MATH 6441: Algebraic Topology I

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Contents

Lecture 1

Jan. 6 — CW-Complexes

1.1 Introduction and Motivation

Algebraic topology builds "functions" (actually *functors*)

$$\{\text{topological spaces, continuous maps}\} \longrightarrow \{\text{algebraic things, algebraic maps}\},$$

where "algebraic things" can be groups, vector spaces, etc. The main objective of algebraic topology is to *distinguish topological spaces*, e.g. showing that $\mathbb{R}^n \not\cong \mathbb{R}^m$ for $n \neq m$. More applications are:

1. Studying maps between spaces.[1]
   - Does a given space M embed in N? For instance, for what m does $\mathbb{R}P^n$ embed in $\mathbb{R}^m$? (This is still not known in general.) Here $\mathbb{R}P^n$ is the real projective space.
   - Lifting maps, i.e. given $f : A \to B$ and $g : E \to B$, does there exist a map $\widetilde{f} : A \to E$ such that $g \circ \widetilde{f} = f$? In other words, is there a map $\widetilde{f}$ such that the following diagram commutes:

$$\begin{tikzcd} & E \arrow[d, "g"] \\ A \arrow[ur, dashed, "\widetilde{f}"] \arrow[r, "f"'] & B \end{tikzcd}$$

   - Fixed point problems: Given $f : X \to X$, does f have a fixed point, i.e. $x \in X$ such that $f(x) = x$? Such theorems are used to prove the existence of solutions to ordinary differential equations, for instance.
2. Group actions, e.g. which finite groups act freely on S^n?
3. Group theory, e.g. showing that every subgroup of a free group is free. Another example is that if F_n is the free group on n generators, then its *commutator* $[F_n, F_n]$ is not finitely generated.
4. Algebra, e.g. proving the fundamental theorem of algebra.

This course will cover the following topics:

1. The *fundamental group* $\pi_1(X, x_0)$ of a space X for $x_0 \in X$, and *covering spaces*.
2. The *homology groups* $H_k(X)$ for $k = 0, 1, 2, \ldots$. These groups are abelian.
3. The *cohomology ring* $H^*(X) = \bigoplus_{k=0}^{\infty} H^k(X)$.

But before getting to this, we need to develop some important ideas.

[1] All maps and functions in this class are continuous unless otherwise specified.

1.2 CW-Complexes

Definition 1.1. Let $D^n \subseteq \mathbb{R}^n$ be the unit disk and $S^{n-1} = \partial D^n$. Given a topological space Y and a continuous map $a : S^{n-1} \to Y$, the space obtained from Y by *attaching* an *n-cell* (*via a*) is

$$Y \cup_a D^n = (Y \sqcup D^n)/\sim,$$

where the equivalence relation $\sim$ is given by $x \sim a(x)$ for $x \in \partial D^n$. Here $\sqcup$ denotes disjoint union.

Definition 1.2. An *n-complex* or an *n-dimensional CW-complex* is defined inductively by:

- A (-1)-complex is the empty set $\varnothing$.
- An n-complex X^n is a space obtained from an $(n-1)$-complex by attaching n-cells.

An n-complex is *finite* if it involves only a finite number of cells. The *k-skeleton* of X is the union of all n-cells in X with $n \leq k$.

Remark. Any CW-complex is Hausdorff. See Hatcher for a proof.

Example 1.2.1. Here are some examples of CW-complexes:

- A 0-complex is a union of points. This is because $D^0 = \{\text{pt}\}$ and $\partial D^0 = \varnothing$.
- A 1-complex is a graph (points and lines connecting them).
- The torus T (a square with opposite sides identified) is a 2-complex. Here the 0-skeleton $T^{(0)}$ is the common corner on the square and the 1-skeleton $T^{(1)}$ is two sides of the square after taking the quotient. The 2-skeleton $T = T^{(2)}$ is the entire torus.
- Another example of a 2-complex is the two-holed torus, which is obtained by identifying the edges of an octagon (pairs of every other edge identified with opposite orientation).[2]
- A third example of a 2-complex is $X^{(1)} \cup_a D^2$ given an attaching map $D^2 \to X^{(1)}$.
- Consider the unit sphere $S^n \subseteq \mathbb{R}^{n+1}$. One way to give S^2 a CW-complex structure is to see the sphere as two disks D^2 glued together, resulting in one 0-cell, one 1-cell, and two 2-cells. Another way is to start with two points, attach two 1-cells to get a circle, and then attaching two disks to get S^2. This results in two 0-cells, two 1-cells, and two 2-cells.

  The second idea generalizes to S^n. We can write

  $$S^n = S^{n-1} \cup_{a_1} D^n \cup_{a_2} D^n,$$

  where S^{n-1} inductively has a CW-complex structure. This yields two k-cells for each $k \leq n$.

  Another way to put a CW-complex structure on S^n is to attach D^n to a point with $\partial D^n \to \{\text{pt}\}$. In particular, notice that a space can in general have several different CW-complex structures.
- Consider the *n-dimensional real projective space*

  $$\mathbb{R}P^n = \{\text{lines through the origin in } \mathbb{R}^{n+1}\}.$$

  Since each line through the origin passes through S^n twice, we can equivalently think of $\mathbb{R}P^n$ as the unit sphere S^n with antipodal points identified.

[2]This CW-decomposition of the two-holed torus results in one 0-cell, four 1-cells, and one 2-cell.

We can also think of this as D^n with antipodal points on ∂D^n identified. Since $\partial D^n = S^{n-1}$, this is simply $\mathbb{R}P^{n-1} \cup_a D^n$, where $a : \partial D^n \to \mathbb{R}P^{n-1}$ is the quotient map. This gives $\mathbb{R}P^n$ a CW-complex structure with one k-cell for each $k \leq n$.

- The complex projective space $\mathbb{C}P^n$ has a similar CW-complex structure with one k-cell for each even $k \leq 2n$. One can verify this as an exercise.
- Any smooth manifold has a CW-complex structure. See Hatcher.

Exercise 1.1. Show the product of CW-complexes is a CW-complex.

Definition 1.3. A *subcomplex* of a CW-complex X is a closed subset $A \subseteq X$ that is a union of cells in X. In particular, A is also a CW-complex and (X, A) is called a *CW-pair*.

1.3 Homotopy

Definition 1.4. Let X and Y be topological spaces. Two maps $f, g : X \to Y$ are *homotopic*, denoted $f \sim g$, if there exists a continuous map $\Phi : X \times [0, 1] \to Y$ such that

$$\Phi(x, 0) = f(x) \quad \text{and} \quad \Phi(x, 1) = g(x)$$

for all $x \in X$. In this case, Φ is called a *homotopy* between f and g.

Remark. We note the following:

- A homotopy Φ gives a family of maps $\phi_{t_0} : X \to Y$ given by $x \mapsto \phi(x, t_0)$ which is continuous in t_0. So maps are homotopic if there is a continuous family of maps between them.
- If $A \subseteq X$ then we say that f is *homotopic to g rel A*, denoted $f \sim_A g$, if there exists Φ as above with the additional property that $\Phi(x, t) = f(x)$ for all $x \in A$, i.e. points in A are fixed.
- If $A \subseteq X$ and $B \subseteq Y$, then the notation $f : (X, A) \to (Y, B)$ means that $f : X \to Y$ and $f(a) \in B$ for each $a \in A$. We say that f is a *map of pairs*. If $f, g : (X, A) \to (Y, B)$, then f, g are *homotopic as maps of pairs* if each $\phi_t : (X, A) \to (Y, B)$ is a map of pairs.

Lecture 2

Jan. 8 — Homotopy

2.1 More on Homotopy

Example 2.0.1. For any space X, any map $f : X \to [0,1]$ is homotopic to the map $g : X \to [0,1]$ given by $x \mapsto 0$. To see this, we have the homotopy $\Phi : X \times [0,1] \times [0,1]$ defined by

$$(x,t) \mapsto (1-t)f(x).$$

We can see that $\Phi(x,0) = f(x)$ and $\Phi(x,1) = 0 = g(x)$.

Exercise 2.1. Show that homotopy is an equivalence relation on maps $X \to Y$.

Definition 2.1. Let $C(X,Y) = \{\text{continuous maps from } X \text{ to } Y\}$. Let $[X,Y] = C(X,Y)/\sim$, i.e. homotopic maps are identified with each other.

Example 2.1.1. We have the following:

1. $[X,[0,1]] = \{g\}$ for any space X, where g is the map $x \mapsto 0$ as above.
2. $[\{*\}, X] = \{\text{path components of } X\}$.

2.2 Homotopy Groups

Definition 2.2. We call a space X *pointed* if there is a designated "base point" $x_0 \in X$. Given two pointed spaces (X,x_0) and (Y,y_0), we define

$$[X,Y]_0 = \{\text{homotopy classes of maps of pairs } (X,\{x_0\}) \to (Y,\{y_0\})\}.$$

Definition 2.3. Let y_0 be the north pole in S^n, i.e. $S^n \subseteq \mathbb{R}^{n+1}$ is the unit sphere and $y_0 = (0,\dots,0,1)$. The *nth homotopy group* of a pointed space (X,x_0) is $\pi_n(X,x_0) = [S^n, X]_0$.

Remark. The homotopy group $\pi_n(X,x_0)$ is in fact a group. We will study $\pi_1(X,x_0)$ next and it is called the *fundamental group* of (X,x_0).

Remark. For which (Y,y_0) is $[Y,X]_0$ "naturally" a group for all (X,x_0)? Similarly, for which (Y,y_0) is $[X,Y]_0$ a group for all (X,x_0)? Here, given a map $f : (X_1,x_1) \to (X_2,x_2)$, there is an obvious *induced map* $f_* : [Y,X_1]_0 \to [Y,X_2]_0$ given by $[g] \mapsto [f \circ g]$. Similarly, there is a map $f^* : [X_2,Y]_0 \to [X_1,Y]_0$ given by $[g] \mapsto [g \circ f]$. In the questions above, "naturally" means that f_* and f^* are homomorphisms. for any (X_1,x_1) and (X_2,x_2). The (perhaps unsatisfying) answer is that a space satisfying the first condition is called an *H-space*, and a space satisfying the second is called an *H'-space*.

2.3 Homotopy Equivalence

Definition 2.4. We say that $f : X \to Y$ is the *homotopy inverse* to a function $g : Y \to X$ if $f \circ g \sim \mathrm{id}_Y$ and $g \circ f \sim \mathrm{id}_X$, where id_X and id_Y are the identity maps on X and Y. If g has a homotopy inverse, then we call g a *homotopy equivalence* from Y to X and we call X, Y *homotopy equivalent.*[1]

Exercise 2.2. Show that homotopy equivalence is an equivalence relation.

Lemma 2.1. *The following are equivalent:*

1. *X and Y are homotopy equivalent.*
2. *For any space Z, there is a one-to-one correspondence $\phi_Z : [X, Z] \to [Y, Z]$ such that for all continuous maps $h : Z \to Z'$, the following diagram commutes:*

$$\begin{array}{ccc} [X,Z] & \xrightarrow{\phi_Z} & [Y,Z] \\ \downarrow{\scriptstyle h_*} & & \downarrow{\scriptstyle h_*} \\ [X,Z'] & \xrightarrow{\phi_{Z'}} & [Y,Z'] \end{array}$$

3. *For any space Z, there is a one-to-one correspondence $\phi^Z : [Z, X] \to [Z, Y]$ such for that all continuous maps $h : Z \to Z'$, the following diagram commutes:*

$$\begin{array}{ccc} [Z',X] & \xrightarrow{\phi^{Z'}} & [Z',Y] \\ \downarrow{\scriptstyle h^*} & & \downarrow{\scriptstyle h^*} \\ [Z,X] & \xrightarrow{\phi^{Z}} & [Z,Y] \end{array}$$

Proof. This is left as an exercise. □

Remark. Based on the previous lemma, two spaces are homotopy equivalent if and only if homotopy classes of maps to and from the space are "naturally equivalent."

Example 2.4.1. We have the following:

- Homeomorphic spaces are homotopy equivalent.
- Let $X = S^1$ and $Y = S^1 \times [0, 1]$. We claim that X is homotopy equivalent to Y.

