MATH 6421: Algebraic Geometry I

Frank Qiang
Instructor: Harold Blum

Georgia Institute of Technology
Fall 2025

Contents

Lecture 1

Aug. 19 — Affine Varieties

1.1 Motivation for Algebraic Geometry

Remark. Why study algebraic geometry? Algebraic geometry connects to many fields of math.

Example 1.0.1. Consider a plane curve $\{f(z_1, z_2) = 0\} \subseteq \mathbb{C}^2$, e.g. an elliptic curve $z_2^2 - z_1^3 + z_1 - 1 = 0$. Compactify and set C to be the closure of C^0 in $\mathbb{CP}^2$, and let $d = \deg f$. There are connections in

1. Topology: $H^1(C, \mathbb{C}) \cong \mathbb{C}^{2g}$, where $g = (d-1)(d-2)/12$;
2. Arithmetic: the number of $\mathbb{Q}$-points is finite if $d > 3$;
3. Complex geometry: We have $C \cong \mathbb{CP}^2$ for $d = 1, 2$, $C \cong \mathbb{C}/\Lambda$ for $d = 3$, and $C \cong \mathbb{H}/\Gamma$ for $d > 3$.

1.2 Affine Varieties

Fix an algebraically closed field k (e.g. $\mathbb{C}$, $\overline{\mathbb{Q}}$, $\overline{\mathbb{F}}_p$, etc.).

Definition 1.1. *Affine space* is the set $\mathbb{A}^n = \mathbb{A}^n_k = \{\vec{a} = (a_1, \ldots, a_n) : a_i \in k\}$.

Remark. Note the following:

1. $\mathbb{A}^n_k$ is the same set as k^n, but forgetting the vector space structure;
2. $f \in k[x_1, \ldots, x_n]$ gives a polynomial function $\mathbb{A}^n_k \to k$ by evaluation: $a \mapsto f(a)$.

Definition 1.2. For a subset $S \subseteq k[x_1, \ldots, x_n]$, its *vanishing set* is

$$V(S) = \{a \in \mathbb{A}^n : f(a) = 0 \text{ for all } f \in S\}.$$

An *affine variety* is a subset of $\mathbb{A}^n_k$ of this form.

Example 1.2.1. Consider the following:

1. $\mathbb{A}^n = V(\varnothing) = V(\{0\})$;
2. $\varnothing = V(1) = V(k[x_1, \ldots, x_n])$;
3. a point $a = (a_1, \ldots, a_n)$ is an affine variety: $V(\{x_1 - a_1, \ldots, x_n - a_n\}) = \{a\}$;
4. a linear space $L \subseteq \mathbb{A}^n$ (it is the kernel of some matrix);
5. plane curves $V(f(x, y)) \subseteq \mathbb{A}^2_{x,y}$;

6. $\mathrm{SL}_n(k) \subseteq \mathbb{A}^{n\times n}$ is an affine variety: $\mathrm{SL}_n(k) = V(\det([x_{i,j}]) - 1)$;
7. $\mathrm{GL}_n(k)$ (as a set) is an affine variety in $\mathbb{A}^{n\times n+1}$: $\mathrm{GL}_n(k) = V(\det([x_{i,j}])y - 1)$;
8. if $X \subseteq \mathbb{A}^m$ and $Y \subseteq \mathbb{A}^n$ are affine varieties, then $X \times Y \subseteq \mathbb{A}^{m+n}$ is an affine variety;
9. the affine varieties $X \subseteq \mathbb{A}^1_k$ are of the form: finite set of points, $\varnothing$, or $\mathbb{A}^1_k$.

Proposition 1.1 (Relation to ideals)**.** *If $S \subseteq k[x_1, \ldots, x_n]$, then $V(S) = V(\langle S\rangle)$, where $\langle S\rangle$ is the ideal generated by S.*

Proof. Since $S \subseteq \langle S\rangle$, we have $V(\langle S\rangle) \subseteq V(S)$. Conversely, if $f, g \in S$ and $h \in k[x_1, \ldots, x_n]$, then $f + g$ and hf vanish on $V(S)$, so we see that $V(S) \subseteq V(\langle S\rangle)$. □

Remark. The statement implies that if $f_1, \ldots, f_r \in k[x_1, \ldots, x_n]$, then $V(f_1, \ldots, f_r) = V((f_1, \ldots, f_n))$. The following are some further applications of the statement:

1. affine varities are vanishing loci of ideals;
2. if $X \subseteq \mathbb{A}^n$ is an affine variety, then X is cut out by finitely many polynomial equations.

To see the second statement, note that $X = V(I)$ for some ideal $I \leq k[x_1, \ldots, x_n]$. By the Hilbert basis theorem that $k[x_1, \ldots, x_n]$ is Noetherian, there are finitely many $f_1, \ldots, f_r \in k[x_1, \ldots, x_n]$ such that $I = (f_1, \ldots, f_r)$. So $X = V(I) = V(f_1, \ldots, f_r)$.

Proposition 1.2 (Properties of the vanishing set)**.** *For ideals I, J of $k[x_1, \ldots, x_n]$,*

1. *if $I \subseteq J$, then $V(J) \subseteq V(I)$;*
2. *$V(I) \cap V(J) = V(I + J)$;*
3. *$V(I) \cup V(J) = V(IJ) = V(I \cap J)$.*

Proof. (1) This follows from definitions and actually holds for general subsets.

(2) Note that $V(I) \cap V(J) = V(I \cap J) = V(\langle I \cup J\rangle) = V(I + J)$.

