MATH 6441: Algebraic Topology I

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Lecture 1

Jan. 6 — CW-Complexes

1.1 Introduction and Motivation

Algebraic topology builds "functions" (actually *functors*)

$$\{\text{topological spaces, continuous maps}\} \longrightarrow \{\text{algebraic things, algebraic maps}\},$$

where "algebraic things" can be groups, vector spaces, etc. The main objective of algebraic topology is to *distinguish topological spaces*, e.g. showing that $\mathbb{R}^n \not\cong \mathbb{R}^m$ for $n \neq m$. More applications are:

1. Studying maps between spaces.[1]

   - Does a given space M embed in N? For instance, for what m does $\mathbb{R}P^n$ embed in $\mathbb{R}^m$? (This is still not known in general.) Here $\mathbb{R}P^n$ is the real projective space.

   - Lifting maps, i.e. given $f : A \to B$ and $g : E \to B$, does there exist a map $\widetilde{f} : A \to E$ such that $g \circ \widetilde{f} = f$? In other words, is there a map $\widetilde{f}$ such that the following diagram commutes:

   $$\begin{tikzcd} & E \arrow[d, "g"] \\ A \arrow[ur, dashed, "\widetilde{f}"] \arrow[r, "f"'] & B \end{tikzcd}$$

   - Fixed point problems: Given $f : X \to X$, does f have a fixed point, i.e. $x \in X$ such that $f(x) = x$? Such theorems are used to prove the existence of solutions to ordinary differential equations, for instance.

2. Group actions, e.g. which finite groups act freely on S^n?

3. Group theory, e.g. showing that every subgroup of a free group is free. Another example is that if F_n is the free group on n generators, then its *commutator* $[F_n, F_n]$ is not finitely generated.

4. Algebra, e.g. proving the fundamental theorem of algebra.

This course will cover the following topics:

1. The *fundamental group* $\pi_1(X, x_0)$ of a space X for $x_0 \in X$, and *covering spaces*.

2. The *homology groups* $H_k(X)$ for $k = 0, 1, 2, \ldots$. These groups are abelian.

3. The *cohomology ring* $H^*(X) = \bigoplus_{k=0}^{\infty} H^k(X)$.

But before getting to this, we need to develop some important ideas.

[1] All maps and functions in this class are continuous unless otherwise specified.

1.2 CW-Complexes

Definition 1.1. Let $D^n \subseteq \mathbb{R}^n$ be the unit disk and $S^{n-1} = \partial D^n$. Given a topological space Y and a continuous map $a : S^{n-1} \to Y$, the space obtained from Y by *attaching* an *n-cell* (*via a*) is

$$Y \cup_a D^n = (Y \sqcup D^n)/\sim,$$

where the equivalence relation $\sim$ is given by $x \sim a(x)$ for $x \in \partial D^n$. Here $\sqcup$ denotes disjoint union.

Definition 1.2. An *n-complex* or an *n-dimensional CW-complex* is defined inductively by:

- A (-1)-complex is the empty set $\varnothing$.
- An n-complex X^n is a space obtained from an $(n-1)$-complex by attaching n-cells.

An n-complex is *finite* if it involves only a finite number of cells. The *k-skeleton* of X is the union of all n-cells in X with $n \leq k$.

Remark. Any CW-complex is Hausdorff. See Hatcher for a proof.

Example 1.2.1. Here are some examples of CW-complexes:

- A 0-complex is a union of points. This is because $D^0 = \{\text{pt}\}$ and $\partial D^0 = \varnothing$.
- A 1-complex is a graph (points and lines connecting them).
- The torus T (a square with opposite sides identified) is a 2-complex. Here the 0-skeleton $T^{(0)}$ is the common corner on the square and the 1-skeleton $T^{(1)}$ is two sides of the square after taking the quotient. The 2-skeleton $T = T^{(2)}$ is the entire torus.
- Another example of a 2-complex is the two-holed torus, which is obtained by identifying the edges of an octagon (pairs of every other edge identified with opposite orientation).[2]
- A third example of a 2-complex is $X^{(1)} \cup_a D^2$ given an attaching map $D^2 \to X^{(1)}$.
- Consider the unit sphere $S^n \subseteq \mathbb{R}^{n+1}$. One way to give S^2 a CW-complex structure is to see the sphere as two disks D^2 glued together, resulting in one 0-cell, one 1-cell, and two 2-cells. Another way is to start with two points, attach two 1-cells to get a circle, and then attaching two disks to get S^2. This results in two 0-cells, two 1-cells, and two 2-cells.

