MATH 4431: Introduction to Topology

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Contents

Lecture 1

Aug. 20 — Review of Metric Spaces

1.1 Metric Spaces

Recall the definition of a *metric space*:

Definition 1.1. Given a set X, a function $d : X \times X \to \mathbb{R}$ is called a *metric* if

(i) (strong positivity) $d(x, y) \geq 0$ for all $x, y \in X$, and $d(x, y) = 0$ if and only if $x = y$,

(ii) (symmetry) $d(x, y) = d(y, x)$,

(iii) and (triangle inequality) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

Example 1.1.1. For any set X, we can define the *discrete metric* by

$$d(x, y) = \begin{cases} 1 & \text{if } x \neq y, \\ 0 & \text{otherwise.} \end{cases}$$

Verify as an exercise that this satisfies the triangle inequality.

Example 1.1.2. The Euclidean metric in $\mathbb{R}^n$ is

$$d(\overline{x}, \overline{y}) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$$

where $\overline{x} = (x_1, \ldots, x_n)$ and $\overline{y} = (y_1, \ldots, y_n)$.

1.2 Open Sets

Definition 1.2. The *open ball* of radius $R > 0$ around $x_0 \in X$ is

$$B_R(x_0) = \{y \in X \mid d(x_0, y) < R\}.$$

Given a set $S \subseteq X$, a point x_0 is called an interior point of S if there exists $r > 0$ such that $B_r(x_0) \subseteq S$. The set S is called *open* if all of its points are interior points.

Proposition 1.1. *The open ball $B_R(x)$ is open.*

Proof. Fix an arbitrary $y \in B_R(x)$, and observe that it suffices to show that y is an interior point. Take $r = R - d(x,y)$, and first note that $r > 0$ since $d(x,y) < R$. Now note that for all $z \in B_r(y)$, we have

$$d(x,z) \leq d(x,y) + d(y,z) < d(x,y) + (R - d(x,y)) = R,$$

so that $z \in B_R(x)$. Thus $B_r(y) \subseteq B_R(x)$, and so y is an interior point. □

Corollary 1.0.1. $B_R(x) = \bigcup_{y \in B_R(x)} B_{r_y}(y)$, *where* $r_y = R - d(x,y)$.

Proof. We have $B_{r_y}(y) \subseteq B_R(x)$ for each $y \in B_R(x)$,[1] and so $\bigcup_{y \in B_R(x)} B_{r_y}(y) \subseteq B_R(x)$. For the reverse inclusion simply observe that $y \in B_{r_y}(y) \subseteq \bigcup_{y \in B_R(x)} B_{r_y}(y)$ for each $y \in B_R(x)$. □

Proposition 1.2. *In a metric space* (X,d), *the following are true:*

(i) $\varnothing, X$ *are open,*

(ii) if $\{S_i\}_{i \in I}$ *are open, then* $\bigcup_{i \in I} S_i$ *is open,*

(iii) and if $\{S_i\}_{i=1}^n$ *are open, then* $\bigcap_{i=1}^n S_i$ *is open.*

Proof. (i) The empty set is open vacuously. To see that X is open, simply take $R = 1$ for any $x \in X$.

(ii) Fix $x \in \bigcup_{i \in I} S_i$ arbitrary, so there exists $i_0 \in I$ with $x \in S_{i_0}$. Since S_{i_0} is open, x is an interior point and thus there exists $r > 0$ such that $B_r(x) \subseteq S_{i_0}$. But then $B_r(x) \subseteq S_{i_0} \subseteq \bigcup_{i \in I} S_i$, so x is an interior point of $\bigcup_{i \in I} S_i$ also and thus $\bigcup_{i \in I} S_i$ is open.

(iii) Now assume $x \in \bigcap_{i=1}^n S_i$. Then for each $1 \leq i \leq n$, there exists $r_i > 0$ such that $B_{r_i}(x) \subseteq S_i$. Then we can choose

$$r = \min\{r_1, \ldots, r_n\} > 0,$$

so that $B_r(x) \subseteq B_{r_i}(x) \subseteq S_i$ for each $1 \leq i \leq n$. Thus $B_r(x) \subseteq \bigcap_{i=1}^n S_i$ and $\bigcap_{i=1}^n S_i$ is open. □

Remark. The above argument for the finite intersection property requires that there are only finitely many r_i. Otherwise it may very well be that $r = \inf\{r_i\} = 0$ and the argument fails.

[1] Using the argument from the previous proposition.

Lecture 2

Aug. 22 — Topology, Basis, Continuity

2.1 Topological Spaces

Definition 2.1. A *topology* $\mathcal{T} \subseteq \mathcal{P}(X)$ is a collection of sets such that

(i) $\varnothing, X \in \mathcal{T}$,

(ii) for any index set I, if $\{s_i\}_{i\in I} \subseteq \mathcal{T}$, then $\bigcup_{i\in I} s_i \in \mathcal{T}$ (closure under arbitrary union),

(iii) and if $\{s_i\}_{i=1}^n \subseteq \mathcal{T}$, then $\bigcap_{i=1}^n s_i \in \mathcal{T}$ (closure under finite intersection).

A set with a topology, i.e. a pair $(X, \mathcal{T})$, is called a *topological space*. Elements of $\mathcal{T}$ are called *open sets*.

Example 2.1.1. The following are examples of topologies on a set X:

- The trivial topology: $\mathcal{T} = \{\varnothing, X\}$.
- The discrete topology: $\mathcal{T} = \mathcal{P}(X)$.[1]
- If (X, d) is a metric space, then $\mathcal{T} = \{\text{collection of metrically open sets}\}$ is a topology on X.

Remark. Not every topology is induced by a metric. For instance consider the trivial topology on $\mathbb{R}$.

2.2 Basis for a Topology

Definition 2.2. A collection $\mathcal{B} \subseteq \mathcal{P}(X)$ is called a *basis* if

(i) $\bigcup_{b\in\mathcal{B}} b = X$, i.e. $\mathcal{B}$ is a covering of X,

(ii) and if $x \in b_1 \cap b_2$ for any $b_1, b_2 \in B$, then there exists $b_3 \in \mathcal{B}$ such that $x \in b_3$ and $b_3 \subseteq b_1 \cap b_2$.

Theorem 2.1. *Given a set X and a basis $\mathcal{B}$, define*

$$\mathcal{T}_{\mathcal{B}} = \left\{ \bigcup_{i\in I} s_i \mid I \textit{ is any index set and } \{s_i\}_{i\in I} \subseteq \mathcal{B} \right\}.$$

Then $\mathcal{T}_{\mathcal{B}}$ is a topology on X.

Proof. First observe that $\varnothing, X \in \mathcal{T}_{\mathcal{B}}$: Picking $I = \varnothing$ gives $\bigcup_{i\in I} s_i = \varnothing \in \mathcal{T}_{\mathcal{B}}$ and picking $I = \mathcal{B}$ gives $\bigcup_{b\in\mathcal{B}} b = X \in \mathcal{T}_{\mathcal{B}}$ by the covering property of a basis.

[1] Note that the discrete topology is induced by the discrete metric.

Now assume $\{s_i\}_{i\in I} \subseteq \mathcal{T}_{\mathcal{B}}$. For each $i \in I$, we have $s_i \in \mathcal{T}_{\mathcal{B}}$ and so there exists an index set J_i such that $s_i = \bigcup_{j\in J_i} b_j$, where the $b_j \in \mathcal{B}$. Then

$$\bigcup_{i\in I} s_i = \bigcup_{i\in I} \bigcup_{j\in J_i} b_j,$$

which is a union of elements of $\mathcal{B}$ and hence is in $\mathcal{T}_{\mathcal{B}}$.

Finally assume $\{s_i\}_{i=1}^n \subseteq \mathcal{T}_{\mathcal{B}}$. Now as each $s_i \in \mathcal{T}_{\mathcal{B}}$, there exists J_i such that $s_i = \bigcup_{j\in J_i} b_j$. Then

$$\bigcap_{i=1}^n s_i = \bigcap_{i=1}^n \bigcup_{j\in J_i} b_j.$$

Now assume $x \in \bigcap_{i=1}^n s_i = \bigcap_{i=1}^n \bigcup_{j\in J_i} b_j$. For each $1 \le i \le n$, there exists $j_i \in J_i$ such that $x \in b_{j_i}$. Hence $x \in \bigcap_{i=1}^n b_{j_i}$. Now by induction on the intersection property of a basis, we can find $b_x \in \mathcal{B}$ with

$$x \in b_x \subseteq \bigcap_{i=1}^n b_{j_i}$$

Also observe that

$$\bigcap_{i=1}^n b_{j_i} \subseteq \bigcap_{i=1}^n \bigcup_{j\in J_i} b_j = \bigcap_{i=1}^n s_i$$

by construction, so we may write

$$\bigcap_{i=1}^n s_i = \bigcup_{x\in\bigcap_{i=1}^n s_i} b_x \in \mathcal{T}_{\mathcal{B}}$$

as a union of elements of $\mathcal{B}$. □

Definition 2.3. A *subbasis* $\mathcal{B} \subseteq \mathcal{P}(X)$ is a collection of sets such that $\bigcup_{b\in\mathcal{B}} b = X$.