(3) We only prove the first equality, the second is similar. Recall that $IJ = \left\{\sum_{i=1}^r f_i g_i : f_i \in I, g_i \in J\right\}$. We have the forwards inclusion $V(I) \cup V(J) \subseteq V(IJ)$ from definitions. For the reverse inclusion, consider a point $x \notin V(I) \cup V(J)$. So there exists $f \in I$ and $g \in J$ such that $f(x), g(x) \neq 0$. So $f(x)g(x) \neq 0$, which implies that $x \notin V(IJ)$. Thus $V(IJ) \subseteq V(I) \cup V(J)$ as well. □

Remark. The above implies that if X and Y are affine varieties in $\mathbb{A}^n_k$, then so are $X \cup Y$ and $X \cap Y$.

Example 1.2.2. Consider $V(y^2 - x^2, y^2 + x^2) \subseteq \mathbb{A}^2$. Note that $(y^2 - x^2, y^2 + x^2) = (x^2, y^2)$, from which we can easily see that $V(y^2 - x^2, y^2 + x^2) = V(x^2, y^2) = \{0\}$.

1.3 Correspondence with Ideals

Remark. Our goal is to build a correspondence between affine varieties in $\mathbb{A}^n_k$ and ideals of $k[x_1, \ldots, x_n]$.

Definition 1.3. For a subset $X \subseteq \mathbb{A}^n_k$, define

$$I(X) = \{f \in k[x_1, \ldots, x_n] : f(a) = 0 \text{ for all } a \in X\}.$$

Remark. Note that $I(X)$ is in fact an ideal of $k[x_1, \ldots, x_n]$.

Example 1.3.1. Consider the following:

1. $I(\varnothing) = k[x_1, \ldots, x_n]$;
2. $I(\mathbb{A}^n_k) = \{0\}$, this will follow from the Hilbert nullstellensatz and relies on $k = \overline{k}$ (for $k = \mathbb{R}$, the polynomial $x^2 + y^2 + 1$ is always nonzero and thus lies in $I(\mathbb{A}^n_{\mathbb{R}})$);
3. for $n = 1$, if $S \subseteq \mathbb{A}^1_k$ be an infinite set, then $I(S) = (0)$.
4. for $n = 1$, we have $I(V(x^2)) = I(\{0\}) = (x)$.

Remark. What properties does $I(X)$ satisfy?

Definition 1.4. Let R be a ring. The *radical* of an ideal $J \leq R$ is

$$\sqrt{J} = \{f \in R : f^n \in J \text{ for some } n > 0\}.$$

An ideal J is *radical* if $J = \sqrt{J}$.

Exercise 1.1. Check the following:

1. $\sqrt{J}$ is always an ideal.
2. $\sqrt{\sqrt{J}} = \sqrt{J}$.
3. An ideal $J \leq R$ is radical if and only if R/J is reduced.[1]

Proposition 1.3. *If $X \subseteq \mathbb{A}^n_k$ is a subset (not necessarily an affine variety), then $I(X)$ is radical.*

Proof. Fix $f \in k[x_1, \ldots, x_n]$. If $f^n \in I(X)$, then $f^n(x) = 0$ for all $x \in X$. This implies $f(x) = 0$ for all $x \in X$, so $f \in I(X)$. Thus we see that $I(X) = \sqrt{I(X)}$. □

Theorem 1.1 (Hilbert's nullstellensatz)**.** *If $J \leq k[x_1, \ldots, x_n]$ is an ideal, then $I(V(J)) = \sqrt{J}$.*

Example 1.4.1. Let $n = 1$, so that $k[x]$ is a PID. Let $f = (x - a_1)^{m_1} \cdots (x - a_r)^{m_r}$. Then

$$I(V(f)) = I(\{a_1, \ldots, a_r\}) = ((x - a_1) \cdots (x - a_r)).$$

[1] Recall that a ring R is *reduced* if for all nonzero $f \in R$ and positive integers n, we have $f^n \neq 0$. It is immediate that an integral domain is reduced.

Lecture 2

Aug. 21 — Hilbert's Nullstellensatz

2.1 Applications of Hilbert's Nullstellensatz

Corollary 2.0.1 (Weak nullstellensatz). *If $J \leq k[x_1, \ldots, x_n]$ is an ideal with $J \neq (1)$, then $V(J) \neq \varnothing$. Equivalently, if $f_1, \ldots, f_r \in k[x_1, \ldots, x_n]$ have no common zeros, then there exist $g_1, \ldots, g_r \in k[x_1, \ldots, x_n]$ such that $\sum_{i=1}^r f_i g_i = 1$.*

Proof. Assume otherwise that $V(J) = \varnothing$. Then $I(V(J)) = I(\varnothing) = (1)$, so by Hilbert's nullstellensatz, we have $\sqrt{J} = (1)$. Then $1^n \in J$ for some $n > 0$, so $1 \in J$, i.e. $J = (1)$. $\square$

Remark. We need k to be algebraically closed. Note that $(1) \neq (x^2+1) \leq \mathbb{R}[x]$ but $V(x^2+1) = \varnothing$.

Corollary 2.0.2. *There is an inclusion-reversing bijection between radical ideals $J \leq k[x_1, \ldots, x_n]$ and affine varieties $X \subseteq \mathbb{A}_k^n$ given by $J \mapsto V(J)$ and $X \mapsto I(X)$.*

Proof. It suffices to show that these maps are inverses. For $J \leq k[x_1, \ldots, x_n]$ a radical ideal, we have

$$I(V(J)) = \sqrt{J} = J$$

by Hilbert's nullstellensatz. For $X \subseteq \mathbb{A}_k^n$ an affine variety, we clearly have $X \subseteq V(I(X))X$. For the reverse inclusion, choose an ideal $J \leq k[x_1, \ldots, x_n]$ such that $V(J) = X$. Then $J \subseteq I(X)$, so we have $V(I(X)) \subseteq V(J) = X$. Thus we also get $V(I(X)) = X$. $\square$

Remark. This implies that maximal ideals in $k[x_1, \ldots, x_n]$ correspond to points in $\mathbb{A}_k^n$, since maximal ideals correspond to minimal varieties under this bijection.

