MATH 6421: Algebraic Geometry I

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Contents

Lecture 1

Aug. 19 — Affine Varieties

1.1 Motivation for Algebraic Geometry

Remark. Why study algebraic geometry? Algebraic geometry connects to many fields of math.

Example 1.0.1. Consider a plane curve $\{f(z_1, z_2) = 0\} \subseteq \mathbb{C}^2$, e.g. an elliptic curve $z_2^2 - z_1^3 + z_1 - 1 = 0$. Compactify and set C to be the closure of C^0 in $\mathbb{CP}^2$, and let $d = \deg f$. There are connections in

1. Topology: $H^1(C, \mathbb{C}) \cong \mathbb{C}^{2g}$, where $g = (d-1)(d-2)/12$;
2. Arithmetic: the number of $\mathbb{Q}$-points is finite if $d > 3$;
3. Complex geometry: We have $C \cong \mathbb{CP}^2$ for $d = 1, 2$, $C \cong \mathbb{C}/\Lambda$ for $d = 3$, and $C \cong \mathbb{H}/\Gamma$ for $d > 3$.

1.2 Affine Varieties

Fix an algebraically closed field k (e.g. $\mathbb{C}$, $\overline{\mathbb{Q}}$, $\overline{\mathbb{F}}_p$, etc.).

Definition 1.1. *Affine space* is the set $\mathbb{A}^n = \mathbb{A}^n_k = \{\vec{a} = (a_1, \ldots, a_n) : a_i \in k\}$.

Remark. Note the following:

1. $\mathbb{A}^n_k$ is the same set as k^n, but forgetting the vector space structure;
2. $f \in k[x_1, \ldots, x_n]$ gives a polynomial function $\mathbb{A}^n_k \to k$ by evaluation: $a \mapsto f(a)$.

Definition 1.2. For a subset $S \subseteq k[x_1, \ldots, x_n]$, its *vanishing set* is

$$V(S) = \{a \in \mathbb{A}^n : f(a) = 0 \text{ for all } f \in S\}.$$

An *affine variety* is a subset of $\mathbb{A}^n_k$ of this form.

Example 1.2.1. Consider the following:

1. $\mathbb{A}^n = V(\varnothing) = V(\{0\})$;
2. $\varnothing = V(1) = V(k[x_1, \ldots, x_n])$;
3. a point $a = (a_1, \ldots, a_n)$ is an affine variety: $V(\{x_1 - a_1, \ldots, x_n - a_n\}) = \{a\}$;
4. a linear space $L \subseteq \mathbb{A}^n$ (it is the kernel of some matrix);
5. plane curves $V(f(x, y)) \subseteq \mathbb{A}^2_{x,y}$;

6. $\mathrm{SL}_n(k) \subseteq \mathbb{A}^{n\times n}$ is an affine variety: $\mathrm{SL}_n(k) = V(\det([x_{i,j}]) - 1)$;
7. $\mathrm{GL}_n(k)$ (as a set) is an affine variety in $\mathbb{A}^{n\times n+1}$: $\mathrm{GL}_n(k) = V(\det([x_{i,j}])y - 1)$;
8. if $X \subseteq \mathbb{A}^m$ and $Y \subseteq \mathbb{A}^n$ are affine varieties, then $X \times Y \subseteq \mathbb{A}^{m+n}$ is an affine variety;
9. the affine varieties $X \subseteq \mathbb{A}^1_k$ are of the form: finite set of points, $\varnothing$, or $\mathbb{A}^1_k$.

Proposition 1.1 (Relation to ideals)**.** *If $S \subseteq k[x_1,\ldots,x_n]$, then $V(S) = V(\langle S\rangle)$, where $\langle S\rangle$ is the ideal generated by S.*

Proof. Since $S \subseteq \langle S\rangle$, we have $V(\langle S\rangle) \subseteq V(S)$. Conversely, if $f, g \in S$ and $h \in k[x_1,\ldots,x_n]$, then $f+g$ and hf vanish on $V(S)$, so we see that $V(S) \subseteq V(\langle S\rangle)$. □

Remark. The statement implies that if $f_1,\ldots,f_r \in k[x_1,\ldots,x_n]$, then $V(f_1,\ldots,f_r) = V((f_1,\ldots,f_n))$. The following are some further applications of the statement:

1. affine varities are vanishing loci of ideals;
2. if $X \subseteq \mathbb{A}^n$ is an affine variety, then X is cut out by finitely many polynomial equations.

To see the second statement, note that $X = V(I)$ for some ideal $I \leq k[x_1,\ldots,x_n]$. By the Hilbert basis theorem that $k[x_1,\ldots,x_n]$ is Noetherian, there are finitely many $f_1,\ldots,f_r \in k[x_1,\ldots,x_n]$ such that $I = (f_1,\ldots,f_r)$. So $X = V(I) = V(f_1,\ldots,f_r)$.

Proposition 1.2 (Properties of the vanishing set)**.** *For ideals I, J of $k[x_1,\ldots,x_n]$,*

1. *if $I \subseteq J$, then $V(J) \subseteq V(I)$;*
2. *$V(I) \cap V(J) = V(I+J)$;*
3. *$V(I) \cup V(J) = V(IJ) = V(I \cap J)$.*

Proof. (1) This follows from definitions and actually holds for general subsets.

(2) Note that $V(I) \cap V(J) = V(I \cap J) = V(\langle I \cup J\rangle) = V(I+J)$.

(3) We only prove the first equality, the second is similar. Recall that $IJ = \left\{\sum_{i=1}^r f_i g_i : f_i \in I, g_i \in J\right\}$. We have the forwards inclusion $V(I) \cup V(J) \subseteq V(IJ)$ from definitions. For the reverse inclusion, consider a point $x \notin V(I) \cup V(J)$. So there exists $f \in I$ and $g \in J$ such that $f(x), g(x) \neq 0$. So $f(x)g(x) \neq 0$, which implies that $x \notin V(IJ)$. Thus $V(IJ) \subseteq V(I) \cup V(J)$ as well. □

Remark. The above implies that if X and Y are affine varieties in $\mathbb{A}^n_k$, then so are $X \cup Y$ and $X \cap Y$.

Example 1.2.2. Consider $V(y^2 - x^2, y^2 + x^2) \subseteq \mathbb{A}^2$. Note that $(y^2 - x^2, y^2 + x^2) = (x^2, y^2)$, from which we can easily see that $V(y^2 - x^2, y^2 + x^2) = V(x^2, y^2) = \{0\}$.

1.3 Correspondence with Ideals

Remark. Our goal is to build a correspondence between affine varieties in $\mathbb{A}^n_k$ and ideals of $k[x_1,\ldots,x_n]$.

Definition 1.3. For a subset $X \subseteq \mathbb{A}^n_k$, define

$$I(X) = \{f \in k[x_1,\ldots,x_n] : f(a) = 0 \text{ for all } a \in X\}.$$

Remark. Note that $I(X)$ is in fact an ideal of $k[x_1, \ldots, x_n]$.

Example 1.3.1. Consider the following:

1. $I(\varnothing) = k[x_1, \ldots, x_n]$;
2. $I(\mathbb{A}^n_k) = \{0\}$, this will follow from the Hilbert nullstellensatz and relies on $k = \overline{k}$ (for $k = \mathbb{R}$, the polynomial $x^2 + y^2 + 1$ is always nonzero and thus lies in $I(\mathbb{A}^n_{\mathbb{R}})$);
3. for $n = 1$, if $S \subseteq \mathbb{A}^1_k$ be an infinite set, then $I(S) = (0)$.
4. for $n = 1$, we have $I(V(x^2)) = I(\{0\}) = (x)$.

Remark. What properties does $I(X)$ satisfy?

Definition 1.4. Let R be a ring. The *radical* of an ideal $J \leq R$ is

$$\sqrt{J} = \{f \in R : f^n \in J \text{ for some } n > 0\}.$$

An ideal J is *radical* if $J = \sqrt{J}$.

Exercise 1.1. Check the following:

1. $\sqrt{J}$ is always an ideal.
2. $\sqrt{\sqrt{J}} = \sqrt{J}$.
3. An ideal $J \leq R$ is radical if and only if R/J is reduced.[1]

Proposition 1.3. *If $X \subseteq \mathbb{A}^n_k$ is a subset (not necessarily an affine variety), then $I(X)$ is radical.*

Proof. Fix $f \in k[x_1, \ldots, x_n]$. If $f^n \in I(X)$, then $f^n(x) = 0$ for all $x \in X$. This implies $f(x) = 0$ for all $x \in X$, so $f \in I(X)$. Thus we see that $I(X) = \sqrt{I(X)}$. □

Theorem 1.1 (Hilbert's nullstellensatz)**.** *If $J \leq k[x_1, \ldots, x_n]$ is an ideal, then $I(V(J)) = \sqrt{J}$.*

Example 1.4.1. Let $n = 1$, so that $k[x]$ is a PID. Let $f = (x - a_1)^{m_1} \cdots (x - a_r)^{m_r}$. Then

$$I(V(f)) = I(\{a_1, \ldots, a_r\}) = ((x - a_1) \cdots (x - a_r)).$$

[1] Recall that a ring R is *reduced* if for all nonzero $f \in R$ and positive integers n, we have $f^n \neq 0$. It is immediate that an integral domain is reduced.