  Define the maps $f : S^1 \to S^1 \times [0, 1]$ by $x \mapsto (x, 0)$ and $g : S^1 \times [0, 1] \to S^1$ by $(x, t) \mapsto x$. Then we can see that $g \circ f : S^1 \to S^1$ maps $x \mapsto x$, so $g \circ f = \mathrm{id}_{S^1}$. On the other hand, the composition $f \circ g : S^1 \times [0, 1] \to S^1 \times [0, 1]$ maps $(x, t) \mapsto (x, 0)$. Now $f \circ g \sim \mathrm{id}_{S^1 \times [0,1]}$ by homotopy. For instance, define $\Phi : (S^1 \times [0, 1]) \times [0, 1] \to (S^1 \times [0, 1])$ by $((x, t), s) \mapsto (x, st)$, so

$$\Phi((x, t), 1) = (x, t) = \mathrm{id}_{S^1 \times [0,1]}(x, t) \quad \text{and} \quad \Phi((x, t), 0) = (x, 0) = f \circ g.$$

  Thus f is a homotopy equivalence from S^1 to $S^1 \times [0, 1]$. Note that $S^1 \times [0, 1]$ is the annulus.

Definition 2.5. A space is called *contractible* if it is homotopy equivalent to a point.

Example 2.5.1. The spaces $[0, 1]$ and $\mathbb{R}^n$ are contractible (exercise).

[1] We will denote homotopy equivalence by $X \simeq Y$ or simply $X \sim Y$.

Definition 2.6. If $A \subseteq X$, then a *retraction of X to A* is a map $r : X \to A$ such that $r(a) = a$ for every $a \in A$. A *deformation retraction* of X to A is a retraction $r : X \to A$ that is homotopic rel A to the identity map id_X, i.e. we can find $\phi_t : X \to X$ for $t \in [0, 1]$ such that $\phi_0(x) = x$ and $\phi_1(X) \subseteq A$ and $\phi_t(x) = x$ for all $x \in A$ and $t \in [0, 1]$.

Remark. If X deformation retracts to A, then X is homotopy equivalent to A. To see this, suppose we have a homotopy $\phi_t : X \to X$ as above, and let $i : A \to X$ be the inclusion map. Then $\phi_1 \circ i = \mathrm{id}_A$ and $i \circ \phi_1 = \phi_1 \sim \phi_0 = \mathrm{id}_X$, so ϕ_1 is a homotopy equivalence from X to A.

Definition 2.7. Given two spaces X, Y and a map $f : X \to Y$, the *mapping cylinder* of f is the space

$$M_f = ((X \times [0, 1]) \cup Y)/\sim,$$

where the equivalence relation $\sim$ is given by by $(x, 1) \sim f(x)$ for $x \in X$.

Remark. The mapping cylinder M_f deformation retracts to Y. To see this, consider the map $\widetilde{\phi}_t$ given by $(x, s) \mapsto (x, (1 - t)s + t)$ on $X \times [0, 1]$ and $y \mapsto y$ on Y. Since $\widetilde{\phi}_t$ respects the equivalence relation, it descends to a map $\phi_t : M_f \to M_f$ on the quotient space. Note that $\phi_0 = \mathrm{id}_{M_f}$ and $\phi_1(M_f) = Y \subseteq M_f$, and $\phi_t = \mathrm{id}_Y$ for all t. Thus ϕ_1 is a deformation retraction. In particular, this means that $M_f \simeq Y$.

Remark. There are obvious inclusions $i : X \to M_f$ given by $x \mapsto (x, 0)$ and $j : Y \to M_f$ given by $y \mapsto y$. Note that ϕ_1 defined above is the homotopy inverse to j. Now we have the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "i"'] & Y \arrow[d, "j"] \\ & M_f \end{tikzcd}$$

where j is a homotopy equivalence and $j \circ f \sim i$ (exercise).

Remark. The above remark shows the following "slogan" of algebraic topology:

> Any map is an inclusion up to homotopy.

Example 2.7.1. Let X be three circles with two enclosed in a third bigger one, and let Y be two circles enclosing the inner two circles of X connected by a line segment. Let Z be the region inside by the outer circle of X and outside the inner two circles of X.

Define $f : X \to Y$ to be the map which sends $x \in X$ to the point in Y at the other end of an interval (points on the inner circles of X are mapped by radial lines to the circles in Y, and points on the outer circle of X are mapped radially to either the circles or the line segment in Y). One can write an explicit formula for f as an exercise.

Then Z is homeomorphic to M_f, and in particular $Z \simeq Y$. Similarly, M_f is homotopy equivalent to two circles joined at a point, or a circle with a diameter. Thus by transitivity, these two spaces and Z are all homotopy equivalent to each other.

Lecture 3

Jan. 13 — Homotopy, Part 2

3.1 More on Homotopy Equivalence

Lemma 3.1. *If (X, A) is a CW pair and A is contractible, then $X/A \simeq X$.*[1]

Exercise 3.1. The following are some applications of this lemma:

1. Let X be a connected graph (i.e. a 1-complex), and let A be an edge in X connecting distinct vertices. Then A is contractible, so $X/A \simeq X$. Continuing this process, let A be a maximal tree in X, which will also be contractible. Then $X \simeq X/A$, so any connected graph is homotopy equivalent to a *wedge of circles* (with number of circles equal to the number of self-loops in the graph).[2]

2. Consider the space X obtained by attaching a 1-cell A connecting the north and south poles on a sphere. Let B be half of a great circle connecting the endpoints of A. Clearly A and B are both contractible. After collapsing A, we see that $X \simeq X/A$, which is S^2 with the north and south poles identified. On the other hand, by contracting B instead we see that $X \simeq X/B$, which is $S^2 \vee S^1$.

   Note that after contracting A, the subset B is actually no longer contractible.

3. Let X be a torus with attached disks A_1, A_2, A_3 in the tube of the torus. Then

   $$X \simeq ((X/A_1)/A_2)/A_3,$$

   which is three spheres lying in a circle, each attached to the next one at a single point.

   We can also obtain this space by considering the space Y of three spheres attached in a line with an extra 1-cell B attached at the ends of the chain of circles. Also let A be the union of halves of great circles going through the chain of circles, with the same endpoints as B. Then we can see that this creates the same space as before, so that $X \simeq Y/B \simeq Y$. On the other hand, by contracting A, we see that $Y \simeq Y/A = S^2 \vee S^2 \vee S^2 \vee S^1$.

   Of course, all of these spaces are then homotopy equivalent to each other by transitivity.

Lemma 3.2. *Let (X, A) be a CW pair and $f, g : A \to Y$ be homotopic maps. Then $X \cup_f Y \simeq X \cup_g Y$.*

Example 3.0.1. Let $Y = S^2$, and $X = D^2$, and $A = \partial D^2$. Let $g : A \to Y$ map A to a great circle, and let $f : A \to Y$ map A to the north pole. One can show as an exercise that $f \sim g$ (e.g. by pulling the equator towards the north pole). So the lemma says that $X \cup_g Y$, which is a sphere with a disk glued along its equator, is homotopy equivalent to $X \cup_f Y = S^2 \vee S^2$.

[1] Here X/A denotes the quotient of X obtained by collapsing all of A to a single point.

[2] A *wedge* of pointed spaces $(X, x_0) \vee (Y, y_0)$ is the space obtained from $X \sqcup Y$ by identifying x_0 and y_0.

3.2 Homotopy Extension Property

Remark. To prove both of these lemmas, we need the *homotopy extension property* (HEP).

Definition 3.1. A space X and a subspace $A \subseteq X$ have the *homotopy extension property* if given $F_0 : X \to Y$ (for any Y and F_0) and a homotopy $f_t : A \to Y$ such that $f_0 = F_0|_A$, then there is a homotopy $F_t : X \to Y$ such that $F_t|_A = f_t$ for every t.

Lemma 3.3. *A pair* (X, A) *has the homotopy extension property if and only if*

$$(X \times \{0\}) \cup (A \times [0, 1])$$

is a retract of $X \times [0, 1]$.

Proof. ($\Leftarrow$) We will assume that A is closed (not necessarily but makes the proof easier, and almost all examples satisfy this). By assumption, we have a retraction

$$r : (X \times [0, 1]) \to (X \times \{0\}) \cup (A \times [0, 1]).$$

Given $F_0 : X \to Y$ and $f_t : A \to Y$ such that $f_0 = F_0|_A$, we can define a map

$$\widetilde{F} : (X \times \{0\}) \cup (A \times [0, 1]) \to Y$$

by $x \mapsto F_0(x)$ on $X \times \{0\}$ and $(a, t) \mapsto f_t(a)$ on $A \times [0, 1]$. This map $\widetilde{F}$ is continuous since the definitions of $\widetilde{F}$ agree on the intersection and the intersection $A \times \{0\}$ is closed. Now define

$$F : X \times [0, 1] \to Y$$

by $F = \widetilde{F} \circ r$, which is a homotopy of F_0 that extends f_t.

($\Rightarrow$) Let $Y = (X \times [0, 1]) \cup (A \times [0, 1])$. Let

$$F_0 : X \to (X \times \{0\}) \cup (A \times [0, 1])$$

be given by $x \mapsto (x, 0)$, and

$$f_t : A \mapsto (X \times \{0\}) \cup (A \times [0, 1])$$

be given by $a \mapsto (a, t)$. Then the homotopy extension property yields an extension

$$F : X \times [0, 1] \to (X \times \{0\}) \cup (A \times [0, 1]),$$

which is a retraction, as desired. □

Lemma 3.4. *If* (X, A) *is a CW pair, then* $(X \times \{0\}) \cup (A \times [0, 1])$ *is a (deformation) retract of* $X \times [0, 1]$. *In particular,* (X, A) *satisfies the homotopy extension property.*

Proof. The main idea is that for any disk D^n, the space $(D^n \times \{0\}) \cup (\partial D^n \times [0, 1])$ is a deformation retract of $D^n \times [0, 1]$. To see this, let $D^n \subseteq \mathbb{R}^n$ be the unit disk and $D^n \times [0, 1] \subseteq \mathbb{R}^{n+1}$. Let

$$p = (0, \ldots, 0, 2).$$

For any $x \in D^n \times [0, 1]$, let ℓ_x be the line through p and x. Note that

$$\ell_x \cap ((D^n \times [0, 1] \cup (\partial D^n \times [0, 1]))$$

is a unique point. Define $\widetilde{r}(x)$ to be this point, which yields a map

$$\widetilde{r} : D^n \times [0,1] \to (D^n \times \{0\}) \cup (\partial D^n \times [0,1])$$

Note that for $x \in (D^n \times \{0\}) \cup (\partial D^n \times [0,1])$, then $\widetilde{r}(x) = x$ since the point of intersection is unique and x is already in the intersection. Show as an exercise that $\widetilde{r}$ is continuous. Then setting

$$\widetilde{r}_t = t\widetilde{r} + (1-t)\,\mathrm{id}_{D^n \times [0,1]}$$

gives a deformation retraction of $D^n \times [0,1]$ onto $(D^n \times \{0\}) \cup (\partial D^n \times [0,1])$.

To show the general case that $X \times [0,1]$ retracts to $(X \times \{0\}) \cup (A \times [0,1])$, we induct on the dimension of cells. Define r on $X^{(0)} \times [0,1] \to (X \times \{0\}) \cup (A \times [0,1])$ by the following:

- if a 0-cell $D^0 \subseteq A$, then let r be the identity on $D^0 \times [0,1]$;
- if a 0-cell D^0 is not in A, then send $D^0 \times [0,1]$ to $D^0 \subseteq X \times \{0\}$.

Now inductively assume r is defined on $X^{(k-1)} \times [0,1] \to (X \times \{0\}) \cup (A \times [0,1])$. For each k-cell D^k,

- if $D^k \subseteq A$, then let r be the identity on $D^k \times [0,1]$;
- if D^k is not in A, then note that $\partial D^k \times [0,1] \to X^{(k-1)} \times [0,1]$ is defined by induction, and we have an "inclusion" (here $a : \partial D^k \to X^{(k-1)}$ is the attaching map for D^k)

$$\begin{tikzcd} D^k \arrow[r, "i"] \arrow[rr, bend right, "j"'] & X^{(k-1)} \cup D^k \arrow[r, "q"] & (X^{(k-1)} \cup D^k)/\{(x \in \partial D^k) \sim (a(x) \in X^{k-1})\} \end{tikzcd}$$

  So we let $D^k \times \{0\} \to (X \times \{0\}) \cup (A \times [0,1])$ be the map j into $X \times \{0\}$. This defines r on

$$(D^k \times [0,1]) \cup (\partial D^k \times [0,1]),$$

  which extends to $D^k \times [0,1]$ by composing with the map $\widetilde{r}$ from above.

This inductively defines the retraction r on all of $X \times [0,1]$. The last claim follows by Lemma 3.3. □

Remark. Now we can prove the first two lemmas from the beginning of the day.