Corollary 2.0.3. *If X_1, X_2 are affine varieties in $\mathbb{A}_k^n$, then*

1. *$I(X_1 \cup X_2) = I(X_1) \cap I(X_2)$;*
2. *$I(X_1 \cap X_2) = \sqrt{I(X_1) + I(X_2)}$.*

Proof. (1) This follows from definitions.

(2) Write $I(X_1 \cap X_2) = I(V(I(X_1)) \cap V(I(X_2))) = I(V(I(X_1) + I(X_2))) = \sqrt{I(X_1) + I(X_2)}$. $\square$

Example 2.0.1. The radical in (2) is necessary. Consider $X_1 = V(y)$ and $X_2 = V(y - x^2)$ in $\mathbb{A}_k^2$. Then $X_1 \cap X_2 = \{(0,0)\} \subseteq \mathbb{A}_k^2$, so $I(X_1 \cap X_2) = (x, y)$. However, $I(X_1) + I(X_2) = (y) + (y - x^2) = (y, x^2)$.

Note that it is sometimes better to consider (y, x^2) anyway as it tracks multiplicities. In particular, we can see the multiplicity in the dimension of $k[x,y]/(x, y^2) \cong \overline{1}k \oplus \overline{y}k$ as a k-vector space.

2.2 Proof of Hilbert's Nullstellensatz

We will assume the following result from commutative algebra without proof:

Theorem 2.1 (Noether normalization)**.** *Let A be a finitely generated algebra over a field k with A a domain. Then there is an injective k-algebra homomorphism $k[z_1, \ldots, z_n] \hookrightarrow A$ that is finite, i.e. A is a finitely generated $k[z_1, \ldots, z_n]$-module.*

Corollary 2.1.1. *If $K \subseteq L$ is a field extension and L is a finitely generated K-algebra, then $K \subseteq L$ is a finite field extension. In particular, if in addition $K = \overline{K}$, then $K = L$.*

Proof. By Noether normalization, there exists a k-algebra homomorphism $K[z_1, \ldots, z_n] \to L$ that is finite. Then by a result from commutative algebra, L is integral over $K[z_1, \ldots, z_n]$, which implies that $K[z_1, \ldots, z_n]$ must also be a field since L is. Thus $n = 0$, so $K \subseteq L$ is a finite extension. □

Proposition 2.1. *If $(1) \neq J \leq R$ is an ideal, then J is contained in some maximal ideal.*

Proof. Consider the set $P = \{I \leq R : J \subseteq I, I \neq (1)\}$ with the partial order given by inclusion. Note that $P \neq \varnothing$ since $J \in P$. Furthermore, every chain in P has an upper bound (for $\{I_\alpha : \alpha \in A\}$ a chain P, we can take $\bigcup_{\alpha \in A} I_\alpha$, which one can check is indeed an ideal that lies in P; note that $1 \notin I_\alpha$ implies $1 \notin \bigcup_{\alpha \in A} I_\alpha$). So Zorn's lemma implies there is a maximal element in P, which is a maximal ideal. □

Proof of Theorem 1.1. We will proceed in the following steps:

1. Show that the maximal ideals of $k[x_1, \ldots, x_n]$ are of the form $(x_1 - a_1, \ldots, x_n - a_n)$ for $a_i \in k$.
2. Prove the weak nullstellensatz: If $1 \neq J \leq k[x_1, \ldots, x_n]$, is an ideal, then $V(J) \neq \varnothing$.
3. Prove the (strong) nullstellensatz: $I(V(J)) = \sqrt{J}$ for $J \leq k[x_1, \ldots, x_n]$.

The most difficult part is the first step and is where we need k to be algebraically closed.[1]

(1) For $a_1, \ldots, a_n \in k$, the ideal $(x_1 - a_1, \ldots, x_n - a_n)$ is maximal (the quotient is k, which is a field). Conversely, fix a maximal ideal $\mathfrak{m} \in k[x_1, \ldots, x_n]$. Since

$$k \xrightarrow{\phi} k[x_1, \ldots, x_n]/\mathfrak{m} = L$$

is a finitely generated k-algebra and k is algebraically closed, ϕ is an isomorphism by Corollary 2.1.1. Choose $a_i \in k$ such that $\phi(a_i) = \overline{x}_i$, so $\overline{x_i - a_i} = 0$ in L Then $(x_1 - a_1, \ldots, x_n - a_n) \subseteq \mathfrak{m}$, so they must be equal since both the left and right hand sides are maximal ideals.

(2) By Proposition 2.1, J is contained in some maximal ideal $\mathfrak{m}$. By (1), $\mathfrak{m} = (x_1 - a_1, \ldots, x_n - a_n)$ for some $a_1, \ldots, a_n \in k$. Since $J \subseteq \mathfrak{m}$, we have $V(J) \supseteq V(\mathfrak{m}) \supseteq \{(a_1, \ldots, a_n)\}$, so $J \neq \varnothing$.

