MATH 4431: Introduction to Topology

Frank Qiang
Instructor: Asaf Katz

Georgia Institute of Technology
Fall 2024

Contents

Lecture 1

Aug. 20 — Review of Metric Spaces

1.1 Metric Spaces

Recall the definition of a *metric space*:

Definition 1.1. Given a set X, a function $d : X \times X \to \mathbb{R}$ is called a *metric* if

(i) (strong positivity) $d(x, y) \geq 0$ for all $x, y \in X$, and $d(x, y) = 0$ if and only if $x = y$,

(ii) (symmetry) $d(x, y) = d(y, x)$,

(iii) and (triangle inequality) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

Example 1.1.1. For any set X, we can define the *discrete metric* by

$$d(x, y) = \begin{cases} 1 & \text{if } x \neq y, \\ 0 & \text{otherwise.} \end{cases}$$

Verify as an exercise that this satisfies the triangle inequality.

Example 1.1.2. The Euclidean metric in $\mathbb{R}^n$ is

$$d(\overline{x}, \overline{y}) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$$

where $\overline{x} = (x_1, \ldots, x_n)$ and $\overline{y} = (y_1, \ldots, y_n)$.

1.2 Open Sets

Definition 1.2. The *open ball* of radius $R > 0$ around $x_0 \in X$ is

$$B_R(x_0) = \{y \in X \mid d(x_0, y) < R\}.$$

Given a set $S \subseteq X$, a point x_0 is called an interior point of S if there exists $r > 0$ such that $B_r(x_0) \subseteq S$. The set S is called *open* if all of its points are interior points.

Proposition 1.1. *The open ball $B_R(x)$ is open.*

Proof. Fix an arbitrary $y \in B_R(x)$, and observe that it suffices to show that y is an interior point. Take $r = R - d(x,y)$, and first note that $r > 0$ since $d(x,y) < R$. Now note that for all $z \in B_r(y)$, we have

$$d(x,z) \le d(x,y) + d(y,z) < d(x,y) + (R - d(x,y)) = R,$$

so that $z \in B_R(x)$. Thus $B_r(y) \subseteq B_R(x)$, and so y is an interior point. $\square$

Corollary 1.0.1. $B_R(x) = \bigcup_{y \in B_R(x)} B_{r_y}(y)$, *where* $r_y = R - d(x,y)$.

Proof. We have $B_{r_y}(y) \subseteq B_R(x)$ for each $y \in B_R(x)$,[1] and so $\bigcup_{y \in B_R(x)} B_{r_y}(y) \subseteq B_R(x)$. For the reverse inclusion simply observe that $y \in B_{r_y}(y) \subseteq \bigcup_{y \in B_R(x)} B_{r_y}(y)$ for each $y \in B_R(x)$. $\square$

Proposition 1.2. *In a metric space* (X,d), *the following are true:*

(i) $\varnothing, X$ *are open,*

(ii) if $\{S_i\}_{i \in I}$ *are open, then* $\bigcup_{i \in I} S_i$ *is open,*

(iii) and if $\{S_i\}_{i=1}^n$ *are open, then* $\bigcap_{i=1}^n S_i$ *is open.*

Proof. (i) The empty set is open vacuously. To see that X is open, simply take $R = 1$ for any $x \in X$.

(ii) Fix $x \in \bigcup_{i \in I} S_i$ arbitrary, so there exists $i_0 \in I$ with $x \in S_{i_0}$. Since S_{i_0} is open, x is an interior point and thus there exists $r > 0$ such that $B_r(x) \subseteq S_{i_0}$. But then $B_r(x) \subseteq S_{i_0} \subseteq \bigcup_{i \in I} S_i$, so x is an interior point of $\bigcup_{i \in I} S_i$ also and thus $\bigcup_{i \in I} S_i$ is open.

(iii) Now assume $x \in \bigcap_{i=1}^n S_i$. Then for each $1 \le i \le n$, there exists $r_i > 0$ such that $B_{r_i}(x) \subseteq S_i$. Then we can choose

$$r = \min\{r_1, \ldots, r_n\} > 0,$$

so that $B_r(x) \subseteq B_{r_i}(x) \subseteq S_i$ for each $1 \le i \le n$. Thus $B_r(x) \subseteq \bigcap_{i=1}^n S_i$ and $\bigcap_{i=1}^n S_i$ is open. $\square$

Remark. The above argument for the finite intersection property requires that there are only finitely many r_i. Otherwise it may very well be that $r = \inf\{r_i\} = 0$ and the argument fails.

[1] Using the argument from the previous proposition.

Lecture 2

Aug. 22 — Topology, Basis, Continuity

2.1 Topological Spaces

Definition 2.1. A *topology* $\mathcal{T} \subseteq \mathcal{P}(X)$ is a collection of sets such that

(i) $\varnothing, X \in \mathcal{T}$,

(ii) for any index set I, if $\{s_i\}_{i\in I} \subseteq \mathcal{T}$, then $\bigcup_{i\in I} s_i \in \mathcal{T}$ (closure under arbitrary union),

(iii) and if $\{s_i\}_{i=1}^n \subseteq \mathcal{T}$, then $\bigcap_{i=1}^n s_i \in \mathcal{T}$ (closure under finite intersection).

A set with a topology, i.e. a pair $(X, \mathcal{T})$, is called a *topological space*. Elements of $\mathcal{T}$ are called *open sets*.

Example 2.1.1. The following are examples of topologies on a set X:

- The trivial topology: $\mathcal{T} = \{\varnothing, X\}$.
- The discrete topology: $\mathcal{T} = \mathcal{P}(X)$.[1]
- If (X, d) is a metric space, then $\mathcal{T} = \{\text{collection of metrically open sets}\}$ is a topology on X.

Remark. Not every topology is induced by a metric. For instance consider the trivial topology on $\mathbb{R}$.

2.2 Basis for a Topology

Definition 2.2. A collection $\mathcal{B} \subseteq \mathcal{P}(X)$ is called a *basis* if

(i) $\bigcup_{b\in\mathcal{B}} b = X$, i.e. $\mathcal{B}$ is a covering of X,

(ii) and if $x \in b_1 \cap b_2$ for any $b_1, b_2 \in B$, then there exists $b_3 \in \mathcal{B}$ such that $x \in b_3$ and $b_3 \subseteq b_1 \cap b_2$.

Theorem 2.1. *Given a set X and a basis $\mathcal{B}$, define*

$$\mathcal{T}_{\mathcal{B}} = \left\{ \bigcup_{i\in I} s_i \mid I \textit{ is any index set and } \{s_i\}_{i\in I} \subseteq \mathcal{B} \right\}.$$

Then $\mathcal{T}_{\mathcal{B}}$ is a topology on X.

Proof. First observe that $\varnothing, X \in \mathcal{T}_{\mathcal{B}}$: Picking $I = \varnothing$ gives $\bigcup_{i\in I} s_i = \varnothing \in \mathcal{T}_{\mathcal{B}}$ and picking $I = \mathcal{B}$ gives $\bigcup_{b\in\mathcal{B}} b = X \in \mathcal{T}_{\mathcal{B}}$ by the covering property of a basis.

[1] Note that the discrete topology is induced by the discrete metric.

Now assume $\{s_i\}_{i\in I} \subseteq \mathcal{T}_{\mathcal{B}}$. For each $i \in I$, we have $s_i \in \mathcal{T}_{\mathcal{B}}$ and so there exists an index set J_i such that $s_i = \bigcup_{j\in J_i} b_j$, where the $b_j \in \mathcal{B}$. Then

$$\bigcup_{i\in I} s_i = \bigcup_{i\in I}\bigcup_{j\in J_i} b_j,$$

which is a union of elements of $\mathcal{B}$ and hence is in $\mathcal{T}_{\mathcal{B}}$.

Finally assume $\{s_i\}_{i=1}^n \subseteq \mathcal{T}_{\mathcal{B}}$. Now as each $s_i \in \mathcal{T}_{\mathcal{B}}$, there exists J_i such that $s_i = \bigcup_{j\in J_i} b_j$. Then

$$\bigcap_{i=1}^n s_i = \bigcap_{i=1}^n \bigcup_{j\in J_i} b_j.$$

Now assume $x \in \bigcap_{i=1}^n s_i = \bigcap_{i=1}^n \bigcup_{j\in J_i} b_j$. For each $1 \le i \le n$, there exists $j_i \in J_i$ such that $x \in b_{j_i}$. Hence $x \in \bigcap_{i=1}^n b_{j_i}$. Now by induction on the intersection property of a basis, we can find $b_x \in \mathcal{B}$ with

$$x \in b_x \subseteq \bigcap_{i=1}^n b_{j_i}$$

Also observe that

$$\bigcap_{i=1}^n b_{j_i} \subseteq \bigcap_{i=1}^n \bigcup_{j\in J_i} b_j = \bigcap_{i=1}^n s_i$$

by construction, so we may write

$$\bigcap_{i=1}^n s_i = \bigcup_{x\in\bigcap_{i=1}^n s_i} b_x \in \mathcal{T}_{\mathcal{B}}$$

as a union of elements of $\mathcal{B}$. □

Definition 2.3. A *subbasis* $\mathcal{B} \subseteq \mathcal{P}(X)$ is a collection of sets such that $\bigcup_{b\in\mathcal{B}} b = X$.

Remark. One may define a basis $\widetilde{\mathcal{B}}$ from a subbasis $\mathcal{B}$ by adding all finite intersections of elements of $\mathcal{B}$. We get the covering property for free and adding the finite intersections gives us the intersection property of a basis.

Example 2.3.1. For $\mathbb{R}$ with the Euclidean metric, the following are bases for the standard topology:

- $\{B_R(x) \mid x \in \mathbb{R}, R > 0\}$.
- $\{B_R(x) \mid x \in \mathbb{R}, R > 0, R \in \mathbb{Q}\}$. For this use the fact that $\mathbb{Q}$ is dense in $\mathbb{R}$.

In particular this shows that a basis for a topology is not unique in general.

2.3 Continuous Functions

Definition 2.4. Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be two topological spaces. A function $f : X \to Y$ is called *continuous* if for any $O \in \mathcal{T}_Y$, we have $f^{-1}(O) \in \mathcal{T}_X$, i.e. the preimage of an open set is open.[2]

Example 2.4.1. Let X be equipped with the trivial topology $\{\varnothing, X\}$ and let $\mathbb{R}$ be equipped with the standard topology. Then the only continuous functions $f : X \to \mathbb{R}$ are the constant functions $f : x \mapsto c$ for fixed $c \in \mathbb{R}$. To see this, observe that

[2] Recall that $f^{-1}(O) = \{x \in X \mid f(x) \in O\}$.

- $x \mapsto c$ is continuous since any open set in $\mathbb{R}$ either contains c or does not, and so the preimage is either X or $\varnothing$.
- Suppose $f(x_1) = y_1$ and $f(x_2) = y_2$. Let $\epsilon = |y_1 - y_2|$ and observe that $x_1 \in f^{-1}(B_\epsilon(y_1))$ while $x_2 \notin f^{-1}(B_\epsilon(y_1))$, so $f^{-1}(B_\epsilon(y_1))$ is not open in X despite $B_\epsilon(y_1)$ being open in $\mathbb{R}$.

Example 2.4.2. Let X have the discrete topology $\mathcal{T} = \mathcal{P}(X)$ and let $\mathbb{R}$ have the standard topology. Then all functions $X \to \mathbb{R}$ are continuous since any preimage is a subset of X and thus in $\mathcal{P}(X)$.

Remark. In a way, the trivial topology has too few open sets while the discrete topology has too many.

Definition 2.5. Two topological spaces $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ are *topologically equivalent* or *homeomorphic* if there exists a bijection $f : X \to Y$ such that f and f^{-1} are continuous.

Remark. A bijective function f being continuous does not necessarily imply that its inverse f^{-1} is.

Example 2.5.1. Consider $(-\pi/2, \pi/2)$ equipped with the Euclidean metric. This is homeomorphic to $\mathbb{R}$ equipped with the Euclidean metric.[3] One homeomorphism is given by $\tan : (-\pi/2, \pi/2) \to \mathbb{R}$.

[3] Note that $(-\pi/2, \pi/2)$ is bounded while $\mathbb{R}$ is not.

Lecture 3

Aug. 27 — Closed Sets, Continuity, the Subspace Topology

3.1 Closed Sets

Definition 3.1. A set $S \subseteq X$ is called a *closed set* if $S^c = X \setminus S$ is open.

Example 3.1.1. In $\mathbb{R}$, observe that $[a, b]^c = (-\infty, a) \cup (b, \infty)$, which is a union of open sets and thus open. Thus the closed intervals $[a, b] \subseteq \mathbb{R}$ are closed.

Remark. This is not a dichotomy. Sets can be both open and closed (*clopen*), or even neither. Trivially, if X is any topological space, then $\varnothing$ and X are both open and closed.

Example 3.1.2. Let $X = \{0, 1\}$ and $\mathcal{T} = \mathcal{P}(X)$. Then $\{0\}$ is both open and closed.

Example 3.1.3. Let $X = \{1, 2, 3\}$ and $\mathcal{T} = \{\varnothing, X, \{1\}, \{1, 2\}\}$. Then $\{2\}$ is neither open nor closed.

Recall the following De Morgan's laws from set theory:

Proposition 3.1 (De Morgan's laws)**.** *Let I be an index set and $\{A_i\}_{i \in I}$ be sets. Then*

$$\left(\bigcup_{i \in I} A_i \right)^c = \bigcap_{i \in I} A_i^c \quad \textit{and} \quad \left(\bigcap_{i \in I} A_i \right)^c = \bigcup_{i \in I} A_i^c.$$

Corollary 3.0.1. *In a topological space $(X, \mathcal{T})$, we have:*

(i) $\varnothing, X$ are closed.

(ii) if $\{A_i\}_{i \in I}$ are closed, then $\bigcap_{i \in I} A_i$ is closed,

(iii) and if $\{A_i\}_{i=1}^n$ are closed, then so is $\bigcup_{i=1}^n A_i$.

This gives a dual characterization of a topology.

Proof. (i) We have $\varnothing^c = X \in \mathcal{T}$ and $X^c = \varnothing \in \mathcal{T}$.

(ii) Note that

$$\left(\bigcap_{i \in I} A_i \right)^c = \bigcup_{i \in I} A_i^c.$$

As each A_i is closed, we have $A_i^c \in \mathcal{T}$ is open, and hence $\bigcup_{i \in I} A_i^c \in \mathcal{T}$ is open. So $\bigcap_{i \in I} A_i$ is closed.

(iii) Observe that

$$\left(\bigcup_{i=1}^{n} A_i\right)^c = \bigcap_{i=1}^{n} A_i^c.$$

Each A_i is closed, so A_i^c is open. Thus $\bigcap_{i=1}^n A_i^c$ is open, and so $\bigcup_{i=1}^n A_i$ is closed. □

3.2 Properties of Continuity

Recall that a function $f : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ is continuous if for every $O \in \mathcal{T}_Y$, we have $f^{-1}(O) \in \mathcal{T}_X$.

Theorem 3.1. *A function $f : X \to Y$ is continuous if and only if for every C closed in Y, $f^{-1}(C)$ is closed in X.*

Proof. ($\Rightarrow$) Let $C \subseteq Y$ be closed. Note that

$$f^{-1}(C) = \{x \in X \mid f(x) \in C\},$$

so we have

$$(f^{-1}(C))^c = \{x \in X \mid f(x) \notin C\} = \{x \in X \mid f(x) \in C^c\} = f^{-1}(C^c).$$

Since C is closed, C^c is open and so $f^{-1}(C^c) = (f^{-1}(C))^c$ is open. Thus $f^{-1}(C)$ is closed.