Remark. One may define a basis $\widetilde{\mathcal{B}}$ from a subbasis $\mathcal{B}$ by adding all finite intersections of elements of $\mathcal{B}$. We get the covering property for free and adding the finite intersections gives us the intersection property of a basis.

Example 2.3.1. For $\mathbb{R}$ with the Euclidean metric, the following are bases for the standard topology:

- $\{B_R(x) \mid x \in \mathbb{R}, R > 0\}$.
- $\{B_R(x) \mid x \in \mathbb{R}, R > 0, R \in \mathbb{Q}\}$. For this use the fact that $\mathbb{Q}$ is dense in $\mathbb{R}$.

In particular this shows that a basis for a topology is not unique in general.

2.3 Continuous Functions

Definition 2.4. Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be two topological spaces. A function $f : X \to Y$ is called *continuous* if for any $O \in \mathcal{T}_Y$, we have $f^{-1}(O) \in \mathcal{T}_X$, i.e. the preimage of an open set is open.[2]

Example 2.4.1. Let X be equipped with the trivial topology $\{\varnothing, X\}$ and let $\mathbb{R}$ be equipped with the standard topology. Then the only continuous functions $f : X \to \mathbb{R}$ are the constant functions $f : x \mapsto c$ for fixed $c \in \mathbb{R}$. To see this, observe that

[2] Recall that $f^{-1}(O) = \{x \in X \mid f(x) \in O\}$.

- $x \mapsto c$ is continuous since any open set in $\mathbb{R}$ either contains c or does not, and so the preimage is either X or $\varnothing$.
- Suppose $f(x_1) = y_1$ and $f(x_2) = y_2$. Let $\epsilon = |y_1 - y_2|$ and observe that $x_1 \in f^{-1}(B_\epsilon(y_1))$ while $x_2 \notin f^{-1}(B_\epsilon(y_1))$, so $f^{-1}(B_\epsilon(y_1))$ is not open in X despite $B_\epsilon(y_1)$ being open in $\mathbb{R}$.

Example 2.4.2. Let X have the discrete topology $\mathcal{T} = \mathcal{P}(X)$ and let $\mathbb{R}$ have the standard topology. Then all functions $X \to \mathbb{R}$ are continuous since any preimage is a subset of X and thus in $\mathcal{P}(X)$.

Remark. In a way, the trivial topology has too few open sets while the discrete topology has too many.

Definition 2.5. Two topological spaces $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ are *topologically equivalent* or *homeomorphic* if there exists a bijection $f : X \to Y$ such that f and f^{-1} are continuous.

Remark. A bijective function f being continuous does not necessarily imply that its inverse f^{-1} is.

Example 2.5.1. Consider $(-\pi/2, \pi/2)$ equipped with the Euclidean metric. This is homeomorphic to $\mathbb{R}$ equipped with the Euclidean metric.[3] One homeomorphism is given by $\tan : (-\pi/2, \pi/2) \to \mathbb{R}$.

[3]Note that $(-\pi/2, \pi/2)$ is bounded while $\mathbb{R}$ is not.

Lecture 3

Aug. 27 — Closed Sets, Continuity, the Subspace Topology

3.1 Closed Sets

Definition 3.1. A set $S \subseteq X$ is called a *closed set* if $S^c = X \setminus S$ is open.

Example 3.1.1. In $\mathbb{R}$, observe that $[a, b]^c = (-\infty, a) \cup (b, \infty)$, which is a union of open sets and thus open. Thus the closed intervals $[a, b] \subseteq \mathbb{R}$ are closed.

Remark. This is not a dichotomy. Sets can be both open and closed (*clopen*), or even neither. Trivially, if X is any topological space, then $\varnothing$ and X are both open and closed.

Example 3.1.2. Let $X = \{0, 1\}$ and $\mathcal{T} = \mathcal{P}(X)$. Then $\{0\}$ is both open and closed.

Example 3.1.3. Let $X = \{1, 2, 3\}$ and $\mathcal{T} = \{\varnothing, X, \{1\}, \{1, 2\}\}$. Then $\{2\}$ is neither open nor closed.

Recall the following De Morgan's laws from set theory:

Proposition 3.1 (De Morgan's laws)**.** *Let I be an index set and $\{A_i\}_{i \in I}$ be sets. Then*

$$\left(\bigcup_{i \in I} A_i\right)^c = \bigcap_{i \in I} A_i^c \quad \textit{and} \quad \left(\bigcap_{i \in I} A_i\right)^c = \bigcup_{i \in I} A_i^c.$$

Corollary 3.0.1. *In a topological space $(X, \mathcal{T})$, we have:*

(i) $\varnothing, X$ are closed.

(ii) if $\{A_i\}_{i \in I}$ are closed, then $\bigcap_{i \in I} A_i$ is closed,

(iii) and if $\{A_i\}_{i=1}^n$ are closed, then so is $\bigcup_{i=1}^n A_i$.

This gives a dual characterization of a topology.

Proof. (i) We have $\varnothing^c = X \in \mathcal{T}$ and $X^c = \varnothing \in \mathcal{T}$.

(ii) Note that

$$\left(\bigcap_{i \in I} A_i\right)^c = \bigcup_{i \in I} A_i^c.$$

As each A_i is closed, we have $A_i^c \in \mathcal{T}$ is open, and hence $\bigcup_{i \in I} A_i^c \in \mathcal{T}$ is open. So $\bigcap_{i \in I} A_i$ is closed.

(iii) Observe that

$$\left(\bigcup_{i=1}^{n} A_i\right)^c = \bigcap_{i=1}^{n} A_i^c.$$

Each A_i is closed, so A_i^c is open. Thus $\bigcap_{i=1}^n A_i^c$ is open, and so $\bigcup_{i=1}^n A_i$ is closed. $\square$

3.2 Properties of Continuity

Recall that a function $f : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ is continuous if for every $O \in \mathcal{T}_Y$, we have $f^{-1}(O) \in \mathcal{T}_X$.

Theorem 3.1. *A function $f : X \to Y$ is continuous if and only if for every C closed in Y, $f^{-1}(C)$ is closed in X.*

Proof. $(\Rightarrow)$ Let $C \subseteq Y$ be closed. Note that

$$f^{-1}(C) = \{x \in X \mid f(x) \in C\},$$

so we have

$$(f^{-1}(C))^c = \{x \in X \mid f(x) \notin C\} = \{x \in X \mid f(x) \in C^c\} = f^{-1}(C^c).$$

Since C is closed, C^c is open and so $f^{-1}(C^c) = (f^{-1}(C))^c$ is open. Thus $f^{-1}(C)$ is closed.

$(\Leftarrow)$ Assume $S \subseteq Y$ is open. Note that

$$(f^{-1}(S))^c = \{x \in X \mid f(x) \in S\}^c = \{x \in X \mid f(x) \notin S\} = \{x \in X \mid f(x) \in S^c\} = f^{-1}(S^c).$$

Since S is open, S^c is closed and so $f^{-1}(S^c) = (f^{-1}(S))^c$ is closed by assumption. Thus $f^{-1}(S)$ is open, and so we see that f is continuous. $\square$

Theorem 3.2 (Composition theorem)**.** *Let $(X, \mathcal{T}_X)$, $(Y, \mathcal{T}_Y)$, and $(Z, \mathcal{T}_Z)$ be topological spaces. Let*

$$f : X \to Y \quad \text{and} \quad g : Y \to Z$$

be continuous functions. Then $g \circ f : X \to Z$ is continuous.

