MATH 6421: Algebraic Geometry I

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Contents

Lecture 1

Aug. 19 — Affine Varieties

1.1 Motivation for Algebraic Geometry

Remark. Why study algebraic geometry? Algebraic geometry connects to many fields of math.

Example 1.0.1. Consider a plane curve $\{f(z_1, z_2) = 0\} \subseteq \mathbb{C}^2$, e.g. an elliptic curve $z_2^2 - z_1^3 + z_1 - 1 = 0$. Compactify and set C to be the closure of C^0 in $\mathbb{CP}^2$, and let $d = \deg f$. There are connections in

1. Topology: $H^1(C, \mathbb{C}) \cong \mathbb{C}^{2g}$, where $g = (d-1)(d-2)/12$;
2. Arithmetic: the number of $\mathbb{Q}$-points is finite if $d > 3$;
3. Complex geometry: We have $C \cong \mathbb{CP}^2$ for $d = 1, 2$, $C \cong \mathbb{C}/\Lambda$ for $d = 3$, and $C \cong \mathbb{H}/\Gamma$ for $d > 3$.

1.2 Affine Varieties

Fix an algebraically closed field k (e.g. $\mathbb{C}$, $\overline{\mathbb{Q}}$, $\overline{\mathbb{F}}_p$, etc.).

Definition 1.1. *Affine space* is the set $\mathbb{A}^n = \mathbb{A}^n_k = \{\vec{a} = (a_1, \ldots, a_n) : a_i \in k\}$.

Remark. Note the following:

1. $\mathbb{A}^n_k$ is the same set as k^n, but forgetting the vector space structure;
2. $f \in k[x_1, \ldots, x_n]$ gives a polynomial function $\mathbb{A}^n_k \to k$ by evaluation: $a \mapsto f(a)$.

Definition 1.2. For a subset $S \subseteq k[x_1, \ldots, x_n]$, its *vanishing set* is

$$V(S) = \{a \in \mathbb{A}^n : f(a) = 0 \text{ for all } f \in S\}.$$

An *affine variety* is a subset of $\mathbb{A}^n_k$ of this form.

Example 1.2.1. Consider the following:

1. $\mathbb{A}^n = V(\varnothing) = V(\{0\})$;
2. $\varnothing = V(1) = V(k[x_1, \ldots, x_n])$;
3. a point $a = (a_1, \ldots, a_n)$ is an affine variety: $V(\{x_1 - a_1, \ldots, x_n - a_n\}) = \{a\}$;
4. a linear space $L \subseteq \mathbb{A}^n$ (it is the kernel of some matrix);
5. plane curves $V(f(x, y)) \subseteq \mathbb{A}^2_{x,y}$;

6. $\mathrm{SL}_n(k) \subseteq \mathbb{A}^{n\times n}$ is an affine variety: $\mathrm{SL}_n(k) = V(\det([x_{i,j}]) - 1)$;
7. $\mathrm{GL}_n(k)$ (as a set) is an affine variety in $\mathbb{A}^{n\times n+1}$: $\mathrm{GL}_n(k) = V(\det([x_{i,j}])y - 1)$;
8. if $X \subseteq \mathbb{A}^m$ and $Y \subseteq \mathbb{A}^n$ are affine varieties, then $X \times Y \subseteq \mathbb{A}^{m+n}$ is an affine variety;
9. the affine varieties $X \subseteq \mathbb{A}^1_k$ are of the form: finite set of points, $\varnothing$, or $\mathbb{A}^1_k$.

Proposition 1.1 (Relation to ideals)**.** *If $S \subseteq k[x_1, \ldots, x_n]$, then $V(S) = V(\langle S\rangle)$, where $\langle S\rangle$ is the ideal generated by S.*

Proof. Since $S \subseteq \langle S\rangle$, we have $V(\langle S\rangle) \subseteq V(S)$. Conversely, if $f, g \in S$ and $h \in k[x_1, \ldots, x_n]$, then $f + g$ and hf vanish on $V(S)$, so we see that $V(S) \subseteq V(\langle S\rangle)$. $\square$

Remark. The statement implies that if $f_1, \ldots, f_r \in k[x_1, \ldots, x_n]$, then $V(f_1, \ldots, f_r) = V((f_1, \ldots, f_n))$. The following are some further applications of the statement:

1. affine varities are vanishing loci of ideals;
2. if $X \subseteq \mathbb{A}^n$ is an affine variety, then X is cut out by finitely many polynomial equations.

