

# MATH 4431: Introduction to Topology

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# Contents

<b>1</b>	<b>Aug. 20 — Review of Metric Spaces</b>	<b>3</b>
1.1	Metric Spaces . . . . .	3
1.2	Open Sets . . . . .	3
<b>2</b>	<b>Aug. 22 — Topology, Basis, Continuity</b>	<b>5</b>
2.1	Topological Spaces . . . . .	5
2.2	Basis for a Topology . . . . .	5
2.3	Continuous Functions . . . . .	6
<b>3</b>	<b>Aug. 27 — Closed Sets, Continuity, the Subspace Topology</b>	<b>8</b>
3.1	Closed Sets . . . . .	8
3.2	Properties of Continuity . . . . .	9
3.3	Subspace Topology . . . . .	10
<b>4</b>	<b>Aug. 29 — Connectedness</b>	<b>12</b>
4.1	Connected Spaces . . . . .	12
4.2	Connected Sets . . . . .	13
<b>5</b>	<b>Sept. 3 — Path-Connectedness</b>	<b>15</b>
5.1	More on Connectedness . . . . .	15
5.2	Path-Connectedness . . . . .	16
<b>6</b>	<b>Sept. 5 — Compactness</b>	<b>18</b>
6.1	Note on the Subspace Topology . . . . .	18
6.2	Compactness . . . . .	18
<b>7</b>	<b>Sept. 10 — More Compactness</b>	<b>21</b>
7.1	The Cantor Set . . . . .	21
7.2	The Heine-Borel Theorem . . . . .	21
7.3	The Bolzano-Weierstrass Theorem . . . . .	22
<b>8</b>	<b>Sept. 12 — Separation Axioms</b>	<b>24</b>
8.1	Separation Axioms . . . . .	24
8.2	Properties of Hausdorff Spaces . . . . .	24
<b>9</b>	<b>Sept. 17 — Compactification</b>	<b>26</b>
9.1	Motivation for Compactification . . . . .	26
9.2	One-Point Compactification . . . . .	26
<b>10</b>	<b>Sept. 19 — Miscellaneous Topics</b>	<b>28</b>
10.1	Local Compactness and One-Point Compactification . . . . .	28

10.2 Distance to a Closed Set . . . . .	28
10.3 Nested Intersections in Compact Hausdorff Spaces . . . . .	29
<b>11 Sept. 24 — Product Spaces</b>	<b>31</b>
11.1 Product Spaces . . . . .	31
11.2 Properties of Product Spaces . . . . .	32

# Lecture 1

## Aug. 20 — Review of Metric Spaces

### 1.1 Metric Spaces

Recall the definition of a *metric space*:

**Definition 1.1.** Given a set  $X$ , a function  $d : X \times X \rightarrow \mathbb{R}$  is called a *metric* if

- (i) (strong positivity)  $d(x, y) \geq 0$  for all  $x, y \in X$ , and  $d(x, y) = 0$  if and only if  $x = y$ ,
- (ii) (symmetry)  $d(x, y) = d(y, x)$ ,
- (iii) and (triangle inequality)  $d(x, z) \leq d(x, y) + d(y, z)$  for all  $x, y, z \in X$ .

**Example 1.1.1.** For any set  $X$ , we can define the *discrete metric* by

$$d(x, y) = \begin{cases} 1 & \text{if } x \neq y, \\ 0 & \text{otherwise.} \end{cases}$$

Verify as an exercise that this satisfies the triangle inequality.

**Example 1.1.2.** The Euclidean metric in  $\mathbb{R}^n$  is

$$d(\bar{x}, \bar{y}) = \sqrt{\sum_{i=1}^n (x_i - y_i)^2}$$

where  $\bar{x} = (x_1, \dots, x_n)$  and  $\bar{y} = (y_1, \dots, y_n)$ .

### 1.2 Open Sets

**Definition 1.2.** The *open ball* of radius  $R > 0$  around  $x_0 \in X$  is

$$B_R(x_0) = \{y \in X \mid d(x_0, y) < R\}.$$

Given a set  $S \subseteq X$ , a point  $x_0$  is called an interior point of  $S$  if there exists  $r > 0$  such that  $B_r(x_0) \subseteq S$ . The set  $S$  is called *open* if all of its points are interior points.

**Proposition 1.1.** *The open ball  $B_R(x)$  is open.*

*Proof.* Fix an arbitrary  $y \in B_R(x)$ , and observe that it suffices to show that  $y$  is an interior point. Take  $r = R - d(x, y)$ , and first note that  $r > 0$  since  $d(x, y) < R$ . Now note that for all  $z \in B_r(y)$ , we have

$$d(x, z) \leq d(x, y) + d(y, z) < d(x, y) + (R - d(x, y)) = R,$$

so that  $z \in B_R(x)$ . Thus  $B_r(y) \subseteq B_R(x)$ , and so  $y$  is an interior point.  $\square$

**Corollary 1.0.1.**  $B_R(x) = \bigcup_{y \in B_R(x)} B_{r_y}(y)$ , where  $r_y = R - d(x, y)$ .

*Proof.* We have  $B_{r_y}(y) \subseteq B_R(x)$  for each  $y \in B_R(x)$ ,<sup>1</sup> and so  $\bigcup_{y \in B_R(x)} B_{r_y}(y) \subseteq B_R(x)$ . For the reverse inclusion simply observe that  $y \in B_{r_y}(y) \subseteq \bigcup_{y \in B_R(x)} B_{r_y}(y)$  for each  $y \in B_R(x)$ .  $\square$

**Proposition 1.2.** *In a metric space  $(X, d)$ , the following are true:*

- (i)  $\emptyset, X$  are open,
- (ii) if  $\{S_i\}_{i \in I}$  are open, then  $\bigcup_{i \in I} S_i$  is open,
- (iii) and if  $\{S_i\}_{i=1}^n$  are open, then  $\bigcap_{i=1}^n S_i$  is open.

*Proof.* (i) The empty set is open vacuously. To see that  $X$  is open, simply take  $R = 1$  for any  $x \in X$ .

(ii) Fix  $x \in \bigcup_{i \in I} S_i$  arbitrary, so there exists  $i_0 \in I$  with  $x \in S_{i_0}$ . Since  $S_{i_0}$  is open,  $x$  is an interior point and thus there exists  $r > 0$  such that  $B_r(x) \subseteq S_{i_0}$ . But then  $B_r(x) \subseteq S_{i_0} \subseteq \bigcup_{i \in I} S_i$ , so  $x$  is an interior point of  $\bigcup_{i \in I} S_i$  also and thus  $\bigcup_{i \in I} S_i$  is open.

(iii) Now assume  $x \in \bigcap_{i=1}^n S_i$ . Then for each  $1 \leq i \leq n$ , there exists  $r_i > 0$  such that  $B_{r_i}(x) \subseteq S_i$ . Then we can choose

$$r = \min\{r_1, \dots, r_n\} > 0,$$

so that  $B_r(x) \subseteq B_{r_i}(x) \subseteq S_i$  for each  $1 \leq i \leq n$ . Thus  $B_r(x) \subseteq \bigcap_{i=1}^n S_i$  and  $\bigcap_{i=1}^n S_i$  is open.  $\square$

**Remark.** The above argument for the finite intersection property requires that there are only finitely many  $r_i$ . Otherwise it may very well be that  $r = \inf\{r_i\} = 0$  and the argument fails.

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<sup>1</sup>Using the argument from the previous proposition.

# Lecture 2

## Aug. 22 — Topology, Basis, Continuity

### 2.1 Topological Spaces

**Definition 2.1.** A *topology*  $\mathcal{T} \subseteq \mathcal{P}(X)$  is a collection of sets such that

- (i)  $\emptyset, X \in \mathcal{T}$ ,
- (ii) for any index set  $I$ , if  $\{s_i\}_{i \in I} \subseteq \mathcal{T}$ , then  $\bigcup_{i \in I} s_i \in \mathcal{T}$  (closure under arbitrary union),
- (iii) and if  $\{s_i\}_{i=1}^n \subseteq \mathcal{T}$ , then  $\bigcap_{i=1}^n s_i \in \mathcal{T}$  (closure under finite intersection).

A set with a topology, i.e. a pair  $(X, \mathcal{T})$ , is called a *topological space*. Elements of  $\mathcal{T}$  are called *open sets*.

**Example 2.1.1.** The following are examples of topologies on a set  $X$ :

- The trivial topology:  $\mathcal{T} = \{\emptyset, X\}$ .
- The discrete topology:  $\mathcal{T} = \mathcal{P}(X)$ .<sup>1</sup>
- If  $(X, d)$  is a metric space, then  $\mathcal{T} = \{\text{collection of metrically open sets}\}$  is a topology on  $X$ .

**Remark.** Not every topology is induced by a metric. For instance consider the trivial topology on  $\mathbb{R}$ .

### 2.2 Basis for a Topology

**Definition 2.2.** A collection  $\mathcal{B} \subseteq \mathcal{P}(X)$  is called a *basis* if

- (i)  $\bigcup_{b \in \mathcal{B}} b = X$ , i.e.  $\mathcal{B}$  is a covering of  $X$ ,
- (ii) and if  $x \in b_1 \cap b_2$  for any  $b_1, b_2 \in \mathcal{B}$ , then there exists  $b_3 \in \mathcal{B}$  such that  $x \in b_3$  and  $b_3 \subseteq b_1 \cap b_2$ .

**Theorem 2.1.** Given a set  $X$  and a basis  $\mathcal{B}$ , define

$$\mathcal{T}_{\mathcal{B}} = \left\{ \bigcup_{i \in I} s_i \mid I \text{ is any index set and } \{s_i\}_{i \in I} \subseteq \mathcal{B} \right\}.$$

Then  $\mathcal{T}_{\mathcal{B}}$  is a topology on  $X$ .

*Proof.* First observe that  $\emptyset, X \in \mathcal{T}_{\mathcal{B}}$ : Picking  $I = \emptyset$  gives  $\bigcup_{i \in I} s_i = \emptyset \in \mathcal{T}_{\mathcal{B}}$  and picking  $I = \mathcal{B}$  gives  $\bigcup_{b \in \mathcal{B}} b = X \in \mathcal{T}_{\mathcal{B}}$  by the covering property of a basis.

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<sup>1</sup>Note that the discrete topology is induced by the discrete metric.

Now assume  $\{s_i\}_{i \in I} \subseteq \mathcal{T}_{\mathcal{B}}$ . For each  $i \in I$ , we have  $s_i \in \mathcal{T}_{\mathcal{B}}$  and so there exists an index set  $J_i$  such that  $s_i = \bigcup_{j \in J_i} b_j$ , where the  $b_j \in \mathcal{B}$ . Then

$$\bigcup_{i \in I} s_i = \bigcup_{i \in I} \bigcup_{j \in J_i} b_j,$$

which is a union of elements of  $\mathcal{B}$  and hence is in  $\mathcal{T}_{\mathcal{B}}$ .

