MATH 4431: Introduction to Topology

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Fall 2024

Contents

Lecture 1

Aug. 20 — Review of Metric Spaces

1.1 Metric Spaces

Recall the definition of a *metric space*:

Definition 1.1. Given a set X, a function $d : X \times X \to \mathbb{R}$ is called a *metric* if

(i) (strong positivity) $d(x,y) \geq 0$ for all $x,y \in X$, and $d(x,y) = 0$ if and only if $x = y$,

(ii) (symmetry) $d(x,y) = d(y,x)$,

(iii) and (triangle inequality) $d(x,z) \leq d(x,y) + d(y,z)$ for all $x,y,z \in X$.

Example 1.1.1. For any set X, we can define the *discrete metric* by

$$d(x,y) = \begin{cases} 1 & \text{if } x \neq y, \\ 0 & \text{otherwise.} \end{cases}$$

Verify as an exercise that this satisfies the triangle inequality.

Example 1.1.2. The Euclidean metric in $\mathbb{R}^n$ is

$$d(\overline{x}, \overline{y}) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$$

where $\overline{x} = (x_1, \ldots, x_n)$ and $\overline{y} = (y_1, \ldots, y_n)$.

1.2 Open Sets

Definition 1.2. The *open ball* of radius $R > 0$ around $x_0 \in X$ is

$$B_R(x_0) = \{y \in X \mid d(x_0, y) < R\}.$$

Given a set $S \subseteq X$, a point x_0 is called an interior point of S if there exists $r > 0$ such that $B_r(x_0) \subseteq S$. The set S is called *open* if all of its points are interior points.

Proposition 1.1. *The open ball $B_R(x)$ is open.*

Proof. Fix an arbitrary $y \in B_R(x)$, and observe that it suffices to show that y is an interior point. Take $r = R - d(x,y)$, and first note that $r > 0$ since $d(x,y) < R$. Now note that for all $z \in B_r(y)$, we have

$$d(x,z) \leq d(x,y) + d(y,z) < d(x,y) + (R - d(x,y)) = R,$$

so that $z \in B_R(x)$. Thus $B_r(y) \subseteq B_R(x)$, and so y is an interior point. □

Corollary 1.0.1. *$B_R(x) = \bigcup_{y \in B_R(x)} B_{r_y}(y)$, where $r_y = R - d(x,y)$.*

Proof. We have $B_{r_y}(y) \subseteq B_R(x)$ for each $y \in B_R(x)$,[1] and so $\bigcup_{y \in B_R(x)} B_{r_y}(y) \subseteq B_R(x)$. For the reverse inclusion simply observe that $y \in B_{r_y}(y) \subseteq \bigcup_{y \in B_R(x)} B_{r_y}(y)$ for each $y \in B_R(x)$. □

Proposition 1.2. *In a metric space (X,d), the following are true:*

(i) $\varnothing, X$ are open,

(ii) if $\{S_i\}_{i \in I}$ are open, then $\bigcup_{i \in I} S_i$ is open,

(iii) and if $\{S_i\}_{i=1}^n$ are open, then $\bigcap_{i=1}^n S_i$ is open.

Proof. (i) The empty set is open vacuously. To see that X is open, simply take $R = 1$ for any $x \in X$.

(ii) Fix $x \in \bigcup_{i \in I} S_i$ arbitrary, so there exists $i_0 \in I$ with $x \in S_{i_0}$. Since S_{i_0} is open, x is an interior point and thus there exists $r > 0$ such that $B_r(x) \subseteq S_{i_0}$. But then $B_r(x) \subseteq S_{i_0} \subseteq \bigcup_{i \in I} S_i$, so x is an interior point of $\bigcup_{i \in I} S_i$ also and thus $\bigcup_{i \in I} S_i$ is open.

(iii) Now assume $x \in \bigcap_{i=1}^n S_i$. Then for each $1 \leq i \leq n$, there exists $r_i > 0$ such that $B_{r_i}(x) \subseteq S_i$. Then we can choose

$$r = \min\{r_1, \ldots, r_n\} > 0,$$

so that $B_r(x) \subseteq B_{r_i}(x) \subseteq S_i$ for each $1 \leq i \leq n$. Thus $B_r(x) \subseteq \bigcap_{i=1}^n S_i$ and $\bigcap_{i=1}^n S_i$ is open. □

Remark. The above argument for the finite intersection property requires that there are only finitely many r_i. Otherwise it may very well be that $r = \inf\{r_i\} = 0$ and the argument fails.

[1] Using the argument from the previous proposition.