($\Leftarrow$) Assume $S \subseteq Y$ is open. Note that

$$(f^{-1}(S))^c = \{x \in X \mid f(x) \in S\}^c = \{x \in X \mid f(x) \notin S\} = \{x \in X \mid f(x) \in S^c\} = f^{-1}(S^c).$$

Since S is open, S^c is closed and so $f^{-1}(S^c) = (f^{-1}(S))^c$ is closed by assumption. Thus $f^{-1}(S)$ is open, and so we see that f is continuous. □

Theorem 3.2 (Composition theorem)**.** *Let $(X, \mathcal{T}_X)$, $(Y, \mathcal{T}_Y)$, and $(Z, \mathcal{T}_Z)$ be topological spaces. Let*

$$f : X \to Y \quad \text{and} \quad g : Y \to Z$$

be continuous functions. Then $g \circ f : X \to Z$ is continuous.

Proof. Let $S \subseteq Z$ be open. It suffices to show that $(g \circ f)^{-1}(S) \subseteq X$ is open. Note that

$$\begin{aligned}(g \circ f)^{-1}(S) = \{x \in X \mid (g \circ f)(x) \in S\} &= \{x \in X \mid f(x) \in g^{-1}(S)\} \\ &= \{x \in X \mid x \in f^{-1}(g^{-1}(S))\} = f^{-1}(g^{-1}(S)).\end{aligned}$$

Now as g is continuous, $g^{-1}(S)$ is open in Y. Finally as f is continuous, $f^{-1}(g^{-1}(S))$ is open in X. □

Theorem 3.3. *Assume $X = \bigcup_{\alpha \in I} U_\alpha$ for open sets U_α and let $f : X \to Y$. Assume that $f|_{U_\alpha} : U_\alpha \to Y$ is continuous for each $\alpha \in I$. Then f is continuous.*

Proof. Let $S \subseteq Y$ be open, and it suffices to show that $f^{-1}(S)$ is open. Observe that

$$f^{-1}(S) = f^{-1}(S) \cap X = f^{-1}(S) \cap \left(\bigcup_{\alpha \in I} U_\alpha\right) = \bigcup_{\alpha \in I}(f^{-1}(S) \cap U_\alpha) = \bigcup_{\alpha \in I} f|_{U_\alpha}^{-1}(S).$$

The $f|_{U_\alpha}$ are continuous, so each $f|_{U_\alpha}^{-1}(S)$ is open. Thus $f^{-1}(S)$ is open as a union of open sets. □

Theorem 3.4 (Pasting lemma)**.** *Assume X, Y are topological spaces and $A, B \subseteq X$ are open. Suppose $f_1 : A \to Y$ and $f_2 : B \to Y$ are continuous, and that $f_1 \equiv f_2$ on $A \cap B$. Then $f : A \cup B \to Y$ defined by*

$$f(x) = \begin{cases} f_1(x) & \text{if } x \in A, \\ f_2(x) & \text{if } x \in B \end{cases}$$

is continuous.

Proof. Let $S \subseteq Y$ be open, it suffices to show that $f^{-1}(S)$ is open. Observe that

$$f^{-1}(S) = f_1^{-1}(S) \cup f_2^{-1}(S).$$

Both $f_1^{-1}(S)$ and $f_2^{-1}(S)$ are open since f_1 and f_2 are continuous, so $f^{-1}(S)$ is open as their union. □

3.3 Subspace Topology

Definition 3.2. Let $(X, \mathcal{T}_X)$ be a topological space and $S \subseteq X$ a set. The *subspace topology* on S is defined as follows: $O \subseteq S$ is open if there exists $U \subseteq X$ open in X such that $U = O \cap S$.

Example 3.2.1. Let $\mathbb{R}$ be given the metric topology and $S = [0, 1]$.

- The set $[0, 1]$ is not open in $\mathbb{R}$, but it is open in the subspace topology on S since $[0, 1] = S \cap (-1, 2)$.
- The set $[0, 1/2)$ is neither open nor closed in $\mathbb{R}$, but $[0, 1/2) = S \cap (-1/2, 1/2)$, so it is open in S.

Theorem 3.5. *The subspace topology is indeed a topology.*

Proof. Let $(X, \mathcal{T}_X)$ be a topological space and $S \subseteq X$ be given the subspace topology.

(i) We have $S = S \cap X$ and $\varnothing = \varnothing \cap X$, so $S, \varnothing$ are open in S.

(ii) Let $\{U_\alpha\}_{\alpha \in I}$ be open in the subspace topology. Then for every $\alpha \in I$, there exists $O_\alpha \in \mathcal{T}$ such that $U_\alpha = S \cap O_\alpha$. Then

$$\bigcup_{\alpha \in I} U_\alpha = \bigcup_{\alpha \in I} (S \cap O_\alpha) = S \cap \left(\bigcup_{\alpha \in I} O_\alpha \right).$$

The $\{O_\alpha\}_{\alpha \in I}$ are open in X, so their union is open in X. Thus $\bigcup_{\alpha \in I} U_\alpha$ is open in the subspace topology.

(iii) Let $\{U_i\}_{i=1}^n$ be open in the subspace topology. Then there are O_i for $1 \le i \le n$ with $U_i = S \cap O_i$. Then we have

$$\bigcap_{i=1}^n U_i = \bigcap_{i=1}^n (S \cap O_i) = S \cap \left(\bigcap_{i=1}^n O_i \right).$$

As the $O_i \in \mathcal{T}$ are open, $\bigcap_{i=1}^n O_i$ is open in X. Thus $\bigcap_{i=1}^n U_i$ is open in the subspace topology. □

Theorem 3.6. *Assume $f : X \to Y$ is a continuous function and $S \subseteq X$ a subspace. Then $f|_S : S \to Y$ is continuous, where S is equipped with the subspace topology.*

Proof. Let $O \subseteq Y$ be an open set, it suffices to show that $f|_S^{-1}(O)$ is open in the subspace topology. But observing that $f|_S^{-1}(O) = f^{-1}(O) \cap S$ immediately shows that $f|_S^{-1}(O)$ is open in S since $f^{-1}(O)$ is open in X due to the continuity of f. □

Remark. The subspace topology is the smallest topology on S such that the inclusion map $i : S \to X$ given by $i(s) = s$ is a continuous function.

Remark. Let X be a topological space with subspaces $Y \subseteq X$ and $Z \subseteq Y$. Then the subspace topology on Z induced by the subspace Y is the same as the subspace topology on Z induced directly by X.

Remark. A topological space can have a subspace homeomorphic to itself. For instance, consider $\mathbb{R}$ and $(-\pi/2, \pi/2)$ with a homemorphism given by the tangent function.

Lecture 4

Aug. 29 — Connectedness

4.1 Connected Spaces

Definition 4.1. A *separation* of a topological space X is two open, nonempty sets $U, V \subseteq X$ such that $X = U \cup V$ and $U \cap V = \varnothing$. A space is called *connected* if there is no separation of the space.

Proposition 4.1. *If X is separated, i.e. $X = U \cup V$ with U, V open and disjoint, then U and V are both open and closed.*

Proof. Observe that U is open by assumption, and we have

$$U^c = X \setminus U = V,$$

which is also open by assumption. Hence U is closed. The case for V is identical. □

Example 4.1.1. Consider the following:

- The singleton space $\{x\}$ is connected. There are no two nonempty, disjoint open sets.
- Consider the space $X = \{0, 1\}$. This case depends on the choice of topology:
  1. With the trivial topology $\mathcal{T} = \{\varnothing, X\}$, the space is connected.
  2. With the discrete topology $\mathcal{T} = \{\varnothing, X, \{1\}, \{0\}\}$, X is disconnected since $X = \{0\} \cup \{1\}$.
  3. With the topology $\mathcal{T} = \{\varnothing, X, \{1\}\}$, the space is connected. The only nonempty sets $\{1\}, X$ are not disjoint and thus there can be no separation.

Theorem 4.1. *A space X is disconnected if and only if there exists a surjective map $f : X \to \{0, 1\}$ with the discrete topology.*

Proof. ($\Rightarrow$) If X is disconnected, then we may write $X = U \cup V$ with U, V open, disjoint, and nonempty. Then define

$$f(x) = \begin{cases} 0 & x \in U, \\ 1 & x \in V, \end{cases}$$

which is surjective as U, V are nonempty. To see that f is continuous, observe that

$$f^{-1}(\varnothing) = \varnothing, \quad f^{-1}(\{0,1\}) = X, \quad f^{-1}(\{0\}) = U, \quad f^{-1}(\{1\}) = V,$$

each of which are open. These are all of the open sets in the discrete topology, so f is continuous.

($\Leftarrow$) Assume there exists a surjective and continuous map $f : X \to \{0, 1\}$. Define

$$U = f^{-1}(\{0\}) \quad \text{and} \quad V = f^{-1}(\{1\}),$$

which are open since f is continuous. Observe that $U, V \neq \varnothing$ since f is surjective. Also $U \cap V = \varnothing$ since if there is any $x \in U \cap V$, then $f(x) = 0$ as $x \in U$ and $f(x) = 1$ as $x \in V$, a contradiction. Finally, $X = U \cup V$ since $f(x) = 0$ or $f(x) = 1$ for every $x \in X$, i.e. $x \in U$ or $x \in V$. So X is disconnected. $\square$

4.2 Connected Sets

Definition 4.2. Let X be a topological space and $S \subseteq X$. Then S is called *connected* if it is connected in the subspace topology.

Theorem 4.2. *If A, B are connected sets and $A \cap B \neq \varnothing$, then $A \cup B$ is connected.*

Proof. Assume not. Then there exists a continuous, surjective map $f : A \cup B \to \{0, 1\}$ with the discrete topology. Consider $f|_A : A \to \{0, 1\}$, which is continuous in the subspace topology. Notice that $f(A)$ cannot be $\{0, 1\}$ since otherwise A is disconnected. Without loss of generality, assume $f(A) = \{0\}$ since A is nonempty. Now consider $f|_B : B \to \{0, 1\}$, which is also continuous. Similarly, notice that $f(B)$ cannot be $\{0, 1\}$. But there exists $p \in A \cap B$, and $f(p) = 0$ as $p \in A$. Then since $p \in B$, we must have $f(B) = \{0\}$. But then we get that $f(A \cup B) = \{0\} \neq \{0, 1\}$, a contradiction to surjectivity. $\square$

Corollary 4.2.1. *A union of connected sets with "common points" is connected.*

Proof. Run induction (transfinite if the union is infinite) using the previous theorem. $\square$

Theorem 4.3. *Closed intervals in $[a, b] \subseteq \mathbb{R}$ with the metric topology are connected.*

Proof. Assume otherwise that $[a, b] = U \cup V$ with U, V disjoint, open, and nonempty. Assume without loss of generality that $a \in U$. Since V is nonempty, there exists $c > a$ such that $c \in V$. Now consider $[a, c] \subseteq [a, b]$ with $U_1 = U \cap [a, c]$ and $V_1 = V \cap [a, c]$. By the least upper bound property of $\mathbb{R}$, since U_1 is nonempty and bounded from above, there exists $s = \sup U_1$ with $s \leq c$. Now either $s \in U_1$ or $s \notin U_1$.

If $s \in U_1$ (note this implies $s \neq c$), then s is an interior point of U_1 since U_1 is open. So one may find a point t such that $t > s$ and $t \in U_1$. But then s is no longer an upper bound of U_1, a contradiction.

Otherwise $s \notin U_1$. Since U_1, V_1 cover $[a, c]$, we must then have $s \in V_1$ (note this implies $s \neq a$. Since V_1 is open, s is an interior point of V_1, and thus there exists $t < s$ such that $t \in V_1$ and t is an upper bound for U_1. This contradicts s being the least upper bound of U_1.

Since both cases lead to contradictions, we conclue that $[a, b]$ must be connected. $\square$

Corollary 4.3.1. *Open intervals in $\mathbb{R}$ are connected, and $\mathbb{R}$ itself is connected.*

Proof. For some $N_0 \geq 1$ (for instance choose $N_0 \geq 2/(b - a)$) we can write

$$(a, b) = \bigcup_{n=N_0}^{\infty} \left[a + \frac{1}{n}, b - \frac{1}{n}\right],$$

Each of these closed intervals is connected by the previous theorem, and thus the union is connected by Corollary 4.2.1 since they overlap. Similarly writing $\mathbb{R} = \bigcup_{n=1}^{\infty} [-n, n]$ shows that $\mathbb{R}$ is connected. $\square$

Corollary 4.3.2 (Intermediate value theorem)**.** *Let $f : [a,b] \to \mathbb{R}$ be a continuous function. Then for any $f(a) < t < f(b)$, there exists $c \in [a,b]$ such that $f(c) = t$.*

Proof. Assume not. We can consider the open sets $(-\infty, t)$ and (t, ∞) in $\mathbb{R}$. Then $f^{-1}((-\infty,t))$ and $f^{-1}((t,\infty))$ are open sets since f is continuous. They are clearly disjoint (since f must be well-defined), and also nonempty since $a \in f^{-1}((-\infty,t))$ and $b \in f^{-1}((t,\infty))$. Also since $f^{-1}(\{t\}) = \varnothing$ by assumption,

$$[a,b] = f^{-1}((-\infty,t)) \cup f^{-1}((t,\infty)).$$

But this is a separation of $[a,b]$, a contradiction since $[a,b]$ is connected. □

Proposition 4.2. *The open interval $(0,1)$ is not homeomorphic to the closed interval $[0,1]$.*

Proof. Removing any point from $(0,1)$ disconnects it, but $[0,1) = [0,1] \setminus \{1\}$ remains connected.[1] □

Proposition 4.3. *The real line $\mathbb{R}$ is not homeomorphic to the plane $\mathbb{R}^n$ for any $n \geq 2$.*

Proof. Removing a point from $\mathbb{R}$ disconnects it but the same is not true for $\mathbb{R}^n$ when $n \geq 2$. □

[1] To see that $[0,1)$ is connected, we can write $[0,1) = \bigcup_{n=2}^{\infty} [0, 1 - 1/n]$.

Lecture 5

Sept. 3 — Path-Connectedness

5.1 More on Connectedness

Remark. The intervals $[a, b] \subseteq \mathbb{R}$ are homeomorphic to $[0, 1]$ for any $a < b$. We can take $f : [a, b] \to [0, 1]$ defined by

$$f(x) = \frac{1}{b-a}(x-a)$$

for instance as a homemorphism.

Lemma 5.1. *If X is connected and $f : X \to Y$ is continuous, then $f(X)$ is connected.*

Proof. This is part of Homework 2. □

Corollary 5.0.1. *The plane $\mathbb{R}^2$ is connected.*

Proof. Express $\mathbb{R}^2$ as the union of horizontal and vertical lines. Each line is the image of $\mathbb{R}$ and is thus connected by Lemma 5.1. Also any pair of horizontal and vertical lines must intersect, so we can use Corollary 4.2.1 to conclude that the union $\mathbb{R}^2$ is connected. □

Remark. We can extend this to $\mathbb{R}^3$ by embedding planes (copies of $\mathbb{R}^2$), and similarly for $\mathbb{R}^n$.

Proposition 5.1. *The unit circle $\mathbb{S}^1 \subseteq \mathbb{R}^2$ is connected.*

Proof. Define $\gamma : [0, 2\pi] \to \mathbb{R}^2$ by $\gamma(t) = (\cos t, \sin t)$. The image of γ is precisely $\mathbb{S}^1$. □

Proposition 5.2. *Define a relation $\sim$ on X by $x \sim y$ if there exists a connected subset $S \subseteq X$ such that $x, y \in S$. Then $\sim$ is an equivalence relation.*

Proof. For reflexivity, fix $x \in X$ and let S be the largest connected set containing x (this exists since we know at least $\{x\}$ must be connected). Then $x \in S$, so $x \sim x$.

For symmetry, fix $x, y \in X$. If $x \sim y$, then there exists a connected set S such that $x, y \in S$. But then $y, x \in S$, so we see that $y \sim x$.

For transitivity, assume that $x \sim y$ and $y \sim z$. Then there exists S_1 connected such that $x, y \in S_1$ and S_2 connected such that $y, z \in S_2$. Notice that $S_1 \cap S_2 \neq \varnothing$ since $y \in S_1 \cap S_2$. Then $S_1 \cup S_2$ is connected by Theorem 4.2 and $x, y, z \in S_1 \cap S_2$. In particular, $x, z \in S_1 \cap S_2$ and thus $x \sim z$.

