

# MATH 6421: Algebraic Geometry I

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# Contents

|          |   |           |
|----------|---|-----------|
| <b>1</b> | <b>Aug. 19 — Affine Varieties</b>                   | <b>3</b>  |
| 1.1      | Motivation for Algebraic Geometry . . . . .         | 3         |
| 1.2      | Affine Varieties . . . . .                          | 3         |
| 1.3      | Correspondence with Ideals . . . . .                | 4         |
| <b>2</b> | <b>Aug. 21 — Hilbert’s Nullstellensatz</b>          | <b>6</b>  |
| 2.1      | Applications of Hilbert’s Nullstellensatz . . . . . | 6         |
| 2.2      | Proof of Hilbert’s Nullstellensatz . . . . .        | 7         |
| <b>3</b> | <b>Aug. 26 — The Zariski Topology</b>               | <b>9</b>  |
| 3.1      | Polynomial Functions and Subvarieties . . . . .     | 9         |
| 3.2      | The Zariski Topology . . . . .                      | 10        |
| <b>4</b> | <b>Aug. 28 — Irreducibility</b>                     | <b>11</b> |
| 4.1      | Properties of Irreducibility . . . . .              | 11        |
| 4.2      | Dimension . . . . .                                 | 11        |
| <b>5</b> | <b>Sept. 2 — Dimension</b>                          | <b>12</b> |
| 5.1      | More on Dimension . . . . .                         | 12        |
| 5.2      | Hypersurfaces . . . . .                             | 13        |
| 5.3      | Regular Functions . . . . .                         | 14        |
| <b>6</b> | <b>Sept. 4 — Regular Functions</b>                  | <b>15</b> |
| 6.1      | Properties of Regular Functions . . . . .           | 15        |
| 6.2      | Distinguished Open Sets . . . . .                   | 15        |
| 6.3      | Localization . . . . .                              | 17        |
| <b>7</b> | <b>Sept. 9 — Germs and Sheaves</b>                  | <b>19</b> |
| 7.1      | More on Localization . . . . .                      | 19        |
| 7.2      | Germs of Functions . . . . .                        | 19        |
| 7.3      | Sheaves . . . . .                                   | 20        |
| <b>8</b> | <b>Sept. 11 — Morphisms</b>                         | <b>22</b> |
| 8.1      | Morphisms of Open Sets . . . . .                    | 22        |
| 8.2      | Relation to Coordinate Rings . . . . .              | 23        |
| <b>9</b> | <b>Sept. 16 — Morphisms, Part 2</b>                 | <b>25</b> |
| 9.1      | An Example of Isomorphisms . . . . .                | 25        |
| 9.2      | Ringed Spaces and Morphisms . . . . .               | 25        |
| 9.3      | Products of Affine Varieties . . . . .              | 26        |
| 9.4      | Tensor Products . . . . .                           | 27        |

|  |           |
|--|-----------|
| <b>10 Sept. 18 — Pre-varieties</b>     | <b>29</b> |
| 10.1 More on Tensor Products . . . . . | 29        |
| 10.2 Pre-varieties . . . . .           | 30        |

# Lecture 1

## Aug. 19 — Affine Varieties

### 1.1 Motivation for Algebraic Geometry

**Remark.** Why study algebraic geometry? Algebraic geometry connects to many fields of math.

**Example 1.0.1.** Consider a plane curve  $\{f(z_1, z_2) = 0\} \subseteq \mathbb{C}^2$ , e.g. an elliptic curve  $z_2^2 - z_1^3 + z_1 - 1 = 0$ . Compactify and set  $C$  to be the closure of  $C^0$  in  $\mathbb{CP}^2$ , and let  $d = \deg f$ . There are connections in

1. Topology:  $H^1(C, \mathbb{C}) \cong \mathbb{C}^{2g}$ , where  $g = (d-1)(d-2)/12$ ;
2. Arithmetic: the number of  $\mathbb{Q}$ -points is finite if  $d > 3$ ;
3. Complex geometry: We have  $C \cong \mathbb{CP}^2$  for  $d = 1, 2$ ,  $C \cong \mathbb{C}/\Lambda$  for  $d = 3$ , and  $C \cong \mathbb{H}/\Gamma$  for  $d > 3$ .

### 1.2 Affine Varieties

Fix an algebraically closed field  $k$  (e.g.  $\mathbb{C}$ ,  $\overline{\mathbb{Q}}$ ,  $\overline{\mathbb{F}}_p$ , etc.).

**Definition 1.1.** *Affine space* is the set  $\mathbb{A}^n = \mathbb{A}_k^n = \{\vec{a} = (a_1, \dots, a_n) : a_i \in k\}$ .

**Remark.** Note the following:

1.  $\mathbb{A}_k^n$  is the same set as  $k^n$ , but forgetting the vector space structure;
2.  $f \in k[x_1, \dots, x_n]$  gives a polynomial function  $\mathbb{A}_k^n \rightarrow k$  by evaluation:  $a \mapsto f(a)$ .

**Definition 1.2.** For a subset  $S \subseteq k[x_1, \dots, x_n]$ , its *vanishing set* is

$$V(S) = \{a \in \mathbb{A}^n : f(a) = 0 \text{ for all } f \in S\}.$$

An *affine variety* is a subset of  $\mathbb{A}_k^n$  of this form.

**Example 1.2.1.** Consider the following:

1.  $\mathbb{A}^n = V(\emptyset) = V(\{0\})$ ;
2.  $\emptyset = V(1) = V(k[x_1, \dots, x_n])$ ;
3. a point  $a = (a_1, \dots, a_n)$  is an affine variety:  $V(\{x_1 - a_1, \dots, x_n - a_n\}) = \{a\}$ ;
4. a linear space  $L \subseteq \mathbb{A}^n$  (it is the kernel of some matrix);
5. plane curves  $V(f(x, y)) \subseteq \mathbb{A}_{x,y}^2$ ;

6.  $\mathrm{SL}_n(k) \subseteq \mathbb{A}^{n \times n}$  is an affine variety:  $\mathrm{SL}_n(k) = V(\det([x_{i,j}]) - 1)$ ;
7.  $\mathrm{GL}_n(k)$  (as a set) is an affine variety in  $\mathbb{A}^{n \times n+1}$ :  $\mathrm{GL}_n(k) = V(\det([x_{i,j}])y - 1)$ ;
8. if  $X \subseteq \mathbb{A}^m$  and  $Y \subseteq \mathbb{A}^n$  are affine varieties, then  $X \times Y \subseteq \mathbb{A}^{m+n}$  is an affine variety;
9. the affine varieties  $X \subseteq \mathbb{A}_k^1$  are of the form: finite set of points,  $\emptyset$ , or  $\mathbb{A}_k^1$ .

**Proposition 1.1** (Relation to ideals). *If  $S \subseteq k[x_1, \dots, x_n]$ , then  $V(S) = V(\langle S \rangle)$ , where  $\langle S \rangle$  is the ideal generated by  $S$ .*

*Proof.* Since  $S \subseteq \langle S \rangle$ , we have  $V(\langle S \rangle) \subseteq V(S)$ . Conversely, if  $f, g \in S$  and  $h \in k[x_1, \dots, x_n]$ , then  $f + g$  and  $hf$  vanish on  $V(S)$ , so we see that  $V(S) \subseteq V(\langle S \rangle)$ .  $\square$

**Remark.** The statement implies that if  $f_1, \dots, f_r \in k[x_1, \dots, x_n]$ , then  $V(f_1, \dots, f_r) = V((f_1, \dots, f_r))$ . The following are some further applications of the statement:

1. affine varieties are vanishing loci of ideals;
2. if  $X \subseteq \mathbb{A}^n$  is an affine variety, then  $X$  is cut out by finitely many polynomial equations.

To see the second statement, note that  $X = V(I)$  for some ideal  $I \leq k[x_1, \dots, x_n]$ . By the Hilbert basis theorem that  $k[x_1, \dots, x_n]$  is Noetherian, there are finitely many  $f_1, \dots, f_r \in k[x_1, \dots, x_n]$  such that  $I = (f_1, \dots, f_r)$ . So  $X = V(I) = V(f_1, \dots, f_r)$ .

**Proposition 1.2** (Properties of the vanishing set). *For ideals  $I, J$  of  $k[x_1, \dots, x_n]$ ,*

1. *if  $I \subseteq J$ , then  $V(J) \subseteq V(I)$ ;*
2.  *$V(I) \cap V(J) = V(I + J)$ ;*
3.  *$V(I) \cup V(J) = V(IJ) = V(I \cap J)$ .*

*Proof.* (1) This follows from definitions and actually holds for general subsets.

(2) Note that  $V(I) \cap V(J) = V(I \cap J) = V(\langle I \cup J \rangle) = V(I + J)$ .

(3) We only prove the first equality, the second is similar. Recall that  $IJ = \{\sum_{i=1}^r f_i g_i : f_i \in I, g_i \in J\}$ . We have the forwards inclusion  $V(I) \cup V(J) \subseteq V(IJ)$  from definitions. For the reverse inclusion, consider a point  $x \notin V(I) \cup V(J)$ . So there exists  $f \in I$  and  $g \in J$  such that  $f(x), g(x) \neq 0$ . So  $f(x)g(x) \neq 0$ , which implies that  $x \notin V(IJ)$ . Thus  $V(IJ) \subseteq V(I) \cup V(J)$  as well.  $\square$

**Remark.** The above implies that if  $X$  and  $Y$  are affine varieties in  $\mathbb{A}_k^n$ , then so are  $X \cup Y$  and  $X \cap Y$ .

**Example 1.2.2.** Consider  $V(y^2 - x^2, y^2 + x^2) \subseteq \mathbb{A}^2$ . Note that  $(y^2 - x^2, y^2 + x^2) = (x^2, y^2)$ , from which we can easily see that  $V(y^2 - x^2, y^2 + x^2) = V(x^2, y^2) = \{0\}$ .

## 1.3 Correspondence with Ideals

**Remark.** Our goal is to build a correspondence between affine varieties in  $\mathbb{A}_k^n$  and ideals of  $k[x_1, \dots, x_n]$ .

**Definition 1.3.** For a subset  $X \subseteq \mathbb{A}_k^n$ , define

$$I(X) = \{f \in k[x_1, \dots, x_n] : f(a) = 0 \text{ for all } a \in X\}.$$

**Remark.** Note that  $I(X)$  is in fact an ideal of  $k[x_1, \dots, x_n]$ .

**Example 1.3.1.** Consider the following:

1.  $I(\emptyset) = k[x_1, \dots, x_n]$ ;
2.  $I(\mathbb{A}_k^n) = \{0\}$ , this will follow from the Hilbert nullstellensatz and relies on  $k = \bar{k}$  (for  $k = \mathbb{R}$ , the polynomial  $x^2 + y^2 + 1$  is always nonzero and thus lies in  $I(\mathbb{A}_{\mathbb{R}}^n)$ );
3. for  $n = 1$ , if  $S \subseteq \mathbb{A}_k^1$  be an infinite set, then  $I(S) = (0)$ .
4. for  $n = 1$ , we have  $I(V(x^2)) = I(\{0\}) = (x)$ .

**Remark.** What properties does  $I(X)$  satisfy?

**Definition 1.4.** Let  $R$  be a ring. The *radical* of an ideal  $J \leq R$  is

$$\sqrt{J} = \{f \in R : f^n \in J \text{ for some } n > 0\}.$$

An ideal  $J$  is *radical* if  $J = \sqrt{J}$ .

**Exercise 1.1.** Check the following:

1.  $\sqrt{J}$  is always an ideal.
2.  $\sqrt{\sqrt{J}} = \sqrt{J}$ .
3. An ideal  $J \leq R$  is radical if and only if  $R/J$  is reduced.<sup>1</sup>

**Proposition 1.3.** If  $X \subseteq \mathbb{A}_k^n$  is a subset (not necessarily an affine variety), then  $I(X)$  is radical.