(3) The reverse inclusion follows from definitions. For the forward inclusion, fix $f \in I(V(J))$, and we want to show that $f^n \in J$ for some $n > 0$. Add a new variable y and consider

$$J_1 = (J, fy - 1) \leq k[x_1, \ldots, x_n, y].$$

Now $V(J_1) = \{(a, b) = (a_1, \ldots, a_n, b) \in \mathbb{A}_k^{n+1} : a \in V(J), f(a)b = 1\} = \varnothing$ since f vanishes on $V(J)$, so $f(a)b = 0$ for any b. Thus by the weak nullstellensatz, $J_1 = (1)$, so $1 = \sum_{i=1}^r g_i f_i + g_0(fy - 1)$ with

[1] The statement is false when k is not algebraically closed: $(x^2 + 1)$ is maximal in $\mathbb{R}[x]$.

$f_1, \ldots, f_r \in J$ and $g_0, \ldots, g_r \in k[x_1, \ldots, x_n, y]$. Let N be the maximal power of y in the g_i. Multiplying by f^N, we get

$$f^N = \sum_{i=1}^{r} G_i(x_1, \ldots, x_n, fy) f_i + G_0(x_1, \ldots, x_n, fy)(fy - 1)$$

with $G_i \in k[x_1, \ldots, x_n, fy]$. So if we set $fy = 1$, then we have

$$f^N = \sum_{i=1}^{r} G_i(x_1, \ldots, x_n, 1) f_i + 0 \in J,$$

which gives $f \in \sqrt{J}$. To justify this substitution, we can consider the quotient $k[x_1, \ldots, x_n, y]/(fy - 1)$. We have a map $k[x_1, \ldots, x_n] \to k[x_1, \ldots, x_n, y]/(fy - 1)$, which is injective since $(fy - 1)$ does not lie in $k[x_1, \ldots, x_n]$, so an equality in the quotient implies an equality in $k[x_1, \ldots, x_n]$. $\square$

Lecture 3

Aug. 26 — The Zariski Topology

3.1 Polynomial Functions and Subvarieties

Remark. Recall that a polynomial $f \in k[x_1, \dots, x_n]$ gives a function $\mathbb{A}^n_k \to k$ by $a \mapsto f(a)$.

Proposition 3.1. *If $f, g \in k[x_1, \dots, x_n]$ give the same function $\mathbb{A}^n_k \to k$, then $f = g$ in $k[x_1, \dots, x_n]$.*

Proof. Assume $f = g$ as polynomial functions. Then $V(f-g) = \mathbb{A}^n_k$, so $\sqrt{(f-g)} = I(\mathbb{A}^n_k) = (0)$ by Hilbert's nullstellensatz (note that we can also prove $I(\mathbb{A}^n_k) = (0)$ directly, it is enough to have k be an infinite field for this part). Thus $f - g = 0$, so $f = g$ in $k[x_1, \dots, x_n]$. □

Remark. In the above proposition, we need k to be an infinite field (e.g. if $k = \overline{k}$): Otherwise, there are only finitely many functions $\mathbb{A}^n_k \to k$, but infinitely many polynomials in $k[x_1, \dots, x_n]$.

Remark. The set of polynomials functions $\mathbb{A}^n_k \to k$ form a ring, and the above proposition implies that this ring is isomorphic to $k[x_1, \dots, x_n]$.

Definition 3.1. A *polynomial function* on an affine variety $X \subseteq \mathbb{A}^n_k$ is a function $\varphi : X \to k$ such that there exists $f \in k[x_1, \dots, x_n]$ with $\varphi(a) = f(a)$ for every $a \in X$.

Definition 3.2. The *coordinate ring* of X is $A(X) = \{f : X \to k \mid f \text{ is a polynomial function}\}$, which is a ring under pointwise addition and multiplication.

Remark. Observe that there exists a surjective ring homomorphism

$$\begin{aligned} k[x_1, \dots, x_n] &\longrightarrow A(X) \\ f &\longmapsto (a \mapsto f(a)) \end{aligned}$$

with kernel $I(X)$. Thus we have $A(X) \cong k[x_1, \dots, x_n]/I(X)$.

Remark. We can now replace $\mathbb{A}^n_k$ and $k[x_1, \dots, x_n]$ by X and $A(X)$ to study *subvarieties* of X.

Definition 3.3. Let $X \subseteq \mathbb{A}^n_k$ be an affine variety. If $S \subseteq A(X)$ is a subset, then define

$$V_X(S) = \{a \in X : f(a) = 0 \text{ for all } f \in S\}.$$

A subset of X of this form is called an *affine subvariety* of X. (Equivalently, these are the same as an affine variety $Y \subseteq \mathbb{A}^n_k$ such that $Y \subseteq X$.) For $Y \subseteq X$ a subvariety, define

$$I_X(Y) = \{f \in A(X) : f(a) = 0 \text{ for all } a \in Y\}.$$

Proposition 3.2. *There is a bijective correspondence between radical ideals in $A(X)$ and affine subvarieties of X given by $J \mapsto V_X(J)$ and $Y \mapsto I_X(Y)$.*

Proof. See Homework 2. □

3.2 The Zariski Topology

Definition 3.4. The *Zariski topology* on $\mathbb{A}^n_k$ is the topology with closed sets $V(I) \subseteq \mathbb{A}^n_k$, where I is an ideal in $k[x_1, \dots, x_n]$. (Equivalently, the closed sets are the affine varieties in $\mathbb{A}^n_k$.)

Remark. Note the following:

1. On $\mathbb{A}^1_k$, the closed sets are of the form: $\varnothing$, $\mathbb{A}^1_k$, or finite collections of points.
2. When $k = \mathbb{C}$, then $X \subseteq \mathbb{A}^n_{\mathbb{C}}$ being Zariski closed implies that X is closed in the analytic topology on $\mathbb{A}^n_{\mathbb{C}}$. In particular, the Zariski topology is coarser than the analytic topology.
3. On $\mathbb{A}^2_k$, the closed sets are of the form: $\varnothing$, $\mathbb{A}^2_k$, finite collections of points, plane curves, and their finite unions.