To see the second statement, note that $X = V(I)$ for some ideal $I \leq k[x_1, \ldots, x_n]$. By the Hilbert basis theorem that $k[x_1, \ldots, x_n]$ is Noetherian, there are finitely many $f_1, \ldots, f_r \in k[x_1, \ldots, x_n]$ such that $I = (f_1, \ldots, f_r)$. So $X = V(I) = V(f_1, \ldots, f_r)$.

Proposition 1.2 (Properties of the vanishing set)**.** *For ideals I, J of $k[x_1, \ldots, x_n]$,*

1. *if $I \subseteq J$, then $V(J) \subseteq V(I)$;*
2. *$V(I) \cap V(J) = V(I + J)$;*
3. *$V(I) \cup V(J) = V(IJ) = V(I \cap J)$.*

Proof. (1) This follows from definitions and actually holds for general subsets.

(2) Note that $V(I) \cap V(J) = V(I \cap J) = V(\langle I \cup J\rangle) = V(I + J)$.

(3) We only prove the first equality, the second is similar. Recall that $IJ = \left\{\sum_{i=1}^r f_i g_i : f_i \in I, g_i \in J\right\}$. We have the forwards inclusion $V(I) \cup V(J) \subseteq V(IJ)$ from definitions. For the reverse inclusion, consider a point $x \notin V(I) \cup V(J)$. So there exists $f \in I$ and $g \in J$ such that $f(x), g(x) \neq 0$. So $f(x)g(x) \neq 0$, which implies that $x \notin V(IJ)$. Thus $V(IJ) \subseteq V(I) \cup V(J)$ as well. $\square$

Remark. The above implies that if X and Y are affine varieties in $\mathbb{A}^n_k$, then so are $X \cup Y$ and $X \cap Y$.

Example 1.2.2. Consider $V(y^2 - x^2, y^2 + x^2) \subseteq \mathbb{A}^2$. Note that $(y^2 - x^2, y^2 + x^2) = (x^2, y^2)$, from which we can easily see that $V(y^2 - x^2, y^2 + x^2) = V(x^2, y^2) = \{0\}$.

1.3 Correspondence with Ideals

Remark. Our goal is to build a correspondence between affine varieties in $\mathbb{A}^n_k$ and ideals of $k[x_1, \ldots, x_n]$.

Definition 1.3. For a subset $X \subseteq \mathbb{A}^n_k$, define

$$I(X) = \{f \in k[x_1, \ldots, x_n] : f(a) = 0 \text{ for all } a \in X\}.$$

Remark. Note that $I(X)$ is in fact an ideal of $k[x_1, \ldots, x_n]$.

Example 1.3.1. Consider the following:

1. $I(\varnothing) = k[x_1, \ldots, x_n]$;
2. $I(\mathbb{A}_k^n) = \{0\}$, this will follow from the Hilbert nullstellensatz and relies on $k = \overline{k}$ (for $k = \mathbb{R}$, the polynomial $x^2 + y^2 + 1$ is always nonzero and thus lies in $I(\mathbb{A}_{\mathbb{R}}^n)$);
3. for $n = 1$, if $S \subseteq \mathbb{A}_k^1$ be an infinite set, then $I(S) = (0)$.
4. for $n = 1$, we have $I(V(x^2)) = I(\{0\}) = (x)$.

Remark. What properties does $I(X)$ satisfy?

Definition 1.4. Let R be a ring. The *radical* of an ideal $J \leq R$ is

$$\sqrt{J} = \{f \in R : f^n \in J \text{ for some } n > 0\}.$$

An ideal J is *radical* if $J = \sqrt{J}$.