Proof of Lemma 3.1. We prove that the result holds for any (X, A) which satisfies the homotopy extension property. We show that the quotient map $q : X \to X/A$ has a homotopy inverse. Since A is contractible, we know there exists a homotopy $f_t : A \to A \subseteq X$, such that $f_0 = \mathrm{id}_A$ and f_1 is constant. Let $F_0 : X \to X$ be the identity (note that $F_0|_A = \mathrm{id}_A$), so that the homotopy extension property gives a homotopy $F_t : X \to X$ extending f_t. Since $F_t(A) \subseteq A$ for every t, we get maps

$$\widetilde{F}_t : X/A \to X/A,$$

which are well defined since points in A are mapped to points in A. Furthermore, $F_1(A) = \{\mathrm{pt}\}$, so F_1 factors through X/A to give a map $h : X/A \to X$ satisfying $F_1 = h \circ q$. This gives the diagram:

$$\begin{tikzcd} X \arrow[r, "F_1"] \arrow[d, "q"'] & X \arrow[d, "q"] \\ X/A \arrow[r, "\overline{F}_1"'] \arrow[ur, "h"] & X/A \end{tikzcd}$$

It is easy to check that $h \circ q = F_1$ and $q \circ h = \overline{F}_1$, so that the diagram commutes. But then

$$h \circ q = F_1 \sim F_0 = \mathrm{id}_X \quad \text{and} \quad q \circ h = \overline{F}_1 \sim \overline{F}_0 = \mathrm{id}_{X/A},$$

so q is a homotopy equivalence. □

Proof of Lemma 3.2. Let $F : A \times [0,1] \to Y$ be a homotopy, which extends to $F : X \times [0,1] \to Y$ by the homotopy extension property. Consider the mapping cylinder

$$M_F = (X \times [0,1]) \cup_F Y,$$

and one can show that $M_F \simeq X \cup_g Y \simeq X \cup_f Y$. □

Lecture 4

Jan. 15 — Fundamental Group

4.1 Fundamental Group

Remark. The basic idea of the *fundamental group* is to study the topology of a space with loops mapped into the space. For instance, intuitively, any loop in S^2 can be "pulled back" to (i.e. is homotopic to) a constant loop. On the other hand, a loop wrapping around the hole in T^2 gets stuck and cannot be "pulled back" to a constant loop. The same issue happens for a loop in T^2 around the cylindrical part. The fundamental group is a way to make this intuition precise and measure the "holes" in a space.

Definition 4.1. The *fundamental group* of a based space (X, x_0) is

$$\pi_1(X, x_0) = [(S^1, n), (X, x_0)]_0,$$

i.e. the homotopy classes of loops in X based at x_0. Here $n = (0, 1)$ is the north pole of S^1.

Exercise 4.1. Let $S^1 \subseteq \mathbb{R}^2$ be the unit circle and let $p : [0, 1] \to S^1$ be given by

$$t \mapsto (\cos 2\pi t, \sin 2\pi t).$$

Show that p is a quotient map, so we can think of S^1 as $[0, 1]$ with $0, 1$ identified. Moreover, show that there is a one-to-one correspondence between maps of the form[1]

$$\gamma : ([0, 1], \{0, 1\}) \to (X, x_0) \quad \text{and} \quad \widetilde{\gamma} : (S^1, \{(1, 0)\}) \to (X, x_0)$$

given by $\widetilde{\gamma} \mapsto \widetilde{\gamma} \circ p = \gamma$, and that homotopies of $\widetilde{\gamma}$ rel $\{0, 1\}$ correspond to homotopies of S^1 rel $\{(1, 0)\}$.

Remark. Using the above exercise, we can think of $\pi_1(X, x_0) = [S^1, X]_0$ instead as

$$\pi_1(X, x_0) = [([0, 1], \{0, 1\}), (X, x_0)]_0.$$

Given a based loop $\gamma : [0, 1] \to X$, we denote its equivalence class in $\pi_1(X, x_0)$ by $[\gamma]$.

Definition 4.2. If γ_1, γ_2 are loops in X based at $x_0 \in X$, their *concatenation* $\gamma_1 * \gamma_2 : [0, 1] \to X$ is

$$t \mapsto \begin{cases} \gamma_1(2t) & \text{if } 0 \leq t \leq 1/2, \\ \gamma_2(2t - 1) & \text{if } 1/2 \leq t \leq 1. \end{cases}$$

Remark. Concatenation of loops indeed yields another loop since $\gamma_1 * \gamma_2(0) = \gamma_1 * \gamma_2(1) = x_0$ and $\gamma_1 * \gamma_2$ is continuous since the definitions agree on the closed set $\{1/2\}$.

[1] Such a loop γ is called a *based loop*.

Remark. We can clearly see that $\gamma_1 * \gamma_2$ is well-defined given γ_1 and γ_2, but can we define $[\gamma_1] * [\gamma_2]$ for homotopy classes of loops in a well-defined manner? We need to check that if $\gamma_1 \sim \gamma_2$ and $\delta_1 \sim \delta_2$ (i.e. $\gamma_1, \gamma_2 \in [\gamma_1]$ and $\delta_1, \delta_2 \in [\delta_1]$), then we also have $\gamma_1 * \delta_1 \sim \gamma_2 * \delta_2$.

To do this, let $H : [0,1] \times [0,1] \to X$ be the homotopy from γ_1 to γ_2 and $G : [0,1] \times [0,1] \to X$ be the homotopy from δ_1 to δ_2. We need to construct a homotopy $\widetilde{H} : [0,1] \times [0,1] \to X$ from $\gamma_1 * \delta_1$ to $\gamma_2 * \delta_2$.

Note that if we think of $[0,1] \times [0,1]$ as the unit square, then $\widetilde{H}$ is already defined on the boundary: the left and right sides are constantly x_0, the top side is $\gamma_2 * \delta_2$, and the bottom side is $\gamma_1 * \delta_1$. So we only need to define it on the interior. For this, note that the vertical line in the middle of the square is also constantly x_0 by construction: This creates two rectangles on each half, which we can fill with H and G.

More formally, we can construct the homotopy $\widetilde{H} : [0,1] \times [0,1] \to X$ explicitly via

$$(t,s) \mapsto \begin{cases} H(2t,s) & \text{if } 0 \le t \le 1/2, \\ G(2t-1,s) & \text{if } 1/2 \le t \le 1. \end{cases}$$

This is continuous since the definitions agree on the closed set $\{t = 1/2\}$. Thus, setting

$$[\gamma_1] * [\delta_1] = [\gamma_1 * \delta_1]$$

gives a well-defined binary operation by the above discussion.

Lemma 4.1. *The pair $(\pi_1(X, x_0), *)$ is a group.*

Proof. For the identity, let $e : [0,1] \to X$ be the constant loop $t \mapsto x_0$. We will show that

$$[e] * [\gamma] = [\gamma] = [\gamma] * [e].$$

The picture is that $[0,1] \times [0,1]$ has γ on the top side and $\gamma * e$ on the bottom. By drawing a line from the midpoint of the bottom side and the top-right corner, we see that we can fill the right portion with just x_0 and the left portion with γ. The equation of this line is $s = 2t - 1$, so $t = (s+1)/2$. Thus from the picture, we can write the explicit homotopy $H : [0,1] \times [0,1] \to X$ via

$$H(t,s) = \begin{cases} \gamma(2/(s+1), t) & \text{if } 0 \le t \le (s+1)/2, \\ x_0 & \text{if } (s+1)/2 \le t \le 1. \end{cases}$$

One can use a similar construction to show that $[e] * [\gamma] = [\gamma]$, so that $[e]$ is an identity element.

Now we show the existence of inverses. Given a loop γ, define $\overline{\gamma}$ via $\overline{\gamma}(t) = \gamma(1-t)$, i.e. γ backwards. Set $\gamma_s(t) = \gamma(st)$. Note that as t goes from 0 to 1, γ_s goes from $\gamma(0)$ to $\gamma(s)$, and also that $\overline{\gamma}_s(t) = \gamma(s - st)$. So we can write the homotopy $H : [0,1] \times [0,1] \to X$ between $\gamma * \overline{\gamma}$ and e by

$$H(t,s) = \begin{cases} \gamma_s(2t) & \text{if } 0 \le t \le 1/2, \\ \overline{\gamma}_s(2t-1) & \text{if } 1/2 \le t \le 1 \end{cases} = \begin{cases} \gamma(2st) & \text{if } 0 \le t \le 1/2, \\ \gamma(s - s(2t-1)) & \text{if } 1/2 \le t \le 1. \end{cases}$$

Thus setting $[\gamma]^{-1} = [\overline{\gamma}]$ gives us inverses.

Finally, for associativity, we need to see that $(\gamma_1 * \gamma_2) * \gamma_3 \sim \gamma_1 * (\gamma_2 * \gamma_3)$. Again by drawing a picture, we see that we can draw two diagonal lines connecting the starting points of γ_2 on top and bottom and the ending points of γ_1 on top and bottom. Write an explicit formula for the homotopy as an exercise. $\square$

4.2 Induced Homomorphisms

Remark. If $f : X \to Y$ and $x_0 \in X$, let $y_0 = f(x_0)$. Then given a based loop $\gamma : [0,1] \to X$, note that the composition $f \circ \gamma : [0,1] \to Y$ is a based loop in Y. Also, if $\gamma \sim \delta$, then $f \circ \gamma \sim f \circ \delta$ (if H is a homotopy from $\gamma \sim \delta$, then $f \circ H$ is a homotopy from $f \circ \gamma$ to $f \circ \delta$). In particular, f induces a map

$$f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0).$$

Lemma 4.2. *The induced map $f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ is a homomorphism.*

Proof. Note that

$$\gamma_1 * \gamma_2(t) = \begin{cases} \gamma_1(2t) & \text{if } 0 \le t \le 1/2, \\ \gamma_2(2t-1) & \text{if } 1/2 \le t \le 1 \end{cases}$$

and that

$$(f \circ \gamma_1) * (f \circ \gamma_2)(t) = \begin{cases} f(\gamma_1(2t)) & \text{if } 0 \le t \le 1/2, \\ f(\gamma_2(2t-1)) & \text{if } 1/2 \le t \le 1, \end{cases}$$

so $f \circ (\gamma_1 * \gamma_2) = (f \circ \gamma_1) * (f \circ \gamma_2)$, which implies $f_*([\gamma_1 * \gamma_2]) = f_*([\gamma_1]) * f_*([\gamma_2])$. □

Exercise 4.2. Check the following as an exercise:

1. $(f \circ g)_* = f_* \circ g_*$;
2. if $f : X \to Y$ is homotopic rel base point to $g : X \to Y$, then $f_* = g_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$.

Remark. How does π_1 depend on the base point? Let $x_0, x_1 \in X$, and suppose that there exists a path $h : [0,1] \to X$ with $h(0) = x_0$ and $h(1) = x_1$. Then if γ is a loop based at x_1, we can get a loop based at x_0 by going from x_0 to x_1 along h, taking γ, and then going back to x_0. More explicitly, this is

$$h * \gamma * \overline{h}(t) = \begin{cases} h(3t) & \text{if } 0 \le t \le 1/3, \\ \gamma(3t-1) & \text{if } 1/3 \le t \le 2/3, \\ \overline{h}(3t-2) & \text{if } 2/3 \le t \le 1. \end{cases}$$

Lemma 4.3. *The path h induces an isomorphism $\phi_h : \pi_1(X, x_1) \to \pi_1(X, x_0)$ by $[\gamma] \mapsto [h * \gamma * \overline{h}]$.*

Proof. Check as an exercise that ϕ_h is a well-defined homomorphism. To show that ϕ_h is an isomorphism, we claim that $\phi_{\overline{h}}$ is an inverse of ϕ_h. To see this, let $[\gamma] \in \pi_1(X, x_0)$. Then

$$\phi_h \circ \phi_{\overline{h}}([\gamma]) = [h * \overline{h} * \gamma * h * \overline{h}] = [h * \overline{h}] * [\gamma] * [h * \overline{h}] = [e] * [\gamma] * [e] = [\gamma],$$

where the second equality follows by the same proof for associativity of $*$. This proves the result. □

Remark. Note the following based on the above lemma:

1. The isomorphism class of $\pi_1(X, x_0)$ only depends on the path component of X containing x_0.
2. The isomorphism *depends on h*. One needs to be careful about using the correct identification.