Lecture 2

Aug. 22 — Topology, Basis, Continuity

2.1 Topological Spaces

Definition 2.1. A *topology* $\mathcal{T} \subseteq \mathcal{P}(X)$ is a collection of sets such that

(i) $\varnothing, X \in \mathcal{T}$,

(ii) for any index set I, if $\{s_i\}_{i\in I} \subseteq \mathcal{T}$, then $\bigcup_{i\in I} s_i \in \mathcal{T}$ (closure under arbitrary union),

(iii) and if $\{s_i\}_{i=1}^n \subseteq \mathcal{T}$, then $\bigcap_{i=1}^n s_i \in \mathcal{T}$ (closure under finite intersection).

A set with a topology, i.e. a pair $(X, \mathcal{T})$, is called a *topological space*. Elements of $\mathcal{T}$ are called *open sets*.

Example 2.1.1. The following are examples of topologies on a set X:

- The trivial topology: $\mathcal{T} = \{\varnothing, X\}$.
- The discrete topology: $\mathcal{T} = \mathcal{P}(X)$.[1]
- If (X, d) is a metric space, then $\mathcal{T} = \{\text{collection of metrically open sets}\}$ is a topology on X.

Remark. Not every topology is induced by a metric. For instance consider the trivial topology on $\mathbb{R}$.

2.2 Basis for a Topology

Definition 2.2. A collection $\mathcal{B} \subseteq \mathcal{P}(X)$ is called a *basis* if

(i) $\bigcup_{b\in\mathcal{B}} b = X$, i.e. $\mathcal{B}$ is a covering of X,

(ii) and if $x \in b_1 \cap b_2$ for any $b_1, b_2 \in B$, then there exists $b_3 \in \mathcal{B}$ such that $x \in b_3$ and $b_3 \subseteq b_1 \cap b_2$.

Theorem 2.1. *Given a set X and a basis $\mathcal{B}$, define*

$$\mathcal{T}_\mathcal{B} = \left\{ \bigcup_{i\in I} s_i \mid I \text{ is any index set and } \{s_i\}_{i\in I} \subseteq \mathcal{B} \right\}.$$

Then $\mathcal{T}_\mathcal{B}$ is a topology on X.

Proof. First observe that $\varnothing, X \in \mathcal{T}_\mathcal{B}$: Picking $I = \varnothing$ gives $\bigcup_{i\in I} s_i = \varnothing \in \mathcal{T}_\mathcal{B}$ and picking $I = \mathcal{B}$ gives $\bigcup_{b\in\mathcal{B}} b = X \in \mathcal{T}_\mathcal{B}$ by the covering property of a basis.

[1] Note that the discrete topology is induced by the discrete metric.

Now assume $\{s_i\}_{i\in I} \subseteq \mathcal{T}_\mathcal{B}$. For each $i \in I$, we have $s_i \in \mathcal{T}_\mathcal{B}$ and so there exists an index set J_i such that $s_i = \bigcup_{j\in J_i} b_j$, where the $b_j \in \mathcal{B}$. Then

$$\bigcup_{i\in I} s_i = \bigcup_{i\in I}\bigcup_{j\in J_i} b_j,$$

which is a union of elements of $\mathcal{B}$ and hence is in $\mathcal{T}_\mathcal{B}$.

Finally assume $\{s_i\}_{i=1}^n \subseteq \mathcal{T}_\mathcal{B}$. Now as each $s_i \in \mathcal{T}_\mathcal{B}$, there exists J_i such that $s_i = \bigcup_{j\in J_i} b_j$. Then

$$\bigcap_{i=1}^n s_i = \bigcap_{i=1}^n \bigcup_{j\in J_i} b_j.$$

Now assume $x \in \bigcap_{i=1}^n s_i = \bigcap_{i=1}^n \bigcup_{j\in J_i} b_j$. For each $1 \le i \le n$, there exists $j_i \in J_i$ such that $x \in b_{j_i}$. Hence $x \in \bigcap_{i=1}^n b_{j_i}$. Now by induction on the intersection property of a basis, we can find $b_x \in \mathcal{B}$ with

$$x \in b_x \subseteq \bigcap_{i=1}^n b_{j_i}$$

Also observe that

$$\bigcap_{i=1}^n b_{j_i} \subseteq \bigcap_{i=1}^n \bigcup_{j\in J_i} b_j = \bigcap_{i=1}^n s_i$$

by construction, so we may write

$$\bigcap_{i=1}^n s_i = \bigcup_{x\in\bigcap_{i=1}^n s_i} b_x \in \mathcal{T}_\mathcal{B}$$

as a union of elements of $\mathcal{B}$. □

Definition 2.3. A *subbasis* $\mathcal{B} \subseteq \mathcal{P}(X)$ is a collection of sets such that $\bigcup_{b\in\mathcal{B}} b = X$.