Finally assume  $\{s_i\}_{i=1}^n \subseteq \mathcal{T}_{\mathcal{B}}$ . Now as each  $s_i \in \mathcal{T}_{\mathcal{B}}$ , there exists  $J_i$  such that  $s_i = \bigcup_{j \in J_i} b_j$ . Then

$$\bigcap_{i=1}^n s_i = \bigcap_{i=1}^n \bigcup_{j \in J_i} b_j.$$

Now assume  $x \in \bigcap_{i=1}^n s_i = \bigcap_{i=1}^n \bigcup_{j \in J_i} b_j$ . For each  $1 \leq i \leq n$ , there exists  $j_i \in J_i$  such that  $x \in b_{j_i}$ . Hence  $x \in \bigcap_{i=1}^n b_{j_i}$ . Now by induction on the intersection property of a basis, we can find  $b_x \in \mathcal{B}$  with

$$x \in b_x \subseteq \bigcap_{i=1}^n b_{j_i}$$

Also observe that

$$\bigcap_{i=1}^n b_{j_i} \subseteq \bigcap_{i=1}^n \bigcup_{j \in J_i} b_j = \bigcap_{i=1}^n s_i$$

by construction, so we may write

$$\bigcap_{i=1}^n s_i = \bigcup_{x \in \bigcap_{i=1}^n s_i} b_x \in \mathcal{T}_{\mathcal{B}}$$

as a union of elements of  $\mathcal{B}$ . □

**Definition 2.3.** A *subbasis*  $\mathcal{B} \subseteq \mathcal{P}(X)$  is a collection of sets such that  $\bigcup_{b \in \mathcal{B}} b = X$ .

**Remark.** One may define a basis  $\tilde{\mathcal{B}}$  from a subbasis  $\mathcal{B}$  by adding all finite intersections of elements of  $\mathcal{B}$ . We get the covering property for free and adding the finite intersections gives us the intersection property of a basis.

**Example 2.3.1.** For  $\mathbb{R}$  with the Euclidean metric, the following are bases for the standard topology:

- $\{B_R(x) \mid x \in \mathbb{R}, R > 0\}$ .
- $\{B_R(x) \mid x \in \mathbb{R}, R > 0, R \in \mathbb{Q}\}$ . For this use the fact that  $\mathbb{Q}$  is dense in  $\mathbb{R}$ .

In particular this shows that a basis for a topology is not unique in general.

## 2.3 Continuous Functions

**Definition 2.4.** Let  $(X, \mathcal{T}_X)$  and  $(Y, \mathcal{T}_Y)$  be two topological spaces. A function  $f : X \rightarrow Y$  is called *continuous* if for any  $O \in \mathcal{T}_Y$ , we have  $f^{-1}(O) \in \mathcal{T}_X$ , i.e. the preimage of an open set is open.<sup>2</sup>

**Example 2.4.1.** Let  $X$  be equipped with the trivial topology  $\{\emptyset, X\}$  and let  $\mathbb{R}$  be equipped with the standard topology. Then the only continuous functions  $f : X \rightarrow \mathbb{R}$  are the constant functions  $f : x \mapsto c$  for fixed  $c \in \mathbb{R}$ . To see this, observe that

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<sup>2</sup>Recall that  $f^{-1}(O) = \{x \in X \mid f(x) \in O\}$ .

- $x \mapsto c$  is continuous since any open set in  $\mathbb{R}$  either contains  $c$  or does not, and so the preimage is either  $X$  or  $\emptyset$ .
- Suppose  $f(x_1) = y_1$  and  $f(x_2) = y_2$ . Let  $\epsilon = |y_1 - y_2|$  and observe that  $x_1 \in f^{-1}(B_\epsilon(y_1))$  while  $x_2 \notin f^{-1}(B_\epsilon(y_1))$ , so  $f^{-1}(B_\epsilon(y_1))$  is not open in  $X$  despite  $B_\epsilon(y_1)$  being open in  $\mathbb{R}$ .

**Example 2.4.2.** Let  $X$  have the discrete topology  $\mathcal{T} = \mathcal{P}(X)$  and let  $\mathbb{R}$  have the standard topology. Then all functions  $X \rightarrow \mathbb{R}$  are continuous since any preimage is a subset of  $X$  and thus in  $\mathcal{P}(X)$ .

**Remark.** In a way, the trivial topology has too few open sets while the discrete topology has too many.

**Definition 2.5.** Two topological spaces  $(X, \mathcal{T}_X)$  and  $(Y, \mathcal{T}_Y)$  are *topologically equivalent* or *homeomorphic* if there exists a bijection  $f : X \rightarrow Y$  such that  $f$  and  $f^{-1}$  are continuous.

**Remark.** A bijective function  $f$  being continuous does not necessarily imply that its inverse  $f^{-1}$  is.

**Example 2.5.1.** Consider  $(-\pi/2, \pi/2)$  equipped with the Euclidean metric. This is homeomorphic to  $\mathbb{R}$  equipped with the Euclidean metric.<sup>3</sup> One homeomorphism is given by  $\tan : (-\pi/2, \pi/2) \rightarrow \mathbb{R}$ .

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<sup>3</sup>Note that  $(-\pi/2, \pi/2)$  is bounded while  $\mathbb{R}$  is not.



# Lecture 3

## Aug. 27 — Closed Sets, Continuity, the Subspace Topology

### 3.1 Closed Sets

**Definition 3.1.** A set  $S \subseteq X$  is called a *closed set* if  $S^c = X \setminus S$  is open.

**Example 3.1.1.** In  $\mathbb{R}$ , observe that  $[a, b]^c = (-\infty, a) \cup (b, \infty)$ , which is a union of open sets and thus open. Thus the closed intervals  $[a, b] \subseteq \mathbb{R}$  are closed.

**Remark.** This is not a dichotomy. Sets can be both open and closed (*clopen*), or even neither. Trivially, if  $X$  is any topological space, then  $\emptyset$  and  $X$  are both open and closed.

**Example 3.1.2.** Let  $X = \{0, 1\}$  and  $\mathcal{T} = \mathcal{P}(X)$ . Then  $\{0\}$  is both open and closed.

**Example 3.1.3.** Let  $X = \{1, 2, 3\}$  and  $\mathcal{T} = \{\emptyset, X, \{1\}, \{1, 2\}\}$ . Then  $\{2\}$  is neither open nor closed.

Recall the following De Morgan's laws from set theory:

**Proposition 3.1** (De Morgan's laws). *Let  $I$  be an index set and  $\{A_i\}_{i \in I}$  be sets. Then*

$$\left( \bigcup_{i \in I} A_i \right)^c = \bigcap_{i \in I} A_i^c \quad \text{and} \quad \left( \bigcap_{i \in I} A_i \right)^c = \bigcup_{i \in I} A_i^c.$$

**Corollary 3.0.1.** *In a topological space  $(X, \mathcal{T})$ , we have:*

- (i)  $\emptyset, X$  are closed.
- (ii) if  $\{A_i\}_{i \in I}$  are closed, then  $\bigcap_{i \in I} A_i$  is closed,
- (iii) and if  $\{A_i\}_{i=1}^n$  are closed, then so is  $\bigcup_{i=1}^n A_i$ .

*This gives a dual characterization of a topology.*

*Proof.* (i) We have  $\emptyset^c = X \in \mathcal{T}$  and  $X^c = \emptyset \in \mathcal{T}$ .

(ii) Note that

$$\left( \bigcap_{i \in I} A_i \right)^c = \bigcup_{i \in I} A_i^c.$$

As each  $A_i$  is closed, we have  $A_i^c \in \mathcal{T}$  is open, and hence  $\bigcup_{i \in I} A_i^c \in \mathcal{T}$  is open. So  $\bigcap_{i \in I} A_i$  is closed.

(iii) Observe that

$$\left( \bigcup_{i=1}^n A_i \right)^c = \bigcap_{i=1}^n A_i^c.$$

Each  $A_i$  is closed, so  $A_i^c$  is open. Thus  $\bigcap_{i=1}^n A_i^c$  is open, and so  $\bigcup_{i=1}^n A_i$  is closed.  $\square$

## 3.2 Properties of Continuity

Recall that a function  $f : (X, \mathcal{T}_X) \rightarrow (Y, \mathcal{T}_Y)$  is continuous if for every  $O \in \mathcal{T}_Y$ , we have  $f^{-1}(O) \in \mathcal{T}_X$ .

**Theorem 3.1.** *A function  $f : X \rightarrow Y$  is continuous if and only if for every  $C$  closed in  $Y$ ,  $f^{-1}(C)$  is closed in  $X$ .*

*Proof.* ( $\Rightarrow$ ) Let  $C \subseteq Y$  be closed. Note that

$$f^{-1}(C) = \{x \in X \mid f(x) \in C\},$$

so we have

$$(f^{-1}(C))^c = \{x \in X \mid f(x) \notin C\} = \{x \in X \mid f(x) \in C^c\} = f^{-1}(C^c).$$

Since  $C$  is closed,  $C^c$  is open and so  $f^{-1}(C^c) = (f^{-1}(C))^c$  is open. Thus  $f^{-1}(C)$  is closed.

( $\Leftarrow$ ) Assume  $S \subseteq Y$  is open. Note that

$$(f^{-1}(S))^c = \{x \in X \mid f(x) \in S\}^c = \{x \in X \mid f(x) \notin S\} = \{x \in X \mid f(x) \in S^c\} = f^{-1}(S^c).$$

Since  $S$  is open,  $S^c$  is closed and so  $f^{-1}(S^c) = (f^{-1}(S))^c$  is closed by assumption. Thus  $f^{-1}(S)$  is open, and so we see that  $f$  is continuous.  $\square$

**Theorem 3.2** (Composition theorem). *Let  $(X, \mathcal{T}_X)$ ,  $(Y, \mathcal{T}_Y)$ , and  $(Z, \mathcal{T}_Z)$  be topological spaces. Let*

$$f : X \rightarrow Y \quad \text{and} \quad g : Y \rightarrow Z$$

*be continuous functions. Then  $g \circ f : X \rightarrow Z$  is continuous.*

*Proof.* Let  $S \subseteq Z$  be open. It suffices to show that  $(g \circ f)^{-1}(S) \subseteq X$  is open. Note that

$$\begin{aligned} (g \circ f)^{-1}(S) &= \{x \in X \mid (g \circ f)(x) \in S\} = \{x \in X \mid f(x) \in g^{-1}(S)\} \\ &= \{x \in X \mid x \in f^{-1}(g^{-1}(S))\} = f^{-1}(g^{-1}(S)). \end{aligned}$$

Now as  $g$  is continuous,  $g^{-1}(S)$  is open in  $Y$ . Finally as  $f$  is continuous,  $f^{-1}(g^{-1}(S))$  is open in  $X$ .  $\square$

**Theorem 3.3.** *Assume  $X = \bigcup_{\alpha \in I} U_\alpha$  for open sets  $U_\alpha$  and let  $f : X \rightarrow Y$ . Assume that  $f|_{U_\alpha} : U_\alpha \rightarrow Y$  is continuous for each  $\alpha \in I$ . Then  $f$  is continuous.*