So we see that $\sim$ is an equivalence relation. □

Definition 5.1. Let the equivalence relation $\sim$ be defined on X as in Proposition 5.2. Then we can write X as the disjoint union of the equivalence classes of $\sim$. These equivalence classes are called the *connected components* of X.

Remark. The connected components of a space are defined solely via topologies, so they must be invariant under homeomorphism.

Example 5.1.1. The letter S, sitting in $\mathbb{R}^2$, is not homeomorphic to the letter T. There is a point we can remove from T to give three connected components, but removing any point from S gives at most two such connected components.

5.2 Path-Connectedness

Remark. Connectedness is usually a very difficult property to verify. This motivates *path-connectedness*.

Definition 5.2. A set S is *path-connected* if for all $x, y \in S$, there exists a continuous map $\gamma : [0,1] \to S$ such that $\gamma(0) = x$ and $\gamma(1) = y$. Here $[0,1]$ is given the usual metric topology.

Lemma 5.2. *If S is path-connected, then S is connected.*

Proof. This is part of Homework 2. $\square$

Remark. Unlike connectedness, it is immediately obvious that $\mathbb{R}^n$ is path-connected. Simply take the line segment between any two points. Then we can conclude connectedness by the previous lemma.

Example 5.2.1. There are spaces which are connected but not path-connected.

- Consider the *topologist's sine curve*, given by the union of the vertical segment $\{(0,y) \mid -1 \le y \le 1\}$ and the image of $(0,\infty)$ under $x \mapsto (x, \sin(1/x))$, is an example of such a space. See Homework 2 for more details.
- Consider the *cone* C in $\mathbb{R}^2$ defined by ($(0,1)$ denotes an open interval unless otherwise specified)

  $$C = ([0,1] \times \{0\}) \cup (K \times [0,1]) \cup (\{0\} \times [0,1]),$$

  where $K = \{1/n : n \in \mathbb{N}\}$. Note that C is clearly path-connected and hence also connected. Then define the space

  $$D = C \setminus (\{0\} \times (0,1)),$$

  which is now not path-connected (consider the point $(0,1) \in D$) but still connected.

Remark. Observe the following:

- One can define *path-connected components* in a similar manner as connected components.
- A continuous image of a path-connected space is path-connected. Simply compose the curve with the continuous map, which is now a path in the image.
- The union of path-connected spaces sharing a point is path-connected. Take two curves to the common point and concatenate them using the pasting lemma.
- In $\mathbb{R}^n$, connectedness is equivalent to path-connectedness. In general, this holds if you can get a basis of only connected sets.

Remark. Recall from homework that if $f : [0,1] \to [0,1]$ is continuous, then f has a fixed point, i.e. there exists $c \in [0,1]$ with $f(c) = c$. This follows from a clever use of the intermediate value theorem. Now consider a more topological perspective. Consider the diagonal $\{(x,x) \mid x \in [0,1]\}$ and look at the graph of f, which is contained in the closed unit square. This graph is path-connected as the image of a path-connected set and so there is a path between the points $(0, f(0))$ and $(1, f(1))$. But then this path must intersect the diagonal at some point, which gives a fixed point.

Theorem 5.1. *(Brouwer fixed point theorem) Let K be a closed, bounded, and convex set in $\mathbb{R}$. Then any continuous map $f : K \to K$ has a fixed point, i.e. there exists $c \in K$ such that $f(c) = c$.*

Remark. One can see the existence of the Nash equilibrium as a consequence of this theorem.

Remark. In $\mathbb{R}^2$, this theorem follows from the following claim. Let $X = \mathrm{maps}(\mathbb{S}^1, \mathbb{S}^1)$ be the set of all continuous maps from $\mathbb{S}^1$ to itself. Then Brouwer's fixed point theorem in $\mathbb{R}^2$ follows from the following:

Theorem 5.2. *The space* $\mathrm{maps}(\mathbb{S}^1, \mathbb{S}^1)$ *is not path connected.*

Lecture 6

Sept. 5 — Compactness

6.1 Note on the Subspace Topology

Remark. Let X be a topological space with topology $\mathcal{T}_X$, and let $Y \subseteq X$ be a subset endowed with a topology τ. Suppose that for any continuous $f : X \to Z$, there exists a continuous $\widetilde{f} : Y \to Z$ such that the following diagram commutes,

$$\begin{tikzcd} X \arrow[r, "f"] & Z \\ Y \arrow[u, "i"] \arrow[ur, dashed, "\widetilde{f}"'] & \end{tikzcd}$$

where $i : Y \to X$ is the inclusion map.[1] Then in Homework 2 we showed that $\mathcal{T}_Y \subseteq \tau$. We can see this as a *universal property* for the subspace topology.

6.2 Compactness

Definition 6.1. A set $C \subseteq X$ is called *compact* if for any *open cover*

$$C \subseteq \bigcup_{\alpha \in I} U_\alpha, \quad \text{each } U_\alpha \text{ is open},$$

there exists a finite subcover $C \subseteq \bigcup_{i=1}^n U_{\alpha_i}$.

Example 6.1.1. Consider the following:

- In a finite topology, any set is compact. This is because any open cover is already finite.
- In a discrete space, i.e. $\mathcal{T} = \mathcal{P}(X)$, compact sets are precisely the finite sets. It is clear that finite sets are compact, for each x choose a single open set in the cover containing x. Conversely, if a set is compact, we can pick our open cover to contain only singletons, and the existence of a finite subcover means that the set has only finitely many elements.

Theorem 6.1 (Heine-Borel)**.** *Let $C \subseteq \mathbb{R}^n$ be a subset, where $\mathbb{R}^n$ is given the metric topology. Then C is compact if and only if C is closed and bounded.*

Proof. We postpone this proof until later. □

Lemma 6.1. *Let X be compact. If $Y \subseteq X$ is closed, then Y is compact.*

[1] Note that at least set-theoretically, this immediately defines $\widetilde{f} = f|_Y$. But a priori we do not know that $\widetilde{f}$ is continuous.

Proof. Let $Y \subseteq X$ closed be given, and assume that $Y \subseteq \bigcup_{\alpha \in I} U_\alpha$ an open cover. Since Y is closed, its complement Y^c is open. Then

$$Y^c \cup \bigcup_{\alpha \in I} U_\alpha$$

is an open cover of X since $X = Y \cup Y^c$. Since X is compact, there exists a finite subcover

$$X \subseteq Y^c \cup \bigcup_{i=1}^{n} U_{\alpha_i}.$$

Now observe that $Y \subseteq X$ and $Y \cap Y^c = \varnothing$, so actually $Y \subseteq \bigcup_{i=1}^n U_{\alpha_i}$, which is a finite subcover. $\square$

Theorem 6.2. *Let X be compact and $f : X \to Y$ continuous. Then $f(X)$ is compact.*

Proof. Consider $f(X) \subseteq Y$ and let $f(X) \subseteq \bigcup_{\alpha \in I} V_\alpha$, an open cover in Y. Notice that

$$X = f^{-1}(f(X)) \subseteq f^{-1}\left(\bigcup_{\alpha \in I} V_\alpha\right) = \bigcup_{\alpha \in I} f^{-1}(V_\alpha).$$

Note that each $f^{-1}(V_\alpha)$ is open in X since f is continuous and V_α is open in Y, so this is in fact an open cover of X. Thus since X is compact, we may extract a finite subcover

$$X \subseteq \bigcup_{i=1}^{n} f^{-1}(V_{\alpha_i}).$$

Then we see that

$$f(X) \subseteq f\left(\bigcup_{i=1}^{n} f^{-1}(V_{\alpha_i})\right) \subseteq \bigcup_{i=1}^{n} V_{\alpha_i},$$

which is a finite subcover of $f(X)$. Therefore $f(X)$ is compact. $\square$

Theorem 6.3. *Assume $\{C_j\}_{j=1}^m$ are compact subsets of X. Then $\bigcup_{j=1}^m C_j$ is compact.*

Proof. Assume $\bigcup_{j=1}^m C_j \subseteq \bigcup_{\alpha \in I} U_\alpha$, an open cover. Observe this is also an open cover of C_j for each $1 \le j \le m$, so we can extract a finite subcover, i.e. we can find $\alpha_{j,1}, \dots, \alpha_{j,n_j}$ with

$$C_j \subseteq \bigcup_{i=1}^{n_j} U_{\alpha_{j,i}}.$$

Then we see that

$$\bigcup_{j=1}^{m} C_j \subseteq \bigcup_{j=1}^{m} \bigcup_{i=1}^{n_j} U_{\alpha_{j,i}},$$

which is still a finite union. This is then a finite subcover of $\bigcup_{j=1}^m C_j$, so $\bigcup_{j=1}^m C_j$ is compact. $\square$

Theorem 6.4 (Weierstrass)**.** *Let $f : [a, b] \to \mathbb{R}$ be continuous. Then $f([a, b])$ is bounded, and moreover there exist $x_{\max}, x_{\min} \in [a, b]$ such that $f(x_{\max}) \ge f(x) \ge f(x_{\min})$ for all $x \in [a, b]$.*

Proof. Since f is continuous and $[a, b]$ is compact (by Heine-Borel), $f([a, b]) \subseteq \mathbb{R}$ is compact. Thus by Heine-Borel, $f([a, b])$ is bounded. In particular, we can find M, m such that

$$m \leq f(x) \leq M \quad \text{for all } x \in [a, b].$$

For the second part, observe that $f([a, b])$ is bounded and nonempty, so $s = \sup f([a, b])$. Since this is the supremum, there must exist $y_i \in f([a, b])$ such that $y_i \to s$ as $i \to \infty$. Now observe that $f([a, b])$ is closed by Heine-Borel, and in particular it contains its limit points. Thus we obtain $s \in f([a, b])$. Then pick $x_{\max} \in f^{-1}(\{s\}) \subseteq [a, b]$, which will satisfy $f(x_{\max}) = s \geq f(x)$ for all $x \in [a, b]$. by construction.

The argument for finding $x_{\min} \in [a, b]$ is similar. □

Theorem 6.5. *Let X be a topological space, $K \subseteq X$ compact, and $f : K \to \mathbb{R}$ a continuous function. Then f is bounded over K and attains its minimum and maximum on K.*

Proof. The same argument goes through, replacing $[a, b]$ by the compact set K. □

Lecture 7

Sept. 10 — More Compactness

7.1 The Cantor Set

Define $I_0 = [0,1]$ and remove the open middle-thirds interval to get

$$I_1 = [0,1] \setminus (1/3, 2/3) = [0,1/3] \cup [2/3,1].$$

Continue by removing the middle thirds of each interval to get $I_2, I_3, \ldots$. Then the *Cantor set* is defined to be $K = \bigcap_{I \geq 0} I_n$. The Cantor set is compact and uncountable. See more on Homework 3.

7.2 The Heine-Borel Theorem

Theorem 7.1 (Heine-Borel)**.** *If $C \subseteq \mathbb{R}$, then C is compact if and only if C is closed and bounded.*

Proof. ($\Rightarrow$) This direction is easy, see Homework 3 for details.

($\Leftarrow$) First we show that $[a,b] \subseteq \mathbb{R}$ is compact. Let $\{U_\alpha\}_{\alpha \in I}$ be an open cover for $[a,b]$, i.e. $[a,b] \subseteq \bigcup_{\alpha \in I} U_\alpha$. Now define

$$R = \{x \in [a,b] \mid [a,x] \text{ has a finite subcover}\}$$

Clearly $a \in R$ since $a \in [a,b] \subseteq \bigcup_{\alpha \in I} U_\alpha$, so picking any single U_α with $a \in U_\alpha$ gives a finite subcover for $[a,a] = \{a\}$. The goal is now to show that $b \in R$. Observe that $a \in R$ implies $R = \varnothing$, and $R \subseteq [a,b]$, so

$$s = \sup R$$

exists by the completeness of $\mathbb{R}$. We proceed to show that $s \in R$ and then $s = b$, which will show that $b \in R$. As $s \in [a,b] \subseteq \bigcup_{\alpha \in I} U_\alpha$, we can find α_s such that $s \in U_{\alpha_s}$. Since U_{α_s} is open, we can find $\delta > 0$ such that $(s-\delta, s+\delta) \subseteq U_{\alpha_s}$. Then since s is a least upper bound of R, we can find $r \in R$ such that $s - \delta < r \leq s$. Now since $r \in R$, $[a,r]$ admits a finite subcover $\{U_{\alpha_i}\}_{i=1}^n$. Then

$$[a,s] = [a,r] \cup (s-\delta, s] \subseteq \left(\bigcup_{i=1}^n U_{\alpha_i}\right) \cup U_{\alpha_s}$$

is a finite subcover for $[a,s]$, so $s \in R$. Now observe that we actually covered

$$\left[a, s + \frac{\delta}{2}\right] = [a,r] \cup (s-\delta, s+\delta) \subseteq \left(\bigcup_{i=1}^n U_{\alpha_i}\right) \cup U_{\alpha_s}$$

in the previous construction. Then $s + \delta/2 \in R$, which contradicts the minimality of s unless $s = b$. Thus $b \in R$, so $[a, b]$ admits a finite subcover and thus $[a, b]$ is compact.

Now let $C \subseteq \mathbb{R}$ be an arbitrary closed and bounded set. Since C is bounded, there exists $I = (a, b)$ such that $C \subseteq I$. But then $C \subseteq I \subseteq \overline{I} = [a, b]$, so C is a closed subset of a compact set, hence compact. □

Remark. The Heine-Borel theorem also holds more generally in $\mathbb{R}^n$. A later theorem will say that the product of compact sets is compact in the product topology, and thus we can run the same argument as above but with boxes in $\mathbb{R}^n$ instead of intervals.

7.3 The Bolzano-Weierstrass Theorem

Definition 7.1. A point x is an *accumulation point* for a set S if for all open sets U containing x, we have $(U \setminus \{x\}) \cap S \neq \varnothing$.

Remark. We disallow constant sequences when talking about accumulation points.

Proposition 7.1. *Let* $\operatorname{Acc}(A)$ *be the set of accumulation points of a set* A*. Then* $\overline{A} = A \cup \operatorname{Acc}(A)$*.*

Proof. We show that $A \cup \operatorname{Acc}(A)$ is closed, which will imply $\overline{A} \subseteq A \cup \operatorname{Acc}(A)$ by the minimality of the closure. Write

$$(A \cup \operatorname{Acc}(A))^c = A^c \cap \operatorname{Acc}(A)^c.$$

Now assume $x \in A^c \cap \operatorname{Acc}(A)^c$. Since $x \notin \operatorname{Acc}(A)$, there exists U_x open such that $x \in U_x$ and

$$(A \setminus \{x\}) \cap U_x = \varnothing.$$

But also $x \notin A$, so $A \setminus \{x\} = A$ and $A \cap U_x = \varnothing$. Then we can write

$$(A \cup \operatorname{Acc}(A))^c = A^c \cap \operatorname{Acc}(A)^c = \bigcup_{x \in A^c \cap \operatorname{Acc}(A)^c} U_x.$$

This is a union of open sets, hence open, and so $A \cup \operatorname{Acc}(A)$ is closed.

For the other direction, assume $x \in A \cup \operatorname{Acc}(A)$. If $x \in A$, we are done, so assume $x \in \operatorname{Acc}(A) \setminus A$. Now assume otherwise that $x \notin \overline{A}$. Then $x \in (\overline{A})^c$, which is open. Set $U = (\overline{A})^c$, so that

$$U \cap (A \setminus \{x\}) = U \cap A = \varnothing.$$

But then this says that x is not an accumulation point, in contradiction. □

Definition 7.2. We say that a topological space X is *sequentially compact* if every bounded sequence has a convergent subsequence.

Theorem 7.2 (Bolzano-Weierstrass)**.** *Any bounded infinite set* $S \subseteq \mathbb{R}^n$ *has an accumulation point.*

Proof. Since S is bounded, find a compact set containing S. Then apply the later Theorem 7.4. □

Remark. In general, compactness is *not* equivalent to sequential compactness, but both imply the Bolzano-Weierstrass theorem. However, in many spaces (including metric spaces, in particular), the two notions coincide (and are also equivalent to the Bolzano-Weierstrass theorem).