Proof. Let $S \subseteq Z$ be open. It suffices to show that $(g \circ f)^{-1}(S) \subseteq X$ is open. Note that

$$\begin{aligned}(g \circ f)^{-1}(S) = \{x \in X \mid (g \circ f)(x) \in S\} &= \{x \in X \mid f(x) \in g^{-1}(S)\} \\ &= \{x \in X \mid x \in f^{-1}(g^{-1}(S))\} = f^{-1}(g^{-1}(S)).\end{aligned}$$

Now as g is continuous, $g^{-1}(S)$ is open in Y. Finally as f is continuous, $f^{-1}(g^{-1}(S))$ is open in X. $\square$

Theorem 3.3. *Assume $X = \bigcup_{\alpha \in I} U_\alpha$ for open sets U_α and let $f : X \to Y$. Assume that $f|_{U_\alpha} : U_\alpha \to Y$ is continuous for each $\alpha \in I$. Then f is continuous.*

Proof. Let $S \subseteq Y$ be open, and it suffices to show that $f^{-1}(S)$ is open. Observe that

$$f^{-1}(S) = f^{-1}(S) \cap X = f^{-1}(S) \cap \left(\bigcup_{\alpha \in I} U_\alpha\right) = \bigcup_{\alpha \in I} (f^{-1}(S) \cap U_\alpha) = \bigcup_{\alpha \in I} f|_{U_\alpha}^{-1}(S).$$

The $f|_{U_\alpha}$ are continuous, so each $f|_{U_\alpha}^{-1}(S)$ is open. Thus $f^{-1}(S)$ is open as a union of open sets. $\square$

Theorem 3.4 (Pasting lemma)**.** *Assume X, Y are topological spaces and $A, B \subseteq X$ are open. Suppose $f_1 : A \to Y$ and $f_2 : B \to Y$ are continuous, and that $f_1 \equiv f_2$ on $A \cap B$. Then $f : A \cup B \to Y$ defined by*

$$f(x) = \begin{cases} f_1(x) & \text{if } x \in A, \\ f_2(x) & \text{if } x \in B \end{cases}$$

is continuous.

Proof. Let $S \subseteq Y$ be open, it suffices to show that $f^{-1}(S)$ is open. Observe that

$$f^{-1}(S) = f_1^{-1}(S) \cup f_2^{-1}(S).$$

Both $f_1^{-1}(S)$ and $f_2^{-1}(S)$ are open since f_1 and f_2 are continuous, so $f^{-1}(S)$ is open as their union. □

3.3 Subspace Topology

Definition 3.2. Let $(X, \mathcal{T}_X)$ be a topological space and $S \subseteq X$ a set. The *subspace topology* on S is defined as follows: $O \subseteq S$ is open if there exists $U \subseteq X$ open in X such that $U = O \cap S$.

Example 3.2.1. Let $\mathbb{R}$ be given the metric topology and $S = [0, 1]$.

- The set $[0, 1]$ is not open in $\mathbb{R}$, but it is open in the subspace topology on S since $[0, 1] = S \cap (-1, 2)$.
- The set $[0, 1/2)$ is neither open nor closed in $\mathbb{R}$, but $[0, 1/2) = S \cap (-1/2, 1/2)$, so it is open in S.

Theorem 3.5. *The subspace topology is indeed a topology.*

Proof. Let $(X, \mathcal{T}_X)$ be a topological space and $S \subseteq X$ be given the subspace topology.

(i) We have $S = S \cap X$ and $\varnothing = \varnothing \cap X$, so $S, \varnothing$ are open in S.

(ii) Let $\{U_\alpha\}_{\alpha \in I}$ be open in the subspace topology. Then for every $\alpha \in I$, there exists $O_\alpha \in \mathcal{T}$ such that $U_\alpha = S \cap O_\alpha$. Then

$$\bigcup_{\alpha \in I} U_\alpha = \bigcup_{\alpha \in I} (S \cap O_\alpha) = S \cap \left(\bigcup_{\alpha \in I} O_\alpha \right).$$

The $\{O_\alpha\}_{\alpha \in I}$ are open in X, so their union is open in X. Thus $\bigcup_{\alpha \in I} U_\alpha$ is open in the subspace topology.

(iii) Let $\{U_i\}_{i=1}^n$ be open in the subspace topology. Then there are O_i for $1 \le i \le n$ with $U_i = S \cap O_i$. Then we have

$$\bigcap_{i=1}^n U_i = \bigcap_{i=1}^n (S \cap O_i) = S \cap \left(\bigcap_{i=1}^n O_i \right).$$

As the $O_i \in \mathcal{T}$ are open, $\bigcap_{i=1}^n O_i$ is open in X. Thus $\bigcap_{i=1}^n U_i$ is open in the subspace topology. □

Theorem 3.6. *Assume $f : X \to Y$ is a continuous function and $S \subseteq X$ a subspace. Then $f|_S : S \to Y$ is continuous, where S is equipped with the subspace topology.*

Proof. Let $O \subseteq Y$ be an open set, it suffices to show that $f|_S^{-1}(O)$ is open in the subspace topology. But observing that $f|_S^{-1}(O) = f^{-1}(O) \cap S$ immediately shows that $f|_S^{-1}(O)$ is open in S since $f^{-1}(O)$ is open in X due to the continuity of f. □

Remark. The subspace topology is the smallest topology on S such that the inclusion map $i : S \to X$ given by $i(s) = s$ is a continuous function.

Remark. Let X be a topological space with subspaces $Y \subseteq X$ and $Z \subseteq Y$. Then the subspace topology on Z induced by the subspace Y is the same as the subspace topology on Z induced directly by X.

Remark. A topological space can have a subspace homeomorphic to itself. For instance, consider $\mathbb{R}$ and $(-\pi/2, \pi/2)$ with a homemorphism given by the tangent function.

Lecture 4

Aug. 29 — Connectedness

4.1 Connected Spaces

Definition 4.1. A *separation* of a topological space X is two open, nonempty sets $U, V \subseteq X$ such that $X = U \cup V$ and $U \cap V = \varnothing$. A space is called *connected* if there is no separation of the space.

Proposition 4.1. *If X is separated, i.e. $X = U \cup V$ with U, V open and disjoint, then U and V are both open and closed.*

Proof. Observe that U is open by assumption, and we have

$$U^c = X \setminus U = V,$$

which is also open by assumption. Hence U is closed. The case for V is identical. □

Example 4.1.1. Consider the following:

- The singleton space $\{x\}$ is connected. There are no two nonempty, disjoint open sets.
- Consider the space $X = \{0, 1\}$. This case depends on the choice of topology:
  1. With the trivial topology $\mathcal{T} = \{\varnothing, X\}$, the space is connected.
  2. With the discrete topology $\mathcal{T} = \{\varnothing, X, \{1\}, \{0\}\}$, X is disconnected since $X = \{0\} \cup \{1\}$.
  3. With the topology $\mathcal{T} = \{\varnothing, X, \{1\}\}$, the space is connected. The only nonempty sets $\{1\}, X$ are not disjoint and thus there can be no separation.

Theorem 4.1. *A space X is disconnected if and only if there exists a surjective map $f : X \to \{0, 1\}$ with the discrete topology.*

Proof. ($\Rightarrow$) If X is disconnected, then we may write $X = U \cup V$ with U, V open, disjoint, and nonempty. Then define

$$f(x) = \begin{cases} 0 & x \in U, \\ 1 & x \in V, \end{cases}$$

which is surjective as U, V are nonempty. To see that f is continuous, observe that

$$f^{-1}(\varnothing) = \varnothing, \quad f^{-1}(\{0, 1\}) = X, \quad f^{-1}(\{0\}) = U, \quad f^{-1}(\{1\}) = V,$$

each of which are open. These are all of the open sets in the discrete topology, so f is continuous.

($\Leftarrow$) Assume there exists a surjective and continuous map $f : X \to \{0, 1\}$. Define

$$U = f^{-1}(\{0\}) \quad \text{and} \quad V = f^{-1}(\{1\}),$$

which are open since f is continuous. Observe that $U, V \neq \varnothing$ since f is surjective. Also $U \cap V = \varnothing$ since if there is any $x \in U \cap V$, then $f(x) = 0$ as $x \in U$ and $f(x) = 1$ as $x \in V$, a contradiction. Finally, $X = U \cup V$ since $f(x) = 0$ or $f(x) = 1$ for every $x \in X$, i.e. $x \in U$ or $x \in V$. So X is disconnected. $\square$

4.2 Connected Sets

Definition 4.2. Let X be a topological space and $S \subseteq X$. Then S is called *connected* if it is connected in the subspace topology.

Theorem 4.2. *If A, B are connected sets and $A \cap B \neq \varnothing$, then $A \cup B$ is connected.*

Proof. Assume not. Then there exists a continuous, surjective map $f : A \cup B \to \{0, 1\}$ with the discrete topology. Consider $f|_A : A \to \{0, 1\}$, which is continuous in the subspace topology. Notice that $f(A)$ cannot be $\{0, 1\}$ since otherwise A is disconnected. Without loss of generality, assume $f(A) = \{0\}$ since A is nonempty. Now consider $f|_B : B \to \{0, 1\}$, which is also continuous. Similarly, notice that $f(B)$ cannot be $\{0, 1\}$. But there exists $p \in A \cap B$, and $f(p) = 0$ as $p \in A$. Then since $p \in B$, we must have $f(B) = \{0\}$. But then we get that $f(A \cup B) = \{0\} \neq \{0, 1\}$, a contradiction to surjectivity. $\square$

Corollary 4.2.1. *A union of connected sets with "common points" is connected.*

Proof. Run induction (transfinite if the union is infinite) using the previous theorem. $\square$

Theorem 4.3. *Closed intervals in $[a, b] \subseteq \mathbb{R}$ with the metric topology are connected.*

Proof. Assume otherwise that $[a, b] = U \cup V$ with U, V disjoint, open, and nonempty. Assume without loss of generality that $a \in U$. Since V is nonempty, there exists $c > a$ such that $c \in V$. Now consider $[a, c] \subseteq [a, b]$ with $U_1 = U \cap [a, c]$ and $V_1 = V \cap [a, c]$. By the least upper bound property of $\mathbb{R}$, since U_1 is nonempty and bounded from above, there exists $s = \sup U_1$ with $s \leq c$. Now either $s \in U_1$ or $s \notin U_1$.