  The second idea generalizes to S^n. We can write

  $$S^n = S^{n-1} \cup_{a_1} D^n \cup_{a_2} D^n,$$

  where S^{n-1} inductively has a CW-complex structure. This yields two k-cells for each $k \leq n$.

  Another way to put a CW-complex structure on S^n is to attach D^n to a point with $\partial D^n \to \{\text{pt}\}$. In particular, notice that a space can in general have several different CW-complex structures.
- Consider the *n-dimensional real projective space*

  $$\mathbb{R}P^n = \{\text{lines through the origin in } \mathbb{R}^{n+1}\}.$$

  Since each line through the origin passes through S^n twice, we can equivalently think of $\mathbb{R}P^n$ as the unit sphere S^n with antipodal points identified.

[2] This CW-decomposition of the two-holed torus results in one 0-cell, four 1-cells, and one 2-cell.

We can also think of this as D^n with antipodal points on ∂D^n identified. Since $\partial D^n = S^{n-1}$, this is simply $\mathbb{R}P^{n-1} \cup_a D^n$, where $a : \partial D^n \to \mathbb{R}P^{n-1}$ is the quotient map. This gives $\mathbb{R}P^n$ a CW-complex structure with one k-cell for each $k \leq n$.

- The complex projective space $\mathbb{C}P^n$ has a similar CW-complex structure with one k-cell for each even $k \leq 2n$. One can verify this as an exercise.
- Any smooth manifold has a CW-complex structure. See Hatcher.

Exercise 1.1. Show the product of CW-complexes is a CW-complex.

Definition 1.3. A *subcomplex* of a CW-complex X is a closed subset $A \subseteq X$ that is a union of cells in X. In particular, A is also a CW-complex and (X, A) is called a *CW-pair*.

1.3 Homotopy

Definition 1.4. Let X and Y be topological spaces. Two maps $f, g : X \to Y$ are *homotopic*, denoted $f \sim g$, if there exists a continuous map $\Phi : X \times [0,1] \to Y$ such that

$$\Phi(x, 0) = f(x) \quad \text{and} \quad \Phi(x, 1) = g(x)$$

for all $x \in X$. In this case, Φ is called a *homotopy* between f and g.

Remark. We note the following:

- A homotopy Φ gives a family of maps $\phi_{t_0} : X \to Y$ given by $x \mapsto \phi(x, t_0)$ which is continuous in t_0. So maps are homotopic if there is a continuous family of maps between them.
- If $A \subseteq X$ then we say that f is *homotopic to g rel A*, denoted $f \sim_A g$, if there exists Φ as above with the additional property that $\Phi(x, t) = f(x)$ for all $x \in A$, i.e. points in A are fixed.
- If $A \subseteq X$ and $B \subseteq Y$, then the notation $f : (X, A) \to (X, B)$ means that $f : X \to Y$ and $f(a) \in B$ for each $a \in A$. We say that f is a *map of pairs*. If $f, g : (X, A) \to (Y, B)$, then f, g are *homotopic as maps of pairs* if each $\phi_t : (X, A) \to (X, B)$ is a map of pairs.

Lecture 2

Jan. 8 — Homotopy

2.1 More on Homotopy

Example 2.0.1. For any space X, any map $f : X \to [0,1]$ is homotopic to the map $g : X \to [0,1]$ given by $x \mapsto 0$. To see this, we have the homotopy $\Phi : X \times [0,1] \times [0,1]$ defined by

$$(x,t) \mapsto (1-t)f(x).$$

We can see that $\Phi(x,0) = f(x)$ and $\Phi(x,1) = 0 = g(x)$.

Exercise 2.1. Show that homotopy is an equivalence relation on maps $X \to Y$.

Definition 2.1. Let $C(X,Y) = \{\text{continuous maps from } X \text{ to } Y\}$. Let $[X,Y] = C(X,Y)/\sim$, i.e. homotopic maps are identified with each other.