Proposition 3.3. *The Zariski topology on $\mathbb{A}^n_k$ is indeed a topology.*

Proof. First note that $\varnothing = V((1))$ and $\mathbb{A}^n_k = V((0))$ are closed. For arbitrary intersections, note that $\bigcap_\alpha V(I_\alpha) = V(\sum_\alpha I_\alpha)$, and for finite unions, note that $\bigcup_{i=1}^r V(I_i) = V(I_1 \cdots I_r)$. □

Example 3.4.1. The Zariski topology on $\mathbb{A}^{n+m}_k$ is in general *not* the product topology of the Zariski topologies on $\mathbb{A}^n_k$ and $\mathbb{A}^m_k$. Consider $V(y - x^2) \subseteq \mathbb{A}^2_k$, which is a closed set in the Zariski topology, but the only closed sets in $\mathbb{A}^1_k$ are either $\varnothing$, $\mathbb{A}^1_k$, or finite.

Definition 3.5. If $X \subseteq \mathbb{A}^n_k$ is an affine variety, then we can define the *Zariski topology* on X in the following two equivalent ways:

1. take the subspace topology from the Zariski topology on $\mathbb{A}^n_k$;
2. take the closed sets of X to be of the form $V_X(I)$ for some ideal $I \leq A(X)$.

This is because an affine subvariety of X is precisely the intersection of X with an affine variety in $\mathbb{A}^n_k$.

Remark. Our goal now is to relate properties of the Zariski topology on X to the ring $A(X)$, and then to the ideal $I(X) \leq k[x_1, \dots, x_n]$.

Definition 3.6. A topological space X is *reducible* if we can write $X = X_1 \cup X_2$ for some closed sets $X_1, X_2 \subsetneq X$. Otherwise, X is called *irreducible*.

Example 3.6.1. The plane curve $X = V(y^2 - x^2 y) = V(y) \cup V(y - x^2)$ is reducible.

Remark. Note the following:

1. A disconnected topological space is reducible.
2. Many topologies are reducible, e.g. $\mathbb{C}^n$, $\mathbb{R}^n$ with the analytic topology.
3. If X is irreducible and $U \subseteq X$ is a nonempty open set, then $\overline{U} = X$ (we have $\overline{U} \cup (X \setminus U) = X$).

Lecture 4

Aug. 28 — Irreducibility

4.1 Properties of Irreducibility

Proposition 4.1. *Let $X \subseteq \mathbb{A}^n$ be an affine variety. Then the following are equivalent:*

1. *X is irreducible;*
2. *$I(X) \leq k[x_1, \dots, x_n]$ is a prime ideal;*
3. *the coordinate ring $A(X)$ is an integral domain.*

Example 4.0.1. We have the following:

1. $\mathbb{A}^n_k$ is irreducible as $A(\mathbb{A}^n_k) = k[x_1, \dots, x_n]$, which is an integral domain.
2. A *hypersurface* $X \subseteq \mathbb{A}^n_k$ is an affine variety with $I(X) = (f)$ for some $f \in k[x_1, \dots, x_n]$. Then A is irreducible if and only if (f) is prime, if and only if f is irreducible.[1]

4.2 Dimension

Definition 4.1. Let X be a topological space.

- The *dimension* of X, denoted $\dim X$, is the supremum of the n such that there exists a chain of irreducible closed subspaces
$$X \supseteq X_0 \supsetneq X_1 \supsetneq \cdots \supsetneq X_n \neq \varnothing.$$
- For $Y \subseteq X$ closed and irreducible, the *codimension* of Y in X, denoted $\operatorname{codim}_X Y$, is the supremum of the n as above such that $X_n = Y$.

[1] Note that any prime ideal is radical.

Lecture 5

Sept. 2 — Dimension

5.1 More on Dimension

Remark. Recall the following correspondence from before: If $X \subseteq \mathbb{A}_k^n$ is an affine variety, then there exists a bijection between the irreducible closed subsets $Y \subseteq X$ and the prime ideals $\mathfrak{p} \leq A(X)$.

Definition 5.1. For a ring A, the *(Krull) dimension* of A, denoted $\dim A$, is the supremum of the n such that there exists a chain of prime ideals

$$A \supseteq \mathfrak{p}_0 \supsetneq \mathfrak{p}_1 \supsetneq \cdots \supsetneq \mathfrak{p}_n.$$

For a prime ideal $\mathfrak{q} \leq A$, the *height* of $\mathfrak{q}$, denoted $\operatorname{ht} \mathfrak{q}$, is the supremum of the n as above with $\mathfrak{p}_0 = \mathfrak{q}$.

Remark. If X is an affine variety, then we have the following:

1. $\dim X = \dim A(X)$;
2. for $Y \subseteq X$ a closed irreducible subset, $\operatorname{codim}_X Y = \operatorname{ht} I_X(Y)$.

These properties follow from the inclusion-reversing correspondence.

Definition 5.2. Let $K \subseteq L$ be a field extension.

1. A collection of elements $\{z_i : i \in I\} \subseteq L$ is a *transcendence basis* of $K \subseteq L$ if the z_i are algebraically independent (i.e. $K(x_i : i \in I) \xrightarrow{\cong} K(z_i : i \in I)$ by $x_i \mapsto z_i$) and $K(z_i : i \in I) \subseteq L$ is algebraic.
2. The *transcendence degree* $\operatorname{tr.deg}_K L$ is the cardinality of a transcendence basis.