*Proof.* Let  $S \subseteq Y$  be open, and it suffices to show that  $f^{-1}(S)$  is open. Observe that

$$f^{-1}(S) = f^{-1}(S) \cap X = f^{-1}(S) \cap \left( \bigcup_{\alpha \in I} U_\alpha \right) = \bigcup_{\alpha \in I} (f^{-1}(S) \cap U_\alpha) = \bigcup_{\alpha \in I} f|_{U_\alpha}^{-1}(S).$$

The  $f|_{U_\alpha}$  are continuous, so each  $f|_{U_\alpha}^{-1}(S)$  is open. Thus  $f^{-1}(S)$  is open as a union of open sets.  $\square$

**Theorem 3.4** (Pasting lemma). *Assume  $X, Y$  are topological spaces and  $A, B \subseteq X$  are open. Suppose  $f_1 : A \rightarrow Y$  and  $f_2 : B \rightarrow Y$  are continuous, and that  $f_1 \equiv f_2$  on  $A \cap B$ . Then  $f : A \cup B \rightarrow Y$  defined by*

$$f(x) = \begin{cases} f_1(x) & \text{if } x \in A, \\ f_2(x) & \text{if } x \in B \end{cases}$$

*is continuous.*

*Proof.* Let  $S \subseteq Y$  be open, it suffices to show that  $f^{-1}(S)$  is open. Observe that

$$f^{-1}(S) = f_1^{-1}(S) \cup f_2^{-1}(S).$$

Both  $f_1^{-1}(S)$  and  $f_2^{-1}(S)$  are open since  $f_1$  and  $f_2$  are continuous, so  $f^{-1}(S)$  is open as their union.  $\square$

### 3.3 Subspace Topology

**Definition 3.2.** Let  $(X, \mathcal{T}_X)$  be a topological space and  $S \subseteq X$  a set. The *subspace topology* on  $S$  is defined as follows:  $O \subseteq S$  is open if there exists  $U \subseteq X$  open in  $X$  such that  $U = O \cap S$ .

**Example 3.2.1.** Let  $\mathbb{R}$  be given the metric topology and  $S = [0, 1]$ .

- The set  $[0, 1]$  is not open in  $\mathbb{R}$ , but it is open in the subspace topology on  $S$  since  $[0, 1] = S \cap (-1, 2)$ .
- The set  $[0, 1/2)$  is neither open nor closed in  $\mathbb{R}$ , but  $[0, 1/2) = S \cap (-1/2, 1/2)$ , so it is open in  $S$ .

**Theorem 3.5.** *The subspace topology is indeed a topology.*

*Proof.* Let  $(X, \mathcal{T}_X)$  be a topological space and  $S \subseteq X$  be given the subspace topology.

(i) We have  $S = S \cap X$  and  $\emptyset = \emptyset \cap X$ , so  $S, \emptyset$  are open in  $S$ .

(ii) Let  $\{U_\alpha\}_{\alpha \in I}$  be open in the subspace topology. Then for every  $\alpha \in I$ , there exists  $O_\alpha \in \mathcal{T}$  such that  $U_\alpha = S \cap O_\alpha$ . Then

$$\bigcup_{\alpha \in I} U_\alpha = \bigcup_{\alpha \in I} (S \cap O_\alpha) = S \cap \left( \bigcup_{\alpha \in I} O_\alpha \right).$$

The  $\{O_\alpha\}_{\alpha \in I}$  are open in  $X$ , so their union is open in  $X$ . Thus  $\bigcup_{\alpha \in I} U_\alpha$  is open in the subspace topology.

(iii) Let  $\{U_i\}_{i=1}^n$  be open in the subspace topology. Then there are  $O_i$  for  $1 \leq i \leq n$  with  $U_i = S \cap O_i$ . Then we have

$$\bigcap_{i=1}^n U_i = \bigcap_{i=1}^n (S \cap O_i) = S \cap \left( \bigcap_{i=1}^n O_i \right).$$

As the  $O_i \in \mathcal{T}$  are open,  $\bigcap_{i=1}^n O_i$  is open in  $X$ . Thus  $\bigcap_{i=1}^n U_i$  is open in the subspace topology.  $\square$

**Theorem 3.6.** *Assume  $f : X \rightarrow Y$  is a continuous function and  $S \subseteq X$  a subspace. Then  $f|_S : S \rightarrow Y$  is continuous, where  $S$  is equipped with the subspace topology.*

*Proof.* Let  $O \subseteq Y$  be an open set, it suffices to show that  $f|_S^{-1}(O)$  is open in the subspace topology. But observing that  $f|_S^{-1}(O) = f^{-1}(O) \cap S$  immediately shows that  $f|_S^{-1}(O)$  is open in  $S$  since  $f^{-1}(O)$  is open in  $X$  due to the continuity of  $f$ .  $\square$

**Remark.** The subspace topology is the smallest topology on  $S$  such that the inclusion map  $i : S \rightarrow X$  given by  $i(s) = s$  is a continuous function.

**Remark.** Let  $X$  be a topological space with subspaces  $Y \subseteq X$  and  $Z \subseteq Y$ . Then the subspace topology on  $Z$  induced by the subspace  $Y$  is the same as the subspace topology on  $Z$  induced directly by  $X$ .

**Remark.** A topological space can have a subspace homeomorphic to itself. For instance, consider  $\mathbb{R}$  and  $(-\pi/2, \pi/2)$  with a homeomorphism given by the tangent function.

# Lecture 4

## Aug. 29 — Connectedness

### 4.1 Connected Spaces

**Definition 4.1.** A *separation* of a topological space  $X$  is two open, nonempty sets  $U, V \subseteq X$  such that  $X = U \cup V$  and  $U \cap V = \emptyset$ . A space is called *connected* if there is no separation of the space.

**Proposition 4.1.** If  $X$  is separated, i.e.  $X = U \cup V$  with  $U, V$  open and disjoint, then  $U$  and  $V$  are both open and closed.

*Proof.* Observe that  $U$  is open by assumption, and we have

$$U^c = X \setminus U = V,$$

which is also open by assumption. Hence  $U$  is closed. The case for  $V$  is identical.  $\square$

**Example 4.1.1.** Consider the following:

- The singleton space  $\{x\}$  is connected. There are no two nonempty, disjoint open sets.
- Consider the space  $X = \{0, 1\}$ . This case depends on the choice of topology:
  1. With the trivial topology  $\mathcal{T} = \{\emptyset, X\}$ , the space is connected.
  2. With the discrete topology  $\mathcal{T} = \{\emptyset, X, \{1\}, \{0\}\}$ ,  $X$  is disconnected since  $X = \{0\} \cup \{1\}$ .
  3. With the topology  $\mathcal{T} = \{\emptyset, X, \{1\}\}$ , the space is connected. The only nonempty sets  $\{1\}, X$  are not disjoint and thus there can be no separation.

**Theorem 4.1.** A space  $X$  is disconnected if and only if there exists a surjective map  $f : X \rightarrow \{0, 1\}$  with the discrete topology.

*Proof.* ( $\Rightarrow$ ) If  $X$  is disconnected, then we may write  $X = U \cup V$  with  $U, V$  open, disjoint, and nonempty. Then define

$$f(x) = \begin{cases} 0 & x \in U, \\ 1 & x \in V, \end{cases}$$

which is surjective as  $U, V$  are nonempty. To see that  $f$  is continuous, observe that

$$f^{-1}(\emptyset) = \emptyset, \quad f^{-1}(\{0, 1\}) = X, \quad f^{-1}(\{0\}) = U, \quad f^{-1}(\{1\}) = V,$$

each of which are open. These are all of the open sets in the discrete topology, so  $f$  is continuous.

( $\Leftarrow$ ) Assume there exists a surjective and continuous map  $f : X \rightarrow \{0, 1\}$ . Define

$$U = f^{-1}(\{0\}) \quad \text{and} \quad V = f^{-1}(\{1\}),$$

which are open since  $f$  is continuous. Observe that  $U, V \neq \emptyset$  since  $f$  is surjective. Also  $U \cap V = \emptyset$  since if there is any  $x \in U \cap V$ , then  $f(x) = 0$  as  $x \in U$  and  $f(x) = 1$  as  $x \in V$ , a contradiction. Finally,  $X = U \cup V$  since  $f(x) = 0$  or  $f(x) = 1$  for every  $x \in X$ , i.e.  $x \in U$  or  $x \in V$ . So  $X$  is disconnected.  $\square$

## 4.2 Connected Sets

**Definition 4.2.** Let  $X$  be a topological space and  $S \subseteq X$ . Then  $S$  is called *connected* if it is connected in the subspace topology.

**Theorem 4.2.** If  $A, B$  are connected sets and  $A \cap B \neq \emptyset$ , then  $A \cup B$  is connected.

*Proof.* Assume not. Then there exists a continuous, surjective map  $f : A \cup B \rightarrow \{0, 1\}$  with the discrete topology. Consider  $f|_A : A \rightarrow \{0, 1\}$ , which is continuous in the subspace topology. Notice that  $f(A)$  cannot be  $\{0, 1\}$  since otherwise  $A$  is disconnected. Without loss of generality, assume  $f(A) = \{0\}$  since  $A$  is nonempty. Now consider  $f|_B : B \rightarrow \{0, 1\}$ , which is also continuous. Similarly, notice that  $f(B)$  cannot be  $\{0, 1\}$ . But there exists  $p \in A \cap B$ , and  $f(p) = 0$  as  $p \in A$ . Then since  $p \in B$ , we must have  $f(B) = \{0\}$ . But then we get that  $f(A \cup B) = \{0\} \neq \{0, 1\}$ , a contradiction to surjectivity.  $\square$

**Corollary 4.2.1.** A union of connected sets with “common points” is connected.

*Proof.* Run induction (transfinite if the union is infinite) using the previous theorem.  $\square$

**Theorem 4.3.** Closed intervals in  $[a, b] \subseteq \mathbb{R}$  with the metric topology are connected.

*Proof.* Assume otherwise that  $[a, b] = U \cup V$  with  $U, V$  disjoint, open, and nonempty. Assume without loss of generality that  $a \in U$ . Since  $V$  is nonempty, there exists  $c > a$  such that  $c \in V$ . Now consider  $[a, c] \subseteq [a, b]$  with  $U_1 = U \cap [a, c]$  and  $V_1 = V \cap [a, c]$ . By the least upper bound property of  $\mathbb{R}$ , since  $U_1$  is nonempty and bounded from above, there exists  $s = \sup U_1$  with  $s \leq c$ . Now either  $s \in U_1$  or  $s \notin U_1$ .

If  $s \in U_1$  (note this implies  $s \neq c$ ), then  $s$  is an interior point of  $U_1$  since  $U_1$  is open. So one may find a point  $t$  such that  $t > s$  and  $t \in U_1$ . But then  $s$  is no longer an upper bound of  $U_1$ , a contradiction.

Otherwise  $s \notin U_1$ . Since  $U_1, V_1$  cover  $[a, c]$ , we must then have  $s \in V_1$  (note this implies  $s \neq a$ ). Since  $V_1$  is open,  $s$  is an interior point of  $V_1$ , and thus there exists  $t < s$  such that  $t \in V_1$  and  $t$  is an upper bound for  $U_1$ . This contradicts  $s$  being the least upper bound of  $U_1$ .

