

# MATH 6441: Algebraic Topology I

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# Lecture 1

## Jan. 6 — CW-Complexes

### 1.1 Introduction and Motivation

Algebraic topology builds “functions” (actually *functors*)

$$\{\text{topological spaces, continuous maps}\} \longrightarrow \{\text{algebraic things, algebraic maps}\},$$

where “algebraic things” can be groups, vector spaces, etc. The main objective of algebraic topology is to *distinguish topological spaces*, e.g. showing that  $\mathbb{R}^n \not\cong \mathbb{R}^m$  for  $n \neq m$ . More applications are:

1. Studying maps between spaces.<sup>1</sup>

- Does a given space  $M$  embed in  $N$ ? For instance, for what  $m$  does  $\mathbb{R}P^n$  embed in  $\mathbb{R}^m$ ? (This is still not known in general.) Here  $\mathbb{R}P^n$  is the real projective space.
- Lifting maps, i.e. given  $f : A \rightarrow B$  and  $g : E \rightarrow B$ , does there exist a map  $\tilde{f} : A \rightarrow E$  such that  $g \circ \tilde{f} = f$ ? In other words, is there a map  $\tilde{f}$  such that the following diagram commutes:

$$\begin{array}{ccc} & & E \\ & \nearrow \tilde{f} & \downarrow g \\ A & \xrightarrow{f} & B \end{array}$$

- Fixed point problems: Given  $f : X \rightarrow X$ , does  $f$  have a fixed point, i.e.  $x \in X$  such that  $f(x) = x$ ? Such theorems are used to prove the existence of solutions to ordinary differential equations, for instance.

2. Group actions, e.g. which finite groups act freely on  $S^n$ ?

3. Group theory, e.g. showing that every subgroup of a free group is free. Another example is that if  $F_n$  is the free group on  $n$  generators, then its *commutator*  $[F_n, F_n]$  is not finitely generated.

4. Algebra, e.g. proving the fundamental theorem of algebra.

This course will cover the following topics:

1. The *fundamental group*  $\pi_1(X, x_0)$  of a space  $X$  for  $x_0 \in X$ , and *covering spaces*.
2. The *homology groups*  $H_k(X)$  for  $k = 0, 1, 2, \dots$ . These groups are abelian.
3. The *cohomology ring*  $H^*(X) = \bigoplus_{k=0}^{\infty} H^k(X)$ .

But before getting to this, we need to develop some important ideas.

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<sup>1</sup>All maps and functions in this class are continuous unless otherwise specified.

## 1.2 CW-Complexes

**Definition 1.1.** Let  $D^n \subseteq \mathbb{R}^n$  be the unit disk and  $S^{n-1} = \partial D^n$ . Given a topological space  $Y$  and a continuous map  $a : S^{n-1} \rightarrow Y$ , the space obtained from  $Y$  by *attaching* an  $n$ -cell (via  $a$ ) is

$$Y \cup_a D^n = (Y \sqcup D^n) / \sim,$$

where the equivalence relation  $\sim$  is given by  $x \sim a(x)$  for  $x \in \partial D^n$ . Here  $\sqcup$  denotes disjoint union.

**Definition 1.2.** An  $n$ -complex or an  $n$ -dimensional CW-complex is defined inductively by:

- A  $(-1)$ -complex is the empty set  $\emptyset$ .
- An  $n$ -complex  $X^n$  is a space obtained from an  $(n-1)$ -complex by attaching  $n$ -cells.

An  $n$ -complex is *finite* if it involves only a finite number of cells. The  $k$ -skeleton of  $X$  is the union of all  $n$ -cells in  $X$  with  $n \leq k$ .

**Remark.** Any CW-complex is Hausdorff. See Hatcher for a proof.

**Example 1.2.1.** Here are some examples of CW-complexes:

- A 0-complex is a union of points. This is because  $D^0 = \{\text{pt}\}$  and  $\partial D^0 = \emptyset$ .
- A 1-complex is a graph (points and lines connecting them).
- The torus  $T$  (a square with opposite sides identified) is a 2-complex. Here the 0-skeleton  $T^{(0)}$  is the common corner on the square and the 1-skeleton  $T^{(1)}$  is two sides of the square after taking the quotient. The 2-skeleton  $T = T^{(2)}$  is the entire torus.
- Another example of a 2-complex is the two-holed torus, which is obtained by identifying the edges of an octagon (pairs of every other edge identified with opposite orientation).<sup>2</sup>
- A third example of a 2-complex is  $X^{(1)} \cup_a D^2$  given an attaching map  $D^2 \rightarrow X^{(1)}$ .
- Consider the unit sphere  $S^n \subseteq \mathbb{R}^{n+1}$ . One way to give  $S^2$  a CW-complex structure is to see the sphere as two disks  $D^2$  glued together, resulting in one 0-cell, one 1-cell, and two 2-cells. Another way is to start with two points, attach two 1-cells to get a circle, and then attaching two disks to get  $S^2$ . This results in two 0-cells, two 1-cells, and two 2-cells.

The second idea generalizes to  $S^n$ . We can write

$$S^n = S^{n-1} \cup_{a_1} D^n \cup_{a_2} D^n,$$

where  $S^{n-1}$  inductively has a CW-complex structure. This yields two  $k$ -cells for each  $k \leq n$ .

Another way to put a CW-complex structure on  $S^n$  is to attach  $D^n$  to a point with  $\partial D^n \rightarrow \{\text{pt}\}$ . In particular, notice that a space can in general have several different CW-complex structures.

- Consider the  $n$ -dimensional real projective space

$$\mathbb{R}P^n = \{\text{lines through the origin in } \mathbb{R}^{n+1}\}.$$

Since each line through the origin passes through  $S^n$  twice, we can equivalently think of  $\mathbb{R}P^n$  as the unit sphere  $S^n$  with antipodal points identified.

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<sup>2</sup>This CW-decomposition of the two-holed torus results in one 0-cell, four 1-cells, and one 2-cell.

We can also think of this as  $D^n$  with antipodal points on  $\partial D^n$  identified. Since  $\partial D^n = S^{n-1}$ , this is simply  $\mathbb{R}P^{n-1} \cup_a D^n$ , where  $a : \partial D^n \rightarrow \mathbb{R}P^{n-1}$  is the quotient map. This gives  $\mathbb{R}P^n$  a CW-complex structure with one  $k$ -cell for each  $k \leq n$ .

- The complex projective space  $\mathbb{C}P^n$  has a similar CW-complex structure with one  $k$ -cell for each even  $k \leq 2n$ . One can verify this as an exercise.
- Any smooth manifold has a CW-complex structure. See Hatcher.

**Exercise 1.1.** Show the product of CW-complexes is a CW-complex.

**Definition 1.3.** A *subcomplex* of a CW-complex  $X$  is a closed subset  $A \subseteq X$  that is a union of cells in  $X$ . In particular,  $A$  is also a CW-complex and  $(X, A)$  is called a *CW-pair*.

## 1.3 Homotopy

**Definition 1.4.** Let  $X$  and  $Y$  be topological spaces. Two maps  $f, g : X \rightarrow Y$  are *homotopic*, denoted  $f \sim g$ , if there exists a continuous map  $\Phi : X \times [0, 1] \rightarrow Y$  such that

$$\Phi(x, 0) = f(x) \quad \text{and} \quad \Phi(x, 1) = g(x)$$

for all  $x \in X$ . In this case,  $\Phi$  is called a *homotopy* between  $f$  and  $g$ .

**Remark.** We note the following:

- A homotopy  $\Phi$  gives a family of maps  $\phi_{t_0} : X \rightarrow Y$  given by  $x \mapsto \phi(x, t_0)$  which is continuous in  $t_0$ . So maps are homotopic if there is a continuous family of maps between them.
- If  $A \subseteq X$  then we say that  $f$  is *homotopic to  $g$  rel  $A$* , denoted  $f \sim_A g$ , if there exists  $\Phi$  as above with the additional property that  $\Phi(x, t) = f(x)$  for all  $x \in A$ , i.e. points in  $A$  are fixed.
- If  $A \subseteq X$  and  $B \subseteq Y$ , then the notation  $f : (X, A) \rightarrow (Y, B)$  means that  $f : X \rightarrow Y$  and  $f(a) \in B$  for each  $a \in A$ . We say that  $f$  is a *map of pairs*. If  $f, g : (X, A) \rightarrow (Y, B)$ , then  $f, g$  are *homotopic as maps of pairs* if each  $\phi_t : (X, A) \rightarrow (Y, B)$  is a map of pairs.

# Lecture 2

## Jan. 8 — Homotopy

### 2.1 More on Homotopy

**Example 2.0.1.** For any space  $X$ , any map  $f : X \rightarrow [0, 1]$  is homotopic to the map  $g : X \rightarrow [0, 1]$  given by  $x \mapsto 0$ . To see this, we have the homotopy  $\Phi : X \times [0, 1] \times [0, 1]$  defined by

$$(x, t) \mapsto (1 - t)f(x).$$

We can see that  $\Phi(x, 0) = f(x)$  and  $\Phi(x, 1) = 0 = g(x)$ .

**Exercise 2.1.** Show that homotopy is an equivalence relation on maps  $X \rightarrow Y$ .

**Definition 2.1.** Let  $C(X, Y) = \{\text{continuous maps from } X \text{ to } Y\}$ . Let  $[X, Y] = C(X, Y)/\sim$ , i.e. homotopic maps are identified with each other.

**Example 2.1.1.** We have the following:

1.  $[X, [0, 1]] = \{g\}$  for any space  $X$ , where  $g$  is the map  $x \mapsto 0$  as above.
2.  $[\{*\}, X] = \{\text{path components of } X\}$ .

### 2.2 Homotopy Groups

**Definition 2.2.** We call a space  $X$  *pointed* if there is a designated “base point”  $x_0 \in X$ . Given two pointed spaces  $(X, x_0)$  and  $(Y, y_0)$ , we define

$$[X, Y]_0 = \{\text{homotopy classes of maps of pairs } (X, \{x_0\}) \rightarrow (Y, \{y_0\})\}.$$

**Definition 2.3.** Let  $y_0$  be the north pole in  $S^n$ , i.e.  $S^n \subseteq \mathbb{R}^{n+1}$  is the unit sphere and  $y_0 = (0, \dots, 0, 1)$ . The  $n$ th homotopy group of a pointed space  $(X, x_0)$  is  $\pi_n(X, x_0) = [S^n, X]_0$ .

**Remark.** The homotopy group  $\pi_n(X, x_0)$  is in fact a group. We will study  $\pi_1(X, x_0)$  next and it is called the *fundamental group* of  $(X, x_0)$ .

**Remark.** For which  $(Y, y_0)$  is  $[Y, X]_0$  “naturally” a group for all  $(X, x_0)$ ? Similarly, for which  $(Y, y_0)$  is  $[X, Y]_0$  a group for all  $(X, x_0)$ ? Here, given a map  $f : (X_1, x_1) \rightarrow (X_2, x_2)$ , there is an obvious *induced map*  $f_* : [Y, X_1]_0 \rightarrow [Y, X_2]_0$  given by  $[g] \mapsto [f \circ g]$ . Similarly, there is a map  $f^* : [X_2, Y]_0 \rightarrow [X_1, Y]_0$  given by  $[g] \mapsto [g \circ f]$ . In the questions above, “naturally” means that  $f_*$  and  $f^*$  are homomorphisms for any  $(X_1, x_1)$  and  $(X_2, x_2)$ . The (perhaps unsatisfying) answer is that a space satisfying the first condition is called an *H-space*, and a space satisfying the second is called an *H'-space*.

## 2.3 Homotopy Equivalence

**Definition 2.4.** We say that  $f : X \rightarrow Y$  is the *homotopy inverse* to a function  $g : Y \rightarrow X$  if  $f \circ g \sim \text{id}_Y$  and  $g \circ f \sim \text{id}_X$ , where  $\text{id}_X$  and  $\text{id}_Y$  are the identity maps on  $X$  and  $Y$ . If  $g$  has a homotopy inverse, then we call  $g$  a *homotopy equivalence* from  $Y$  to  $X$  and we call  $X, Y$  *homotopy equivalent*.<sup>1</sup>

**Exercise 2.2.** Show that homotopy equivalence is an equivalence relation.

**Lemma 2.1.** *The following are equivalent:*

1.  $X$  and  $Y$  are homotopy equivalent.
2. For any space  $Z$ , there is a one-to-one correspondence  $\phi_Z : [X, Z] \rightarrow [Y, Z]$  such that for all continuous maps  $h : Z \rightarrow Z'$ , the following diagram commutes:

$$\begin{array}{ccc} [X, Z] & \xrightarrow{\phi_Z} & [Y, Z] \\ \downarrow h_* & & \downarrow h_* \\ [X, Z'] & \xrightarrow{\phi_{Z'}} & [Y, Z'] \end{array}$$

3. For any space  $Z$ , there is a one-to-one correspondence  $\phi^Z : [Z, X] \rightarrow [Z, Y]$  such that for all continuous maps  $h : Z \rightarrow Z'$ , the following diagram commutes:

$$\begin{array}{ccc} [Z', X] & \xrightarrow{\phi^{Z'}} & [Z', Y] \\ \downarrow h^* & & \downarrow h^* \\ [Z, X] & \xrightarrow{\phi^Z} & [Z, Y] \end{array}$$

*Proof.* This is left as an exercise. □

**Remark.** Based on the previous lemma, two spaces are homotopy equivalent if and only if homotopy classes of maps to and from the space are “naturally equivalent.”

**Example 2.4.1.** We have the following:

- Homeomorphic spaces are homotopy equivalent.
- Let  $X = S^1$  and  $Y = S^1 \times [0, 1]$ . We claim that  $X$  is homotopy equivalent to  $Y$ .

Define the maps  $f : S^1 \rightarrow S^1 \times [0, 1]$  by  $x \mapsto (x, 0)$  and  $g : S^1 \times [0, 1] \rightarrow S^1$  by  $(x, t) \mapsto x$ . Then we can see that  $g \circ f : S^1 \rightarrow S^1$  maps  $x \mapsto x$ , so  $g \circ f = \text{id}_{S^1}$ . On the other hand, the composition  $f \circ g : S^1 \times [0, 1] \rightarrow S^1 \times [0, 1]$  maps  $(x, t) \mapsto (x, 0)$ . Now  $f \circ g \sim \text{id}_{S^1 \times [0, 1]}$  by homotopy. For instance, define  $\Phi : (S^1 \times [0, 1]) \times [0, 1] \rightarrow (S^1 \times [0, 1])$  by  $((x, t), s) \mapsto (x, st)$ , so

$$\Phi((x, t), 1) = (x, t) = \text{id}_{S^1 \times [0, 1]}(x, t) \quad \text{and} \quad \Phi((x, t), 0) = (x, 0) = f \circ g.$$

Thus  $f$  is a homotopy equivalence from  $S^1$  to  $S^1 \times [0, 1]$ . Note that  $S^1 \times [0, 1]$  is the annulus.