Theorem 7.3. *A sequentially compact space has the Bolzano-Weierstrass property, namely that any bounded infinite set has an accumulation point.*

Proof. This is easy, pick a countable subset (i.e. a sequence) and apply sequential compactness. □

Theorem 7.4. *A compact space has the Bolzano-Weierstrass property, namely that any infinite set has an accumulation point.*

Proof. Let A be an infinite set in X, where X is compact. Assume otherwise that A has no accumulation points in X. Then there is no accumulation point for A outside of A, so $\mathrm{Acc}(A) \subseteq A$. This gives

$$\overline{A} = A \cup \mathrm{Acc}(A) = A,$$

so A is closed. Thus A is a closed subset of a compact space, hence compact. Now for any $a \in A$, pick an open set U_a such that $a \in U_a$ and $U_a \cap (A \setminus \{a\}) = \varnothing$. Write $A \subseteq \bigcup_{a \in A} U_a$, and by compactness we can find a finite subcover $A \subseteq \bigcup_{i=1}^{n} U_{a_i}$. Then observe that

$$A = A \cap \bigcup_{i=1}^{n} U_{a_i} = \bigcup_{i=1}^{n} (A \cap U_{a_i}) = \bigcup_{i=1}^{n} \{a_i\} = \{a_1, \ldots, a_n\},$$

This is in contradiction with A being infinite. □

Remark. Usually, this proof goes by showing that compactness implies sequential compactness, which then implies the Bolzano-Weierstrass property. But this proof avoids going through convergent sequences.

Lecture 8

Sept. 12 — Separation Axioms

8.1 Separation Axioms

Definition 8.1. A topological space is said to satisfy the T_0 *axiom* if the following holds: For every $a, b \in X$ with $a \neq b$, there exists U open such that either $a \in U$, $b \notin U$ or $b \in U$, $a \notin U$.

Remark. With the T_0 axiom, we cannot choose which point is in U and which is not. For instance take $X = \{a, b\}$ with topology $\mathcal{T} = \{\varnothing, \{a\}, X\}$. This space is T_0, but we can only choose U to contain a.

Definition 8.2. A space is said to satisfy the T_1 *axiom* if for every $a, b \in X$ with $a \neq b$, there exist U_a, U_b open such that $a \in U_a$, $b \notin U_a$ and $b \in U_b$, $a \notin U_b$.

Remark. With the T_1 axiom, U_a and U_b need not be disjoint.

Definition 8.3. A space is said to be T_2 or *Hausdorff* if the following holds: For every $a, b \in X$ with $a \neq b$, there exist U_a, U_b open such that $a \in U_a$, $b \in U_b$ and $U_a \cap U_b = \varnothing$.

Example 8.3.1. Metric spaces are Hausdorff. For any $a \neq b$, pick balls with radius $d(a, b)/2$ around a, b.

Theorem 8.1. *We have the proper inclusion* $T_2 \subsetneq T_1 \subsetneq T_0$.

Proof. The inclusions and $T_0 \neq T_1$ is clear (e.g. above). For $T_1 \neq T_2$ take the *line with two origins*[1]. $\square$

Theorem 8.2. *In a* T_1 *space, every singleton* $\{x\}$ *is closed.*

Proof. Fix $x \in X$. For every $y \neq x$, by the T_1 axiom we can find U_y open such that $y \in U_y$ and $x \notin U_y$. In particular, this means that $U_y \subseteq \{x\}^c$. Then we can write

$$\{x\}^c \subseteq \bigcup_{y \in \{x\}^c} U_y \subseteq \{x\}^c.$$

So $\{x\}^c = \bigcup_{y \in \{x\}^c} U_y$, which is open as a union of open sets. Thus $\{x\}$ is closed. $\square$

8.2 Properties of Hausdorff Spaces

Theorem 8.3. *In a Hausdorff space, a point* x *is an accumulation point of a set* A *if and only if every neighborhood of* x *contains infinitely many elements of* A.

[1] The *line with two origins* is $X = \mathbb{R} \cup \{p\}$ with topology generated by the open sets in $\mathbb{R}$ (with the metric topology), and adding $\widetilde{U} = (U \setminus \{0\}) \cup \{p\}$ for each open set $U \subseteq \mathbb{R}$ containing 0. One can separate 0 and p but not with disjoint sets.

Proof. ($\Leftarrow$) This is clear.

($\Rightarrow$) Pick x an accumulation point of A, and assume otherwise that there exists a neighborhood U of x with only finitely many elements of A, i.e. $|(U \setminus \{x\}) \cap A| < \infty$. Then we can write

$$(U \setminus \{x\}) \cap A = \{a_1, \ldots, a_n\} = \bigcup_{i=1}^{n} \{a_i\}.$$

Since our space is Hausdorff and thus also T_1, these singletons $\{a_i\}$ are closed. Then $(U \setminus \{x\}) \cap A$ is closed as a finite union of closed sets. Now since our space is Hausdorff, for every $1 \leq i \leq n$ we can separate a_i from x, i.e. there exists U_{x_i}, U_{a_i} open such that $x \in U_{x_i}$, $a_i \in U_{a_i}$ and $U_{x_i} \cap U_{a_i} = \varnothing$. Then

$$U' = U \cap \bigcap_{i=1}^{n} U_{x_i}$$

is open as a finite intersection of open sets. Also $x \in U'$ since $x \in U$ and $x \in U_{x_i}$ for each i. But

$$(U \setminus \{x\}) \cap A = \{a_1, \ldots, a_n\} \subseteq \bigcup_{i=1}^{n} U_{a_i}$$

and $U_{a_j} \cap \bigcap_{i=1}^{n} U_{x_i} = \varnothing$ for all j, so $(U' \setminus \{x\}) \cap A = \varnothing$. Contradiction. □

Remark. Maybe just the T_1 axiom is enough for this theorem. Think more about this.

Definition 8.4. A sequence $\{x_n\}_{n=1}^{\infty} \subseteq (X, \mathcal{T})$ *converges* to a point $x \in X$, written $x_n \to x$, if for any open set U containing x, there exists $N_0 \in \mathbb{N}$ such that $x_n \in U$ for every $n \geq N_0$.

Theorem 8.4. *In a Hausdorff space, a convergent sequence has a unique limit.*

Proof. Assume otherwise that $x_n \to L_1$ and $x_n \to L_2$ with $L_1 \neq L_2$. Then since our space is Hausdorff, we can find U_{L_1}, U_{L_2} open such that $L_1 \in U_{L_1}$, $L_2 \in U_{L_2}$ and $U_{L_1} \cap U_{L_2} = \varnothing$. Since $x_n \to L_1$, there exists $N_0 \in \mathbb{N}$ such that $x_n \in U_{L_1}$ for all $n \geq N_0$. Similarly we can find $N_0' \in \mathbb{N}$ with $x_n \in U_{L_2}$ for all $n \geq N_0'$ since $x_n \to L_2$. But then for $N = \max\{N_0, N_0'\}$, we have $x_N \in U_{L_1} \cap U_{L_2}$, a contradiction. □

Theorem 8.5. *In a Hausdorff space, every compact set is closed.*

Proof. Let $C \subseteq X$ be compact, and we show that C^c is open. So fix $y \in C^c$. For any $x \in C$, since our space is Hausdorff, we can find U_x, U_y open such that $x \in U_x$, $y \in U_y$ and $U_x \cap U_y = \varnothing$. Now consider $\bigcup_{x \in C} U_x$. This is an open cover of C, so we can find a finite subcover $C \subseteq \bigcup_{i=1}^{n} U_{x_i}$ since C is compact. Then the finite intersection $\bigcap_{i=1}^{n} U_{y_i}$ is an open set contain y, and it is disjoint from C by construction since $U_{x_i} \cap U_{y_i} = \varnothing$ for each i. Now set $\widetilde{U}_y = \bigcap_{i=1}^{n} U_{y_i}$, so that

$$C^c \subseteq \bigcup_{y \in C^c} \widetilde{U}_y \subseteq C^c.$$

Thus $C^c = \bigcup_{y \in C^c} \widetilde{U}_y$, which is open as the union of open sets, so we conclude that C is closed. □

Lecture 9

Sept. 17 — Compactification

9.1 Motivation for Compactification

Let $(X, \mathcal{T})$ be a topological space which is not compact. Usually we assume X is Hausdorff, and the goal is to find a compact space which looks like X, i.e. compactify X.

Remark. The naive idea is to take the trivial topology on X in place of $\mathcal{T}$, getting X_{trivial}. This is compact, the identity map $\mathrm{id} : X \to X_{\text{trivial}}$ is continuous and bijective, but it is not a homeomorphism. This is bad because we forget all the topological structure on X, for instance every sequence converges to every point in X_{trivial}. We would like to compactify X while keeping as much structure as possible.

Example 9.0.1. Let $X = (0, 1)$ with the metric topology. Take $Y = [0, 1]$ with the metric topology, so X embeds into Y by the inclusion map.[1] Note that Y is compact by Heine-Borel.

Example 9.0.2. Let $X = (0, 1)$ with the metric topology. Take $Y = \mathbb{S}^1 \subseteq \mathbb{R}^2$ to be the unit circle, where $\mathbb{R}^2$ has the metric topology. Then X embeds into Y by the stereographic projection (technically $\mathbb{R}$ is embedded but $\mathbb{R}$ is homeomorphic to $(0, 1)$ by the arctangent) by adding one point at the north pole.

Example 9.0.3. For the open unit disk in $\mathbb{R}^2$, we can add its boundary to get the closed unit disk as a compactification (the closed unit disk is compact by Heine-Borel). This adds uncountably many points. An alternative is to add only a single point at infinity, and identify the boundary with this point.

9.2 One-Point Compactification

Definition 9.1. For a topological space X, the *one-point compactification* (or *Alexandroff compactification*) of X is the set $X^+ = X \cup \{\infty\}$ with topology generated by the basis

$$\{U \subseteq X \text{ open}\} \cup \{K^c \mid K \subseteq X \text{ is compact}\}.$$

Remark. One way to think about the one-point compactification is that we are forcing all unbounded sequences in X to converge to to the new point ∞.

Theorem 9.1. *For a Hausdorff topological space X, its one-point compactification X^+ is compact.*

Proof. Let $\{O_\alpha\}$ be an open cover of X^+, i.e. $X^+ = \bigcup_{\alpha \in I} O_\alpha$. Some open set must contain the point ∞, and each open set contains a basis element, so there exists $\alpha' \in I$ such that $O_{\alpha'}$ contains K^c, for some $K \subseteq X$ compact. Now since $K \subseteq X \subseteq X^+$, we see that $K \subseteq \bigcup_{\alpha \in I}(O_\alpha \cap X)$ is an open cover of K

[1] By X *embeds into* Y, we mean that there is a continuous injection from X to Y.

(note that the Hausdorff condition implies that every compact K is closed in X, and thus the K^c basis elements are open in X). So by compactness, there exists a finite subcover $K \subseteq \bigcup_{\alpha \in I_{\text{finite}}} (O_\alpha \cap X)$. Then

$$X^+ = O_{\alpha'} \cup K \subseteq O_{\alpha'} \cup \bigcup_{\alpha \in I_{\text{finite}}} (O_\alpha \cap X) \subseteq \bigcup_{\alpha \in I_{\text{finite}}} O_\alpha$$

since $K^c \subseteq O_{\alpha'}$. This is a finite subcover, so X is compact. □

Remark. In analysis, we sometimes speak of a sequence diverging to ∞, i.e. the sequence eventually escapes any compact set. This is precisely convergence to ∞ in the one-point compactification.

Theorem 9.2. *Assume X is a Hausdorff, locally compact but not compact topological space.*[2] *Then the inclusion map* $\mathrm{id} : X \to X^+$ *is a dense, continuous embedding.*[3]

Proof. First clearly $\mathrm{id} : X \to \mathrm{id}(X) \subseteq X^+$ is injective. Now we show that id is continuous. It is enough to show that the preimage of basis elements of X^+ is open in X. If $U \subseteq X$ is open, then $\mathrm{id}^{-1}(U) = U \subseteq X$ is clearly open. Otherwise consider $K^c \cup \{\infty\}$ for $K \subseteq X$ compact. Then we have

$$\mathrm{id}^{-1}(K^c \cup \{\infty\}) = K^c \subseteq X.$$

Since X is Hausdorff, the compact set K is closed, and so K^c is open. Thus id is continuous.

Finally we show density, i.e. $\overline{X} = X^+$, where the closure is taken in X^+. To do this, suppose otherwise that $\overline{X} \neq X^+$. Clearly $X \subseteq \overline{X}$, so if $\overline{X} \neq X^+$, we must have $\overline{X} = X$ (since $X^+ = X \cup \{\infty\}$). But then X is closed in X^+, which is compact, so X is also compact in X^+ since X and thus X^+ is Hausdorff (since X is locally compact). This implies (exercise) that X itself is compact. Contradiction. □

Remark. If X is already compact, then $X^c = \{\infty\}$ is open in X^+. Obviously X^+ is compact since X is and every sequence converging to ∞ must eventually be constant. In particular, X^+ must be disconnected, and the extra point ∞ just sits there to the side.

Remark. Due to the above, we must assume that X is not compact in order to get a dense embedding.

Example 9.1.1. Consider the space $X = [0, 1)$. Then the one-point compactification of X is $[0, 1]$.

[2]A space X is *locally compact* if for every $x \in X$, there exists U open with $x \in U$ such that $\overline{U}$ is compact.
[3]The embedding is *dense* if $\overline{X} = X^+$.

Lecture 10

Sept. 19 — Miscellaneous Topics

10.1 Local Compactness and One-Point Compactification

Definition 10.1. A map $f : X \to Y$ is called *open* if for every $U \subseteq X$ open, $f(U)$ is open. A *topological embedding* is an injective, continuous, and open map.

Remark. This ensures that f with codomain restricted to $f(U)$ is a homeomorphism.

Example 10.1.1. Demanding only that f is injective and continuous is not enough. Let $(X, \mathcal{T})$ be any topological space and let X_{trivial} be X with the trivial topology. Then the identity map $\text{id} : X \to X_{\text{trivial}}$ is injective, continuous, but not open is general if $\mathcal{T}$ is not trivial. So this is *not* a topological embedding.

Theorem 10.1. *If X is Hausdorff and locally compact, then $X^+ = X \cup \{\infty\}$ is Hausdorff.*

Proof. Pick $y, y' \in X^+$ with $y \neq y'$. If $y, y' \in X$, then since X is Hausdorff, there are disjoint open sets $U, U' \subseteq X$ with $y \in U$ and $y' \in U$. But then $U, U' \subseteq X^+$ are still open and disjoint in X^+ since the identity map embeds X into X^+. Thus U, U' are disjoint open sets separating y, y'.

Now assume without loss of generality that $y' = \infty$. By local compactness, there is a open set $U \subseteq X$ containing y with compact closure. Then observe that $(\overline{U})^c \cup \{\infty\}$ is open in X^+ since $\overline{U}$ is compact. Since U is also open in X^+ and clearly disjoint from $(\overline{U})^c \cup \{\infty\}$, we have separated y, ∞. $\square$

10.2 Distance to a Closed Set

Definition 10.2. Let (X, d) be a metric space. For $x \in X$ and $A \subseteq X$, the *distance from x to A* is

$$d(x, A) = \inf\{d(x, a) : a \in A\}.$$

Remark. This infimum is achieved in $\mathbb{R}^n$, but not necessarily in a more general metric space.