*Proof.* Fix  $f \in k[x_1, \dots, x_n]$ . If  $f^n \in I(X)$ , then  $f^n(x) = 0$  for all  $x \in X$ . This implies  $f(x) = 0$  for all  $x \in X$ , so  $f \in I(X)$ . Thus we see that  $I(X) = \sqrt{I(X)}$ .  $\square$

**Theorem 1.1** (Hilbert's nullstellensatz). If  $J \leq k[x_1, \dots, x_n]$  is an ideal, then  $I(V(J)) = \sqrt{J}$ .

**Example 1.4.1.** Let  $n = 1$ , so that  $k[x]$  is a PID. Let  $f = (x - a_1)^{m_1} \cdots (x - a_r)^{m_r}$ . Then

$$I(V(f)) = I(\{a_1, \dots, a_r\}) = ((x - a_1) \cdots (x - a_r)).$$

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<sup>1</sup>Recall that a ring  $R$  is *reduced* if for all nonzero  $f \in R$  and positive integers  $n$ , we have  $f^n \neq 0$ . It is immediate that an integral domain is reduced.

# Lecture 2

## Aug. 21 — Hilbert's Nullstellensatz

### 2.1 Applications of Hilbert's Nullstellensatz

**Corollary 2.0.1** (Weak nullstellensatz). *If  $J \leq k[x_1, \dots, x_n]$  is an ideal with  $J \neq (1)$ , then  $V(J) \neq \emptyset$ . Equivalently, if  $f_1, \dots, f_r \in k[x_1, \dots, x_n]$  have no common zeros, then there exist  $g_1, \dots, g_r \in k[x_1, \dots, x_n]$  such that  $\sum_{i=1}^r f_i g_i = 1$ .*

*Proof.* Assume otherwise that  $V(J) = \emptyset$ . Then  $I(V(J)) = I(\emptyset) = (1)$ , so by Hilbert's nullstellensatz, we have  $\sqrt{J} = (1)$ . Then  $1^n \in J$  for some  $n > 0$ , so  $1 \in J$ , i.e.  $J = (1)$ .  $\square$

**Remark.** We need  $k$  to be algebraically closed. Note that  $(1) \neq (x^2 + 1) \leq \mathbb{R}[x]$  but  $V(x^2 + 1) = \emptyset$ .

**Corollary 2.0.2.** *There is an inclusion-reversing bijection between radical ideals  $J \leq k[x_1, \dots, x_n]$  and affine varieties  $X \subseteq \mathbb{A}_k^n$  given by  $J \mapsto V(J)$  and  $X \mapsto I(X)$ .*

*Proof.* It suffices to show that these maps are inverses. For  $J \leq k[x_1, \dots, x_n]$  a radical ideal, we have

$$I(V(J)) = \sqrt{J} = J$$

by Hilbert's nullstellensatz. For  $X \subseteq \mathbb{A}_k^n$  an affine variety, we clearly have  $X \subseteq V(I(X))$ . For the reverse inclusion, choose an ideal  $J \leq k[x_1, \dots, x_n]$  such that  $V(J) = X$ . Then  $J \subseteq I(X)$ , so we have  $V(I(X)) \subseteq V(J) = X$ . Thus we also get  $V(I(X)) = X$ .  $\square$

**Remark.** This implies that maximal ideals in  $k[x_1, \dots, x_n]$  correspond to points in  $\mathbb{A}_k^n$ , since maximal ideals correspond to minimal varieties under this bijection.

**Corollary 2.0.3.** *If  $X_1, X_2$  are affine varieties in  $\mathbb{A}_k^n$ , then*

1.  $I(X_1 \cup X_2) = I(X_1) \cap I(X_2)$ ;
2.  $I(X_1 \cap X_2) = \sqrt{I(X_1) + I(X_2)}$ .

*Proof.* (1) This follows from definitions.

(2) Write  $I(X_1 \cap X_2) = I(V(I(X_1)) \cap V(I(X_2))) = I(V(I(X_1) + I(X_2))) = \sqrt{I(X_1) + I(X_2)}$ .  $\square$

**Example 2.0.1.** The radical in (2) is necessary. Consider  $X_1 = V(y)$  and  $X_2 = V(y - x^2)$  in  $\mathbb{A}_k^2$ . Then  $X_1 \cap X_2 = \{(0, 0)\} \subseteq \mathbb{A}_k^2$ , so  $I(X_1 \cap X_2) = (x, y)$ . However,  $I(X_1) + I(X_2) = (y) + (y - x^2) = (y, x^2)$ .

Note that it is sometimes better to consider  $(y, x^2)$  anyway as it tracks multiplicities. In particular, we can see the multiplicity in the dimension of  $k[x, y]/(x, y^2) \cong \bar{1}k \oplus \bar{y}k$  as a  $k$ -vector space.

## 2.2 Proof of Hilbert's Nullstellensatz

We will assume the following result from commutative algebra without proof:

**Theorem 2.1** (Noether normalization). *Let  $A$  be a finitely generated algebra over a field  $k$  with  $A$  a domain. Then there is an injective  $k$ -algebra homomorphism  $k[z_1, \dots, z_n] \hookrightarrow A$  that is finite, i.e.  $A$  is a finitely generated  $k[z_1, \dots, z_n]$ -module.*

**Corollary 2.1.1.** *If  $K \subseteq L$  is a field extension and  $L$  is a finitely generated  $K$ -algebra, then  $K \subseteq L$  is a finite field extension. In particular, if in addition  $K = \overline{K}$ , then  $K = L$ .*

*Proof.* By Noether normalization, there exists a  $k$ -algebra homomorphism  $K[z_1, \dots, z_n] \rightarrow L$  that is finite. Then by a result from commutative algebra,  $L$  is integral over  $K[z_1, \dots, z_n]$ , which implies that  $K[z_1, \dots, z_n]$  must also be a field since  $L$  is. Thus  $n = 0$ , so  $K \subseteq L$  is a finite extension.  $\square$

**Proposition 2.1.** *If  $(1) \neq J \leq R$  is an ideal, then  $J$  is contained in some maximal ideal.*

*Proof.* Consider the set  $P = \{I \leq R : J \subseteq I, I \neq (1)\}$  with the partial order given by inclusion. Note that  $P \neq \emptyset$  since  $J \in P$ . Furthermore, every chain in  $P$  has an upper bound (for  $\{I_\alpha : \alpha \in A\}$  a chain  $P$ , we can take  $\bigcup_{\alpha \in A} I_\alpha$ , which one can check is indeed an ideal that lies in  $P$ ; note that  $1 \notin I_\alpha$  implies  $1 \notin \bigcup_{\alpha \in A} I_\alpha$ ). So Zorn's lemma implies there is a maximal element in  $P$ , which is a maximal ideal.  $\square$

*Proof of Theorem 1.1.* We will proceed in the following steps:

1. Show that the maximal ideals of  $k[x_1, \dots, x_n]$  are of the form  $(x_1 - a_1, \dots, x_n - a_n)$  for  $a_i \in k$ .
2. Prove the weak nullstellensatz: If  $(1) \neq J \leq k[x_1, \dots, x_n]$ , is an ideal, then  $V(J) \neq \emptyset$ .
3. Prove the (strong) nullstellensatz:  $I(V(J)) = \sqrt{J}$  for  $J \leq k[x_1, \dots, x_n]$ .

The most difficult part is the first step and is where we need  $k$  to be algebraically closed.<sup>1</sup>

(1) For  $a_1, \dots, a_n \in k$ , the ideal  $(x_1 - a_1, \dots, x_n - a_n)$  is maximal (the quotient is  $k$ , which is a field). Conversely, fix a maximal ideal  $\mathfrak{m} \in k[x_1, \dots, x_n]$ . Since

$$k \xrightarrow{\phi} k[x_1, \dots, x_n]/\mathfrak{m} = L$$

is a finitely generated  $k$ -algebra and  $k$  is algebraically closed,  $\phi$  is an isomorphism by Corollary 2.1.1. Choose  $a_i \in k$  such that  $\phi(a_i) = \overline{x_i}$ , so  $\overline{x_i - a_i} = 0$  in  $L$ . Then  $(x_1 - a_1, \dots, x_n - a_n) \subseteq \mathfrak{m}$ , so they must be equal since both the left and right hand sides are maximal ideals.

(2) By Proposition 2.1,  $J$  is contained in some maximal ideal  $\mathfrak{m}$ . By (1),  $\mathfrak{m} = (x_1 - a_1, \dots, x_n - a_n)$  for some  $a_1, \dots, a_n \in k$ . Since  $J \subseteq \mathfrak{m}$ , we have  $V(J) \supseteq V(\mathfrak{m}) \supseteq \{(a_1, \dots, a_n)\}$ , so  $J \neq \emptyset$ .

(3) The reverse inclusion follows from definitions. For the forward inclusion, fix  $f \in I(V(J))$ , and we want to show that  $f^n \in J$  for some  $n > 0$ . Add a new variable  $y$  and consider

$$J_1 = (J, fy - 1) \leq k[x_1, \dots, x_n, y].$$

Now  $V(J_1) = \{(a, b) = (a_1, \dots, a_n, b) \in \mathbb{A}_k^{n+1} : a \in V(J), f(a)b = 1\} = \emptyset$  since  $f$  vanishes on  $V(J)$ , so  $f(a)b = 0$  for any  $b$ . Thus by the weak nullstellensatz,  $J_1 = (1)$ , so  $1 = \sum_{i=1}^r g_i f_i + g_0(fy - 1)$  with

<sup>1</sup>The statement is false when  $k$  is not algebraically closed:  $(x^2 + 1)$  is maximal in  $\mathbb{R}[x]$ .



$f_1, \dots, f_r \in J$  and  $g_0, \dots, g_r \in k[x_1, \dots, x_n, y]$ . Let  $N$  be the maximal power of  $y$  in the  $g_i$ . Multiplying by  $f^N$ , we get

$$f^N = \sum_{i=1}^r G_i(x_1, \dots, x_n, fy) f_i + G_0(x_1, \dots, x_n, fy)(fy - 1)$$

with  $G_i \in k[x_1, \dots, x_n, fy]$ . So if we set  $fy = 1$ , then we have

$$f^N = \sum_{i=1}^r G_i(x_1, \dots, x_n, 1) f_i + 0 \in J,$$

which gives  $f \in \sqrt{J}$ . To justify this substitution, we can consider the quotient  $k[x_1, \dots, x_n, y]/(fy - 1)$ . We have a map  $k[x_1, \dots, x_n] \rightarrow k[x_1, \dots, x_n, y]/(fy - 1)$ , which is injective since  $(fy - 1)$  does not lie in  $k[x_1, \dots, x_n]$ , so an equality in the quotient implies an equality in  $k[x_1, \dots, x_n]$ .  $\square$

# Lecture 3

## Aug. 26 — The Zariski Topology

### 3.1 Polynomial Functions and Subvarieties

**Remark.** Recall that a polynomial  $f \in k[x_1, \dots, x_n]$  gives a function  $\mathbb{A}_k^n \rightarrow k$  by  $a \mapsto f(a)$ .

**Proposition 3.1.** *If  $f, g \in k[x_1, \dots, x_n]$  give the same function  $\mathbb{A}_k^n \rightarrow k$ , then  $f = g$  in  $k[x_1, \dots, x_n]$ .*

*Proof.* Assume  $f = g$  as polynomial functions. Then  $V(f - g) = \mathbb{A}_k^n$ , so  $\sqrt{(f - g)} = I(\mathbb{A}_k^n) = (0)$  by Hilbert's nullstellensatz (note that we can also prove  $I(\mathbb{A}_k^n) = (0)$  directly, it is enough to have  $k$  be an infinite field for this part). Thus  $f - g = 0$ , so  $f = g$  in  $k[x_1, \dots, x_n]$ .  $\square$

**Remark.** In the above proposition, we need  $k$  to be an infinite field (e.g. if  $k = \bar{k}$ ): Otherwise, there are only finitely many functions  $\mathbb{A}_k^n \rightarrow k$ , but infinitely many polynomials in  $k[x_1, \dots, x_n]$ .

**Remark.** The set of polynomial functions  $\mathbb{A}_k^n \rightarrow k$  form a ring, and the above proposition implies that this ring is isomorphic to  $k[x_1, \dots, x_n]$ .

**Definition 3.1.** A *polynomial function* on an affine variety  $X \subseteq \mathbb{A}_k^n$  is a function  $\varphi : X \rightarrow k$  such that there exists  $f \in k[x_1, \dots, x_n]$  with  $\varphi(a) = f(a)$  for every  $a \in X$ .