**Definition 2.5.** A space is called *contractible* if it is homotopy equivalent to a point.

**Example 2.5.1.** The spaces  $[0, 1]$  and  $\mathbb{R}^n$  are contractible (exercise).

<sup>1</sup>We will denote homotopy equivalence by  $X \simeq Y$  or simply  $X \sim Y$ .



**Definition 2.6.** If  $A \subseteq X$ , then a *retraction of  $X$  to  $A$*  is a map  $r : X \rightarrow A$  such that  $r(a) = a$  for every  $a \in A$ . A *deformation retraction of  $X$  to  $A$*  is a retraction  $r : X \rightarrow A$  that is homotopic rel  $A$  to the identity map  $\text{id}_X$ , i.e. we can find  $\phi_t : X \rightarrow X$  for  $t \in [0, 1]$  such that  $\phi_0(x) = x$  and  $\phi_1(X) \subseteq A$  and  $\phi_t(x) = x$  for all  $x \in A$  and  $t \in [0, 1]$ .

**Remark.** If  $X$  deformation retracts to  $A$ , then  $X$  is homotopy equivalent to  $A$ . To see this, suppose we have a homotopy  $\phi_t : X \rightarrow X$  as above, and let  $i : A \rightarrow X$  be the inclusion map. Then  $\phi_1 \circ i = \text{id}_A$  and  $i \circ \phi_1 = \phi_1 \sim \phi_0 = \text{id}_X$ , so  $\phi_1$  is a homotopy equivalence from  $X$  to  $A$ .

**Definition 2.7.** Given two spaces  $X, Y$  and a map  $f : X \rightarrow Y$ , the *mapping cylinder of  $f$*  is the space

$$M_f = ((X \times [0, 1]) \cup Y) / \sim,$$

where the equivalence relation  $\sim$  is given by  $(x, 1) \sim f(x)$  for  $x \in X$ .

**Remark.** The mapping cylinder  $M_f$  deformation retracts to  $Y$ . To see this, consider the map  $\tilde{\phi}_t$  given by  $(x, s) \mapsto (x, (1-t)s + t)$  on  $X \times [0, 1]$  and  $y \mapsto y$  on  $Y$ . Since  $\tilde{\phi}_t$  respects the equivalence relation, it descends to a map  $\phi_t : M_f \rightarrow M_f$  on the quotient space. Note that  $\phi_0 = \text{id}_{M_f}$  and  $\phi_1(M_f) = Y \subseteq M_f$ , and  $\phi_t = \text{id}_Y$  for all  $t$ . Thus  $\phi_1$  is a deformation retraction. In particular, this means that  $M_f \simeq Y$ .

**Remark.** There are obvious inclusions  $i : X \rightarrow M_f$  given by  $x \mapsto (x, 0)$  and  $j : Y \rightarrow M_f$  given by  $y \mapsto y$ . Note that  $\phi_1$  defined above is the homotopy inverse to  $j$ . Now we have the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \searrow i & \downarrow j \\ & & M_f \end{array}$$

where  $j$  is a homotopy equivalence and  $j \circ f \sim i$  (exercise).

**Remark.** The above remark shows the following “slogan” of algebraic topology:

Any map is an inclusion up to homotopy.

**Example 2.7.1.** Let  $X$  be three circles with two enclosed in a third bigger one, and let  $Y$  be two circles enclosing the inner two circles of  $X$  connected by a line segment. Let  $Z$  be the region inside by the outer circle of  $X$  and outside the inner two circles of  $X$ .

Define  $f : X \rightarrow Y$  to be the map which sends  $x \in X$  to the point in  $Y$  at the other end of an interval (points on the inner circles of  $X$  are mapped by radial lines to the circles in  $Y$ , and points on the outer circle of  $X$  are mapped radially to either the circles or the line segment in  $Y$ ). One can write an explicit formula for  $f$  as an exercise.

Then  $Z$  is homeomorphic to  $M_f$ , and in particular  $Z \simeq Y$ . Similarly,  $M_f$  is homotopy equivalent to two circles joined at a point, or a circle with a diameter. Thus by transitivity, these two spaces and  $Z$  are all homotopy equivalent to each other.

# Lecture 3

## Jan. 13 — Homotopy, Part 2

### 3.1 More on Homotopy Equivalence

**Lemma 3.1.** *If  $(X, A)$  is a CW pair and  $A$  is contractible, then  $X/A \simeq X$ .<sup>1</sup>*

**Exercise 3.1.** The following are some applications of this lemma:

1. Let  $X$  be a connected graph (i.e. a 1-complex), and let  $A$  be an edge in  $X$  connecting distinct vertices. Then  $A$  is contractible, so  $X/A \simeq X$ . Continuing this process, let  $A$  be a maximal tree in  $X$ , which will also be contractible. Then  $X \simeq X/A$ , so any connected graph is homotopy equivalent to a *wedge of circles* (with number of circles equal to the number of self-loops in the graph).<sup>2</sup>
2. Consider the space  $X$  obtained by attaching a 1-cell  $A$  connecting the north and south poles on a sphere. Let  $B$  be half of a great circle connecting the endpoints of  $A$ . Clearly  $A$  and  $B$  are both contractible. After collapsing  $A$ , we see that  $X \simeq X/A$ , which is  $S^2$  with the north and south poles identified. On the other hand, by contracting  $B$  instead we see that  $X \simeq X/B$ , which is  $S^2 \vee S^1$ .

Note that after contracting  $A$ , the subset  $B$  is actually no longer contractible.

3. Let  $X$  be a torus with attached disks  $A_1, A_2, A_3$  in the tube of the torus. Then

$$X \simeq ((X/A_1)/A_2)/A_3,$$

which is three spheres lying in a circle, each attached to the next one at a single point.

We can also obtain this space by considering the space  $Y$  of three spheres attached in a line with an extra 1-cell  $B$  attached at the ends of the chain of circles. Also let  $A$  be the union of halves of great circles going through the chain of circles, with the same endpoints as  $B$ . Then we can see that this creates the same space as before, so that  $X \simeq Y/B \simeq Y$ . On the other hand, by contracting  $A$ , we see that  $Y \simeq Y/A = S^2 \vee S^2 \vee S^2 \vee S^1$ .

Of course, all of these spaces are then homotopy equivalent to each other by transitivity.

**Lemma 3.2.** *Let  $(X, A)$  be a CW pair and  $f, g : A \rightarrow Y$  be homotopic maps. Then  $X \cup_f Y \simeq X \cup_g Y$ .*

**Example 3.0.1.** Let  $Y = S^2$ , and  $X = D^2$ , and  $A = \partial D^2$ . Let  $g : A \rightarrow Y$  map  $A$  to a great circle, and let  $f : A \rightarrow Y$  map  $A$  to the north pole. One can show as an exercise that  $f \sim g$  (e.g. by pulling the equator towards the north pole). So the lemma says that  $X \cup_g Y$ , which is a sphere with a disk glued along its equator, is homotopy equivalent to  $X \cup_f Y = S^2 \vee S^2$ .

<sup>1</sup>Here  $X/A$  denotes the quotient of  $X$  obtained by collapsing all of  $A$  to a single point.

<sup>2</sup>A *wedge of pointed spaces*  $(X, x_0) \vee (Y, y_0)$  is the space obtained from  $X \sqcup Y$  by identifying  $x_0$  and  $y_0$ .

## 3.2 Homotopy Extension Property

**Remark.** To prove both of these lemmas, we need the *homotopy extension property* (HEP).

**Definition 3.1.** A space  $X$  and a subspace  $A \subseteq X$  have the *homotopy extension property* if given  $F_0 : X \rightarrow Y$  (for any  $Y$  and  $F_0$ ) and a homotopy  $f_t : A \rightarrow Y$  such that  $f_0 = F_0|_A$ , then there is a homotopy  $F_t : X \rightarrow Y$  such that  $F_t|_A = f_t$  for every  $t$ .

**Lemma 3.3.** A pair  $(X, A)$  has the homotopy extension property if and only if

$$(X \times \{0\}) \cup (A \times [0, 1])$$

is a retract of  $X \times [0, 1]$ .

*Proof.* ( $\Leftarrow$ ) We will assume that  $A$  is closed (not necessarily but makes the proof easier, and almost all examples satisfy this). By assumption, we have a retraction

$$r : (X \times [0, 1]) \rightarrow (X \times \{0\}) \cup (A \times [0, 1]).$$

Given  $F_0 : X \rightarrow Y$  and  $f_t : A \rightarrow Y$  such that  $f_0 = F_0|_A$ , we can define a map

$$\tilde{F} : (X \times \{0\}) \cup (A \times [0, 1]) \rightarrow Y$$

by  $x \mapsto F_0(x)$  on  $X \times \{0\}$  and  $(a, t) \mapsto f_t(a)$  on  $A \times [0, 1]$ . This map  $\tilde{F}$  is continuous since the definitions of  $\tilde{F}$  agree on the intersection and the intersection  $A \times \{0\}$  is closed. Now define

$$F : X \times [0, 1] \rightarrow Y$$

by  $F = \tilde{F} \circ r$ , which is a homotopy of  $F_0$  that extends  $f_t$ .

( $\Rightarrow$ ) Let  $Y = (X \times [0, 1]) \cup (A \times [0, 1])$ . Let

$$F_0 : X \rightarrow (X \times \{0\}) \cup (A \times [0, 1])$$

be given by  $x \mapsto (x, 0)$ , and

$$f_t : A \mapsto (X \times \{0\}) \cup (A \times [0, 1])$$

be given by  $a \mapsto (a, t)$ . Then the homotopy extension property yields an extension

$$F : X \times [0, 1] \rightarrow (X \times \{0\}) \cup (A \times [0, 1]),$$

which is a retraction, as desired. □

**Lemma 3.4.** If  $(X, A)$  is a CW pair, then  $(X \times \{0\}) \cup (A \times [0, 1])$  is a (deformation) retract of  $X \times [0, 1]$ . In particular,  $(X, A)$  satisfies the homotopy extension property.

*Proof.* The main idea is that for any disk  $D^n$ , the space  $(D^n \times \{0\}) \cup (\partial D^n \times [0, 1])$  is a deformation retract of  $D^n \times [0, 1]$ . To see this, let  $D^n \subseteq \mathbb{R}^n$  be the unit disk and  $D^n \times [0, 1] \subseteq \mathbb{R}^{n+1}$ . Let

$$p = (0, \dots, 0, 2).$$

For any  $x \in D^n \times [0, 1]$ , let  $\ell_x$  be the line through  $p$  and  $x$ . Note that

$$\ell_x \cap ((D^n \times [0, 1]) \cup (\partial D^n \times [0, 1]))$$

is a unique point. Define  $\tilde{r}(x)$  to be this point, which yields a map

$$\tilde{r} : D^n \times [0, 1] \rightarrow (D^n \times \{0\}) \cup (\partial D^n \times [0, 1])$$

Note that for  $x \in (D^n \times \{0\}) \cup (\partial D^n \times [0, 1])$ , then  $\tilde{r}(x) = x$  since the point of intersection is unique and  $x$  is already in the intersection. Show as an exercise that  $\tilde{r}$  is continuous. Then setting

$$\tilde{r}_t = t\tilde{r} + (1 - t)\text{id}_{D^n \times [0, 1]}$$

gives a deformation retraction of  $D^n \times [0, 1]$  onto  $(D^n \times \{0\}) \cup (\partial D^n \times [0, 1])$ .

To show the general case that  $X \times [0, 1]$  retracts to  $(X \times \{0\}) \cup (A \times [0, 1])$ , we induct on the dimension of cells. Define  $r$  on  $X^{(0)} \times [0, 1] \rightarrow (X \times \{0\}) \cup (A \times [0, 1])$  by the following:

- if a 0-cell  $D^0 \subseteq A$ , then let  $r$  be the identity on  $D^0 \times [0, 1]$ ;
- if a 0-cell  $D^0$  is not in  $A$ , then send  $D^0 \times [0, 1]$  to  $D^0 \subseteq X \times \{0\}$ .

Now inductively assume  $r$  is defined on  $X^{(k-1)} \times [0, 1] \rightarrow (X \times \{0\}) \cup (A \times [0, 1])$ . For each  $k$ -cell  $D^k$ ,

- if  $D^k \subseteq A$ , then let  $r$  be the identity on  $D^k \times [0, 1]$ ;
- if  $D^k$  is not in  $A$ , then note that  $\partial D^k \times [0, 1] \rightarrow X^{(k-1)} \times [0, 1]$  is defined by induction, and we have an “inclusion” (here  $a : \partial D^k \rightarrow X^{(k-1)}$  is the attaching map for  $D^k$ )

$$\begin{array}{ccc} D^k & \xrightarrow{i} & X^{(k-1)} \cup D^k \xrightarrow{q} (X^{(k-1)} \cup D^k) / \{(x \in \partial D^k) \sim (a(x) \in X^{(k-1)})\} \\ & \searrow j & \nearrow \\ & & \end{array}$$

So we let  $D^k \times \{0\} \rightarrow (X \times \{0\}) \cup (A \times [0, 1])$  be the map  $j$  into  $X \times \{0\}$ . This defines  $r$  on

$$(D^k \times [0, 1]) \cup (\partial D^k \times [0, 1]),$$

which extends to  $D^k \times [0, 1]$  by composing with the map  $\tilde{r}$  from above.