Since both cases lead to contradictions, we conclude that  $[a, b]$  must be connected.  $\square$

**Corollary 4.3.1.** Open intervals in  $\mathbb{R}$  are connected, and  $\mathbb{R}$  itself is connected.

*Proof.* For some  $N_0 \geq 1$  (for instance choose  $N_0 \geq 2/(b-a)$ ) we can write

$$(a, b) = \bigcup_{n=N_0}^{\infty} \left[ a + \frac{1}{n}, b - \frac{1}{n} \right],$$

Each of these closed intervals is connected by the previous theorem, and thus the union is connected by Corollary 4.2.1 since they overlap. Similarly writing  $\mathbb{R} = \bigcup_{n=1}^{\infty} [-n, n]$  shows that  $\mathbb{R}$  is connected.  $\square$

**Corollary 4.3.2** (Intermediate value theorem). *Let  $f : [a, b] \rightarrow \mathbb{R}$  be a continuous function. Then for any  $f(a) < t < f(b)$ , there exists  $c \in [a, b]$  such that  $f(c) = t$ .*

*Proof.* Assume not. We can consider the open sets  $(-\infty, t)$  and  $(t, \infty)$  in  $\mathbb{R}$ . Then  $f^{-1}((-\infty, t))$  and  $f^{-1}((t, \infty))$  are open sets since  $f$  is continuous. They are clearly disjoint (since  $f$  must be well-defined), and also nonempty since  $a \in f^{-1}((-\infty, t))$  and  $b \in f^{-1}((t, \infty))$ . Also since  $f^{-1}(\{t\}) = \emptyset$  by assumption,

$$[a, b] = f^{-1}((-\infty, t)) \cup f^{-1}((t, \infty)).$$

But this is a separation of  $[a, b]$ , a contradiction since  $[a, b]$  is connected. □

**Proposition 4.2.** *The open interval  $(0, 1)$  is not homeomorphic to the closed interval  $[0, 1]$ .*

*Proof.* Removing any point from  $(0, 1)$  disconnects it, but  $[0, 1] = [0, 1] \setminus \{1\}$  remains connected.<sup>1</sup> □

**Proposition 4.3.** *The real line  $\mathbb{R}$  is not homeomorphic to the plane  $\mathbb{R}^n$  for any  $n \geq 2$ .*

*Proof.* Removing a point from  $\mathbb{R}$  disconnects it but the same is not true for  $\mathbb{R}^n$  when  $n \geq 2$ . □

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<sup>1</sup>To see that  $[0, 1]$  is connected, we can write  $[0, 1] = \bigcup_{n=2}^{\infty} [0, 1 - 1/n]$ .

# Lecture 5

## Sept. 3 — Path-Connectedness

### 5.1 More on Connectedness

**Remark.** The intervals  $[a, b] \subseteq \mathbb{R}$  are homeomorphic to  $[0, 1]$  for any  $a < b$ . We can take  $f : [a, b] \rightarrow [0, 1]$  defined by

$$f(x) = \frac{1}{b-a}(x-a)$$

for instance as a homeomorphism.

**Lemma 5.1.** *If  $X$  is connected and  $f : X \rightarrow Y$  is continuous, then  $f(X)$  is connected.*

*Proof.* This is part of Homework 2. □

**Corollary 5.0.1.** *The plane  $\mathbb{R}^2$  is connected.*

*Proof.* Express  $\mathbb{R}^2$  as the union of horizontal and vertical lines. Each line is the image of  $\mathbb{R}$  and is thus connected by Lemma 5.1. Also any pair of horizontal and vertical lines must intersect, so we can use Corollary 4.2.1 to conclude that the union  $\mathbb{R}^2$  is connected. □

**Remark.** We can extend this to  $\mathbb{R}^3$  by embedding planes (copies of  $\mathbb{R}^2$ ), and similarly for  $\mathbb{R}^n$ .

**Proposition 5.1.** *The unit circle  $\mathbb{S}^1 \subseteq \mathbb{R}^2$  is connected.*

*Proof.* Define  $\gamma : [0, 2\pi] \rightarrow \mathbb{R}^2$  by  $\gamma(t) = (\cos t, \sin t)$ . The image of  $\gamma$  is precisely  $\mathbb{S}^1$ . □

**Proposition 5.2.** *Define a relation  $\sim$  on  $X$  by  $x \sim y$  if there exists a connected subset  $S \subseteq X$  such that  $x, y \in S$ . Then  $\sim$  is an equivalence relation.*

*Proof.* For reflexivity, fix  $x \in X$  and let  $S$  be the largest connected set containing  $x$  (this exists since we know at least  $\{x\}$  must be connected). Then  $x \in S$ , so  $x \sim x$ .

For symmetry, fix  $x, y \in X$ . If  $x \sim y$ , then there exists a connected set  $S$  such that  $x, y \in S$ . But then  $y, x \in S$ , so we see that  $y \sim x$ .

For transitivity, assume that  $x \sim y$  and  $y \sim z$ . Then there exists  $S_1$  connected such that  $x, y \in S_1$  and  $S_2$  connected such that  $y, z \in S_2$ . Notice that  $S_1 \cap S_2 \neq \emptyset$  since  $y \in S_1 \cap S_2$ . Then  $S_1 \cup S_2$  is connected by Theorem 4.2 and  $x, y, z \in S_1 \cup S_2$ . In particular,  $x, z \in S_1 \cup S_2$  and thus  $x \sim z$ .

So we see that  $\sim$  is an equivalence relation. □



**Definition 5.1.** Let the equivalence relation  $\sim$  be defined on  $X$  as in Proposition 5.2. Then we can write  $X$  as the disjoint union of the equivalence classes of  $\sim$ . These equivalence classes are called the *connected components* of  $X$ .

**Remark.** The connected components of a space are defined solely via topologies, so they must be invariant under homeomorphism.

**Example 5.1.1.** The letter  $S$ , sitting in  $\mathbb{R}^2$ , is not homeomorphic to the letter  $T$ . There is a point we can remove from  $T$  to give three connected components, but removing any point from  $S$  gives at most two such connected components.

## 5.2 Path-Connectedness

**Remark.** Connectedness is usually a very difficult property to verify. This motivates *path-connectedness*.

**Definition 5.2.** A set  $S$  is *path-connected* if for all  $x, y \in S$ , there exists a continuous map  $\gamma : [0, 1] \rightarrow S$  such that  $\gamma(0) = x$  and  $\gamma(1) = y$ . Here  $[0, 1]$  is given the usual metric topology.

**Lemma 5.2.** *If  $S$  is path-connected, then  $S$  is connected.*

*Proof.* This is part of Homework 2. □

**Remark.** Unlike connectedness, it is immediately obvious that  $\mathbb{R}^n$  is path-connected. Simply take the line segment between any two points. Then we can conclude connectedness by the previous lemma.

**Example 5.2.1.** There are spaces which are connected but not path-connected.

- Consider the *topologist's sine curve*, given by the union of the vertical segment  $\{(0, y) \mid -1 \leq y \leq 1\}$  and the image of  $(0, \infty)$  under  $x \mapsto (x, \sin(1/x))$ , is an example of such a space. See Homework 2 for more details.
- Consider the *cone*  $C$  in  $\mathbb{R}^2$  defined by ( $(0, 1)$  denotes an open interval unless otherwise specified)

$$C = ([0, 1] \times \{0\}) \cup (K \times [0, 1]) \cup (\{0\} \times [0, 1]),$$

where  $K = \{1/n : n \in \mathbb{N}\}$ . Note that  $C$  is clearly path-connected and hence also connected. Then define the space

$$D = C \setminus (\{0\} \times (0, 1)),$$

which is now not path-connected (consider the point  $(0, 1) \in D$ ) but still connected.

**Remark.** Observe the following:

- One can define *path-connected components* in a similar manner as connected components.
- A continuous image of a path-connected space is path-connected. Simply compose the curve with the continuous map, which is now a path in the image.
- The union of path-connected spaces sharing a point is path-connected. Take two curves to the common point and concatenate them using the pasting lemma.
- In  $\mathbb{R}^n$ , connectedness is equivalent to path-connectedness. In general, this holds if you can get a basis of only connected sets.

**Remark.** Recall from homework that if  $f : [0, 1] \rightarrow [0, 1]$  is continuous, then  $f$  has a fixed point, i.e. there exists  $c \in [0, 1]$  with  $f(c) = c$ . This follows from a clever use of the intermediate value theorem. Now consider a more topological perspective. Consider the diagonal  $\{(x, x) \mid x \in [0, 1]\}$  and look at the graph of  $f$ , which is contained in the closed unit square. This graph is path-connected as the image of a path-connected set and so there is a path between the points  $(0, f(0))$  and  $(1, f(1))$ . But then this path must intersect the diagonal at some point, which gives a fixed point.

**Theorem 5.1.** (*Brouwer fixed point theorem*) *Let  $K$  be a closed, bounded, and convex set in  $\mathbb{R}$ . Then any continuous map  $f : K \rightarrow K$  has a fixed point, i.e. there exists  $c \in K$  such that  $f(c) = c$ .*

**Remark.** One can see the existence of the Nash equilibrium as a consequence of this theorem.

**Remark.** In  $\mathbb{R}^2$ , this theorem follows from the following claim. Let  $X = \text{maps}(\mathbb{S}^1, \mathbb{S}^1)$  be the set of all continuous maps from  $\mathbb{S}^1$  to itself. Then Brouwer's fixed point theorem in  $\mathbb{R}^2$  follows from the following:

**Theorem 5.2.** *The space  $\text{maps}(\mathbb{S}^1, \mathbb{S}^1)$  is not path connected.*

# Lecture 6

## Sept. 5 — Compactness

### 6.1 Note on the Subspace Topology

**Remark.** Let  $X$  be a topological space with topology  $\mathcal{T}_X$ , and let  $Y \subseteq X$  be a subset endowed with a topology  $\tau$ . Suppose that for any continuous  $f : X \rightarrow Z$ , there exists a continuous  $\tilde{f} : Y \rightarrow Z$  such that the following diagram commutes,

$$\begin{array}{ccc} X & \xrightarrow{f} & Z \\ i \uparrow & \nearrow \tilde{f} & \\ Y & & \end{array}$$

where  $i : Y \rightarrow X$  is the inclusion map.<sup>1</sup> Then in Homework 2 we showed that  $\mathcal{T}_Y \subseteq \tau$ . We can see this as a *universal property* for the subspace topology.

### 6.2 Compactness

**Definition 6.1.** A set  $C \subseteq X$  is called *compact* if for any *open cover*

$$C \subseteq \bigcup_{\alpha \in I} U_{\alpha}, \quad \text{each } U_{\alpha} \text{ is open,}$$

there exists a finite subcover  $C \subseteq \bigcup_{i=1}^n U_{\alpha_i}$ .