**Definition 3.2.** The *coordinate ring* of  $X$  is  $A(X) = \{f : X \rightarrow k \mid f \text{ is a polynomial function}\}$ , which is a ring under pointwise addition and multiplication.

**Remark.** Observe that there exists a surjective ring homomorphism

$$\begin{aligned} k[x_1, \dots, x_n] &\longrightarrow A(X) \\ f &\longmapsto (a \mapsto f(a)) \end{aligned}$$

with kernel  $I(X)$ . Thus we have  $A(X) \cong k[x_1, \dots, x_n]/I(X)$ .

**Remark.** We can now replace  $\mathbb{A}_k^n$  and  $k[x_1, \dots, x_n]$  by  $X$  and  $A(X)$  to study *subvarieties* of  $X$ .

**Definition 3.3.** Let  $X \subseteq \mathbb{A}_k^n$  be an affine variety. If  $S \subseteq A(X)$  is a subset, then define

$$V_X(S) = \{a \in X : f(a) = 0 \text{ for all } f \in S\}.$$

A subset of  $X$  of this form is called an *affine subvariety* of  $X$ . (Equivalently, these are the same as an affine variety  $Y \subseteq \mathbb{A}_k^n$  such that  $Y \subseteq X$ .) For  $Y \subseteq X$  a subvariety, define

$$I_X(Y) = \{f \in A(X) : f(a) = 0 \text{ for all } a \in Y\}.$$

**Proposition 3.2.** *There is a bijective correspondence between radical ideals in  $A(X)$  and affine subvarieties of  $X$  given by  $J \mapsto V_X(J)$  and  $Y \mapsto I_X(Y)$ .*

*Proof.* See Homework 2. □

## 3.2 The Zariski Topology

**Definition 3.4.** The *Zariski topology* on  $\mathbb{A}_k^n$  is the topology with closed sets  $V(I) \subseteq \mathbb{A}_k^n$ , where  $I$  is an ideal in  $k[x_1, \dots, x_n]$ . (Equivalently, the closed sets are the affine varieties in  $\mathbb{A}_k^n$ .)

**Remark.** Note the following:

1. On  $\mathbb{A}_k^1$ , the closed sets are of the form:  $\emptyset$ ,  $\mathbb{A}_k^1$ , or finite collections of points.
2. When  $k = \mathbb{C}$ , then  $X \subseteq \mathbb{A}_{\mathbb{C}}^n$  being Zariski closed implies that  $X$  is closed in the analytic topology on  $\mathbb{A}_{\mathbb{C}}^n$ . In particular, the Zariski topology is coarser than the analytic topology.
3. On  $\mathbb{A}_k^2$ , the closed sets are of the form:  $\emptyset$ ,  $\mathbb{A}_k^2$ , finite collections of points, plane curves, and their finite unions.

**Proposition 3.3.** *The Zariski topology on  $\mathbb{A}_k^n$  is indeed a topology.*

*Proof.* First note that  $\emptyset = V((1))$  and  $\mathbb{A}_k^n = V((0))$  are closed. For arbitrary intersections, note that  $\bigcap_{\alpha} V(I_{\alpha}) = V(\sum_{\alpha} I_{\alpha})$ , and for finite unions, note that  $\bigcup_{i=1}^r V(I_i) = V(I_1 \cdots I_r)$ . □

**Example 3.4.1.** The Zariski topology on  $\mathbb{A}_k^{n+m}$  is in general *not* the product topology of the Zariski topologies on  $\mathbb{A}_k^n$  and  $\mathbb{A}_k^m$ . Consider  $V(y - x^2) \subseteq \mathbb{A}_k^2$ , which is a closed set in the Zariski topology, but the only closed sets in  $\mathbb{A}_k^1$  are either  $\emptyset$ ,  $\mathbb{A}_k^1$ , or finite.

**Definition 3.5.** If  $X \subseteq \mathbb{A}_k^n$  is an affine variety, then we can define the *Zariski topology* on  $X$  in the following two equivalent ways:

1. take the subspace topology from the Zariski topology on  $\mathbb{A}_k^n$ ;
2. take the closed sets of  $X$  to be of the form  $V_X(I)$  for some ideal  $I \leq A(X)$ .

This is because an affine subvariety of  $X$  is precisely the intersection of  $X$  with an affine variety in  $\mathbb{A}_k^n$ .

**Remark.** Our goal now is to relate properties of the Zariski topology on  $X$  to the ring  $A(X)$ , and then to the ideal  $I(X) \leq k[x_1, \dots, x_n]$ .

**Definition 3.6.** A topological space  $X$  is *reducible* if we can write  $X = X_1 \cup X_2$  for some closed sets  $X_1, X_2 \subsetneq X$ . Otherwise,  $X$  is called *irreducible*.

**Example 3.6.1.** The plane curve  $X = V(y^2 - x^2y) = V(y) \cup V(y - x^2)$  is reducible.

**Remark.** Note the following:

1. A disconnected topological space is reducible.
2. Many topologies are reducible, e.g.  $\mathbb{C}^n$ ,  $\mathbb{R}^n$  with the analytic topology.
3. If  $X$  is irreducible and  $U \subseteq X$  is a nonempty open set, then  $\overline{U} = X$  (we have  $\overline{U} \cup (X \setminus U) = X$ ).

# Lecture 4

## Aug. 28 — Irreducibility

### 4.1 Properties of Irreducibility

**Proposition 4.1.** *Let  $X \subseteq \mathbb{A}^n$  be an affine variety. Then the following are equivalent:*

1.  $X$  is irreducible;
2.  $I(X) \leq k[x_1, \dots, x_n]$  is a prime ideal;
3. the coordinate ring  $A(X)$  is an integral domain.

**Example 4.0.1.** We have the following:

1.  $\mathbb{A}_k^n$  is irreducible as  $A(\mathbb{A}_k^n) = k[x_1, \dots, x_n]$ , which is an integral domain.
2. A *hypersurface*  $X \subseteq \mathbb{A}_k^n$  is an affine variety with  $I(X) = (f)$  for some  $f \in k[x_1, \dots, x_n]$ . Then  $A$  is irreducible if and only if  $(f)$  is prime, if and only if  $f$  is irreducible.<sup>1</sup>

### 4.2 Dimension

**Definition 4.1.** Let  $X$  be a topological space.

- The *dimension* of  $X$ , denoted  $\dim X$ , is the supremum of the  $n$  such that there exists a chain of irreducible closed subspaces

$$X \supseteq X_0 \supsetneq X_1 \supsetneq \cdots \supsetneq X_n \neq \emptyset.$$

- For  $Y \subseteq X$  closed and irreducible, the *codimension* of  $Y$  in  $X$ , denoted  $\operatorname{codim}_X Y$ , is the supremum of the  $n$  as above such that  $X_n = Y$ .

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<sup>1</sup>Note that any prime ideal is radical.

# Lecture 5

## Sept. 2 — Dimension

### 5.1 More on Dimension

**Remark.** Recall the following correspondence from before: If  $X \subseteq \mathbb{A}_k^n$  is an affine variety, then there exists a bijection between the irreducible closed subsets  $Y \subseteq X$  and the prime ideals  $\mathfrak{p} \leq A(X)$ .

**Definition 5.1.** For a ring  $A$ , the *(Krull) dimension* of  $A$ , denoted  $\dim A$ , is the supremum of the  $n$  such that there exists a chain of prime ideals

$$A \supseteq \mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n.$$

For a prime ideal  $\mathfrak{q} \leq A$ , the *height* of  $\mathfrak{q}$ , denoted  $\text{ht } \mathfrak{q}$ , is the supremum of the  $n$  as above with  $\mathfrak{p}_0 = \mathfrak{q}$ .

**Remark.** If  $X$  is an affine variety, then we have the following:

1.  $\dim X = \dim A(X)$ ;
2. for  $Y \subseteq X$  a closed irreducible subset,  $\text{codim}_X Y = \text{ht } I_X(Y)$ .

These properties follow from the inclusion-reversing correspondence.

**Definition 5.2.** Let  $K \subseteq L$  be a field extension.

1. A collection of elements  $\{z_i : i \in I\} \subseteq L$  is a *transcendence basis* of  $K \subseteq L$  if the  $z_i$  are algebraically independent (i.e.  $K(x_i : i \in I) \xrightarrow{\cong} K(z_i : i \in I)$  by  $x_i \mapsto z_i$ ) and  $K(z_i : i \in I) \subseteq L$  is algebraic.
2. The *transcendence degree*  $\text{tr.deg}_K L$  is the cardinality of a transcendence basis.

**Theorem 5.1** (Dimension theory). *Let  $A$  be a finitely generated  $k$ -algebra that is a domain. Then*

1.  $\dim A = \text{tr.deg}_k \text{Frac}(A)$ ;
2. for any prime ideal  $\mathfrak{p} \leq A$ , we have  $\text{ht } \mathfrak{p} + \dim A/\mathfrak{p} = \dim A$ ;
3. all maximal chains of prime ideals  $A \supseteq \mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n$  are of the same length.

**Remark.** The following are consequences of the above result from commutative algebra:

1.  $\dim_k \mathbb{A}_k^n = \dim k[x_1, \dots, x_n] = \text{tr.deg}_k k(x_1, \dots, x_n) = n$ .
2. If  $X$  is irreducible, then  $A(X)$  is a domain, so for  $x \in X$ , we have

$$\text{codim}_X \{x\} = \text{ht } I(\{x\}) = \dim A(X) - \dim A(X)/I(\{x\}) = \dim A(X) = \dim X,$$

where we note that  $A(X)/I(\{x\}) \cong k$  is a field.

3. If  $X$  is an irreducible affine variety and  $U \subseteq X$  is a nonempty open subset, then

$$\dim U = \sup_{x \in U} \operatorname{codim}_U \{x\} = \sup_{x \in U} \operatorname{codim}_X \{x\} = \dim X.$$

This follows since we can pass from a chain in  $U$  to a chain in  $X$  by taking closures.

4. If  $X$  is an irreducible affine variety and  $Z \subseteq X$  is an irreducible closed subset, then

$$\dim Z = \dim X - \operatorname{codim}_X Z.$$

Note that (2)-(4) can be false if  $X$  is not irreducible. To contradict (4), let  $X = V(x, y) \cup V(z) \subseteq \mathbb{A}_k^3$  with  $Z = V(x, y)$ . Then we have  $\dim X = 2$ ,  $\dim Z = 1$ ,  $\operatorname{codim}_X Z = 0$ .

## 5.2 Hypersurfaces

**Remark.** We now want to study hypersurfaces.

**Theorem 5.2** (Krull's Hauptidealsatz). *If  $A$  is a Noetherian ring and  $f \in A$  is nonzero and a non-unit, then every minimal prime ideal containing  $f$  has height 1.*

**Corollary 5.2.1.** *If  $X \subseteq \mathbb{A}_k^n$  is an irreducible affine variety and  $f \in A(X)$  is a nonzero non-unit, then*

$$\dim Z = \dim X - 1$$

*for every irreducible component  $Z$  of  $V_X(f)$ .*

*Proof.* Since  $X$  is irreducible,  $A(X)$  is a domain. So there is a correspondence between the minimal prime ideals  $\mathfrak{p} \subsetneq A(X)$  and the minimal irreducible closed subsets  $Z \supseteq V_X(f)$ , which corresponds to the irreducible components  $Z$  of  $V_X(f)$ . For such a component  $Z$ , we know

$$\dim Z = \dim Z - \operatorname{codim}_X Z = \dim X - \operatorname{ht} I(Z) = \dim X - 1$$

by Krull's Hauptidealsatz, which is the desired result.  $\square$

**Example 5.2.1.** Corollary 5.2.1 implies that if  $f \in k[x_1, \dots, x_n]$  is non-constant, then

$$\dim V(f) = \dim \mathbb{A}_k^n - 1 = n - 1.$$

**Theorem 5.3.** *An irreducible affine variety  $Y \subseteq \mathbb{A}_k^n$  has  $\dim Y = n - 1$  if and only if  $Y = V(f)$  for some non-constant polynomial  $f \in k[x_1, \dots, x_n]$ .*

*Proof.* ( $\Leftarrow$ ) This was Corollary 5.2.1.