Lecture 5

Jan. 22 — Simple Computations

5.1 Fundamental Groups and Homotopy Equivalence

Lemma 5.1. *Suppose $f_0, f_1 : X \to Y$ are homotopic by the homotopy $H : X \times [0,1] \to Y$. Let $x_0 \in X$ be a basepoint and define $h : [0,1] \to Y$ by $t \mapsto H(x_0, t)$. Then the following diagram commutes:*

$$\begin{tikzcd}
\pi_1(X, x_0) \arrow[r, "(f_0)_*"] \arrow[dr, "(f_1)_*"'] & \pi_1(Y, f_0(x_0)) \\
& \pi_1(Y, f_1(x_0)) \arrow[u, "\phi_h"']
\end{tikzcd}$$

Proof. Fix an arbitrary $[\gamma] \in \pi_1(X, x_0)$, and we construct a homotopy from $h * (f_1 \circ \gamma) * \overline{h}$ to $f_0 \circ \gamma$. The picture is the following: we have $h * (f_1 \circ \gamma) * \overline{h}$ at the top and $f_0 \circ \gamma$ at the bottom, where t parametrizes the horizontal direction and s parametrizes the vertical direction. Draw a trapezoidal shape by connecting the middle two points on the top edge to the two bottom corners.

Define the homotopy as follows: For a fixed s, define $H'(t,s)$ first by $h_s(t)$ in the first third, $H(\gamma(t), s)$ in the second third, and then $\overline{h}_s(t)$ in the last third. Construct the explicit homotopy as an exercise. $\square$

Theorem 5.1. *If $f : X \to Y$ is a homotopy equivalence, then the induced map*

$$f_* : \pi_1(X, x_0) \to \pi_1(Y, f(x_0))$$

on fundamental groups is an isomorphism.

Proof. Let g be a homotopy inverse to f, so we have the composition:

$$\pi_1(X, x_0) \xrightarrow{f_*} \pi_1(Y, f(x_0)) \xrightarrow{g_*} \pi_1(X, g(f(x_0)))$$

But we know $g \circ f = \mathrm{id}_X$, so by the lemma there exists a path $h : X \to X$ from x_0 to $g(f(x_0))$ such that $g_* \circ f_* = \phi_h$ is an isomorphism. So f_* is injective. Similarly, $f_* \circ g_*$ is an isomorphism, and therefore f_* is surjective. Since we already know f_* is a homomorphism, this shows that f_* is an isomorphism. $\square$

Remark. Recall that we have defined a "functor"

$$\{\text{pointed topological spaces, pointed maps}\} \to \{\text{groups, homomorphisms}\},$$

where homotopy equivalent spaces are mapped to isomorphic groups and homotopic maps give rise to the "same" homomorphism. We will finally make some computations of fundamental groups next.

5.2 Simple Computations of Fundamental Groups

Lemma 5.2. *If X is contractible, then $\pi_1(X, x_0) = \{1\}$ for all $x_0 \in X$, where $\{1\}$ is the trivial group.*

Proof. If $Y = \{y_0\}$ is a one-point space, then there exists a unique loop $\gamma : [0,1] \to X$ given by $t \mapsto y_0$. So $\pi_1(Y, y_0) = \{1\}$. Since X is contractible, it is homotopy equivalent to Y and so $\pi_1(X, x_0) = \{1\}$. □

Definition 5.1. We say that a space X is *simply connected* if

1. X is path-connected, and
2. $\pi_1(X, x_0) = \{1\}$ for some $x_0 \in X$.

Remark. Simply connected means that "points in X are connected in a very simple way."

Lemma 5.3. *A space X is simply connected if and only if every two points in X are connected by a unique homotopy class of paths in X.*

Proof. ($\Leftarrow$) Clearly X is path-connected. Furthermore, any loop based at x_0 is a path from x_0 to itself, and the constant is as well. Thus any loop is homotopic to the constant loop, i.e. $\pi_1(X, x_0) = \{1\}$.

($\Rightarrow$) For any $a, b \in X$, there exists a path from a to b (since X is path-connected). Now suppose $\gamma, \delta : [0,1] \to X$ are paths from a to b. By $\pi_1(X, a) = \{1\}$ we know that $\gamma * \overline{\delta} \sim e_a$, so

$$\gamma \sim \gamma * (\overline{\delta} * \delta) \sim (\gamma * \overline{\delta}) * \delta \sim e_a * \delta \sim \delta,$$

i.e. γ and δ are in the same homotopy class. □

Lemma 5.4. *Let $X = A \cup B$, where $A, B, A \cap B$ are open and path-connected. Let $x_0 \in A \cap B$. Then any loop $\gamma : [0,1] \to X$ based at x_0 can be written as*

$$\gamma \sim \gamma_1 * \gamma_2 * \cdots * \gamma_n,$$

where each γ_i is a loop in A or B based at x_0.

Proof. We first claim that there exist $0 = t_0 < t_1 < \cdots < t_n = 1$ such that $\operatorname{im} \gamma|_{[t_{i-1}, t_i]} \subseteq A$ or B, and $\gamma(t_i) \in A \cap B$ for every i. The proof of this will use the following topology fact:

> **Lemma** (Lebesgue number lemma)**.** Let X be a compact metric space and $\{U_\alpha\}_{\alpha \in A}$ be an open cover. Then there exists a *Lebesgue number* $\delta > 0$ such that for all sets S with $\operatorname{diam}(S) = \sup_{x,y \in S} d(x,y) < \delta$, there exists $\alpha \in A$ such that $S \subseteq U_\alpha$.

To prove the claim, let $U_1 = \gamma^{-1}(A)$ and $U_2 = \gamma^{-1}(B)$, which is an open cover of $[0,1]$. So there exists $\delta > 0$ such that if $|b - a| < \delta$, then $[a, b] \subseteq U_i$ for $i = 1$ or 2. Thus $\gamma([a,b]) \subseteq A$ or B. Now let n be a positive integer such that $1/n < \delta$, so that for each i we have

$$\operatorname{im} \gamma|_{[i/n, (i+1)/n]} \subseteq A \text{ or } B.$$

So we start with $t_i = 1/n$ for $i = 0, \ldots, n$. Now if $\gamma|_{[t_{i-1}, t_i]}$ and $\gamma|_{[t_i, t_{i+1}]}$ both have image in A (or both in B), then throw out t_i. Then $\gamma|_{[t_{i-1}, t_i]}$ will have image in A or B and $\gamma(t_i) \in A \cap B$, as desired.

Now given the claim, let $\delta_i : [0,1] \to A \cap B$ connect x_0 to $\gamma(t_1)$, and set $\gamma_i = \gamma|_{[t_{i-1}, t_i]}$. Then

$$\gamma \sim \gamma_1 * \gamma_2 * \cdots * \gamma_n \sim (\gamma_1 * \overline{\delta}_1) * (\delta_1 * \gamma_2 * \overline{\delta}_2) * \cdots * (\delta_{n-1} * \gamma_n),$$

where each of the above loops is either in A or B. □

Theorem 5.2. *We have $\pi_1(S^n, x_0) = \{1\}$ for all $n \geq 2$.*

Proof. We have $\pi_1(S^n, x_0) = \{1\}$ for all $n \geq 2$. Let $A = S^n \setminus \{(0, \dots, 0, 1)\}$ and $B = S^n \setminus \{(0, \dots, 0, -1)\}$. Note that $A \cong B \cong \mathbb{R}^n$, so they are path-connected. Furthermore, $A \cap B = S^{n-1} \times \mathbb{R}$, which is also path-connected if $n \geq 2$.[1] Now take any $x_0 \in A \cap B$. Then any $[\gamma] \in \pi_1(S^n, x_0)$ can be written as

$$[\gamma] = [\gamma_1] * [\gamma_2] * \cdots * [\gamma_n],$$

where $[\gamma_i] \in \pi_1(A, x_0)$ or $\pi_1(B, x_0)$ by the lemma. But we have

$$\pi_1(A, x_0) = \pi_1(B, x_0) = \{1\},$$

so $[\gamma] = [e_{x_0}]$ and hence $\pi_1(S^n, x_0) = \{1\}$. □

Theorem 5.3. *Given two spaces X and Y, we have $\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0)$.*

Proof. The map $\Phi : \pi_1(X, x_0) \times \pi_1(Y, y_0) \to \pi_1(X \times Y, (x_0, y_0))$ given by

$$([\gamma], [\delta]) \mapsto [\gamma \times \delta],$$

where $(\gamma \times \delta)(t) = (\gamma(t), \delta(t))$, is an isomorphism. Check as an exercise that Φ is well-defined and a bijection (for the second part, consider the projections $p_X : X \times Y \to X$ and $p_Y : X \times Y \to Y$). □

5.3 Fundamental Group of the Circle

Our next objective is the following computation:

Theorem 5.4. *We have $\pi_1(S^1, (1, 0)) \cong \mathbb{Z}$. In particular, the map sending $n \in \mathbb{Z}$ to*

$$\gamma_n : [0, 1] \to S^1 : t \mapsto (\cos 2\pi nt, \sin 2\pi nt)$$

is an isomorphism $\mathbb{Z} \to \pi_1(S^1, (1, 0))$.

Remark. The proof is an example of a very important technique that we will see again soon. The proof involves studying the map

$$p : \mathbb{R} \to S^1 : t \mapsto (\cos 2\pi t, \sin 2\pi t).$$

Note that $p^{-1}((1, 0)) = \mathbb{Z}$. This is a particular example of a *covering map*, which we will study later.

Definition 5.2. If $\gamma : [0, 1] \to S^1$ is a path based at the point $(1, 0)$, then a *lift of γ based at $n \in \mathbb{Z}$* is a map $\widetilde{\gamma}_n : [0, 1] \to \mathbb{R}$ such that $\widetilde{\gamma}_n(0) = n$ and $p \circ \widetilde{\gamma}_n = \gamma$.

Lemma 5.5. *We have the following:*

(a) For each $n \in \mathbb{Z}$, each loop $\gamma : [0, 1] \to S^1$ based at $(0, 1)$ lifts to a unique path $\widetilde{\gamma}_n$ based at n.

(b) If $\gamma \sim \gamma'$ are loops in S^1 based at $(0, 1)$ and $\widetilde{\gamma}_n, \widetilde{\gamma}'_n$ are their lifts based at n, then $\widetilde{\gamma}_n \sim \widetilde{\gamma}'_n$ rel $\{0, 1\}$.

The above properties are called path lifting *and* homotopy lifting.

Proof. We will prove this next class. □

[1] Note that this does not work for $n = 1$: the intersection $S^0 \times \mathbb{R} = \{\pm 1\} \times \mathbb{R}$ is not path-connected.

Proof of Theorem 5.4. Given $[\gamma] \in \pi_1(S^1, (1,0))$, part (a) of Lemma 5.5 says that there is a unique lift $\widetilde{\gamma}_0 : [0,1] \to \mathbb{R}$. Since $\widetilde{\gamma}_0(1) \in p^{-1}((1,0)) = \mathbb{Z}$, we can define $\Phi : \pi_1(S^1, (1,0)) \to \mathbb{Z}$ by $[\gamma] \mapsto \widetilde{\gamma}_0(1)$. Part (b) of Lemma 5.5 says that Φ is well-defined. We need to show the following:

1. Φ is surjective:

   Let $\widetilde{\delta}^n(t) = nt$ for $t \in [0,1]$ and $\delta^n = p \circ \widetilde{\delta}$. Then $\widetilde{\delta}^n$ is the lift of δ^n based at 0, and $\Phi([\delta^n]) = n$.

2. Φ is injective:

   Suppose γ, γ' are loops in S^1 such that

   $$\Phi([\gamma]) = \widetilde{\gamma}_0(1) = \widetilde{\gamma}'_0(1) = \Phi([\gamma']).$$

   Set $\widetilde{H}(s,t) = (1-t)\widetilde{\gamma}_0(s) + t\widetilde{\gamma}'_0(s)$ and $H = p \circ \widetilde{H}$. Then H is a homotopy from γ to γ' (exercise).

3. Φ is a homomorphism:

   Let $[\gamma], [\gamma'] \in \pi_1(S^1, (1,0))$ and let $\widetilde{\gamma}_0, \widetilde{\gamma}'_0$ be their lifts based at 0. Then

   $$\Phi([\gamma]) = \widetilde{\gamma}_0(1) = n \in \mathbb{Z} \quad \text{and} \quad \Phi([\gamma']) = \widetilde{\gamma}'_0(1) = m \in \mathbb{Z}.$$

   Then note the following:

   (a) $t \mapsto n + \widetilde{\gamma}'_0(t)$ is a lift of γ' and starts at n, so by uniqueness it is $\widetilde{\gamma}'_n$, and

   (b) $\widetilde{\gamma}_0 * \widetilde{\gamma}'_n$ is a lift of $\gamma * \gamma'$.

   So $\Phi([\gamma] * [\gamma']) = \widetilde{\gamma} * \widetilde{\gamma}'_0(1) = n + m = \Phi([\gamma]) + \Phi([\gamma'])$, i.e. Φ is a homomorphism.