Example 2.1.1. We have the following:

1. $[X,[0,1]] = \{g\}$ for any space X, where g is the map $x \mapsto 0$ as above.
2. $[\{*\},X] = \{\text{path components of } X\}$.

2.2 Homotopy Groups

Definition 2.2. We call a space X *pointed* if there is a designated "base point" $x_0 \in X$. Given two pointed spaces (X,x_0) and (Y,y_0), we define

$$[X,Y]_0 = \{\text{homotopy classes of maps of pairs } (X,\{x_0\}) \to (Y,\{y_0\})\}.$$

Definition 2.3. Let y_0 be the north pole in S^n, i.e. $S^n \subseteq \mathbb{R}^{n+1}$ is the unit sphere and $y_0 = (0,\dots,0,1)$. The *nth homotopy group* of a pointed space (X,x_0) is $\pi_n(X,x_0) = [S^n,X]_0$.

Remark. The homotopy group $\pi_n(X,x_0)$ is in fact a group. We will study $\pi_1(X,x_0)$ next and it is called the *fundamental group* of (X,x_0).

Remark. For which (Y,y_0) is $[Y,X]_0$ "naturally" a group for all (X,x_0)? Similarly, for which (Y,y_0) is $[X,Y]_0$ a group for all (X,x_0)? Here, given a map $f : (X_1,x_1) \to (X_2,x_2)$, there is an obvious *induced map* $f_* : [Y,X_1]_0 \to [Y,X_2]_0$ given by $[g] \mapsto [f \circ g]$. Similarly, there is a map $f^* : [X_2,Y]_0 \to [X_1,Y]_0$ given by $[g] \mapsto [g \circ f]$. In the questions above, "naturally" means that f_* and f^* are homomorphisms. for any (X_1,x_1) and (X_2,x_2). The (perhaps unsatisfying) answer is that a space satisfying the first condition is called an *H-space*, and a space satisfying the second is called an *H'-space*.

2.3 Homotopy Equivalence

Definition 2.4. We say that $f : X \to Y$ is the *homotopy inverse* to a function $g : Y \to X$ if $f \circ g \sim \mathrm{id}_Y$ and $g \circ f \sim \mathrm{id}_X$, where id_X and id_Y are the identity maps on X and Y. If g has a homotopy inverse, then we call g a *homotopy equivalence* from Y to X and we call X, Y *homotopy equivalent.*[1]

Exercise 2.2. Show that homotopy equivalence is an equivalence relation.

Lemma 2.1. *The following are equivalent:*

1. *X and Y are homotopy equivalent.*
2. *For any space Z, there is a one-to-one correspondence $\phi_Z : [X, Z] \to [Y, Z]$ such that for all continuous maps $h : Z \to Z'$, the following diagram commutes:*

$$\begin{array}{ccc} [X, Z] & \xrightarrow{\phi_Z} & [Y, Z] \\ \downarrow{\scriptstyle h_*} & & \downarrow{\scriptstyle h_*} \\ [X, Z'] & \xrightarrow{\phi_{Z'}} & [Y, Z'] \end{array}$$

3. *For any space Z, there is a one-to-one correspondence $\phi^Z : [Z, X] \to [Z, Y]$ such for that all continuous maps $h : Z \to Z'$, the following diagram commutes:*

$$\begin{array}{ccc} [Z', X] & \xrightarrow{\phi^{Z'}} & [Z', Y] \\ \downarrow{\scriptstyle h^*} & & \downarrow{\scriptstyle h^*} \\ [Z, X] & \xrightarrow{\phi^Z} & [Z, Y] \end{array}$$

Proof. This is left as an exercise. □

Remark. Based on the previous lemma, two spaces are homotopy equivalent if and only if homotopy classes of maps to and from the space are "naturally equivalent."

Example 2.4.1. We have the following:

- Homeomorphic spaces are homotopy equivalent.
- Let $X = S^1$ and $Y = S^1 \times [0, 1]$. We claim that X is homotopy equivalent to Y.