( $\Rightarrow$ ) We will use that  $A(\mathbb{A}_k^n) = k[x_1, \dots, x_n]$  is a UFD. Since  $Y$  is irreducible and  $\dim Y = n - 1$ ,

$$\operatorname{ht} I(Y) = \operatorname{codim}_{\mathbb{A}_k^n} Y = \dim \mathbb{A}_k^n - \dim Y = 1.$$

Since  $(0) \subsetneq I(Y) \subsetneq k[x_1, \dots, x_n]$ , there exists a non-constant  $f \in k[x_1, \dots, x_n]$  with  $f \in I(Y)$ . Write

$$f = f_1 \cdots f_r$$

with  $f_i$  irreducible by unique factorization, and note that the  $f_i$  are also prime since we are in a UFD. Since  $I(Y)$  is prime, some  $f_i$  is in  $I(Y)$ , so we have the inclusions

$$(0) \subsetneq (f_i) \subseteq I(Y).$$

Since  $\operatorname{ht} I(Y) = 1$ , we must have  $(f_i) = I(Y)$ , so  $Y = V(I(Y)) = V(f_i)$ .  $\square$

## 5.3 Regular Functions

**Definition 5.3.** Let  $X$  be an affine variety and  $U \subseteq X$  open. A function  $\varphi : U \rightarrow k$  is *regular* if for each  $a \in U$ , there exists an open neighborhood  $a \in U_a \subseteq U$  and  $f, g \in A(X)$  such that

$$\varphi(x) = \frac{g(x)}{f(x)}, \quad f(x) \neq 0, \quad \text{for all } x \in U_a.$$

Define  $\mathcal{O}_X(U) = \{\varphi : U \rightarrow k \mid \varphi \text{ is a regular function on } U\}$ .

**Exercise 5.1.** Check that  $\mathcal{O}_X(U)$  is a ring under pointwise addition and multiplication of outputs.

**Remark.** To patch open sets together, we will later need the notion of a *morphism*, and a morphism  $U \rightarrow Y \subseteq \mathbb{A}_k^m$  should be given by

$$x \longmapsto (\varphi_1(x), \dots, \varphi_m(x))$$

with  $\varphi_i$  regular functions on  $U$ .

**Example 5.3.1.** We have the following:

1. If  $X \subseteq \mathbb{A}_k^n$  is an affine variety, then any  $\varphi \in A(X)$  is regular. Furthermore, we get an injective ring homomorphism  $A(X) \rightarrow \mathcal{O}_X(X)$ . We will see that this is an isomorphism.
2. If  $X = \mathbb{A}_k^1$  and  $U = \mathbb{A}_k^1 \setminus \{0\}$ , then for any  $n \geq 0$  and  $g \in k[x]$ , the function  $g/x^n$  is regular on  $U$ . In general, if we fix  $f, g \in A(X)$  and set  $U = X \setminus V(f)$ , then the map  $g/f^m$  is regular on  $U$ .
3. Let  $X = V(x_1x_4 - x_2x_3) \subseteq \mathbb{A}_k^4$  and  $U = X \setminus V(x_2, x_4)$ . Then the following map is regular:

$$\begin{aligned} \varphi : U &\longrightarrow k \\ (x_1, x_2, x_3, x_4) &\longmapsto \begin{cases} x_1/x_2, & \text{if } x_2 \neq 0, \\ x_3/x_4, & \text{if } x_4 \neq 0. \end{cases} \end{aligned}$$

Note that on  $U \setminus V(x_2x_4)$ , we have  $x_1/x_2 = x_3/x_4$  since  $x_1x_4 = x_2x_3$  on  $X$ .

# Lecture 6

## Sept. 4 — Regular Functions

### 6.1 Properties of Regular Functions

**Proposition 6.1.** *Let  $X$  be an affine variety and  $U \subseteq X$  open. Then:*

1. *if  $\varphi \in \mathcal{O}_X(U)$ , then  $V(\varphi) = \{x \in U : \varphi(x) = 0\}$  is closed in  $U$ ;*
2. *(identity principle) If  $X$  is irreducible,  $U \subseteq X$  is nonempty and open, and  $\varphi, \psi \in \mathcal{O}_X(U)$  with  $\varphi|_W = \psi|_W$  for some  $W \subseteq U$  nonempty and open, then  $\varphi = \psi$  in  $\mathcal{O}_X(U)$ .*

*Proof.* (1) It suffices to show that  $U \setminus V(\varphi)$  is open in  $U$ . Fix  $a \in U \setminus V(\varphi)$ . Since  $\varphi$  is regular, there exists an open neighborhood  $a \in U_a \subseteq U$  and  $f_a, g_a \in A(X)$  such that

$$\varphi|_{U_a} = \frac{g_a}{f_a}.$$

So  $a \in \{g_a \neq 0\} \cap U_a \subseteq U \setminus V(\varphi)$ . This is an open set containing  $a$  in  $U \setminus V(\varphi)$ , so  $U \setminus V(\varphi)$  is open.

(2) Since  $X$  is irreducible,  $U$  is also irreducible. The locus  $\{x \in U : \varphi(x) = \psi(x)\} = V(\varphi - \psi)$  is closed in  $U$  by (1). It also contains  $W$ . Since  $W$  is dense (it is a nonempty open set in an irreducible topological space), we must have  $V(\varphi - \psi) = U$ . This proves the claim.  $\square$

**Example 6.0.1.** In (2) of Proposition 6.1, the assumption that  $X$  is irreducible is necessary. Consider

$$U = X = V(xy) \subseteq \mathbb{A}_k^2 \quad \text{and} \quad W = V(xy) \setminus V(x).$$

Then the regular functions  $\varphi = x$  and  $\psi = x + y$  agree on  $W$  but are not equal on  $U$ .

### 6.2 Distinguished Open Sets

**Remark.** We will see that an affine variety has a basis of open sets on which we can compute  $\mathcal{O}_X(U)$ .

**Definition 6.1.** A *distinguished open set* of an affine variety  $X$  is a subset of the form

$$D(f) = X \setminus V(f)$$

for some polynomial function  $f \in A(X)$ .

**Remark.** We have the following:

1. The  $D(f)$  are closed under (finite) intersection:  $D(fg) = D(f) \cap D(g)$ .



2. The  $D(f)$  form a basis for the Zariski topology on  $X$ : If  $U \subseteq X$  is open, then  $U = X \setminus V(f_1, \dots, f_r)$  for some  $f_1, \dots, f_r \in A(X)$  (since  $X$  is Noetherian). So  $U = D(f_1) \cup \dots \cup D(f_r)$ .

**Remark.** We will view  $D(f)$  as “small open sets” (under mild assumptions,  $\text{codim}_X(X \setminus D(f)) = 1$ ).

**Theorem 6.1.** *If  $X$  is an affine variety and  $f \in A(X)$ , then*

$$\mathcal{O}_X(D(f)) = \left\{ \frac{g}{f^m} : g \in A(X), m \geq 0 \right\}.$$

*Proof.* We have an injective ring homomorphism

$$\left\{ \frac{g}{f^m} : g \in A(X), m \geq 0 \right\} \longrightarrow \mathcal{O}_X(D(f)),$$

it suffices to show this map is surjective. Fix  $\varphi \in \mathcal{O}_X(D(f))$ . For any  $a \in D(f)$ , there exists an open neighborhood  $a \in U_a \subseteq D(f)$  and  $f_a, g_a \in A(X)$  such that  $\varphi|_{U_a} = g_a/f_a$ . We may further assume that

1.  $U_a = D(h_a)$  for some  $h_a \in A(X)$  (by shrinking  $U_a$  if necessary, since the  $D(h)$  form a basis);
2.  $h_a = f_a$  (by rewriting  $g_a/f_a = g_a h_a / f_a h_a$  and replacing  $h_a, f_a$  with  $f_a h_a$ ).

Then for  $a, b \in D(f)$ , we have  $f_a g_b = f_b g_a$  on  $D(f_a) \cap D(f_b)$ . Since both the left and right hand sides vanish on  $X \setminus (D(f_a) \cap D(f_b))$ , we have  $f_a g_b = f_b g_a$  in  $A(X)$ . Now we can write

$$V(f) = \bigcap_{a \in D(f)} V(f_a) = V(f_a : a \in D(f)),$$

so  $f \in I(V(f_a : a \in D(f)))$ . By the Nullstellensatz, there exists  $n \geq 0$  such that

$$f^n = \sum_{a \in D(f)} k_a f_a, \quad k_a \in A(X),$$

where only finitely many of the  $k_a$  are nonzero. Set  $g = \sum_{a \in D(f)} k_a g_a$ , and we claim that  $\varphi = g/f^n$ . To see this, note that on  $U_b$ , we have  $\varphi|_{U_b} = g_b/f_b$ . Now since  $f_a g_b = f_b g_a$ , we have

$$g f_b = \sum_{a \in D(f)} k_a g_a f_b = \sum_{a \in D(f)} k_a f_a g_b = f^n g_b,$$

which shows that  $\varphi|_{U_b} = (g/f^n)|_{U_b}$ . Since this holds for any  $U_b$ , we have  $\varphi = g/f^n$  in  $\mathcal{O}_X(D(f))$ .  $\square$

**Remark.** Theorem 6.1 has the following consequences:

1. The  $f = 1$  case implies that the natural ring homomorphism  $A(X) \rightarrow \mathcal{O}_X(X)$  is surjective and hence an isomorphism (note that  $D(1) = X$ ).
2. We will see that  $\mathcal{O}_X(D(f)) \cong A(X)_f$ , the *localization* of  $A(X)$  at  $f$ .

**Example 6.1.1.** How do we compute  $\mathcal{O}_X(U)$  on non-distinguished open sets? Consider

$$X = \mathbb{A}_k^2 \quad \text{and} \quad U = \mathbb{A}_k^2 \setminus \{(0, 0)\}.$$

Note that  $U$  is never a distinguished open set. We claim that the ring homomorphism

$$k[x, y] \longrightarrow \mathcal{O}_{\mathbb{A}_k^2}(\mathbb{A}_k^2 \setminus \{(0, 0)\})$$

is an isomorphism. The map is injective by the identity principle, so it suffices to show surjectivity. The strategy is use  $U = D(x) \cup D(y)$  (in general, cover  $U$  by basis elements). Fix  $\varphi : U \rightarrow k$  regular, so

$$\begin{aligned}\varphi|_{D(x)} &= \frac{f}{x^m} \quad \text{for some } f \in k[x, y], m \geq 0 \\ \varphi|_{D(y)} &= \frac{g}{y^n} \quad \text{for some } g \in k[x, y], n \geq 0.\end{aligned}$$

Since we are in a UFD, we may assume that  $x \nmid f$  and  $y \nmid g$ . Now  $fy^n = gx^m$  on  $D(y) \cap D(x)$ , so by the identity principle,  $fy^n = gx^m$  on  $\mathbb{A}_k^2$ , so  $fy^n = gx^m$  in  $k[x, y]$ . Using that  $y \nmid g$ ,  $x \nmid f$ , and that  $k[x, y]$  is a UFD, we must have  $n = m = 0$ , hence  $f = g$ . In particular, we have

$$\varphi|_{D(x)} = \varphi|_{D(y)} = f,$$

so the map  $k[x, y] \rightarrow \mathcal{O}_X(U)$  is surjective.

## 6.3 Localization

**Remark.** We want to invert a subset of a ring, in particular *multiplicative systems*.

**Definition 6.2.** A *multiplicative system* of a ring  $A$  is a subset such that

1.  $1 \in S$ ;
2.  $S$  is closed under multiplication.

**Example 6.2.1.** The following examples of  $S$  are multiplicative systems:

1.  $S = A$  or  $S = \{1\}$ ;
2. if  $\mathfrak{p} \leq A$  is a prime ideal, then  $S = A \setminus \mathfrak{p}$ ;
3. if  $f \in A$ , then  $S = \{f^m : m \geq 0\}$ .