This inductively defines the retraction  $r$  on all of  $X \times [0, 1]$ . The last claim follows by Lemma 3.3.  $\square$

**Remark.** Now we can prove the first two lemmas from the beginning of the day.

*Proof of Lemma 3.1.* We prove that the result holds for any  $(X, A)$  which satisfies the homotopy extension property. We show that the quotient map  $q : X \rightarrow X/A$  has a homotopy inverse. Since  $A$  is contractible, we know there exists a homotopy  $f_t : A \rightarrow A \subseteq X$ , such that  $f_0 = \text{id}_A$  and  $f_1$  is constant. Let  $F_0 : X \rightarrow X$  be the identity (note that  $F_0|_A = \text{id}_A$ ), so that the homotopy extension property gives a homotopy  $F_t : X \rightarrow X$  extending  $f_t$ . Since  $F_t(A) \subseteq A$  for every  $t$ , we get maps

$$\tilde{F}_t : X/A \rightarrow X/A,$$

which are well defined since points in  $A$  are mapped to points in  $A$ . Furthermore,  $F_1(A) = \{\text{pt}\}$ , so  $F_1$  factors through  $X/A$  to give a map  $h : X/A \rightarrow X$  satisfying  $F_1 = h \circ q$ . This gives the diagram:

$$\begin{array}{ccc} X & \xrightarrow{F_1} & X \\ q \downarrow & \nearrow h & \downarrow q \\ X/A & \xrightarrow{\tilde{F}_1} & X/A \end{array}$$

It is easy to check that  $h \circ q = F_1$  and  $q \circ h = \overline{F}_1$ , so that the diagram commutes. But then

$$h \circ q = F_1 \sim F_0 = \text{id}_X \quad \text{and} \quad q \circ h = \overline{F}_1 \sim \overline{F}_0 = \text{id}_{X/A},$$

so  $q$  is a homotopy equivalence. □

*Proof of Lemma 3.2.* Let  $F : A \times [0, 1] \rightarrow Y$  be a homotopy, which extends to  $F : X \times [0, 1] \rightarrow Y$  by the homotopy extension property. Consider the mapping cylinder

$$M_F = (X \times [0, 1]) \cup_F Y,$$

and one can show that  $M_F \simeq X \cup_g Y \simeq X \cup_f Y$ . □

# Lecture 4

## Jan. 15 — Fundamental Group

### 4.1 Fundamental Group

**Remark.** The basic idea of the *fundamental group* is to study the topology of a space with loops mapped into the space. For instance, intuitively, any loop in  $S^2$  can be “pulled back” to (i.e. is homotopic to) a constant loop. On the other hand, a loop wrapping around the hole in  $T^2$  gets stuck and cannot be “pulled back” to a constant loop. The same issue happens for a loop in  $T^2$  around the cylindrical part. The fundamental group is a way to make this intuition precise and measure the “holes” in a space.

**Definition 4.1.** The *fundamental group* of a based space  $(X, x_0)$  is

$$\pi_1(X, x_0) = [(S^1, n), (X, x_0)]_0,$$

i.e. the homotopy classes of loops in  $X$  based at  $x_0$ . Here  $n = (0, 1)$  is the north pole of  $S^1$ .

**Exercise 4.1.** Let  $S^1 \subseteq \mathbb{R}^2$  be the unit circle and let  $p : [0, 1] \rightarrow S^1$  be given by

$$t \mapsto (\cos 2\pi t, \sin 2\pi t).$$

Show that  $p$  is a quotient map, so we can think of  $S^1$  as  $[0, 1]$  with  $0, 1$  identified. Moreover, show that there is a one-to-one correspondence between maps of the form<sup>1</sup>

$$\gamma : ([0, 1], \{0, 1\}) \rightarrow (X, x_0) \quad \text{and} \quad \tilde{\gamma} : (S^1, \{(1, 0)\}) \rightarrow (X, x_0)$$

given by  $\tilde{\gamma} \mapsto \tilde{\gamma} \circ p = \gamma$ , and that homotopies of  $\tilde{\gamma}$  rel  $\{(1, 0)\}$  correspond to homotopies of  $S^1$  rel  $\{(1, 0)\}$ .

**Remark.** Using the above exercise, we can think of  $\pi_1(X, x_0) = [S^1, X]_0$  instead as

$$\pi_1(X, x_0) = [[0, 1], \{0, 1\}], (X, x_0)]_0.$$

Given a based loop  $\gamma : [0, 1] \rightarrow X$ , we denote its equivalence class in  $\pi_1(X, x_0)$  by  $[\gamma]$ .

**Definition 4.2.** If  $\gamma_1, \gamma_2$  are loops in  $X$  based at  $x_0 \in X$ , their *concatenation*  $\gamma_1 * \gamma_2 : [0, 1] \rightarrow X$  is

$$t \mapsto \begin{cases} \gamma_1(2t) & \text{if } 0 \leq t \leq 1/2, \\ \gamma_2(2t - 1) & \text{if } 1/2 \leq t \leq 1. \end{cases}$$

**Remark.** Concatenation of loops indeed yields another loop since  $\gamma_1 * \gamma_2(0) = \gamma_1 * \gamma_2(1) = x_0$  and  $\gamma_1 * \gamma_2$  is continuous since the definitions agree on the closed set  $\{1/2\}$ .

---

<sup>1</sup>Such a loop  $\gamma$  is called a *based loop*.

**Remark.** We can clearly see that  $\gamma_1 * \gamma_2$  is well-defined given  $\gamma_1$  and  $\gamma_2$ , but can we define  $[\gamma_1] * [\gamma_2]$  for homotopy classes of loops in a well-defined manner? We need to check that if  $\gamma_1 \sim \gamma_2$  and  $\delta_1 \sim \delta_2$  (i.e.  $\gamma_1, \gamma_2 \in [\gamma_1]$  and  $\delta_1, \delta_2 \in [\delta_1]$ ), then we also have  $\gamma_1 * \delta_1 \sim \gamma_2 * \delta_2$ .

To do this, let  $H : [0, 1] \times [0, 1] \rightarrow X$  be the homotopy from  $\gamma_1$  to  $\gamma_2$  and  $G : [0, 1] \times [0, 1] \rightarrow X$  be the homotopy from  $\delta_1$  to  $\delta_2$ . We need to construct a homotopy  $\tilde{H} : [0, 1] \times [0, 1] \rightarrow X$  from  $\gamma_1 * \delta_1$  to  $\gamma_2 * \delta_2$ .

Note that if we think of  $[0, 1] \times [0, 1]$  as the unit square, then  $\tilde{H}$  is already defined on the boundary: the left and right sides are constantly  $x_0$ , the top side is  $\gamma_2 * \delta_2$ , and the bottom side is  $\gamma_1 * \delta_1$ . So we only need to define it on the interior. For this, note that the vertical line in the middle of the square is also constantly  $x_0$  by construction: This creates two rectangles on each half, which we can fill with  $H$  and  $G$ .

More formally, we can construct the homotopy  $\tilde{H} : [0, 1] \times [0, 1] \rightarrow X$  explicitly via

$$(t, s) \mapsto \begin{cases} H(2t, s) & \text{if } 0 \leq t \leq 1/2, \\ G(2t - 1, s) & \text{if } 1/2 \leq t \leq 1. \end{cases}$$

This is continuous since the definitions agree on the closed set  $\{t = 1/2\}$ . Thus, setting

$$[\gamma_1] * [\delta_1] = [\gamma_1 * \delta_1]$$

gives a well-defined binary operation by the above discussion.

**Lemma 4.1.** *The pair  $(\pi_1(X, x_0), *)$  is a group.*

*Proof.* For the identity, let  $e : [0, 1] \rightarrow X$  be the constant loop  $t \mapsto x_0$ . We will show that

$$[e] * [\gamma] = [\gamma] = [\gamma] * [e].$$

The picture is that  $[0, 1] \times [0, 1]$  has  $\gamma$  on the top side and  $\gamma * e$  on the bottom. By drawing a line from the midpoint of the bottom side and the top-right corner, we see that we can fill the right portion with just  $x_0$  and the left portion with  $\gamma$ . The equation of this line is  $s = 2t - 1$ , so  $t = (s + 1)/2$ . Thus from the picture, we can write the explicit homotopy  $H : [0, 1] \times [0, 1] \rightarrow X$  via

$$H(t, s) = \begin{cases} \gamma(2/(s + 1), t) & \text{if } 0 \leq t \leq (s + 1)/2, \\ x_0 & \text{if } (s + 1)/2 \leq t \leq 1. \end{cases}$$

One can use a similar construction to show that  $[e] * [\gamma] = [\gamma]$ , so that  $[e]$  is an identity element.

Now we show the existence of inverses. Given a loop  $\gamma$ , define  $\bar{\gamma}$  via  $\bar{\gamma}(t) = \gamma(1 - t)$ , i.e.  $\gamma$  backwards. Set  $\gamma_s(t) = \gamma(st)$ . Note that as  $t$  goes from 0 to 1,  $\gamma_s$  goes from  $\gamma(0)$  to  $\gamma(s)$ , and also that  $\bar{\gamma}_s(t) = \gamma(s - st)$ . So we can write the homotopy  $H : [0, 1] \times [0, 1] \rightarrow X$  between  $\gamma * \bar{\gamma}$  and  $e$  by

$$H(t, s) = \begin{cases} \gamma_s(2t) & \text{if } 0 \leq t \leq 1/2, \\ \bar{\gamma}_s(2t - 1) & \text{if } 1/2 \leq t \leq 1 \end{cases} = \begin{cases} \gamma(2st) & \text{if } 0 \leq t \leq 1/2, \\ \gamma(s - s(2t - 1)) & \text{if } 1/2 \leq t \leq 1. \end{cases}$$

Thus setting  $[\gamma]^{-1} = [\bar{\gamma}]$  gives us inverses.

Finally, for associativity, we need to see that  $(\gamma_1 * \gamma_2) * \gamma_3 \sim \gamma_1 * (\gamma_2 * \gamma_3)$ . Again by drawing a picture, we see that we can draw two diagonal lines connecting the starting points of  $\gamma_2$  on top and bottom and the ending points of  $\gamma_1$  on top and bottom. Write an explicit formula for the homotopy as an exercise.  $\square$

## 4.2 Induced Homomorphisms

**Remark.** If  $f : X \rightarrow Y$  and  $x_0 \in X$ , let  $y_0 = f(x_0)$ . Then given a based loop  $\gamma : [0, 1] \rightarrow X$ , note that the composition  $f \circ \gamma : [0, 1] \rightarrow Y$  is a based loop in  $Y$ . Also, if  $\gamma \sim \delta$ , then  $f \circ \gamma \sim f \circ \delta$  (if  $H$  is a homotopy from  $\gamma \sim \delta$ , then  $f \circ H$  is a homotopy from  $f \circ \gamma$  to  $f \circ \delta$ ). In particular,  $f$  induces a map

$$f_* : \pi_1(X, x_0) \rightarrow \pi_1(Y, y_0).$$

**Lemma 4.2.** *The induced map  $f_* : \pi_1(X, x_0) \rightarrow \pi_1(Y, y_0)$  is a homomorphism.*

*Proof.* Note that

$$\gamma_1 * \gamma_2(t) = \begin{cases} \gamma_1(2t) & \text{if } 0 \leq t \leq 1/2, \\ \gamma_2(2t - 1) & \text{if } 1/2 \leq t \leq 1 \end{cases}$$

and that

$$(f \circ \gamma_1) * (f \circ \gamma_2)(t) = \begin{cases} f(\gamma_1(2t)) & \text{if } 0 \leq t \leq 1/2, \\ f(\gamma_2(2t - 1)) & \text{if } 1/2 \leq t \leq 1, \end{cases}$$

so  $f \circ (\gamma_1 * \gamma_2) = (f \circ \gamma_1) * (f \circ \gamma_2)$ , which implies  $f_*([\gamma_1 * \gamma_2]) = f_*([\gamma_1]) * f_*([\gamma_2])$ .  $\square$

**Exercise 4.2.** Check the following as an exercise:

1.  $(f \circ g)_* = f_* \circ g_*$ ;
2. if  $f : X \rightarrow Y$  is homotopic rel base point to  $g : X \rightarrow Y$ , then  $f_* = g_* : \pi_1(X, x_0) \rightarrow \pi_1(Y, y_0)$ .

**Remark.** How does  $\pi_1$  depend on the base point? Let  $x_0, x_1 \in X$ , and suppose that there exists a path  $h : [0, 1] \rightarrow X$  with  $h(0) = x_0$  and  $h(1) = x_1$ . Then if  $\gamma$  is a loop based at  $x_1$ , we can get a loop based at  $x_0$  by going from  $x_0$  to  $x_1$  along  $h$ , taking  $\gamma$ , and then going back to  $x_0$ . More explicitly, this is

$$h * \gamma * \bar{h}(t) = \begin{cases} h(3t) & \text{if } 0 \leq t \leq 1/3, \\ \gamma(3t - 1) & \text{if } 1/3 \leq t \leq 2/3, \\ \bar{h}(3t - 2) & \text{if } 2/3 \leq t \leq 1. \end{cases}$$

**Lemma 4.3.** *The path  $h$  induces an isomorphism  $\phi_h : \pi_1(X, x_1) \rightarrow \pi_1(X, x_0)$  by  $[\gamma] \mapsto [h * \gamma * \bar{h}]$ .*

*Proof.* Check as an exercise that  $\phi_h$  is a well-defined homomorphism. To show that  $\phi_h$  is an isomorphism, we claim that  $\phi_{\bar{h}}$  is an inverse of  $\phi_h$ . To see this, let  $[\gamma] \in \pi_1(X, x_0)$ . Then

$$\phi_h \circ \phi_{\bar{h}}([\gamma]) = [h * \bar{h} * \gamma * h * \bar{h}] = [h * \bar{h}] * [\gamma] * [h * \bar{h}] = [e] * [\gamma] * [e] = [\gamma],$$

where the second equality follows by the same proof for associativity of  $*$ . This proves the result.  $\square$

**Remark.** Note the following based on the above lemma:

1. The isomorphism class of  $\pi_1(X, x_0)$  only depends on the path component of  $X$  containing  $x_0$ .
2. The isomorphism *depends on*  $h$ . One needs to be careful about using the correct identification.



# Lecture 5

## Jan. 22 — Simple Computations

### 5.1 Fundamental Groups and Homotopy Equivalence

**Lemma 5.1.** *Suppose  $f_0, f_1 : X \rightarrow Y$  are homotopic by the homotopy  $H : X \times [0, 1] \rightarrow Y$ . Let  $x_0 \in X$  be a basepoint and define  $h : [0, 1] \rightarrow Y$  by  $t \mapsto H(x_0, t)$ . Then the following diagram commutes:*

$$\begin{array}{ccc} \pi_1(X, x_0) & \xrightarrow{(f_0)_*} & \pi_1(Y, f_0(x_0)) \\ & \searrow (f_1)_* & \uparrow \phi_h \\ & & \pi_1(Y, f_1(x_0)) \end{array}$$

*Proof.* Fix an arbitrary  $[\gamma] \in \pi_1(X, x_0)$ , and we construct a homotopy from  $h * (f_1 \circ \gamma) * \bar{h}$  to  $f_0 \circ \gamma$ . The picture is the following: we have  $h * (f_1 \circ \gamma) * \bar{h}$  at the top and  $f_0 \circ \gamma$  at the bottom, where  $t$  parametrizes the horizontal direction and  $s$  parametrizes the vertical direction. Draw a trapezoidal shape by connecting the middle two points on the top edge to the two bottom corners.