Thus Φ is an isomorphism, and we have $\pi_1(S^1, (1,0)) \cong \mathbb{Z}$. $\square$

Lecture 6

Jan. 27 — Fundamental Group of the Circle

6.1 Proof of Path Lifting

Proof of Lemma 5.5. (a) Let $A = S^1 \setminus \{(1,0)\}$, and note that

$$p^{-1}(A) = \bigcup_{i \in \mathbb{Z}} (i, i+1) = \bigcup_{i \in \mathbb{Z}} A_i.$$

Notice that each restriction $p|_{A_i} : A_i \to A$ is a homeomorphism. Now let $B = S^1 \setminus \{(-1,0)\}$, so

$$p^{-1}(B) = \bigcup_{i \in \mathbb{Z}} \left(i - \frac{1}{2}, i + \frac{1}{2} \right) = \bigcup_{i \in \mathbb{Z}} B_i.$$

Similarly, each $p|_{B_i} : B_i \to B$ is a homeomorphism. Now if $\gamma : [0,1] \to S^1$ is contained in A (or B), we can choose any $i \in \mathbb{Z}$ and let $\widetilde{\gamma} = (p|_{A_i})^{-1} \circ \gamma$, giving a lift of γ. Then for a general $\gamma : [0,1] \to S^1$ with $\gamma(0) = (1,0)$, the set $\{\gamma^{-1}(A), \gamma^{-1}(B)\}$ is an open cover of the compact metric space $[0,1]$, so there exists a Lebesgue number $\delta > 0$ such that any interval $[a,b]$ with $b - a < \delta$ lies in either $\gamma^{-1}(A)$ or $\gamma^{-1}(B)$. Choose n such that $1/n < \delta$. If $t_n = i/n$ for $i = 0, \dots, n$, then

$$\gamma([t_i, t_{i+1}]) \subseteq A \text{ or } B$$

for every i. Again for convenience, if $[t_{i-1}, t_i]$ and $[t_i, t_{i+1}]$ are both in $\gamma^{-1}(A)$ or $f^{-1}(B)$, then discard t_i. So we have a partition $0 = t_0 < t_1 < \cdots < t_k = 1$ such that (note that γ starts at $(1,0) \notin A$)

$$\gamma([t_i, t_{i+1}]) \subseteq \begin{cases} A & \text{if } i \text{ is odd,} \\ B & \text{if } i \text{ is even.} \end{cases}$$

Then we want to build $\widetilde{\gamma}_n$. Define $\widetilde{\gamma}_n$ on $[t_0, t_1]$ to be $(p|_{B_n})^{-1} \circ \gamma|_{[t_0,t_1]}$. Now $\widetilde{\gamma}_n(t_1) \in A_i$ for a unique i, so define $\widetilde{\gamma}_n$ on $[t_1, t_2]$ by $(p|_{A_i})^{-1} \circ \gamma|_{[t_1,t_2]}$. Note that $\widetilde{\gamma}_n$ is continuous on $[t_0, t_2]$ since the two definitions agree at $t = t_1$. Inductively continue to define the lift $\widetilde{\gamma}_n$ on all of $[0,1]$.

(b) The proof is very similar to path lifting. Given a homotopy $H : [0,1] \times [0,1] \to S^1$, we can find a Lebesgue number $\delta > 0$ for $\{H^{-1}(A), H^{-1}(B)\}$. Pick n such that $\sqrt{2}/n < \delta$ and break $[0,1] \times [0,1]$ into n^2 squares of side length $1/n$. The diameter of each square is at most $\sqrt{2}/n$, so each square can be lifted as above. Finish the construction as an exercise to lift H on all of $[0,1] \times [0,1]$. $\square$

6.2 Applications of the Fundamental Group of S^1

Corollary 6.0.1. *There is no retraction $D^2 \to \partial D^2$.*

Proof. Suppose there was a retraction $r : D^2 \to \partial D^2$, and let $i : S^1 \to D^2$ be the inclusion of S^1 as the boundary of D^2. Then we have the composition:

$$S^1 \xrightarrow{i} D^2 \xrightarrow{r} S^1$$

Noting that $r \circ i = S^1 \to S^1$ is the identity, so $(r \circ i)_* : \pi_1(S^1, (1,0)) \to \pi_1(S^1, (1,0))$ is the identity map. In particular, $r_* \circ i_* = (r \circ i)_*$ is the identity map, hence i_* must be injective. But

$$i_* : \pi_1(S^1, (1,0)) \to \pi_1(D^2, (1,0))$$

where $\pi_1(S^1, (1,0)) \cong \mathbb{Z}$ and $\pi_1(D^2, (1,0)) = \{1\}$, so i_* cannot be injective. Contradiction. $\square$

Corollary 6.0.2. *Any map $f : D^2 \to D^2$ has a fixed point, i.e. $x \in D^2$ such that $f(x) = x$.*

Proof. Suppose otherwise that $f : D^2 \to D^2$ has no fixed points. Then for each $x \in D^2$, there is a unique ray R_x starting at $f(x)$ and going through x. Note that $R_x \cap \partial D^2$ in a unique point (on the interior of R_x). Define $r : D^2 \to S^1$ by $x \mapsto R_x \cap \partial D^2$. Show that r is continuous as an exercise (e.g. parametrize the line). But then r is a retraction $D^2 \to \partial D^2$, a contradiction. $\square$

Remark. There are more applications such as the fundamental theorem of algebra, the ham sandwich theorem, and the Borsuk-Ulam theorem. See Hatcher for more details.

6.3 Free Products of Groups

Definition 6.1. Let G_1 and G_2 be groups. A *word* in $G_1 \sqcup G_2$ is a finite sequence

$$x = (x_1, x_2, \ldots, x_n)$$

for some n, where each x_i is in G_1 or G_2. Define an equivalence relation on words in $G_1 \sqcup G_2$ which is generated by (show as an exercise that this is in fact an equivalence relation):

1. replace a, b in a sequence by ab if a, b are in the same group (or the reverse of this), and
2. if e_i (the identity in G_i) is in a sequence, then remove it (or add it in any place in a sequence).

Denote the equivalence class of a word x by $[x]$. Call a word $x = (x_1, \ldots, x_n)$ *reduced* if

1. $x_j \neq e_i$ for any j or i, and
2. x_i and x_{i+1} are from different groups.

Show that each $[x]$ contains a unique reduced word (note: uniqueness is hard). The *free product* of G_1 and G_2 is the group $G_1 * G_2$ of all equivalence classes of words in $G_1 \sqcup G_2$, with multiplication

$$[x_1, \ldots, x_n] \cdot [y_1, \ldots, y_m] = [x_1, \ldots, x_n, y_1, \ldots, y_m].$$

Remark. Note that in $G_1 * G_2$, the identity e is the empty word and the inverse is given by

$$[x_1, \ldots, x_n]^{-1} = [x_n^{-1}, \ldots, x_1^{-1}].$$

Check as an exercise that $G_1 * G_2$ is in fact a group (really only need to check associativity).

Proposition 6.1. *Let $j_i : G_i \to G_1 * G_2$ be the inclusion of G_i into $G_1 * G_2$. Given any homomorphisms $\phi_i : G_i \to H$ where H is any group, there exists a unique homomorphism $\phi : G_1 * G_2 \to H$ such that $\phi \circ j_i = \phi_i$, i.e. the following diagram commutes:*

$$\begin{tikzcd} & G_1 \arrow[d, "j_1"] \arrow[ddr, bend left, "\phi_1"] & \\ G_2 \arrow[r, "j_2"] \arrow[drr, bend right, "\phi_2"'] & G_1 * G_2 \arrow[dr, dashed, "\phi"] & \\ & & H \end{tikzcd}$$

Proof. If the x_i are reduced and $x_1 \in G_1$, then define

$$\phi_1(x_1, x_2, \ldots, x_n) = \phi_1(x_1) \cdot \phi_2(x_2). \cdot \phi_1(x_3) \cdot \ldots.$$

Then check the following as an exercise:

1. Show such that ϕ exists and is unique.
2. Show this property *defines* the free product, i.e. if D is another group satisfying the property in the proposition, then $D \cong G_1 * G_2$.

The second part above says that this is the *universal property* of the free product. □

Example 6.1.1. Represent $\mathbb{Z}$ in product notation via $\{x^n\}$, where $x^n x^m = x^{n+m}$. Then

$$\mathbb{Z} * \mathbb{Z} = \{x^n\} * \{y^m\} = \{e, x^{n_1} y^{m_1} \ldots x^{n_k}, x^{n_1} y^{m_1} \ldots y^{m_k}, y^{m_1} x^{n_1} \ldots y^{m_k}, y^{m_1} x^{n_1} \ldots x^{n_k}\}.$$

This group is called the *free group on two generators*, and $\mathbb{Z}$ is the *free group on one generator*.

Lecture 7

Jan. 29 — Some Group Theory

7.1 Group Presentations

Definition 7.1. The *free group* on n generators, denoted F_n, is defined inductively via

$$F_n = F_{n-1} * \mathbb{Z},$$

where $F_1 = \mathbb{Z}$. (One can also consider F_∞.)

Remark. Note that any homomorphism $\phi : \mathbb{Z} \to G$ (for any group G) is determined by $\phi(1)$. Moreover, given any $g \in G$, there is a unique homomorphism which maps $1 \mapsto g$. Thus by the universal property of the free product, a homomorphism $F_n \to G$ is determined uniquely by a choice of $g_1, \dots, g_n \in G$.

Definition 7.2. A *group presentation* is a group $\langle X | R \rangle$ defined as follows:

- X is some set (of generators);
- R is a set of words (relations) in $X \cup X^{-1}$ (formally denote $x \in X$ as $x^{-1} \in X^{-1}$);
- let $n = |X|$ and F_n be the free group on n generators, so that we can think of $R \subseteq F_n$;
- let $\langle R \rangle$ be the smallest normal subgroup of F_n containing R;
- define the group $\langle X | R \rangle = F_n / \langle R \rangle$.

We say that $\langle X | R \rangle$ is a *presentation* of a group G if $G \cong \langle X | R \rangle$.

Example 7.2.1. The group $\langle g | g^n \rangle$ is all the words in g, g^{-1}:

$$\dots, g^{-2}, g^{-1}, e, g, g^2, g^3, \dots,$$

but $g^n = e$, so $g^{n+1} = g^n g = eg = g$ and thus we have $g^{-1} = eg^{-1} = g^n g^{-1} = g^{n-1} g g^{-1} = g^{n-1}$. So there is a one-to-one correspondence between elements of $\langle g | g^n \rangle$ and g^k for $k = 0, \dots, n-1$.

Exercise 7.1. Show that $\langle g | g^n \rangle \cong \mathbb{Z}/n$, so that $\langle g | g^n \rangle$ is a presentation of $\mathbb{Z}/n$.

Lemma 7.1. *Every group has a presentation.*

Proof. Let G be a group. Let $X \subseteq G$ be a collection of elements of G that generate G (e.g. take $X = G$ itself). Let $n = |X|$, so there exists a unique $\phi : F_n \to G$ sending the generators of F_n to the $g_i \in X$. Let $N = \ker \varphi$, so the first isomorphism theorem says that $G \cong F_n / N$ (note that ϕ is clearly surjective). Let R be a subset of N that generates N (e.g. take $R = N$). Then $G \cong \langle X | R \rangle$. $\square$

Remark. Using fewer generators to write G or N may give more less complicated presentations of G.

Definition 7.3. We say that G is *finitely generated* if $G \cong \langle X|R\rangle$ such that $|X| < \infty$. We say that G is *finitely presented* if both $|X|, |R| < \infty$.

Exercise 7.2. Show the following:

1. If $G = \langle g_1, \dots, g_n | r_1, \dots, r_m\rangle$, then for any group H and any map $h : \{g_1, \dots, g_n\} \to H$ satisfying $h(r_i) = e_H$ (the notation $h(r_i)$ means to replace any letters g_j in r_i by $h(g_j)$), there exists a unique homomorphism $\phi_h : G \to H$ such that $\phi_h(g_i) = h(g_i)$.

2. If $G_1 = \langle g_1, \dots, g_n | r_1, \dots, r_m\rangle$ and $G_2 = \langle h_1, \dots, h_k | s_1, \dots, s_\ell\rangle$, then
$$G_1 * G_2 = \langle g_1, \dots, g_n, h_1, \dots, h_k | r_1, \dots, r_m, s_1, \dots, s_\ell\rangle.$$

Definition 7.4. Given groups G_1, G_2 and K, and homomorphisms $\psi_i : K \to G_i$, the *free product with amalgamation* is
$$G_1 *_K G_2 = \frac{G_1 * G_2}{\langle\{\psi_1(k)\psi_2(k)^{-1}\}_{k\in K}\rangle},$$
where $\langle\{\psi_1(k)\psi_2(k)^{-1}\}_{k\in K}\rangle$ is the smallest normal subgroup of $G_1 * G_2$ containing $\{\psi_1(k)\psi_2(k)^{-1}\}_{k\in K}$.