**Definition 6.3.** The *localization* of a ring  $A$  at a multiplicative system  $S$  is the ring

$$S^{-1}A = \left\{ \frac{a}{s} : a \in A, s \in S \right\} / \sim$$

where the  $a/s$  are formal symbols with  $a/s \sim a'/s'$  if  $t(as' - a's) = 0$  for some  $t \in S$ .<sup>1</sup> The operations are given by the usual addition and multiplication of fractions:

$$\frac{a}{s} \cdot \frac{a'}{s'} = \frac{aa'}{ss'} \quad \text{and} \quad \frac{a}{s} + \frac{a'}{s'} = \frac{as' + a's}{ss'}.$$

Check as an exercise that these operations respect the equivalence relation.

**Example 6.3.1.** The following are examples of localization:

1. If  $A$  is a domain and  $S = A \setminus \{0\}$ , then  $S^{-1}A = \text{Frac } A$ .
2. If  $S = \langle f \rangle = \{1, f, f^2, \dots\}$ , then we will write  $A_f = S^{-1}A$ .
3. If  $S = A \setminus \mathfrak{p}$  for a prime ideal  $\mathfrak{p}$ , then we will write  $A_{\mathfrak{p}} = S^{-1}A$ .

---

<sup>1</sup>Note that if  $A$  is a domain and  $0 \notin S$ , then this condition is equivalent to  $as' = a's$ .

**Proposition 6.2.** *We have the following properties of localization:*

1. (Universal property of localization) *For any ring homomorphism  $\varphi : A \rightarrow B$  such that  $\varphi(s)$  for all  $s \in S$ , then there exists a unique ring homomorphism which makes the following diagram commute:*

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & B \\ \pi: a \mapsto a/1 \searrow & & \nearrow \exists! \\ & S^{-1}A & \end{array}$$

2. *There is a bijection between the prime ideals  $\mathfrak{p} \leq A$  with  $\mathfrak{p} \cap S = \emptyset$  and the prime ideals  $\mathfrak{q} \leq S^{-1}A$  given by  $\mathfrak{p} \mapsto \pi(\mathfrak{p})S^{-1}A$  with inverse  $\mathfrak{q} \mapsto \pi^{-1}(\mathfrak{q})$ , where  $\pi : A \rightarrow S^{-1}A$  is the map  $a \mapsto a/1$ .*

**Remark.** In more generality, for an  $A$ -module  $M$ , we can define the localization  $S^{-1}M$ , which is an  $S^{-1}A$ -module. This gives a functor  $\text{Mod}_A \rightarrow \text{Mod}_{S^{-1}A}$  which is exact.

# Lecture 7

## Sept. 9 — Germs and Sheaves

### 7.1 More on Localization

**Proposition 7.1.** *If  $X$  is an affine variety and  $f \in A(X)$  is nonzero, then  $\mathcal{O}_X(D(f)) \cong A(X)_f$ .*

*Proof.* We define a ring homomorphism as follows:

$$\begin{aligned} A(X)_f &\longrightarrow \mathcal{O}_X(D(f)) \\ \frac{g}{f^m} &\longmapsto \left( x \mapsto \frac{g(x)}{f^m(x)} \right). \end{aligned}$$

To check that this is well-defined, assume  $g/f^m \sim h/f^n$  in  $A(X)_f$ . So there exists  $k \geq 0$  such that

$$f^k(gf^n - hf^m) = 0 \quad \text{in } A(X).$$

So  $gf^n - hf^m = 0$  as functions  $D(f) \rightarrow k$ , so  $g/f^m = h/f^n$  as functions  $D(f) \rightarrow k$ . Thus their images agree in  $\mathcal{O}_X(D(f))$ , so the map is well-defined.

Surjectivity follows from the argument from last time. For injectivity, assume  $g/f^m = 0$  as functions  $D(f) \rightarrow k$  with  $g \in A(X)$ . Then  $fg = 0$  in  $A(X)$ , so  $g/f^m \sim 0/1$  in  $A(X)_f$ .  $\square$

### 7.2 Germs of Functions

**Definition 7.1.** Let  $p \in X$  be a point on an affine variety.

1. A *germ* of a regular function of  $X$  at  $p$  is a pair  $(U, f)$  such that  $x \in U \subseteq X$  is open and  $f$  is a regular function  $U \rightarrow k$ , up to the equivalence relation  $(U, \varphi) \sim (V, \psi)$  if there exists an open set  $x \in W \subseteq U \cap V$  such that  $\varphi|_W = \psi|_W$ .
2. Define  $\mathcal{O}_{X,p} = \{\text{germs of regular functions of } X \text{ at } p\}$ .

**Exercise 7.1.** Check that  $\mathcal{O}_{X,p}$  is a ring with operations

$$\begin{aligned} (U, \varphi) \cdot (V, \psi) &= (U \cap V, \varphi|_{U \cap V} \cdot \psi|_{U \cap V}), \\ (U, \varphi) + (V, \psi) &= (U \cap V, \varphi|_{U \cap V} + \psi|_{U \cap V}), \end{aligned}$$

with the zero function as the zero element and the constant 1 function as the unit element.

**Lemma 7.1.**  $\mathcal{O}_{X,p}$  is a local ring with unique maximal ideal  $\mathfrak{m}_p = \{(U, \varphi) \in \mathcal{O}_{X,p} : \varphi(p) = 0\}$ .

*Proof.* It suffices to show that the units of  $\mathcal{O}_{X,p}$  are precisely  $\mathcal{O}_{X,p} \setminus \mathfrak{m}_p$ . To see the reverse inclusion, fix  $(U, \varphi) \in \mathcal{O}_{X,p}$  with  $\varphi(p) \neq 0$ . So there exists an open neighborhood  $p \in W \subseteq U$  such that  $\varphi|_W$  never vanishes. Then

$$(U, \varphi) \cdot (W, 1/\varphi|_W) = (W, \varphi|_W) \cdot (W, 1/\varphi|_W) = (W, 1),$$

so  $(U, \varphi)$  is a unit in  $\mathcal{O}_{X,p}$ . The forward inclusion is similar.  $\square$

**Proposition 7.2.** *With the above setup, there is an isomorphism*

$$\begin{aligned} A(X)_{I(p)} &\longrightarrow \mathcal{O}_{X,p} \\ \frac{f}{g} &\longmapsto \left( D(g), x \mapsto \frac{f(x)}{g(x)} \right) \end{aligned}$$

with  $I(p) = \{f \in A(X) : f(p) = 0\}$ .

*Proof.* To see that this is well-defined, let  $f/g \sim f'/g' \in A(X)_{I(p)}$ . Then  $h(fg' - f'g) = 0$  for some  $h \in A(X)$  with  $h(p) \neq 0$ . So  $f/g = f'/g'$  as functions  $D(h) \cap D(g) \rightarrow k$ , which means that  $f/g = f'/g'$  as elements in  $\mathcal{O}_{X,p}$ . Thus the map is well-defined.

Injectivity is similar to before. For surjectivity, choose  $(U, \varphi) \in \mathcal{O}_{X,p}$ . Since  $\varphi : U \rightarrow k$  is a regular function, there exists an open set  $p \in U_p \subseteq U$  and  $f, g \in A(X)$  such that  $g$  does not vanish on  $U_p$  and  $\varphi(x) = f(x)/g(x)$  for all  $x \in U_p$ . So  $(U, \varphi) \sim (D(g), f/g)$  in  $\mathcal{O}_{X,p}$ , i.e.  $(U, \varphi)$  is in the image.  $\square$

**Example 7.1.1.** If  $X = \mathbb{A}_k^n$  and  $p = 0$ , then

$$\mathcal{O}_{\mathbb{A}_k^n, 0} \cong k[x_1, \dots, x_n]_{(x_1, \dots, x_n)} = \left\{ \frac{f}{g} : f \in k[x_1, \dots, x_n], g \in k[x_1, \dots, x_n] \setminus (x_1, \dots, x_n) \right\}.$$

**Remark.** We will relate the local properties of  $X$  at  $p$  to properties of  $\mathcal{O}_{X,p}$ . We will use the following statements from commutative algebra: Let  $A$  be a ring and  $\mathfrak{p} \subseteq A$  a prime ideal. Then

1.  $A_{\mathfrak{p}}$  is a local ring with unique maximal ideal  $\mathfrak{p}A_{\mathfrak{p}}$ .
2. There is a bijection from the prime ideals of  $A_{\mathfrak{p}}$  to the prime ideals of  $A$  contained in  $\mathfrak{p}$ .
3.  $\text{ht}_A \mathfrak{p} = \dim A_{\mathfrak{p}}$  (this follows from (2)).

This has the following consequence: If  $X$  is an affine variety and  $p \in X$ , then

$$\text{codim}_X \{p\} = \text{ht}_{A(X)} I(p) = \dim A(X)_{I(p)} = \dim \mathcal{O}_{X,p}.$$

## 7.3 Sheaves

**Remark.** We will now formalize the structures  $\mathcal{O}_X(U)$  and  $\mathcal{O}_{X,p}$  that we have seen before.

**Definition 7.2.** A *presheaf (of rings)*  $\mathcal{F}$  on a topological space  $X$  is the data of

1. for every open set  $U \subseteq X$ , a ring  $\mathcal{F}(U)$ ;
2. for every inclusion of open sets  $U \subseteq V \subseteq X$ , a ring homomorphism  $\rho_{V,U} : \mathcal{F}(V) \rightarrow \mathcal{F}(U)$

satisfying the following properties:

1.  $\mathcal{F}(\emptyset) = 0$ ;
2.  $\rho_{U,U}$  is the identity map;
3. for inclusions of open sets  $U \subseteq V \subseteq W \subseteq X$ , we have  $\rho_{W,U} = \rho_{V,U} \circ \rho_{W,V}$ .

**Example 7.2.1.** If  $X$  is an affine variety, then  $\mathcal{O}_X$  gives a presheaf of rings with

1. for  $U \subseteq X$ , the ring is  $\mathcal{O}_X(U) = \{\text{regular functions } \varphi : U \rightarrow k\}$ ;
2. for  $U \subseteq V \subseteq X$ , the map  $\mathcal{O}_X(V) \rightarrow \mathcal{O}_X(U)$  is given by  $\varphi \mapsto \varphi|_U$ .

**Remark.** We often call  $s \in \mathcal{F}(U)$  a *section*, and for  $U \subseteq V$ , we call  $s|_U = \rho_{V,U}(s)$  the *restriction*.

**Remark.** A presheaf is the same thing as a functor  $\text{Open}_X^{\text{op}} \rightarrow \text{Rings}$ , where  $\text{Open}_X$  is the category with objects the nonempty open sets of  $X$  and morphisms corresponding to the inclusions  $U \subseteq V$ .

**Definition 7.3.** A presheaf  $\mathcal{F}$  on  $X$  is a *sheaf* if it satisfies the *gluing property*: For any  $U \subseteq X$  open, an open cover  $\{U_i\}_{i \in I}$  of  $U$ , and  $\varphi_i \in \mathcal{F}(U_i)$  with  $\varphi_i|_{U_i \cap U_j} = \varphi_j|_{U_i \cap U_j}$  for all  $i, j \in I$ , there exists a unique  $\varphi \in \mathcal{F}(U)$  such that  $\varphi|_{U_i} = \varphi_i$  for all  $i \in I$ .

**Example 7.3.1.** We have the following:

1. If  $X$  is an affine variety, then  $\mathcal{O}_X$  is a sheaf (if we take  $\varphi_i \in \mathcal{O}_X(U_i)$  that agree on the overlaps, then we get  $\varphi : U \rightarrow k$ , which is regular since regularity is a local property).
2. If  $M$  is a smooth manifold, then we can define a sheaf (on open subsets  $U \subseteq M$ ) by

$$U \mapsto \mathcal{F}^{\text{sm}}(U) = \{\text{smooth functions } U \rightarrow \mathbb{R}\}.$$

We may also consider  $\mathcal{F}^{\text{cont}}$ ,  $\mathcal{F}^{\text{diff}}$ ,  $\mathcal{F}^{\text{loc, const}}$ , etc. However,  $\mathcal{F}^{\text{const}}$  is a presheaf, but not a sheaf in general: We can take  $U = U_1 \cup U_2$  with  $U_1 \cap U_2 = \emptyset$ , and we will only get a locally constant function. Similarly,  $\mathcal{F}^{\text{bounded}}$  is only a presheaf but not a sheaf.