If $s \in U_1$ (note this implies $s \neq c$), then s is an interior point of U_1 since U_1 is open. So one may find a point t such that $t > s$ and $t \in U_1$. But then s is no longer an upper bound of U_1, a contradiction.

Otherwise $s \notin U_1$. Since U_1, V_1 cover $[a, c]$, we must then have $s \in V_1$ (note this implies $s \neq a$. Since V_1 is open, s is an interior point of V_1, and thus there exists $t < s$ such that $t \in V_1$ and t is an upper bound for U_1. This contradicts s being the least upper bound of U_1.

Since both cases lead to contradictions, we conclue that $[a, b]$ must be connected. $\square$

Corollary 4.3.1. *Open intervals in $\mathbb{R}$ are connected, and $\mathbb{R}$ itself is connected.*

Proof. For some $N_0 \geq 1$ (for instance choose $N_0 \geq 2/(b - a)$) we can write

$$(a, b) = \bigcup_{n=N_0}^{\infty} \left[a + \frac{1}{n}, b - \frac{1}{n}\right],$$

Each of these closed intervals is connected by the previous theorem, and thus the union is connected by Corollary 4.2.1 since they overlap. Similarly writing $\mathbb{R} = \bigcup_{n=1}^{\infty} [-n, n]$ shows that $\mathbb{R}$ is connected. $\square$

Corollary 4.3.2 (Intermediate value theorem). *Let $f : [a,b] \to \mathbb{R}$ be a continuous function. Then for any $f(a) < t < f(b)$, there exists $c \in [a,b]$ such that $f(c) = t$.*

Proof. Assume not. We can consider the open sets $(-\infty, t)$ and (t, ∞) in $\mathbb{R}$. Then $f^{-1}((-\infty,t))$ and $f^{-1}((t,\infty))$ are open sets since f is continuous. They are clearly disjoint (since f must be well-defined), and also nonempty since $a \in f^{-1}((-\infty,t))$ and $b \in f^{-1}((t,\infty))$. Also since $f^{-1}(\{t\}) = \varnothing$ by assumption,

$$[a,b] = f^{-1}((-\infty,t)) \cup f^{-1}((t,\infty)).$$

But this is a separation of $[a,b]$, a contradiction since $[a,b]$ is connected. □

Proposition 4.2. *The open interval $(0,1)$ is not homeomorphic to the closed interval $[0,1]$.*

Proof. Removing any point from $(0,1)$ disconnects it, but $[0,1) = [0,1] \setminus \{1\}$ remains connected.[1] □

Proposition 4.3. *The real line $\mathbb{R}$ is not homeomorphic to the plane $\mathbb{R}^n$ for any $n \geq 2$.*

Proof. Removing a point from $\mathbb{R}$ disconnects it but the same is not true for $\mathbb{R}^n$ when $n \geq 2$. □

[1] To see that $[0,1)$ is connected, we can write $[0,1) = \bigcup_{n=2}^{\infty} [0, 1-1/n]$.

Lecture 5

Sept. 3 — Path-Connectedness

5.1 More on Connectedness

Remark. The intervals $[a, b] \subseteq \mathbb{R}$ are homeomorphic to $[0, 1]$ for any $a < b$. We can take $f : [a, b] \to [0, 1]$ defined by

$$f(x) = \frac{1}{b-a}(x-a)$$

for instance as a homemorphism.

Lemma 5.1. *If X is connected and $f : X \to Y$ is continuous, then $f(X)$ is connected.*

Proof. This is part of Homework 2. □

Corollary 5.0.1. *The plane $\mathbb{R}^2$ is connected.*

Proof. Express $\mathbb{R}^2$ as the union of horizontal and vertical lines. Each line is the image of $\mathbb{R}$ and is thus connected by Lemma 5.1. Also any pair of horizontal and vertical lines must intersect, so we can use Corollary 4.2.1 to conclude that the union $\mathbb{R}^2$ is connected. □

Remark. We can extend this to $\mathbb{R}^3$ by embedding planes (copies of $\mathbb{R}^2$), and similarly for $\mathbb{R}^n$.

Proposition 5.1. *The unit circle $\mathbb{S}^1 \subseteq \mathbb{R}^2$ is connected.*

Proof. Define $\gamma : [0, 2\pi] \to \mathbb{R}^2$ by $\gamma(t) = (\cos t, \sin t)$. The image of γ is precisely $\mathbb{S}^1$. □

Proposition 5.2. *Define a relation $\sim$ on X by $x \sim y$ if there exists a connected subset $S \subseteq X$ such that $x, y \in S$. Then $\sim$ is an equivalence relation.*

Proof. For reflexivity, fix $x \in X$ and let S be the largest connected set containing x (this exists since we know at least $\{x\}$ must be connected). Then $x \in S$, so $x \sim x$.

For symmetry, fix $x, y \in X$. If $x \sim y$, then there exists a connected set S such that $x, y \in S$. But then $y, x \in S$, so we see that $y \sim x$.

For transitivity, assume that $x \sim y$ and $y \sim z$. Then there exists S_1 connected such that $x, y \in S_1$ and S_2 connected such that $y, z \in S_2$. Notice that $S_1 \cap S_2 \neq \varnothing$ since $y \in S_1 \cap S_2$. Then $S_1 \cup S_2$ is connected by Theorem 4.2 and $x, y, z \in S_1 \cap S_2$. In particular, $x, z \in S_1 \cap S_2$ and thus $x \sim z$.

So we see that $\sim$ is an equivalence relation. □

Definition 5.1. Let the equivalence relation $\sim$ be defined on X as in Proposition 5.2. Then we can write X as the disjoint union of the equivalence classes of $\sim$. These equivalence classes are called the *connected components* of X.

Remark. The connected components of a space are defined solely via topologies, so they must be invariant under homeomorphism.

Example 5.1.1. The letter S, sitting in $\mathbb{R}^2$, is not homeomorphic to the letter T. There is a point we can remove from T to give three connected components, but removing any point from S gives at most two such connected components.

5.2 Path-Connectedness

Remark. Connectedness is usually a very difficult property to verify. This motivates *path-connectedness*.

Definition 5.2. A set S is *path-connected* if for all $x, y \in S$, there exists a continuous map $\gamma : [0,1] \to S$ such that $\gamma(0) = x$ and $\gamma(1) = y$. Here $[0,1]$ is given the usual metric topology.

Lemma 5.2. *If S is path-connected, then S is connected.*

Proof. This is part of Homework 2. □

Remark. Unlike connectedness, it is immediately obvious that $\mathbb{R}^n$ is path-connected. Simply take the line segment between any two points. Then we can conclude connectedness by the previous lemma.

Example 5.2.1. There are spaces which are connected but not path-connected.

- Consider the *topologist's sine curve*, given by the union of the vertical segment $\{(0,y) \mid -1 \leq y \leq 1\}$ and the image of $(0,\infty)$ under $x \mapsto (x, \sin(1/x))$, is an example of such a space. See Homework 2 for more details.
- Consider the *cone* C in $\mathbb{R}^2$ defined by ($(0,1)$ denotes an open interval unless otherwise specified)

  $$C = ([0,1] \times \{0\}) \cup (K \times [0,1]) \cup (\{0\} \times [0,1]),$$

  where $K = \{1/n : n \in \mathbb{N}\}$. Note that C is clearly path-connected and hence also connected. Then define the space

  $$D = C \setminus (\{0\} \times (0,1)),$$

  which is now not path-connected (consider the point $(0,1) \in D$) but still connected.

Remark. Observe the following:

- One can define *path-connected components* in a similar manner as connected components.
- A continuous image of a path-connected space is path-connected. Simply compose the curve with the continuous map, which is now a path in the image.
- The union of path-connected spaces sharing a point is path-connected. Take two curves to the common point and concatenate them using the pasting lemma.
- In $\mathbb{R}^n$, connectedness is equivalent to path-connectedness. In general, this holds if you can get a basis of only connected sets.

Remark. Recall from homework that if $f : [0, 1] \to [0, 1]$ is continuous, then f has a fixed point, i.e. there exists $c \in [0, 1]$ with $f(c) = c$. This follows from a clever use of the intermediate value theorem. Now consider a more topological perspective. Consider the diagonal $\{(x, x) \mid x \in [0, 1]\}$ and look at the graph of f, which is contained in the closed unit square. This graph is path-connected as the image of a path-connected set and so there is a path between the points $(0, f(0))$ and $(1, f(1))$. But then this path must intersect the diagonal at some point, which gives a fixed point.