Remark. The idea is that $G_1 *_K G_2$ is the set of all words in $G_1 \cup G_2$ but if we see $\psi_1(k)$ in a word, we can replace it by $\psi_2(k)$ and vice versa. In terms of group presentations, if
$$\begin{aligned} G_1 &= \langle g_1, \dots, g_n | r_1, \dots, r_m\rangle, \\ G_2 &= \langle g'_1, \dots, g'_{n'} | r'_1, \dots, r'_{m'}\rangle, \\ K &= \langle h_1, \dots, h_k | s_1, \dots, s_\ell\rangle, \end{aligned}$$
then we can write
$$G_1 *_K G_2 = \langle g_1, \dots, g_n, g'_1, \dots, g'_{n'} | r_1, \dots, r_m, r'_1, \dots, r'_{m'}, \psi_1(h_1)(\psi_2(h_1))^{-1}, \dots, \psi_1(h_k)(\psi_2(h_k))^{-1}\rangle.$$
Note that the s_i show up in the above since ψ_1, ψ_2 are homomorphisms.

Exercise 7.3. Show the following:

1. Check that the above presentation for $G_1 *_K G_2$ is correct.

2. Let $\iota_i : G_i \to G_1 * G_2$ be the inclusions and $j_i : G_i \to G_1 *_K G_2$ be the induced maps. Then given any homomorphisms $\phi_i : G_i \to H$ (where H is any group) such that
$$\phi_1 \circ \psi_1(k) = \phi_2 \circ \psi_2(k) \quad \text{for all } k \in K,$$
then there exists a unique homomorphism $\Phi : G_1 *_K G_2 \to H$ such that $\Phi \circ j_i = \phi_i$, i.e.
$$\begin{tikzcd}
& G_1 \arrow[dr, "j_1"] \arrow[drr, bend left, "\phi_1"] & & \\
K, \arrow[ur, "\psi_1"] \arrow[dr, "\psi_2"'] & & G_1 *_K G_2 \arrow[r, dashed, "\Phi"] & H \\
& G_2 \arrow[ur, "j_2"'] \arrow[urr, bend right, "\phi_2"'] & &
\end{tikzcd}$$
Show that this is the *universal property* for the free product with amalgamation.

7.2 Seifert-van Kampen Theorem

Theorem 7.1 (Seifert-van Kampen). *Let X be a topological space with base point x_0. Let $A, B \subseteq X$ be open sets with $X = A \cup B$ such that $A, B, A \cap B$ are path-connected and $x_0 \in A \cap B$. Let*

$$\psi_A : \pi_1(A, x_0) \to \pi_1(X, x_0) \quad \text{and} \quad \psi_B : \pi_1(B, x_0) \to \pi_1(X, x_0)$$

be the homomorphisms induced from the inclusions $A \cap B \to A$ and $A \cap B \to B$. Then

$$\pi_1(X, x_0) \cong \pi_1(A, x_0) *_{\pi_1(A \cap B, x_0)} \pi_1(B, x_0).$$

Remark. There is a more general version where $X = \bigcup_{\alpha \in A} U_\alpha$. See Hatcher for more details.

Example 7.4.1. Consider $W_2 = S^1 \vee S^1$, a wedge of two circles. Let x_0 be the point of intersection of the circles. Let A be an open neighborhood of the left circle and B be an open neighborhood of the right circle. Note that $A \simeq B \simeq S^1$ and $A \cap B \simeq \{\text{pt}\}$. So we see that

$$\pi_1(A, x_0) \cong \mathbb{Z} \cong \langle g_1 | \ \rangle, \quad \pi_1(B, x_0) \cong \mathbb{Z} \cong \langle g_2 | \ \rangle, \quad \pi_1(A \cap B, x_0) = \{e\},$$

and so $\psi_A : \pi_1(A \cap B, x_0) \to \pi_1(A, x_0)$ and $\psi_B : \pi_1(A \cap B, x_0) \to \pi_1(B, x_0)$ must both map $e \mapsto e$. Thus

$$\pi_1(W_2, x_0) \cong \pi_1(A, x_0) *_{\pi_1(A \cap B, x_0)} \pi_1(B, x_0) \cong \langle g_1, g_2 | \psi_A(e)(\psi_B(e))^{-1} \rangle \cong \langle g_1, g_2 | \ \rangle \cong F_2,$$

by the Seifert-van Kampen theorem.

Exercise 7.4. Show the following:

1. If W_n is a wedge of n circles, then $\pi_1(W_n, x_0) \cong F_n$.
2. We have $\pi_1(\text{any connected graph}) = F_n$ for some n.

Example 7.4.2. Consider the torus $T^2 = S^1 \times S^1$. Recall that for a product, we know

$$\pi_1(T^2) \cong \pi_1(S^1) \times \pi_1(S^1) \cong \mathbb{Z} \times \mathbb{Z}.$$

We can also use van Kampen's theorem to see this. Think of T^2 as a square with opposite sides identified. Let A be the square with a circle missing in the circle, which deformation retracts to the boundary of the square. Let B be a big disk in the middle, so that $A \cap B$ deformation retracts to a circle. So

$$A \simeq W_2, \quad B \simeq \{\text{pt}\}, \quad A \cap B \simeq S^1.$$

Thus by our previous computations, we have

$$\pi_1(A, x_0) \cong \langle g_1, g_2 | \ \rangle, \quad \pi_1(B, x_0) \cong \{e\}, \quad \pi_1(A \cap B, x_0) \cong \langle h | \ \rangle.$$

The inclusion $\psi_B : \pi_1(A \cap B, x_0) \to \pi_1(B, x_0)$ sends $h \to e$, and $\psi_A : \pi_1(A \cap B, x_0) \to \pi_1(A, x_0)$ sends $h \mapsto g_1 g_2 g_1^{-1} g_2^{-1}$. To see the last claim, push a loop in $A \cap B$ to the boundary of the square (note that under the homotopy equivalence of A and W_2, we can assume x_0 maps to a corner point), where it follows the four edges of the square. These are the loops $g_1, g_2, g_1^{-1}, g_2^{-1}$, where the last two loops are oriented in the opposite direction of the first two. Thus by van Kampen's theorem,

$$\pi_1(T^2, x_0) \cong \pi_1(A, x_0) *_{\pi_1(A \cap B, x_0)} \pi_1(B, x_0) \cong \langle g_1, g_2 | \psi_A(h)(\psi_B(h))^{-1} \rangle = \langle g_1, g_2 | g_1 g_2 g_1^{-1} g_2^{-1} \rangle.$$

Exercise 7.5. Check the following:

1. Show that $\mathbb{Z} \times \mathbb{Z} \cong \langle g_1, g_2 | g_1 g_2 g_1^{-1} g_2^{-1} \rangle$, so that $\pi_1(T^2) \cong \mathbb{Z} \times \mathbb{Z}$ again.
2. Compute $\pi_1(\Sigma_g, x_0)$, where Σ_g is the surface of genus g. Show that $\pi_1(\Sigma_g)$ is not abelian if $g > 1$.

Lecture 8

Feb. 3 — Seifert-van Kampen Theorem

8.1 Applications of the Seifert-van Kampen Theorem

Theorem 8.1. *Let X be a path-connected space $f : \partial D^n \to X$ be continuous with $x_0 \in \partial D^n$. Set*

$$Y = X \cup_f D^n = X \sqcup D^n / \{(x \in D^n) \sim (f(x) \in X)\}.$$

Then (for $n = 1$, we need X to have a base point $f(x_0)$ with an open neighborhood $U \simeq \{f(x_0)\}$)

$$\pi_1(Y, y_0) = \begin{cases} \pi_1(X, f(x_0)) * \mathbb{Z} & \text{if } n = 1, \\ \pi_1(X, f(x_0))/\langle r \rangle & \text{if } n = 2, \\ \pi_1(X, f(x_0)) & \text{if } n \geq 3, \end{cases}$$

where $r = f_*(g)$ where g generates $\pi_1(\partial D^2, x_0) \cong \mathbb{Z}$.

Proof. We proof this in the case $n = 2$. Let

$$A = X \cup_f (D^2 \setminus \{0\}) = X \cup_f (S^1 \times (0, 1]) \simeq X$$

and B be the interior of D^2 (so $B \simeq \{\mathrm{pt}\}$. Then we can see that

$$A \cap B = (\operatorname{int} D^2) \setminus \{0\} = S^1 \times (0, 1) \simeq S^1.$$

Note that we can choose y_0 to be $f(x_0) \in X$ because the that is where it is sent under the deformation retraction from A to X. Now $\psi_A : \pi_1(A \cap B, y_0) \to \pi_1(A, y_0)$

$$\pi_1(A \cap B, y_0) \cong \langle g | \, \rangle \quad \text{and} \quad \pi_1(A, y_0) \cong \pi_1(X, f(x_0))$$

is given by $g \mapsto f_*(g)$, and $\psi_B(g) = e$. Thus the Seifert-van Kampen theorem implies

$$\pi_1(Y, y_0) \cong \pi_1(A, y_0) *_{\pi_1(A \cap B, y_0)} \pi_1(B, y_0) \cong \frac{\pi_1(A, y_0) * \{e\}}{\langle \psi_A(g)(\psi_B(g))^{-1} \rangle} \cong \frac{\pi_1(A, y_0)}{\langle r \rangle} \cong \frac{\pi_1(X, f(x_0))}{\langle r \rangle},$$

which is the desired result for $n = 2$. The $n \geq 3$ case is similar (except the intersection is now contractible in this case). Check the $n = 1$ case as an exercise. □

Remark. This allows us to compute the fundamental group of any CW complex (hence any manifold).

Corollary 8.1.1. *Let G be a finitely presented group. Then there exists a topological space X (in fact, a compact CW complex) such that $\pi_1(X, x_0) \cong G$.*

Proof. Let $G = \langle g_1, \dots, g_n | r_1, \dots, r_m$ and W_n be the wedge of n circles, so that

$$\pi_1(W_n, x_0) \cong F_n \cong \langle g_1, \dots, g_n | \ \rangle.$$

Now for each r_i let $f_i : S^1 \to W_n$ be a map such that $(f_i)_*(g) = r_i$ (show as an exercise that such f_i exists; essentially for each word, take the loops corresponding to each letter in order). Let

$$X = W_n \cup_{f_i} \left(\bigsqcup_{i=1}^{m} D^2 \right),$$

and the previous theorem tells us that $\pi_1(X, x_0) \cong \langle g_1, \dots, g_n | r_1, \dots, r_m \rangle \cong G$. $\square$

Remark. The topological space realizing G as its fundamental group is not unique. For instance, we can take the above construction and add a 5-cell, which does not change the fundamental group.

8.2 Proof of the Seifert-van Kampen Theorem

Proof of Theorem 7.1. We have the inclusions $A \subseteq X$ and $B \subseteq X$, which induce maps

$$\phi_A : \pi_1(A, x_0) \to \pi_1(X, x_0) \quad \text{and} \quad \phi_B : \pi_1(B, x_0) \to \pi_1(X, x_0).$$

By the universal property of free products, we get a map $\Phi : \pi_1(A, x_0) * \pi_1(B, x_0) \to \pi_1(X, x_0)$ by

$$([\gamma_1], [\delta_1], [\gamma_2], \dots) \mapsto \phi_A([\gamma_1])\phi_B([\delta_1])\phi_A([\gamma_2]) \dots.$$

Note that if $[\gamma] \in \pi_1(A \cap B, x_0)$, then $\psi_A([\gamma]) = [\gamma] = \psi_B([\gamma])$, so

$$\phi_A \circ \psi_A([\gamma]) = [\gamma] = \phi_B \circ \psi_B([\gamma]).$$

This tells us that $\Phi(\psi_A([\gamma])(\psi_B([\gamma]))^{-1}) = e$, so we see that

$$K = \langle \psi_A([\gamma])(\psi_B([\gamma]))^{-1} \rangle_{[\gamma] \in \pi_1(A \cap B, x_0)}$$

lies in the kernel of Φ. This gives us an induced map (still called it Φ)

$$\Phi : \pi_1(A, x_0) *_{\pi_1(A \cap B, x_0)} \pi_1(B, x_0) \to \pi_1(X, x_0).$$

Lemma 5.4 says that Φ is surjective, so it suffices to check injectivity. To do this, let $[\gamma_i] \in \pi_1(A, x_0)$ and $[\eta_i] \in \pi_1(B, x_0)$ with

$$\Phi([\gamma_1][\eta_1] \dots, [\gamma_n], [\eta_n]) = [\gamma_1 * \eta_1 * \dots * \gamma_n * \eta_n] = e. \tag{$*$}$$

We need to see that we can get from the word $[\gamma_1][\eta_1] \dots [\gamma_n][\eta_n]$ to the empty word by a sequence of:

1. replace a, b by $a \cdot b$ if a, b are in the same group (and the reverse of this);
2. if we see $\psi_A(k)$ in the word, we can replace it with $\psi_B(k)$ (and the reverse of this).