**Example 6.1.1.** Consider the following:

- In a finite topology, any set is compact. This is because any open cover is already finite.
- In a discrete space, i.e.  $\mathcal{T} = \mathcal{P}(X)$ , compact sets are precisely the finite sets. It is clear that finite sets are compact, for each  $x$  choose a single open set in the cover containing  $x$ . Conversely, if a set is compact, we can pick our open cover to contain only singletons, and the existence of a finite subcover means that the set has only finitely many elements.

**Theorem 6.1** (Heine-Borel). *Let  $C \subseteq \mathbb{R}^n$  be a subset, where  $\mathbb{R}^n$  is given the metric topology. Then  $C$  is compact if and only if  $C$  is closed and bounded.*

*Proof.* We postpone this proof until later. □

**Lemma 6.1.** *Let  $X$  be compact. If  $Y \subseteq X$  is closed, then  $Y$  is compact.*

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<sup>1</sup>Note that at least set-theoretically, this immediately defines  $\tilde{f} = f|_Y$ . But a priori we do not know that  $\tilde{f}$  is continuous.

*Proof.* Let  $Y \subseteq X$  closed be given, and assume that  $Y \subseteq \bigcup_{\alpha \in I} U_\alpha$  an open cover. Since  $Y$  is closed, its complement  $Y^c$  is open. Then

$$Y^c \cup \bigcup_{\alpha \in I} U_\alpha$$

is an open cover of  $X$  since  $X = Y \cup Y^c$ . Since  $X$  is compact, there exists a finite subcover

$$X \subseteq Y^c \cup \bigcup_{i=1}^n U_{\alpha_i}.$$

Now observe that  $Y \subseteq X$  and  $Y \cap Y^c = \emptyset$ , so actually  $Y \subseteq \bigcup_{i=1}^n U_{\alpha_i}$ , which is a finite subcover.  $\square$

**Theorem 6.2.** *Let  $X$  be compact and  $f : X \rightarrow Y$  continuous. Then  $f(X)$  is compact.*

*Proof.* Consider  $f(X) \subseteq Y$  and let  $f(X) \subseteq \bigcup_{\alpha \in I} V_\alpha$ , an open cover in  $Y$ . Notice that

$$X = f^{-1}(f(X)) \subseteq f^{-1}\left(\bigcup_{\alpha \in I} V_\alpha\right) = \bigcup_{\alpha \in I} f^{-1}(V_\alpha).$$

Note that each  $f^{-1}(V_\alpha)$  is open in  $X$  since  $f$  is continuous and  $V_\alpha$  is open in  $Y$ , so this is in fact an open cover of  $X$ . Thus since  $X$  is compact, we may extract a finite subcover

$$X \subseteq \bigcup_{i=1}^n f^{-1}(V_{\alpha_i}).$$

Then we see that

$$f(X) \subseteq f\left(\bigcup_{i=1}^n f^{-1}(V_{\alpha_i})\right) \subseteq \bigcup_{i=1}^n V_{\alpha_i},$$

which is a finite subcover of  $f(X)$ . Therefore  $f(X)$  is compact.  $\square$

**Theorem 6.3.** *Assume  $\{C_j\}_{j=1}^m$  are compact subsets of  $X$ . Then  $\bigcup_{j=1}^m C_j$  is compact.*

*Proof.* Assume  $\bigcup_{j=1}^m C_j \subseteq \bigcup_{\alpha \in I} U_\alpha$ , an open cover. Observe this is also an open cover of  $C_j$  for each  $1 \leq j \leq m$ , so we can extract a finite subcover, i.e. we can find  $\alpha_{j,1}, \dots, \alpha_{j,n_j}$  with

$$C_j \subseteq \bigcup_{i=1}^{n_j} U_{\alpha_{j,i}}.$$

Then we see that

$$\bigcup_{j=1}^m C_j \subseteq \bigcup_{j=1}^m \bigcup_{i=1}^{n_j} U_{\alpha_{j,i}},$$

which is still a finite union. This is then a finite subcover of  $\bigcup_{j=1}^m C_j$ , so  $\bigcup_{j=1}^m C_j$  is compact.  $\square$

**Theorem 6.4 (Weierstrass).** *Let  $f : [a, b] \rightarrow \mathbb{R}$  be continuous. Then  $f([a, b])$  is bounded, and moreover there exist  $x_{\max}, x_{\min} \in [a, b]$  such that  $f(x_{\max}) \geq f(x) \geq f(x_{\min})$  for all  $x \in [a, b]$ .*

*Proof.* Since  $f$  is continuous and  $[a, b]$  is compact (by Heine-Borel),  $f([a, b]) \subseteq \mathbb{R}$  is compact. Thus by Heine-Borel,  $f([a, b])$  is bounded. In particular, we can find  $M, m$  such that

$$m \leq f(x) \leq M \quad \text{for all } x \in [a, b].$$

For the second part, observe that  $f([a, b])$  is bounded and nonempty, so  $s = \sup f([a, b])$ . Since this is the supremum, there must exist  $y_i \in f([a, b])$  such that  $y_i \rightarrow s$  as  $i \rightarrow \infty$ . Now observe that  $f([a, b])$  is closed by Heine-Borel, and in particular it contains its limit points. Thus we obtain  $s \in f([a, b])$ . Then pick  $x_{\max} \in f^{-1}(\{s\}) \subseteq [a, b]$ , which will satisfy  $f(x_{\max}) = s \geq f(x)$  for all  $x \in [a, b]$ . by construction.

The argument for finding  $x_{\min} \in [a, b]$  is similar. □

**Theorem 6.5.** *Let  $X$  be a topological space,  $K \subseteq X$  compact, and  $f : K \rightarrow \mathbb{R}$  a continuous function. Then  $f$  is bounded over  $K$  and attains its minimum and maximum on  $K$ .*

*Proof.* The same argument goes through, replacing  $[a, b]$  by the compact set  $K$ . □

# Lecture 7

## Sept. 10 — More Compactness

### 7.1 The Cantor Set

Define  $I_0 = [0, 1]$  and remove the open middle-thirds interval to get

$$I_1 = [0, 1] \setminus (1/3, 2/3) = [0, 1/3] \cup [2/3, 1].$$

Continue by removing the middle thirds of each interval to get  $I_2, I_3, \dots$ . Then the *Cantor set* is defined to be  $K = \bigcap_{n \geq 0} I_n$ . The Cantor set is compact and uncountable. See more on Homework 3.

### 7.2 The Heine-Borel Theorem

**Theorem 7.1** (Heine-Borel). *If  $C \subseteq \mathbb{R}$ , then  $C$  is compact if and only if  $C$  is closed and bounded.*

*Proof.* ( $\Rightarrow$ ) This direction is easy, see Homework 3 for details.

( $\Leftarrow$ ) First we show that  $[a, b] \subseteq \mathbb{R}$  is compact. Let  $\{U_\alpha\}_{\alpha \in I}$  be an open cover for  $[a, b]$ , i.e.  $[a, b] \subseteq \bigcup_{\alpha \in I} U_\alpha$ . Now define

$$R = \{x \in [a, b] \mid [a, x] \text{ has a finite subcover}\}$$

Clearly  $a \in R$  since  $a \in [a, b] \subseteq \bigcup_{\alpha \in I} U_\alpha$ , so picking any single  $U_\alpha$  with  $a \in U_\alpha$  gives a finite subcover for  $[a, a] = \{a\}$ . The goal is now to show that  $b \in R$ . Observe that  $a \in R$  implies  $R \neq \emptyset$ , and  $R \subseteq [a, b]$ , so

$$s = \sup R$$

exists by the completeness of  $\mathbb{R}$ . We proceed to show that  $s \in R$  and then  $s = b$ , which will show that  $b \in R$ . As  $s \in [a, b] \subseteq \bigcup_{\alpha \in I} U_\alpha$ , we can find  $\alpha_s$  such that  $s \in U_{\alpha_s}$ . Since  $U_{\alpha_s}$  is open, we can find  $\delta > 0$  such that  $(s - \delta, s + \delta) \subseteq U_{\alpha_s}$ . Then since  $s$  is a least upper bound of  $R$ , we can find  $r \in R$  such that  $s - \delta < r \leq s$ . Now since  $r \in R$ ,  $[a, r]$  admits a finite subcover  $\{U_{\alpha_i}\}_{i=1}^n$ . Then

$$[a, s] = [a, r] \cup (s - \delta, s] \subseteq \left( \bigcup_{i=1}^n U_{\alpha_i} \right) \cup U_{\alpha_s}$$

is a finite subcover for  $[a, s]$ , so  $s \in R$ . Now observe that we actually covered

$$\left[ a, s + \frac{\delta}{2} \right] = [a, r] \cup (s - \delta, s + \delta) \subseteq \left( \bigcup_{i=1}^n U_{\alpha_i} \right) \cup U_{\alpha_s}$$

in the previous construction. Then  $s + \delta/2 \in R$ , which contradicts the minimality of  $s$  unless  $s = b$ . Thus  $b \in R$ , so  $[a, b]$  admits a finite subcover and thus  $[a, b]$  is compact.

Now let  $C \subseteq \mathbb{R}$  be an arbitrary closed and bounded set. Since  $C$  is bounded, there exists  $I = (a, b)$  such that  $C \subseteq I$ . But then  $C \subseteq I \subseteq \bar{I} = [a, b]$ , so  $C$  is a closed subset of a compact set, hence compact.  $\square$

**Remark.** The Heine-Borel theorem also holds more generally in  $\mathbb{R}^n$ . A later theorem will say that the product of compact sets is compact in the product topology, and thus we can run the same argument as above but with boxes in  $\mathbb{R}^n$  instead of intervals.

### 7.3 The Bolzano-Weierstrass Theorem

**Definition 7.1.** A point  $x$  is an *accumulation point* for a set  $S$  if for all open sets  $U$  containing  $x$ , we have  $(U \setminus \{x\}) \cap S \neq \emptyset$ .

**Remark.** We disallow constant sequences when talking about accumulation points.

**Proposition 7.1.** Let  $\text{Acc}(A)$  be the set of accumulation points of a set  $A$ . Then  $\bar{A} = A \cup \text{Acc}(A)$ .

*Proof.* We show that  $A \cup \text{Acc}(A)$  is closed, which will imply  $\bar{A} \subseteq A \cup \text{Acc}(A)$  by the minimality of the closure. Write

$$(A \cup \text{Acc}(A))^c = A^c \cap \text{Acc}(A)^c.$$

Now assume  $x \in A^c \cap \text{Acc}(A)^c$ . Since  $x \notin \text{Acc}(A)$ , there exists  $U_x$  open such that  $x \in U_x$  and

$$(A \setminus \{x\}) \cap U_x = \emptyset.$$

But also  $x \notin A$ , so  $A \setminus \{x\} = A$  and  $A \cap U_x = \emptyset$ . Then we can write

$$(A \cup \text{Acc}(A))^c = A^c \cap \text{Acc}(A)^c = \bigcup_{x \in A^c \cap \text{Acc}(A)^c} U_x.$$

This is a union of open sets, hence open, and so  $A \cup \text{Acc}(A)$  is closed.