Define the homotopy as follows: For a fixed  $s$ , define  $H'(t, s)$  first by  $h_s(t)$  in the first third,  $H(\gamma(t), s)$  in the second third, and then  $\bar{h}_s(t)$  in the last third. Construct the explicit homotopy as an exercise.  $\square$

**Theorem 5.1.** *If  $f : X \rightarrow Y$  is a homotopy equivalence, then the induced map*

$$f_* : \pi_1(X, x_0) \rightarrow \pi_1(Y, f(x_0))$$

*on fundamental groups is an isomorphism.*

*Proof.* Let  $g$  be a homotopy inverse to  $f$ , so we have the composition:

$$\pi_1(X, x_0) \xrightarrow{f_*} \pi_1(Y, f(x_0)) \xrightarrow{g_*} \pi_1(X, g(f(x_0)))$$

But we know  $g \circ f = \text{id}_X$ , so by the lemma there exists a path  $h : X \rightarrow X$  from  $x_0$  to  $g(f(x_0))$  such that  $g_* \circ f_* = \phi_h$  is an isomorphism. So  $f_*$  is injective. Similarly,  $f_* \circ g_*$  is an isomorphism, and therefore  $f_*$  is surjective. Since we already know  $f_*$  is a homomorphism, this shows that  $f_*$  is an isomorphism.  $\square$

**Remark.** Recall that we have defined a “functor”

$$\{\text{pointed topological spaces, pointed maps}\} \rightarrow \{\text{groups, homomorphisms}\},$$

where homotopy equivalent spaces are mapped to isomorphic groups and homotopic maps give rise to the “same” homomorphism. We will finally make some computations of fundamental groups next.

## 5.2 Simple Computations of Fundamental Groups

**Lemma 5.2.** *If  $X$  is contractible, then  $\pi_1(X, x_0) = \{1\}$  for all  $x_0 \in X$ , where  $\{1\}$  is the trivial group.*

*Proof.* If  $Y = \{y_0\}$  is a one-point space, then there exists a unique loop  $\gamma : [0, 1] \rightarrow Y$  given by  $t \mapsto y_0$ . So  $\pi_1(Y, y_0) = \{1\}$ . Since  $X$  is contractible, it is homotopy equivalent to  $Y$  and so  $\pi_1(X, x_0) = \{1\}$ .  $\square$

**Definition 5.1.** We say that a space  $X$  is *simply connected* if

1.  $X$  is path-connected, and
2.  $\pi_1(X, x_0) = \{1\}$  for some  $x_0 \in X$ .

**Remark.** Simply connected means that “points in  $X$  are connected in a very simple way.”

**Lemma 5.3.** *A space  $X$  is simply connected if and only if every two points in  $X$  are connected by a unique homotopy class of paths in  $X$ .*

*Proof.* ( $\Leftarrow$ ) Clearly  $X$  is path-connected. Furthermore, any loop based at  $x_0$  is a path from  $x_0$  to itself, and the constant is as well. Thus any loop is homotopic to the constant loop, i.e.  $\pi_1(X, x_0) = \{1\}$ .

( $\Rightarrow$ ) For any  $a, b \in X$ , there exists a path from  $a$  to  $b$  (since  $X$  is path-connected). Now suppose  $\gamma, \delta : [0, 1] \rightarrow X$  are paths from  $a$  to  $b$ . By  $\pi_1(X, a) = \{1\}$  we know that  $\gamma * \bar{\delta} \sim e_a$ , so

$$\gamma \sim \gamma * (\bar{\delta} * \delta) \sim (\gamma * \bar{\delta}) * \delta \sim e_a * \delta \sim \delta,$$

i.e.  $\gamma$  and  $\delta$  are in the same homotopy class.  $\square$

**Lemma 5.4.** *Let  $X = A \cup B$ , where  $A, B, A \cap B$  are open and path-connected. Let  $x_0 \in A \cap B$ . Then any loop  $\gamma : [0, 1] \rightarrow X$  based at  $x_0$  can be written as*

$$\gamma \sim \gamma_1 * \gamma_2 * \cdots * \gamma_n,$$

where each  $\gamma_i$  is a loop in  $A$  or  $B$  based at  $x_0$ .

*Proof.* We first claim that there exist  $0 = t_0 < t_1 < \cdots < t_n = 1$  such that  $\text{im } \gamma|_{[t_{i-1}, t_i]} \subseteq A$  or  $B$ , and  $\gamma(t_i) \in A \cap B$  for every  $i$ . The proof of this will use the following topology fact:

**Lemma** (Lebesgue number lemma). Let  $X$  be a compact metric space and  $\{U_\alpha\}_{\alpha \in A}$  be an open cover. Then there exists a *Lebesgue number*  $\delta > 0$  such that for all sets  $S$  with  $\text{diam}(S) = \sup_{x, y \in S} d(x, y) < \delta$ , there exists  $\alpha \in A$  such that  $S \subseteq U_\alpha$ .

To prove the claim, let  $U_1 = \gamma^{-1}(A)$  and  $U_2 = \gamma^{-1}(B)$ , which is an open cover of  $[0, 1]$ . So there exists  $\delta > 0$  such that if  $|b - a| < \delta$ , then  $[a, b] \subseteq U_i$  for  $i = 1$  or  $2$ . Thus  $\gamma([a, b]) \subseteq A$  or  $B$ . Now let  $n$  be a positive integer such that  $1/n < \delta$ , so that for each  $i$  we have

$$\text{im } \gamma|_{[i/n, (i+1)/n]} \subseteq A \text{ or } B.$$

So we start with  $t_i = i/n$  for  $i = 0, \dots, n$ . Now if  $\gamma|_{[t_{i-1}, t_i]}$  and  $\gamma|_{[t_i, t_{i+1}]}$  both have image in  $A$  (or both in  $B$ ), then throw out  $t_i$ . Then  $\gamma|_{[t_{i-1}, t_i]}$  will have image in  $A$  or  $B$  and  $\gamma(t_i) \in A \cap B$ , as desired.

Now given the claim, let  $\delta_i : [0, 1] \rightarrow A \cap B$  connect  $x_0$  to  $\gamma(t_i)$ , and set  $\gamma_i = \gamma|_{[t_{i-1}, t_i]}$ . Then

$$\gamma \sim \gamma_1 * \gamma_2 * \cdots * \gamma_n \sim (\gamma_1 * \bar{\delta}_1) * (\delta_1 * \gamma_2 * \bar{\delta}_2) * \cdots * (\delta_{n-1} * \gamma_n),$$

where each of the above loops is either in  $A$  or  $B$ .  $\square$

**Theorem 5.2.** *We have  $\pi_1(S^n, x_0) = \{1\}$  for all  $n \geq 2$ .*

*Proof.* We have  $\pi_1(S^n, x_0) = \{1\}$  for all  $n \geq 2$ . Let  $A = S^n \setminus \{(0, \dots, 0, 1)\}$  and  $B = S^n \setminus \{(0, \dots, 0, -1)\}$ . Note that  $A \cong B \cong \mathbb{R}^n$ , so they are path-connected. Furthermore,  $A \cap B = S^{n-1} \times \mathbb{R}$ , which is also path-connected if  $n \geq 2$ .<sup>1</sup> Now take any  $x_0 \in A \cap B$ . Then any  $[\gamma] \in \pi_1(S^n, x_0)$  can be written as

$$[\gamma] = [\gamma_1] * [\gamma_2] * \cdots * [\gamma_n],$$

where  $[\gamma_i] \in \pi_1(A, x_0)$  or  $\pi_1(B, x_0)$  by the lemma. But we have

$$\pi_1(A, x_0) = \pi_1(B, x_0) = \{1\},$$

so  $[\gamma] = [e_{x_0}]$  and hence  $\pi_1(S^n, x_0) = \{1\}$ . □

**Theorem 5.3.** *Given two spaces  $X$  and  $Y$ , we have  $\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0)$ .*

*Proof.* The map  $\Phi : \pi_1(X, x_0) \times \pi_1(Y, y_0) \rightarrow \pi_1(X \times Y, (x_0, y_0))$  given by

$$([\gamma], [\delta]) \mapsto [\gamma \times \delta],$$

where  $(\gamma \times \delta)(t) = (\gamma(t), \delta(t))$ , is an isomorphism. Check as an exercise that  $\Phi$  is well-defined and a bijection (for the second part, consider the projections  $p_X : X \times Y \rightarrow X$  and  $p_Y : X \times Y \rightarrow Y$ ). □

## 5.3 Fundamental Group of the Circle

Our next objective is the following computation:

**Theorem 5.4.** *We have  $\pi_1(S^1, (1, 0)) \cong \mathbb{Z}$ . In particular, the map sending  $n \in \mathbb{Z}$  to*

$$\gamma_n : [0, 1] \rightarrow S^1 : t \mapsto (\cos 2\pi nt, \sin 2\pi nt)$$

*is an isomorphism  $\mathbb{Z} \rightarrow \pi_1(S^1, (1, 0))$ .*

**Remark.** The proof is an example of a very important technique that we will see again soon. The proof involves studying the map

$$p : \mathbb{R} \rightarrow S^1 : t \mapsto (\cos 2\pi t, \sin 2\pi t).$$

Note that  $p^{-1}((1, 0)) = \mathbb{Z}$ . This is a particular example of a *covering map*, which we will study later.

**Definition 5.2.** If  $\gamma : [0, 1] \rightarrow S^1$  is a path based at the point  $(1, 0)$ , then a *lift of  $\gamma$  based at  $n \in \mathbb{Z}$*  is a map  $\tilde{\gamma}_n : [0, 1] \rightarrow \mathbb{R}$  such that  $\tilde{\gamma}_n(0) = n$  and  $p \circ \tilde{\gamma}_n = \gamma$ .

**Lemma 5.5.** *We have the following:*

- (a) *For each  $n \in \mathbb{Z}$ , each loop  $\gamma : [0, 1] \rightarrow S^1$  based at  $(1, 0)$  lifts to a unique path  $\tilde{\gamma}_n$  based at  $n$ .*
- (b) *If  $\gamma \sim \gamma'$  are loops in  $S^1$  based at  $(1, 0)$  and  $\tilde{\gamma}_n, \tilde{\gamma}'_n$  are their lifts based at  $n$ , then  $\tilde{\gamma}_n \sim \tilde{\gamma}'_n \text{ rel } \{0, 1\}$ .*

*The above properties are called path lifting and homotopy lifting.*

*Proof.* We will prove this next class. □

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<sup>1</sup>Note that this does not work for  $n = 1$ : the intersection  $S^0 \times \mathbb{R} = \{\pm 1\} \times \mathbb{R}$  is not path-connected.

*Proof of Theorem 5.4.* Given  $[\gamma] \in \pi_1(S^1, (1, 0))$ , part (a) of Lemma 5.5 says that there is a unique lift  $\tilde{\gamma}_0 : [0, 1] \rightarrow \mathbb{R}$ . Since  $\tilde{\gamma}_0(1) \in p^{-1}((1, 0)) = \mathbb{Z}$ , we can define  $\Phi : \pi_1(S^1, (1, 0)) \rightarrow \mathbb{Z}$  by  $[\gamma] \mapsto \tilde{\gamma}_0(1)$ . Part (b) of Lemma 5.5 says that  $\Phi$  is well-defined. We need to show the following:

1.  $\Phi$  is surjective:

Let  $\tilde{\delta}^n(t) = nt$  for  $t \in [0, 1]$  and  $\delta^n = p \circ \tilde{\delta}^n$ . Then  $\tilde{\delta}^n$  is the lift of  $\delta^n$  based at 0, and  $\Phi([\delta^n]) = n$ .

2.  $\Phi$  is injective:

Suppose  $\gamma, \gamma'$  are loops in  $S^1$  such that

$$\Phi([\gamma]) = \tilde{\gamma}_0(1) = \tilde{\gamma}'_0(1) = \Phi([\gamma']).$$

Set  $\tilde{H}(s, t) = (1 - t)\tilde{\gamma}_0(s) + t\tilde{\gamma}'_0(s)$  and  $H = p \circ \tilde{H}$ . Then  $H$  is a homotopy from  $\gamma$  to  $\gamma'$  (exercise).

3.  $\Phi$  is a homomorphism:

Let  $[\gamma], [\gamma'] \in \pi_1(S^1, (1, 0))$  and let  $\tilde{\gamma}_0, \tilde{\gamma}'_0$  be their lifts based at 0. Then

$$\Phi([\gamma]) = \tilde{\gamma}_0(1) = n \in \mathbb{Z} \quad \text{and} \quad \Phi([\gamma']) = \tilde{\gamma}'_0(1) = m \in \mathbb{Z}.$$

Then note the following:

(a)  $t \mapsto n + \tilde{\gamma}'_0(t)$  is a lift of  $\gamma'$  and starts at  $n$ , so by uniqueness it is  $\tilde{\gamma}'_n$ , and

(b)  $\tilde{\gamma}_0 * \tilde{\gamma}'_n$  is a lift of  $\gamma * \gamma'$ .

So  $\Phi([\gamma] * [\gamma']) = \tilde{\gamma} * \tilde{\gamma}'_0(1) = n + m = \Phi([\gamma]) + \Phi([\gamma'])$ , i.e.  $\Phi$  is a homomorphism.