Remark. One may define a basis $\widetilde{\mathcal{B}}$ from a subbasis $\mathcal{B}$ by adding all finite intersections of elements of $\mathcal{B}$. We get the covering property for free and adding the finite intersections gives us the intersection property of a basis.

Example 2.3.1. For $\mathbb{R}$ with the Euclidean metric, the following are bases for the standard topology:

- $\{B_R(x) \mid x \in \mathbb{R}, R > 0\}$.
- $\{B_R(x) \mid x \in \mathbb{R}, R > 0, R \in \mathbb{Q}\}$. For this use the fact that $\mathbb{Q}$ is dense in $\mathbb{R}$.

In particular this shows that a basis for a topology is not unique in general.

2.3 Continuous Functions

Definition 2.4. Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be two topological spaces. A function $f : X \to Y$ is called *continuous* if for any $O \in \mathcal{T}_Y$, we have $f^{-1}(O) \in \mathcal{T}_X$, i.e. the preimage of an open set is open.[2]

Example 2.4.1. Let X be equipped with the trivial topology $\{\varnothing, X\}$ and let $\mathbb{R}$ be equipped with the standard topology. Then the only continuous functions $f : X \to \mathbb{R}$ are the constant functions $f : x \mapsto c$ for fixed $c \in \mathbb{R}$. To see this, observe that

[2] Recall that $f^{-1}(O) = \{x \in X \mid f(x) \in O\}$.

- $x \mapsto c$ is continuous since any open set in $\mathbb{R}$ either contains c or does not, and so the preimage is either X or $\varnothing$.
- Suppose $f(x_1) = y_1$ and $f(x_2) = y_2$. Let $\epsilon = |y_1 - y_2|$ and observe that $x_1 \in f^{-1}(B_\epsilon(y_1))$ while $x_2 \notin f^{-1}(B_\epsilon(y_1))$, so $f^{-1}(B_\epsilon(y_1))$ is not open in X despite $B_\epsilon(y_1)$ being open in $\mathbb{R}$.

Example 2.4.2. Let X have the discrete topology $\mathcal{T} = \mathcal{P}(X)$ and let $\mathbb{R}$ have the standard topology. Then all functions $X \to \mathbb{R}$ are continuous since any preimage is a subset of X and thus in $\mathcal{P}(X)$.

Remark. In a way, the trivial topology has too few open sets while the discrete topology has too many.

Definition 2.5. Two topological spaces $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ are *topologically equivalent* or *homeomorphic* if there exists a bijection $f : X \to Y$ such that f and f^{-1} are continuous.

Remark. A bijective function f being continuous does not necessarily imply that its inverse f^{-1} is.

Example 2.5.1. Consider $(-\pi/2, \pi/2)$ equipped with the Euclidean metric. This is homeomorphic to $\mathbb{R}$ equipped with the Euclidean metric.[3] One homeomorphism is given by $\tan : (-\pi/2, \pi/2) \to \mathbb{R}$.

[3] Note that $(-\pi/2, \pi/2)$ is bounded while $\mathbb{R}$ is not.

Lecture 3

Aug. 27 — Closed Sets, Continuity, the Subspace Topology

3.1 Closed Sets

Definition 3.1. A set $S \subseteq X$ is called a *closed set* if $S^c = X \setminus S$ is open.

Example 3.1.1. In $\mathbb{R}$, observe that $[a,b]^c = (-\infty, a) \cup (b, \infty)$, which is a union of open sets and thus open. Thus the closed intervals $[a,b] \subseteq \mathbb{R}$ are closed.

Remark. This is not a dichotomy. Sets can be both open and closed (*clopen*), or even neither. Trivially, if X is any topological space, then $\varnothing$ and X are both open and closed.

Example 3.1.2. Let $X = \{0,1\}$ and $\mathcal{T} = \mathcal{P}(X)$. Then $\{0\}$ is both open and closed.

Example 3.1.3. Let $X = \{1,2,3\}$ and $\mathcal{T} = \{\varnothing, X, \{1\}, \{1,2\}\}$. Then $\{2\}$ is neither open nor closed.

