_{x,C_2}, V_{x,C_2} such that $x \in U_{x,C_2}$, $C_2 \subseteq V_{x,C_2}$, and $U_{x,C_2} \cap V_{x,C_2} = \varnothing$. Then we make the same argument. We have

$$C_1 \subseteq \bigcup_{x \in C_1} U_{x,C_2},$$

an open cover of C_1, so by compactness there is a finite subcover $C_1 \subseteq U_1 = \bigcup_{i=1}^{m} U_{x_i,C_2}$. Then

$$C_2 \subseteq U_2 = \bigcap_{i=1}^{m} V_{x_i,C_2},$$

which is open as a finite intersection of open sets. As before, $U_1 \cap U_2 = \varnothing$ since each U_{x_i,C_2} is disjoint from V_{x_i,C_2}. Thus we get a separation by disjoint compact sets. $\square$

Lecture 11

Sept. 24 — Product Spaces

11.1 Product Spaces

Definition 11.1. For two sets X, Y, define their *Cartesian product* as

$$X \times Y = \{(x, y) \mid x \in X, y \in Y\}.$$

Similarly we can define[1]

$$\prod_{i \in I} X_i = \{(x) \mid (x)_i \in X_i\}.$$

If each X_i is a topological space with topology $\mathcal{T}_i$, then we define the following topologies on $\prod_{i \in I} X_i$:

- The *box topology*: Take as a basis sets of the form $\prod_{i \in I} U_i$ where $U_i \subseteq X_i$ are open sets.

  Suppose B_1, B_2 are basis sets and $x \in B_1 \cap B_2$. Then by definition $B_1 = \prod_{i \in I} U_i$ and $B_2 \in \prod_{i \in I} V_i$, where $U_i, V_i \subseteq X_i$ are open. Then set $B_3 = \prod_{i \in I}(U_i \cap V_i)$. Clearly $x \in B_3$ and $B_3 \subseteq B_1 \cap B_2$ is a basis element (each $U_i \cap V_i$ is a finite intersection of open sets and thus open), so this is a basis.

- The *product topology*: Take as a subbasis the sets $\pi_i^{-1}(U_i) \subseteq \prod_{i \in I} X_i$ for each $i \in I$ and $U_i \subseteq X_i$ open. Here $\pi_i : \prod_{j \in I} X_j \to X_i$ is the projection onto the ith factor.

  The subbasis sets here are of the form $U_i \times \prod_{j \neq i} X_j$. The general basis sets will be finite intersections of these sets, i.e.

  $$U_{i_1} \times \cdots \times U_{i_n} \times \prod_{j \neq i_1, \ldots, i_n} X_j.$$

  These are called the *cylindrical sets*. Think of this as having only finitely many restrictions on the factors, whereas we get to choose arbitrarily many restrictions with the box topology.

Remark. For finite products, the box and product topologies coincide. They differ for infinite products: The box topology is finer than the product topology (so the box topology has more open sets).

Example 11.1.1. Consider the following product spaces:

- The power set $\mathcal{P}(X) = 2^X = \{0, 1\}^X$. Think of the elements as functions $X \to \{0, 1\}$, which pick whether or not to include each element of X in the corresponding subset. Thus the power set comes with a natural topology (box or product) if $\{0, 1\}$ is given the discrete topology.

- The space $\{0, 1\}^{\mathbb{N}}$ is the Cantor set, if $\{0, 1\}$ is given the discrete topology. The Cantor set with the metric topology inherited from $\mathbb{R}$ is homeomorphic to $\{0, 1\}^{\mathbb{N}}$ with the product topology. Think of the sequence $\{x_n\} \subseteq \{0, 1\}^{\mathbb{N}}$ as choosing whether to pick the left or right third at each step.

[1] One can also think of an element of the product as a function $I \to \bigcup_{i \in I} X_i$, where $f(i) \in X_i$.

- The space $[0,1]^{\mathbb{N}}$ is called *Hilbert's cube*. Note that $[0,1] \times [0,1]^{\mathbb{N}} \cong [0,1]^{\mathbb{N}}$. For a homeomorphism, simply shift the sequence one to the right, putting a 0 in the first slot. Then forget about the 0.

Remark. Always assume $\prod_{i \in I} X_i$ is given the product topology, unless otherwise specified.

11.2 Properties of Product Spaces

Theorem 11.1. *Assume $(X_i, \mathcal{T}_i)_{i \in I}$ are each T_0, each T_1, or each Hausdorff. Then the product $\prod_{i \in I} X_i$ is also T_0, T_1, or Hausdorff, respectively.*

Proof. We prove only the Hausdorff case. Let $(x), (y) \in \prod_{i \in I} X_i$ be distinct. As $(x) \neq (y)$, there is $i \in I$ with $x_i \neq y_i$, where $x_i, y_i \in X_i$. Since X_i is Hausdorff, there exist $A, B \subseteq X_i$ open, disjoint with $x_i \in A$ and $y_i \in B$. Then set

$$U = A \times \prod_{j \neq i} X_j \quad \text{and} \quad V = B \times \prod_{j \neq i} X_j.$$

These sets are open since they are cylindrical, and clearly $(x) \in U$, $(y) \in V$ since $x_i \in A$, $y_i \in B$. Also U, V are disjoint since their ith components A, B are disjoint. So this is a separation of (x) and (y) by disjoint, open sets, and we conclude that $\prod_{i \in I} X_i$ is Hausdorff. □

Corollary 11.1.1. *Assume $(X_i, \mathcal{T}_i)_{i \in I}$ are each T_0, each T_1, or each Hausdorff. Then $\prod_{i \in I} X_i$ with the box topology is also T_0, T_1, or Hausdorff, respectively.*

Proof. The box topology is finer than the product topology. □

Remark. In the product topology, the projections $\pi_i : \prod_{j \in I} X_j \to X_i$ are continuous, onto, and open. The continuity and surjectivity of π_i is essentially by construction of the product topology. To see that π_i is open, consider a basis element of $\prod_{j \in I} X_j$, which is of the form

$$U = U_{i_1} \times \cdots \times U_{i_n} \times \prod_{j \neq i_1, \ldots, i_n} X_j.$$

Then $\pi_i(U)$ is either one of the U_i or one of the X_i, which are both open.