Proposition 10.1. *For any $x \in \mathbb{R}^n$ and closed set $A \subseteq \mathbb{R}^n$, there exists $a \in A$ with $d(x, A) = d(x, a)$.*

Proof. Pick $a_0 \in A$ and note that $d(x, a_0) < \infty$. Then set $R = 2d(x, a_0) > 0$. Consider $\overline{B_R(x)} \cap A$, which is a closed and bounded set containing a_0, hence compact by Heine-Borel. Now apply the compact case to $\overline{B_R(x)} \cap A$ to get $a_{\min}$ with $d(x, a_{\min}) = d(x, \overline{B_R(x)} \cap A)$. Now any $a \in A$ with $a \notin \overline{B_R(x)}$ satisfies

$$d(x, a) \geq R > d(x, a_0) \geq d(x, a_{\min}),$$

and hence it cannot be the minimum. Thus we must have $d(x, a_{\min}) = d(x, A)$. $\square$

Remark. Define

$$\ell^2(\mathbb{N}) = \left\{ \{a_n\}_{n=1}^\infty \subseteq \mathbb{R} : \sum_{n=1}^\infty a_n^2 < \infty \right\}.$$

This is an inner product space over $\mathbb{R}$, and in particular we can induce a metric

$$d(\{a_n\},\{b_n\}) = \sqrt{\sum_{n=1}^\infty |a_n - b_n|^2}$$

to turn (X,d) into a complete metric space. Also notice that $\{e_i\}_{i=1}^\infty$, where e_i is the sequence with 1 in the ith position and 0 everywhere else, is an orthonormal basis for $\ell^2(\mathbb{N})$.

Lemma 10.1. *Define the set*

$$A = \left\{ \left(1 + \frac{1}{i}\right) e_i \right\}_{i=1}^\infty .$$

Then $d(\{0\}, A) = 1$. In particular, this is an example of a metric space where the infimum of the distance from a point to a closed set is not achieved.

Proof. Observe that

$$d\left(\{0\},(1+1/i)e_i\right) = 1 + \frac{1}{i},$$

and so

$$d(\{0\},A) = \inf\{d(\{0\},(1+1/i)e_i : i \in \mathbb{N}\} = \inf\left\{1 + \frac{1}{i} : i \in \mathbb{N}\right\} = 1.$$

In particular, this infimum is clearly not achieved since $d(\{0\},(1+1/i)e_i) = 1 + 1/i > 1$ for each $i \in \mathbb{N}$. Now we show that A is closed by showing that it contains its limit points. For this, first observe that $d(a,a') \geq 1$ for any $a, a' \in A$ with $a \neq a'$. To see this, we can compute that

$$d\left((1+1/i)e_i,(1+1/j)e_j\right) = \sqrt{\left(1+\frac{1}{i}\right)^2 + \left(1+\frac{1}{j}\right)^2} \geq \sqrt{2} \geq 1$$

whenever $i \neq j$. Now assume we have $\{a_n\} \subseteq A$ with $a_n \to x$. In particular, $\{a_n\}$ must be a Cauchy sequence, and so for $\epsilon = 1/2$, there exists $N_0 \in \mathbb{N}$ such that for all $n, m \geq N_0$, we have $d(a_n, a_m) < 1/2$. But $d(a,a') \geq 1$, so the sequence must stabilize after N_0, and hence $x = a_n$ for $n \geq N_0$. In particular, $x \in A$, so we conclude that A is closed. This finishes the example. □

10.3 Nested Intersections in Compact Hausdorff Spaces

Proposition 10.2. *Let X be a compact Hausdorff space, and $Y_i \subseteq X$ be closed and connected for $i \in I$. Assume the $\{Y_i\}$ are totally ordered, i.e. $Y_i \subseteq Y_j$ or $Y_j \subseteq Y_i$ for all $i,j \in I$. Then $\bigcap_{i\in I} Y_i$ is connected.*

Proof. Let U, V be a separation of $Y = \bigcap_{i\in I} Y_i$. Then $Y = U \cup V$ and U, V are open in Y, disjoint, and nonempty. In particular, we can find U', V' open in X such that $U = U' \cap Y$ and $V = V' \cap Y$. However, U', V' may no longer separate Y. This is why we need the Hausdorff condition. Use the next lemma to fix the proof from here, see more details in Homework 4. □

Lemma 10.2. *In a compact Hausdorff space, if C_1, C_2 are two compact disjoint sets, then there exist U_1, U_2 open and disjoint such that $C_1 \subseteq U_1, C_2 \subseteq U_2$ and $U_1 \cap U_2 = \varnothing$.*

Proof. First we show this in the case where $C_1 = \{x\}$ is a singleton and $C_2 = C$. For all $y \in C$, consider the pair x and y. Then there exists $U_{x,y}, V_{x,y}$ open such that $x \in U_{x,y}$, $y \in V_{x,y}$, and $U_{x,y} \cap V_{x,y} = \varnothing$. Observe that $\bigcup_{y \in C} V_{x,y}$ is an open cover of C, so by compactness there exists a finite subcover $C \subseteq V = \bigcup_{i=1}^n V_{x,y_i}$. Then $x \in U = \bigcap_{i=1}^n U_{x,y_i}$, which is open as a finite intersection of open sets. Also each U_{x,y_i} is disjoint from V_{x,y_i}, so U is disjoint from V. Then $U_1 = U$ and $U_2 = V$ are the desired open sets.

Now let C_1, C_2 be any two disjoint compact sets. For all $x \in C_1$, there are open sets U_{x,C_2}, V_{x,C_2} such that $x \in U_{x,C_2}$, $C_2 \subseteq V_{x,C_2}$, and $U_{x,C_2} \cap V_{x,C_2} = \varnothing$. Then we make the same argument. We have

$$C_1 \subseteq \bigcup_{x \in C_1} U_{x,C_2},$$

an open cover of C_1, so by compactness there is a finite subcover $C_1 \subseteq U_1 = \bigcup_{i=1}^m U_{x_i,C_2}$. Then

$$C_2 \subseteq U_2 = \bigcap_{i=1}^m V_{x_i,C_2},$$

which is open as a finite intersection of open sets. As before, $U_1 \cap U_2 = \varnothing$ since each U_{x_i,C_2} is disjoint from V_{x_i,C_2}. Thus we get a separation by disjoint compact sets. $\square$

Lecture 11

Sept. 24 — Product Spaces

11.1 Product Spaces

Definition 11.1. For two sets X, Y, define their *Cartesian product* as

$$X \times Y = \{(x, y) \mid x \in X, y \in Y\}.$$

Similarly we can define[1]

$$\prod_{i \in I} X_i = \{(x) \mid (x)_i \in X_i\}.$$

If each X_i is a topological space with topology $\mathcal{T}_i$, then we define the following topologies on $\prod_{i \in I} X_i$:

- The *box topology*: Take as a basis sets of the form $\prod_{i \in I} U_i$ where $U_i \subseteq X_i$ are open sets.

  Suppose B_1, B_2 are basis sets and $x \in B_1 \cap B_2$. Then by definition $B_1 = \prod_{i \in I} U_i$ and $B_2 \in \prod_{i \in I} V_i$, where $U_i, V_i \subseteq X_i$ are open. Then set $B_3 = \prod_{i \in I} (U_i \cap V_i)$. Clearly $x \in B_3$ and $B_3 \subseteq B_1 \cap B_2$ is a basis element (each $U_i \cap V_i$ is a finite intersection of open sets and thus open), so this is a basis.

- The *product topology*: Take as a subbasis the sets $\pi_i^{-1}(U_i) \subseteq \prod_{i \in I} X_i$ for each $i \in I$ and $U_i \subseteq X_i$ open. Here $\pi_i : \prod_{j \in I} X_j \to X_i$ is the projection onto the ith factor.

  The subbasis sets here are of the form $U_i \times \prod_{j \neq i} X_j$. The general basis sets will be finite intersections of these sets, i.e.

  $$U_{i_1} \times \cdots \times U_{i_n} \times \prod_{j \neq i_1, \ldots, i_n} X_j.$$

  These are called the *cylindrical sets*. Think of this as having only finitely many restrictions on the factors, whereas we get to choose arbitrarily many restrictions with the box topology.

Remark. For finite products, the box and product topologies coincide. They differ for infinite products: The box topology is finer than the product topology (so the box topology has more open sets).

Example 11.1.1. Consider the following product spaces:

- The power set $\mathcal{P}(X) = 2^X = \{0, 1\}^X$. Think of the elements as functions $X \to \{0, 1\}$, which pick whether or not to include each element of X in the corresponding subset. Thus the power set comes with a natural topology (box or product) if $\{0, 1\}$ is given the discrete topology.

- The space $\{0, 1\}^{\mathbb{N}}$ is the Cantor set, if $\{0, 1\}$ is given the discrete topology. The Cantor set with the metric topology inherited from $\mathbb{R}$ is homeomorphic to $\{0, 1\}^{\mathbb{N}}$ with the product topology. Think of the sequence $\{x_n\} \subseteq \{0, 1\}^{\mathbb{N}}$ as choosing whether to pick the left or right third at each step.

[1] One can also think of an element of the product as a function $I \to \bigcup_{i \in I} X_i$, where $f(i) \in X_i$.

- The space $[0,1]^{\mathbb{N}}$ is called *Hilbert's cube*. Note that $[0,1] \times [0,1]^{\mathbb{N}} \cong [0,1]^{\mathbb{N}}$. For a homeomorphism, simply shift the sequence one to the right, putting a 0 in the first slot. Then forget about the 0.

Remark. Always assume $\prod_{i \in I} X_i$ is given the product topology, unless otherwise specified.

11.2 Properties of Product Spaces

Theorem 11.1. *Assume $(X_i, \mathcal{T}_i)_{i \in I}$ are each T_0, each T_1, or each Hausdorff. Then the product $\prod_{i \in I} X_i$ is also T_0, T_1, or Hausdorff, respectively.*

Proof. We prove only the Hausdorff case. Let $(x), (y) \in \prod_{i \in I} X_i$ be distinct. As $(x) \neq (y)$, there is $i \in I$ with $x_i \neq y_i$, where $x_i, y_i \in X_i$. Since X_i is Hausdorff, there exist $A, B \subseteq X_i$ open, disjoint with $x_i \in A$ and $y_i \in B$. Then set

$$U = A \times \prod_{j \neq i} X_j \quad \text{and} \quad V = B \times \prod_{j \neq i} X_j.$$

These sets are open since they are cylindrical, and clearly $(x) \in U$, $(y) \in V$ since $x_i \in A$, $y_i \in B$. Also U, V are disjoint since their ith components A, B are disjoint. So this is a separation of (x) and (y) by disjoint, open sets, and we conclude that $\prod_{i \in I} X_i$ is Hausdorff. □

Corollary 11.1.1. *Assume $(X_i, \mathcal{T}_i)_{i \in I}$ are each T_0, each T_1, or each Hausdorff. Then $\prod_{i \in I} X_i$ with the box topology is also T_0, T_1, or Hausdorff, respectively.*

Proof. The box topology is finer than the product topology. □

Remark. In the product topology, the projections $\pi_i : \prod_{j \in I} X_j \to X_i$ are continuous, onto, and open. The continuity and surjectivity of π_i is essentially by construction of the product topology. To see that π_i is open, consider a basis element of $\prod_{j \in I} X_j$, which is of the form

$$U = U_{i_1} \times \cdots \times U_{i_n} \times \prod_{j \neq i_1, \ldots, i_n} X_j.$$

Then $\pi_i(U)$ is either one of the U_i or one of the X_i, which are both open.

Theorem 11.2 (Universal property of the product topology)**.** *The following diagram commutes:*

$$\begin{tikzcd} Z \arrow[r, "f_i"] \arrow[dr, dashed, "f"'] & X_i \\ & \prod_{i \in I} X_i \arrow[u, "\pi_i"'] \end{tikzcd}$$

In particular, there exists a unique continuous map $f : Z \to \prod_{i \in I} X_i$ such that $f_i = \pi_i \circ f$ for each $i \in I$.

Proof. Define $f : Z \to \prod_{i \in I} X_i$ on the set theory level by $(f(z))_i = f_i(z)$. Now we show the continuity of f. Fix a basis element

$$B = U_{i_1} \times \cdots \times U_{i_n} \times \prod_{j \neq i_1, \ldots, i_n} X_j.$$

Then we can write

$$f^{-1}(B) = \{z \mid f(z) \in B\}.$$

Now observe that $f(z) \in B = U_{i_1} \times \cdots \times U_{i_n} \times \prod_{j \neq i_1, \ldots, i_n} X_j$ is equivalent to

$$z \in \bigcap_{k=1}^{n} f_{i_k}^{-1}(U_{i_k}).$$

Since U_{i_k} is open in X_{i_k} and each f_i is continuous, each $f_{i_k}^{-1}(U_{i_k})$ is open in Z. Then this is a finite intersection of open sets in Z, hence open. Thus f is continuous. $\square$

Remark. If we had the box topology, this argument would not work. In particular, the intersection that we get could be infinite, which would not necessarily be open in Z.

Remark. This universal property formalizes the notion that we define functions from products by defining a function on each factor. Additionally, this generalizes the result from multivariable calculus that a vector-valued function is continuous precisely when each component function is continuous.

Lecture 12

Sept. 26 — Products and Topological Properties

12.1 Connectedness and Path-Connectedness

Theorem 12.1. *Assume that each X_i is path-connected, then $\prod_{i \in I} X_i$ is path-connected.*

Proof. Fix $x, y \in \prod_{i \in I} X_i$, and define $x_i = \pi_i(x)$ and $y_i = \pi_i(y)$. Since $x_i, y_i \in X_i$ and X_i is path-connected, there exists $\gamma_i : [0,1] \to X_i$ continuous such that $\gamma_i(0) = x_i$ and $\gamma_i(1) = y_i$. By the universal property of products, there exists $\gamma : [0,1] \to \prod_{i \in I} X_i$ continuous such that $\pi_i \circ \gamma = \gamma_i$. This implies

$$(\gamma(0))_i = \gamma_i(0) = x_i,$$

so $\gamma(0) = x$. Similarly $\gamma(1) = y$, so γ is a path from x to y. This says that $\prod_{i \in I} X_i$ is path-connected. $\square$

Theorem 12.2. *If $\{X_i\}$ are connected, then $\prod_{i \in I} X_i$ is connected.*

Proof. Suppose otherwise that $\prod_{i \in I} X_i$ is not connected, i.e. there exists a separation of $\prod_{i \in I} X_i$. So let U, V be two disjoint, nonempty, open subsets of $\prod_{i \in I} X_i$ such that $U \cup V = \prod_{i \in I} X_i$.

First we claim that there exist $x \in U$ and $y \in V$ such $x_i \neq y_i$ only at a single index i. To see this, note that U and V contain basis elements $U' \subseteq U$ and $V' \subseteq V$. Then these basis elements look like

$$U' = U_{i_1} \times \cdots \times U_{i_n} \times \prod_{j \neq i_1, \ldots, i_n} X_j \quad \text{and} \quad V' = V_{j_1} \times \cdots \times V_{j_m} \times \prod_{k \neq j_1, \ldots, j_m} X_k.$$

Then clearly we may choose $x \in U'$ and $y \in V'$ such that they differ in only finitely many coordinates. Then we have

$$x = (x_1, \ldots, x_n, \ldots) \quad \text{and} \quad y = (y_1, \ldots, y_n, \ldots),$$

where $x_i = y_i$ for $i > n$. Now consider x_i, y_i for $1 \leq i \leq n$. If $x_i = y_i$, then do nothing. Otherwise if $x_i \neq y_i$, define

$$y' = (x_1, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots).$$

Then $y' \in \prod_{i \in I} X_i = U \cup V$, so either $y' \in U$ or $y' \in V$ since $U \cap V = \varnothing$. If $y' \in V$, continue with $y = y'$, and if $y' \in U$, change $x = y'$. Do this for the finitely many $1 \leq i < n$, and we obtain $x_i = y_i$ except for a single index i. Assume without loss of generality that $x_1 \neq y_1$.