Theorem 5.1. *(Brouwer fixed point theorem) Let K be a closed, bounded, and convex set in $\mathbb{R}$. Then any continuous map $f : K \to K$ has a fixed point, i.e. there exists $c \in K$ such that $f(c) = c$.*

Remark. One can see the existence of the Nash equilibrium as a consequence of this theorem.

Remark. In $\mathbb{R}^2$, this theorem follows from the following claim. Let $X = \text{maps}(\mathbb{S}^1, \mathbb{S}^1)$ be the set of all continuous maps from $\mathbb{S}^1$ to itself. Then Brouwer's fixed point theorem in $\mathbb{R}^2$ follows from the following:

Theorem 5.2. *The space* $\text{maps}(\mathbb{S}^1, \mathbb{S}^1)$ *is not path connected.*

Lecture 6

Sept. 5 — Compactness

6.1 Note on the Subspace Topology

Remark. Let X be a topological space with topology $\mathcal{T}_X$, and let $Y \subseteq X$ be a subset endowed with a topology τ. Suppose that for any continuous $f : X \to Z$, there exists a continuous $\widetilde{f} : Y \to Z$ such that the following diagram commutes,

$$\begin{tikzcd} X \arrow[r, "f"] & Z \\ Y \arrow[u, "i"] \arrow[ur, dashed, "\widetilde{f}"'] & \end{tikzcd}$$

where $i : Y \to X$ is the inclusion map.[1] Then in Homework 2 we showed that $\mathcal{T}_Y \subseteq \tau$. We can see this as a *universal property* for the subspace topology.

6.2 Compactness

Definition 6.1. A set $C \subseteq X$ is called *compact* if for any *open cover*

$$C \subseteq \bigcup_{\alpha \in I} U_\alpha, \quad \text{each } U_\alpha \text{ is open,}$$

there exists a finite subcover $C \subseteq \bigcup_{i=1}^{n} U_{\alpha_i}$.

Example 6.1.1. Consider the following:

- In a finite topology, any set is compact. This is because any open cover is already finite.
- In a discrete space, i.e. $\mathcal{T} = \mathcal{P}(X)$, compact sets are precisely the finite sets. It is clear that finite sets are compact, for each x choose a single open set in the cover containing x. Conversely, if a set is compact, we can pick our open cover to contain only singletons, and the existence of a finite subcover means that the set has only finitely many elements.

Theorem 6.1 (Heine-Borel)**.** *Let $C \subseteq \mathbb{R}^n$ be a subset, where $\mathbb{R}^n$ is given the metric topology. Then C is compact if and only if C is closed and bounded.*

Proof. We postpone this proof until later. □

Lemma 6.1. *Let X be compact. If $Y \subseteq X$ is closed, then Y is compact.*

[1] Note that at least set-theoretically, this immediately defines $\widetilde{f} = f|_Y$. But a priori we do not know that $\widetilde{f}$ is continuous.

Proof. Let $Y \subseteq X$ closed be given, and assume that $Y \subseteq \bigcup_{\alpha \in I} U_\alpha$ an open cover. Since Y is closed, its complement Y^c is open. Then

$$Y^c \cup \bigcup_{\alpha \in I} U_\alpha$$

is an open cover of X since $X = Y \cup Y^c$. Since X is compact, there exists a finite subcover

$$X \subseteq Y^c \cup \bigcup_{i=1}^{n} U_{\alpha_i}.$$

Now observe that $Y \subseteq X$ and $Y \cap Y^c = \varnothing$, so actually $Y \subseteq \bigcup_{i=1}^{n} U_{\alpha_i}$, which is a finite subcover. $\square$

Theorem 6.2. *Let X be compact and $f : X \to Y$ continuous. Then $f(X)$ is compact.*

Proof. Consider $f(X) \subseteq Y$ and let $f(X) \subseteq \bigcup_{\alpha \in I} V_\alpha$, an open cover in Y. Notice that

$$X = f^{-1}(f(X)) \subseteq f^{-1}\left(\bigcup_{\alpha \in I} V_\alpha\right) = \bigcup_{\alpha \in I} f^{-1}(V_\alpha).$$

Note that each $f^{-1}(V_\alpha)$ is open in X since f is continuous and V_α is open in Y, so this is in fact an open cover of X. Thus since X is compact, we may extract a finite subcover

$$X \subseteq \bigcup_{i=1}^{n} f^{-1}(V_{\alpha_i}).$$

Then we see that

$$f(X) \subseteq f\left(\bigcup_{i=1}^{n} f^{-1}(V_{\alpha_i})\right) \subseteq \bigcup_{i=1}^{n} V_{\alpha_i},$$

which is a finite subcover of $f(X)$. Therefore $f(X)$ is compact. $\square$

Theorem 6.3. *Assume $\{C_j\}_{j=1}^{m}$ are compact subsets of X. Then $\bigcup_{j=1}^{m} C_j$ is compact.*

Proof. Assume $\bigcup_{j=1}^{m} C_j \subseteq \bigcup_{\alpha \in I} U_\alpha$, an open cover. Observe this is also an open cover of C_j for each $1 \le j \le m$, so we can extract a finite subcover, i.e. we can find $\alpha_{j,1}, \dots, \alpha_{j,n_j}$ with

$$C_j \subseteq \bigcup_{i=1}^{n_j} U_{\alpha_{j,i}}.$$

Then we see that

$$\bigcup_{j=1}^{m} C_j \subseteq \bigcup_{j=1}^{m} \bigcup_{i=1}^{n_j} U_{\alpha_{j,i}},$$

which is still a finite union. This is then a finite subcover of $\bigcup_{j=1}^{m} C_j$, so $\bigcup_{j=1}^{m} C_j$ is compact. $\square$

Theorem 6.4 (Weierstrass)**.** *Let $f : [a,b] \to \mathbb{R}$ be continuous. Then $f([a,b])$ is bounded, and moreover there exist $x_{\max}, x_{\min} \in [a,b]$ such that $f(x_{\max}) \ge f(x) \ge f(x_{\min})$ for all $x \in [a,b]$.*

Proof. Since f is continuous and $[a,b]$ is compact (by Heine-Borel), $f([a,b]) \subseteq \mathbb{R}$ is compact. Thus by Heine-Borel, $f([a,b])$ is bounded. In particular, we can find M, m such that

$$m \leq f(x) \leq M \quad \text{for all } x \in [a,b].$$

For the second part, observe that $f([a,b])$ is bounded and nonempty, so $s = \sup f([a,b])$. Since this is the supremum, there must exist $y_i \in f([a,b])$ such that $y_i \to s$ as $i \to \infty$. Now observe that $f([a,b])$ is closed by Heine-Borel, and in particular it contains its limit points. Thus we obtain $s \in f([a,b])$. Then pick $x_{\max} \in f^{-1}(\{s\}) \subseteq [a,b]$, which will satisfy $f(x_{\max}) = s \geq f(x)$ for all $x \in [a,b]$. by construction.

The argument for finding $x_{\min} \in [a,b]$ is similar. □

Theorem 6.5. *Let X be a topological space, $K \subseteq X$ compact, and $f : K \to \mathbb{R}$ a continuous function. Then f is bounded over K and attains its minimum and maximum on K.*

Proof. The same argument goes through, replacing $[a,b]$ by the compact set K. □

Lecture 7

Sept. 10 — More Compactness

7.1 The Cantor Set

Define $I_0 = [0,1]$ and remove the open middle-thirds interval to get

$$I_1 = [0,1] \setminus (1/3, 2/3) = [0,1/3] \cup [2/3,1].$$

Continue by removing the middle thirds of each interval to get $I_2, I_3, \ldots$. Then the *Cantor set* is defined to be $K = \bigcap_{I \geq 0} I_n$. The Cantor set is compact and uncountable. See more on Homework 3.

7.2 The Heine-Borel Theorem

Theorem 7.1 (Heine-Borel)**.** *If $C \subseteq \mathbb{R}$, then C is compact if and only if C is closed and bounded.*

Proof. ($\Rightarrow$) This direction is easy, see Homework 3 for details.