Exercise 1.1. Check the following:

1. $\sqrt{J}$ is always an ideal.
2. $\sqrt{\sqrt{J}} = \sqrt{J}$.
3. An ideal $J \leq R$ is radical if and only if R/J is reduced.[1]

Proposition 1.3. *If $X \subseteq \mathbb{A}_k^n$ is a subset (not necessarily an affine variety), then $I(X)$ is radical.*

Proof. Fix $f \in k[x_1, \ldots, x_n]$. If $f^n \in I(X)$, then $f^n(x) = 0$ for all $x \in X$. This implies $f(x) = 0$ for all $x \in X$, so $f \in I(X)$. Thus we see that $I(X) = \sqrt{I(X)}$. □

Theorem 1.1 (Hilbert's nullstellensatz)**.** *If $J \leq k[x_1, \ldots, x_n]$ is an ideal, then $I(V(J)) = \sqrt{J}$.*

Example 1.4.1. Let $n = 1$, so that $k[x]$ is a PID. Let $f = (x - a_1)^{m_1} \cdots (x - a_r)^{m_r}$. Then

$$I(V(f)) = I(\{a_1, \ldots, a_r\}) = ((x - a_1) \cdots (x - a_r)).$$

[1] Recall that a ring R is *reduced* if for all nonzero $f \in R$ and positive integers n, we have $f^n \neq 0$. It is immediate that an integral domain is reduced.

Lecture 2

Aug. 21 — Hilbert's Nullstellensatz

2.1 Applications of Hilbert's Nullstellensatz

Corollary 2.0.1 (Weak nullstellensatz)**.** *If $J \leq k[x_1, \dots, x_n]$ is an ideal with $J \neq (1)$, then $V(J) \neq \varnothing$. Equivalently, if $f_1, \dots, f_r \in k[x_1, \dots, x_n]$ have no common zeros, then there exist $g_1, \dots, g_r \in k[x_1, \dots, x_n]$ such that $\sum_{i=1}^r f_i g_i = 1$.*

Proof. Assume otherwise that $V(J) = \varnothing$. Then $I(V(J)) = I(\varnothing) = (1)$, so by Hilbert's nullstellensatz, we have $\sqrt{J} = (1)$. Then $1^n \in J$ for some $n > 0$, so $1 \in J$, i.e. $J = (1)$. □

Remark. We need k to be algebraically closed. Note that $(1) \neq (x^2 + 1) \leq \mathbb{R}[x]$ but $V(x^2 + 1) = \varnothing$.

Corollary 2.0.2. *There is an inclusion-reversing bijection between radical ideals $J \leq k[x_1, \dots, x_n]$ and affine varieties $X \subseteq \mathbb{A}_k^n$ given by $J \mapsto V(J)$ and $X \mapsto I(X)$.*

Proof. It suffices to show that these maps are inverses. For $J \leq k[x_1, \dots, x_n]$ a radical ideal, we have

$$I(V(J)) = \sqrt{J} = J$$

by Hilbert's nullstellensatz. For $X \subseteq \mathbb{A}_k^n$ an affine variety, we clearly have $X \subseteq V(I(X))X$. For the reverse inclusion, choose an ideal $J \leq k[x_1, \dots, x_n]$ such that $V(J) = X$. Then $J \subseteq I(X)$, so we have $V(I(X)) \subseteq V(J) = X$. Thus we also get $V(I(X)) = X$. □

Remark. This implies that maximal ideals in $k[x_1, \dots, x_n]$ correspond to points in $\mathbb{A}_k^n$, since maximal ideals correspond to minimal varieties under this bijection.

Corollary 2.0.3. *If X_1, X_2 are affine varieties in $\mathbb{A}_k^n$, then*

1. $I(X_1 \cup X_2) = I(X_1) \cap I(X_2)$;
2. $I(X_1 \cap X_2) = \sqrt{I(X_1) + I(X_2)}$.

Proof. (1) This follows from definitions.

(2) Write $I(X_1 \cap X_2) = I(V(I(X_1)) \cap V(I(X_2))) = I(V(I(X_1) + I(X_2))) = \sqrt{I(X_1) + I(X_2)}$. □

Example 2.0.1. The radical in (2) is necessary. Consider $X_1 = V(y)$ and $X_2 = V(y - x^2)$ in $\mathbb{A}_k^2$. Then $X_1 \cap X_2 = \{(0, 0)\} \subseteq \mathbb{A}_k^2$, so $I(X_1 \cap X_2) = (x, y)$. However, $I(X_1) + I(X_2) = (y) + (y - x^2) = (y, x^2)$.