For the other direction, assume  $x \in A \cup \text{Acc}(A)$ . If  $x \in A$ , we are done, so assume  $x \in \text{Acc}(A) \setminus A$ . Now assume otherwise that  $x \notin \bar{A}$ . Then  $x \in (\bar{A})^c$ , which is open. Set  $U = (\bar{A})^c$ , so that

$$U \cap (A \setminus \{x\}) = U \cap A = \emptyset.$$

But then this says that  $x$  is not an accumulation point, in contradiction.  $\square$

**Definition 7.2.** We say that a topological space  $X$  is *sequentially compact* if every bounded sequence has a convergent subsequence.

**Theorem 7.2** (Bolzano-Weierstrass). Any bounded infinite set  $S \subseteq \mathbb{R}^n$  has an accumulation point.

*Proof.* Since  $S$  is bounded, find a compact set containing  $S$ . Then apply the later Theorem 7.4.  $\square$

**Remark.** In general, compactness is *not* equivalent to sequential compactness, but both imply the Bolzano-Weierstrass theorem. However, in many spaces (including metric spaces, in particular), the two notions coincide (and are also equivalent to the Bolzano-Weierstrass theorem).

**Theorem 7.3.** *A sequentially compact space has the Bolzano-Weierstrass property, namely that any bounded infinite set has an accumulation point.*

*Proof.* This is easy, pick a countable subset (i.e. a sequence) and apply sequential compactness.  $\square$

**Theorem 7.4.** *A compact space has the Bolzano-Weierstrass property, namely that any infinite set has an accumulation point.*

*Proof.* Let  $A$  be an infinite set in  $X$ , where  $X$  is compact. Assume otherwise that  $A$  has no accumulation points in  $X$ . Then there is no accumulation point for  $A$  outside of  $A$ , so  $\text{Acc}(A) \subseteq A$ . This gives

$$\overline{A} = A \cup \text{Acc}(A) = A,$$

so  $A$  is closed. Thus  $A$  is a closed subset of a compact space, hence compact. Now for any  $a \in A$ , pick an open set  $U_a$  such that  $a \in U_a$  and  $U_a \cap (A \setminus \{a\}) = \emptyset$ . Write  $A \subseteq \bigcup_{a \in A} U_a$ , and by compactness we can find a finite subcover  $A \subseteq \bigcup_{i=1}^n U_{a_i}$ . Then observe that

$$A = A \cap \bigcup_{i=1}^n U_{a_i} = \bigcup_{i=1}^n (A \cap U_{a_i}) = \bigcup_{i=1}^n \{a_i\} = \{a_1, \dots, a_n\},$$

This is in contradiction with  $A$  being infinite.  $\square$

**Remark.** Usually, this proof goes by showing that compactness implies sequential compactness, which then implies the Bolzano-Weierstrass property. But this proof avoids going through convergent sequences.



# Lecture 8

## Sept. 12 — Separation Axioms

### 8.1 Separation Axioms

**Definition 8.1.** A topological space is said to satisfy the  $T_0$  axiom if the following holds: For every  $a, b \in X$  with  $a \neq b$ , there exists  $U$  open such that either  $a \in U, b \notin U$  or  $b \in U, a \notin U$ .

**Remark.** With the  $T_0$  axiom, we cannot choose which point is in  $U$  and which is not. For instance take  $X = \{a, b\}$  with topology  $\mathcal{T} = \{\emptyset, \{a\}, X\}$ . This space is  $T_0$ , but we can only choose  $U$  to contain  $a$ .

**Definition 8.2.** A space is said to satisfy the  $T_1$  axiom if for every  $a, b \in X$  with  $a \neq b$ , there exist  $U_a, U_b$  open such that  $a \in U_a, b \notin U_a$  and  $b \in U_b, a \notin U_b$ .

**Remark.** With the  $T_1$  axiom,  $U_a$  and  $U_b$  need not be disjoint.

**Definition 8.3.** A space is said to be  $T_2$  or Hausdorff if the following holds: For every  $a, b \in X$  with  $a \neq b$ , there exist  $U_a, U_b$  open such that  $a \in U_a, b \in U_b$  and  $U_a \cap U_b = \emptyset$ .

**Example 8.3.1.** Metric spaces are Hausdorff. For any  $a \neq b$ , pick balls with radius  $d(a, b)/2$  around  $a, b$ .

**Theorem 8.1.** We have the proper inclusion  $T_2 \subsetneq T_1 \subsetneq T_0$ .

*Proof.* The inclusions and  $T_0 \neq T_1$  is clear (e.g. above). For  $T_1 \neq T_2$  take the line with two origins<sup>1</sup>.  $\square$

**Theorem 8.2.** In a  $T_1$  space, every singleton  $\{x\}$  is closed.

*Proof.* Fix  $x \in X$ . For every  $y \neq x$ , by the  $T_1$  axiom we can find  $U_y$  open such that  $y \in U_y$  and  $x \notin U_y$ . In particular, this means that  $U_y \subseteq \{x\}^c$ . Then we can write

$$\{x\}^c \subseteq \bigcup_{y \in \{x\}^c} U_y \subseteq \{x\}^c.$$

So  $\{x\}^c = \bigcup_{y \in \{x\}^c} U_y$ , which is open as a union of open sets. Thus  $\{x\}$  is closed.  $\square$

### 8.2 Properties of Hausdorff Spaces

**Theorem 8.3.** In a Hausdorff space, a point  $x$  is an accumulation point of a set  $A$  if and only if every neighborhood of  $x$  contains infinitely many elements of  $A$ .

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<sup>1</sup>The line with two origins is  $X = \mathbb{R} \cup \{p\}$  with topology generated by the open sets in  $\mathbb{R}$  (with the metric topology), and adding  $\tilde{U} = (U \setminus \{0\}) \cup \{p\}$  for each open set  $U \subseteq \mathbb{R}$  containing 0. One can separate 0 and  $p$  but not with disjoint sets.

*Proof.* ( $\Leftarrow$ ) This is clear.

( $\Rightarrow$ ) Pick  $x$  an accumulation point of  $A$ , and assume otherwise that there exists a neighborhood  $U$  of  $x$  with only finitely many elements of  $A$ , i.e.  $|(U \setminus \{x\}) \cap A| < \infty$ . Then we can write

$$(U \setminus \{x\}) \cap A = \{a_1, \dots, a_n\} = \bigcup_{i=1}^n \{a_i\}.$$

Since our space is Hausdorff and thus also  $T_1$ , these singletons  $\{a_i\}$  are closed. Then  $(U \setminus \{x\}) \cap A$  is closed as a finite union of closed sets. Now since our space is Hausdorff, for every  $1 \leq i \leq n$  we can separate  $a_i$  from  $x$ , i.e. there exists  $U_{x_i}, U_{a_i}$  open such that  $x \in U_{x_i}$ ,  $a_i \in U_{a_i}$  and  $U_{x_i} \cap U_{a_i} = \emptyset$ . Then

$$U' = U \cap \bigcap_{i=1}^n U_{x_i}$$

is open as a finite intersection of open sets. Also  $x \in U'$  since  $x \in U$  and  $x \in U_{x_i}$  for each  $i$ . But

$$(U \setminus \{x\}) \cap A = \{a_1, \dots, a_n\} \subseteq \bigcup_{i=1}^n U_{a_i}$$

and  $U_{a_j} \cap \bigcap_{i=1}^n U_{x_i} = \emptyset$  for all  $j$ , so  $(U' \setminus \{x\}) \cap A = \emptyset$ . Contradiction.  $\square$

**Remark.** Maybe just the  $T_1$  axiom is enough for this theorem. Think more about this.

**Definition 8.4.** A sequence  $\{x_n\}_{n=1}^\infty \subseteq (X, \mathcal{T})$  converges to a point  $x \in X$ , written  $x_n \rightarrow x$ , if for any open set  $U$  containing  $x$ , there exists  $N_0 \in \mathbb{N}$  such that  $x_n \in U$  for every  $n \geq N_0$ .

**Theorem 8.4.** In a Hausdorff space, a convergent sequence has a unique limit.

*Proof.* Assume otherwise that  $x_n \rightarrow L_1$  and  $x_n \rightarrow L_2$  with  $L_1 \neq L_2$ . Then since our space is Hausdorff, we can find  $U_{L_1}, U_{L_2}$  open such that  $L_1 \in U_{L_1}$ ,  $L_2 \in U_{L_2}$  and  $U_{L_1} \cap U_{L_2} = \emptyset$ . Since  $x_n \rightarrow L_1$ , there exists  $N_0 \in \mathbb{N}$  such that  $x_n \in U_{L_1}$  for all  $n \geq N_0$ . Similarly we can find  $N'_0 \in \mathbb{N}$  with  $x_n \in U_{L_2}$  for all  $n \geq N'_0$  since  $x_n \rightarrow L_2$ . But then for  $N = \max\{N_0, N'_0\}$ , we have  $x_N \in U_{L_1} \cap U_{L_2}$ , a contradiction.  $\square$

**Theorem 8.5.** In a Hausdorff space, every compact set is closed.

*Proof.* Let  $C \subseteq X$  be compact, and we show that  $C^c$  is open. So fix  $y \in C^c$ . For any  $x \in C$ , since our space is Hausdorff, we can find  $U_x, U_y$  open such that  $x \in U_x$ ,  $y \in U_y$  and  $U_x \cap U_y = \emptyset$ . Now consider  $\bigcup_{x \in C} U_x$ . This is an open cover of  $C$ , so we can find a finite subcover  $C \subseteq \bigcup_{i=1}^n U_{x_i}$  since  $C$  is compact. Then the finite intersection  $\bigcap_{i=1}^n U_{y_i}$  is an open set contain  $y$ , and it is disjoint from  $C$  by construction since  $U_{x_i} \cap U_{y_i} = \emptyset$  for each  $i$ . Now set  $\tilde{U}_y = \bigcap_{i=1}^n U_{y_i}$ , so that

$$C^c \subseteq \bigcup_{y \in C^c} \tilde{U}_y \subseteq C^c.$$

Thus  $C^c = \bigcup_{y \in C^c} \tilde{U}_y$ , which is open as the union of open sets, so we conclude that  $C$  is closed.  $\square$

# Lecture 9

## Sept. 17 — Compactification

### 9.1 Motivation for Compactification

Let  $(X, \mathcal{T})$  be a topological space which is not compact. Usually we assume  $X$  is Hausdorff, and the goal is to find a compact space which looks like  $X$ , i.e. compactify  $X$ .

**Remark.** The naive idea is to take the trivial topology on  $X$  in place of  $\mathcal{T}$ , getting  $X_{\text{trivial}}$ . This is compact, the identity map  $\text{id} : X \rightarrow X_{\text{trivial}}$  is continuous and bijective, but it is not a homeomorphism. This is bad because we forget all the topological structure on  $X$ , for instance every sequence converges to every point in  $X_{\text{trivial}}$ . We would like to compactify  $X$  while keeping as much structure as possible.

**Example 9.0.1.** Let  $X = (0, 1)$  with the metric topology. Take  $Y = [0, 1]$  with the metric topology, so  $X$  embeds into  $Y$  by the inclusion map.<sup>1</sup> Note that  $Y$  is compact by Heine-Borel.