($\Leftarrow$) First we show that $[a,b] \subseteq \mathbb{R}$ is compact. Let $\{U_\alpha\}_{\alpha \in I}$ be an open cover for $[a,b]$, i.e. $[a,b] \subseteq \bigcup_{\alpha \in I} U_\alpha$. Now define

$$R = \{x \in [a,b] \mid [a,x] \text{ has a finite subcover}\}$$

Clearly $a \in R$ since $a \in [a,b] \subseteq \bigcup_{\alpha \in I} U_\alpha$, so picking any single U_α with $a \in U_\alpha$ gives a finite subcover for $[a,a] = \{a\}$. The goal is now to show that $b \in R$. Observe that $a \in R$ implies $R = \varnothing$, and $R \subseteq [a,b]$, so

$$s = \sup R$$

exists by the completeness of $\mathbb{R}$. We proceed to show that $s \in R$ and then $s = b$, which will show that $b \in R$. As $s \in [a,b] \subseteq \bigcup_{\alpha \in I} U_\alpha$, we can find α_s such that $s \in U_{\alpha_s}$. Since U_{α_s} is open, we can find $\delta > 0$ such that $(s-\delta, s+\delta) \subseteq U_{\alpha_s}$. Then since s is a least upper bound of R, we can find $r \in R$ such that $s - \delta < r \leq s$. Now since $r \in R$, $[a,r]$ admits a finite subcover $\{U_{\alpha_i}\}_{i=1}^n$. Then

$$[a,s] = [a,r] \cup (s-\delta, s] \subseteq \left(\bigcup_{i=1}^n U_{\alpha_i} \right) \cup U_{\alpha_s}$$

is a finite subcover for $[a,s]$, so $s \in R$. Now observe that we actually covered

$$\left[a, s + \frac{\delta}{2}\right] = [a,r] \cup (s-\delta, s+\delta) \subseteq \left(\bigcup_{i=1}^n U_{\alpha_i} \right) \cup U_{\alpha_s}$$

in the previous construction. Then $s + \delta/2 \in R$, which contradicts the minimality of s unless $s = b$. Thus $b \in R$, so $[a, b]$ admits a finite subcover and thus $[a, b]$ is compact.

Now let $C \subseteq \mathbb{R}$ be an arbitrary closed and bounded set. Since C is bounded, there exists $I = (a, b)$ such that $C \subseteq I$. But then $C \subseteq I \subseteq \overline{I} = [a, b]$, so C is a closed subset of a compact set, hence compact. □

Remark. The Heine-Borel theorem also holds more generally in $\mathbb{R}^n$. A later theorem will say that the product of compact sets is compact in the product topology, and thus we can run the same argument as above but with boxes in $\mathbb{R}^n$ instead of intervals.

7.3 The Bolzano-Weierstrass Theorem

Definition 7.1. A point x is an *accumulation point* for a set S if for all open sets U containing x, we have $(U \setminus \{x\}) \cap S \neq \varnothing$.

Remark. We disallow constant sequences when talking about accumulation points.

Proposition 7.1. *Let* $\mathrm{Acc}(A)$ *be the set of accumulation points of a set* A*. Then* $\overline{A} = A \cup \mathrm{Acc}(A)$*.*

Proof. We show that $A \cup \mathrm{Acc}(A)$ is closed, which will imply $\overline{A} \subseteq A \cup \mathrm{Acc}(A)$ by the minimality of the closure. Write

$$(A \cup \mathrm{Acc}(A))^c = A^c \cap \mathrm{Acc}(A)^c.$$

Now assume $x \in A^c \cap \mathrm{Acc}(A)^c$. Since $x \notin \mathrm{Acc}(A)$, there exists U_x open such that $x \in U_x$ and

$$(A \setminus \{x\}) \cap U_x = \varnothing.$$

But also $x \notin A$, so $A \setminus \{x\} = A$ and $A \cap U_x = \varnothing$. Then we can write

$$(A \cup \mathrm{Acc}(A))^c = A^c \cap \mathrm{Acc}(A)^c = \bigcup_{x \in A^c \cap \mathrm{Acc}(A)^c} U_x.$$

This is a union of open sets, hence open, and so $A \cup \mathrm{Acc}(A)$ is closed.

For the other direction, assume $x \in A \cup \mathrm{Acc}(A)$. If $x \in A$, we are done, so assume $x \in \mathrm{Acc}(A) \setminus A$. Now assume otherwise that $x \notin \overline{A}$. Then $x \in (\overline{A})^c$, which is open. Set $U = (\overline{A})^c$, so that

$$U \cap (A \setminus \{x\}) = U \cap A = \varnothing.$$

But then this says that x is not an accumulation point, in contradiction. □

Definition 7.2. We say that a topological space X is *sequentially compact* if every bounded sequence has a convergent subsequence.

Theorem 7.2 (Bolzano-Weierstrass)**.** *Any bounded infinite set* $S \subseteq \mathbb{R}^n$ *has an accumulation point.*

Proof. Since S is bounded, find a compact set containing S. Then apply the later Theorem 7.4. □

Remark. In general, compactness is *not* equivalent to sequential compactness, but both imply the Bolzano-Weierstrass theorem. However, in many spaces (including metric spaces, in particular), the two notions coincide (and are also equivalent to the Bolzano-Weierstrass theorem).

Theorem 7.3. *A sequentially compact space has the Bolzano-Weierstrass property, namely that any bounded infinite set has an accumulation point.*

Proof. This is easy, pick a countable subset (i.e. a sequence) and apply sequential compactness. □

Theorem 7.4. *A compact space has the Bolzano-Weierstrass property, namely that any infinite set has an accumulation point.*

Proof. Let A be an infinite set in X, where X is compact. Assume otherwise that A has no accumulation points in X. Then there is no accumulation point for A outside of A, so $\mathrm{Acc}(A) \subseteq A$. This gives

$$\overline{A} = A \cup \mathrm{Acc}(A) = A,$$

so A is closed. Thus A is a closed subset of a compact space, hence compact. Now for any $a \in A$, pick an open set U_a such that $a \in U_a$ and $U_a \cap (A \setminus \{a\}) = \varnothing$. Write $A \subseteq \bigcup_{a \in A} U_a$, and by compactness we can find a finite subcover $A \subseteq \bigcup_{i=1}^{n} U_{a_i}$. Then observe that

$$A = A \cap \bigcup_{i=1}^{n} U_{a_i} = \bigcup_{i=1}^{n} (A \cap U_{a_i}) = \bigcup_{i=1}^{n} \{a_i\} = \{a_1, \ldots, a_n\},$$

This is in contradiction with A being infinite. □

Remark. Usually, this proof goes by showing that compactness implies sequential compactness, which then implies the Bolzano-Weierstrass property. But this proof avoids going through convergent sequences.

Lecture 8

Sept. 12 — Separation Axioms

8.1 Separation Axioms

Definition 8.1. A topological space is said to satisfy the T_0 *axiom* if the following holds: For every $a, b \in X$ with $a \neq b$, there exists U open such that either $a \in U$, $b \notin U$ or $b \in U$, $a \notin U$.

Remark. With the T_0 axiom, we cannot choose which point is in U and which is not. For instance take $X = \{a, b\}$ with topology $\mathcal{T} = \{\varnothing, \{a\}, X\}$. This space is T_0, but we can only choose U to contain a.

Definition 8.2. A space is said to satisfy the T_1 *axiom* if for every $a, b \in X$ with $a \neq b$, there exist U_a, U_b open such that $a \in U_a$, $b \notin U_a$ and $b \in U_b$, $a \notin U_b$.

Remark. With the T_1 axiom, U_a and U_b need not be disjoint.

Definition 8.3. A space is said to be T_2 or *Hausdorff* if the following holds: For every $a, b \in X$ with $a \neq b$, there exist U_a, U_b open such that $a \in U_a$, $b \in U_b$ and $U_a \cap U_b = \varnothing$.

Example 8.3.1. Metric spaces are Hausdorff. For any $a \neq b$, pick balls with radius $d(a, b)/2$ around a, b.

Theorem 8.1. *We have the proper inclusion* $T_2 \subsetneq T_1 \subsetneq T_0$.

Proof. The inclusions and $T_0 \neq T_1$ is clear (e.g. above). For $T_1 \neq T_2$ take the *line with two origins*[1]. $\square$

Theorem 8.2. *In a* T_1 *space, every singleton* $\{x\}$ *is closed.*

Proof. Fix $x \in X$. For every $y \neq x$, by the T_1 axiom we can find U_y open such that $y \in U_y$ and $x \notin U_y$. In particular, this means that $U_y \subseteq \{x\}^c$. Then we can write

$$\{x\}^c \subseteq \bigcup_{y \in \{x\}^c} U_y \subseteq \{x\}^c.$$

So $\{x\}^c = \bigcup_{y \in \{x\}^c} U_y$, which is open as a union of open sets. Thus $\{x\}$ is closed. $\square$

8.2 Properties of Hausdorff Spaces

Theorem 8.3. *In a Hausdorff space, a point* x *is an accumulation point of a set* A *if and only if every neighborhood of* x *contains infinitely many elements of* A.

[1] The *line with two origins* is $X = \mathbb{R} \cup \{p\}$ with topology generated by the open sets in $\mathbb{R}$ (with the metric topology), and adding $\widetilde{U} = (U \setminus \{0\}) \cup \{p\}$ for each open set $U \subseteq \mathbb{R}$ containing 0. One can separate 0 and p but not with disjoint sets.