Now define a map $f : X_1 \to \prod_{i \in I} X_i$ by $f(\widetilde{x}) = (\widetilde{x}, x_2, x_3, \ldots)$. Note that f is continuous by the universal property of products (the component maps $X_1 \to X_i$ are either the identity if $X_i = X_1$ or constant otherwise). Then $f(X_1)$ is connected since X_1 is connected and f is continuous. But now $x \in f(X_1) \cap U$ and $y \in f(X_1) \cap V$, so $U \cap V \neq \varnothing$. Contradiction. $\square$

Lecture 13

Oct. 1 — Tychonoff's Theorem

13.1 Revisiting the Cantor Set

Proposition 13.1. *We have the following properties of the Cantor set C:*

(a) $C \neq \varnothing$*,*

(b) $|C| = \aleph$*,*

(c) C *is closed,*

(d) C *is compact,*

(e) and C *is nowhere dense.*

Proof. This was done in Homework 3. □

Proposition 13.2. *We have* $C \cong \{0,1\}^{\mathbb{N}}$*, where* $\{0,1\}$ *has the discrete topology.*

Proof. Observe that to obtain $x \in C = \bigcap_{n=1}^{\infty} C_n$, at each step I_n we must choose the left or right part of the current subinterval. So define the map $f : C \to \{0,1\}^{\mathbb{N}}$ by

$$(f(x))_i = \begin{cases} 0 & \text{if } x \text{ belongs to the left third,} \\ 1 & \text{if } x \text{ belongs to the right third.} \end{cases}$$

For injectivity, observe that if $a, b \in C$ with $a \neq b$, then there exists some $N \in \mathbb{N}$ such that $3^{-N} < |a - b|$. In particular, this means that a and b cannot be in the same subinterval in I_N, so they must have taken different paths at some step. This implies that $f(a) \neq f(b)$. For surjectivity, given a string $\{0,1\}^{\mathbb{N}}$, take the intersection of subintervals encoded by it. This intersection consists of a unique point and is a subset of C, so map the string to this point. So f is bijective.

Now we show that f is continuous. Let U be an open set in $\{0,1\}^{\mathbb{N}}$. Note that we may assume U is a basis element, i.e.

$$U = U_{\alpha_1} \times \cdots \times U_{\alpha_n} \times \prod_{i \neq \alpha_1, \ldots, \alpha_n} \{0,1\},$$

where the $U_{\alpha_i} \subseteq \{0,1\}$ are open. Since there are only finitely many U_{α_i}, we may assume that

$$U = U_1 \times \cdots \times U_m \times \{0,1\}^{\mathbb{N}},$$

where $U_i = \{0\}$ or $U_i = \{1\}$. The general basis elements may be written as finite unions and intersections of sets of this form. Thus observe that $f^{-1}(U)$ is the subinterval obtained fixing the path given by the U_i, terminating at the mth step and taking all remaining choices. This gives some subinterval $C \cap I \subseteq C \cap I_{m+1}$. Note that the subintervals of I_{m+1} are separated by gaps of width $3^{-(n+1)}$, so we can find an open set $V \subseteq \mathbb{R}$ containing I, disjoint from the other subintervals of I_{m+1}. Then $C \cap I = C \cap V$ is open in C, so $f^{-1}(U)$ is open in C and thus f is continuous.

Now C is compact, $\{0,1\}^{\mathbb{N}}$ is Hausdorff, and f is a continuous bijection, so f is a homeomorphism. □

Proposition 13.3. *For the Cantor set C, we have $C^{\mathbb{N}} \cong C$.*

Proof. This is a homework exercise. For the bijection, first show that $\mathbb{N} \times \mathbb{N} \cong \mathbb{N}$ (traverse $\mathbb{N} \times \mathbb{N}$ in diagonals). For continuity, note that an open set in $C^{\mathbb{N}}$ has finitely many constraints, and an open set in each component of $C^{\mathbb{N}}$ has finitely many constraints. This is still finite many constraints in total. □

13.2 Tychonoff's Theorem

Theorem 13.1 (Tychonoff). *Let $(X_i, \mathcal{T}_i)$ be a compact space for each $i \in I$. Then the product space $\prod_{i \in I} X_i$, with the product topology, is compact.*

Corollary 13.1.1 (Heine-Borel). *A set $C \subseteq \mathbb{R}^n$ is compact if and only if C is closed and bounded.*

Proof. ($\Rightarrow$) This direction is easy. Immediately C is closed since $\mathbb{R}^n$ is Hausdorff. For boundedness, cover C by open balls of radius R centered at the origin, then use compactness.

($\Leftarrow$) We previously showed that $[a,b] \subseteq \mathbb{R}$ is compact. Since C is bounded, there is some box in $\mathbb{R}^n$ with

$$C \subseteq \prod_{i=1}^{n} [a_i, b_i],$$

which is compact by Tychonoff's theorem. Thus C is a closed subset of a compact set, so C is compact. □

13.3 Set Theory and the Axiom of Choice

Example 13.0.1 (Russell's paradox). Define the set

$$S = \{\text{all sets } R \mid R \text{ does not contain itself}\}.$$

Then consider whether S contains itself. If $S \in S$, then S cannot contain itself by the definition of S. But if $S \notin S$, then S must contain itself by the definition of S again. Both cases lead to contradictions.

Remark. The above paradox led to the creation of the *Zermelo-Fraenkel* (ZF) axioms of set theory. More commonly in mathematics, we use ZFC set theory, adding the *axiom of choice* (AC) to ZF.

Axiom 13.1 (Axiom of choice). *Let X_i be a collection of nonempty sets. Then there exists a choice function $f : I \to \bigcup_{i \in I} X_i$ such that $f(i) \in X_i$ for all $i \in I$.*

Theorem 13.2 (P. Cohen). *The systems $\mathsf{ZF} \cup \mathsf{AC}$ and $\mathsf{ZF} \cup \neg\mathsf{AC}$ are both sound, i.e. they do not lead to contradictions. In other words, the axiom of choice is independent of* ZF.

Theorem 13.3. *Tychonoff's theorem is equivalent to the axiom of choice, i.e. they imply each other.*

Lecture 14

Oct. 3 — Tychonoff's Theorem, Part 2

14.1 Zorn's Lemma

Definition 14.1. A *partially ordered set* is a set X with a relation $\leq$ such that

(i) (reflexivity) $x \leq x$ for all $x \in X$,

(ii) (anti-symmetry) for all $x, y \in X$, if $x \leq y$ and $y \leq x$, then $x = y$,

(iii) and (transitivity) for all $x, y, z \in X$, if $x \leq y$ and $y \leq z$, then $x \leq z$.

Definition 14.2. A *chain* in a partial ordering is a completely ordered subset.

Definition 14.3. An *upper bound* of a chain $C \subseteq X$ is an element $u \in X$ such that $a \leq u$ for every $a \in C$. A *maximum* of X is an element $z \in X$ such that there is no $x \in X$, $x \neq z$, with $x \leq z$.

Lemma 14.1 (Zorn's lemma). *Let $(X, \leq)$ be a partially ordered set. If every chain in X has an upper bound, then X attains a maximum.*

Remark. Zorn's lemma is equivalent to the axiom of choice. Assuming Zorn's lemma, consider the set of partial choice functions on the index set I, ordered by inclusion of their domains of definition. Show an upper bound and obtain a full choice function as a maximal element by Zorn's lemma.

Remark. When applying Zorn's lemma, the partial ordering $\leq$ is often taken to be inclusion or containment of sets.[1] In this case, taking the union over chains typically yields an upper bound.[2]

Example 14.3.1. Every vector space has a basis. Here a basis is a maximal linearly independent set. Let X be the set of linearly independent sets, ordered by inclusion. Argue that the union over a chain in X is again a linearly independent set and thus an upper bound. Finish using Zorn's lemma.

Example 14.3.2. Every ring has maximal ideals. This can also be shown via Zorn's lemma.

14.2 Proof of Tychonoff's Theorem

Remark. Recall from Homework 4 the following problem: A space X is compact if and only if it has the *finite intersection property*, i.e. for every collection $\{C_i\}_{i \in I}$ of closed sets such that $\bigcap_{i \in I_{\text{fin}}} \neq \varnothing$ for every finite set $I_{\text{fin}} \subseteq I$, we have $\bigcap_{i \in I} C_i \neq \varnothing$. This is some description of compactness via closed sets instead of open sets. We will use this characterization of compactness for the proof of Tychonoff's theorem.

[1] One can easily verify that this satisfies the properties of a partial order.

[2] In many cases, one shows that this is indeed an upper bound by projecting to some element of the chain.

Theorem 14.1 (Tychonoff)**.** *Let $(X_i, \mathcal{T}_i)$ be compact for each $i \in I$. Then $\prod_{i \in I} X_i$ is compact.*

Proof. Let $\{C_\alpha\}_{\alpha \in J}$ be a collection of closed subsets of $\prod_{i \in I} X_i$ such that for every finite subset $J_{\text{fin}} \subseteq J$, we have $\bigcap_{\alpha \in J_{\text{fin}}} C_\alpha \neq \varnothing$. We then wish to show that $\bigcap_{\alpha \in J} C_\alpha \neq \varnothing$.

We say that a collection of sets of $\prod_{i \in I} X_i$ has the FI property if for every finite subcollection, the intersection is nonempty. Let S be the collection of all FI-subcollections of the product space. Define a partial order over S by containment. We want maximal elements in S, which we will find by applying Zorn's lemma. Let R be a chain in S, and define $T = \bigcup_{r \in R} r$. Clearly $r \leq T$ for every $r \in R$, so it suffices to show that T satisfies FI. To do this, pick J_{fin}, some finite set of indices for elements of T. For each $\alpha \in J_{\text{fin}}$, there exists $r_\alpha \in R$ such that $T_\alpha \in r_\alpha$. Pick the largest r_α out of the $\alpha \in J_{\text{fin}}$ (we can do this since R is a chain and J_{fin} is finite, so we can run a sorting algorithm, for instance). Let r_β be this largest element. Then $T_\alpha \in r_\beta$ for all $\alpha \in J_{\text{fin}}$, and r_β satisfies FI. Thus we obtain

$$\bigcap_{\alpha \in J_{\text{fin}}} T_\alpha \neq \varnothing$$

by the FI property of r_β. So T also satisfies FI, and thus T is an upper bound for R. Then by Zorn's lemma, there is a maximum M in the set S of FI-subcollections. We may assume M contains $\{C_\alpha\}_{\alpha \in J}$.

At this point, we make a few observations:

(i) If $\{S_j\}_{j=1}^m \in M$, then $\bigcap_{j=1}^m S_j \in M$.

To see this, assume not. Then consider $M \cup \{\bigcap_{j=1}^m S_j\}$. Any finite subset looks like

$$m_1 \cap \cdots \cap m_k \cap S_{j_1} \cap \cdots \cap S_{j_\ell},$$

which are all in M, so this intersection is nonempty. Thus $M \cup \{\bigcap_{j=1}^m S_j\}$ is a larger FI collection containing M, which contradicts the maximality of M.

(ii) If $S \subseteq X$ such that $S \cap m \neq \varnothing$ for every $m \in M$, then $S \in M$.

Similarly assume not. Then consider $M \cup \{S\}$. Any finite subset looks like

$$m_1 \cap \cdots \cap m_k \cap S = S \cap \bigcap_{j=1}^k m_j.$$

By the previous observation, $\bigcap_{j=1}^k m_j \in M$, so $S \cap \bigcap_{j=1}^k m_j \neq \varnothing$. Thus $M \cup \{S\}$ is a larger FI collection containing M, which again contradicts the maximality of M.

Now consider the M that contained $\{C_\alpha\}_{\alpha \in J}$. For each $i \in I$, consider $\{\pi_i(m)\}_{m \in M}$. Note that $\pi_i(m)$ need not be closed. So look at the closures $\overline{\pi_i(m)} \subseteq X_i$ instead. Then for any J_{fin} finite,

$$\bigcap_{j \in J_{\text{fin}}} \overline{\pi_i(m_j)} \neq \varnothing$$

because we have that

$$\varnothing \neq \pi_i\left(\bigcap_{j \in J_{\text{fin}}} m_j\right) \subseteq \bigcap_{j \in J_{\text{fin}}} \overline{\pi_i(m_j)}.$$

Here we know $\bigcap_{j\in J_{\text{fin}}} m_j \neq \varnothing$ by the FI property of M. Then by the compactness of the X_i, we have

$$\bigcap_{j\in J} \overline{\pi_i(m_j)} \neq \varnothing.$$

So choose $x_i \in \bigcap_{j\in J} \overline{\pi_i(m_j)}$, and set $x = (y_i)_{i\in I}$. Now we claim that (note that the C_α are already closed)

$$x \in \bigcap_{m\in M} \overline{m} \subseteq \bigcap_{\alpha\in J} C_\alpha \subseteq \prod_{i\in I} X_i,$$

which immediately implies Tychonoff's theorem. We finish the proof of this claim next class. $\square$

Lecture 15

Oct. 8 — Tychonoff's Theorem, Part 3

15.1 Proof of Tychonoff's Theorem, Continued

Theorem 15.1 (Tychonoff)**.** *Let* $(X_i, \mathcal{T}_i)$ *be compact for each* $i \in I$*. Then* $\prod_{i\in I} X_i$ *is compact.*

Proof. We left off with the claim that

$$x \in \bigcap_{m\in M} \overline{m} \subseteq \bigcap_{\alpha\in J} C_\alpha \subseteq \prod_{i\in I} X_i.$$

To show this, we will show that $x \in \overline{m}$ for every $m \in M$. We need to show:

- For every $m \in M$ and open $U \subseteq \prod_{i\in I} X_i$ containing x, we have $U \cap m \neq \varnothing$.

  Pick U a basis element (containing x), i.e.

$$\begin{aligned} U = U_{j_1} \times \cdots \times U_{j_n} \times \prod_{k\neq j_1,\ldots,j_n} X_k &= \left(U_{j_1} \times \prod_{r\neq j_1} X_r\right) \cap \cdots \cap \left(U_{j_n} \times \prod_{r'\neq j_n} X_{r'}\right) \\ &= \pi_{j_1}^{-1}(U_{j_1}) \cap \cdots \cap \pi_{j_n}^{-1}(U_{j_n}). \end{aligned}$$

  By property (i) of M, it is enough to show that $\prod_{j_\ell}^{-1} U_{j_\ell} \in M$ for each ℓ, since this implies that $U \in M$, so that $U \cap m \neq \varnothing$ for every $m \in M$ by the FI property of M. Now consider $\pi_{j_\ell}^{-1}(U_{j_\ell})$. By property (ii) of M, it suffices to show that $\pi_{j_\ell}^{-1}(U_{j_\ell}) \cap m \neq \varnothing$ for every $m \in M$. This happens if and only if $U_{j_\ell} \cap \pi_{j_\ell}(m) \neq \varnothing$, so it is enough to show this. As $x \in U$, we have $x_{j_\ell} \in U_{j_\ell}$. But also

$$x_{j_\ell} \in \bigcap_{m\in M} \overline{\pi_{j_\ell}(m)},$$

  so $x_{j_\ell} \in \overline{\pi_{j_\ell}(m)}$. Since U_{j_ℓ} is open and contains x_{j_ℓ}, this means that $U_{j_\ell} \cap \pi_{j_\ell}(m) \neq \varnothing$.

Thus $\bigcap_{\alpha\in J} \neq \varnothing$, so $\prod_{i\in I} X_i$ has the finite intersection property, i.e. $\prod_{i\in I} X_i$ is compact. □

Remark. If we have Tychonoff's theorem, why does Heine-Borel not work for a product of infinitely many factors $\mathbb{R}$? We run into issues with boundedness: It is nontrivial to even define a metric on an infinite product of metric spaces. An analogue of Heine-Borel for infinite dimensions is the Arzela-Ascoli theorem, which requires an additional property on top of closedness and boundedness (equicontinuity).

15.2 Quotient Spaces

Definition 15.1. A continuous and surjective function $p : X \to Y$ is called a *quotient map* if $U \subseteq Y$ is open if and only if $p^{-1}(U) \subseteq X$ is open.

Example 15.1.1. If a continuous surjection p is also an open map, then p is a quotient map.

Definition 15.2. An *equivalence relation* on a set X is a relation $\sim$ which is

(i) (reflexivity) $x \sim x$ for all $x \in X$,

(ii) (symmetry) $x \sim y$ if and only if $y \sim x$,

(iii) and (transitivity) if $x \sim y$ and $y \sim z$, then $x \sim z$.