Theorem 5.1 (Dimension theory)**.** *Let A be a finitely generated k-algebra that is a domain. Then*

1. $\dim A = \operatorname{tr.deg}_k \operatorname{Frac}(A)$*;*
2. *for any prime ideal $\mathfrak{p} \leq A$, we have* $\operatorname{ht} \mathfrak{p} + \dim A/\mathfrak{p} = \dim A$*;*
3. *all maximal chains of prime ideals $A \supseteq \mathfrak{p}_0 \supsetneq \mathfrak{p}_1 \supsetneq \cdots \supsetneq \mathfrak{p}_n$ are of the same length.*

Remark. The following are consequences of the above result from commutative algebra:

1. $\dim_k \mathbb{A}_k^n = \dim k[x_1, \ldots, x_n] = \operatorname{tr.deg}_k k(x_1, \ldots, x_n) = n$.
2. If X is irreducible, then $A(X)$ is a domain, so for $x \in X$, we have

$$\operatorname{codim}_X \{x\} = \operatorname{ht} I(\{x\}) = \dim A(X) - \dim A(X)/I(\{x\}) = \dim A(X) = \dim X,$$

where we note that $A(X)/I(\{x\}) \cong k$ is a field.

3. If X is an irreducible affine variety and $U \subseteq X$ is a nonempty open subset, then
$$\dim U = \sup_{x \in U} \operatorname{codim}_U \{x\} = \sup_{x \in U} \operatorname{codim}_X \{x\} = \dim X.$$
This follows since we can pass from a chain in U to a chain in X by taking closures.
4. If X is an irreducible affine variety and $Z \subseteq X$ is an irreducible closed subset, then
$$\dim Z = \dim X - \operatorname{codim}_X Z.$$

Note that (2)-(4) can be false if X is not irreducible. To contradict (4), let $X = V(x, y) \cup V(z) \subseteq \mathbb{A}^3_k$ with $Z = V(x, y)$. Then we have $\dim X = 2$, $\dim Z = 1$, $\operatorname{codim}_X Z = 0$.

5.2 Hypersurfaces

Remark. We now want to study hypersurfaces.

Theorem 5.2 (Krull's Hauptidealsatz)**.** *If A is a Noetherian ring and $f \in A$ is nonzero and a non-unit, then every minimal prime ideal containing f has height 1.*

Corollary 5.2.1. *If $X \subseteq \mathbb{A}^n_k$ is an irreducible affine variety and $f \in A(X)$ is a nonzero non-unit, then*
$$\dim Z = \dim X - 1$$
for every irreducible component Z of $V_X(f)$.

Proof. Since X is irreducible, $A(X)$ is a domain. So there is a correspondence between the minimal prime ideals $f \in \mathfrak{p} \subsetneq A(X)$ and the minimal irreducible closed subsets $Z \supseteq V_X(f)$, which corresponds to the irreducible components Z of $V_X(f)$. For such a component Z, we know
$$\dim Z = \dim Z - \operatorname{codim}_X Z = \dim X - \operatorname{ht} I(Z) = \dim X - 1$$
by Krull's Hauptidealsatz, which is the desired result. □

Example 5.2.1. Corollary 5.2.1 implies that if $f \in k[x_1, \ldots, x_n]$ is non-constant, then
$$\dim V(f) = \dim \mathbb{A}^n_k - 1 = n - 1.$$

Theorem 5.3. *An irreducible affine variety $Y \subseteq \mathbb{A}^n_k$ has $\dim Y = n - 1$ if and only if $Y = V(f)$ for some non-constant polynomial $f \in k[x_1, \ldots, x_n]$.*

Proof. ($\Leftarrow$) This was Corollary 5.2.1.

($\Rightarrow$) We will use that $A(\mathbb{A}^n_k) = k[x_1, \ldots, x_n]$ is a UFD. Since Y is irreducible and $\dim Y = n - 1$,
$$\operatorname{ht} I(Y) = \operatorname{codim}_{\mathbb{A}^n_k} Y = \dim \mathbb{A}^n_k - \dim Y = 1.$$
Since $(0) \subsetneq I(Y) \subsetneq k[x_1, \ldots, x_n]$, there exists a non-constant $f \in k[x_1, \ldots, x_n]$ with $f \in I(Y)$. Write
$$f = f_1 \cdots f_r$$
with f_i irreducible by unique factorization, and note that the f_i are also prime since we are in a UFD. Since $I(Y)$ is prime, some f_i is in $I(Y)$, so we have the inclusions
$$(0) \subsetneq (f_i) \subseteq I(Y).$$
Since $\operatorname{ht} I(Y) = 1$, we must have $(f_i) = I(Y)$, so $Y = V(I(Y)) = V(f_i)$. □

5.3 Regular Functions

Definition 5.3. Let X be an affine variety and $U \subseteq X$ open. A function $\varphi : U \to k$ is *regular* if for each $a \in U$, there exists an open neighborhood $a \in U_a \subseteq U$ and $f, g \in A(X)$ such that

$$\varphi(x) = \frac{g(x)}{f(x)}, \quad f(x) \neq 0, \quad \text{for all } x \in U_a.$$

Define $\mathcal{O}_X(U) = \{\varphi : U \to k \mid \varphi \text{ is a regular function on } U\}$.

Exercise 5.1. Check that $\mathcal{O}_X(U)$ is a ring under pointwise addition and multiplication of outputs.

Remark. To patch open sets together, we will later need the notion of a *morphism*, and a morphism $U \to Y \subseteq \mathbb{A}_k^m$ should be given by

$$x \longmapsto (\varphi_1(x), \dots, \varphi_m(x))$$

with φ_i regular functions on U.