Note that it is sometimes better to consider (y, x^2) anyway as it tracks multiplicities. In particular, we can see the multiplicity in the dimension of $k[x, y]/(x, y^2) \cong \overline{1}k \oplus \overline{y}k$ as a k-vector space.

2.2 Proof of Hilbert's Nullstellensatz

We will assume the following result from commutative algebra without proof:

Theorem 2.1 (Noether normalization)**.** *Let A be a finitely generated algebra over a field k with A a domain. Then there is an injective k-algebra homomorphism $k[z_1, \ldots, z_n] \hookrightarrow A$ that is finite, i.e. A is a finitely generated $k[z_1, \ldots, z_n]$-module.*

Corollary 2.1.1. *If $K \subseteq L$ is a field extension and L is a finitely generated K-algebra, then $K \subseteq L$ is a finite field extension. In particular, if in addition $K = \overline{K}$, then $K = L$.*

Proof. By Noether normalization, there exists a k-algebra homomorphism $K[z_1, \ldots, z_n] \to L$ that is finite. Then by a result from commutative algebra, L is integral over $K[z_1, \ldots, z_n]$, which implies that $K[z_1, \ldots, z_n]$ must also be a field since L is. Thus $n = 0$, so $K \subseteq L$ is a finite extension. $\square$

Proposition 2.1. *If $(1) \neq J \leq R$ is an ideal, then J is contained in some maximal ideal.*

Proof. Consider the set $P = \{I \leq R : J \subseteq I, I \neq (1)\}$ with the partial order given by inclusion. Note that $P \neq \varnothing$ since $J \in P$. Furthermore, every chain in P has an upper bound (for $\{I_\alpha : \alpha \in A\}$ a chain P, we can take $\bigcup_{\alpha \in A} I_\alpha$, which one can check is indeed an ideal that lies in P; note that $1 \notin I_\alpha$ implies $1 \notin \bigcup_{\alpha \in A} I_\alpha$). So Zorn's lemma implies there is a maximal element in P, which is a maximal ideal. $\square$

Proof of Theorem 1.1. We will proceed in the following steps:

1. Show that the maximal ideals of $k[x_1, \ldots, x_n]$ are of the form $(x_1 - a_1, \ldots, x_n - a_n)$ for $a_i \in k$.
2. Prove the weak nullstellensatz: If $1 \neq J \leq k[x_1, \ldots, x_n]$, is an ideal, then $V(J) \neq \varnothing$.
3. Prove the (strong) nullstellensatz: $I(V(J)) = \sqrt{J}$ for $J \leq k[x_1, \ldots, x_n]$.

The most difficult part is the first step and is where we need k to be algebraically closed.[1]

(1) For $a_1, \ldots, a_n \in k$, the ideal $(x_1 - a_1, \ldots, x_n - a_n)$ is maximal (the quotient is k, which is a field). Conversely, fix a maximal ideal $\mathfrak{m} \in k[x_1, \ldots, x_n]$. Since

$$k \xrightarrow{\phi} k[x_1, \ldots, x_n]/\mathfrak{m} = L$$

is a finitely generated k-algebra and k is algebraically closed, ϕ is an isomorphism by Corollary 2.1.1. Choose $a_i \in k$ such that $\phi(a_i) = \overline{x}_i$, so $\overline{x_i - a_i} = 0$ in L Then $(x_1 - a_1, \ldots, x_n - a_n) \subseteq \mathfrak{m}$, so they must be equal since both the left and right hand sides are maximal ideals.

(2) By Proposition 2.1, J is contained in some maximal ideal $\mathfrak{m}$. By (1), $\mathfrak{m} = (x_1 - a_1, \ldots, x_n - a_n)$ for some $a_1, \ldots, a_n \in k$. Since $J \subseteq \mathfrak{m}$, we have $V(J) \supseteq V(\mathfrak{m}) \supseteq \{(a_1, \ldots, a_n)\}$, so $J \neq \varnothing$.

(3) We will prove this next class. $\square$

[1]The statement is false when k is not algebraically closed: $(x^2 + 1)$ is maximal in $\mathbb{R}[x]$.