Proof. ($\Leftarrow$) This is clear.

($\Rightarrow$) Pick x an accumulation point of A, and assume otherwise that there exists a neighborhood U of x with only finitely many elements of A, i.e. $|(U \setminus \{x\}) \cap A| < \infty$. Then we can write

$$(U \setminus \{x\}) \cap A = \{a_1, \dots, a_n\} = \bigcup_{i=1}^{n} \{a_i\}.$$

Since our space is Hausdorff and thus also T_1, these singletons $\{a_i\}$ are closed. Then $(U \setminus \{x\}) \cap A$ is closed as a finite union of closed sets. Now since our space is Hausdorff, for every $1 \leq i \leq n$ we can separate a_i from x, i.e. there exists U_{x_i}, U_{a_i} open such that $x \in U_{x_i}$, $a_i \in U_{a_i}$ and $U_{x_i} \cap U_{a_i} = \varnothing$. Then

$$U' = U \cap \bigcap_{i=1}^{n} U_{x_i}$$

is open as a finite intersection of open sets. Also $x \in U'$ since $x \in U$ and $x \in U_{x_i}$ for each i. But

$$(U \setminus \{x\}) \cap A = \{a_1, \dots, a_n\} \subseteq \bigcup_{i=1}^{n} U_{a_i}$$

and $U_{a_j} \cap \bigcap_{i=1}^{n} U_{x_i} = \varnothing$ for all j, so $(U' \setminus \{x\}) \cap A = \varnothing$. Contradiction. □

Remark. Maybe just the T_1 axiom is enough for this theorem. Think more about this.

Definition 8.4. A sequence $\{x_n\}_{n=1}^{\infty} \subseteq (X, \mathcal{T})$ *converges* to a point $x \in X$, written $x_n \to x$, if for any open set U containing x, there exists $N_0 \in \mathbb{N}$ such that $x_n \in U$ for every $n \geq N_0$.

Theorem 8.4. *In a Hausdorff space, a convergent sequence has a unique limit.*

Proof. Assume otherwise that $x_n \to L_1$ and $x_n \to L_2$ with $L_1 \neq L_2$. Then since our space is Hausdorff, we can find U_{L_1}, U_{L_2} open such that $L_1 \in U_{L_1}$, $L_2 \in U_{L_2}$ and $U_{L_1} \cap U_{L_2} = \varnothing$. Since $x_n \to L_1$, there exists $N_0 \in \mathbb{N}$ such that $x_n \in U_{L_1}$ for all $n \geq N_0$. Similarly we can find $N_0' \in \mathbb{N}$ with $x_n \in U_{L_2}$ for all $n \geq N_0'$ since $x_n \to L_2$. But then for $N = \max\{N_0, N_0'\}$, we have $x_N \in U_{L_1} \cap U_{L_2}$, a contradiction. □

Theorem 8.5. *In a Hausdorff space, every compact set is closed.*

Proof. Let $C \subseteq X$ be compact, and we show that C^c is open. So fix $y \in C^c$. For any $x \in C$, since our space is Hausdorff, we can find U_x, U_y open such that $x \in U_x$, $y \in U_y$ and $U_x \cap U_y = \varnothing$. Now consider $\bigcup_{x \in C} U_x$. This is an open cover of C, so we can find a finite subcover $C \subseteq \bigcup_{i=1}^{n} U_{x_i}$ since C is compact. Then the finite intersection $\bigcap_{i=1}^{n} U_{y_i}$ is an open set contain y, and it is disjoint from C by construction since $U_{x_i} \cap U_{y_i} = \varnothing$ for each i. Now set $\widetilde{U}_y = \bigcap_{i=1}^{n} U_{y_i}$, so that

$$C^c \subseteq \bigcup_{y \in C^c} \widetilde{U}_y \subseteq C^c.$$

Thus $C^c = \bigcup_{y \in C^c} \widetilde{U}_y$, which is open as the union of open sets, so we conclude that C is closed. □

Lecture 9

Sept. 17 — Compactification

9.1 Motivation for Compactification

Let $(X, \mathcal{T})$ be a topological space which is not compact. Usually we assume X is Hausdorff, and the goal is to find a compact space which looks like X, i.e. compactify X.

Remark. The naive idea is to take the trivial topology on X in place of $\mathcal{T}$, getting X_{trivial}. This is compact, the identity map $\mathrm{id} : X \to X_{\text{trivial}}$ is continuous and bijective, but it is not a homeomorphism. This is bad because we forget all the topological structure on X, for instance every sequence converges to every point in X_{trivial}. We would like to compactify X while keeping as much structure as possible.

Example 9.0.1. Let $X = (0, 1)$ with the metric topology. Take $Y = [0, 1]$ with the metric topology, so X embeds into Y by the inclusion map.[1] Note that Y is compact by Heine-Borel.

Example 9.0.2. Let $X = (0, 1)$ with the metric topology. Take $Y = \mathbb{S}^1 \subseteq \mathbb{R}^2$ to be the unit circle, where $\mathbb{R}^2$ has the metric topology. Then X embeds into Y by the stereographic projection (technically $\mathbb{R}$ is embedded but $\mathbb{R}$ is homeomorphic to $(0, 1)$ by the arctangent) by adding one point at the north pole.

Example 9.0.3. For the open unit disk in $\mathbb{R}^2$, we can add its boundary to get the closed unit disk as a compactification (the closed unit disk is compact by Heine-Borel). This adds uncountably many points. An alternative is to add only a single point at infinity, and identify the boundary with this point.

9.2 One-Point Compactification

Definition 9.1. For a topological space X, the *one-point compactification* (or *Alexandroff compactification*) of X is the set $X^+ = X \cup \{\infty\}$ with topology generated by the basis

$$\{U \subseteq X \text{ open}\} \cup \{K^c \mid K \subseteq X \text{ is compact}\}.$$

Remark. One way to think about the one-point compactification is that we are forcing all unbounded sequences in X to converge to to the new point ∞.

Theorem 9.1. *For a Hausdorff topological space X, its one-point compactification X^+ is compact.*

Proof. Let $\{O_\alpha\}$ be an open cover of X^+, i.e. $X^+ = \bigcup_{\alpha \in I} O_\alpha$. Some open set must contain the point ∞, and each open set contains a basis element, so there exists $\alpha' \in I$ such that $O_{\alpha'}$ contains K^c, for some $K \subseteq X$ compact. Now since $K \subseteq X \subseteq X^+$, we see that $K \subseteq \bigcup_{\alpha \in I}(O_\alpha \cap X)$ is an open cover of K

[1]By X *embeds into* Y, we mean that there is a continuous injection from X to Y.

(note that the Hausdorff condition implies that every compact K is closed in X, and thus the K^c basis elements are open in X). So by compactness, there exists a finite subcover $K \subseteq \bigcup_{\alpha \in I_{\text{finite}}} (O_\alpha \cap X)$. Then

$$X^+ = O_{\alpha'} \cup K \subseteq O_{\alpha'} \cup \bigcup_{\alpha \in I_{\text{finite}}} (O_\alpha \cap X) \subseteq \bigcup_{\alpha \in I_{\text{finite}}} O_\alpha$$

since $K^c \subseteq O_{\alpha'}$. This is a finite subcover, so X is compact. □

Remark. In analysis, we sometimes speak of a sequence diverging to ∞, i.e. the sequence eventually escapes any compact set. This is precisely convergence to ∞ in the one-point compactification.

Theorem 9.2. *Assume X is a Hausdorff, locally compact but not compact topological space.*[2] *Then the inclusion map* $\mathrm{id} : X \to X^+$ *is a dense, continuous embedding.*[3]

Proof. First clearly $\mathrm{id} : X \to \mathrm{id}(X) \subseteq X^+$ is injective. Now we show that id is continuous. It is enough to show that the preimage of basis elements of X^+ is open in X. If $U \subseteq X$ is open, then $\mathrm{id}^{-1}(U) = U \subseteq X$ is clearly open. Otherwise consider $K^c \cup \{\infty\}$ for $K \subseteq X$ compact. Then we have

$$\mathrm{id}^{-1}(K^c \cup \{\infty\}) = K^c \subseteq X.$$

Since X is Hausdorff, the compact set K is closed, and so K^c is open. Thus id is continuous.

Finally we show density, i.e. $\overline{X} = X^+$, where the closure is taken in X^+. To do this, suppose otherwise that $\overline{X} \neq X^+$. Clearly $X \subseteq \overline{X}$, so if $\overline{X} \neq X^+$, we must have $\overline{X} = X$ (since $X^+ = X \cup \{\infty\}$). But then X is closed in X^+, which is compact, so X is also compact in X^+ since X and thus X^+ is Hausdorff (since X is locally compact). This implies (exercise) that X itself is compact. Contradiction. □

Remark. If X is already compact, then $X^c = \{\infty\}$ is open in X^+. Obviously X^+ is compact since X is and every sequence converging to ∞ must eventually be constant. In particular, X^+ must be disconnected, and the extra point ∞ just sits there to the side.