Lecture 2

Aug. 21 — Hilbert's Nullstellensatz

2.1 Applications of Hilbert's Nullstellensatz

Corollary 2.0.1 (Weak nullstellensatz). *If $J \leq k[x_1, \dots, x_n]$ is an ideal with $J \neq (1)$, then $V(J) \neq \varnothing$. Equivalently, if $f_1, \dots, f_r \in k[x_1, \dots, x_n]$ have no common zeros, then there exist $g_1, \dots, g_r \in k[x_1, \dots, x_n]$ such that $\sum_{i=1}^r f_i g_i = 1$.*

Proof. Assume otherwise that $V(J) = \varnothing$. Then $I(V(J)) = I(\varnothing) = (1)$, so by Hilbert's nullstellensatz, we have $\sqrt{J} = (1)$. Then $1^n \in J$ for some $n > 0$, so $1 \in J$, i.e. $J = (1)$. □

Remark. We need k to be algebraically closed. Note that $(1) \neq (x^2+1) \leq \mathbb{R}[x]$ but $V(x^2+1) = \varnothing$.

Corollary 2.0.2. *There is an inclusion-reversing bijection between radical ideals $J \leq k[x_1, \dots, x_n]$ and affine varieties $X \subseteq \mathbb{A}_k^n$ given by $J \mapsto V(J)$ and $X \mapsto I(X)$.*

Proof. It suffices to show that these maps are inverses. For $J \leq k[x_1, \dots, x_n]$ a radical ideal, we have

$$I(V(J)) = \sqrt{J} = J$$

by Hilbert's nullstellensatz. For $X \subseteq \mathbb{A}_k^n$ an affine variety, we clearly have $X \subseteq V(I(X))X$. For the reverse inclusion, choose an ideal $J \leq k[x_1, \dots, x_n]$ such that $V(J) = X$. Then $J \subseteq I(X)$, so we have $V(I(X)) \subseteq V(J) = X$. Thus we also get $V(I(X)) = X$. □

Remark. This implies that maximal ideals in $k[x_1, \dots, x_n]$ correspond to points in $\mathbb{A}_k^n$, since maximal ideals correspond to minimal varieties under this bijection.

Corollary 2.0.3. *If X_1, X_2 are affine varieties in $\mathbb{A}_k^n$, then*

1. $I(X_1 \cup X_2) = I(X_1) \cap I(X_2)$;
2. $I(X_1 \cap X_2) = \sqrt{I(X_1) + I(X_2)}$.

Proof. (1) This follows from definitions.

(2) Write $I(X_1 \cap X_2) = I(V(I(X_1)) \cap V(I(X_2))) = I(V(I(X_1) + I(X_2))) = \sqrt{I(X_1) + I(X_2)}$. □

Example 2.0.1. The radical in (2) is necessary. Consider $X_1 = V(y)$ and $X_2 = V(y - x^2)$ in $\mathbb{A}_k^2$. Then $X_1 \cap X_2 = \{(0,0)\} \subseteq \mathbb{A}_k^2$, so $I(X_1 \cap X_2) = (x, y)$. However, $I(X_1) + I(X_2) = (y) + (y - x^2) = (y, x^2)$.

Note that it is sometimes better to consider (y, x^2) anyway as it tracks multiplicities. In particular, we can see the multiplicity in the dimension of $k[x,y]/(x, y^2) \cong \overline{1}k \oplus \overline{y}k$ as a k-vector space.

2.2 Proof of Hilbert's Nullstellensatz

We will assume the following result from commutative algebra without proof:

Theorem 2.1 (Noether normalization). *Let A be a finitely generated algebra over a field k with A a domain. Then there is an injective k-algebra homomorphism $k[z_1, \ldots, z_n] \hookrightarrow A$ that is finite, i.e. A is a finitely generated $k[z_1, \ldots, z_n]$-module.*

Corollary 2.1.1. *If $K \subseteq L$ is a field extension and L is a finitely generated K-algebra, then $K \subseteq L$ is a finite field extension. In particular, if in addition $K = \overline{K}$, then $K = L$.*

Proof. By Noether normalization, there exists a k-algebra homomorphism $K[z_1, \ldots, z_n] \to L$ that is finite. Then by a result from commutative algebra, L is integral over $K[z_1, \ldots, z_n]$, which implies that $K[z_1, \ldots, z_n]$ must also be a field since L is. Thus $n = 0$, so $K \subseteq L$ is a finite extension. □

Proposition 2.1. *If $(1) \neq J \leq R$ is an ideal, then J is contained in some maximal ideal.*

Proof. Consider the set $P = \{I \leq R : J \subseteq I, I \neq (1)\}$ with the partial order given by inclusion. Note that $P \neq \varnothing$ since $J \in P$. Furthermore, every chain in P has an upper bound (for $\{I_\alpha : \alpha \in A\}$ a chain P, we can take $\bigcup_{\alpha \in A} I_\alpha$, which one can check is indeed an ideal that lies in P; note that $1 \notin I_\alpha$ implies $1 \notin \bigcup_{\alpha \in A} I_\alpha$). So Zorn's lemma implies there is a maximal element in P, which is a maximal ideal. □

Proof of Theorem 1.1. We will proceed in the following steps:

1. Show that the maximal ideals of $k[x_1, \ldots, x_n]$ are of the form $(x_1 - a_1, \ldots, x_n - a_n)$ for $a_i \in k$.
2. Prove the weak nullstellensatz: If $1 \neq J \leq k[x_1, \ldots, x_n]$, is an ideal, then $V(J) \neq \varnothing$.
3. Prove the (strong) nullstellensatz: $I(V(J)) = \sqrt{J}$ for $J \leq k[x_1, \ldots, x_n]$.

The most difficult part is the first step and is where we need k to be algebraically closed.[1]

(1) For $a_1, \ldots, a_n \in k$, the ideal $(x_1 - a_1, \ldots, x_n - a_n)$ is maximal (the quotient is k, which is a field). Conversely, fix a maximal ideal $\mathfrak{m} \in k[x_1, \ldots, x_n]$. Since

$$k \xrightarrow{\phi} k[x_1, \ldots, x_n]/\mathfrak{m} = L$$

is a finitely generated k-algebra and k is algebraically closed, ϕ is an isomorphism by Corollary 2.1.1. Choose $a_i \in k$ such that $\phi(a_i) = \overline{x}_i$, so $\overline{x_i - a_i} = 0$ in L Then $(x_1 - a_1, \ldots, x_n - a_n) \subseteq \mathfrak{m}$, so they must be equal since both the left and right hand sides are maximal ideals.

(2) By Proposition 2.1, J is contained in some maximal ideal $\mathfrak{m}$. By (1), $\mathfrak{m} = (x_1 - a_1, \ldots, x_n - a_n)$ for some $a_1, \ldots, a_n \in k$. Since $J \subseteq \mathfrak{m}$, we have $V(J) \supseteq V(\mathfrak{m}) \supseteq \{(a_1, \ldots, a_n)\}$, so $J \neq \varnothing$.

(3) The reverse inclusion follows from definitions. For the forward inclusion, fix $f \in I(V(J))$, and we want to show that $f^n \in J$ for some $n > 0$. Add a new variable y and consider

$$J_1 = (J, fy - 1) \leq k[x_1, \ldots, x_n, y].$$

Now $V(J_1) = \{(a, b) = (a_1, \ldots, a_n, b) \in \mathbb{A}_k^{n+1} : a \in V(J), f(a)b = 1\} = \varnothing$ since f vanishes on $V(J)$, so $f(a)b = 0$ for any b. Thus by the weak nullstellensatz, $J_1 = (1)$, so $1 = \sum_{i=1}^r g_i f_i + g_0(fy - 1)$ with

[1] The statement is false when k is not algebraically closed: $(x^2 + 1)$ is maximal in $\mathbb{R}[x]$.

$f_1, \ldots, f_r \in J$ and $g_0, \ldots, g_r \in k[x_1, \ldots, x_n, y]$. Let N be the maximal power of y in the g_i. Multiplying by f^N, we get

$$f^N = \sum_{i=1}^{r} G_i(x_1, \ldots, x_n, fy) f_i + G_0(x_1, \ldots, x_n, fy)(fy - 1)$$

with $G_i \in k[x_1, \ldots, x_n, fy]$. So if we set $fy = 1$, then we have

$$f^N = \sum_{i=1}^{r} G_i(x_1, \ldots, x_n, 1) f_i + 0 \in J,$$

which gives $f \in \sqrt{J}$. To justify this substitution, we can consider the quotient $k[x_1, \ldots, x_n, y]/(fy - 1)$. We have a map $k[x_1, \ldots, x_n] \to k[x_1, \ldots, x_n, y]/(fy - 1)$, which is injective since $(fy - 1)$ does not lie in $k[x_1, \ldots, x_n]$, so an equality in the quotient implies an equality in $k[x_1, \ldots, x_n]$. $\square$

Lecture 3

Aug. 26 — The Zariski Topology

3.1 Polynomial Functions and Subvarieties

Remark. Recall that a polynomial $f \in k[x_1, \dots, x_n]$ gives a function $\mathbb{A}_k^n \to k$ by $a \mapsto f(a)$.

Proposition 3.1. *If $f, g \in k[x_1, \dots, x_n]$ give the same function $\mathbb{A}_k^n \to k$, then $f = g$ in $k[x_1, \dots, x_n]$.*

Proof. Assume $f = g$ as polynomial functions. Then $V(f - g) = \mathbb{A}_k^n$, so $\sqrt{(f-g)} = I(\mathbb{A}_k^n) = (0)$ by Hilbert's nullstellensatz (note that we can also prove $I(\mathbb{A}_k^n) = (0)$ directly, it is enough to have k be an infinite field for this part). Thus $f - g = 0$, so $f = g$ in $k[x_1, \dots, x_n]$. $\square$

Remark. In the above proposition, we need k to be an infinite field (e.g. if $k = \overline{k}$): Otherwise, there are only finitely many functions $\mathbb{A}_k^n \to k$, but infinitely many polynomials in $k[x_1, \dots, x_n]$.