Thus  $\Phi$  is an isomorphism, and we have  $\pi_1(S^1, (1, 0)) \cong \mathbb{Z}$ . □

# Lecture 6

## Jan. 27 — Fundamental Group of the Circle

### 6.1 Path Lifting

*Proof of Lemma 5.5.* (a) Let  $A = S^1 \setminus \{(1, 0)\}$ , and note that

$$p^{-1}(A) = \bigcup_{i \in \mathbb{Z}} (i, i+1) = \bigcup_{i \in \mathbb{Z}} A_i.$$

Notice that each restriction  $p|_{A_i} : A_i \rightarrow A$  is a homeomorphism. Now let  $B = S^1 \setminus \{(-1, 0)\}$ , so

$$p^{-1}(B) = \bigcup_{i \in \mathbb{Z}} \left(i - \frac{1}{2}, i + \frac{1}{2}\right) = \bigcup_{i \in \mathbb{Z}} B_i.$$

Similarly, each  $p|_{B_i} : B_i \rightarrow B$  is a homeomorphism. Now if  $\gamma : [0, 1] \rightarrow S^1$  is contained in  $A$  (or  $B$ ), we can choose any  $i \in \mathbb{Z}$  and let  $\tilde{\gamma} = (p|_{A_i})^{-1} \circ \gamma$ , giving a lift of  $\gamma$ . Then for a general  $\gamma : [0, 1] \rightarrow S^1$  with  $\gamma(0) = (1, 0)$ , the set  $\{\gamma^{-1}(A), \gamma^{-1}(B)\}$  is an open cover of the compact metric space  $[0, 1]$ , so there exists a Lebesgue number  $\delta > 0$  such that any interval  $[a, b]$  with  $b - a < \delta$  lies in either  $\gamma^{-1}(A)$  or  $\gamma^{-1}(B)$ . Choose  $n$  such that  $1/n < \delta$ . If  $t_n = i/n$  for  $i = 0, \dots, n$ , then

$$\gamma([t_i, t_{i+1}]) \subseteq A \text{ or } B$$

for every  $i$ . Again for convenience, if  $[t_{i-1}, t_i]$  and  $[t_i, t_{i+1}]$  are both in  $\gamma^{-1}(A)$  or  $\gamma^{-1}(B)$ , then discard  $t_i$ . So we have a partition  $0 = t_0 < t_1 < \dots < t_k = 1$  such that (note that  $\gamma$  starts at  $(1, 0) \notin A$ )

$$\gamma([t_i, t_{i+1}]) \subseteq \begin{cases} A & \text{if } i \text{ is odd,} \\ B & \text{if } i \text{ is even.} \end{cases}$$

Then we want to build  $\tilde{\gamma}_n$ . Define  $\tilde{\gamma}_n$  on  $[t_0, t_1]$  to be  $(p|_{B_n})^{-1} \circ \gamma|_{[t_0, t_1]}$ . Now  $\tilde{\gamma}_n(t_1) \in A_i$  for a unique  $i$ , so define  $\tilde{\gamma}_n$  on  $[t_1, t_2]$  by  $(p|_{A_i})^{-1} \circ \gamma|_{[t_1, t_2]}$ . Note that  $\tilde{\gamma}_n$  is continuous on  $[t_0, t_2]$  since the two definitions agree at  $t = t_1$ . Inductively continue to define the lift  $\tilde{\gamma}_n$  on all of  $[0, 1]$ .

(b) The proof is very similar to path lifting. Given a homotopy  $H : [0, 1] \times [0, 1] \rightarrow S^1$ , we can find a Lebesgue number  $\delta > 0$  for  $\{H^{-1}(A), H^{-1}(B)\}$ . Pick  $n$  such that  $\sqrt{2}/n < \delta$  and break  $[0, 1] \times [0, 1]$  into  $n^2$  squares of side length  $1/n$ . The diameter of each square is at most  $\sqrt{2}/n$ , so each square can be lifted as above. Finish the construction as an exercise to lift  $H$  on all of  $[0, 1] \times [0, 1]$ .  $\square$

### 6.2 Applications of the Fundamental Group of $S^1$

**Corollary 6.0.1.** *There is no retraction  $D^2 \rightarrow \partial D^2$ .*

*Proof.* Suppose there was a retraction  $r : D^2 \rightarrow \partial D^2$ , and let  $i : S^1 \rightarrow D^2$  be the inclusion of  $S^1$  as the boundary of  $D^2$ . Then we have the composition:

$$S^1 \xrightarrow{i} D^2 \xrightarrow{r} S^1$$

Noting that  $r \circ i = S^1 \rightarrow S^1$  is the identity, so  $(r \circ i)_* : \pi_1(S^1, (1, 0)) \rightarrow \pi_1(S^1, (1, 0))$  is the identity map. In particular,  $r_* \circ i_* = (r \circ i)_*$  is the identity map, hence  $i_*$  must be injective. But

$$i_* : \pi_1(S^1, (1, 0)) \rightarrow \pi_1(D^2, (1, 0))$$

where  $\pi_1(S^1, (1, 0)) \cong \mathbb{Z}$  and  $\pi_1(D^2, (1, 0)) = \{1\}$ , so  $i_*$  cannot be injective. Contradiction.  $\square$

**Corollary 6.0.2.** *Any map  $f : D^2 \rightarrow D^2$  has a fixed point, i.e.  $x \in D^2$  such that  $f(x) = x$ .*

*Proof.* Suppose otherwise that  $f : D^2 \rightarrow D^2$  has no fixed points. Then for each  $x \in D^2$ , there is a unique ray  $R_x$  starting at  $f(x)$  and going through  $x$ . Note that  $R_x \cap \partial D^2$  in a unique point (on the interior of  $R_x$ ). Define  $r : D^2 \rightarrow S^1$  by  $x \mapsto R_x \cap \partial D^2$ . Show that  $r$  is continuous as an exercise (e.g. parametrize the line). But then  $r$  is a retraction  $D^2 \rightarrow \partial D^2$ , a contradiction.  $\square$

**Remark.** There are more applications such as the fundamental theorem of algebra, the ham sandwich theorem, and the Borsuk-Ulam theorem. See Hatcher for more details.

## 6.3 Free Products of Groups

**Definition 6.1.** Let  $G_1$  and  $G_2$  be groups. A *word* in  $G_1 \sqcup G_2$  is a finite sequence

$$x = (x_1, x_2, \dots, x_n)$$

for some  $n$ , where each  $x_i$  is in  $G_1$  or  $G_2$ . Define an equivalence relation on words in  $G_1 \sqcup G_2$  which is generated by (show as an exercise that this is in fact an equivalence relation):

1. replace  $a, b$  in a sequence by  $ab$  if  $a, b$  are in the same group (or the reverse of this), and
2. if  $e_i$  (the identity in  $G_i$ ) is in a sequence, then remove it (or add it in any place in a sequence).

Denote the equivalence class of a word  $x$  by  $[x]$ . Call a word  $x = (x_1, \dots, x_n)$  *reduced* if

1.  $x_j \neq e_i$  for any  $j$  or  $i$ , and
2.  $x_i$  and  $x_{i+1}$  are from different groups.

Show that each  $[x]$  contains a unique reduced word (note: uniqueness is hard). The *free product* of  $G_1$  and  $G_2$  is the group  $G_1 * G_2$  of all equivalence classes of words in  $G_1 \sqcup G_2$ , with multiplication

$$[x_1, \dots, x_n] \cdot [y_1, \dots, y_m] = [x_1, \dots, x_n, y_1, \dots, y_m].$$

**Remark.** Note that in  $G_1 * G_2$ , the identity  $e$  is the empty word and the inverse is given by

$$[x_1, \dots, x_n]^{-1} = [x_n^{-1}, \dots, x_1^{-1}].$$

Check as an exercise that  $G_1 * G_2$  is in fact a group (really only need to check associativity).

**Proposition 6.1.** *Let  $j_i : G_i \rightarrow G_1 * G_2$  be the inclusion of  $G_i$  into  $G_1 * G_2$ . Given any homomorphisms  $\phi_i : G_i \rightarrow H$  where  $H$  is any group, there exists a unique homomorphism  $\phi : G_1 * G_2 \rightarrow H$  such that  $\phi \circ j_i = \phi_i$ , i.e. the following diagram commutes:*

$$\begin{array}{ccccc}
 & & G_1 & & \\
 & & \downarrow j_1 & \searrow \phi_1 & \\
 G_2 & \xrightarrow{j_2} & G_1 * G_2 & \xrightarrow{\phi} & H \\
 & \searrow \phi_2 & & & 
 \end{array}$$

*Proof.* If the  $x_i$  are reduced and  $x_1 \in G_1$ , then define

$$\phi_1(x_1, x_2, \dots, x_n) = \phi_1(x_1) \cdot \phi_2(x_2) \cdot \phi_1(x_3) \cdot \dots$$

Then check the following as an exercise:

1. Show such that  $\phi$  exists and is unique.
2. Show this property *defines* the free product, i.e. if  $D$  is another group satisfying the property in the proposition, then  $D \cong G_1 * G_2$ .

The second part above says that this is the *universal property* of the free product. □

**Example 6.1.1.** Represent  $\mathbb{Z}$  in product notation via  $\{x^n\}$ , where  $x^n x^m = x^{n+m}$ . Then

$$\mathbb{Z} * \mathbb{Z} = \{x^n\} * \{y^m\} = \{e, x^{n_1} y^{m_1} \dots x^{n_k}, x^{n_1} y^{m_1} \dots y^{m_k}, y^{m_1} x^{n_1} \dots y^{m_k}, y^{m_1} x^{n_1} \dots x^{n_k}\}.$$

This group is called the *free group on two generators*, and  $\mathbb{Z}$  is the *free group on one generator*.

# Lecture 7

## Jan. 29 — Some Group Theory

### 7.1 Group Presentations

**Definition 7.1.** The *free group* on  $n$  generators, denoted  $F_n$ , is defined inductively via

$$F_n = F_{n-1} * \mathbb{Z},$$

where  $F_1 = \mathbb{Z}$ . (One can also consider  $F_\infty$ .)

**Remark.** Note that any homomorphism  $\phi : \mathbb{Z} \rightarrow G$  (for any group  $G$ ) is determined by  $\phi(1)$ . Moreover, given any  $g \in G$ , there is a unique homomorphism which maps  $1 \mapsto g$ . Thus by the universal property of the free product, a homomorphism  $F_n \rightarrow G$  is determined uniquely by a choice of  $g_1, \dots, g_n \in G$ .

**Definition 7.2.** A *group presentation* is a group  $\langle X|R \rangle$  defined as follows:

- $X$  is some set (of generators);
- $R$  is a set of words (relations) in  $X \cup X^{-1}$  (formally denote  $x \in X$  as  $x^{-1} \in X^{-1}$ );
- let  $n = |X|$  and  $F_n$  be the free group on  $n$  generators, so that we can think of  $R \subseteq F_n$ ;
- let  $\langle R \rangle$  be the smallest normal subgroup of  $F_n$  containing  $R$ ;
- define the group  $\langle X|R \rangle = F_n / \langle R \rangle$ .

We say that  $\langle X|R \rangle$  is a *presentation* of a group  $G$  if  $G \cong \langle X|R \rangle$ .

**Example 7.2.1.** The group  $\langle g|g^n \rangle$  is all the words in  $g, g^{-1}$ :

$$\dots, g^{-2}, g^{-1}, e, g, g^2, g^3, \dots,$$

but  $g^n = e$ , so  $g^{n+1} = g^n g = eg = g$  and thus we have  $g^{-1} = eg^{-1} = g^n g^{-1} = g^{n-1} g g^{-1} = g^{n-1}$ . So there is a one-to-one correspondence between elements of  $\langle g|g^n \rangle$  and  $g^k$  for  $k = 0, \dots, n-1$ .

**Exercise 7.1.** Show that  $\langle g|g^n \rangle \cong \mathbb{Z}/n$ , so that  $\langle g|g^n \rangle$  is a presentation of  $\mathbb{Z}/n$ .

**Lemma 7.1.** *Every group has a presentation.*

*Proof.* Let  $G$  be a group. Let  $X \subseteq G$  be a collection of elements of  $G$  that generate  $G$  (e.g. take  $X = G$  itself). Let  $n = |X|$ , so there exists a unique  $\phi : F_n \rightarrow G$  sending the generators of  $F_n$  to the  $g_i \in X$ . Let  $N = \ker \phi$ , so the first isomorphism theorem says that  $G \cong F_n / N$  (note that  $\phi$  is clearly surjective). Let  $R$  be a subset of  $N$  that generates  $N$  (e.g. take  $R = N$ ). Then  $G \cong \langle X|R \rangle$ .  $\square$

**Remark.** Using fewer generators to write  $G$  or  $N$  may give more less complicated presentations of  $G$ .



**Definition 7.3.** We say that  $G$  is *finitely generated* if  $G \cong \langle X | R \rangle$  such that  $|X| < \infty$ . We say that  $G$  is *finitely presented* if both  $|X|, |R| < \infty$ .

**Exercise 7.2.** Show the following:

1. If  $G = \langle g_1, \dots, g_n | r_1, \dots, r_m \rangle$ , then for any group  $H$  and any map  $h : \{g_1, \dots, g_n\} \rightarrow H$  satisfying  $h(r_i) = e_H$  (the notation  $h(r_i)$  means to replace any letters  $g_j$  in  $r_i$  by  $h(g_j)$ ), there exists a unique homomorphism  $\phi_h : G \rightarrow H$  such that  $\phi_h(g_i) = h(g_i)$ .
2. If  $G_1 = \langle g_1, \dots, g_n | r_1, \dots, r_m \rangle$  and  $G_2 = \langle h_1, \dots, h_k | s_1, \dots, s_\ell \rangle$ , then

$$G_1 * G_2 = \langle g_1, \dots, g_n, h_1, \dots, h_k | r_1, \dots, r_m, s_1, \dots, s_\ell \rangle.$$

**Definition 7.4.** Given groups  $G_1, G_2$  and  $K$ , and homomorphisms  $\psi_i : K \rightarrow G_i$ , the *free product with amalgamation* is

$$G_1 *_K G_2 = \frac{G_1 * G_2}{\langle \{\psi_1(k)\psi_2(k)^{-1}\}_{k \in K} \rangle},$$

where  $\langle \{\psi_1(k)\psi_2(k)^{-1}\}_{k \in K} \rangle$  is the smallest normal subgroup of  $G_1 * G_2$  containing  $\{\psi_1(k)\psi_2(k)^{-1}\}_{k \in K}$ .

**Remark.** The idea is that  $G_1 *_K G_2$  is the set of all words in  $G_1 \cup G_2$  but if we see  $\psi_1(k)$  in a word, we can replace it by  $\psi_2(k)$  and vice versa. In terms of group presentations, if

$$\begin{aligned} G_1 &= \langle g_1, \dots, g_n | r_1, \dots, r_m \rangle, \\ G_2 &= \langle g'_1, \dots, g'_{n'} | r'_1, \dots, r'_{m'} \rangle, \\ K &= \langle h_1, \dots, h_k | s_1, \dots, s_\ell \rangle, \end{aligned}$$

then we can write

$$G_1 *_K G_2 = \langle g_1, \dots, g_n, g'_1, \dots, g'_{n'} | r_1, \dots, r_m, r'_1, \dots, r'_{m'}, \psi_1(h_1)(\psi_2(h_1))^{-1}, \dots, \psi_1(h_k)(\psi_2(h_k))^{-1} \rangle.$$

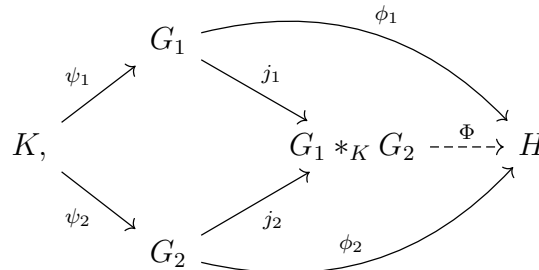
Note that the  $s_i$  show up in the above since  $\psi_1, \psi_2$  are homomorphisms.