Theorem 11.2 (Universal property of the product topology)**.** *The following diagram commutes:*

$$\begin{tikzcd} Z \arrow[r, "f_i"] \arrow[dr, dashed, "f"'] & X_i \\ & \prod_{i \in I} X_i \arrow[u, "\pi_i"'] \end{tikzcd}$$

In particular, there exists a unique continuous map $f : Z \to \prod_{i \in I} X_i$ such that $f_i = \pi_i \circ f$ for each $i \in I$.

Proof. Define $f : Z \to \prod_{i \in I} X_i$ on the set theory level by $(f(z))_i = f_i(z)$. Now we show the continuity of f. Fix a basis element

$$B = U_{i_1} \times \cdots \times U_{i_n} \times \prod_{j \neq i_1, \ldots, i_n} X_j.$$

Then we can write

$$f^{-1}(B) = \{z \mid f(z) \in B\}.$$

Now observe that $f(z) \in B = U_{i_1} \times \cdots \times U_{i_n} \times \prod_{j \neq i_1, \ldots, i_n} X_j$ is equivalent to

$$z \in \bigcap_{k=1}^{n} f_{i_k}^{-1}(U_{i_k}).$$

Since U_{i_k} is open in X_{i_k} and each f_i is continuous, each $f_{i_k}^{-1}(U_{i_k})$ is open in Z. Then this is a finite intersection of open sets in Z, hence open. Thus f is continuous. $\square$

Remark. If we had the box topology, this argument would not work. In particular, the intersection that we get could be infinite, which would not necessarily be open in Z.

Remark. This universal property formalizes the notion that we define functions from products by defining a function on each factor. Additionally, this generalizes the result from multivariable calculus that a vector-valued function is continuous precisely when each component function is continuous.

Lecture 12

Sept. 26 — Products and Topological Properties

12.1 Connectedness and Path-Connectedness

Theorem 12.1. *Assume that each X_i is path-connected, then $\prod_{i\in I} X_i$ is path-connected.*

Proof. Fix $x, y \in \prod_{i\in I} X_i$, and define $x_i = \pi_i(x)$ and $y_i = \pi_i(y)$. Since $x_i, y_i \in X_i$ and X_i is path-connected, there exists $\gamma_i : [0,1] \to X_i$ continuous such that $\gamma_i(0) = x_i$ and $\gamma_i(1) = y_i$. By the universal property of products, there exists $\gamma : [0,1] \to \prod_{i\in I} X_i$ continuous such that $\pi_i \circ \gamma = \gamma_i$. This implies

$$(\gamma(0))_i = \gamma_i(0) = x_i,$$

so $\gamma(0) = x$. Similarly $\gamma(1) = y$, so γ is a path from x to y. This says that $\prod_{i\in I} X_i$ is path-connected. □

Theorem 12.2. *If $\{X_i\}$ are connected, then $\prod_{i\in I} X_i$ is connected.*

Proof. Suppose otherwise that $\prod_{i\in I} X_i$ is not connected, i.e. there exists a separation of $\prod_{i\in I} X_i$. So let U, V be two disjoint, nonempty, open subsets of $\prod_{i\in I} X_i$ such that $U \cup V = \prod_{i\in I} X_i$.

First we claim that there exist $x \in U$ and $y \in V$ such $x_i \neq y_i$ only at a single index i. To see this, note that U and V contain basis elements $U' \subseteq U$ and $V' \subseteq V$. Then these basis elements look like

$$U' = U_{i_1} \times \cdots \times U_{i_n} \times \prod_{j \neq i_1, \ldots, i_n} X_j \quad \text{and} \quad V' = V_{j_1} \times \cdots \times V_{j_m} \times \prod_{k \neq j_1, \ldots, j_m} X_k.$$

Then clearly we may choose $x \in U'$ and $y \in V'$ such that they differ in only finitely many coordinates. Then we have

$$x = (x_1, \ldots, x_n, \ldots) \quad \text{and} \quad y = (y_1, \ldots, y_n, \ldots),$$

where $x_i = y_i$ for $i > n$. Now consider x_i, y_i for $1 \leq i \leq n$. If $x_i = y_i$, then do nothing. Otherwise if $x_i \neq y_i$, define

$$y' = (x_1, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots).$$

Then $y' \in \prod_{i\in I} X_i = U \cup V$, so either $y' \in U$ or $y' \in V$ since $U \cap V = \varnothing$. If $y' \in V$, continue with $y = y'$, and if $y' \in U$, change $x = y'$. Do this for the finitely many $1 \leq i < n$, and we obtain $x_i = y_i$ except for a single index i. Assume without loss of generality that $x_1 \neq y_1$.

Now define a map $f : X_1 \to \prod_{i\in I} X_i$ by $f(\widetilde{x}) = (\widetilde{x}, x_2, x_3, \ldots)$. Note that f is continuous by the universal property of products (the component maps $X_1 \to X_i$ are either the identity if $X_i = X_1$ or constant otherwise). Then $f(X_1)$ is connected since X_1 is connected and f is continuous. But now $x \in f(X_1) \cap U$ and $y \in f(X_1) \cap V$, so $U \cap V \neq \varnothing$. Contradiction. □