Remark. The set of polynomials functions $\mathbb{A}_k^n \to k$ form a ring, and the above proposition implies that this ring is isomorphic to $k[x_1, \dots, x_n]$.

Definition 3.1. A *polynomial function* on an affine variety $X \subseteq \mathbb{A}_k^n$ is a function $\varphi : X \to k$ such that there exists $f \in k[x_1, \dots, x_n]$ with $\varphi(a) = f(a)$ for every $a \in X$.

Definition 3.2. The *coordinate ring* of X is $A(X) = \{f : X \to k \mid f \text{ is a polynomial function}\}$, which is a ring under pointwise addition and multiplication.

Remark. Observe that there exists a surjective ring homomorphism

$$\begin{aligned} k[x_1, \dots, x_n] &\longrightarrow A(X) \\ f &\longmapsto (a \mapsto f(a)) \end{aligned}$$

with kernel $I(X)$. Thus we have $A(X) \cong k[x_1, \dots, x_n]/I(X)$.

Remark. We can now replace $\mathbb{A}_k^n$ and $k[x_1, \dots, x_n]$ by X and $A(X)$ to study *subvarieties* of X.

Definition 3.3. Let $X \subseteq \mathbb{A}_k^n$ be an affine variety. If $S \subseteq A(X)$ is a subset, then define

$$V_X(S) = \{a \in X : f(a) = 0 \text{ for all } f \in S\}.$$

A subset of X of this form is called an *affine subvariety* of X. (Equivalently, these are the same as an affine variety $Y \subseteq \mathbb{A}_k^n$ such that $Y \subseteq X$.) For $Y \subseteq X$ a subvariety, define

$$I_X(Y) = \{f \in A(X) : f(a) = 0 \text{ for all } a \in Y\}.$$

Proposition 3.2. *There is a bijective correspondence between radical ideals in $A(X)$ and affine subvarieties of X given by $J \mapsto V_X(J)$ and $Y \mapsto I_X(Y)$.*

Proof. See Homework 2. $\square$

3.2 The Zariski Topology

Definition 3.4. The *Zariski topology* on $\mathbb{A}^n_k$ is the topology with closed sets $V(I) \subseteq \mathbb{A}^n_k$, where I is an ideal in $k[x_1, \ldots, x_n]$. (Equivalently, the closed sets are the affine varieties in $\mathbb{A}^n_k$.)

Remark. Note the following:

1. On $\mathbb{A}^1_k$, the closed sets are of the form: $\varnothing$, $\mathbb{A}^1_k$, or finite collections of points.
2. When $k = \mathbb{C}$, then $X \subseteq \mathbb{A}^n_{\mathbb{C}}$ being Zariski closed implies that X is closed in the analytic topology on $\mathbb{A}^n_{\mathbb{C}}$. In particular, the Zariski topology is coarser than the analytic topology.
3. On $\mathbb{A}^2_k$, the closed sets are of the form: $\varnothing$, $\mathbb{A}^2_k$, finite collections of points, plane curves, and their finite unions.

Proposition 3.3. *The Zariski topology on $\mathbb{A}^n_k$ is indeed a topology.*

Proof. First note that $\varnothing = V((1))$ and $\mathbb{A}^n_k = V((0))$ are closed. For arbitrary intersections, note that $\bigcap_\alpha V(I_\alpha) = V(\sum_\alpha I_\alpha)$, and for finite unions, note that $\bigcup_{i=1}^r V(I_i) = V(I_1 \cdots I_r)$. $\square$

Example 3.4.1. The Zariski topology on $\mathbb{A}^{n+m}_k$ is in general *not* the product topology of the Zariski topologies on $\mathbb{A}^n_k$ and $\mathbb{A}^m_k$. Consider $V(y - x^2) \subseteq \mathbb{A}^2_k$, which is a closed set in the Zariski topology, but the only closed sets in $\mathbb{A}^1_k$ are either $\varnothing$, $\mathbb{A}^1_k$, or finite.

Definition 3.5. If $X \subseteq \mathbb{A}^n_k$ is an affine variety, then we can define the *Zariski topology* on X in the following two equivalent ways:

1. take the subspace topology from the Zariski topology on $\mathbb{A}^n_k$;
2. take the closed sets of X to be of the form $V_X(I)$ for some ideal $I \leq A(X)$.

This is because an affine subvariety of X is precisely the intersection of X with an affine variety in $\mathbb{A}^n_k$.

Remark. Our goal now is to relate properties of the Zariski topology on X to the ring $A(X)$, and then to the ideal $I(X) \leq k[x_1, \ldots, x_n]$.

Definition 3.6. A topological space X is *reducible* if we can write $X = X_1 \cup X_2$ for some closed sets $X_1, X_2 \subsetneq X$. Otherwise, X is called *irreducible*.

Example 3.6.1. The plane curve $X = V(y^2 - x^2 y) = V(y) \cup V(y - x^2)$ is reducible.

Remark. Note the following:

1. A disconnected topological space is reducible.
2. Many topologies are reducible, e.g. $\mathbb{C}^n$, $\mathbb{R}^n$ with the analytic topology.
3. If X is irreducible and $U \subseteq X$ is a nonempty open set, then $\overline{U} = X$ (we have $\overline{U} \cup (X \setminus U) = X$).

Lecture 4

Aug. 28 — Irreducibility

4.1 Properties of Irreducibility

Proposition 4.1. *Let $X \subseteq \mathbb{A}^n$ be an affine variety. Then the following are equivalent:*

1. *X is irreducible;*
2. *$I(X) \leq k[x_1, \ldots, x_n]$ is a prime ideal;*
3. *the coordinate ring $A(X)$ is an integral domain.*

Example 4.0.1. We have the following:

1. $\mathbb{A}_k^n$ is irreducible as $A(\mathbb{A}_k^n) = k[x_1, \ldots, x_n]$, which is an integral domain.
2. A *hypersurface* $X \subseteq \mathbb{A}_k^n$ is an affine variety with $I(X) = (f)$ for some $f \in k[x_1, \ldots, x_n]$. Then A is irreducible if and only if (f) is prime, if and only if f is irreducible.[1]

4.2 Dimension

Definition 4.1. Let X be a topological space.

- The *dimension* of X, denoted $\dim X$, is the supremum of the n such that there exists a chain of irreducible closed subspaces
$$X \supseteq X_0 \supsetneq X_1 \supsetneq \cdots \supsetneq X_n \neq \varnothing.$$
- For $Y \subseteq X$ closed and irreducible, the *codimension* of Y in X, denoted $\operatorname{codim}_X Y$, is the supremum of the n as above such that $X_n = Y$.

[1]Note that any prime ideal is radical.

Lecture 5

Sept. 2 — Dimension

5.1 More on Dimension

Remark. Recall the following correspondence from before: If $X \subseteq \mathbb{A}_k^n$ is an affine variety, then there exists a bijection between the irreducible closed subsets $Y \subseteq X$ and the prime ideals $\mathfrak{p} \leq A(X)$.

Definition 5.1. For a ring A, the *(Krull) dimension* of A, denoted $\dim A$, is the supremum of the n such that there exists a chain of prime ideals

$$A \supseteq \mathfrak{p}_0 \supsetneq \mathfrak{p}_1 \supsetneq \cdots \supsetneq \mathfrak{p}_n.$$

For a prime ideal $\mathfrak{q} \leq A$, the *height* of $\mathfrak{q}$, denoted $\operatorname{ht} \mathfrak{q}$, is the supremum of the n as above with $\mathfrak{p}_0 = \mathfrak{q}$.

Remark. If X is an affine variety, then we have the following:

1. $\dim X = \dim A(X)$;
2. for $Y \subseteq X$ a closed irreducible subset, $\operatorname{codim}_X Y = \operatorname{ht} I_X(Y)$.

These properties follow from the inclusion-reversing correspondence.

Definition 5.2. Let $K \subseteq L$ be a field extension.

1. A collection of elements $\{z_i : i \in I\} \subseteq L$ is a *transcendence basis* of $K \subseteq L$ if the z_i are algebraically independent (i.e. $K(x_i : i \in I) \xrightarrow{\cong} K(z_i : i \in I)$ by $x_i \mapsto z_i$) and $K(z_i : i \in I) \subseteq L$ is algebraic.
2. The *transcendence degree* $\operatorname{tr.deg}_K L$ is the cardinality of a transcendence basis.

Theorem 5.1 (Dimension theory)**.** *Let A be a finitely generated k-algebra that is a domain. Then*

1. $\dim A = \operatorname{tr.deg}_k \operatorname{Frac}(A)$*;*
2. *for any prime ideal $\mathfrak{p} \leq A$, we have $\operatorname{ht} \mathfrak{p} + \dim A/\mathfrak{p} = \dim A$;*
3. *all maximal chains of prime ideals $A \supseteq \mathfrak{p}_0 \supsetneq \mathfrak{p}_1 \supsetneq \cdots \supsetneq \mathfrak{p}_n$ are of the same length.*

Remark. The following are consequences of the above result from commutative algebra:

1. $\dim_k \mathbb{A}_k^n = \dim k[x_1, \ldots, x_n] = \operatorname{tr.deg}_k k(x_1, \ldots, x_n) = n$.
2. If X is irreducible, then $A(X)$ is a domain, so for $x \in X$, we have

$$\operatorname{codim}_X\{x\} = \operatorname{ht} I(\{x\}) = \dim A(X) - \dim A(X)/I(\{x\}) = \dim A(X) = \dim X,$$

where we note that $A(X)/I(\{x\}) \cong k$ is a field.