We will prove the theorem only for $n = 2$. Now $(*)$ says that there exists a homotopy H between x_0 and $\gamma_1 * \eta_1$. As before, we can use the Lebesgue number lemma to find n such that squares of side length $1/n$ are mapped by H into either A or B (we can assume that the number of γ_i, η_i divides n). Check as an exercise that we can assume $H(i/n, j/n) = x_0$, i.e. that we can change H and γ_i, η_i by a homotopy such that the homotopy of γ_i is in A and the homotopy of η_i is in B (hint: consider radial lines around $(i/n, j/n)$). So we have an $n \times n$ grid where each corner point is x_0, and the bottom edges are

$$\gamma_1 \sim \gamma_1' * \gamma_1'' \quad \text{and} \quad \eta_1 \sim \eta_1' * \eta_1'',$$

where these homotopies take place in A or B, respectively. Let a_1, a_2, a_3, a_4 be the four edges lying above $\gamma_1', \gamma_1'', \eta_1', \eta_1''$ on the grid (recall that $n = 2$ in this case) We will show that we can go from $[\gamma_1][\eta_1]$ to $[a_1][a_2][a_3][a_4]$ using (1) and (2). Then we can inductively push this to the top, which is the empty word.

Let $\delta_1, \delta_2, \delta_3$ be the three interior edges which connect the bottom row and the second-to-last row. Since each square maps to A or B, the first three squares lie in A and the last one lies in B. Let

$$G = \pi_1(A, x_0) \quad \text{and} \quad H = \pi_1(B, x_0).$$

Note that we have

$$\delta_1 * \eta_1' \sim \delta_1 * \eta' * \delta_3 * \overline{\delta}_3 \text{ in } G, \quad \delta_1 * \eta_1' * \delta_3 \sim a_1 * a_2 * a_3 \text{ in } G, \quad \delta_3 * \eta_1'' \sim a_4 \text{ in } H,$$

and thus we can write

$$\begin{aligned}
[\gamma_1]^G[\eta_1]^H = [\gamma_1]^G([\eta_1'][\eta_1''])^H &\overset{(1)}{=} [\gamma_1]^G[\eta_1']^H[\eta_1'']^H \\
&\overset{(2)}{=} [\gamma_1]^G[\eta_1']^G[\eta_1'']^H \\
&\overset{(1)}{=} ([\gamma_1][\eta_1'])^G[\eta_1'']^H = ([\gamma_1][\eta_1'][\delta_3][\overline{\delta}_3])^G[\eta_1'']^H \\
&\overset{(2)}{=} ([\gamma_1][\eta_1'][\delta_3])^G[\overline{\delta}_3]^H[\eta_1'']^H \\
&\overset{(1)}{=} ([a_1][a_2][a_3])^G([\overline{\delta}_3][\eta_1''])^H = ([a_1][a_2][a_3])^G[a_4]^H \\
&\overset{(1)}{=} [a_1]^G[a_2]^G[a_3]^G[a_4]^H,
\end{aligned}$$

which proves the statement for $n = 2$. See Hatcher for the general case. □

8.3 Covering Spaces

Definition 8.1. A *covering space* of a space X is a pair $(\widetilde{X}, p)$ where $\widetilde{X}$ is a space and $p : \widetilde{X} \to X$ such that every point $x \in X$ has an *evenly covered neighborhood.* An open set U is *evenly covered* if

$$p^{-1}(U) = \text{disjoint union of open sets } \{U_\alpha\} \text{ in } \widetilde{X}$$

such that $p|_{U_\alpha} : U_\alpha \to U$ is a homeomorphism for every α.

Example 8.1.1. The following are examples of covering maps:

1. If $p : \widetilde{X} \to X$ is a homeomorphism, then it is a covering map.
2. We saw that $p : \mathbb{R} \to S^1$ given by $t \mapsto (\cos 2\pi t, \sin 2\pi t)$ is a covering map.

Exercise 8.1. If $(\widetilde{X}, p)$ is a covering space of X and $(\widetilde{Y}, p')$ is a covering space of Y, then show that

$$p \times p' : \widetilde{X} \times \widetilde{Y} \to X \times Y, \quad (x, y) \mapsto (p(x), p'(y))$$

is a covering map. This gives a covering map $\mathbb{R}^2 \to T^2$ by $(t, s) \mapsto (\cos 2\pi t, \sin 2\pi t, \cos 2\pi s, \sin 2\pi s)$.

Lecture 9

Feb. 5 — Covering Spaces

9.1 Examples of Covering Spaces

Example 9.0.1. The following are more examples of covering maps:

3. Define $p_n : S^1 \to S^1$ by $\theta \mapsto n\theta$. These are covering maps for each $n \in \mathbb{Z}$.
4. Let X be a wedge of two circles (corresponding to a, b), and let $\widetilde{X}$ be a circle with three outer circles attached, evenly spaced (the inner circle corresponding to a_1, a_2, a_3 and the three outer circles corresponding to b_1, b_2, b_3). Let p map a_i to a and b_i to b. This is a covering map.

   Write out a formula for p and really check that p is a covering map as an exercise.
5. Again let X be a wedge of two circles, labeled a, b. Let $\widetilde{X}$ be a wedge of two circles, with an extra circle attached on either side. Let a_3, b_3 be the extra circles on the left and right, let b_1, b_2 be the top and bottom halves of the circle on the left, and let a_1, a_2 be the top and bottom halves of the circles on the right. Let p map a_i to a and b_i to b. Then p is a covering map.
6. Consider the quotient map $p : S^2 \to \mathbb{R}P^2$. This map is a covering map. Note that each point in $\mathbb{R}P^2$ has a neighborhood whose preimage is two open sets on opposite sides of S^2.

9.2 Covering Spaces and Lifting

Lemma 9.1. *Let $(\widetilde{X}, p)$ be a covering space of a connected space X. Then the cardinality $|p^{-1}(x)|$ is independent of $x \in X$.*

Proof. Fix $x_0 \in X$ and let $n = |p^{-1}(x_0)|$. Let $A = \{x \in X : |p^{-1}(x)| = n\}$, and note that $A \neq \varnothing$ since $x_0 \in A$. We will show that A is both open and closed, which implies that $A = X$ by connectedness.

To see that A is open, let $x \in A$. By the definition of a covering space, there exists an open set U in X such that $x \in U$ and $p^{-1}(U) = \{U_1, \ldots, U_n\}$. Thus for any $x' \in U$, we have

$$p^{-1}(x') \subseteq p^{-1}(U) = \{U_1, \ldots, U_n\},$$

and $p^{-1}(x') \cap U_i = \{\text{pt}\}$ since $p|_{U_i} : U_i \to U$ is a homeomorphism. Thus $|p^{-1}(x')| = n$, so $x' \in A$. This holds for each $x' \in U$, so $U \subseteq A$, i.e. A is an open set.

One can make a similar argument to check that if $x \notin A$, then there exists a open set U about x such that $U \cap A = \varnothing$. This shows that $X \setminus A$ is open, i.e. A is closed, which completes the proof. $\square$

Definition 9.1. We call $|p^{-1}(x)|$ the *degree* of the covering space.

Definition 9.2. If $(\widetilde{X}, p)$ is a covering space for X and $f : Y \to X$ is a continuous map, then a *lift* of f to $\widetilde{X}$ is a map $\widetilde{f} : Y \to \widetilde{X}$ such that $p \circ \widetilde{f} = f$, i.e. the following diagram commutes:

$$\begin{tikzcd} & \widetilde{X} \arrow[d, "p"] \\ Y \arrow[ur, dashed, "\widetilde{f}"] \arrow[r, "f"] & X \end{tikzcd}$$

If $f(y_0) = x_0$ and $\widetilde{x}_0 \in \widetilde{X}$ with $p(\widetilde{x}_0) = x_0$, then $\widetilde{f}$ is a *lift of f based at $\widetilde{x}_0$* if $\widetilde{f}$ is a lift and $\widetilde{f}(y_0) = \widetilde{x}_0$.

Lemma 9.2. *Let $(\widetilde{X}, p)$ be a covering space of X, $x_0 \in X$, and $\widetilde{x}_0 \in p^{-1}(x_0)$. Then*

(a) each path $\gamma : [0,1] \to X$ based at x_0 has a unique lift $\widetilde{\gamma} : [0,1] \to X$ based at $\widetilde{x}_0$.

(b) if $H : Y \times [0,1] \to X$ is a homotopy with $h_0(y) = H(y,0)$ and $\widetilde{h}_0 : Y \to \widetilde{X}$ a lift of h_0, then there is a unique lift $\widetilde{H} : Y \times [0,1] \to \widetilde{X}$ of H such that $\widetilde{H}(y,0) = h_0(y)$.

The above properties are called path lifting *and* homotopy lifting.

Proof. (a) The proof of this part is exactly the proof of part (a) in Lemma 5.5.

(b) The proof from Lemma 5.5 works if $Y = [0,1]$. For the general case, see proof of Theorem 10.1. □

9.3 Connections to the Fundamental Group

Lemma 9.3. *If $(\widetilde{X}, p)$ is a path connected covering space of X and $x_0 \in X$, $\widetilde{x}_0 \in p^{-1}(x_0)$, then the map $p_* : \pi_1(\widetilde{X}, \widetilde{x}_0) \to \pi_1(X, x_0)$ satisfies the following:*

1. *p_* is injective;*
2. *the image of p_* is the set of loops in $\pi_1(X, x_0)$ that when lifted are loops in $\widetilde{X}$ based at $\widetilde{x}_0$;*
3. *the index $[\pi_1(X, x_0) : p_*(\pi_1(\widetilde{X}, \widetilde{x}_0))]$ is the degree of $(\widetilde{X}, p)$.*

Proof. (1) Suppose that $p_*([\gamma]) = [e]$, so there exists a homotopy H in X between x_0 and $p \circ \gamma$. Note that γ is a lift of $H(t,0)$, so Lemma 9.2 says H lifts to a homotopy $\widetilde{H}$ starting at γ in $\widetilde{X}$. Note that $H|_{\{0\}\times[0,1]}$ is a constant loop and the loop $t \mapsto \widetilde{x}_0$ is a lift of $H|_{\{0\}\times[0,1]}$ based at $\widetilde{x}_0$, so by uniqueness we see that $\widetilde{H}|_{\{0\}\times[0,1]} = \widetilde{x}_0$. Similarly, we see that $\widetilde{H}_{\{1\}\times[0,1]} = \widetilde{x}_0$ and $\widetilde{H}_{[0,1]\times\{1\}} = \widetilde{x}_0$. Thus $\widetilde{H}$ is a homotopy of loops based at $\widetilde{x}_0$ from γ to the constant loop, i.e. $[\gamma] = [e_{x_0}]$. So p_* is injective.

(2) Clearly if $[\gamma] \in \pi_1(X, x_0)$ lifts to a loop $\widetilde{\gamma}$ based at $\widetilde{x}_0$, then $[\gamma] = p_*(\widetilde{\gamma})$, so $[\gamma]$ is in the image of p_*. Now if $[\eta] = p_*([\gamma])$, then $\eta \sim p \circ \gamma$ in X. Let $\widetilde{\eta}$ be the lift of η based at $\widetilde{x}_0$. By Lemma 9.2, the homotopy $\eta \sim p \circ \gamma$ lifts to a homotopy $\widetilde{\eta} \sim \gamma$ rel endpoints. But γ is a loop, so $\widetilde{\eta}$ must be a loop.

(3) Let $H = p_*(\pi_1(\widetilde{X}, \widetilde{x}_0)) \leq \pi_1(X, x_0)$. If $[\gamma] \in \pi_1(X, x_0)$ and $[\delta] \in H$, then note that by part (2), δ lifts to a loop $\widetilde{\delta}$ based at $\widetilde{x}_0$. Let $\widetilde{\delta * \gamma}$ be a lift of $\delta * \gamma$ based at $\widetilde{x}_0$, and note that $\widetilde{\gamma}(1) = \widetilde{\delta * \gamma}(1) = \widetilde{\delta} * \widetilde{\gamma}(1)$. This allows us to define

$$\phi : \{\text{right cosets of } H\} \to p^{-1}(x_0)$$

by $H[\gamma] \mapsto \widetilde{\gamma}(1)$, which is well-defined by the above arguments.

