MATH 6421: Algebraic Geometry I

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Contents

Lecture 1

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1.1 Motivation for Algebraic Geometry

Remark. Why study algebraic geometry? Algebraic geometry connects to many fields of math.

Example 1.0.1. Consider a plane curve $\{f(z_1, z_2) = 0\} \subseteq \mathbb{C}^2$, e.g. an elliptic curve $z_2^2 - z_1^3 + z_1 - 1 = 0$. If we compactify and set C to be the closure of C^0 in $\mathbb{CP}^2$, then $d = \deg f$. There are connections in

1. Topology: $H^1(C, \mathbb{C}) \cong \mathbb{C}^{2g}$, where $g = (d-1)(d-2)/12$;
2. Arithmetic: the number of $\mathbb{Q}$-points is finite if $d > 3$;
3. Complex geometry: We have $C \cong \mathbb{CP}^2$ for $d = 1, 2$, $C \cong \mathbb{C}/\Lambda$ for $d = 3$, and $C \cong \mathbb{H}/\Gamma$ for $d > 3$.

1.2 Affine Varieties

Fix an algebraically closed field k (e.g. $\mathbb{C}$, $\overline{\mathbb{Q}}$, $\overline{\mathbb{F}}_p$, etc.).

Definition 1.1. *Affine space* is the set $\mathbb{A}^n = \mathbb{A}_k^n = \{\vec{a} = (a_1, \ldots, a_n) : a_i \in k\}$.

Remark. Note the following:

1. $\mathbb{A}_k^n$ is the same set as k^n, but forgetting the vector space structure;
2. $f \in k[x_1, \ldots, x_n]$ gives a polynomial function $\mathbb{A}_k^n \to k$ by evaluation: $a \mapsto f(a)$.

Definition 1.2. For a subset $S \subseteq k[x_1, \ldots, x_n]$, its *vanishing set* is

$$V(S) = \{a \in \mathbb{A}^n : f(a) = 0 \text{ for all } f \in S\}.$$

An *affine variety* is a subset of $\mathbb{A}_k^n$ of this form.

Example 1.2.1. Consider the following:

1. $\mathbb{A}^n = V(\varnothing) = V(\{0\})$;
2. $\varnothing = V(1) = V(k[x_1, \ldots, x_n])$;
3. a point $a = (a_1, \ldots, a_n)$ is an affine variety: $V(\{x_1 - a_1, \ldots, x_n - a_n\}) = \{a\}$;
4. a linear space $L \subseteq \mathbb{A}^n$ (it is the kernel of some matrix);
5. plane curves $V(f(x, y)) \subseteq \mathbb{A}_{x,y}^2$;

6. $\mathrm{SL}_n(k) \subseteq \mathbb{A}^{n\times n}$ is an affine variety: $\mathrm{SL}_n(k) = V(\det([x_{i,j}]) - 1)$;
7. $\mathrm{GL}_n(k)$ (as a set) is an affine variety in $\mathbb{A}^{n\times n+1}$: $\mathrm{GL}_n(k) = V(\det([x_{i,j}])y - 1)$;
8. if $X \subseteq \mathbb{A}^m$ and $Y \subseteq \mathbb{A}^n$ are affine varieties, then $X \times Y \subseteq \mathbb{A}^{m+n}$ is an affine variety;
9. the affine varieties $X \subseteq \mathbb{A}^1_k$ are of the form: finite set of points, $\varnothing$, or $\mathbb{A}^1_k$.

Proposition 1.1 (Relation to ideals)**.** *If $S \subseteq k[x_1, \ldots, x_n]$, then $V(S) = V(\langle S\rangle)$, where $\langle S\rangle$ is the ideal generated by S.*

Proof. Since $S \subseteq \langle S\rangle$, we have $V(\langle S\rangle) \subseteq V(S)$. Conversely, if $f, g \in S$ and $h \in k[x_1, \ldots, x_n]$, then $f + g$ and hf vanish on $V(S)$, so we see that $V(S) \subseteq V(\langle S\rangle)$. $\square$

Remark. The statement implies that if $f_1, \ldots, f_r \in k[x_1, \ldots, x_n]$, then $V(f_1, \ldots, f_r) = V((f_1, \ldots, f_n))$. The following are some further applications of the statement:

1. affine varities are vanishing loci of ideals;
2. if $X \subseteq \mathbb{A}^n$ is an affine variety, then X is cut out by finitely many polynomial equations.

To see the second statement, note that $X = V(I)$ for some ideal $I \leq k[x_1, \ldots, x_n]$. By the Hilbert basis theorem that $k[x_1, \ldots, x_n]$ is Noetherian, there are finitely many $f_1, \ldots, f_r \in k[x_1, \ldots, x_n]$ such that $I = (f_1, \ldots, f_r)$. So $X = V(I) = V(f_1, \ldots, f_r)$.