3. If X is an irreducible affine variety and $U \subseteq X$ is a nonempty open subset, then
$$\dim U = \sup_{x \in U} \operatorname{codim}_U\{x\} = \sup_{x \in U} \operatorname{codim}_X\{x\} = \dim X.$$
This follows since we can pass from a chain in U to a chain in X by taking closures.

4. If X is an irreducible affine variety and $Z \subseteq X$ is an irreducible closed subset, then
$$\dim Z = \dim X - \operatorname{codim}_X Z.$$

Note that (2)-(4) can be false if X is not irreducible. To contradict (4), let $X = V(x,y) \cup V(z) \subseteq \mathbb{A}_k^3$ with $Z = V(x,y)$. Then we have $\dim X = 2$, $\dim Z = 1$, $\operatorname{codim}_X Z = 0$.

5.2 Hypersurfaces

Remark. We now want to study hypersurfaces.

Theorem 5.2 (Krull's Hauptidealsatz)**.** *If A is a Noetherian ring and $f \in A$ is nonzero and a non-unit, then every minimal prime ideal containing f has height* 1*.*

Corollary 5.2.1. *If $X \subseteq \mathbb{A}_k^n$ is an irreducible affine variety and $f \in A(X)$ is a nonzero non-unit, then*
$$\dim Z = \dim X - 1$$
for every irreducible component Z of $V_X(f)$.

Proof. Since X is irreducible, $A(X)$ is a domain. So there is a correspondence between the minimal prime ideals $f \in \mathfrak{p} \subsetneq A(X)$ and the minimal irreducible closed subsets $Z \supseteq V_X(f)$, which corresponds to the irreducible components Z of $V_X(f)$. For such a component Z, we know
$$\dim Z = \dim Z - \operatorname{codim}_X Z = \dim X - \operatorname{ht} I(Z) = \dim X - 1$$
by Krull's Hauptidealsatz, which is the desired result. □

Example 5.2.1. Corollary 5.2.1 implies that if $f \in k[x_1, \ldots, x_n]$ is non-constant, then
$$\dim V(f) = \dim \mathbb{A}_k^n - 1 = n - 1.$$

Theorem 5.3. *An irreducible affine variety $Y \subseteq \mathbb{A}_k^n$ has $\dim Y = n-1$ if and only if $Y = V(f)$ for some non-constant polynomial $f \in k[x_1, \ldots, x_n]$.*

Proof. ($\Leftarrow$) This was Corollary 5.2.1.

($\Rightarrow$) We will use that $A(\mathbb{A}_k^n) = k[x_1, \ldots, x_n]$ is a UFD. Since Y is irreducible and $\dim Y = n-1$,
$$\operatorname{ht} I(Y) = \operatorname{codim}_{\mathbb{A}_k^n} Y = \dim \mathbb{A}_k^n - \dim Y = 1.$$
Since $(0) \subsetneq I(Y) \subsetneq k[x_1, \ldots, x_n]$, there exists a non-constant $f \in k[x_1, \ldots, x_n]$ with $f \in I(Y)$. Write
$$f = f_1 \cdots f_r$$
with f_i irreducible by unique factorization, and note that the f_i are also prime since we are in a UFD. Since $I(Y)$ is prime, some f_i is in $I(Y)$, so we have the inclusions
$$(0) \subsetneq (f_i) \subseteq I(Y).$$
Since $\operatorname{ht} I(Y) = 1$, we must have $(f_i) = I(Y)$, so $Y = V(I(Y)) = V(f_i)$. □

5.3 Regular Functions

Definition 5.3. Let X be an affine variety and $U \subseteq X$ open. A function $\varphi : U \to k$ is *regular* if for each $a \in U$, there exists an open neighborhood $a \in U_a \subseteq U$ and $f, g \in A(X)$ such that

$$\varphi(x) = \frac{g(x)}{f(x)}, \quad f(x) \neq 0, \quad \text{for all } x \in U_a.$$

Define $\mathcal{O}_X(U) = \{\varphi : U \to k \mid \varphi \text{ is a regular function on } U\}$.

Exercise 5.1. Check that $\mathcal{O}_X(U)$ is a ring under pointwise addition and multiplication of outputs.

Remark. To patch open sets together, we will later need the notion of a *morphism*, and a morphism $U \to Y \subseteq \mathbb{A}_k^m$ should be given by

$$x \longmapsto (\varphi_1(x), \ldots, \varphi_m(x))$$

with φ_i regular functions on U.

Example 5.3.1. We have the following:

1. If $X \subseteq \mathbb{A}_k^n$ is an affine variety, then any $\varphi \in A(X)$ is regular. Furthermore, we get an injective ring homomorphism $A(X) \to \mathcal{O}_X(X)$. We will see that this is an isomorphism.
2. If $X = \mathbb{A}_k^1$ and $U = \mathbb{A}_k^1 \setminus \{0\}$, then for any $n \geq 0$ and $g \in k[x]$, the function g/x^n is regular on U. In general, if we fix $f, g \in A(X)$ and set $U = X \setminus V(f)$, then the map g/f^m is regular on U.
3. Let $X = V(x_1x_4 - x_2x_3) \subseteq \mathbb{A}_k^4$ and $U = X \setminus V(x_2, x_4)$. Then the following map is regular:

$$\begin{aligned} \varphi : U &\longrightarrow k \\ (x_1, x_2, x_3, x_4) &\longmapsto \begin{cases} x_1/x_2, & \text{if } x_2 \neq 0, \\ x_3/x_4, & \text{if } x_4 \neq 0. \end{cases} \end{aligned}$$

   Note that on $U \setminus V(x_2x_4)$, we have $x_1/x_2 = x_3/x_4$ since $x_1x_4 = x_2x_3$ on X.

Lecture 6

Sept. 4 — Regular Functions

6.1 Properties of Regular Functions

Proposition 6.1. *Let X be an affine variety and $U \subseteq X$ open. Then:*

1. *if $\varphi \in \mathcal{O}_X(U)$, then $V(\varphi) = \{x \in U : \varphi(x) = 0\}$ is closed in U;*
2. *(identity principle) If X is irreducible, $U \subseteq X$ is nonempty and open, and $\varphi, \psi \in \mathcal{O}_X(U)$ with $\varphi|_W = \psi|_W$ for some $W \subseteq U$ nonempty and open, then $\varphi = \psi$ in $\mathcal{O}_X(U)$.*

Proof. (1) It suffices to show that $U \setminus V(\varphi)$ is open in U. Fix $a \in U \setminus V(\varphi)$. Since φ is regular, there exists an open neighborhood $a \in U_a \subseteq U$ and $f_a, g_a \in A(X)$ such that

$$\varphi|_{U_a} = \frac{g_a}{f_a}.$$

So $a \in \{g_a \neq 0\} \cap U_a \subseteq U \setminus V(\varphi)$. This is an open set containing a in $U \setminus V(\varphi)$, so $U \setminus V(\varphi)$ is open.

(2) Since X is irreducible, U is also irreducible. The locus $\{x \in U : \varphi(x) = \psi(x)\} = V(\varphi - \psi)$ is closed in U by (1). It also contains W. Since W is dense (it is a nonempty open set in an irreducible topological space), we must have $V(\varphi - \psi) = U$. This proves the claim. $\square$

Example 6.0.1. In (2) of Proposition 6.1, the assumption that X is irreducible is necessary. Consider

$$U = X = V(xy) \subseteq \mathbb{A}^2_k \quad \text{and} \quad W = V(xy) \setminus V(x).$$

Then the regular functions $\varphi = x$ and $\psi = x + y$ agree on W but are not equal on U.

6.2 Distinguished Open Sets

Remark. We will see that an affine variety has a basis of open sets on which we can compute $\mathcal{O}_X(U)$.

Definition 6.1. A *distinguished open set* of an affine variety X is a subset of the form

$$D(f) = X \setminus V(f)$$

for some polynomial function $f \in A(X)$.

Remark. We have the following:

1. The $D(f)$ are closed under (finite) intersection: $D(fg) = D(f) \cap D(g)$.

2. The $D(f)$ form a basis for the Zariski topology on X: If $U \subseteq X$ is open, then $U = X \setminus V(f_1, \ldots, f_r)$ for some $f_1, \ldots, f_r \in A(X)$ (since X is Noetherian). So $U = D(f_1) \cup \cdots \cup D(f_r)$.

Remark. We will view $D(f)$ as "small open sets" (under mild assumptions, $\operatorname{codim}_X(X \setminus D(f)) = 1$).

Theorem 6.1. *If X is an affine variety and $f \in A(X)$, then*

$$\mathcal{O}_X(D(f)) = \left\{ \frac{g}{f^m} : g \in A(X), m \geq 0 \right\}.$$

Proof. We have an injective ring homomorphism

$$\left\{ \frac{g}{f^m} : g \in A(X), m \geq 0 \right\} \longrightarrow \mathcal{O}_X(D(f)),$$

it suffices to show this map is surjective. Fix $\varphi \in \mathcal{O}_X(D(f))$. For any $a \in D(f)$, there exists an open neighborhood $a \in U_a \subseteq D(f)$ and $f_a, g_a \in A(X)$ such that $\varphi|_{U_a} = g_a/f_a$. We may further assume that

1. $U_a = D(h_a)$ for some $h_a \in A(X)$ (by shrinking U_a if necessary, since the $D(h)$ form a basis);
2. $h_a = f_a$ (by rewriting $g_a/f_a = g_a h_a / f_a h_a$ and replacing h_a, f_a with $f_a h_a$).