Define the *equivalence class* of $x \in X$ as $[x] = \{y \in X \mid y \sim x\}$, and $X/{\sim}$ to be the set of equivalence classes. There is a standard map $p : X \to X/{\sim}$ given by $x \mapsto [x]$.[1]

Definition 15.3. We define the *quotient topology* on $X/{\sim}$ by giving it the smallest topology such that $p : X \to X/{\sim}$ is a quotient map, i.e. we define $U \subseteq X/{\sim}$ to be open if $p^{-1}(U) \subseteq X$ is open.

Example 15.3.1. Consider the interval $X = [0, 1]$. Define $\sim$ via the equivalence classes $[0] = \{0, 1\} = [1]$ and $[x] = \{x\}$ for $x \neq 0, 1$. Then the resulting quotient space $X/{\sim}$ is homeomorphic to the circle.[2]

[1] Note that p is surjective by construction. For any equivalence class, pick any member of it as a preimage.
[2] This an example of *gluing*.

Lecture 16

Oct. 10 — Quotient Spaces

16.1 Examples of Quotient Spaces

Example 16.0.1 (Line with two origins)**.** Take two copies of $\mathbb{R}$, i.e. $X = \mathbb{R} \times \{0, 1\}$, and define the equivalence relation $(x, 0) \sim (x, 1)$ for all $x \neq 0$. Then $X/{\sim}$ is not Hausdorff (consider the points $x = (0, 0)$ and $y = (0, 1)$, any neighborhoods around x, y must intersect), even though X is.

Remark. The above example shows that the quotient of a Hausdorff space need not be Hausdorff.

Example 16.0.2 (Cylinder)**.** Take a square in $\mathbb{R}^2$ and identify the left and right sides (in the same direction). Then the quotient space is homeomorphic to a cylinder.

Example 16.0.3 (Möbius strip)**.** Now take a square in $\mathbb{R}^2$ and identify the left and right sides in opposite directions. The resulting quotient space is still a 2D surface, but it is not homeomorphic to a cylinder. In particular, the Möbius strip is not orientable.

Example 16.0.4 (Torus)**.** Take a square in $\mathbb{R}^2$ and identify the left and right sides, as well as the top and bottom sides (both in the same direction). The resulting quotient space is homeomorphic to a torus. In particular, the torus has no boundary.

Example 16.0.5 (Klein bottle)**.** Take the square in $\mathbb{R}^2$, identify the left and right (in the same direction), but identify the top and bottom in opposite directions. This quotient space is the Klein bottle, which cannot be embedded in $\mathbb{R}^3$ (but can embed in $\mathbb{R}^4$). The Klein bottle is also not orientable.

Example 16.0.6 (Cone)**.** Take the square in $\mathbb{R}^2$, identify the left and right sides (in the same direction), and collapse the bottom side to a single point. The quotient is a cone, which is homeomorphic to a disk.

Remark. We can define the *connected sum* $S_1 \# S_2$ of two surfaces S_1, S_2 by removing a disk from each surface, and then gluing the boundaries of these removed disks together.

16.2 Properties of Quotient Spaces

Theorem 16.1. *Given a surjective map $p : X \to Y$, the set of $U \subseteq Y$ such that $p^{-1}(U)$ is open defines a topology on Y.*

Proof. First, for $\varnothing \in Y$, we have $p^{-1}(\varnothing) = \varnothing \in X$ is open, so $\varnothing \in Y$ is open. Similarly, $p^{-1}(Y) = X$ is open, so Y is open. Thus $\varnothing, Y$ are both open.

Now fix an arbitrary union $\bigcup_{\alpha\in I} U_\alpha \subseteq Y$ of open sets. Then we have

$$p^{-1}\left(\bigcup_{\alpha\in I} U_\alpha\right) = \bigcup_{\alpha\in I} p^{-1}(U_\alpha).$$

These sets are open in X by assumption, so their union is open as well. Thus $\bigcup_{\alpha\in I} U_\alpha$ is also open. Finally, consider a finite intersection $\bigcap_{i=1}^n U_i \subseteq Y$ of open sets. Similarlly we see that

$$p^{-1}\left(\bigcap_{i=1}^n U_i\right) = \bigcap_{i=1}^n p^{-1}(U_i).$$

Again these are open in X by assumption, so their finite intersection is open. So $\bigcap_{i=1}^n U_i$ is also open. □

Corollary 16.1.1. *If X is compact, then $X/\sim$ with the quotient topology is compact.*

Proof. We have $X/\sim = p(X)$, so $X/\sim$ is the continuous image of a compact set. □

Theorem 16.2 (Universal property of quotient spaces)**.** *Let X, Y be topological spaces and $\sim$ an equivalence relation on X. Let $f : X \to Y$ be a continuous map and further suppose that $f(x) = f(y)$ whenever $x \sim y$. Then the following diagram commutes:*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "p"'] & Y \\ X/\sim \arrow[ur, dashed, "\widetilde{f}"'] & \end{tikzcd}$$

i.e. there exists a unique function $\widetilde{f} : X/\sim \to Y$ such that $f = \widetilde{f} \circ p$.

Proof. For $[x] \in X/\sim$, define $\widetilde{f}([x]) = f(x)$. This is well-defined by assumption since f respects $\sim$. To show that $\widetilde{f}$ is continuous, consider an open set $U \subseteq Y$. Then

$$\widetilde{f}^{-1}(U) = p(f^{-1}(U)),$$

which is open since f is continuous and p is a quotient map (notice f^{-1} gives all of $[x]$ if it gives x). □

Lecture 17

Oct. 22 — Urysohn Metrization

17.1 More Properties of Quotient Spaces

Recall the universal property of the quotient space. The following is a corollary:

Corollary 17.0.1. *Assume $g : X \to Y$ is continuous, and define $x \sim y$ if $g(x) = g(y)$. Then the following diagram commutes:*

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "\pi"'] & Y \\ X/{\sim} \arrow[ur, dashed, "\widetilde{g}"'] & \end{tikzcd}$$

i.e. there exists a unique continuous map $\widetilde{g} : X/{\sim} \to Y$ such that $g = \widetilde{g} \circ \pi$.

Remark. Note that in the above corollary, $\widetilde{g}$ is injective by construction.

17.2 More on Separation Axioms

Remark. For now we will assume all spaces are T_1, i.e. every singleton is closed.

Definition 17.1. A space is *regular* if for every $x \in X$ and $A \subseteq X$ closed with $x \notin A$, there exists U_x, V_A open such that $x \in U_x$, $A \subseteq V_A$, and $U_x \cap V_A = \varnothing$.

Definition 17.2. A space is *normal* if for every $A, B \subseteq X$ closed with $A \cap B = \varnothing$, there exist U_A, V_B open such that $A \subseteq U_A$, $B \subseteq V_B$, and $U_A \cap V_B = \varnothing$.

Remark. Since we assume T_1, singletons are closed and thus we can think of x as the closed set $\{x\}$. This says that normal is stronger than regular. Similarly, we can set $A = \{y\}$, which is closed by the T_1 condition. This says that regular is stronger than Hausdorff. All of these relationships are strict.

Proposition 17.1. *Every metric space is normal.*

Proof. Take $A, B \subseteq (X, d)$ closed with $A \cap B = \varnothing$. Define $f : X \to \mathbb{R}$ by

$$f(x) = \frac{d(x, A)}{d(x, A) + d(x, B)}.$$

Recall from Homework 4 that

- if $d(x, A) = 0$ and A is closed, then $x \in A$,

- and if $x \in A$, B is closed, and $A \cap B = \varnothing$, then $d(x, B) > 0$.

So for all $x \in A$, we have
$$f(x) = \frac{d(x,A)}{d(x,A)+d(x,B)} = \frac{0}{d(x,B)} = 0$$
since $d(x, A) = 0$ and $d(x, B) > 0$. On the other hand, if $x \in B$, then
$$f(x) = \frac{d(x,A)}{d(x,A)+d(x,B)} = \frac{d(x,A)}{d(x,A)} = 1$$
since $d(x, B) = 0$ and $d(x, A) > 0$. Now note that we actually have $f : X \to [0, 1]$, where $f|_A \equiv 0$ and $f|_B \equiv 1$. Also note that f is continuous since $d(x, A)$ and $d(x, B)$ are continuous and $d(x, A)$ and $d(x, B)$ are never simultaneously zero. So define $U = f^{-1}([0, 1/4))$ and $V = f^{-1}((3/4, 1])$, which are open since f is continuous. Then $A \subseteq U$ and $B \subseteq V$, and $U \cap V = \varnothing$ since they are preimages of disjoint sets. □

17.3 Urysohn's Metrization Theorem

Definition 17.3. A space X has *locally countable basis* if for every $x \in X$, there exists a countable sequence of open neighborhoods $\{U_n\}$, each with $x \in U_n$, such that for any basis element B containing x, there exists U_n such that $x \in U_n \subseteq B$. A space X has *countable basis* if X has a countable basis.

Remark. Metric spaces have locally countable bases, e.g. take $U_n = B(x, 1/n)$.

Remark. A space with locally countable basis is sometimes called *first countable*, and a space with countable basis (not local) is sometimes called *second countable*. Note that $\mathbb{R}^n$ is second countable (for instance take open balls with rational centers and rational radii).

Theorem 17.1 (Urysohn metrization theorem)**.** *If X is a normal topological space with countable basis, then X is metrizable, i.e. there exists a metric $d : X \times X \to \mathbb{R}$ such that the induced metric topology coincides with the given topology on X.*

Lemma 17.1 (Urysohn's lemma)**.** *Let X be a normal topological space. Let $A, B \subseteq X$ be closed with $A \cap B = \varnothing$. Then there exists $f : X \to [0, 1]$ continuous such that $f|_A \equiv 0$ and $f|_B \equiv 1$.*[1]

Proof. Define $P = \mathbb{Q} \cap [0, 1] = \{p_0, p_1, \dots\}$ such that $p_0 = 1$ and $p_0 = 0$ (note that P is countable, so we may enumerate it). We want to find open sets $U_p \subseteq X$ for $p \in P$ such that $A \subseteq U_p$ and

1. $U_1 = U_{p_0} = B^c$,
2. and for all $p < q$ in P, we have $\overline{U_p} \subseteq U_q$.

We define these sets via induction. First set $U_1 = U_{p_0} = B^c$ as a base case, which is open since B is closed and contains A since $A \cap B = \varnothing$. Now for the induction step, assume we have $U_{p_0}, \dots, U_{p_n}$ open such that $\overline{U_{p_i}} \subseteq U_{p_j}$ for $i < j$. Consider p_{n+1}. We have the following cases:

- Suppose $p_{n+1} > p = \max\{p_0, \dots, p_n\}$. In this case, note that $\overline{U_p}$ and B are closed and disjoint (B is separated from U_p by an open set, so B cannot intersect the closure of U_p). Then normality guarantees open sets $U \supseteq \overline{U_p}$ and $V \supseteq B$ such that $U \cap V = \varnothing$. Set $U_{p_{n+1}} = U$.
- Now suppose $p_{n+1} < p = \min\{p_0, \dots, p_n\}$. Here note that U_p^c is closed, so A, U_p^c are closed and disjoint. Use the same normality argument as above to find $U_{p_{n+1}}$.

[1] A function f satisfying these properties is called an *Urysohn function*.

- Otherwise $p_i < p_{n+1} < p_j$ for some $0 \leq i, j \leq n$ (we have finitely many p_k at each step, so we can order them and find the right location for p_{n+1}). Then use the normality argument on $\overline{U_{p_i}}$ and $U_{p_j}^c$.

After the induction, we obtain the countable collection of open sets U_p for $p \in P$. Also for $p \in \mathbb{Q} \setminus P$, define $U_p = \varnothing$ if $p < 0$ and $U_p = X$ if $p > 1$. This defines U_p for all $p \in \mathbb{Q}$. Now define $f : X \to \mathbb{R}$ by

$$f(x) = \inf\{p \in \mathbb{Q} \mid x \in U_p\}.$$

First note that if $x \in A$, then $x \in A \subseteq U_p$ for every $p \in P$, so $f(x) = \inf\{p \in P\} = 0$. On the other hand, if $x \in B$, then $x \in B = U_1^c \subseteq U_p^c$ for every $p \leq 1$, i.e. $x \notin U_p$ for $p \leq 1$. But $x \in U_p = X$ for $p > 1$, so $f(x) = \inf\{q \in \mathbb{Q} \mid q > 1\} = 1$. So f satisfies the desired properties of an Urysohn function.

It only remains to show that f is continuous, which we will do next class. $\square$

Lecture 18

Oct. 24 — Urysohn Metrization, Part 2

18.1 Urysohn's Metrization Theorem, Continued

Lemma 18.1 (Urysohn's lemma). *Let X be a normal space, and $A, B \subseteq X$ closed and disjoint. Then there exists a continuous map $f : X \to [0,1]$ such that $f|_A \equiv 0$ and $f|_B \equiv 1$.*

Proof. We continue from last time. We had open sets U_p for $p \in P = \mathbb{Q} \cap [0,1]$ and $f : X \to \mathbb{R}$ given by

$$f(x) = \inf\{p \in P : x \in U_p\},$$

and we must now show that f is continuous. To do this, fix $x \in X$ and let $\epsilon > 0$. We must find an open set $U \ni x$ such that $f(U) \subseteq (f(x) - \epsilon, f(x) + \epsilon)$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, we can find $r, s \in \mathbb{Q}$ such that

$$f(x) - \epsilon < r < f(x) < s < f(x) + \epsilon.$$

Set $V = U_s \setminus \overline{U_r}$, which is open since U_s is open and $\overline{U_r}$ is closed. Note that $x \in U_s$ since $f(x) < s$. Now also $x \notin \overline{U_r}$ since otherwise we have $x \in \overline{U_r} \subseteq U_t$ for some $r < t < f(x)$, which is impossible since this implies $f(x) \leq t < f(x)$. So we see that $x \in V$. Now pick any $y \in V$, and we show that

$$f(x) - \epsilon < r \leq f(y) \leq s < f(x) + \epsilon.$$

Since $y \in V = U_s \setminus \overline{U_r}$, we have $r \leq f(y)$ since $y \notin \overline{U_r}$ and $f(y) \leq s$ since $y \in U_s$. This implies the desired inequality, so V is the desired open neighborhood containing x. Thus f is continuous. □

Theorem 18.1 (Urysohn metrization theorem). *Let X be a normal space which is second-countable, i.e. X has countable basis. Then X is metrizable.*

Remark. The converse does not hold: Not all metric spaces are second-countable (although most are).

Theorem 18.2 (Tietze extension theorem). *Let X be a normal space, $A \subseteq X$ closed, and $f : A \to \mathbb{R}$ continuous. Then one may extend f to all of X in a continuous fashion (preserving the same bound as f if f is bounded on A).*

Proof. Prove this as an exercise. Use Urysohn's lemma. □

Remark. The Tietze extension theorem is a generalization of Urysohn's lemma: For disjoint closed sets $A, B \subseteq X$, define $f : A \cup B \to \mathbb{R}$ by $f|_A \equiv 0$ and $f|_B \equiv 1$. Then Tietze extension defines f on all of X.

Example 18.0.1. Consider Hilbert's cube $X = [0,1]^{\mathbb{N}}$, which is compact by Tychonoff's theorem. We can define a metric on X by $d_X(x, y) = \sum_{n=1}^{\infty} 2^{-n} d(x_n, y_n)$, where d is any metric on $[0,1]$. Note that d is bounded since d is continuous and $[0,1] \times [0,1]$ is compact. To see that d_X is indeed a metric, note that:

(i) We clearly have $d_X(x, y) \geq 0$. If $d_X(x, y) = 0$, then $d(x_n, y_n) = 0$ for every n, so $x = y$.

(ii) Symmetry is clear from the definition of d_X.