**Exercise 7.3.** Show the following:

1. Check that the above presentation for  $G_1 *_K G_2$  is correct.
2. Let  $\iota_i : G_i \rightarrow G_1 * G_2$  be the inclusions and  $j_i : G_i \rightarrow G_1 *_K G_2$  be the induced maps. Then given any homomorphisms  $\phi_i : G_i \rightarrow H$  (where  $H$  is any group) such that

$$\phi_1 \circ \psi_1(k) = \phi_2 \circ \psi_2(k) \quad \text{for all } k \in K,$$

then there exists a unique homomorphism  $\Phi : G_1 *_K G_2 \rightarrow H$  such that  $\Phi \circ j_i = \phi_i$ , i.e.



Show that this is the *universal property* for the free product with amalgamation.

## 7.2 Seifert-van Kampen Theorem

**Theorem 7.1** (Seifert-van Kampen). *Let  $X$  be a topological space with base point  $x_0$ . Let  $A, B \subseteq X$  be open sets with  $X = A \cup B$  such that  $A, B, A \cap B$  are path-connected and  $x_0 \in A \cap B$ . Let*

$$\psi_A : \pi_1(A, x_0) \rightarrow \pi_1(X, x_0) \quad \text{and} \quad \psi_B : \pi_1(B, x_0) \rightarrow \pi_1(X, x_0)$$

*be the homomorphisms induced from the inclusions  $A \cap B \rightarrow A$  and  $A \cap B \rightarrow B$ . Then*

$$\pi_1(X, x_0) \cong \pi_1(A, x_0) *_{\pi_1(A \cap B, x_0)} \pi_1(B, x_0).$$

**Remark.** There is a more general version where  $X = \bigcup_{\alpha \in A} U_\alpha$ . See Hatcher for more details.

**Example 7.4.1.** Consider  $W_2 = S^1 \vee S^1$ , a wedge of two circles. Let  $x_0$  be the point of intersection of the circles. Let  $A$  be an open neighborhood of the left circle and  $B$  be an open neighborhood of the right circle. Note that  $A \simeq B \simeq S^1$  and  $A \cap B \simeq \{\text{pt}\}$ . So we see that

$$\pi_1(A, x_0) \cong \mathbb{Z} \cong \langle g_1 | \rangle, \quad \pi_1(B, x_0) \cong \mathbb{Z} \cong \langle g_2 | \rangle, \quad \pi_1(A \cap B, x_0) = \{e\},$$

and so  $\psi_A : \pi_1(A \cap B, x_0) \rightarrow \pi_1(A, x_0)$  and  $\psi_B : \pi_1(A \cap B, x_0) \rightarrow \pi_1(B, x_0)$  must both map  $e \mapsto e$ . Thus

$$\pi_1(W_2, x_0) \cong \pi_1(A, x_0) *_{\pi_1(A \cap B, x_0)} \pi_1(B, x_0) \cong \langle g_1, g_2 | \psi_A(e)(\psi_B(e))^{-1} \rangle \cong \langle g_1, g_2 | \rangle \cong F_2,$$

by the Seifert-van Kampen theorem.

**Exercise 7.4.** Show the following:

1. If  $W_n$  is a wedge of  $n$  circles, then  $\pi_1(W_n, x_0) \cong F_n$ .
2. We have  $\pi_1(\text{any connected graph}) = F_n$  for some  $n$ .

**Example 7.4.2.** Consider the torus  $T^2 = S^1 \times S^1$ . Recall that for a product, we know

$$\pi_1(T^2) \cong \pi_1(S^1) \times \pi_1(S^1) \cong \mathbb{Z} \times \mathbb{Z}.$$

We can also use van Kampen's theorem to see this. Think of  $T^2$  as a square with opposite sides identified. Let  $A$  be the square with a circle missing in the circle, which deformation retracts to the boundary of the square. Let  $B$  be a big disk in the middle, so that  $A \cap B$  deformation retracts to a circle. So

$$A \simeq W_2, \quad B \simeq \{\text{pt}\}, \quad A \cap B \simeq S^1.$$

Thus by our previous computations, we have

$$\pi_1(A, x_0) \cong \langle g_1, g_2 | \rangle, \quad \pi_1(B, x_0) \cong \{e\}, \quad \pi_1(A \cap B, x_0) \cong \langle h | \rangle.$$

The inclusion  $\psi_B : \pi_1(A \cap B, x_0) \rightarrow \pi_1(B, x_0)$  sends  $h \rightarrow e$ , and  $\psi_A : \pi_1(A \cap B, x_0) \rightarrow \pi_1(A, x_0)$  sends  $h \mapsto g_1 g_2 g_1^{-1} g_2^{-1}$ . To see the last claim, push a loop in  $A \cap B$  to the boundary of the square (note that under the homotopy equivalence of  $A$  and  $W_2$ , we can assume  $x_0$  maps to a corner point), where it follows the four edges of the square. These are the loops  $g_1, g_2, g_1^{-1}, g_2^{-1}$ , where the last two loops are oriented in the opposite direction of the first two. Thus by van Kampen's theorem,

$$\pi_1(T^2, x_0) \cong \pi_1(A, x_0) *_{\pi_1(A \cap B, x_0)} \pi_1(B, x_0) \cong \langle g_1, g_2 | \psi_A(h)(\psi_B(h))^{-1} \rangle = \langle g_1, g_2 | g_1 g_2 g_1^{-1} g_2^{-1} \rangle.$$

**Exercise 7.5.** Check the following:

1. Show that  $\mathbb{Z} \times \mathbb{Z} \cong \langle g_1, g_2 | g_1 g_2 g_1^{-1} g_2^{-1} \rangle$ , so that  $\pi_1(T^2) \cong \mathbb{Z} \times \mathbb{Z}$  again.
2. Compute  $\pi_1(\Sigma_g, x_0)$ , where  $\Sigma_g$  is the surface of genus  $g$ . Show that  $\pi_1(\Sigma_g)$  is not abelian if  $g > 1$ .

# Lecture 8

## Feb. 3 — Seifert-van Kampen Theorem

### 8.1 Applications of the Seifert-van Kampen Theorem

**Theorem 8.1.** *Let  $X$  be a path-connected space  $f : \partial D^n \rightarrow X$  be continuous with  $x_0 \in \partial D^n$ . Set*

$$Y = X \cup_f D^n = X \sqcup D^n / \{(x \in D^n) \sim (f(x) \in X)\}.$$

*Then (for  $n = 1$ , we need  $X$  to have a base point  $f(x_0)$  with an open neighborhood  $U \simeq \{f(x_0)\}$ )*

$$\pi_1(Y, y_0) = \begin{cases} \pi_1(X, f(x_0)) * \mathbb{Z} & \text{if } n = 1, \\ \pi_1(X, f(x_0)) / \langle r \rangle & \text{if } n = 2, \\ \pi_1(X, f(x_0)) & \text{if } n \geq 3, \end{cases}$$

*where  $r = f_*(g)$  where  $g$  generates  $\pi_1(\partial D^2, x_0) \cong \mathbb{Z}$ .*

*Proof.* We proof this in the case  $n = 2$ . Let

$$A = X \cup_f (D^2 \setminus \{0\}) = X \cup_f (S^1 \times (0, 1]) \simeq X$$

and  $B$  be the interior of  $D^2$  (so  $B \simeq \{\text{pt}\}$ ). Then we can see that

$$A \cap B = (\text{int } D^2) \setminus \{0\} = S^1 \times (0, 1) \simeq S^1.$$

Note that we can choose  $y_0$  to be  $f(x_0) \in X$  because the that is where it is sent under the deformation retraction from  $A$  to  $X$ . Now  $\psi_A : \pi_1(A \cap B, y_0) \rightarrow \pi_1(A, y_0)$

$$\pi_1(A \cap B, y_0) \cong \langle g \rangle \quad \text{and} \quad \pi_1(A, y_0) \cong \pi_1(X, f(x_0))$$

is given by  $g \mapsto f_*(g)$ , and  $\psi_B(g) = e$ . Thus the Seifert-van Kampen theorem implies

$$\pi_1(Y, y_0) \cong \pi_1(A, y_0) *_{\pi_1(A \cap B, y_0)} \pi_1(B, y_0) \cong \frac{\pi_1(A, y_0) * \{e\}}{\langle \psi_A(g)(\psi_B(g))^{-1} \rangle} \cong \frac{\pi_1(A, y_0)}{\langle r \rangle} \cong \frac{\pi_1(X, f(x_0))}{\langle r \rangle},$$

which is the desired result for  $n = 2$ . The  $n \geq 3$  case is similar (except the intersection is now contractible in this case). Check the  $n = 1$  case as an exercise.  $\square$

**Remark.** This allows us to compute the fundamental group of any CW complex (hence any manifold).

**Corollary 8.1.1.** *Let  $G$  be a finitely presented group. Then there exists a topological space  $X$  (in fact, a compact CW complex) such that  $\pi_1(X, x_0) \cong G$ .*

*Proof.* Let  $G = \langle g_1, \dots, g_n | r_1, \dots, r_m \rangle$  and  $W_n$  be the wedge of  $n$  circles, so that

$$\pi_1(W_n, x_0) \cong F_n \cong \langle g_1, \dots, g_n \rangle.$$

Now for each  $r_i$  let  $f_i : S^1 \rightarrow W_n$  be a map such that  $(f_i)_*(g) = r_i$  (show as an exercise that such  $f_i$  exists; essentially for each word, take the loops corresponding to each letter in order). Let

$$X = W_n \cup_{f_i} \left( \bigsqcup_{i=1}^m D^2 \right),$$

and the previous theorem tells us that  $\pi_1(X, x_0) \cong \langle g_1, \dots, g_n | r_1, \dots, r_m \rangle \cong G$ .  $\square$

**Remark.** The topological space realizing  $G$  as its fundamental group is not unique. For instance, we can take the above construction and add a 5-cell, which does not change the fundamental group.

## 8.2 Proof of the Seifert-van Kampen Theorem

*Proof of Theorem 7.1.* We have the inclusions  $A \subseteq X$  and  $B \subseteq X$ , which induce maps

$$\phi_A : \pi_1(A, x_0) \rightarrow \pi_1(X, x_0) \quad \text{and} \quad \phi_B : \pi_1(B, x_0) \rightarrow \pi_1(X, x_0).$$

By the universal property of free products, we get a map  $\Phi : \pi_1(A, x_0) * \pi_1(B, x_0) \rightarrow \pi_1(X, x_0)$  by

$$([\gamma_1], [\delta_1], [\gamma_2], \dots) \mapsto \phi_A([\gamma_1])\phi_B([\delta_1])\phi_A([\gamma_2]) \dots$$

Note that if  $[\gamma] \in \pi_1(A \cap B, x_0)$ , then  $\psi_A([\gamma]) = [\gamma] = \psi_B([\gamma])$ , so

$$\phi_A \circ \psi_A([\gamma]) = [\gamma] = \phi_B \circ \psi_B([\gamma]).$$

This tells us that  $\Phi(\psi_A([\gamma])(\psi_B([\gamma]))^{-1}) = e$ , so we see that

$$K = \langle \psi_A([\gamma])(\psi_B([\gamma]))^{-1} \rangle_{[\gamma] \in \pi_1(A \cap B, x_0)}$$

lies in the kernel of  $\Phi$ . This gives us an induced map (still called it  $\Phi$ )

$$\Phi : \pi_1(A, x_0) *_{\pi_1(A \cap B, x_0)} \pi_1(B, x_0) \rightarrow \pi_1(X, x_0).$$

Lemma 5.4 says that  $\Phi$  is surjective, so it suffices to check injectivity. To do this, let  $[\gamma_i] \in \pi_1(A, x_0)$  and  $[\eta_i] \in \pi_1(B, x_0)$  with

$$\Phi([\gamma_1][\eta_1] \dots [\gamma_n][\eta_n]) = [\gamma_1 * \eta_1 * \dots * \gamma_n * \eta_n] = e. \quad (*)$$

We need to see that we can get from the word  $[\gamma_1][\eta_1] \dots [\gamma_n][\eta_n]$  to the empty word by a sequence of:

1. replace  $a, b$  by  $a \cdot b$  if  $a, b$  are in the same group (and the reverse of this);
2. if we see  $\psi_A(k)$  in the word, we can replace it with  $\psi_B(k)$  (and the reverse of this).

We will prove the theorem only for  $n = 2$ . Now  $(*)$  says that there exists a homotopy  $H$  between  $x_0$  and  $\gamma_1 * \eta_1$ . As before, we can use the Lebesgue number lemma to find  $n$  such that squares of side length  $1/n$  are mapped by  $H$  into either  $A$  or  $B$  (we can assume that the number of  $\gamma_i, \eta_i$  divides  $n$ ). Check as an exercise that we can assume  $H(i/n, j/n) = x_0$ , i.e. that we can change  $H$  and  $\gamma_i, \eta_i$  by a homotopy

such that the homotopy of  $\gamma_i$  is in  $A$  and the homotopy of  $\eta_i$  is in  $B$  (hint: consider radial lines around  $(i/n, j/n)$ ). So we have an  $n \times n$  grid where each corner point is  $x_0$ , and the bottom edges are

$$\gamma_1 \sim \gamma'_1 * \gamma''_1 \quad \text{and} \quad \eta_1 \sim \eta'_1 * \eta''_1,$$

where these homotopies take place in  $A$  or  $B$ , respectively. Let  $a_1, a_2, a_3, a_4$  be the four edges lying above  $\gamma'_1, \gamma''_1, \eta'_1, \eta''_1$  on the grid (recall that  $n = 2$  in this case) We will show that we can go from  $[\gamma_1][\eta_1]$  to  $[a_1][a_2][a_3][a_4]$  using (1) and (2). Then we can inductively push this to the top, which is the empty word.