3. If  $\mathcal{F}$  is a sheaf on a topological space  $X$  and  $U \subseteq X$  is open, then we get a sheaf  $\mathcal{F}|_U$  on  $U$  defined by  $\mathcal{F}|_U(V) = \mathcal{F}(V)$  for  $V \subseteq U$  open.

**Definition 7.4.** The *stalk* of a sheaf  $\mathcal{F}$  on a topological space  $X$  at  $x \in X$  is

$$\mathcal{F}_x = \{(U, \varphi) : U \subseteq X \text{ open and } \varphi \in \mathcal{F}(U)\} / \sim,$$

where  $(U, \varphi) \sim (V, \psi)$  if there exists an open set  $x \in W \subseteq U \cap V$  such that  $\varphi|_W = \psi|_W$ .

**Example 7.4.1.** If  $X$  is an affine variety and  $p \in X$ , then  $\mathcal{O}_{X,p} \cong (\mathcal{O}_X)_p$ .

**Remark.** As before with  $\mathcal{O}_{X,p}$ , one can check that  $\mathcal{F}_x$  naturally has the structure of a ring.

**Remark.** An alternative perspective is to define the stalk as a direct limit:

$$\mathcal{F}_x = \varinjlim_{U \ni x} \mathcal{F}(U),$$

where the limit is taken over all open  $x \in U \subseteq X$  with respect to the ordering  $U \leq V$  if  $V \subseteq U$ .

# Lecture 8

## Sept. 11 — Morphisms

### 8.1 Morphisms of Open Sets

**Remark.** Recall that a continuous map  $f : \mathbb{R}^m \rightarrow \mathbb{R}^n$  is *smooth* if it satisfies either of the following equivalent conditions:

1. there exist smooth functions  $f_1, \dots, f_n : \mathbb{R}^m \rightarrow \mathbb{R}$  such that  $f(x) = (f_1(x), \dots, f_n(x))$ ;
2. for each open set  $U \subseteq \mathbb{R}^n$  and smooth  $\varphi : U \rightarrow \mathbb{R}$ , the function  $f^*\varphi := \varphi \circ f : \mathbb{R}^m \rightarrow \mathbb{R}$  is smooth.

The implication  $(1 \Rightarrow 2)$  follows by the chain rule. To see  $(2 \Rightarrow 1)$ , take  $y_i : \mathbb{R}^n \rightarrow \mathbb{R}$  defined by  $f_i := f^*y_i$ . We want a similar definition in algebraic geometry.

**Definition 8.1.** Let  $X$  and  $Y$  be open sets of affine varieties. A *morphism*  $f : X \rightarrow Y$  is a continuous map such that for every  $U \subseteq Y$  open and  $\varphi \in \mathcal{O}_Y(U)$ , the map

$$\begin{array}{ccccc} & & f^*\varphi & & \\ & \nearrow & & \searrow & \\ f^{-1}(U) & \xrightarrow{f} & U & \xrightarrow{\varphi} & k \end{array}$$

satisfies  $f^*\varphi \in \mathcal{O}_X(f^{-1}(U))$ . A morphism is an *isomorphism* if it has a two-sided inverse (equivalently,  $f$  is a bijection and  $f^{-1}$  is a morphism).

**Remark.** We have the following properties of morphisms:

1. (Composition) If  $f : X \rightarrow Y$  and  $g : Y \rightarrow Z$  are morphisms of open sets of affine varieties, then so is  $g \circ f : X \rightarrow Z$ .
2. (Local on target) If  $X \rightarrow Y$  is a map of open sets of affine varieties such that there exists an open cover  $\{U_i\}_{i \in I}$  of  $Y$  with  $f|_{f^{-1}(U_i)} : f^{-1}(U_i) \rightarrow U_i$  a morphism for all  $i \in I$ , then  $f$  is a morphism.

**Proposition 8.1.** Let  $X \subseteq \mathbb{A}^m$  and  $Y \subseteq \mathbb{A}^n$  be affine varieties. Let  $U \subseteq X$  and  $V \subseteq Y$  be open sets. A map  $f : U \rightarrow V$  is a morphism if and only if there exist  $\varphi_1, \dots, \varphi_n \in \mathcal{O}_X(U)$  such that

$$f(x) = (\varphi_1(x), \dots, \varphi_n(x)).$$

*Proof.* ( $\Rightarrow$ ) Let  $U \subseteq \mathbb{A}^m_{x_i}$  and  $V \subseteq \mathbb{A}^n_{y_i}$ . By the definition of a morphism,  $y_i : V \rightarrow k$  satisfies

$$\varphi_i := f^*y_i \in \mathcal{O}_X(U),$$

so we can write  $f(x) = (\varphi_1(x), \dots, \varphi_n(x))$ .

( $\Leftarrow$ ) Assume there exist  $\varphi_1, \dots, \varphi_n \in \mathcal{O}_X(U)$  such that  $f(x) = (\varphi_1(x), \dots, \varphi_n(x))$ .

We first show that  $f$  is continuous. Let  $Z \subseteq V$  be a closed set. So we can write  $Z = V(g_1, \dots, g_r)$  for some  $g_1, \dots, g_r \in A(\mathbb{A}^n) \cong k[y_1, \dots, y_n]$ . Now we have

$$\begin{aligned} f^{-1}(Z) &= \{x \in U : f(x) \in Z\} = \{x \in U : g_i(f(x)) = 0 \text{ for } i = 1, \dots, r\} \\ &= \{x \in U : (f^*g_i)(x) = 0 \text{ for } i = 1, \dots, r\}. \end{aligned}$$

Note that  $f^*g_i = g_i(\varphi_1, \dots, \varphi_n)$ , which is regular since a composition of a polynomial with fractions of polynomials is again a fraction of polynomials. So  $f^{-1}(Z)$  is closed in  $U$ .

Now to show that  $f$  is a morphism, it suffices to show that for any  $W \subseteq Y$  open and  $\varphi \in \mathcal{O}_Y(W)$ , we have  $f^*\varphi \in \mathcal{O}_X(f^{-1}(W))$ . The proof of this is similar to before.  $\square$

**Example 8.1.1.** We have the following:

1. Morphisms  $\mathbb{A}^m \rightarrow \mathbb{A}^n$  are of the form

$$x \mapsto (f_1(x), \dots, f_n(x))$$

with  $f_1, \dots, f_n \in \mathcal{O}_{\mathbb{A}^m}(\mathbb{A}^m) = k[x_1, \dots, x_m]$ .

2. Write  $\mathbb{A}_t^1$  to mean  $\mathbb{A}^1$  with variable  $t$ . Then we can define  $\mathbb{A}_t^1 \rightarrow V(y - x^2) \subseteq \mathbb{A}_{x,y}^2$  by  $t \mapsto (t, t^2)$ . We can get an inverse  $V(y - x^2) \rightarrow \mathbb{A}_t^1$  by  $(x, y) \mapsto x$ , so  $\mathbb{A}_t^1$  and  $V(y - x^2)$  are isomorphic.
3. Consider the map  $g : \mathbb{A}_t^1 \rightarrow V(x^2 - y^3) \subseteq \mathbb{A}_{x,y}^2$  given by  $t \mapsto (t^3, t^2)$ . This map is bijective, but it is not an isomorphism. To see this, we can show that  $(g^{-1})^*\varphi$  is not regular for some regular function  $\varphi$  on  $\mathbb{A}_1^1$ . For instance, we can take  $\varphi = t$ , so that

$$(g^{-1})^*(t) = (x, y) \mapsto \begin{cases} x/y & \text{if } y \neq 0, \\ 0 & \text{otherwise,} \end{cases}$$

which we can see is not regular.

## 8.2 Relation to Coordinate Rings

**Remark.** Let  $X \subseteq \mathbb{A}^m$  and  $Y \subseteq \mathbb{A}^n$  be affine varieties. Then a morphism  $f : X \rightarrow Y$  of affine varieties induces a  $k$ -algebra morphism (called the *pullback* of  $f$ )

$$\begin{aligned} f^* : A(Y) &\longrightarrow A(X) \\ \varphi &\longmapsto f^*\varphi = \varphi \circ f \end{aligned}$$

with the properties  $(g \circ f)^* = f^* \circ g^*$  and  $(\text{id}_X)^* = \text{id}_{A(X)}$ , i.e.  $X \mapsto A(X)$  is a contravariant functor.

**Proposition 8.2.** *The following map is a bijection:*

$$\begin{aligned} \text{Hom}_{\text{aff, var}}(X, Y) &\xrightarrow{\Phi} \text{Hom}_{k\text{-alg}}(A(Y), A(X)) \\ f &\longmapsto f^* \end{aligned}$$

*Proof.* Note that  $A(X) \cong k[x_1, \dots, x_m]/I(X)$  and  $A(Y) \cong k[y_1, \dots, y_n]/I(Y)$ . Given a morphism

$$\begin{aligned} f : X &\longrightarrow Y \\ x &\longmapsto (\varphi_1(x), \dots, \varphi_n(x)), \end{aligned}$$



we can define  $f^*\bar{y}_i = \varphi_i$ . Conversely, given a  $k$ -algebra homomorphism  $\phi : A(Y) \rightarrow A(X)$ , we can set  $\varphi_i = \phi(\bar{y}_i)$ . Now consider the morphism defined by

$$\begin{aligned} f : X &\longrightarrow \mathbb{A}_{y_i}^n \\ x &\longmapsto (\varphi_1(x), \dots, \varphi_n(x)). \end{aligned}$$

We claim that  $f(X) \subseteq Y$ . To see this, fix  $x \in X$ . If  $h \in I(Y)$ , then

$$h(f(x)) = h(\varphi_1(x), \dots, \varphi_n(x)) = \phi(h)(x) = 0(x) = 0,$$

so  $f(X) \subseteq Y$ . Thus we get a morphism  $f : X \rightarrow Y$  by  $x \mapsto (\varphi_1(x), \dots, \varphi_n(x))$  with  $f^*y_i = \varphi_i$ . One can check that this gives a map  $\Psi : \text{Hom}_{k\text{-alg}}(A(Y), A(X)) \rightarrow \text{Hom}_{\text{aff, var}}(X, Y)$  which is inverse to  $\Phi$ .  $\square$

**Example 8.1.2.** We have the following:

1. Recall the morphism  $g : \mathbb{A}_t^1 \rightarrow V(y - x^2) \subseteq \mathbb{A}_{x,y}^2$  given by  $t \mapsto (t, t^2)$ . The pullback is given by

$$\begin{aligned} g^* : \frac{k[x, y]}{(y - x^2)} &\longmapsto k[t] \\ x &\longmapsto t \\ y &\longmapsto t^2. \end{aligned}$$

Note that  $g^*$  is an isomorphism of  $k$ -algebras, so  $g$  is an isomorphism of affine varieties. This gives an alternative way of seeing this without writing down an inverse to  $g$ .

2. Recall the morphism  $h : \mathbb{A}_t^1 \rightarrow V(x^2 - y^3) \subseteq \mathbb{A}_{x,y}^2$  given by  $t \mapsto (t^3, t^2)$ . The pullback is

$$\begin{aligned} h^* : \frac{k[x, y]}{(x^2 - y^3)} &\longmapsto k[t] \\ x &\longmapsto t^3 \\ y &\longmapsto t^2. \end{aligned}$$

Note that  $t \notin \text{Im } h^*$ , so  $h^*$  is not an isomorphism, so  $h$  is not an isomorphism.

**Remark.** There is a one-to-one correspondence between affine varieties (up to isomorphism) and finitely generated reduced  $k$ -algebras (up to isomorphism).

To see this, observe that if  $X \subseteq \mathbb{A}^n$  is an affine variety, then  $A(X) \cong k[x_1, \dots, x_n]/I(X)$ . This is finitely generated, and reduced since  $I(X)$  is radical. Conversely, let  $A$  be a reduced finitely generated  $k$ -algebra. Then  $A \cong k[y_1, \dots, y_m]/I$  since  $A$  is finitely generated, and  $I$  is radical since  $A$  is reduced. Thus by Hilbert's nullstellensatz,  $Y = V(I)$  satisfies  $I(Y) = I(V(I)) = I$ , so  $A \cong A(Y)$ .