Remark. Due to the above, we must assume that X is not compact in order to get a dense embedding.

Example 9.1.1. Consider the space $X = [0, 1)$. Then the one-point compactification of X is $[0, 1]$.

[2] A space X is *locally compact* if for every $x \in X$, there exists U open with $x \in U$ such that $\overline{U}$ is compact.
[3] The embedding is *dense* if $\overline{X} = X^+$.

Lecture 10

Sept. 19 — Miscellaneous Topics

10.1 Local Compactness and One-Point Compactification

Definition 10.1. A map $f : X \to Y$ is called *open* if for every $U \subseteq X$ open, $f(U)$ is open. A *topological embedding* is an injective, continuous, and open map.

Remark. This ensures that f with codomain restricted to $f(U)$ is a homeomorphism.

Example 10.1.1. Demanding only that f is injective and continuous is not enough. Let $(X, \mathcal{T})$ be any topological space and let X_{trivial} be X with the trivial topology. Then the identity map $\text{id} : X \to X_{\text{trivial}}$ is injective, continuous, but not open is general if $\mathcal{T}$ is not trivial. So this is *not* a topological embedding.

Theorem 10.1. *If X is Hausdorff and locally compact, then $X^+ = X \cup \{\infty\}$ is Hausdorff.*

Proof. Pick $y, y' \in X^+$ with $y \neq y'$. If $y, y' \in X$, then since X is Hausdorff, there are disjoint open sets $U, U' \subseteq X$ with $y \in U$ and $y' \in U$. But then $U, U' \subseteq X^+$ are still open and disjoint in X^+ since the identity map embeds X into X^+. Thus U, U' are disjoint open sets separating y, y'.

Now assume without loss of generality that $y' = \infty$. By local compactness, there is a open set $U \subseteq X$ containing y with compact closure. Then observe that $(\overline{U})^c \cup \{\infty\}$ is open in X^+ since $\overline{U}$ is compact. Since U is also open in X^+ and clearly disjoint from $(\overline{U})^c \cup \{\infty\}$, we have separated y, ∞. □

10.2 Distance to a Closed Set

Definition 10.2. Let (X, d) be a metric space. For $x \in X$ and $A \subseteq X$, the *distance from x to A* is

$$d(x, A) = \inf\{d(x, a) : a \in A\}.$$

Remark. This infimum is achieved in $\mathbb{R}^n$, but not necessarily in a more general metric space.

Proposition 10.1. *For any $x \in \mathbb{R}^n$ and closed set $A \subseteq \mathbb{R}^n$, there exists $a \in A$ with $d(x, A) = d(x, a)$.*

Proof. Pick $a_0 \in A$ and note that $d(x, a_0) < \infty$. Then set $R = 2d(x, a_0) > 0$. Consider $\overline{B_R(x)} \cap A$, which is a closed and bounded set containing a_0, hence compact by Heine-Borel. Now apply the compact case to $\overline{B_R(x)} \cap A$ to get $a_{\min}$ with $d(x, a_{\min}) = d(x, \overline{B_R(x)} \cap A)$. Now any $a \in A$ with $a \notin \overline{B_R(x)}$ satisfies

$$d(x, a) \geq R > d(x, a_0) \geq d(x, a_{\min}),$$

and hence it cannot be the minimum. Thus we must have $d(x, a_{\min}) = d(x, A)$. □

Remark. Define

$$\ell^2(\mathbb{N}) = \left\{ \{a_n\}_{n=1}^\infty \subseteq \mathbb{R} : \sum_{n=1}^\infty a_n^2 < \infty \right\}.$$

This is an inner product space over $\mathbb{R}$, and in particular we can induce a metric

$$d(\{a_n\}, \{b_n\}) = \sqrt{\sum_{n=1}^\infty |a_n - b_n|^2}$$

to turn (X, d) into a complete metric space. Also notice that $\{e_i\}_{i=1}^\infty$, where e_i is the sequence with 1 in the ith position and 0 everywhere else, is an orthonormal basis for $\ell^2(\mathbb{N})$.

Lemma 10.1. *Define the set*

$$A = \left\{ \left(1 + \frac{1}{i}\right) e_i \right\}_{i=1}^\infty .$$

Then $d(\{0\}, A) = 1$. In particular, this is an example of a metric space where the infimum of the distance from a point to a closed set is not achieved.

Proof. Observe that

$$d\left(\{0\}, (1 + 1/i)e_i\right) = 1 + \frac{1}{i},$$

and so

$$d(\{0\}, A) = \inf\{d(\{0\}, (1 + 1/i)e_i : i \in \mathbb{N}\} = \inf \left\{ 1 + \frac{1}{i} : i \in \mathbb{N} \right\} = 1.$$

In particular, this infimum is clearly not achieved since $d(\{0\}, (1 + 1/i)e_i) = 1 + 1/i > 1$ for each $i \in \mathbb{N}$. Now we show that A is closed by showing that it contains its limit points. For this, first observe that $d(a, a') \geq 1$ for any $a, a' \in A$ with $a \neq a'$. To see this, we can compute that

$$d\left((1 + 1/i)e_i, (1 + 1/j)e_j\right) = \sqrt{\left(1 + \frac{1}{i}\right)^2 + \left(1 + \frac{1}{j}\right)^2} \geq \sqrt{2} \geq 1$$

whenever $i \neq j$. Now assume we have $\{a_n\} \subseteq A$ with $a_n \to x$. In particular, $\{a_n\}$ must be a Cauchy sequence, and so for $\epsilon = 1/2$, there exists $N_0 \in \mathbb{N}$ such that for all $n, m \geq N_0$, we have $d(a_n, a_m) < 1/2$. But $d(a, a') \geq 1$, so the sequence must stabilize after N_0, and hence $x = a_n$ for $n \geq N_0$. In particular, $x \in A$, so we conclude that A is closed. This finishes the example. □

10.3 Nested Intersections in Compact Hausdorff Spaces

Proposition 10.2. *Let X be a compact Hausdorff space, and $Y_i \subseteq X$ be closed and connected for $i \in I$. Assume the $\{Y_i\}$ are totally ordered, i.e. $Y_i \subseteq Y_j$ or $Y_j \subseteq Y_i$ for all $i, j \in I$. Then $\bigcap_{i \in I} Y_i$ is connected.*

Proof. Let U, V be a separation of $Y = \bigcap_{i \in I} Y_i$. Then $Y = U \cup V$ and U, V are open in Y, disjoint, and nonempty. In particular, we can find U', V' open in X such that $U = U' \cap Y$ and $V = V' \cap Y$. However, U', V' may no longer separate Y. This is why we need the Hausdorff condition. Use the next lemma to fix the proof from here, see more details in Homework 4. □

Lemma 10.2. *In a compact Hausdorff space, if C_1, C_2 are two compact disjoint sets, then there exist U_1, U_2 open and disjoint such that $C_1 \subseteq U_1, C_2 \subseteq U_2$ and $U_1 \cap U_2 = \varnothing$.*

Proof. First we show this in the case where $C_1 = \{x\}$ is a singleton and $C_2 = C$. For all $y \in C$, consider the pair x and y. Then there exists $U_{x,y}, V_{x,y}$ open such that $x \in U_{x,y}$, $y \in V_{x,y}$, and $U_{x,y} \cap V_{x,y} = \varnothing$. Observe that $\bigcup_{y \in C} V_{x,y}$ is an open cover of C, so by compactness there exists a finite subcover $C \subseteq V = \bigcup_{i=1}^n V_{x,y_i}$. Then $x \in U = \bigcap_{i=1}^n U_{x,y_i}$, which is open as a finite intersection of open sets. Also each U_{x,y_i} is disjoint from V_{x,y_i}, so U is disjoint from V. Then $U_1 = U$ and $U_2 = V$ are the desired open sets.

Now let C_1, C_2 be any two disjoint compact sets. For all $x \in C_1$, there are open sets U_{x,C_2}, V_{x,C_2} such that $x \in U_{x,C_2}$, $C_2 \subseteq V_{x,C_2}$, and $U_{x,C_2} \cap V_{x,C_2} = \varnothing$. Then we make the same argument. We have

$$C_1 \subseteq \bigcup_{x \in C_1} U_{x,C_2},$$

an open cover of C_1, so by compactness there is a finite subcover $C_1 \subseteq U_1 = \bigcup_{i=1}^m U_{x_i,C_2}$. Then

$$C_2 \subseteq U_2 = \bigcap_{i=1}^m V_{x_i,C_2},$$

which is open as a finite intersection of open sets. As before, $U_1 \cap U_2 = \varnothing$ since each U_{x_i,C_2} is disjoint from V_{x_i,C_2}. Thus we get a separation by disjoint compact sets. $\square$