Then for $a, b \in D(f)$, we have $f_a g_b = f_b g_a$ on $D(f_a) \cap D(f_b)$. Since both the left and right hand sides vanish on $X \setminus (D(f_a) \cap D(f_b))$, we have $f_a g_b = f_b g_a$ in $A(X)$. Now we can write

$$V(f) = \bigcap_{a \in D(f)} V(f_a) = V(f_a : a \in D(f)),$$

so $f \in I(V(f_a : a \in D(f)))$. By the Nullstellensatz, there exists $n \geq 0$ such that

$$f^n = \sum_{a \in D(f)} k_a f_a, \quad k_a \in A(X),$$

where only finitely many of the k_a are nonzero. Set $g = \sum_{a \in D(f)} k_a g_a$, and we claim that $\varphi = g/f^n$. To see this, note that on U_b, we have $\varphi|_{U_b} = g_b/f_b$. Now since $f_a g_b = f_b g_a$, we have

$$g f_b = \sum_{a \in D(f)} k_a g_a f_b = \sum_{a \in D(f)} k_a f_a g_b = f^n g_b,$$

which shows that $\varphi|_{U_b} = (g/f^n)|_{U_b}$. Since this holds for any U_b, we have $\varphi = g/f^n$ in $\mathcal{O}_X(D(f))$. $\square$

Remark. Theorem 6.1 has the following consequences:

1. The $f = 1$ case implies that the natural ring homomorphism $A(X) \to \mathcal{O}_X(X)$ is surjective and hence an isomorphism (note that $D(1) = X$).
2. We will see that $\mathcal{O}_X(D(f)) \cong A(X)_f$, the *localization* of $A(X)$ at f.

Example 6.1.1. How do we compute $\mathcal{O}_X(U)$ on non-distinguished open sets? Consider

$$X = \mathbb{A}^2_k \quad \text{and} \quad U = \mathbb{A}^2_k \setminus \{(0,0)\}.$$

Note that U is never a distinguished open set. We claim that the ring homomorphism

$$k[x,y] \longrightarrow \mathcal{O}_{\mathbb{A}^2_k}(\mathbb{A}^2_k \setminus \{(0,0)\})$$

is an isomorphism. The map is injective by the identity principle, so it suffices to show surjectivity. The strategy is use $U = D(x) \cup D(y)$ (in general, cover U by basis elements). Fix $\varphi : U \to k$ regular, so

$$\varphi|_{D(x)} = \frac{f}{x^m} \quad \text{for some } f \in k[x,y], m \geq 0$$
$$\varphi|_{D(y)} = \frac{g}{y^n} \quad \text{for some } g \in k[x,y], n \geq 0.$$

Since we are in a UFD, we may assume that $x \nmid f$ and $y \nmid g$. Now $fy^n = gx^m$ on $D(y) \cap D(x)$, so by the identity principle, $fy^n = gx^m$ on $\mathbb{A}^2_k$, so $fy^n = gx^m$ in $k[x,y]$. Using that $y \nmid g$, $x \nmid f$, and that $k[x,y]$ is a UFD, we must have $n = m = 0$, hence $f = g$. In particular, we have

$$\varphi|_{D(x)} = \varphi|_{D(y)} = f,$$

so the map $k[x,y] \to \mathcal{O}_X(U)$ is surjective.

6.3 Localization

Remark. We want to invert a subset of a ring, in particular *multiplicative systems.*

Definition 6.2. A *multiplicative system* of a ring A is a subset such that

1. $1 \in S$;
2. S is closed under multiplication.

Example 6.2.1. The following examples of S are multiplicative systems:

1. $S = A$ or $S = \{1\}$;
2. if $\mathfrak{p} \leq A$ is a prime ideal, then $S = A \setminus \mathfrak{p}$;
3. if $f \in A$, then $S = \{f^m : m \geq 0\}$.

Definition 6.3. The *localization* of a ring A at a multiplicative system S is the ring

$$S^{-1}A = \left\{ \frac{a}{s} : a \in A, s \in S \right\} / \sim$$

where the a/s are formal symbols with $a/s \sim a'/s'$ if $t(as' - a's) = 0$ for some $t \in S$.[1] The operations are given by the usual addition and multiplication of fractions:

$$\frac{a}{s} \cdot \frac{a'}{s'} = \frac{aa'}{ss'} \quad \text{and} \quad \frac{a}{s} + \frac{a'}{s'} = \frac{as' + a's}{ss'}.$$

Check as an exercise that these operations respect the equivalence relation.

Example 6.3.1. The following are examples of localization:

1. If A is a domain and $S = A \setminus \{0\}$, then $S^{-1}A = \operatorname{Frac} A$.
2. If $S = \langle f \rangle = \{1, f, f^2, \dots\}$, then we will write $A_f = S^{-1}A$.
3. If $S = A \setminus \mathfrak{p}$ for a prime ideal $\mathfrak{p}$, then we will write $A_\mathfrak{p} = S^{-1}A$.

[1] Note that if A is a domain and $0 \notin S$, then this condition is equivalent to $as' = a's$.

Proposition 6.2. *We have the following properties of localization:*

1. *(Universal property of localization) For any ring homomorphism $\varphi : A \to B$ such that $\varphi(s)$ for all $s \in S$, then there exists a unique ring homomorphism which makes the following diagram commute:*

$$\begin{tikzcd} A \arrow[rr, "\varphi"] \arrow[dr, "\pi : a \mapsto a/1"'] & & B \\ & S^{-1}A \arrow[ur, dashed, "\exists!"'] & \end{tikzcd}$$

2. *There is a bijection between the prime ideals $\mathfrak{p} \leq A$ with $\mathfrak{p} \cap S = \varnothing$ and the prime ideals $\mathfrak{q} \leq S^{-1}A$ given by $\mathfrak{p} \mapsto \pi(\mathfrak{p})S^{-1}A$ with inverse $\mathfrak{q} \mapsto \pi^{-1}(\mathfrak{q})$, where $\pi : A \to S^{-1}A$ is the map $a \mapsto a/1$.*

Remark. In more generality, for an A-module M, we can define the localization $S^{-1}M$, which is an $S^{-1}A$-module. This gives a functor $\mathrm{Mod}_A \to \mathrm{Mod}_{S^{-1}A}$ which is exact.

Lecture 7

Sept. 9 — Germs and Sheaves

7.1 More on Localization

Proposition 7.1. *If X is an affine variety and $f \in A(X)$ is nonzero, then $\mathcal{O}_X(D(f)) \cong A(X)_f$.*

Proof. We define a ring homomorphism as follows:

$$A(X)_f \longrightarrow \mathcal{O}_X(D(f))$$
$$\frac{g}{f^m} \longmapsto \left(x \mapsto \frac{g(x)}{f^m(x)}\right).$$

To check that this is well-defined, assume $g/f^m \sim h/f^n$ in $A(X)_f$. So there exists $k \geq 0$ such that

$$f^k(gf^n - hf^m) = 0 \quad \text{in } A(X).$$

So $gf^n - hf^m = 0$ as functions $D(f) \to k$, so $g/f^m = h/f^n$ as functions $D(f) \to k$. Thus their images agree in $\mathcal{O}_X(D(f))$, so the map is well-defined.

Surjectivity follows from the argument from last time. For injectivity, assume $g/f^m = 0$ as functions $D(f) \to k$ with $g \in A(X)$. Then $fg = 0$ in $A(X)$, so $g/f^m \sim 0/1$ in $A(X)_f$. □

7.2 Germs of Functions

Definition 7.1. Let $p \in X$ be a point on an affine variety.

1. A *germ* of a regular function of X at p is a pair (U, f) such that $x \in U \subseteq X$ is open and f is a regular function $U \to k$, up to the equivalence relation $(U, \varphi) \sim (V, \psi)$ if there exists an open set $x \in W \subseteq U \cap V$ such that $\varphi|_W = \psi|_W$.
2. Define $\mathcal{O}_{X,p} = \{\text{germs of regular functions of } X \text{ at } p\}$.

Exercise 7.1. Check that $\mathcal{O}_{X,p}$ is a ring with operations

$$(U, \varphi) \cdot (V, \psi) = (U \cap V, \varphi|_{U\cap V} \cdot \psi|_{U\cap V}),$$
$$(U, \varphi) + (V, \psi) = (U \cap V, \varphi|_{U\cap V} + \psi|_{U\cap V}),$$

with the zero function as the zero element and the constant 1 function as the unit element.

Lemma 7.1. *$\mathcal{O}_{X,p}$ is a local ring with unique maximal ideal $\mathfrak{m}_p = \{(U, \varphi) \in \mathcal{O}_{X,p} : \varphi(p) = 0\}$.*

Proof. It suffices to show that the units of $\mathcal{O}_{X,p}$ are precisely $\mathcal{O}_{X,p} \setminus \mathfrak{m}_p$. To see the reverse inclusion, fix $(U, \varphi) \in \mathcal{O}_{X,p}$ with $\varphi(p) \neq 0$. So there exists an open neighborhood $p \in W \subseteq U$ such that $\varphi|_W$ never vanishes. Then

$$(U, \varphi) \cdot (W, 1/\varphi|_W) = (W, \varphi|_W) \cdot (W, 1/\varphi|_W)) = (W, 1),$$

so (U, φ) is a unit in $\mathcal{O}_{X,p}$. The forward inclusion is similar. □

Proposition 7.2. *With the above setup, there is an isomorphism*

$$\begin{aligned} A(X)_{I(p)} &\longrightarrow \mathcal{O}_{X,p} \\ \frac{f}{g} &\longmapsto \left(D(g), x \mapsto \frac{f(x)}{g(x)}\right) \end{aligned}$$

with $I(p) = \{f \in A(X) : f(p) = 0\}$.