  Define the maps $f : S^1 \to S^1 \times [0, 1]$ by $x \mapsto (x, 0)$ and $g : S^1 \times [0, 1] \to S^1$ by $(x, t) \mapsto x$. Then we can see that $g \circ f : S^1 \to S^1$ maps $x \mapsto x$, so $g \circ f = \mathrm{id}_{S^1}$. On the other hand, the composition $f \circ g : S^1 \times [0, 1] \to S^1 \times [0, 1]$ maps $(x, t) \mapsto (x, 0)$. Now $f \circ g \sim \mathrm{id}_{S^1 \times [0,1]}$ by homotopy. For instance, define $\Phi : (S^1 \times [0, 1]) \times [0, 1] \to (S^1 \times [0, 1])$ by $((x, t), s) \mapsto (x, st)$, so

$$\Phi((x, t), 1) = (x, t) = \mathrm{id}_{S^1 \times [0,1]}(x, t) \quad \text{and} \quad \Phi((x, t), 0) = (x, 0) = f \circ g.$$

  Thus f is a homotopy equivalence from S^1 to $S^1 \times [0, 1]$. Note that $S^1 \times [0, 1]$ is the annulus.

Definition 2.5. A space is called *contractible* if it is homotopy equivalent to a point.

Example 2.5.1. The spaces $[0, 1]$ and $\mathbb{R}^n$ are contractible (exercise).

[1] We will denote homotopy equivalence by $X \simeq Y$ or simply $X \sim Y$.

Definition 2.6. If $A \subseteq X$, then a *retraction of X to A* is a map $r : X \to A$ such that $r(a) = a$ for every $a \in A$. A *deformation retraction* of X to A is a retraction $r : X \to A$ that is homotopic rel A to the identity map id_X, i.e. we can find $\phi_t : X \to X$ for $t \in [0,1]$ such that $\phi_0(x) = x$ and $\phi_1(X) \subseteq A$ and $\phi_t(x) = x$ for all $x \in A$ and $t \in [0,1]$.

Remark. If X deformation retracts to A, then X is homotopy equivalent to A. To see this, suppose we have a homotopy $\phi_t : X \to X$ as above, and let $i : A \to X$ be the inclusion map. Then $\phi_1 \circ i = \mathrm{id}_A$ and $i \circ \phi_1 = \phi_1 \sim \phi_0 = \mathrm{id}_X$, so ϕ_1 is a homotopy equivalence from X to A.

Definition 2.7. Given two spaces X, Y and a map $f : X \to Y$, the *mapping cylinder* of f is the space

$$M_f = ((X \times [0,1]) \cup Y)/\sim,$$

where the equivalence relation $\sim$ is given by by $(x, 1) \sim f(x)$ for $x \in X$.

Remark. The mapping cylinder M_f deformation retracts to Y. To see this, consider the map $\widetilde{\phi_t}$ given by $(x, s) \mapsto (x, (1-t)s + t)$ on $X \times [0,1]$ and $y \mapsto y$ on Y. Since $\widetilde{\phi_t}$ respects the equivalence relation, it descends to a map $\phi_t : M_f \to M_f$ on the quotient space. Note that $\phi_0 = \mathrm{id}_{M_f}$ and $\phi_1(M_f) = Y \subseteq M_f$, and $\phi_t = \mathrm{id}_Y$ for all t. Thus ϕ_1 is a deformation retraction. In particular, this means that $M_f \simeq Y$.

Remark. There are obvious inclusions $i : X \to M_f$ given by $x \mapsto (x, 0)$ and $j : Y \to M_f$ given by $y \mapsto y$. Note that ϕ_1 defined above is the homotopy inverse to j. Now we have the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "i"'] & Y \arrow[d, "j"] \\ & M_f \end{tikzcd}$$

where j is a homotopy equivalence and $j \circ f \sim i$ (exercise).

Remark. The above remark shows the following "slogan" of algebraic topology:

> Any map is an inclusion up to homotopy.

Example 2.7.1. Let X be three circles with two enclosed in a third bigger one, and let Y be two circles enclosing the inner two circles of X connected by a line segment. Let Z be the region inside by the outer circle of X and outside the inner two circles of X.

Define $f : X \to Y$ to be the map which sends $x \in X$ to the point in Y at the other end of an interval (points on the inner circles of X are mapped by radial lines to the circles in Y, and points on the outer circle of X are mapped radially to either the circles or the line segment in Y). One can write an explicit formula for f as an exercise.

Then Z is homeomorphic to M_f, and in particular $Z \simeq Y$. Similarly, M_f is homotopy equivalent to two circles joined at a point, or a circle with a diameter.