**Example 9.0.2.** Let  $X = (0, 1)$  with the metric topology. Take  $Y = \mathbb{S}^1 \subseteq \mathbb{R}^2$  to be the unit circle, where  $\mathbb{R}^2$  has the metric topology. Then  $X$  embeds into  $Y$  by the stereographic projection (technically  $\mathbb{R}$  is embedded but  $\mathbb{R}$  is homeomorphic to  $(0, 1)$  by the arctangent) by adding one point at the north pole.

**Example 9.0.3.** For the open unit disk in  $\mathbb{R}^2$ , we can add its boundary to get the closed unit disk as a compactification (the closed unit disk is compact by Heine-Borel). This adds uncountably many points. An alternative is to add only a single point at infinity, and identify the boundary with this point.

### 9.2 One-Point Compactification

**Definition 9.1.** For a topological space  $X$ , the *one-point compactification* (or *Alexandroff compactification*) of  $X$  is the set  $X^+ = X \cup \{\infty\}$  with topology generated by the basis

$$\{U \subseteq X \text{ open}\} \cup \{K^c \mid K \subseteq X \text{ is compact}\}.$$

**Remark.** One way to think about the one-point compactification is that we are forcing all unbounded sequences in  $X$  to converge to the new point  $\infty$ .

**Theorem 9.1.** For a Hausdorff topological space  $X$ , its one-point compactification  $X^+$  is compact.

*Proof.* Let  $\{O_\alpha\}$  be an open cover of  $X^+$ , i.e.  $X^+ = \bigcup_{\alpha \in I} O_\alpha$ . Some open set must contain the point  $\infty$ , and each open set contains a basis element, so there exists  $\alpha' \in I$  such that  $O_{\alpha'}$  contains  $K^c$ , for some  $K \subseteq X$  compact. Now since  $K \subseteq X \subseteq X^+$ , we see that  $K \subseteq \bigcup_{\alpha \in I} (O_\alpha \cap X)$  is an open cover of  $K$

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<sup>1</sup>By  $X$  embeds into  $Y$ , we mean that there is a continuous injection from  $X$  to  $Y$ .

(note that the Hausdorff condition implies that every compact  $K$  is closed in  $X$ , and thus the  $K^c$  basis elements are open in  $X$ ). So by compactness, there exists a finite subcover  $K \subseteq \bigcup_{\alpha \in I_{\text{finite}}} (O_\alpha \cap X)$ . Then

$$X^+ = O_{\alpha'} \cup K \subseteq O_{\alpha'} \cup \bigcup_{\alpha \in I_{\text{finite}}} (O_\alpha \cap X) \subseteq \bigcup_{\alpha \in I_{\text{finite}}} O_\alpha$$

since  $K^c \subseteq O_{\alpha'}$ . This is a finite subcover, so  $X$  is compact.  $\square$

**Remark.** In analysis, we sometimes speak of a sequence diverging to  $\infty$ , i.e. the sequence eventually escapes any compact set. This is precisely convergence to  $\infty$  in the one-point compactification.

**Theorem 9.2.** *Assume  $X$  is a Hausdorff, locally compact but not compact topological space.<sup>2</sup> Then the inclusion map  $\text{id} : X \rightarrow X^+$  is a dense, continuous embedding.<sup>3</sup>*

*Proof.* First clearly  $\text{id} : X \rightarrow \text{id}(X) \subseteq X^+$  is injective. Now we show that  $\text{id}$  is continuous. It is enough to show that the preimage of basis elements of  $X^+$  is open in  $X$ . If  $U \subseteq X$  is open, then  $\text{id}^{-1}(U) = U \subseteq X$  is clearly open. Otherwise consider  $K^c \cup \{\infty\}$  for  $K \subseteq X$  compact. Then we have

$$\text{id}^{-1}(K^c \cup \{\infty\}) = K^c \subseteq X.$$

Since  $X$  is Hausdorff, the compact set  $K$  is closed, and so  $K^c$  is open. Thus  $\text{id}$  is continuous.

Finally we show density, i.e.  $\overline{X} = X^+$ , where the closure is taken in  $X^+$ . To do this, suppose otherwise that  $\overline{X} \neq X^+$ . Clearly  $X \subseteq \overline{X}$ , so if  $\overline{X} \neq X^+$ , we must have  $\overline{X} = X$  (since  $X^+ = X \cup \{\infty\}$ ). But then  $X$  is closed in  $X^+$ , which is compact, so  $X$  is also compact in  $X^+$  since  $X$  and thus  $X^+$  is Hausdorff (since  $X$  is locally compact). This implies (exercise) that  $X$  itself is compact. Contradiction.  $\square$

**Remark.** If  $X$  is already compact, then  $X^c = \{\infty\}$  is open in  $X^+$ . Obviously  $X^+$  is compact since  $X$  is and every sequence converging to  $\infty$  must eventually be constant. In particular,  $X^+$  must be disconnected, and the extra point  $\infty$  just sits there to the side.

**Remark.** Due to the above, we must assume that  $X$  is not compact in order to get a dense embedding.

**Example 9.1.1.** Consider the space  $X = [0, 1)$ . Then the one-point compactification of  $X$  is  $[0, 1]$ .

<sup>2</sup>A space  $X$  is *locally compact* if for every  $x \in X$ , there exists  $U$  open with  $x \in U$  such that  $\overline{U}$  is compact.

<sup>3</sup>The embedding is *dense* if  $\overline{X} = X^+$ .

# Lecture 10

## Sept. 19 — Miscellaneous Topics

### 10.1 Local Compactness and One-Point Compactification

**Definition 10.1.** A map  $f : X \rightarrow Y$  is called *open* if for every  $U \subseteq X$  open,  $f(U)$  is open. A *topological embedding* is an injective, continuous, and open map.

**Remark.** This ensures that  $f$  with codomain restricted to  $f(U)$  is a homeomorphism.

**Example 10.1.1.** Demanding only that  $f$  is injective and continuous is not enough. Let  $(X, \mathcal{T})$  be any topological space and let  $X_{\text{trivial}}$  be  $X$  with the trivial topology. Then the identity map  $\text{id} : X \rightarrow X_{\text{trivial}}$  is injective, continuous, but not open in general if  $\mathcal{T}$  is not trivial. So this is *not* a topological embedding.

**Theorem 10.1.** If  $X$  is Hausdorff and locally compact, then  $X^+ = X \cup \{\infty\}$  is Hausdorff.

*Proof.* Pick  $y, y' \in X^+$  with  $y \neq y'$ . If  $y, y' \in X$ , then since  $X$  is Hausdorff, there are disjoint open sets  $U, U' \subseteq X$  with  $y \in U$  and  $y' \in U'$ . But then  $U, U' \subseteq X^+$  are still open and disjoint in  $X^+$  since the identity map embeds  $X$  into  $X^+$ . Thus  $U, U'$  are disjoint open sets separating  $y, y'$ .

Now assume without loss of generality that  $y' = \infty$ . By local compactness, there is a open set  $U \subseteq X$  containing  $y$  with compact closure. Then observe that  $(\overline{U})^c \cup \{\infty\}$  is open in  $X^+$  since  $\overline{U}$  is compact. Since  $U$  is also open in  $X^+$  and clearly disjoint from  $(\overline{U})^c \cup \{\infty\}$ , we have separated  $y, \infty$ .  $\square$

### 10.2 Distance to a Closed Set

**Definition 10.2.** Let  $(X, d)$  be a metric space. For  $x \in X$  and  $A \subseteq X$ , the *distance from  $x$  to  $A$*  is

$$d(x, A) = \inf\{d(x, a) : a \in A\}.$$

**Remark.** This infimum is achieved in  $\mathbb{R}^n$ , but not necessarily in a more general metric space.

**Proposition 10.1.** For any  $x \in \mathbb{R}^n$  and closed set  $A \subseteq \mathbb{R}^n$ , there exists  $a \in A$  with  $d(x, A) = d(x, a)$ .

*Proof.* Pick  $a_0 \in A$  and note that  $d(x, a_0) < \infty$ . Then set  $R = 2d(x, a_0) > 0$ . Consider  $\overline{B_R(x)} \cap A$ , which is a closed and bounded set containing  $a_0$ , hence compact by Heine-Borel. Now apply the compact case to  $\overline{B_R(x)} \cap A$  to get  $a_{\min}$  with  $d(x, a_{\min}) = d(x, \overline{B_R(x)} \cap A)$ . Now any  $a \in A$  with  $a \notin \overline{B_R(x)}$  satisfies

$$d(x, a) \geq R > d(x, a_0) \geq d(x, a_{\min}),$$

and hence it cannot be the minimum. Thus we must have  $d(x, a_{\min}) = d(x, A)$ .  $\square$

**Remark.** Define

$$\ell^2(\mathbb{N}) = \left\{ \{a_n\}_{n=1}^\infty \subseteq \mathbb{R} : \sum_{n=1}^\infty a_n^2 < \infty \right\}.$$

This is an inner product space over  $\mathbb{R}$ , and in particular we can induce a metric

$$d(\{a_n\}, \{b_n\}) = \sqrt{\sum_{n=1}^\infty |a_n - b_n|^2}$$

to turn  $(X, d)$  into a complete metric space. Also notice that  $\{e_i\}_{i=1}^\infty$ , where  $e_i$  is the sequence with 1 in the  $i$ th position and 0 everywhere else, is an orthonormal basis for  $\ell^2(\mathbb{N})$ .

**Lemma 10.1.** *Define the set*

$$A = \left\{ \left(1 + \frac{1}{i}\right) e_i \right\}_{i=1}^\infty.$$

*Then  $d(\{0\}, A) = 1$ . In particular, this is an example of a metric space where the infimum of the distance from a point to a closed set is not achieved.*

*Proof.* Observe that

$$d(\{0\}, (1 + 1/i)e_i) = 1 + \frac{1}{i},$$

and so

$$d(\{0\}, A) = \inf\{d(\{0\}, (1 + 1/i)e_i) : i \in \mathbb{N}\} = \inf\left\{1 + \frac{1}{i} : i \in \mathbb{N}\right\} = 1.$$

In particular, this infimum is clearly not achieved since  $d(\{0\}, (1 + 1/i)e_i) = 1 + 1/i > 1$  for each  $i \in \mathbb{N}$ . Now we show that  $A$  is closed by showing that it contains its limit points. For this, first observe that  $d(a, a') \geq 1$  for any  $a, a' \in A$  with  $a \neq a'$ . To see this, we can compute that

$$d((1 + 1/i)e_i, (1 + 1/j)e_j) = \sqrt{\left(1 + \frac{1}{i}\right)^2 + \left(1 + \frac{1}{j}\right)^2} \geq \sqrt{2} \geq 1$$

whenever  $i \neq j$ . Now assume we have  $\{a_n\} \subseteq A$  with  $a_n \rightarrow x$ . In particular,  $\{a_n\}$  must be a Cauchy sequence, and so for  $\epsilon = 1/2$ , there exists  $N_0 \in \mathbb{N}$  such that for all  $n, m \geq N_0$ , we have  $d(a_n, a_m) < 1/2$ . But  $d(a, a') \geq 1$ , so the sequence must stabilize after  $N_0$ , and hence  $x = a_n$  for  $n \geq N_0$ . In particular,  $x \in A$ , so we conclude that  $A$  is closed. This finishes the example.  $\square$