Example 5.3.1. We have the following:

1. If $X \subseteq \mathbb{A}_k^n$ is an affine variety, then any $\varphi \in A(X)$ is regular. Furthermore, we get an injective ring homomorphism $A(X) \to \mathcal{O}_X(X)$. We will see that this is an isomorphism.
2. If $X = \mathbb{A}_k^1$ and $U = \mathbb{A}_k^1 \setminus \{0\}$, then for any $n \geq 0$ and $g \in k[x]$, the function g/x^n is regular on U. In general, if we fix $f, g \in A(X)$ and set $U = X \setminus V(f)$, then the map g/f^m is regular on U.
3. Let $X = V(x_1x_4 - x_2x_3) \subseteq \mathbb{A}_k^4$ and $U = X \setminus V(x_2, x_4)$. Then the following map is regular:

$$\begin{aligned} \varphi : U &\longrightarrow k \\ (x_1, x_2, x_3, x_4) &\longmapsto \begin{cases} x_1/x_2, & \text{if } x_2 \neq 0, \\ x_3/x_4, & \text{if } x_4 \neq 0. \end{cases} \end{aligned}$$

   Note that on $U \setminus V(x_2x_4)$, we have $x_1/x_2 = x_3/x_4$ since $x_1x_4 = x_2x_3$ on X.

Lecture 6

Sept. 4 — Regular Functions

6.1 Properties of Regular Functions

Proposition 6.1. *Let X be an affine variety and $U \subseteq X$ open. Then:*

1. *if $\varphi \in \mathcal{O}_X(U)$, then $V(\varphi) = \{x \in U : \varphi(x) = 0\}$ is closed in U;*
2. *(identity principle) If X is irreducible, $U \subseteq X$ is nonempty and open, and $\varphi, \psi \in \mathcal{O}_X(U)$ with $\varphi|_W = \psi|_W$ for some $W \subseteq U$ nonempty and open, then $\varphi = \psi$ in $\mathcal{O}_X(U)$.*

Proof. (1) It suffices to show that $U \setminus V(\varphi)$ is open in U. Fix $a \in U \setminus V(\varphi)$. Since φ is regular, there exists an open neighborhood $a \in U_a \subseteq U$ and $f_a, g_a \in A(X)$ such that

$$\varphi|_{U_a} = \frac{g_a}{f_a}.$$

So $a \in \{g_a \neq 0\} \cap U_a \subseteq U \setminus V(\varphi)$. This is an open set containing a in $U \setminus V(\varphi)$, so $U \setminus V(\varphi)$ is open.

(2) Since X is irreducible, U is also irreducible. The locus $\{x \in U : \varphi(x) = \psi(x)\} = V(\varphi - \psi)$ is closed in U by (1). It also contains W. Since W is dense (it is a nonempty open set in an irreducible topological space), we must have $V(\varphi - \psi) = U$. This proves the claim. $\square$

Example 6.0.1. In (2) of Proposition 6.1, the assumption that X is irreducible is necessary. Consider

$$U = X = V(xy) \subseteq \mathbb{A}^2_k \quad \text{and} \quad W = V(xy) \setminus V(x).$$

Then the regular functions $\varphi = x$ and $\psi = x + y$ agree on W but are not equal on U.

6.2 Distinguished Open Sets

Remark. We will see that an affine variety has a basis of open sets on which we can compute $\mathcal{O}_X(U)$.

Definition 6.1. A *distinguished open set* of an affine variety X is a subset of the form

$$D(f) = X \setminus V(f)$$

for some polynomial function $f \in A(X)$.

Remark. We have the following:

1. The $D(f)$ are closed under (finite) intersection: $D(fg) = D(f) \cap D(g)$.

2. The $D(f)$ form a basis for the Zariski topology on X: If $U \subseteq X$ is open, then $U = X \setminus V(f_1, \dots, f_r)$ for some $f_1, \dots, f_r \in A(X)$ (since X is Noetherian). So $U = D(f_1) \cup \cdots \cup D(f_r)$.

Remark. We will view $D(f)$ as "small open sets" (under mild assumptions, $\operatorname{codim}_X(X \setminus D(f)) = 1$).

Theorem 6.1. *If X is an affine variety and $f \in A(X)$, then*

$$\mathcal{O}_X(D(f)) = \left\{ \frac{g}{f^m} : g \in A(X), m \geq 0 \right\}.$$

Proof. We have an injective ring homomorphism

$$\left\{ \frac{g}{f^m} : g \in A(X), m \geq 0 \right\} \longrightarrow \mathcal{O}_X(D(f)),$$

it suffices to show this map is surjective. Fix $\varphi \in \mathcal{O}_X(D(f))$. For any $a \in D(f)$, there exists an open neighborhood $a \in U_a \subseteq D(f)$ and $f_a, g_a \in A(X)$ such that $\varphi|_{U_a} = g_a/f_a$. We may further assume that

1. $U_a = D(h_a)$ for some $h_a \in A(X)$ (by shrinking U_a if necessary, since the $D(h)$ form a basis);
2. $h_a = f_a$ (by rewriting $g_a/f_a = g_a h_a / f_a h_a$ and replacing h_a, f_a with $f_a h_a$).