Recall the following De Morgan's laws from set theory:

Proposition 3.1 (De Morgan's laws)**.** *Let I be an index set and $\{A_i\}_{i\in I}$ be sets. Then*

$$\left(\bigcup_{i\in I} A_i\right)^c = \bigcap_{i\in I} A_i^c \quad \textit{and} \quad \left(\bigcap_{i\in I} A_i\right)^c = \bigcup_{i\in I} A_i^c.$$

Corollary 3.0.1. *In a topological space $(X, \mathcal{T})$, we have:*

(i) $\varnothing, X$ are closed.

(ii) if $\{A_i\}_{i\in I}$ are closed, then $\bigcap_{i\in I} A_i$ is closed,

(iii) and if $\{A_i\}_{i=1}^n$ are closed, then so is $\bigcup_{i=1}^n A_i$.

This gives a dual characterization of a topology.

Proof. (i) We have $\varnothing^c = X \in \mathcal{T}$ and $X^c = \varnothing \in \mathcal{T}$.

(ii) Note that

$$\left(\bigcap_{i\in I} A_i\right)^c = \bigcup_{i\in I} A_i^c.$$

As each A_i is closed, we have $A_i^c \in \mathcal{T}$ is open, and hence $\bigcup_{i\in I} A_i^c \in \mathcal{T}$ is open. So $\bigcap_{i\in I} A_i$ is closed.

(iii) Observe that

$$\left(\bigcup_{i=1}^{n} A_i\right)^c = \bigcap_{i=1}^{n} A_i^c.$$

Each A_i is closed, so A_i^c is open. Thus $\bigcap_{i=1}^n A_i^c$ is open, and so $\bigcup_{i=1}^n A_i$ is closed. □

3.2 Properties of Continuity

Recall that a function $f : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ is continuous if for every $O \in \mathcal{T}_Y$, we have $f^{-1}(O) \in \mathcal{T}_X$.

Theorem 3.1. *A function $f : X \to Y$ is continuous if and only if for every C closed in Y, $f^{-1}(C)$ is closed in X.*

Proof. ($\Rightarrow$) Let $C \subseteq Y$ be closed. Note that

$$f^{-1}(C) = \{x \in X \mid f(x) \in C\},$$

so we have

$$(f^{-1}(C))^c = \{x \in X \mid f(x) \notin C\} = \{x \in X \mid f(x) \in C^c\} = f^{-1}(C^c).$$

Since C is closed, C^c is open and so $f^{-1}(C^c) = (f^{-1}(C))^c$ is open. Thus $f^{-1}(C)$ is closed.

($\Leftarrow$) Assume $S \subseteq Y$ is open. Note that

$$(f^{-1}(S))^c = \{x \in X \mid f(x) \in S\}^c = \{x \in X \mid f(x) \notin S\} = \{x \in X \mid f(x) \in S^c\} = f^{-1}(S^c).$$

Since S is open, S^c is closed and so $f^{-1}(S^c) = (f^{-1}(S))^c$ is closed by assumption. Thus $f^{-1}(S)$ is open, and so we see that f is continuous. □

Theorem 3.2 (Composition theorem). *Let $(X, \mathcal{T}_X)$, $(Y, \mathcal{T}_Y)$, and $(Z, \mathcal{T}_Z)$ be topological spaces. Let*

$$f : X \to Y \quad \text{and} \quad g : Y \to Z$$

be continuous functions. Then $g \circ f : X \to Z$ is continuous.

Proof. Let $S \subseteq Z$ be open. It suffices to show that $(g \circ f)^{-1}(S) \subseteq X$ is open. Note that

$$\begin{aligned}(g \circ f)^{-1}(S) = \{x \in X \mid (g \circ f)(x) \in S\} &= \{x \in X \mid f(x) \in g^{-1}(S)\} \\ &= \{x \in X \mid x \in f^{-1}(g^{-1}(S))\} = f^{-1}(g^{-1}(S)).\end{aligned}$$

Now as g is continuous, $g^{-1}(S)$ is open in Y. Finally as f is continuous, $f^{-1}(g^{-1}(S))$ is open in X. □

Theorem 3.3. *Assume $X = \bigcup_{\alpha \in I} U_\alpha$ for open sets U_α and let $f : X \to Y$. Assume that $f|_{U_\alpha} : U_\alpha \to Y$ is continuous for each $\alpha \in I$. Then f is continuous.*

Proof. Let $S \subseteq Y$ be open, and it suffices to show that $f^{-1}(S)$ is open. Observe that

$$f^{-1}(S) = f^{-1}(S) \cap X = f^{-1}(S) \cap \left(\bigcup_{\alpha \in I} U_\alpha\right) = \bigcup_{\alpha \in I} (f^{-1}(S) \cap U_\alpha) = \bigcup_{\alpha \in I} f|_{U_\alpha}^{-1}(S).$$