(iii) For the triangle inequality, fix $x, y, z \in X$ and we have

$$d_X(x, z) = \sum_{n=1}^{\infty} 2^{-n} d(x_n, z_n) \leq \sum_{n=1}^{\infty} 2^{-n} (d(x_n, y_n) + d(y_n, z_n)).$$

Since $d_X(x, y)$ and $d_X(y, z)$ are finite ($d(x, y) \geq 0$ is bounded and $\sum_{n=1}^{\infty} 2^{-n}$ converges), the above series converges absolutely and thus we can split the above series into the sum of two series:

$$d_X(x, z) \leq \sum_{n=1}^{\infty} 2^{-n} d(x_n, y_n) + \sum_{n=1}^{\infty} 2^{-n} d(y_n, z_n) = d_X(x, y) + d_X(y, z).$$

This gives the triangle inequality.

The proof of the Urysohn metrization theorem will utilize this observation.

Lecture 19

Oct. 29 — Urysohn Metrization, Part 3

19.1 Proof of Urysohn's Metrization Theorem

Theorem 19.1 (Urysohn metrization theorem)**.** *If a topological space X is normal and second countable, i.e. it has a countable basis, then X is metrizable.*

Proof. The idea is to embed X into the Hilbert cube $[0,1]^{\mathbb{N}}$ with the product topology. Note that $[0,1]^{\mathbb{N}}$ is metrizable, e.g. take the metric

$$d(x,y) = \sum_{i=1}^{\infty} \frac{1}{2^i} d_i(x_i, y_i),$$

where d_i is a metric on $[0,1]$. We saw last time that d is a metric, and it also induces the product topology (check this as an exercise, we can throw away the tail since the series converges).

Now we give the embedding. Since X is second countable, let $\{B_n\}_{n=1}^{\infty}$ be a basis for the topology on X. For each pair of indices $n < m$ with $\overline{B_n} \subseteq B_m$, note that $\overline{B_n} \cap B_m^c = \varnothing$. Now $\overline{B_n}$ and B_m^c are also closed, so by Urysohn's lemma there exists a continuous function $g_{n,m} : X \to [0,1]$ such that $g_{n,m}(\overline{B_n}) = \{1\}$ and $g_{n,m}(B_m^c) = \{0\}$. Now define $f : X \to f(X) \subseteq [0,1]^{\mathbb{N}}$ (with the subspace topology) by

$$f(x) = (g_{n,m}(x))_{n,m\in\mathbb{N},\overline{B_n}\subseteq B_m}.$$

Note that these pairs n, m exist since there is some basis element B_m which contains x. Also note that $f(X)$ is metrizable (simply obtain a metric from $[0,1]^{\mathbb{N}}$ by restricting d). Then $\{x\}$ is closed (we assume T_1) and B_m^c is closed, so by normality there exists an open set U_n containing x such that $\overline{U_n} \subseteq B_m^c$. Then there exists a basis element $B_n \subseteq U_n$ containing x with the same property.

Now we show that $f : X \to f(X)$ is a topological embedding. We show that:

(i) f is injective: Assume $x \neq y$, and we want to find a pair n, m such that $g_{n,m}(x) \neq g_{n,m}(y)$. Since we assume X is T_1, normal implies Hausdorff and thus we can find disjoint open sets U_x, U_y with $x \in U_x$ and $y \in U_y$. From this we can find V_x, V_y open such that

$$x \in V_x \subseteq \overline{V_x} \subseteq U_x \quad \text{and} \quad y \in V_y \subseteq \overline{V_y} \subseteq U_y.$$

We can get such sets by considering $\{x\}$ and U_x^c and using normality. Now there exists a basis element B_n such that $x \in B_n \subseteq V_x$. Then by considering $\{x\}$ and B_n^c, using normality we can find a basis element B_m such that

$$x \in B_m \subseteq \overline{B_m} \subseteq B_n.$$

Then $g_{n,m}(x) = 1$ (we have $x \in \overline{B_m}$) but $g_{n,m}(y) = 0$ (we have $y \notin B_n$), so we get $f(x) \neq f(y)$.

(ii) f is continuous: The function $f : X \to [0,1]^{\mathbb{N}}$ is continuous by the universal property of products since each $g_{n,m}$ is continuous. Now $f(X)$ has the subspace topology, so $f : X \to f(X)$ is continuous.

(iii) f is open (on $f(X)$ with the subspace topology): Fix an open set $B \subseteq X$, and we would like to show that $f(B) \subseteq f(X) \subseteq [0,1]^{\mathbb{N}}$. Fix $y = f(x) \in f(B)$. Since $f(X)$ is metrizable, it suffices to find $\epsilon > 0$ such that whenever $z \in f(X)$ and $d(z,y) < \epsilon$), then we have $z \in f(B)$.

Assume otherwise that this does not hold, i.e. for every $\epsilon > 0$ there exists z_ϵ such that $d(z_\epsilon, y) < \epsilon$ and $z_\epsilon \notin f(B)$. Now set $\epsilon_n = 1/n$, which gives a sequence $z_n \in f(X) \subseteq [0,1]^{\mathbb{N}}$. Since $[0,1]^{\mathbb{N}}$ is compact metric space, it is sequentially compact, so there exists a convergent subsequence $z_{n_k} \to z \in [0,1]^{\mathbb{N}}$. Since $d(z_{n_k}, y) < 1/n_k \leq 1/k$ for every k, so $d(z,y) = 0$, i.e. $z = y \in f(X)$.

Now let $z_n = f(w_n)$ for some point $w_n \in X$, which is unique for each z_n since f is injective. Note that by continuity and injectivity, we have $w_n \to x$. Since $z_n \notin f(B)$, we must have $w_n \notin B$. Then $w_n \in B^c$ and x is a limit point of B^c, so we must have $x \in B^c$ since B^c is closed. Contradiction.

Then f is a topological embedding, and so the metric on $f(X)$ gives a metric on X. □

Remark. The idea in the previous proof is to generate bump functions similar to

$$f(x) = \begin{cases} \exp(-1/(1-x^2)^2) & \text{if } |x| < 1, \\ 0 & \text{otherwise} \end{cases}$$

in analysis. Then the peaks roughly indicate where the points in X are.

Remark. Normality can sometimes be difficult to show, but any compact Hausdorff space is normal.

19.2 Omissions in Point-Set Topology

Here are some topics we omitted but may be of interest:

- More on metrization: The Nagata-Smirnov and Bing metrization theorems.
- Dimension theory: The *topological dimension* of a space, for instance we would expect that the dimension of $\mathbb{R}^n$ is n. This is distinct from the notion of *Hausdorff dimension.*
- Topological manifolds: We can cover a space with an *atlas* consisting of of *charts* each homeomorphic to $\mathbb{R}^n$. The charts should be compatible on their intersection, i.e. $(f \circ g^{-1})|_{A \cap B} = \mathrm{id}$.
- The Baire category theorem.

Lecture 20

Oct. 31 — Algebraic Topology

20.1 Introduction to Algebraic Topology

Remark. We previously showed, using connectedness, that $\mathbb{R}$ cannot be homeomorphic to $\mathbb{R}^n$ for any $n \geq 2$. How can we show that $\mathbb{R}^2$ and $\mathbb{R}^3$ are not homeomorphic?

One attempt might be to use a similar idea and try removing a line from $\mathbb{R}^2$, thus disconnecting it. But the image of a line need not still be a line (unlike a point, whose image is always a point). For instance, space-filling curves exist, so it is not obvious that the image cannot also disconnect $\mathbb{R}^3$.

Another example is the cylinder and the Möbius strip. How can we show that these surfaces are not homeomorphic? They are both of dimension 2, connected, compact, etc. One way in this case is to consider normal maps, and show that the cylinder is orientable whereas the Möbius strip is not.

Algebraic topology provides new algebraic invariants (e.g. the fundamental group) that allow us to distinguish spaces in these cases. Distinguishing these algebraic invariants is often a much easier task.

20.2 Homotopy

Definition 20.1. Two maps $f, g : X \to Y$ are *homotopic* if there exists a *homotopy* $H : X \times [0,1] \to Y$ which satisfies $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$ for all $x \in X$.

Remark. It is helpful to think of the parameter $t \in [0,1]$ as time, and we move from f to g.

Example 20.1.1. Here are some examples:

1. In $\mathbb{R}$, the maps $f(x) = x$ and $g(x) = 0$ are homotopic. Define $H : \mathbb{R} \times [0,1] \to \mathbb{R}$ by
$$H(x,t) = (1-t)f(x) + tg(x) = (1-t)x.$$
In fact, this method (linear interpolation) works for any continuous $f : \mathbb{R} \to \mathbb{R}$, since $\mathbb{R}$ is *convex*.
2. If Y is path-connected, then any two constant maps $f_1, f_2 : X \to Y$ are homotopic. By path-connectedness, there is a curve $\gamma : [0,1] \to Y$ with $\gamma(0) = f_1(x)$ and $\gamma(1) = f_2(x)$, then
$$H(x,t) = \gamma(t)$$
is a homotopy between f_1 and f_2.
3. Let $f : X \to X$ be the identity map, which we will use to describe the shape of X. Now we would like a *skeleton* (like a wireframe) $g : X \to Y \subseteq X$ such that $g|_Y \equiv \mathrm{id}_Y$ and g is homotopic to f. Note that skeletons need not be unique or homeormorphic (consider a two-holed donut in $\mathbb{R}^2$).

4. A circle in $\mathbb{R}^2$ with a line segment protruding from it is homotopic to a normal circle. In some sense, the important feature of the circle from the perspective of homotopy is that it has a hole.

Remark. Homotopy allows us to capture the idea of holes in a space.

Definition 20.2. Two spaces X, Y are said to be *homotopy equivalent* if there are continuous maps $f : X \to Y$ and $g : Y \to X$ such that $f \circ g$ is homotopic to id_Y and $g \circ f$ is homotopic to id_X.

Remark. Homotopy equivalence is a strictly weaker (more general) notion than homeomorphism.

Lecture 21

Nov. 5 — The Fundamental Group

21.1 Homotopy

Remark. We will always write $I = [0, 1]$, so that a homotopy is a map $H : X \times I \to Y$. We will also write $f \sim g$ to indicate that f and g are homotopic.

Remark. Homotopy is an equivalence relation on the set of continuous maps from X to Y, and homotopy equivalence is an equivalence relation on the set of topological spaces.

Proposition 21.1. *Let $S \subseteq \mathbb{R}^n$ be a convex set with non-empty interior. Then S is homotopic to the closed n-dimensional ball $\overline{B^n}$.*

Proof. Since S has non-empty interior, there exists an open ball B_r^n inside it. We may assume $\overline{B_r^n}$ is contained inside S (e.g. by halving the radius). Denote the center of $\overline{B_r^n}$ by x_0. For any $y \in S$, by convexity the entire line segment from x_0 to y is contained in S. Using this idea, we define $H : S \times I \to S$ by

$$H(y, t) = x_0 + (y - x_0)g(y, t),$$

where

$$f(y) = \min\left\{\frac{r}{\|y - x_0\|}, 1\right\} \quad \text{and} \quad g(y, t) = \begin{cases} 1 + t(f(y) - 1)/f(y) & 0 \le t \le f(y), \\ f(y) & t \ge f(y). \end{cases}$$

Note that these functions are continuous as a composition of continuous maps. We can also verify

$$H(y, 0) = x_0 + (y - x_0)g(y, 0) = x_0 + (y - x_0) = y = \mathrm{id}_S(y)$$

and

$$H(y, 1) = x_0 + (y - x_0)g(y, 1) = x_0 + (y - x_0)\min\left\{1, \frac{r}{\|y - x_0\|}\right\}.$$

If the minimum is 1, then $r \ge \|y - x_0\|$ and so $y \in \overline{B_r^n}$. Thus we get

$$H(y, 1) = x_0 + (y - x_0) = y = \mathrm{id}_{\overline{B_r^n}}(y).$$

Otherwise the minimum is $r/\|y - x_0\|$, so we get

$$H(y, 1) = x_0 + (y - x_0)\frac{r}{\|y - x_0\|} \in \overline{B_r^n}.$$

So H is a homotopy between S and $\overline{B_r^n}$, as desired. $\square$

Remark. A simpler way to show this is to argue using that homotopy is an equivalence relation. Shrink both S and $\overline{B_r^n}$ to a point (both are convex, so simply shrink along the line segment $x_0 + (y - x_0)(1-t)$), so they must be homotopic to each other by transitivity.

Definition 21.1. A space X is called *contractible* if $\mathrm{id}_X \sim$ constant.

Example 21.1.1. The space $\mathbb{R}^n$ is contractible. Define $H(x,t) = (1-t)x$ as a homotopy to the origin.

Theorem 21.1. *The unit circle $S^1 \subseteq \mathbb{R}^2$ is not contractible.*

Remark. This theorem is actually quite hard to show, we will do this eventually.

Remark. This theorem also leads to a proof of the *Jordan curve theorem*, which states that any simple closed curve in $\mathbb{R}^2$ divides the plane into two regions, one bounded and one unbounded. The idea is to first show that the image of any such curve in is homotopic to the circle (this itself is difficult, and is what Jordan originally proved), and then show the result only for the circle.

21.2 The Fundamental Group

Remark. We will be interested in homotopy of curves $\gamma : [0,1] \to X$.

Definition 21.2. Given two curves $\gamma_1, \gamma_2 : [0,1] \to X$ with $\gamma_1(1) = \gamma_2(0)$, their *concatenation* is

$$\gamma_1 * \gamma_2(t) = \begin{cases} \gamma_1(2t) & 0 \le t \le 1/2, \\ \gamma_2(2t-1) & 1/2 \le t \le 1. \end{cases}$$

Definition 21.3. For a space X, its *fundamental group (with base point $x_0 \in X$)* is

$$\pi_1(X, x_0) = \{\text{continuous curves } \gamma : I \to X \text{ such that } \gamma(0) = \gamma(1) = x_0\}/\sim,$$

where $\sim$ is the homotopy of curves preserving x_0. The group operation on $\pi_1(X, x_0)$ is defined by

$$[\gamma_1] * [\gamma_2] = [\gamma_1 * \gamma_2],$$

where $[\gamma]$ denotes the equivalence class of γ under $\sim$.

Proposition 21.2. *The group operation $*$ on $\pi_1(X, x_0)$ is well-defined.*

Proof. First we show that the group operation is well-defined. Replace γ_1 with $\widetilde{\gamma_1}$ and γ_2 with $\widetilde{\gamma_2}$, where $\gamma_1 \sim \widetilde{\gamma_1}$ with homotopy H_1 and $\gamma_2 \sim \widetilde{\gamma_2}$ with homotopy H_2. We want $\gamma_1 * \gamma_2 \sim \widetilde{\gamma_1} * \widetilde{\gamma_2}$. Define $H(t,s)$ by

$$H(t,s) = \begin{cases} H_1(t, 2s) * \gamma_2 & 0 \le s \le 1/2, \\ \widetilde{\gamma_1} * H_2(t, 2s-1) & 1/2 \le s \le 1. \end{cases}$$

The idea is that we take the homotopy H_1 first followed by the homotopy H_2. Note that for $s \neq 1/2$, the map H is a composition of continuous maps, hence continuous. At $s = 1/2$, we have

$$H_1(t,1) * \gamma_2 = \widetilde{\gamma_1} * \gamma_2 = \widetilde{\gamma_1} * H_2(t,0)$$

since H_1 is a homotopy from γ_1 to $\widetilde{\gamma_1}$ and H_2 is a homotopy from γ_2 to $\widetilde{\gamma_2}$. Now we check that

$$H(t,0) = H_1(t,0) * \gamma_2 = \gamma_1 * \gamma_2 \quad \text{and} \quad H(t,1) = \widetilde{\gamma_1} * H_2(t,1) = \widetilde{\gamma_1} * \widetilde{\gamma_2}.$$

So H is a homotopy from $\gamma_1 * \gamma_2$ to $\widetilde{\gamma_1} * \widetilde{\gamma_2}$, and the group operation on $\pi_1(X, x_0)$ is well-defined. $\square$

Theorem 21.2. *The fundamental group $\pi_1(X, x_0)$ with the operation $*$ is a group.*

Proof. We have already shown that $*$ is well-defined, so it suffices to show the group axioms.

The identity element is the constant curve $e(t) = x_0$ for all $t \in [0, 1]$. We will finish this next class. $\square$