Let  $\delta_1, \delta_2, \delta_3$  be the three interior edges which connect the bottom row and the second-to-last row. Since each square maps to  $A$  or  $B$ , the first three squares lie in  $A$  and the last one lies in  $B$ . Let

$$G = \pi_1(A, x_0) \quad \text{and} \quad H = \pi_1(B, x_0).$$

Note that we have

$$\delta_1 * \eta'_1 \sim \delta_1 * \eta' * \delta_3 * \bar{\delta}_3 \text{ in } G, \quad \delta_1 * \eta'_1 * \delta_3 \sim a_1 * a_2 * a_3 \text{ in } G, \quad \delta_3 * \eta''_1 \sim a_4 \text{ in } H,$$

and thus we can write

$$\begin{aligned} [\gamma_1]^G [\eta_1]^H &= [\gamma_1]^G ([\eta'_1][\eta''_1])^H \stackrel{(1)}{=} [\gamma_1]^G [\eta'_1]^H [\eta''_1]^H \\ &\stackrel{(2)}{=} [\gamma_1]^G [\eta'_1]^G [\eta''_1]^H \\ &\stackrel{(1)}{=} ([\gamma_1][\eta'_1])^G [\eta''_1]^H = ([\gamma_1][\eta'_1][\delta_3][\bar{\delta}_3])^G [\eta''_1]^H \\ &\stackrel{(2)}{=} ([\gamma_1][\eta'_1][\delta_3])^G [\bar{\delta}_3]^H [\eta''_1]^H \\ &\stackrel{(1)}{=} ([a_1][a_2][a_3])^G ([\bar{\delta}_3][\eta''_1])^H = ([a_1][a_2][a_3])^G [a_4]^H \\ &\stackrel{(1)}{=} [a_1]^G [a_2]^G [a_3]^G [a_4]^H, \end{aligned}$$

which proves the statement for  $n = 2$ . See Hatcher for the general case.  $\square$

### 8.3 Covering Spaces

**Definition 8.1.** A *covering space* of a space  $X$  is a pair  $(\tilde{X}, p)$  where  $\tilde{X}$  is a space and  $p : \tilde{X} \rightarrow X$  such that every point  $x \in X$  has an *evenly covered neighborhood*. An open set  $U$  is *evenly covered* if

$$p^{-1}(U) = \text{disjoint union of open sets } \{U_\alpha\} \text{ in } \tilde{X}$$

such that  $p|_{U_\alpha} : U_\alpha \rightarrow U$  is a homeomorphism for every  $\alpha$ .

**Example 8.1.1.** The following are examples of covering maps:

1. If  $p : \tilde{X} \rightarrow X$  is a homeomorphism, then it is a covering map.
2. We saw that  $p : \mathbb{R} \rightarrow S^1$  given by  $t \mapsto (\cos 2\pi t, \sin 2\pi t)$  is a covering map.

**Exercise 8.1.** If  $(\tilde{X}, p)$  is a covering space of  $X$  and  $(\tilde{Y}, p')$  is a covering space of  $Y$ , then show that

$$p \times p' : \tilde{X} \times \tilde{Y} \rightarrow X \times Y, \quad (x, y) \mapsto (p(x), p'(y))$$

is a covering map. This gives a covering map  $\mathbb{R}^2 \rightarrow T^2$  by  $(t, s) \mapsto (\cos 2\pi t, \sin 2\pi t, \cos 2\pi s, \sin 2\pi s)$ .

# Lecture 9

## Feb. 5 — Covering Spaces

### 9.1 More on Covering Spaces

**Example 9.0.1.** The following are more examples of covering maps:

3. Define  $p_n : S^1 \rightarrow S^1$  by  $\theta \mapsto n\theta$ . These are covering maps for each  $n \in \mathbb{Z}$ .
4. Let  $X$  be a wedge of two circles (corresponding to  $a, b$ ), and let  $\tilde{X}$  be a circle with three outer circles attached, evenly spaced (the inner circle corresponding to  $a_1, a_2, a_3$  and the three outer circles corresponding to  $b_1, b_2, b_3$ ). Let  $p$  map  $a_i$  to  $a$  and  $b_i$  to  $b$ . This is a covering map.

Write out a formula for  $p$  and really check that  $p$  is a covering map as an exercise.

5. Again let  $X$  be a wedge of two circles, labeled  $a, b$ . Let  $\tilde{X}$  be a wedge of two circles, with an extra circle attached on either side. Let  $a_3, b_3$  be the extra circles on the left and right, let  $b_1, b_2$  be the top and bottom halves of the circle on the left, and let  $a_1, a_2$  be the top and bottom halves of the circles on the right. Let  $p$  map  $a_i$  to  $a$  and  $b_i$  to  $b$ . Then  $p$  is a covering map.
6. Consider the quotient map  $p : S^2 \rightarrow \mathbb{R}P^2$ . This map is a covering map. Note that each point in  $\mathbb{R}P^2$  has a neighborhood whose preimage is two open sets on opposite sides of  $S^2$ .

**Lemma 9.1.** *Let  $(\tilde{X}, p)$  be a covering space of a connected space  $X$ . Then the cardinality  $|p^{-1}(x)|$  is independent of  $x \in X$ .*

*Proof.* Fix  $x_0 \in X$  and let  $n = |p^{-1}(x_0)|$ . Let  $A = \{x \in X : |p^{-1}(x)| = n\}$ , and note that  $A \neq \emptyset$  since  $x_0 \in A$ . We will show that  $A$  is both open and closed, which implies that  $A = X$  by connectedness.

To see that  $A$  is open, let  $x \in A$ . By the definition of a covering space, there exists an open set  $U$  in  $X$  such that  $x \in U$  and  $p^{-1}(U) = \{U_1, \dots, U_n\}$ . Thus for any  $x' \in U$ , we have

$$p^{-1}(x') \subseteq p^{-1}(U) = \{U_1, \dots, U_n\},$$

and  $p^{-1}(x') \cap U_i = \{\text{pt}\}$  since  $p|_{U_i} : U_i \rightarrow U$  is a homeomorphism. Thus  $|p^{-1}(x')| = n$ , so  $x' \in A$ . This holds for each  $x' \in U$ , so  $U \subseteq A$ , i.e.  $A$  is an open set.

One can make a similar argument to check that if  $x \notin A$ , then there exists a open set  $U$  about  $x$  such that  $U \cap A = \emptyset$ . This shows that  $X \setminus A$  is open, i.e.  $A$  is closed, which completes the proof.  $\square$

**Definition 9.1.** We call  $|p^{-1}(x)|$  the *degree* of the covering space.

**Definition 9.2.** If  $(\tilde{X}, p)$  is a covering space for  $X$  and  $f : Y \rightarrow X$  is a continuous map, then a *lift* of  $f$  to  $\tilde{X}$  is a map  $\tilde{f} : Y \rightarrow \tilde{X}$  such that  $p \circ \tilde{f} = f$ , i.e. the following diagram commutes:

$$\begin{array}{ccc}
 & & \tilde{X} \\
 & \nearrow \tilde{f} & \downarrow p \\
 Y & \xrightarrow{f} & X
 \end{array}$$

If  $f(y_0) = x_0$  and  $\tilde{x}_0 \in \tilde{X}$  with  $p(\tilde{x}_0) = x_0$ , then  $\tilde{f}$  is a *lift of  $f$  based at  $\tilde{x}_0$*  if  $\tilde{f}$  is a lift and  $\tilde{f}(y_0) = \tilde{x}_0$ .

**Lemma 9.2.** *Let  $(\tilde{X}, p)$  be a covering space of  $X$ ,  $x_0 \in X$ , and  $\tilde{x}_0 \in p^{-1}(x_0)$ . Then*

- (a) *each path  $\gamma : [0, 1] \rightarrow X$  based at  $x_0$  has a unique lift  $\tilde{\gamma} : [0, 1] \rightarrow \tilde{X}$  based at  $\tilde{x}_0$ .*
- (b) *if  $H : Y \times [0, 1] \rightarrow X$  is a homotopy with  $h_0(y) = H(y, 0)$  and  $\tilde{h}_0 : Y \rightarrow \tilde{X}$  a lift of  $h_0$ , then there is a unique lift  $\tilde{H} : Y \times [0, 1] \rightarrow \tilde{X}$  of  $H$  such that  $\tilde{H}(y, 0) = \tilde{h}_0(y)$ .*

The above properties are called *path lifting* and *homotopy lifting*.

*Proof.* (a) The proof of this part is exactly the proof of part (a) in Lemma 5.5.

(b) The proof from Lemma 5.5 works if  $Y = [0, 1]$ . For the general case, see the proof of Theorem 10.1.  $\square$

**Lemma 9.3.** *If  $(\tilde{X}, p)$  is a path connected covering space of  $X$  and  $x_0 \in X$ ,  $\tilde{x}_0 \in p^{-1}(x_0)$ , then the map  $p_* : \pi_1(\tilde{X}, \tilde{x}_0) \rightarrow \pi_1(X, x_0)$  satisfies the following:*

- 1.  *$p_*$  is injective;*
- 2. *the image of  $p_*$  is the set of loops in  $\pi_1(X, x_0)$  that when lifted are loops in  $\tilde{X}$  based at  $\tilde{x}_0$ ;*
- 3. *the index  $[\pi_1(X, x_0) : p_*(\pi_1(\tilde{X}, \tilde{x}_0))]$  is the degree of  $(\tilde{X}, p)$ .*

*Proof.* (1) Suppose that  $p_*([\gamma]) = [e]$ , so there exists a homotopy  $H$  in  $X$  between  $x_0$  and  $p \circ \gamma$ . Note that  $\gamma$  is a lift of  $H(t, 0)$ , so Lemma 9.2 says  $H$  lifts to a homotopy  $\tilde{H}$  starting at  $\gamma$  in  $\tilde{X}$ . Note that  $H|_{\{0\} \times [0, 1]}$  is a constant loop and the loop  $t \mapsto \tilde{x}_0$  is a lift of  $H|_{\{0\} \times [0, 1]}$  based at  $\tilde{x}_0$ , so by uniqueness we see that  $\tilde{H}|_{\{0\} \times [0, 1]} = \tilde{x}_0$ . Similarly, we see that  $\tilde{H}|_{\{1\} \times [0, 1]} = \tilde{x}_0$  and  $\tilde{H}|_{[0, 1] \times \{1\}} = \tilde{x}_0$ . Thus  $\tilde{H}$  is a homotopy of loops based at  $\tilde{x}_0$  from  $\gamma$  to the constant loop, i.e.  $[\gamma] = [e_{\tilde{x}_0}]$ . So  $p_*$  is injective.

(2) Clearly if  $[\gamma] \in \pi_1(X, x_0)$  lifts to a loop  $\tilde{\gamma}$  based at  $\tilde{x}_0$ , then  $[\gamma] = p_*([\tilde{\gamma}])$ , so  $[\gamma]$  is in the image of  $p_*$ . Now if  $[\eta] = p_*([\gamma])$ , then  $\eta \sim p \circ \gamma$  in  $X$ . Let  $\tilde{\eta}$  be the lift of  $\eta$  based at  $\tilde{x}_0$ . By Lemma 9.2, the homotopy  $\eta \sim p \circ \gamma$  lifts to a homotopy  $\tilde{\eta} \sim \gamma$  rel endpoints. But  $\gamma$  is a loop, so  $\tilde{\eta}$  must be a loop.

(3) Let  $H = p_*(\pi_1(\tilde{X}, \tilde{x}_0)) \leq \pi_1(X, x_0)$ . If  $[\gamma] \in \pi_1(X, x_0)$  and  $[\delta] \in H$ , then note that by part (2),  $\delta$  lifts to a loop  $\tilde{\delta}$  based at  $\tilde{x}_0$ . Let  $\tilde{\delta * \gamma}$  be a lift of  $\delta * \gamma$  based at  $\tilde{x}_0$ , and note that  $\tilde{\gamma}(1) = \tilde{\delta * \gamma}(1) = \tilde{\delta} * \tilde{\gamma}(1)$ . This allows us to define

$$\phi : \{\text{right cosets of } H\} \rightarrow p^{-1}(x_0)$$

by  $H[\gamma] \mapsto \tilde{\gamma}(1)$ , which is well-defined by the above arguments.

If  $\tilde{x}_1 \in p^{-1}(x_0)$ , then let  $\tilde{\gamma}$  be a path in  $\tilde{X}$  from  $\tilde{x}_0$  to  $\tilde{x}_1$ . Let  $\gamma = p \circ \tilde{\gamma}$ , which is a loop in  $X$  based at  $x_0$ . Clearly  $\phi(H[\gamma]) = \tilde{\gamma}(1) = \tilde{x}_1$ , so  $\phi$  is onto. Now suppose that  $\phi(H[\gamma]) = \phi(H[\eta])$ . If  $\tilde{\gamma}, \tilde{\eta}$  are lifts of  $\gamma$  based at  $\tilde{x}_0$ , then  $\tilde{\gamma}(1) = \tilde{\eta}(1)$ . Thus  $\tilde{\gamma} * \tilde{\eta}^{-1}$  is a loop in  $\tilde{X}$ , so

$$p_*([\tilde{\gamma} * \tilde{\eta}^{-1}]) = [\gamma] * [\eta]^{-1} = [\gamma] * [\eta]^{-1} \in H.$$

This gives  $H[\gamma] = H[\eta]$ , so  $\phi$  is injective. Thus  $\phi$  is a bijection, so  $[\pi_1(X, x_0) : H] = |p^{-1}(x_0)|$ .  $\square$

**Example 9.2.1.** Recall the following examples from before:

1. For the covering map  $p : \mathbb{R} \rightarrow S^1$ , we have  $p_* : \pi_1(\mathbb{R}, 0) \rightarrow \pi_1(S^1, (1, 0))$  which sends  $e \mapsto 0$ , if we view  $p_*$  as  $p_* : \{e\} \rightarrow \mathbb{Z}$ . One can see all three properties of the above lemma in this example, in particular that  $[\pi_1(S^1) : \pi_1(\mathbb{R})] = \infty = |p^{-1}((1, 0))|$ .
2. Let  $p_n : S^1 \rightarrow S^1$  send  $\theta \mapsto n\theta$ . Then  $(p_n)_* : \pi_1(S^1, (1, 0)) \rightarrow \pi_1(S^1, (1, 0))$ , which we can view as a map  $(p_n)_* : \mathbb{Z} \rightarrow \mathbb{Z}$ , or  $(p_n)_* : \langle g \rangle \rightarrow \langle h \rangle$ . Then  $(p_n)_*$  maps  $g \mapsto h^n$ , so the subgroup of  $\mathbb{Z}$  corresponding to this covering space is  $n\mathbb{Z}$ . Again we can see all three properties of the above lemma, in particular that we have  $[\mathbb{Z} : n\mathbb{Z}] = n = |p_n^{-1}((1, 0))|$ .