Proposition 1.2 (Properties of the vanishing set)**.** *For ideals I, J of $k[x_1, \ldots, x_n]$,*

1. *if $I \subseteq J$, then $V(J) \subseteq V(I)$;*
2. *$V(I) \cap V(J) = V(I + J)$;*
3. *$V(I) \cup V(J) = V(IJ) = V(I \cap J)$.*

Proof. (1) This follows from definitions and actually holds for general subsets.

(2) Note that $V(I) \cap V(J) = V(I \cap J) = V(\langle I \cup J\rangle) = V(I + J)$.

(3) We only prove the first equality, the second is similar. Recall that $IJ = \left\{\sum_{i=1}^r f_i g_i : f_i \in I, g_i \in J\right\}$. We have the forwards inclusion $V(I) \cup V(J) \subseteq V(IJ)$ from definitions. For the reverse inclusion, consider a point $x \notin V(I) \cup V(J)$. So there exists $f \in I$ and $g \in J$ such that $f(x), g(x) \neq 0$. So $f(x)g(x) \neq 0$, which implies that $x \notin V(IJ)$. Thus $V(IJ) \subseteq V(I) \cup V(J)$ as well. $\square$

Remark. The above implies that if X and Y are affine varieties in $\mathbb{A}^n_k$, then so are $X \cup Y$ and $X \cap Y$.

Example 1.2.2. Consider $V(y^2 - x^2, y^2 + x^2) \subseteq \mathbb{A}^2$. Note that $(y^2 - x^2, y^2 + x^2) = (x^2, y^2)$, from which we can easily see that $V(y^2 - x^2, y^2 + x^2) = V(x^2, y^2) = \{0\}$.

1.3 Correspondence with Ideals

Remark. Our goal is to build a correspondence between affine varieties in $\mathbb{A}^n_k$ and ideals of $k[x_1, \ldots, x_n]$.

Definition 1.3. For a subset $X \subseteq \mathbb{A}^n_k$, define

$$I(X) = \{f \in k[x_1, \ldots, x_n] : f(a) = 0 \text{ for all } a \in X\}.$$

Remark. Note that $I(X)$ is in fact an ideal of $k[x_1, \dots, x_n]$.

Example 1.3.1. Consider the following:

1. $I(\varnothing) = k[x_1, \dots, x_n]$;
2. $I(\mathbb{A}^n_k) = \{0\}$, this will follow from the Hilbert nullstellensatz and relies on $k = \overline{k}$ (for $k = \mathbb{R}$, the polynomial $x^2 + y^2 + 1$ is always nonzero and thus lies in $I(\mathbb{A}^n_\mathbb{R})$);
3. for $n = 1$, if $S \subseteq \mathbb{A}^1_k$ be an infinite set, then $I(S) = (0)$.
4. for $n = 1$, we have $I(V(x^2)) = I(\{0\}) = (x)$.

Remark. What properties does $I(X)$ satisfy?

Definition 1.4. Let R be a ring. The *radical* of an ideal $J \leq R$ is

$$\sqrt{J} = \{f \in R : f^n \in J \text{ for some } n > 0\}.$$

An ideal J is *radical* if $J = \sqrt{J}$.

Exercise 1.1. Check the following:

1. $\sqrt{J}$ is always an ideal.
2. $\sqrt{\sqrt{J}} = \sqrt{J}$.
3. An ideal $J \leq R$ is radical if and only if R/J is reduced.[1]

Proposition 1.3. *If $X \subseteq \mathbb{A}^n_k$ is a subset (not necessarily an affine variety), then $I(X)$ is radical.*

Proof. Fix $f \in k[x_1, \dots, x_n]$. If $f^n \in I(X)$, then $f^n(x) = 0$ for all $x \in X$. This implies $f(x) = 0$ for all $x \in X$, so $f \in I(X)$. Thus we see that $I(X) = \sqrt{I(X)}$. □

Theorem 1.1 (Hilbert's nullstellensatz). *If $J \leq k[x_1, \dots, x_n]$ is an ideal, then $I(V(J)) = \sqrt{J}$.*

Example 1.4.1. Let $n = 1$, so that $k[x]$ is a PID. Let $f = (x - a_1)^{m_1} \cdots (x - a_r)^{m_r}$. Then

$$I(V(f)) = I(\{a_1, \dots, a_r\}) = ((x - a_1)^{m_1} \cdots (x - a_r)^{m_r}).$$

[1] Recall that a ring R is *reduced* if for all nonzero $f \in R$ and positive integers n, we have $f^n \neq 0$. It is immediate that an integral domain is reduced.