In more abstract language, this means that there is an equivalence of categories

$$\text{AffVar} \longleftrightarrow \text{RedFGAlg}_k^{\text{op}}.$$

# Lecture 9

## Sept. 16 — Morphisms, Part 2

### 9.1 An Example of Isomorphisms

**Example 9.0.1.** What of the following are isomorphic over  $\mathbb{C}$ ?

1.  $\mathbb{A}^1 \setminus \{1\}$ ;
2.  $V(x^2 + y^2) \subseteq \mathbb{A}^2$ ;
3.  $V(y - x^2, z - x^3) \subseteq \mathbb{A}^3$ ;
4.  $V(xy) \subseteq \mathbb{A}^2$ ;
5.  $V(y^2 - x^2 - x^3) \subseteq \mathbb{A}^2$ ;
6.  $V(x^2 - y^2 - 1) \subseteq \mathbb{A}^2$ .

Note that (2) and (4) are not irreducible. In fact, they are isomorphic since we can write (2) as

$$V(x^2 + y^2) = V((x + iy)(x - iy)) \cong V(xy).$$

We have seen (3) previously on homework, and we have an isomorphism  $\mathbb{A}^1 \rightarrow Y = V(y - x^2, z - x^3)$  by  $t \mapsto (t, t^2, t^3)$ . We can also see this by noting that  $A(Y) \cong \mathbb{C}[x] \cong A(\mathbb{A}^1)$ . For (1), note that

$$\mathbb{A}^1 \setminus \{1\} \cong \mathbb{A}^1 \setminus \{0\}$$

and  $A(\mathbb{A}^1 \setminus \{0\}) \cong \mathbb{C}[x^{\pm 1}]$ , whereas  $A(\mathbb{A}^1) \cong \mathbb{C}[x]$ . So  $\mathbb{A}^1 \setminus \{1\} \not\cong \mathbb{A}^1$ . For (6), note that

$$V(x^2 - y^2 - 1) = V((x - y)(x + y) - 1) \cong V(uv - 1) \cong \mathbb{A}^1 \setminus \{0\}$$

by the map  $V(uv - 1) \rightarrow \mathbb{A}^1 \setminus \{0\}$  given by  $(u, v) \mapsto u$ , with inverse  $t \mapsto (t, 1/t)$ . Finally, letting  $C$  be the curve in 6, one can show that there is a singularity at the origin with  $\dim_{\mathbb{C}}(\mathcal{O}_{C,0}/\mathfrak{m}_0) = 2$ , which is different than the other examples. So the isomorphism classes are  $\{2, 4\}$ ,  $\{1, 6\}$ ,  $\{3\}$ , and  $\{5\}$ .

### 9.2 Ringed Spaces and Morphisms

**Definition 9.1.** A *ringed space*  $(X, \mathcal{O}_X)$  is a topological space  $X$  with a sheaf of rings  $\mathcal{O}_X$  on  $X$ .

**Example 9.1.1.** If  $X$  is an affine variety and  $\mathcal{O}_X$  is the sheaf of regular functions, then  $(X, \mathcal{O}_X)$  is a ringed space. Similarly, if  $M$  is a complex manifold and  $\mathcal{O}_M$  is the sheaf of holomorphic functions on  $M$ , then  $(M, \mathcal{O}_M)$  is a ringed space.

**Remark.** From now on, for a ringed space  $(X, \mathcal{O}_X)$ , we will always assume  $\mathcal{O}_X$  is a sheaf of  $k$ -valued functions on  $X$ . With this assumption, we can make sense of pullbacks.

**Definition 9.2.** A *morphism* of ringed spaces

$$(X, \mathcal{O}_X) \xrightarrow{f} (Y, \mathcal{O}_Y)$$

is a continuous map  $f : X \rightarrow Y$  such that for every  $U \subseteq Y$  open and  $\varphi \in \mathcal{O}_Y(U)$ ,

$$\begin{array}{ccccc} & & f^*\varphi & & \\ & \nearrow & & \searrow & \\ f^{-1}(U) & \xrightarrow{f} & U & \xrightarrow{\varphi} & k \end{array}$$

is an element of  $\mathcal{O}_X(f^{-1}(U))$ . A morphism is an *isomorphism* if it has a two-sided inverse.

**Remark.** A one-sided inverse need not be two-sided: Consider  $f : \mathbb{A}^2 \rightarrow \mathbb{A}^1$  given by  $(x, y) \mapsto x$  and  $g : \mathbb{A}^1 \rightarrow \mathbb{A}^2$  given by  $x \mapsto (x, 0)$ . Then  $f \circ g = \text{id}_{\mathbb{A}^1}$ , but  $g \circ f$  is not the identity on  $\mathbb{A}^2$ .

**Remark.** If  $(X, \mathcal{O}_X) \rightarrow (Y, \mathcal{O}_Y)$  is a morphism of ringed spaces, then for  $V \subseteq U \subseteq Y$  open, we get

$$\begin{array}{ccc} \mathcal{O}_Y(U) & \longrightarrow & \mathcal{O}_X(f^{-1}(U)) \\ \downarrow \text{res.} & & \downarrow \text{res.} \\ \mathcal{O}_Y(V) & \longrightarrow & \mathcal{O}_X(f^{-1}(V)) \end{array}$$

which is a commutative diagram of ring homomorphisms.

**Remark.** If  $X$  and  $Y$  are open sets of affine varieties, then a map  $f : X \rightarrow Y$  is a morphism of open sets of affine varieties if and only if it is a morphism of ringed spaces.

**Definition 9.3** (Redefinition of affine variety). An *affine variety*  $(X, \mathcal{O}_X)$  is a ringed space isomorphic to an affine variety in the original sense (as ringed spaces).

**Remark.** We will often write just  $X$  for the affine variety instead of the pair  $(X, \mathcal{O}_X)$ .

**Example 9.3.1.** Recall that  $\mathbb{A}^1 \setminus \{0\} \cong V(xy - 1) \subseteq \mathbb{A}^2$  from Example 9.0.1. In particular,  $\mathbb{A}^1 \setminus \{0\}$  is an affine variety in the new sense (but not in the old sense).

**Proposition 9.1.** If  $X$  is an affine variety (in the old sense) and  $f \in A(X)$ , then  $D(f)$  is an affine variety.

*Proof.* Write  $X = V(I) \subseteq \mathbb{A}_{x_i}^n$  and consider the map

$$\begin{aligned} D(f) &\longrightarrow V(I, fy - 1) \subseteq \mathbb{A}_{x_i}^n \times \mathbb{A}_y^1 \\ x &\longmapsto (x, 1/f(x)). \end{aligned}$$

This has an inverse  $V(I, fy - 1) \rightarrow D(f)$  given by  $(x, y) \mapsto x$ . So  $D(f) \cong V(I, fy - 1)$  as ringed spaces. Thus  $D(f)$  is an affine variety (in the new sense).  $\square$

## 9.3 Products of Affine Varieties

**Remark.** If  $X \subseteq \mathbb{A}_{x_i}^m$  and  $Y \subseteq \mathbb{A}_{y_i}^n$  are affine varieties, then

$$X \times Y = V(I(X), I(Y)) \subseteq \mathbb{A}^{m+n},$$

viewing  $I(X), I(Y)$  as ideals in  $k[x_1, \dots, x_m, y_1, \dots, y_n]$ . So  $X \times Y$  is an affine variety with morphisms

$$\begin{array}{ccc} & X \times Y & \\ (x,y) \mapsto x \swarrow p_1 & & \searrow p_2 (x,y) \mapsto y \\ X & & Y \end{array}$$

**Proposition 9.2.** *For every affine variety  $Z$  and diagram of morphisms*

$$\begin{array}{ccccc} Z & & & & \\ & \searrow f & & \nearrow f_Y & \\ & & X \times Y & \xrightarrow{p_2} & Y \\ & \nearrow f_X & \downarrow p_1 & & \\ & & X & & \end{array}$$

*there is a unique morphism  $f$  which makes the diagram commute.*

*Proof.* We already know that there is a unique set theoretic map which makes the diagram commute. Then since  $f_X$  and  $f_Y$  are given as regular functions, so is  $f$ . So  $f$  is a morphism.  $\square$

**Remark.** We will now try to understand the isomorphism  $A(X \times Y) \cong A(X) \otimes_k A(Y)$ .

## 9.4 Tensor Products

**Definition 9.4.** Let  $A$  be a (commutative) ring and  $M, N$  be  $A$ -modules. The *tensor product*  $M \otimes_A N$  is the  $A$ -module generated by the symbols  $m \otimes n$  for  $m \in M$  and  $n \in N$ , subject to the relations

1. (distributive law):  $(m + m') \otimes n = m \otimes n + m' \otimes n$ ,
2. (multiplication with scalars):  $a(m \otimes n) = (am) \otimes n = m \otimes (an)$ .

To make this precise,  $M \otimes_A N = A^{M \times N} / R$ , where  $R$  is the submodule generated by these relations.

**Example 9.4.1.** We have  $\mathbb{Z}/3\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} = 0$ . We can compute

$$1 \otimes 1 = (3 - 2) \otimes 1 = 3 \otimes 1 - 2 \otimes 1 = 3 \otimes 1 + 1 \otimes (-2) = 0 \otimes 1 + 1 \otimes 0 = 0 \otimes 0,$$

and similarly for the other elements. In general, if  $\gcd(m, n) = 1$ , then  $\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} = 0$ .

**Proposition 9.3** (Universal property of the tensor product). *For any bilinear map  $\Phi : M \times N \rightarrow P$  to an  $A$ -module  $P$  (i.e.  $n \mapsto \Phi(m, n)$  is  $A$ -linear for each  $m \in M$  and the same for  $m \mapsto \Phi(m, n)$ ),*

$$\begin{array}{ccc} M \times N & \xrightarrow{\Phi} & P \\ (m,n) \mapsto m \otimes n \downarrow & \nearrow \Psi & \\ M \otimes N & & \end{array}$$

*there exists a unique  $A$ -module homomorphism  $\Psi : M \otimes N \rightarrow P$  such that the above diagram commutes.*

**Remark.** We have the following properties of the tensor product:

1.  $A \otimes M \cong M$ ;

2.  $M \otimes N \cong N \otimes M$ ;
3.  $(M \otimes N) \otimes P \cong M \otimes (N \otimes P)$ ;
4.  $(M \oplus N) \otimes P \cong (M \otimes P) \oplus (N \otimes P)$ .

The way to prove these is to use the universal property to construct maps in either direction and show that they compose to the identity.

5. For a fixed  $A$ -module  $M$  and an exact sequence

$$N' \xrightarrow{f} N \xrightarrow{g} N'' \longrightarrow 0,$$

the sequence (where  $F$  is defined by  $m \otimes n' \mapsto m \otimes f(n')$  and  $G$  is defined by  $m \otimes n \mapsto m \otimes g(n)$ )

$$M \otimes N' \xrightarrow{F} M \otimes N \xrightarrow{G} M \otimes N'' \longrightarrow 0$$

is also exact. In particular,  $\otimes M$  induces a right exact functor  $\text{Mod}_A \rightarrow \text{Mod}_A$  by  $N \mapsto M \otimes N$ .

**Example 9.4.2.** The functor  $\otimes M$  is in general not left exact. Consider

$$0 \longrightarrow \mathbb{Z} \xrightarrow{1 \mapsto 2} \mathbb{Z} \xrightarrow{1 \mapsto 1} \mathbb{Z}/2\mathbb{Z} \longrightarrow 0.$$

After tensoring with  $\mathbb{Z}/2\mathbb{Z}$ , we get the sequence

$$\mathbb{Z}/2\mathbb{Z} \xrightarrow{1 \mapsto 0} \mathbb{Z}/2\mathbb{Z} \xrightarrow{1 \mapsto 1 \otimes 1} \mathbb{Z}/2\mathbb{Z} \otimes \mathbb{Z}/2\mathbb{Z} \longrightarrow 0,$$

where the first map is not injective. Note that right exactness gives  $\mathbb{Z}/2\mathbb{Z} \otimes \mathbb{Z}/2\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z}$ .