**Exercise 9.1.** Check the properties of the above lemma explicitly for Example 9.0.1(4). Note that picking different base points can yield different images of  $p_*$ .



# Lecture 10

## Feb. 10 — Covering Spaces, Part 2

### 10.1 Covering Spaces, Continued

**Lemma 10.1.** *If  $(\tilde{X}, p)$  is a path connected covering space of  $X$  and  $x_0 \in X$ , then*

$$\{p_*(\pi_1(\tilde{X}, \tilde{x}))\}_{\tilde{x} \in p^{-1}(x_0)}$$

*is a conjugacy class of subgroups of  $\pi_1(X, x_0)$ .*

*Proof.* Let  $\tilde{x}_0, \tilde{x}_1 \in p^{-1}(x_0)$  and  $H_i = p_*(\pi_1(\tilde{X}, \tilde{x}_i))$ . Since  $\tilde{X}$  is path connected, we can find a path  $h$  in  $\tilde{X}$  from  $\tilde{x}_0$  to  $\tilde{x}_1$ . Clearly  $\gamma = p \circ h$  is a loop in  $X$  based at  $x_0$ . If  $[\eta] \in H_1$ , then by Lemma 9.3  $\eta$  lifts to a loop  $\tilde{\eta}$  based at  $\tilde{x}_1$ . Note that  $h * \tilde{\eta} * \bar{h}$  is a loop based at  $\tilde{x}_0$  in  $\tilde{X}$ . Then

$$[\gamma] \cdot [\eta] \cdot [\gamma]^{-1} = [(p \circ h) * (p \circ \tilde{\eta}) * (p \circ \bar{h})] = p_*([h * \tilde{\eta} * \bar{h}]) \in H_0.$$

This says that  $[\gamma]H_1[\gamma]^{-1} \subseteq H_0$ . The same argument says  $[\gamma]^{-1}H_0[\gamma] \subseteq H_1$ , so

$$H_0[\gamma][\gamma]^{-1}H_0[\gamma][\gamma]^{-1} \subseteq [\gamma]H_1[\gamma]^{-1}.$$

This implies that  $H_0 = [\gamma]H_1[\gamma]^{-1}$ , so  $H_0, H_1$  are conjugate.

We still need to show we get the full conjugacy class. For this, suppose  $H \leq \pi_1(X, x_0)$  such that there exists  $[\alpha] \in \pi_1(X, x_0)$  with  $[\alpha]H[\alpha]^{-1} = H_0$ . If  $[\alpha] \in H_0$ , then  $H = H_0$  and we are done. If  $[\alpha] \notin H_0$ , then  $\alpha$  lifts to a path  $\tilde{\alpha}$  (not necessarily a loop) based at  $\tilde{x}_0$ . Let  $\tilde{x}_2 = \tilde{\alpha}(1)$ , and set  $H_2 = p_*(\pi_1(\tilde{X}, \tilde{x}_2))$ . From above, we see that  $H = H_2$ , which completes the proof.  $\square$

**Definition 10.1.** A space  $X$  is *locally path connected* if for every  $y \in Y$  and open set  $U$  containing  $y$ , there exists an open set  $V$  such that  $y \in V \subseteq U$  and  $V$  is path connected.

**Example 10.1.1.** An example of a space which is path connected but not locally path connected is

$$C = (\{1/n\} \times [0, 1])_{n \in \mathbb{N}} \times (\{0\} \times [0, 1]) \cup ([0, 1] \times \{0\}),$$

also known as the *comb space*. We can see that  $C$  is path connected, but it is not locally path connected, e.g. take  $y = (0, 1)$  and  $U$  to be any small enough open neighborhood containing  $y$ .

**Theorem 10.1.** *Let  $(\tilde{X}, p)$  be a covering space of  $X$ . Let  $x_0 \in X$  and  $\tilde{x}_0 \in p^{-1}(x_0)$ . Suppose  $f : Y \rightarrow X$  is any map with  $Y$  path connected and locally path connected, with  $y_0 \in Y$  such that  $f(y_0) = x_0$ . Then there exists a lift  $\tilde{f} : Y \rightarrow \tilde{X}$  of  $f$  such that  $\tilde{f}(y_0) = \tilde{x}_0$  if and only if  $f_*(\pi_1(Y, y_0)) \subseteq p_*(\pi_1(\tilde{X}, \tilde{x}_0))$ .*

*Proof.* ( $\Rightarrow$ ) If  $\tilde{f} : Y \rightarrow \tilde{X}$  exists, then  $p \circ \tilde{f} = f$ , and so

$$f_*(\pi_1(Y, y_0)) = (p \circ \tilde{f})_*(\pi_1(Y, y_0)) = p_*(\tilde{f}_*(\pi_1(Y, y_0))) \subseteq p_*(\pi_1(\tilde{X}, \tilde{x}_0))$$

since  $f_*(\pi_1(Y, y_0)) \subseteq \pi_1(X, x_0)$ . This proves the first direction.

( $\Leftarrow$ ) Suppose that  $f_*(\pi_1(Y, y_0)) \subseteq p_*(\pi_1(\tilde{X}, \tilde{x}_0))$ . Since  $Y$  is path connected, for any  $y \in Y$ , there exists a path  $\gamma : [0, 1] \rightarrow Y$  from  $y_0$  to  $y$ . Then there is a unique lift  $\widetilde{f \circ \gamma} : [0, 1] \rightarrow \tilde{X}$  based at  $\tilde{x}_0$ . Define

$$\tilde{f}(y) = \widetilde{f \circ \gamma}(1).$$

Note that if  $\tilde{f}$  is well-defined, then it is clear that  $p \circ \tilde{f} = f$ .

To see that  $\tilde{f}$  is well-defined, we must show that  $\widetilde{f \circ \gamma_1}(1)$  is independent of the choice of path  $\gamma$ . Let  $\gamma, \eta$  be paths from  $y_0$  to  $y$ . Note that  $\gamma * \bar{\eta}$  is a loop in  $Y$  based at  $y_0$ , so

$$(f \circ \gamma) * (f \circ \bar{\eta}) = f_*(\gamma * \bar{\eta}) \in p_*(\pi_1(\tilde{X}, \tilde{x}_0)).$$

Thus by Lemma 9.3, this loop lifts to a loop in  $\tilde{X}$  based at  $\tilde{x}_0$ . Then

$$(f \circ \gamma) * (f \circ \bar{\eta}) = \widetilde{(f \circ \gamma) * (f \circ \bar{\eta})},$$

where  $\widetilde{f \circ \gamma}$  is a lift based at  $\tilde{x}_0$  and  $\widetilde{f \circ \bar{\eta}}$  is a lift based at  $\widetilde{f \circ \gamma}(1)$ . But  $(f \circ \gamma) * (f \circ \bar{\eta})$  is a loop, so

$$\widetilde{f \circ \bar{\eta}}(1) = \tilde{x}_0.$$

Then  $\widetilde{f \circ \bar{\eta}}$  is the lift of  $\eta$  based at  $\tilde{x}_0$ , so  $\widetilde{f \circ \bar{\eta}}(1) = \widetilde{f \circ \bar{\eta}}(0) = \widetilde{f \circ \gamma}(1)$ . Thus,  $\widetilde{f \circ \eta}(1) = \widetilde{f \circ \gamma}(1)$ .

Now it just remains to show that  $\tilde{f}$  is continuous. To see this, take any open set  $U \subseteq \tilde{X}$ , and we will show that for all  $y \in \tilde{f}^{-1}(U)$ , there exists an open set  $V$  in  $Y$  such that  $y \in V \subseteq \tilde{f}^{-1}(U)$  (this implies that  $\tilde{f}^{-1}(U)$  is open). Let  $W$  be an evenly covered open set containing  $f(y)$ , and  $\tilde{W} \subseteq p^{-1}(W)$  such that  $p|_{\tilde{W}} : \tilde{W} \rightarrow W$  is a homeomorphism (by possibly shrinking  $W$ , e.g. by intersecting  $\tilde{W}$  with  $U$  and taking the image under  $p$ , we can assume that  $\tilde{W} \subseteq U$ ). Since  $Y$  is locally path connected, there exists a path connected open set  $V$  in  $Y$  containing  $y$  and  $V \subseteq f^{-1}(W)$ .

Fix a path  $\gamma$  from  $y_0$  to  $y$ . For any point  $y' \in V$ , there exists a path  $\eta$  from  $y$  to  $y'$  in  $V$ . By definition,

$$\tilde{f}(y') = \widetilde{f \circ (\gamma * \eta)}(1).$$

But if  $\widetilde{f \circ \eta}$  is a lift of  $f \circ \eta$  based at  $\widetilde{f \circ \gamma}(1) = \tilde{f}(y)$ , then

$$f \circ (\gamma * \eta)(1) = \widetilde{f \circ \eta}(1).$$

We know that  $\widetilde{f \circ \eta} = (p|_{\tilde{W}})^{-1} \circ f \circ \eta$ , so we see that

$$\tilde{f}(y') = \widetilde{f \circ \eta}(1) \in \tilde{W} \subseteq U.$$

This shows that  $V \subseteq \tilde{f}^{-1}(U)$ , so  $\tilde{f}$  is continuous, which completes the proof.  $\square$

**Remark.** It is fairly common in algebraic topology for a topological hypothesis to imply some type of algebraic conclusion. The opposite, however, as in the above theorem, is much rarer.

**Lemma 10.2.** *Let  $(\tilde{X}, p)$  be a covering space of  $X$ , and let  $\tilde{f}_1, \tilde{f}_2 : Y \rightarrow \tilde{X}$  be two lifts of  $f : Y \rightarrow X$ . If  $Y$  is connected and  $\tilde{f}_1$  and  $\tilde{f}_2$  agree at one point, then  $\tilde{f}_1 = \tilde{f}_2$ .*

*Proof.* Let  $A = \{y \in Y : \tilde{f}_1(y) = \tilde{f}_2(y)\}$ . Note that  $A \neq \emptyset$  by assumption. If  $y \in A$ , then let  $U$  be an evenly covered neighborhood of  $f(y)$  and  $\tilde{U} \subseteq p^{-1}(U)$  such that  $\tilde{f}_1(y) = \tilde{f}_2(y) \in \tilde{U}$  and  $p|_{\tilde{U}} : \tilde{U} \rightarrow U$  is a homeomorphism. Since  $f$  is continuous, there exists an open neighborhood  $V$  of  $y$  in  $Y$  such that  $f(V) \subseteq U$ . Now we have

$$\tilde{f}_1|_V = (p|_{\tilde{U}})^{-1} \circ f|_V = \tilde{f}_2|_V,$$

so  $V \subseteq A$ , which shows that  $A$  is open. A similar argument shows that  $Y \setminus A$  is open, so  $A$  is closed. Since  $Y$  is connected and  $A$  is open, closed, nonempty, we must have  $A = Y$ , hence  $\tilde{f}_1 = \tilde{f}_2$ .  $\square$

**Definition 10.2.** We say that two covering spaces  $(\tilde{X}_1, p_1)$  and  $(\tilde{X}_2, p_2)$  of  $X$  are *isomorphic* if there exists a homeomorphism  $h : \tilde{X}_1 \rightarrow \tilde{X}_2$  such that  $p_2 \circ h = p_1$ , i.e. the following diagram commutes:

$$\begin{array}{ccc} \tilde{X}_1 & \xrightarrow{\quad h \quad} & \tilde{X}_2 \\ & \searrow p_1 & \downarrow p_2 \\ & & X \end{array}$$

**Corollary 10.1.1.** *Suppose  $(\tilde{X}_1, p_1)$  and  $(\tilde{X}_2, p_2)$  are path connected, locally path connected covering spaces of  $X$  and  $x_0 \in X$ ,  $\tilde{x}_i \in p_i^{-1}(x_0)$ , then*

- (a) *if  $(p_1)_*(\pi_1(\tilde{X}_1, \tilde{x}_1)) \subseteq (p_2)_*(\pi_1(\tilde{X}_2, \tilde{x}_2))$ , then  $p_1$  lifts to a covering map  $p : \tilde{X}_1 \rightarrow \tilde{X}_2$  which takes the base point  $\tilde{x}_1$  to the base point  $\tilde{x}_2$ ;*
- (b)  *$(\tilde{X}_1, \tilde{x}_1)$  and  $(\tilde{X}_2, \tilde{x}_2)$  are isomorphic covering spaces taking  $\tilde{x}_1$  to  $\tilde{x}_2$  if and only if*

$$(p_1)_*(\pi_1(\tilde{X}_1, \tilde{x}_1)) = (p_2)_*(\pi_1(\tilde{X}_2, \tilde{x}_2));$$

- (c)  *$(\tilde{X}_1, \tilde{x}_1)$  and  $(\tilde{X}_2, \tilde{x}_2)$  are isomorphic covering spaces of  $X$  if and only if  $(p_1)_*(\pi_1(\tilde{X}_1, \tilde{x}_1))$  is conjugate to  $(p_2)_*(\pi_1(\tilde{X}_2, \tilde{x}_2))$ .*

*Proof.* (a) By Theorem 10.1, we get a lift  $p : \tilde{X}_1 \rightarrow \tilde{X}_2$  of  $p_1$  taking  $\tilde{x}_1$  to  $\tilde{x}_2$ . We need to show that  $p$  is a covering map. Let  $x \in \tilde{X}_2$ , and we show that  $x$  has an evenly covered neighborhood. Let  $U$  be a neighborhood of  $p_2(x)$  in  $X$  which is evenly covered by both  $p_1$  and  $p_2$  (e.g. take the intersection of the two neighborhoods evenly covered by  $p_1$  and  $p_2$ ), so there exists a unique  $\tilde{U} \subseteq \tilde{X}_2$  such that  $x \in \tilde{U}$  and  $p_2|_{\tilde{U}} : \tilde{U} \rightarrow U$  is a homeomorphism. Write  $p^{-1}(\tilde{U}) = \bigcup_{\alpha} U_{\alpha}$ . Clearly  $\bigcup_{\alpha} U_{\alpha} \subseteq p^{-1}(U)$ , so  $p_1|_{U_{\alpha}} : U_{\alpha} \rightarrow U$  is a homeomorphism. Thus  $p|_{U_{\alpha}} : p_2^{-1}|_{\tilde{U}} \circ p_1|_{U_{\alpha}}$  is a homeomorphism  $U_{\alpha} \rightarrow \tilde{U}$ , so  $\tilde{U}$  is evenly covered.

We finish the proof of (b) and (c) from (a) next time.  $\square$