The $f|_{U_\alpha}$ are continuous, so each $f|_{U_\alpha}^{-1}(S)$ is open. Thus $f^{-1}(S)$ is open as a union of open sets. □

Theorem 3.4 (Gluing)**.** *Assume X, Y are topological spaces and $A, B \subseteq X$ are open. Suppose $f_1 : A \to Y$ and $f_2 : B \to Y$ are continuous, and that $f_1 \equiv f_2$ on $A \cap B$. Then $f : A \cup B \to Y$ defined by*

$$f(x) = \begin{cases} f_1(x) & \text{if } x \in A, \\ f_2(x) & \text{if } x \in B \end{cases}$$

is continuous.

Proof. Let $S \subseteq Y$ be open, it suffices to show that $f^{-1}(S)$ is open. Observe that

$$f^{-1}(S) = f_1^{-1}(S) \cup f_2^{-1}(S).$$

Both $f_1^{-1}(S)$ and $f_2^{-1}(S)$ are open since f_1 and f_2 are continuous, so $f^{-1}(S)$ is open as their union. □

3.3 Subspace Topology

Definition 3.2. Let $(X, \mathcal{T}_X)$ be a topological space and $S \subseteq X$ a set. The *subspace topology* on S is defined as follows: $O \subseteq S$ is open if there exists $U \subseteq X$ open in X such that $U = O \cap S$.

Example 3.2.1. Let $\mathbb{R}$ be given the metric topology and $S = [0, 1]$.

- The set $[0, 1]$ is not open in $\mathbb{R}$, but it is open in the subspace topology on S since $[0, 1] = S \cap (-1, 2)$.
- The set $[0, 1/2)$ is neither open nor closed in $\mathbb{R}$, but $[0, 1/2) = S \cap (-1/2, 1/2)$, so it is open in S.

Theorem 3.5. *The subspace topology is indeed a topology.*

Proof. Let $(X, \mathcal{T}_X)$ be a topological space and $S \subseteq X$ be given the subspace topology.

(i) We have $S = S \cap X$ and $\varnothing = \varnothing \cap X$, so $S, \varnothing$ are open in S.

(ii) Let $\{U_\alpha\}_{\alpha \in I}$ be open in the subspace topology. Then for every $\alpha \in I$, there exists $O_\alpha \in \mathcal{T}$ such that $U_\alpha = S \cap O_\alpha$. Then

$$\bigcup_{\alpha \in I} U_\alpha = \bigcup_{\alpha \in I} (S \cap O_\alpha) = S \cap \left(\bigcup_{\alpha \in I} O_\alpha \right).$$

The $\{O_\alpha\}_{\alpha \in I}$ are open in X, so their union is open in X. Thus $\bigcup_{\alpha \in I} U_\alpha$ is open in the subspace topology.

(iii) Let $\{U_i\}_{i=1}^n$ be open in the subspace topology. Then there are O_i for $1 \le i \le n$ with $U_i = S \cap O_i$. Then we have

$$\bigcap_{i=1}^n U_i = \bigcap_{i=1}^n (S \cap O_i) = S \cap \left(\bigcap_{i=1}^n O_i \right).$$

As the $O_i \in \mathcal{T}$ are open, $\bigcap_{i=1}^n O_i$ is open in X. Thus $\bigcap_{i=1}^n U_i$ is open in the subspace topology. □

Theorem 3.6. *Assume $f : X \to Y$ is a continuous function and $S \subseteq X$ a subspace. Then $f|_S : S \to Y$ is continuous, where S is equipped with the subspace topology.*

Proof. Let $O \subseteq Y$ be an open set, it suffices to show that $f|_S^{-1}(O)$ is open in the subspace topology. But observing that $f|_S^{-1}(O) = f^{-1}(O) \cap S$ immediately shows that $f|_S^{-1}(O)$ is open in S since $f^{-1}(O)$ is open in X due to the continuity of f. □

Remark. The subspace topology is the smallest topology on S such that the inclusion map $i : S \to X$ given by $i(s) = s$ is a continuous function.

Remark. Let X be a topological space with subspaces $Y \subseteq X$ and $Z \subseteq Y$. Then the subspace topology on Z induced by the subspace Y is the same as the subspace topology on Z induced directly by X.

Remark. A topological space can have a subspace homeomorphic to itself. For instance, consider $\mathbb{R}$ and $(-\pi/2, \pi/2)$ with a homemorphism given by the tangent function.