**Exercise 9.1.** Show that  $\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}/(m, n)\mathbb{Z}$ .

# Lecture 10

## Sept. 18 — Pre-varieties

### 10.1 More on Tensor Products

**Proposition 10.1.** *If  $B$  and  $C$  are  $A$ -algebras (i.e. there are ring homomorphisms  $f : A \rightarrow B$  and  $g : A \rightarrow C$  which give  $a \cdot b := f(a)b$  and  $a \cdot c = g(a)c$ , then  $B \otimes_A C$  is also an  $A$ -algebra with*

$$(b \otimes c) \cdot (b' \otimes c') := (bb') \otimes (cc')$$

*and ring homomorphism  $A \rightarrow B \otimes_A C$  given by  $a \mapsto a \otimes 1$  (equivalently,  $1 \otimes a$ ).*

**Proposition 10.2.**  $k[x_1, \dots, x_m] \otimes_k k[y_1, \dots, y_n] \cong k[x_1, \dots, x_m, y_1, \dots, y_n]$ .

**Proposition 10.3.**  $(k[x_1, \dots, x_m]/I) \otimes_k (k[y_1, \dots, y_n]/J) \cong k[x_1, \dots, x_m, y_1, \dots, y_n]/\langle I, J \rangle$ .

*Proof.* Set  $R = k[x_1, \dots, x_m]$  and  $S = k[y_1, \dots, y_n]$ . We have a short exact sequence

$$0 \longrightarrow I \longrightarrow R \longrightarrow R/I \longrightarrow 0.$$

Applying the right exact functor  $\otimes_k (S/J)$  (and vice versa with  $J$  and  $\otimes R$ ) gives an exact sequence

$$\begin{array}{ccccccc} & & R \otimes_k J & & & & \\ & & \downarrow & & & & \\ & & R \otimes_k S & & & & \\ & & \downarrow & & & & \\ I \otimes_k (S/J) & \longrightarrow & R \otimes_k (S/J) & \longrightarrow & (R/I) \otimes_k (S/J) & \longrightarrow & 0 \\ & & \downarrow & & & & \\ & & 0 & & & & \end{array}$$

So we have

$$(R/I) \otimes_k (S/J) \cong \frac{R \otimes_k (S/J)}{\text{Im}(I \otimes_k (S/J) \rightarrow R \otimes_k (S/J))} \cong \frac{R \otimes_k S}{I \otimes_k S + R \otimes_k J},$$

which is the desired result since  $I \otimes_k S + R \otimes_k J = \langle I, J \rangle$  in  $R \otimes_k S$ . □

**Proposition 10.4** (Milne). *Let  $B$  and  $C$  be finitely generated  $k$ -algebras with  $k = \bar{k}$ .*

1. *If  $B$  and  $C$  are reduced, then so is  $B \otimes_k C$ .*

2. If  $B$  and  $C$  are domains, then so is  $B \otimes_k C$ .

**Remark.** We need  $k = \bar{k}$  in Proposition 10.4. Consider the domains  $\mathbb{R}[x]/(x^2 + 1)$ ,  $\mathbb{R}[y]/(y^2 + 1)$ . Then

$$\frac{\mathbb{R}[x]}{(x^2 + 1)} \otimes_{\mathbb{R}} \frac{\mathbb{R}[y]}{(y^2 + 1)} \cong \frac{\mathbb{R}[x, y]}{(x^2 + 1, y^2 + 1)},$$

which is not a domain since  $(\overline{x - y})(\overline{x + y}) = \overline{x^2 - y^2} = \overline{-1 - (-1)} = 0$ .

**Corollary 10.0.1.** If  $X \subseteq \mathbb{A}^m$  and  $Y \subseteq \mathbb{A}^n$  are affine varieties, then

1.  $I(X \times Y) = \langle I(X), I(Y) \rangle \subseteq k[x_1, \dots, x_m, y_1, \dots, y_n]$ .
2.  $A(X \times Y) \cong A(X) \otimes_k A(Y)$ .
3. If  $X$  and  $Y$  are irreducible, then  $X \times Y$  is irreducible.

*Proof.* Observe that  $V(I(X), I(Y)) = X \times Y \subseteq \mathbb{A}^{m+n}$ , so  $I(X \times Y) = \sqrt{\langle I(X), I(Y) \rangle}$ . Now we know that  $I(X)$  and  $I(Y)$  are radical in  $k[x_1, \dots, x_m]$  and  $k[y_1, \dots, y_n]$ , respectively, so

$$\frac{k[x_1, \dots, x_m]}{I(X)} \quad \text{and} \quad \frac{k[y_1, \dots, y_n]}{I(Y)}$$

are reduced. By Proposition 10.4, we get that

$$\frac{k[x_1, \dots, x_m, y_1, \dots, y_n]}{\langle I(X), I(Y) \rangle} \cong \frac{k[x_1, \dots, x_m]}{I(X)} \otimes_k \frac{k[y_1, \dots, y_n]}{I(Y)}$$

is reduced, so  $\langle I(X), I(Y) \rangle$  is radical. Thus  $I(X \times Y) = \langle I(X), I(Y) \rangle$ , so (1) holds.

Now (1) implies (2), and (3) follows since  $X$  and  $Y$  being irreducible implies  $A(X)$  and  $A(Y)$  are domains, which implies  $A(X \times Y)$  is a domain by Proposition 10.4 and (2), so  $X \times Y$  is irreducible.  $\square$

## 10.2 Pre-varieties

**Remark.** We will now head towards defining a *variety*, which is roughly finitely many affine varieties glued together (a *pre-variety*) with a separation condition (an algebraic version of Hausdorff).

**Definition 10.1.** A *pre-variety* is a ringed space  $(X, \mathcal{O}_X)$  such that there exists a finite open cover  $X = \bigcup_{i=1}^s U_i$  with  $(U_i, \mathcal{O}_X|_{U_i})$  being an affine variety for all  $i = 1, \dots, s$ . A *morphism* of pre-varieties

$$f : (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$$

is a morphism of the ringed spaces. We will often just write  $X$  for  $(X, \mathcal{O}_X)$ .

**Remark.** We call  $\varphi \in \mathcal{O}_X(U)$  with  $U \subseteq X$  open and  $\varphi : U \rightarrow k$  a *regular function* on  $U$ .

**Example 10.1.1.** Consider the following:

1. An affine variety  $X$  is a pre-variety. However, we have multiple choices for the open cover: We can take  $X = X$ , or  $X = \bigcup_{i=1}^s D(f_i)$  with  $f_i \in \mathcal{O}_X(X)$  and  $(f_1, \dots, f_s) = (1)$  in  $\mathcal{O}_X(X)$ .
2.  $\mathbb{P}_k^n = (\mathbb{A}^{n+1} \setminus \{0\})/k^*$  is a pre-variety. We will see that  $\mathbb{P}_k^1 = \mathbb{A}_k^1 \cup \{\text{pt}\}$ .

3. Let  $X = V(I) \subseteq \mathbb{A}^n$  be an affine variety and  $U \subseteq X$  open. Set

$$\mathcal{O}_U(V) = \{\varphi : V \rightarrow k \mid \varphi \text{ is regular}\}.$$

Then  $(U, \mathcal{O}_U)$  is a pre-variety. To see this, note that  $U = \bigcup_{f \in I(X \setminus U)} D(f)$ . Since  $U$  is Noetherian (hence is compact), we can find a finite subcover, so  $U = \bigcup_{i=1}^s D(f_i)$  for some  $f_i \in A(X)$ .

4. (Gluing) Let  $X_1$  and  $X_2$  be affine varieties, and  $U_{1,2} \subseteq X_1$ ,  $U_{2,1} \subseteq X_2$  open, with an isomorphism

$$f : U_{1,2} \longrightarrow U_{2,1}.$$

Then we get a pre-variety by setting  $X = (X_1 \sqcup X_2)/\sim$ , where  $a \sim f(a)$  for all  $a \in U_{1,2}$ ,  $f(a) \sim a$  for all  $a \in U_{2,1}$ , and  $b \sim b$  for all  $b \in X_1 \sqcup X_2$ . We have quotient maps

$$j_1 : X_1 \longrightarrow X \quad \text{and} \quad j_2 : X_2 \longrightarrow X.$$

Now  $X$  is a topological space with the quotient topology, and  $j_1, j_2$  are open embeddings (i.e. have open images and are homeomorphisms onto their images). Define a sheaf of rings  $\mathcal{O}_X$  on  $X$  by

$$\mathcal{O}_X(U) = \{\varphi : U \rightarrow k \mid j_1^* \varphi \in \mathcal{O}_{X_1}(j_1^{-1}(U)) \text{ and } j_2^* \varphi \in \mathcal{O}_{X_2}(j_2^{-1}(U))\}.$$

One can check  $X = j_1(X_1) \cup j_2(X_2)$  and  $(j(X_i), \mathcal{O}_X|_{j(X_i)}) \cong (X_i, \mathcal{O}_{X_i})$ , so  $(X, \mathcal{O}_X)$  is a pre-variety.

**Example 10.1.2.** Consider  $X_1 = \mathbb{A}_x^1$  and  $X_2 = \mathbb{A}_y^1$ , with  $U_{1,2} = \mathbb{A}_x^1 \setminus \{0\}$  and  $U_{2,1} = \mathbb{A}_y^1 \setminus \{0\}$ . Define

$$\begin{aligned} f : U_{1,2} &\longrightarrow U_{2,1} \\ x &\longmapsto 1/x. \end{aligned}$$

Then we can take  $\mathbb{P}_k^1 = (X_1 \sqcup X_2)/\sim$ . What are the regular functions  $\mathbb{P}_k^1 \rightarrow k$ ? We should get only the constant functions (When  $k = \mathbb{C}$ ,  $\mathbb{P}_{\mathbb{C}}^1$  is compact, so a holomorphic function  $f : \mathbb{P}_{\mathbb{C}}^1 \rightarrow \mathbb{C}$  is bounded. By restricting to  $X_1$ , we get a bounded map  $f : \mathbb{C} \rightarrow \mathbb{C}$ , so  $f$  is constant by Liouville's theorem).

In general, let  $j_i : X_i \rightarrow \mathbb{P}_k^1$  be the quotient maps. Fix  $\varphi \in \mathcal{O}_{\mathbb{P}^1}(\mathbb{P}^1)$ . Now

$$\varphi|_{X_1} := j_1^* \varphi = \sum_{i \geq 0} a_i x^i \quad \text{and} \quad \varphi|_{X_2} := j_2^* \varphi = \sum_{i \geq 0} b_i y^i$$

for some  $a_i, b_i \in k$ . They must agree on the overlap, so

$$\sum_{i \geq 0} a_i x^i = \sum_{i \geq 0} b_i (1/x)^i$$

as functions on  $\mathbb{A}^1 \setminus \{0\}$ . Since  $\mathcal{O}_{\mathbb{A}^1}(\mathbb{A}^1 \setminus \{0\}) = k[x^{\pm 1}]$ , we have  $a_i = b_i = 0$  for  $i > 0$  and  $a_0 = b_0$  (since the powers of  $x^{\pm 1}$  are  $k$ -linearly independent), so  $\varphi$  is a constant function.

If we instead took  $f : U_{1,2} \rightarrow U_{2,1}$  to be  $x \mapsto x$ , then  $X = (X_1 \sqcup X_2)/\sim$  is the “bug-eyed line” with two points  $0, 0'$  at the origin (this is the *line with two origins* when  $k = \mathbb{R}$ , which is not Hausdorff.) Note that  $X \setminus \{0, 0'\} \cong \mathbb{A}^1 \setminus \{0\}$ . In our case, the bad property is that there exist two morphisms

$$g_1, g_2 : \mathbb{A}^1 \longrightarrow X$$

such that  $g_1|_{\mathbb{A}^1 \setminus \{0\}} = g_2|_{\mathbb{A}^1 \setminus \{0\}}$  and  $g_1 \neq g_2$ , i.e. “limits are not unique” on  $X$ . Note that a similar computation shows  $\mathcal{O}_X(X) \cong k[x]$ , so in particular,  $X \not\cong \mathbb{P}_k^1$ .

**Example 10.1.3** (General gluing procedure). Let  $I$  be a finite index set,  $(X_i, \mathcal{O}_{X_i})$  affine varieties,  $U_{i,j} \subseteq X_i$  open sets, and  $f_{i,j} : U_{i,j} \rightarrow U_{j,i}$  isomorphisms for each  $i, j \in I$ . With some compatibility conditions, we can also glue the  $X_i$  together to get a pre-variety.