Proof. To see that this is well-defined, let $f/g \sim f'/g' \in A(X)_{I(p)}$. Then $h(fg' - f'g) = 0$ for some $h \in A(X)$ with $h(p) \neq 0$. So $f/g = f'/g'$ as functions $D(h) \cap D(g) \to k$, which means that $f/g = f'/g'$ as elements in $\mathcal{O}_{X,p}$. Thus the map is well-defined.

Injectivity is similar to before. For surjectivity, choose $(U, \varphi) \in \mathcal{O}_{X,p}$. Since $\varphi : U \to k$ is a regular function, there exists an open set $p \in U_p \subseteq U$ and $f, g \in A(X)$ such that g does not vanish on U_p and $\varphi(x) = f(x)/g(x)$ for all $x \in U_p$. So $(U, \varphi) \sim (D(g), f/g)$ in $\mathcal{O}_{X,p}$, i.e. (U, φ) is in the image. □

Example 7.1.1. If $X = \mathbb{A}^n_k$ and $p = 0$, then

$$\mathcal{O}_{\mathbb{A}^n_k,0} \cong k[x_1, \ldots, x_n]_{(x_1,\ldots,x_n)} = \left\{ \frac{f}{g} : f \in k[x_1, \ldots, x_n], g \in k[x_1, \ldots, x_n] \setminus (x_1, \ldots, x_n) \right\}.$$

Remark. We will relate the local properties of X at p to properties of $\mathcal{O}_{X,p}$. We will use the following statements from commutative algebra: Let A be a ring and $\mathfrak{p} \subseteq A$ a prime ideal. Then

1. $A_\mathfrak{p}$ is a local ring with unique maximal ideal $\mathfrak{p}A_\mathfrak{p}$.
2. There is a bijection from the prime ideals of $A_\mathfrak{p}$ to the prime ideals of A contained in $\mathfrak{p}$.
3. $\operatorname{ht}_A \mathfrak{p} = \dim A_\mathfrak{p}$ (this follows from (2)).

This has the following consequence: If X is an affine variety and $p \in X$, then

$$\operatorname{codim}_X\{p\} = \operatorname{ht}_{A(X)} I(p) = \dim A(X)_{I(p)} = \dim \mathcal{O}_{X,p}.$$

7.3 Sheaves

Remark. We will now formalize the structures $\mathcal{O}_X(U)$ and $\mathcal{O}_{X,p}$ that we have seen before.

Definition 7.2. A *presheaf (of rings)* $\mathcal{F}$ on a topological space X is the data of

1. for every open set $U \subseteq X$, a ring $\mathcal{F}(U)$;
2. for every inclusion of open sets $U \subseteq V \subseteq X$, a ring homomorphism $\rho_{V,U} : \mathcal{F}(V) \to \mathcal{F}(U)$

satisfying the following properties:

1. $\mathcal{F}(\varnothing) = 0$;
2. $\rho_{U,U}$ is the identity map;
3. for inclusions of open sets $U \subseteq V \subseteq W \subseteq X$, we have $\rho_{W,U} = \rho_{V,U} \circ \rho_{W,V}$.

Example 7.2.1. If X is an affine variety, then $\mathcal{O}_X$ gives a presheaf of rings with

1. for $U \subseteq X$, the ring is $\mathcal{O}_X(U) = \{\text{regular functions } \varphi : U \to k\}$;
2. for $U \subseteq V \subseteq X$, the map $\mathcal{O}_X(V) \to \mathcal{O}_X(U)$ is given by $\varphi \mapsto \varphi|_U$.

Remark. We often call $s \in \mathcal{F}(U)$ a *section*, and for $U \subseteq V$, we call $s|_U = \rho_{V,U}(s)$ the *restriction*.

Remark. A presheaf is the same thing as a functor $\mathrm{Open}_X^{\mathrm{op}} \to \mathrm{Rings}$, where Open_X is the category with objects the nonempty open sets of X and morphisms corresponding to the inclusions $U \subseteq V$.

Definition 7.3. A presheaf $\mathcal{F}$ on X is a *sheaf* if it satisfies the *gluing property*: For any $U \subseteq X$ open, an open cover $\{U_i\}_{i \in I}$ of U, and $\varphi_i \in \mathcal{F}(U_i)$ with $\varphi_i|_{U_i \cap U_j} = \varphi_j|_{U_i \cap U_j}$ for all $i, j \in I$, there exists a unique $\varphi \in \mathcal{F}(U)$ such that $\varphi|_{U_i} = \varphi_i$ for all $i \in I$.

Example 7.3.1. We have the following:

1. If X is an affine variety, then $\mathcal{O}_X$ is a sheaf (if we take $\varphi_i \in \mathcal{O}_X(U_i)$ that agree on the overlaps, then we get $\varphi : U \to k$, which is regular since regularity is a local property).
2. If M is a smooth manifold, then we can define a sheaf (on open subsets $U \subseteq M$) by
$$U \longmapsto \mathcal{F}^{\mathrm{sm}}(U) = \{\text{smooth functions } U \to \mathbb{R}\}.$$
We may also consider $\mathcal{F}^{\mathrm{cont}}$, $\mathcal{F}^{\mathrm{diff}}$, $\mathcal{F}^{\mathrm{loc,const}}$, etc. However, $\mathcal{F}^{\mathrm{const}}$ is a presheaf, but not a sheaf in general: We can take $U = U_1 \cup U_2$ with $U_1 \cap U_2 = \varnothing$, and we will only get a locally constant function. Similarly, $\mathcal{F}^{\mathrm{bounded}}$ is only a presheaf but not a sheaf.
3. If $\mathcal{F}$ is a sheaf on a topological space X and $U \subseteq X$ is open, then we get a sheaf $\mathcal{F}|_U$ on U defined by $\mathcal{F}|_U(V) = \mathcal{F}(V)$ for $V \subseteq U$ open.

Definition 7.4. The *stalk* of a sheaf $\mathcal{F}$ on a topological space X at $x \in X$ is
$$\mathcal{F}_x = \{(U, \varphi) : U \subseteq X \text{ open and } \varphi \in \mathcal{F}(U)\}/\sim,$$
where $(U, \varphi) \sim (V, \psi)$ if there exists an open set $x \in W \subseteq U \cap V$ such that $\varphi|_W = \psi|_W$.

Example 7.4.1. If X is an affine variety and $p \in X$, then $\mathcal{O}_{X,p} \cong (\mathcal{O}_X)_p$.

Remark. As before with $\mathcal{O}_{X,p}$, one can check that $\mathcal{F}_x$ naturally has the structure of a ring.

Remark. An alternative perspective is to define the stalk as a direct limit:
$$\mathcal{F}_x = \varinjlim_{U \ni x} \mathcal{F}(U),$$
where the limit is taken over all open $x \in U \subseteq X$ with respect to the ordering $U \leq V$ if $V \subseteq U$.

Lecture 8

Sept. 11 — Morphisms

8.1 Morphisms of Open Sets

Remark. Recall that a continuous map $f : \mathbb{R}^m \to \mathbb{R}^n$ is *smooth* if it satisfies either of the following equivalent conditions:

1. there exist smooth functions $f_1, \ldots, f_n : \mathbb{R}^m \to \mathbb{R}$ such that $f(x) = (f_1(x), \ldots, f_n(x))$;
2. for each open set $U \subseteq \mathbb{R}^n$ and smooth $\varphi : U \to \mathbb{R}$, the function $f^*\varphi := \varphi \circ f : \mathbb{R}^m \to \mathbb{R}$ is smooth.

The implication $(1 \Rightarrow 2)$ follows by the chain rule. To see $(2 \Rightarrow 1)$, take $y_i : \mathbb{R}^n \to \mathbb{R}$ defined by $f_i := f^*y_i$. We want a similar definition in algebraic geometry.

Definition 8.1. Let X and Y be open sets of affine varieties. A *morphism* $f : X \to Y$ is a continuous map such that for every $U \subseteq Y$ open and $\varphi \in \mathcal{O}_Y(U)$, the map

$$\begin{tikzcd} f^{-1}(U) \arrow[r, "f"] \arrow[rr, bend left, "f^*\varphi"] & U \arrow[r, "\varphi"] & k \end{tikzcd}$$

satisfies $f^*\varphi \in \mathcal{O}_X(f^{-1}(U))$. A morphism is an *isomorphism* if it has a two-sided inverse (equivalently, f is a bijection and f^{-1} is a morphism).

Remark. We have the following properties of morphisms:

1. (Composition) If $f : X \to Y$ and $g : Y \to Z$ are morphisms of open sets of affine varieties, then so is $g \circ f : X \to Z$.
2. (Local on target) If $X \to Y$ is a map of open sets of affine varieties such that there exists an open cover $\{U_i\}_{i \in I}$ of Y with $f|_{f^{-1}(U_i)} : f^{-1}(U_i) \to U_i$ a morphism for all $i \in I$, then f is a morphism.