### 10.3 Nested Intersections in Compact Hausdorff Spaces

**Proposition 10.2.** *Let  $X$  be a compact Hausdorff space, and  $Y_i \subseteq X$  be closed and connected for  $i \in I$ . Assume the  $\{Y_i\}$  are totally ordered, i.e.  $Y_i \subseteq Y_j$  or  $Y_j \subseteq Y_i$  for all  $i, j \in I$ . Then  $\bigcap_{i \in I} Y_i$  is connected.*

*Proof.* Let  $U, V$  be a separation of  $Y = \bigcap_{i \in I} Y_i$ . Then  $Y = U \cup V$  and  $U, V$  are open in  $Y$ , disjoint, and nonempty. In particular, we can find  $U', V'$  open in  $X$  such that  $U = U' \cap Y$  and  $V = V' \cap Y$ . However,  $U', V'$  may no longer separate  $Y$ . This is why we need the Hausdorff condition. Use the next lemma to fix the proof from here, see more details in Homework 4.  $\square$

**Lemma 10.2.** *In a compact Hausdorff space, if  $C_1, C_2$  are two compact disjoint sets, then there exist  $U_1, U_2$  open and disjoint such that  $C_1 \subseteq U_1, C_2 \subseteq U_2$  and  $U_1 \cap U_2 = \emptyset$ .*

*Proof.* First we show this in the case where  $C_1 = \{x\}$  is a singleton and  $C_2 = C$ . For all  $y \in C$ , consider the pair  $x$  and  $y$ . Then there exists  $U_{x,y}, V_{x,y}$  open such that  $x \in U_{x,y}, y \in V_{x,y}$ , and  $U_{x,y} \cap V_{x,y} = \emptyset$ . Observe that  $\bigcup_{y \in C} V_{x,y}$  is an open cover of  $C$ , so by compactness there exists a finite subcover  $C \subseteq V = \bigcup_{i=1}^n V_{x,y_i}$ . Then  $x \in U = \bigcap_{i=1}^n U_{x,y_i}$ , which is open as a finite intersection of open sets. Also each  $U_{x,y_i}$  is disjoint from  $V_{x,y_i}$ , so  $U$  is disjoint from  $V$ . Then  $U_1 = U$  and  $U_2 = V$  are the desired open sets.

Now let  $C_1, C_2$  be any two disjoint compact sets. For all  $x \in C_1$ , there are open sets  $U_{x,C_2}, V_{x,C_2}$  such that  $x \in U_{x,C_2}, C_2 \subseteq V_{x,C_2}$ , and  $U_{x,C_2} \cap V_{x,C_2} = \emptyset$ . Then we make the same argument. We have

$$C_1 \subseteq \bigcup_{x \in C_1} U_{x,C_2},$$

an open cover of  $C_1$ , so by compactness there is a finite subcover  $C_1 \subseteq U_1 = \bigcup_{i=1}^m U_{x_i,C_2}$ . Then

$$C_2 \subseteq U_2 = \bigcap_{i=1}^m V_{x_i,C_2},$$

which is open as a finite intersection of open sets. As before,  $U_1 \cap U_2 = \emptyset$  since each  $U_{x_i,C_2}$  is disjoint from  $V_{x_i,C_2}$ . Thus we get a separation by disjoint compact sets.  $\square$

# Lecture 11

## Sept. 24 — Product Spaces

### 11.1 Product Spaces

**Definition 11.1.** For two sets  $X, Y$ , define their *Cartesian product* as

$$X \times Y = \{(x, y) \mid x \in X, y \in Y\}.$$

Similarly we can define<sup>1</sup>

$$\prod_{i \in I} X_i = \{(x) \mid (x)_i \in X_i\}.$$

If each  $X_i$  is a topological space with topology  $\mathcal{T}_i$ , then we define the following topologies on  $\prod_{i \in I} X_i$ :

- The *box topology*: Take as a basis sets of the form  $\prod_{i \in I} U_i$  where  $U_i \subseteq X_i$  are open sets.

Suppose  $B_1, B_2$  are basis sets and  $x \in B_1 \cap B_2$ . Then by definition  $B_1 = \prod_{i \in I} U_i$  and  $B_2 = \prod_{i \in I} V_i$ , where  $U_i, V_i \subseteq X_i$  are open. Then set  $B_3 = \prod_{i \in I} (U_i \cap V_i)$ . Clearly  $x \in B_3$  and  $B_3 \subseteq B_1 \cap B_2$  is a basis element (each  $U_i \cap V_i$  is a finite intersection of open sets and thus open), so this is a basis.

- The *product topology*: Take as a subbasis the sets  $\pi_i^{-1}(U_i) \subseteq \prod_{i \in I} X_i$  for each  $i \in I$  and  $U_i \subseteq X_i$  open. Here  $\pi_i : \prod_{j \in I} X_j \rightarrow X_i$  is the projection onto the  $i$ th factor.

The subbasis sets here are of the form  $U_i \times \prod_{j \neq i} X_j$ . The general basis sets will be finite intersections of these sets, i.e.

$$U_{i_1} \times \cdots \times U_{i_n} \times \prod_{j \neq i_1, \dots, i_n} X_j.$$

These are called the *cylindrical sets*. Think of this as having only finitely many restrictions on the factors, whereas we get to choose arbitrarily many restrictions with the box topology.

**Remark.** For finite products, the box and product topologies coincide. They differ for infinite products: The box topology is finer than the product topology (so the box topology has more open sets).

**Example 11.1.1.** Consider the following product spaces:

- The power set  $\mathcal{P}(X) = 2^X = \{0, 1\}^X$ . Think of the elements as functions  $X \rightarrow \{0, 1\}$ , which pick whether or not to include each element of  $X$  in the corresponding subset. Thus the power set comes with a natural topology (box or product) if  $\{0, 1\}$  is given the discrete topology.
- The space  $\{0, 1\}^{\mathbb{N}}$  is the Cantor set, if  $\{0, 1\}$  is given the discrete topology. The Cantor set with the metric topology inherited from  $\mathbb{R}$  is homeomorphic to  $\{0, 1\}^{\mathbb{N}}$  with the product topology. Think of the sequence  $\{x_n\} \subseteq \{0, 1\}^{\mathbb{N}}$  as choosing whether to pick the left or right third at each step.

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<sup>1</sup>One can also think of an element of the product as a function  $I \rightarrow \bigcup_{i \in I} X_i$ , where  $f(i) \in X_i$ .



- The space  $[0, 1]^{\mathbb{N}}$  is called *Hilbert's cube*. Note that  $[0, 1] \times [0, 1]^{\mathbb{N}} \cong [0, 1]^{\mathbb{N}}$ . For a homeomorphism, simply shift the sequence one to the right, putting a 0 in the first slot. Then forget about the 0.

**Remark.** Always assume  $\prod_{i \in I} X_i$  is given the product topology, unless otherwise specified.

## 11.2 Properties of Product Spaces

**Theorem 11.1.** Assume  $(X_i, \mathcal{T}_i)_{i \in I}$  are each  $T_0$ , each  $T_1$ , or each Hausdorff. Then the product  $\prod_{i \in I} X_i$  is also  $T_0$ ,  $T_1$ , or Hausdorff, respectively.

*Proof.* We prove only the Hausdorff case. Let  $(x), (y) \in \prod_{i \in I} X_i$  be distinct. As  $(x) \neq (y)$ , there is  $i \in I$  with  $x_i \neq y_i$ , where  $x_i, y_i \in X_i$ . Since  $X_i$  is Hausdorff, there exist  $A, B \subseteq X_i$  open, disjoint with  $x_i \in A$  and  $y_i \in B$ . Then set

$$U = A \times \prod_{j \neq i} X_j \quad \text{and} \quad V = B \times \prod_{j \neq i} X_j.$$

These sets are open since they are cylindrical, and clearly  $(x) \in U$ ,  $(y) \in V$  since  $x_i \in A$ ,  $y_i \in B$ . Also  $U, V$  are disjoint since their  $i$ th components  $A, B$  are disjoint. So this is a separation of  $(x)$  and  $(y)$  by disjoint, open sets, and we conclude that  $\prod_{i \in I} X_i$  is Hausdorff.  $\square$

**Corollary 11.1.1.** Assume  $(X_i, \mathcal{T}_i)_{i \in I}$  are each  $T_0$ , each  $T_1$ , or each Hausdorff. Then  $\prod_{i \in I} X_i$  with the box topology is also  $T_0$ ,  $T_1$ , or Hausdorff, respectively.

*Proof.* The box topology is finer than the product topology.  $\square$

**Remark.** In the product topology, the projections  $\pi_i : \prod_{j \in I} X_j \rightarrow X_i$  are continuous, onto, and open. The continuity and surjectivity of  $\pi_i$  is essentially by construction of the product topology. To see that  $\pi_i$  is open, consider a basis element of  $\prod_{j \in I} X_j$ , which is of the form

$$U = U_{i_1} \times \cdots \times U_{i_n} \times \prod_{j \neq i_1, \dots, i_n} X_j.$$

Then  $\pi_i(U)$  is either one of the  $U_i$  or one of the  $X_i$ , which are both open.

**Theorem 11.2** (Universal property of the product topology). *The following diagram commutes:*

$$\begin{array}{ccc} Z & \xrightarrow{f_i} & X_i \\ & \searrow f & \uparrow \pi_i \\ & & \prod_{i \in I} X_i \end{array}$$

In particular, there exists a unique continuous map  $f : Z \rightarrow \prod_{i \in I} X_i$  such that  $f_i = \pi_i \circ f$  for each  $i \in I$ .

*Proof.* Define  $f : Z \rightarrow \prod_{i \in I} X_i$  on the set theory level by  $(f(z))_i = f_i(z)$ . Now we show the continuity of  $f$ . Fix a basis element

$$B = U_{i_1} \times \cdots \times U_{i_n} \times \prod_{j \neq i_1, \dots, i_n} X_j.$$

Then we can write

$$f^{-1}(B) = \{z \mid f(z) \in B\}.$$

Now observe that  $f(z) \in B = U_{i_1} \times \cdots \times U_{i_n} \times \prod_{j \neq i_1, \dots, i_n} X_j$  is equivalent to

$$z \in \bigcap_{k=1}^n f_{i_k}^{-1}(U_{i_k}).$$

Since  $U_{i_k}$  is open in  $X_{i_k}$  and each  $f_i$  is continuous, each  $f_{i_k}^{-1}(U_{i_k})$  is open in  $Z$ . Then this is a finite intersection of open sets in  $Z$ , hence open. Thus  $f$  is continuous.  $\square$

**Remark.** If we had the box topology, this argument would not work. In particular, the intersection that we get could be infinite, which would not necessarily be open in  $Z$ .

**Remark.** This universal property formalizes the notion that we define functions from products by defining a function on each factor. Additionally, this generalizes the result from multivariable calculus that a vector-valued function is continuous precisely when each component function is continuous.