Then for $a, b \in D(f)$, we have $f_a g_b = f_b g_a$ on $D(f_a) \cap D(f_b)$. Since both the left and right hand sides vanish on $X \setminus (D(f_a) \cap D(f_b))$, we have $f_a g_b = f_b g_a$ in $A(X)$. Now we can write

$$V(f) = \bigcap_{a \in D(f)} V(f_a) = V(f_a : a \in D(f)),$$

so $f \in I(V(f_a : a \in D(f)))$. By the Nullstellensatz, there exists $n \geq 0$ such that

$$f^n = \sum_{a \in D(f)} k_a f_a, \quad k_a \in A(X),$$

where only finitely many of the k_a are nonzero. Set $g = \sum_{a \in D(f)} k_a g_a$, and we claim that $\varphi = g/f^n$. To see this, note that on U_b, we have $\varphi|_{U_b} = g_b/f_b$. Now since $f_a g_b = f_b g_a$, we have

$$g f_b = \sum_{a \in D(f)} k_a g_a f_b = \sum_{a \in D(f)} k_a f_a g_b = f^n g_b,$$

which shows that $\varphi|_{U_b} = (g/f^n)|_{U_b}$. Since this holds for any U_b, we have $\varphi = g/f^n$ in $\mathcal{O}_X(D(f))$. $\square$

Remark. Theorem 6.1 has the following consequences:

1. The $f = 1$ case implies that the natural ring homomorphism $A(X) \to \mathcal{O}_X(X)$ is surjective and hence an isomorphism (note that $D(1) = X$).
2. We will see that $\mathcal{O}_X(D(f)) \cong A(X)_f$, the *localization* of $A(X)$ at f.

Example 6.1.1. How do we compute $\mathcal{O}_X(U)$ on non-distinguished open sets? Consider

$$X = \mathbb{A}^2_k \quad \text{and} \quad U = \mathbb{A}^2_k \setminus \{(0,0)\}.$$

Note that U is never a distinguished open set. We claim that the ring homomorphism

$$k[x,y] \longrightarrow \mathcal{O}_{\mathbb{A}^2_k}(\mathbb{A}^2_k \setminus \{(0,0)\})$$

is an isomorphism. The map is injective by the identity principle, so it suffices to show surjectivity. The strategy is use $U = D(x) \cup D(y)$ (in general, cover U by basis elements). Fix $\varphi : U \to k$ regular, so

$$\varphi|_{D(x)} = \frac{f}{x^m} \quad \text{for some } f \in k[x,y], m \geq 0$$
$$\varphi|_{D(y)} = \frac{g}{y^n} \quad \text{for some } g \in k[x,y], n \geq 0.$$

Since we are in a UFD, we may assume that $x \nmid f$ and $y \nmid g$. Now $fy^n = gx^m$ on $D(y) \cap D(x)$, so by the identity principle, $fy^n = gx^m$ on $\mathbb{A}^2_k$, so $fy^n = gx^m$ in $k[x,y]$. Using that $y \nmid g$, $x \nmid f$, and that $k[x,y]$ is a UFD, we must have $n = m = 0$, hence $f = g$. In particular, we have

$$\varphi|_{D(x)} = \varphi|_{D(y)} = f,$$

so the map $k[x,y] \to \mathcal{O}_X(U)$ is surjective.

6.3 Localization

Remark. We want to invert a subset of a ring, in particular *multiplicative systems.*

Definition 6.2. A *multiplicative system* of a ring A is a subset such that

1. $1 \in S$;
2. S is closed under multiplication.

Example 6.2.1. The following examples of S are multiplicative systems:

1. $S = A$ or $S = \{1\}$;
2. if $\mathfrak{p} \leq A$ is a prime ideal, then $S = A \setminus \mathfrak{p}$;
3. if $f \in A$, then $S = \{f^m : m \geq 0\}$.

Definition 6.3. The *localization* of a ring A at a multiplicative system S is the ring

$$S^{-1}A = \left\{ \frac{a}{s} : a \in A, s \in S \right\} / \sim$$

where the a/s are formal symbols with $a/s \sim a'/s'$ if $t(as' - a's) = 0$ for some $t \in S$.[1] The operations are given by the usual addition and multiplication of fractions:

$$\frac{a}{s} \cdot \frac{a'}{s'} = \frac{aa'}{ss'} \quad \text{and} \quad \frac{a}{s} + \frac{a'}{s'} = \frac{as' + a's}{ss'}.$$

Check as an exercise that these operations respect the equivalence relation.

Example 6.3.1. The following are examples of localization:

1. If A is a domain and $S = A \setminus \{0\}$, then $S^{-1}A = \operatorname{Frac} A$.
2. If $S = \langle f \rangle = \{1, f, f^2, \dots\}$, then we will write $A_f = S^{-1}A$.
3. If $S = A \setminus \mathfrak{p}$ for a prime ideal $\mathfrak{p}$, then we will write $A_\mathfrak{p} = S^{-1}A$.

[1]Note that if A is a domain and $0 \notin S$, then this condition is equivalent to $as' = a's$.

Proposition 6.2. *We have the following properties of localization:*

1. *(Universal property of localization) For any ring homomorphism $\varphi : A \to B$ such that $\varphi(s)$ for all $s \in S$, then there exists a unique ring homomorphism which makes the following diagram commute:*

$$\begin{tikzcd} A \arrow[rr, "\varphi"] \arrow[dr, "\pi : a \mapsto a/1"'] & & B \\ & S^{-1}A \arrow[ur, dashed, "\exists!"'] & \end{tikzcd}$$

2. *There is a bijection between the prime ideals $\mathfrak{p} \leq A$ with $\mathfrak{p} \cap S = \varnothing$ and the prime ideals $\mathfrak{q} \leq S^{-1}A$ given by $\mathfrak{p} \mapsto \pi(\mathfrak{p})S^{-1}A$ with inverse $\mathfrak{q} \mapsto \pi^{-1}(\mathfrak{q})$, where $\pi : A \to S^{-1}A$ is the map $a \mapsto a/1$.*

Remark. In more generality, for an A-module M, we can define the localization $S^{-1}M$, which is an $S^{-1}A$-module. This gives a functor $\mathrm{Mod}_A \to \mathrm{Mod}_{S^{-1}A}$ which is exact.