Proposition 8.1. *Let $X \subseteq \mathbb{A}^m$ and $Y \subseteq \mathbb{A}^n$ be affine varieties. Let $U \subseteq X$ and $V \subseteq Y$ be open sets. A map $f : U \to V$ is a morphism if and only if there exist $\varphi_1, \ldots, \varphi_n \in \mathcal{O}_X(U)$ such that*

$$f(x) = (\varphi_1(x), \ldots, \varphi_n(x)).$$

Proof. $(\Rightarrow)$ Let $U \subseteq \mathbb{A}^m_{x_i}$ and $V \subseteq \mathbb{A}^n_{y_i}$. By the definition of a morphism, $y_i : V \to k$ satisfies

$$\varphi_i := f^*y_i \in \mathcal{O}_X(U),$$

so we can write $f(x) = (\varphi_1(x), \ldots, \varphi_n(x))$.

$(\Leftarrow)$ Assume there exist $\varphi_1, \ldots, \varphi_n \in \mathcal{O}_X(U)$ such that $f(x) = (\varphi_1(x), \ldots, \varphi_n(x))$.

We first show that f is continuous. Let $Z \subseteq V$ be a closed set. So we can write $Z = V(g_r, \ldots, g_r)$ for some $g_1, \ldots, g_r \in A(\mathbb{A}^n) \cong k[y_1, \ldots, y_n]$. Now we have

$$\begin{aligned} f^{-1}(Z) = \{x \in U : f(x) \in Z\} &= \{x \in U : g_i(f(x)) = 0 \text{ for } i = 1, \ldots, r\} \\ &= \{x \in U : (f^* g_i)(x) = 0 \text{ for } i = 1, \ldots, r\}. \end{aligned}$$

Note that $f^* g_i = g_i(\varphi_1, \ldots, \varphi_n)$, which is regular since a composition of a polynomial with fractions of polynomials is again a fraction of polynomials. So $f^{-1}(Z)$ is closed in U.

Now to show that f is a morphism, it suffices to show that for any $W \subseteq Y$ open and $\varphi \in \mathcal{O}_Y(W)$, we have $f^*\varphi \in \mathcal{O}_X(f^{-1}(W))$. The proof of this is similar to before. □

Example 8.1.1. We have the following:

1. Morphisms $\mathbb{A}^m \to \mathbb{A}^n$ are of the form

$$x \longmapsto (f_1(x), \ldots, f_n(x))$$

   with $f_1, \ldots, f_n \in \mathcal{O}_{\mathbb{A}^m}(\mathbb{A}^m) = k[x_1, \ldots, x_m]$.

2. Write $\mathbb{A}^1_t$ to mean $\mathbb{A}^1$ with variable t. Then we can define $\mathbb{A}^1_t \to V(y - x^2) \subseteq \mathbb{A}^2_{x,y}$ by $t \mapsto (t, t^2)$. We can get an inverse $V(y - x^2) \to \mathbb{A}^1_t$ by $(x, y) \mapsto x$, so $\mathbb{A}^1_t$ and $V(y - x^2)$ are isomorphic.

3. Consider the map $g : \mathbb{A}^1_t \to V(x^2 - y^3) \subseteq \mathbb{A}^2_{x,y}$ given by $t \mapsto (t^3, t^2)$. This map is bijective, but it is not an isomorphism. To see this, we can show that $(g^{-1})^*\varphi$ is not regular for some regular function φ on $\mathbb{A}^t_1$. For instance, we can take $\varphi = t$, so that

$$(g^{-1})^*(t) = (x, y) \mapsto \begin{cases} x/y & \text{if } y \neq 0, \\ 0 & \text{otherwise}, \end{cases}$$

   which we can see is not regular.

8.2 Relation to Coordinate Rings

Remark. Let $X \subseteq \mathbb{A}^m$ and $Y \subseteq \mathbb{A}^n$ be affine varieties. Then a morphism $f : X \to Y$ of affine varieties induces a k-algebra morphism (called the *pullback* of f)

$$\begin{aligned} f^* : A(Y) &\longrightarrow A(X) \\ \varphi &\longmapsto f^* = \varphi \circ f \end{aligned}$$

with the properties $(g \circ f)^* = f^* \circ g^*$ and $(\mathrm{id}_X)^* = \mathrm{id}_{A(X)}$, i.e. $X \mapsto A(X)$ is a contravariant functor.

Proposition 8.2. *The following map is a bijection:*

$$\begin{aligned} \mathrm{Hom}_{\mathrm{aff,var}}(X, Y) &\xrightarrow{\Phi} \mathrm{Hom}_{k\text{-alg}}(A(Y), A(X)) \\ f &\longmapsto f^* \end{aligned}$$

Proof. Note that $A(X) \cong k[x_1, \ldots, x_m]/I(X)$ and $A(Y) \cong k[y_1, \ldots, y_n]/I(Y)$. Given a morphism

$$\begin{aligned} f : X &\longrightarrow Y \\ x &\longmapsto (\varphi_1(x), \ldots, \varphi_n(x)), \end{aligned}$$

we can define $f^*\overline{y}_i = \varphi_i$. Conversely, given a k-algebra homomorphism $\phi : A(Y) \to A(X)$, we can set $\varphi_i = \phi(\overline{y}_i)$. Now consider the morphism defined by

$$\begin{aligned} f : X &\longrightarrow \mathbb{A}^n_{y_i} \\ x &\longmapsto (\varphi_1(x), \dots, \varphi_n(x)). \end{aligned}$$

We claim that $f(X) \subseteq Y$. To see this, fix $x \in X$. If $h \in I(Y)$, then

$$h(f(x)) = h(\varphi_1(x), \dots, \varphi_n(x)) = \phi(h)(x) = 0(x) = 0,$$

so $f(X) \subseteq Y$. Thus we get a morphism $f : X \to Y$ by $x \mapsto (\varphi_1(x), \dots, \varphi_n(x))$ with $f^*y_i = \varphi_i$. One can check that this gives a map $\Psi : \mathrm{Hom}_{k\text{-alg}}(A(Y), A(X)) \to \mathrm{Hom}_{\mathrm{aff,var}}(X, Y)$ which is inverse to Φ. □

Example 8.1.2. We have the following:

1. Recall the morphism $g : \mathbb{A}^1_t \to V(y - x^2) \subseteq \mathbb{A}^2_{x,y}$ given by $t \mapsto (t, t^2)$. The pullback is given by

$$\begin{aligned} g^* : \frac{k[x,y]}{(y - x^2)} &\longmapsto k[t] \\ x &\longmapsto t \\ y &\longmapsto t^2. \end{aligned}$$

   Note that g^* is an isomorphism of k-algebras, so g is an isomorphism of affine varieties. This gives an alternative way of seeing this without writing down an inverse to g.

2. Recall the morphism $h : \mathbb{A}^1_t \to V(x^2 - y^3) \subseteq \mathbb{A}^2_{x,y}$ given by $t \mapsto (t^3, t^2)$. The pullback is

$$\begin{aligned} h^* : \frac{k[x,y]}{(x^2 - y^3)} &\longmapsto k[t] \\ x &\longmapsto t^3 \\ y &\longmapsto t^2. \end{aligned}$$

   Note that $t \notin \operatorname{Im} h^*$, so h^* is not an isomorphism, so h is not an isomorphism.

Remark. There is a one-to-one correspondence between affine varieties (up to isomorphism) and finitely generated reduced k-algebras (up to isomorphism).

To see this, observe that if $X \subseteq \mathbb{A}^n$ is an affine variety, then $A(X) \cong k[x_1, \dots, x_n]/I(X)$. This is finitely generated, and reduced since $I(X)$ is radical. Conversely, let A be a reduced finitely generated k-algebra. Then $A \cong k[y_1, \dots, y_m]/I$ since A is finitely generated, and I is radical since A is reduced. Thus by Hilbert's nullstellensatz, $Y = V(I)$ satisfies $I(Y) = I(V(I)) = I$, so $A \cong A(Y)$.

In more abstract language, this means that there is an equivalence of categories

$$\mathrm{AffVar} \longleftrightarrow \mathrm{RedFGAlg}_k^{\mathrm{op}}.$$

Lecture 9

Sept. 16 — Morphisms, Part 2

9.1 An Example of Isomorphisms

Example 9.0.1. What of the following are isomorphic over $\mathbb{C}$?

1. $\mathbb{A}^1 \setminus \{1\}$;
2. $V(x^2 + y^2) \subseteq \mathbb{A}^2$;
3. $V(y - x^2, z - x^3) \subseteq \mathbb{A}^3$;
4. $V(xy) \subseteq \mathbb{A}^2$;
5. $V(y^2 - x^2 - x^3) \subseteq \mathbb{A}^2$;
6. $V(x^2 - y^2 - 1) \subseteq \mathbb{A}^2$.

Note that (2) and (4) are not irreducible. In fact, they are isomorphic since we can write (2) as

$$V(x^2 + y^2) = V((x + iy)(x - iy)) \cong V(xy).$$

We have seen (3) previously on homework, and we have an isomorphism $\mathbb{A}^1 \to Y = V(y - x^2, z - x^3)$ by $t \mapsto (t, t^2, t^3)$. We can also see this by noting that $A(Y) \cong \mathbb{C}[x] \cong A(\mathbb{A}^1)$. For (1), note that

$$\mathbb{A}^1 \setminus \{1\} \cong \mathbb{A}^1 \setminus \{0\}$$

and $A(\mathbb{A}^1 \setminus \{0\}) \cong \mathbb{C}[x^{\pm 1}]$, whereas $A(\mathbb{A}^1) \cong \mathbb{C}[x]$. So $A \setminus \{1\} \not\cong \mathbb{A}^1$. For (6), note that

$$V(x^2 - y^2 - 1) = V((x - y - 1)(x + y + 1)) \cong V(uv - 1) \cong \mathbb{A}^1 \setminus \{0\}$$

by the map $V(uv - 1) \to \mathbb{A}^1 \setminus \{0\}$ given by $(u, v) \mapsto u$, with inverse $t \mapsto (t, 1/t)$. Finally, letting C be the curve in 6, one can show that there is a singularity at the origin with $\dim_{\mathbb{C}}(\mathcal{O}_{C,0}/\mathfrak{m}_0) = 1$, which is different than the other examples. So the isomorphism classes are $\{2, 4\}$, $\{1, 6\}$, $\{3\}$, and $\{5\}$.

9.2 Ringed Spaces and Morphisms

Definition 9.1. A *ringed space* $(X, \mathcal{O}_X)$ is a topological space X with a sheaf of rings $\mathcal{O}_X$ on X.

Example 9.1.1. If X is an affine variety and $\mathcal{O}_X$ is the sheaf of regular functions, then $(X, \mathcal{O}_X)$ is a ringed space. Similarly, if M is a complex manifold and $\mathcal{O}_M$ is the sheaf of holomorphic functions on M, then $(M, \mathcal{O}_M)$ is a ringed space.

Remark. From now on, for a ringed space $(X, \mathcal{O}_X)$, we will always assume $\mathcal{O}_X$ is a sheaf of k-valued functions on X. With this assumption, we can make sense of pullbacks.

Definition 9.2. A *morphism* of ringed spaces

$$(X, \mathcal{O}_X) \xrightarrow{f} (Y, \mathcal{O}_Y)$$

is a continuous map $f : X \to Y$ such that for every $U \subseteq Y$ open and $\varphi \in \mathcal{O}_Y(U)$,

$$\begin{tikzcd} f^{-1}(U) \arrow[r, "f"] \arrow[rr, bend left, "f^*\varphi"] & U \arrow[r, "\varphi"] & k \end{tikzcd}$$

is an element of $\mathcal{O}_X(f^{-1}(U))$. A morphism is an *isomorphism* if it has a two-sided inverse.

Remark. A one-sided inverse need not be two-sided: Consider $f : \mathbb{A}^2 \to \mathbb{A}^1$ given by $(x, y) \mapsto x$ and $g : \mathbb{A}^1 \to \mathbb{A}^2$ given by $x \mapsto (x, 0)$. Then $f \circ g = \mathrm{id}_{\mathbb{A}^1}$, but $g \circ f$ is not the identity on $\mathbb{A}^2$.

Remark. If $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a morphism of ringed spaces, then for $V \subseteq U \subseteq Y$ open, we get

$$\begin{tikzcd} \mathcal{O}_Y(U) \arrow[r] \arrow[d, "\text{res.}"] & \mathcal{O}_X(f^{-1}(U)) \arrow[d, "\text{res.}"] \\ \mathcal{O}_Y(V) \arrow[r] & \mathcal{O}_X(f^{-1}(V)) \end{tikzcd}$$

which is a commutative diagram of ring homomorphisms.

Remark. If X and Y are open sets of affine varieties, then a map $f : X \to Y$ is a morphism of open sets of affine varieties if and only if it is a morphism of ringed spaces.

Definition 9.3 (Redefinition of affine variety)**.** An *affine variety* $(X, \mathcal{O}_X)$ is a ringed space isomorphic to an affine variety in the original sense (as ringed spaces).

Remark. We will often write just X for the affine variety instead of the pair $(X, \mathcal{O}_X)$.

Example 9.3.1. Recall that $\mathbb{A}^1 \setminus \{0\} \cong V(xy - 1) \subseteq \mathbb{A}^2$ from Example 9.0.1. In particular, $\mathbb{A}^1 \setminus \{0\}$ is an affine variety in the new sense (but not in the old sense).

Proposition 9.1. *If X is an affine variety (in the old sense) and $f \in A(X)$, then $D(f)$ is an affine variety.*

Proof. Write $X = V(I) \subseteq \mathbb{A}^n_{x_i}$ and consider the map

$$\begin{aligned} D(f) &\longrightarrow V(I, fy - 1) \subseteq \mathbb{A}^n_{x_i} \times \mathbb{A}^1_y \\ x &\longmapsto (x, 1/f(x)). \end{aligned}$$

This has an inverse $V(I, fy - 1) \to D(f)$ given by $(x, y) \mapsto x$. So $D(f) \cong V(I, fy - 1)$ as ringed spaces. Thus $D(f)$ is an affine variety (in the new sense). □

9.3 Products of Affine Varieties

Remark. If $X \subseteq \mathbb{A}^m_{x_i}$ and $Y \subseteq \mathbb{A}^n_{y_i}$ are affine varieties, then

$$X \times Y = V(I(X), I(Y)) \subseteq \mathbb{A}^{m+n},$$

viewing $I(X), I(Y)$ as ideals in $k[x_1, \ldots, x_m, y_1, \ldots, y_n]$. So $X \times Y$ is an affine variety with morphisms

$$\begin{tikzcd} & X \times Y \arrow[dl, "(x,y)\mapsto x"', "p_1"] \arrow[dr, "(x,y)\mapsto y", "p_2"'] & \\ X & & Y \end{tikzcd}$$

Proposition 9.2. *For every affine variety Z and diagram of morphisms*

$$\begin{tikzcd} Z \arrow[dr, dashed, "f"] \arrow[drr, bend left, "f_Y"] \arrow[ddr, bend right, "f_X"'] & & \\ & X \times Y \arrow[r, "p_2"] \arrow[d, "p_1"] & Y \\ & X & \end{tikzcd}$$

there is a unique morphism f which makes the diagram commute.

Proof. We already know that there is a unique set theoretic map which makes the diagram commute. Then since f_X and f_Y are given as regular functions, so is f. So f is a morphism. □

Remark. We will now try to understand the isomorphism $A(X \times Y) \cong A(X) \otimes_k A(Y)$.

9.4 Tensor Products

Definition 9.4. Let A be a (commutative) ring and M, N be A-modules. The *tensor product* $M \otimes_A N$ is the A-module generated by the symbols $m \otimes n$ for $m \in M$ and $n \in N$, subject to the relations

1. (distributive law): $(m + m') \otimes n = m \otimes n + m' \otimes n$,
2. (multiplication with scalars): $a(m \otimes n) = (am) \otimes n = m \otimes (an)$.

To make this precise, $M \otimes_A N = A^{M \times N}/R$, where R is the submodule generated by these relations.

Example 9.4.1. We have $\mathbb{Z}/3\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} = 0$. We can compute

$$1 \otimes 1 = (3-2) \otimes 1 = 3 \otimes 1 - 2 \otimes 1 = 3 \otimes 1 + 1 \otimes (-2) = 0 \otimes 1 + 1 \otimes 0 = 0 \otimes 0,$$

and similarly for the other elements. In general, if $\gcd(m, n) = 1$, then $\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} = 0$.

Proposition 9.3 (Universal property of the tensor product)**.** *For any bilinear map $\Phi : M \times N \to P$ to an A-module P (i.e. $n \mapsto \Phi(m, n)$ is A-linear for each $m \in M$ and the same for $m \mapsto \Phi(m, n)$),*

$$\begin{tikzcd} M \times N \arrow[r, "\Phi"] \arrow[d, "(m,n)\mapsto m\otimes n"'] & P \\ M \otimes N \arrow[ur, "\Psi"'] & \end{tikzcd}$$

there exists a unique A-module homomorphism $\Psi : M \otimes N \to P$ such that the above diagram commutes.

Remark. We have the following properties of the tensor product:

1. $A \otimes M \cong M$;

2. $M \otimes N \cong N \otimes M$;
3. $(M \otimes N) \otimes P \cong M \otimes (N \otimes P)$;
4. $(M \oplus N) \otimes P \cong (M \otimes P) \oplus (N \otimes P)$.

The way to prove these is to use the universal property to construct maps in either direction and show that they compose to the identity.

5. For a fixed A-module M and an exact sequence

$$N' \xrightarrow{f} N \xrightarrow{g} N'' \longrightarrow 0,$$

the sequence (where F is defined by $m \otimes n' \mapsto m \otimes f(n')$ and G is defined by $m \otimes n \mapsto m \otimes g(n)$)

$$M \otimes N' \xrightarrow{F} M \otimes N \xrightarrow{G} M \otimes N'' \longrightarrow 0$$

is also exact. In particular, $\otimes M$ induces a right exact functor $\mathrm{Mod}_A \to \mathrm{Mod}_A$ by $N \mapsto M \otimes N$.

Example 9.4.2. The functor $\otimes M$ is in general not left exact. Consider

$$0 \longrightarrow \mathbb{Z} \xrightarrow{1 \mapsto 2} \mathbb{Z} \xrightarrow{1 \mapsto 1} \mathbb{Z}/2\mathbb{Z} \longrightarrow 0.$$

After tensoring with $\mathbb{Z}/2\mathbb{Z}$, we get the sequence

$$\mathbb{Z}/2\mathbb{Z} \xrightarrow{1 \mapsto 0} \mathbb{Z}/2\mathbb{Z} \xrightarrow{1 \mapsto 1 \otimes 1} \mathbb{Z}/2\mathbb{Z} \otimes \mathbb{Z}/2\mathbb{Z} \longrightarrow 0,$$

where the first map is not injective. Note that right exactness gives $\mathbb{Z}/2\mathbb{Z} \otimes \mathbb{Z}/2\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z}$.

Exercise 9.1. Show that $\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}/(m,n)\mathbb{Z}$.

Proposition 9.4. *If B and C are A-algebras, then $B \otimes_A C$ is also an A-algebra.*