If $\widetilde{x}_1 \in p^{-1}(x_0)$, then let $\widetilde{\gamma}$ be a path in $\widetilde{X}$ from $\widetilde{x}_0$ to $\widetilde{x}_1$. Let $\gamma = p \circ \widetilde{\gamma}$, which is a loop in X based at x_0. Clearly $\phi(H[\gamma]) = \widetilde{\gamma}(1) = \widetilde{x}_1$, so ϕ is onto. Now suppose that $\phi(H[\gamma]) = \phi(H[\eta])$. If $\widetilde{\gamma}, \widetilde{\eta}$ are lifts of γ based at $\widetilde{x}_0$, then $\widetilde{\gamma}(1) = \widetilde{\eta}(1)$. Thus $\widetilde{\gamma} * \overline{\widetilde{\eta}}$ is a loop in $\widetilde{X}$, so

$$p_*([\widetilde{\gamma} * \overline{\widetilde{\eta}}]) = [\gamma] * [\overline{\eta}] = [\gamma] * [\eta]^{-1} \in H.$$

This gives $H[\gamma] = H[\eta]$, so ϕ is injective. Thus ϕ is a bijection, so $[\pi_1(X, x_0) : H] = |p^{-1}(x_0)|$. $\square$

Example 9.2.1. Recall the following examples from before:

1. For the covering map $p : \mathbb{R} \to S^1$, we have $p_* : \pi_1(\mathbb{R}, 0) \to \pi_1(S^1, (1,0))$ which sends $e \mapsto 0$, if we view p_* as $p_* : \{e\} \to \mathbb{Z}$. One can see all three properties of the above lemma in this example, in particular that $[\pi_1(S^1) : \pi_1(\mathbb{R})] = \infty = |p^{-1}((1,0))|$.
2. Let $p_n : S^1 \to S^1$ send $\theta \mapsto n\theta$. Then $(p_n)_* : \pi_1(S^1, (1,0)) \to \pi_1(S^1, (1,0))$, which we can view as a map $(p_n)_* : \mathbb{Z} \to \mathbb{Z}$, or $(p_n)_* : \langle g | \ \rangle \to \langle h | \ \rangle$. Then $(p_n)_*$ maps $g \mapsto h^n$, so the subgroup of $\mathbb{Z}$ corresponding to this covering space is $n\mathbb{Z}$. Again we can see all three properties of the above lemma, in particular that we have $[\mathbb{Z} : n\mathbb{Z}] = n = |p_n^{-1}((1,0))|$.

Exercise 9.1. Check the properties of the above lemma explicitly for Example 9.0.1(4). Note that picking different base points can yield different images of p_*.

Lecture 10

Feb. 10 — Covering Spaces, Part 2

10.1 More Connections to the Fundamental Group

Lemma 10.1. *If $(\widetilde{X}, p)$ is a path connected covering space of X and $x_0 \in X$, then*

$$\{p_*(\pi_1(\widetilde{X}, \widetilde{x}))\}_{\widetilde{x} \in p^{-1}(x_0)}$$

is a conjugacy class of subgroups of $\pi_1(X, x_0)$.

Proof. Let $\widetilde{x}_0, \widetilde{x}_1 \in p^{-1}(x_0)$ and $H_i = p_*(\pi_1(\widetilde{X}, \widetilde{x}_i))$. Since $\widetilde{X}$ is path connected, we can find a path h in $\widetilde{X}$ from $\widetilde{x}_0$ to $\widetilde{x}_1$. Clearly $\gamma = p \circ h$ is a loop in X based at x_0. If $[\eta] \in H_1$, then by Lemma 9.3 η lifts to a loop $\widetilde{\eta}$ based at $\widetilde{x}_1$. Note that $h * \widetilde{\eta} * \overline{h}$ is a loop based at $\widetilde{x}_0$ in $\widetilde{X}$. Then

$$[\gamma] \cdot [\eta] \cdot [\gamma]^{-1} = [(p \circ h) * (p \circ \widetilde{\eta}) * (p \circ \overline{h})] = p_*([h * \widetilde{\eta} * \overline{h}]) \in H_0.$$

This says that $[\gamma] H_1 [\gamma]^{-1} \subseteq H_0$. The same argument says $[\gamma]^{-1} H_0 [\gamma] \subseteq H_1$, so

$$H_0 = [\gamma][\gamma]^{-1} H_0 [\gamma][\gamma]^{-1} \subseteq [\gamma] H_1 [\gamma]^{-1}.$$

This implies that $H_0 = [\gamma] H_1 [\gamma]^{-1}$, so H_0, H_1 are conjugate.

We still need to show we get the full conjugacy class. For this, suppose $H \leq \pi_1(X, x_0)$ such that there exists $[\alpha] \in \pi_1(X, x_0)$ with $[\alpha] H [\alpha]^{-1} = H_0$. If $[\alpha] \in H_0$, then $H = H_0$ are we are done. If $[\alpha] \notin H_0$, then α lifts to a path $\widetilde{\alpha}$ (not necessarily a loop) based at $\widetilde{x}_0$. Let $\widetilde{x}_2 = \widetilde{\alpha}(1)$, and set $H_2 = p_*(\pi_1(\widetilde{X}, \widetilde{x}_2))$. From above, we see that $H = H_2$, which completes the proof. $\square$

Definition 10.1. A space X is *locally path connected* if for every $y \in Y$ and open set U containing y, there exists an open set V such that $y \in V \subseteq U$ and V is path connected.

Example 10.1.1. An example of a space which is path connected but not locally path connected is

$$C = (\{1/n\} \times [0,1])_{n \in \mathbb{N}} \times (\{0\} \times [0,1]) \cup ([0,1] \times \{0\}),$$

also known as the *comb space*. We can see that C is path connected, but it is not locally path connected, e.g. take $y = (0,1)$ and U to be any small enough open neighborhood containing y.

Theorem 10.1. *Let $(\widetilde{X}, p)$ be a covering space of X. Let $x_0 \in X$ and $\widetilde{x}_0 \in p^{-1}(x_0)$. Suppose $f : Y \to X$ is any map with Y path connected and locally path connected, with $y_0 \in Y$ such that $f(y_0) = x_0$. Then there exists a lift $\widetilde{f} : Y \to \widetilde{X}$ of f such that $\widetilde{f}(y_0) = \widetilde{x}_0$ if and only if $f_*(\pi_1(Y, y_0)) \subseteq p_*(\pi_1(\widetilde{X}, \widetilde{x}_0))$.*

Proof. ($\Rightarrow$) If $\widetilde{f} : Y \to \widetilde{X}$ exists, then $p \circ \widetilde{f} = f$, and so

$$f_*(\pi_1(Y, y_0)) = (p \circ \widetilde{f})_*(\pi_1(Y, y_0)) = p_*(\widetilde{f}_*(\pi_1(Y, y_0))) \subseteq p_*(\pi_1(\widetilde{X}, \widetilde{x}_0))$$

since $f_*(\pi_1(Y, y_0)) \subseteq \pi_1(X, x_0)$. This proves the first direction.

($\Leftarrow$) Suppose that $f_*(\pi_1(Y, y_0)) \subseteq p_*(\pi_1(\widetilde{X}, \widetilde{x}_0))$. Since Y is path connected, for any $y \in Y$, there exists a path $\gamma : [0, 1] \to Y$ from y_0 to y. Then there is a unique lift $\widetilde{f \circ \gamma} : [0, 1] \to \widetilde{X}$ based at $\widetilde{x}_0$. Define

$$\widetilde{f}(y) = \widetilde{f \circ \gamma}(1).$$

Note that if $\widetilde{f}$ is well-defined, then it is clear that $p \circ \widetilde{f} = f$.

To see that $\widetilde{f}$ is well-defined, we must show that $\widetilde{f \circ \gamma_1}(1)$ is independent of the choice of path γ. Let γ, η be paths from y_0 to y. Note that $\gamma * \overline{\eta}$ is a loop in Y based at y_0, so

$$(f \circ \gamma) * (f \circ \overline{\eta}) = f_*(\gamma * \overline{\eta}) \in p_*(\pi_1(\widetilde{X}, \widetilde{x}_0)).$$

Thus by Lemma 9.3, this loop lifts to a loop in $\widetilde{X}$ based at $\widetilde{x}_0$. Then

$$\widetilde{(f \circ \gamma) * (f \circ \overline{\eta})} = (\widetilde{f \circ \gamma}) * (\widetilde{f \circ \overline{\eta}}),$$

where $\widetilde{f \circ \gamma}$ is a lift based at $\widetilde{x}_0$ and $\widetilde{f \circ \overline{\eta}}$ is a lift based at $\widetilde{f \circ \gamma}(1)$. But $\widetilde{(f \circ \gamma) * (f \circ \overline{\eta})}$ is a loop, so

$$\widetilde{f \circ \overline{\eta}}(1) = \widetilde{x}_0.$$

Then $\overline{\widetilde{f \circ \overline{\eta}}}$ is the lift of η based at $\widetilde{x}_0$, so $\overline{\widetilde{f \circ \overline{\eta}}}(1) = \widetilde{f \circ \overline{\eta}}(0) = \widetilde{f \circ \gamma}(1)$. Thus, $\widetilde{f \circ \eta}(1) = \widetilde{f \circ \gamma}(1)$.

Now it just remains to show that $\widetilde{f}$ is continuous. To see this, take any open set $U \subseteq \widetilde{X}$, and we will show that for all $y \in \widetilde{f}^{-1}(U)$, there exists an open set V in Y such that $y \in V \subseteq \widetilde{f}^{-1}(U)$ (this implies that $\widetilde{f}^{-1}(U)$ is open). Let W be an evenly covered open set containing $f(y)$, and $\widetilde{W} \subseteq p^{-1}(W)$ such that $p|_{\widetilde{W}} : \widetilde{W} \to W$ is a homeomorphism (by possibly shrinking W, e.g. by intersecting $\widetilde{W}$ with U and taking the image under p, we can assume that $\widetilde{W} \subseteq U$). Since Y is locally path connected, there exists a path connected open set V in Y containing y and $V \subseteq f^{-1}(W)$

Fix a path γ from y_0 to y. For any point $y' \in V$, there exists a path η from y to y' in V. By definition,

$$\widetilde{f}(y') = f \circ \widetilde{(\gamma * \eta)}(1).$$

But if $\widetilde{f \circ \eta}$ is a lift of $f \circ \eta$ based at $\widetilde{f \circ \gamma}(1) = \widetilde{f}(y)$, then

$$f \circ \widetilde{(\gamma * \eta)}(1) = \widetilde{f \circ \eta}(1).$$

We know that $\widetilde{f \circ \eta} = (p|_{\widetilde{W}})^{-1} \circ f \circ \eta$, so we see that

$$\widetilde{f}(y') = \widetilde{f \circ \eta}(1) \in \widetilde{W} \subseteq U.$$

This shows that $V \subseteq \widetilde{f}^{-1}(U)$, so $\widetilde{f}$ is continuous, which completes the proof. $\square$

Remark. It is fairly commmon in algebraic topology for a topological hypothesis to imply some type of algebraic conclusion. The opposite, however, as in the above theorem, is much rarer.

Lemma 10.2. *Let $(\widetilde{X}, p)$ be a covering space of X, and let $\widetilde{f}_1, \widetilde{f}_2 : Y \to \widetilde{X}$ be two lifts of $f : Y \to X$. If Y is connected and $\widetilde{f}_1$ and $\widetilde{f}_2$ agree at one point, then $\widetilde{f}_1 = \widetilde{f}_2$.*

Proof. Let $A = \{y \in Y : \widetilde{f}_1(y) = \widetilde{f}_2(y)\}$. Note that $A \neq \varnothing$ by assumption. If $y \in A$, then let U be an evenly covered neighborhood of $f(y)$ and $\widetilde{U} \subseteq p^{-1}(U)$ such that $\widetilde{f}_1(y) = \widetilde{f}_2(y) \in \widetilde{U}$ and $p|_{\widetilde{U}} : \widetilde{U} \to U$ is a homeomorphism. Since f is continuous, there exists an open neighborhood V of y in Y such that $f(V) \subseteq U$. Now we have

$$\widetilde{f}_1|_V = (p|_{\widetilde{U}})^{-1} \circ f|_V = \widetilde{f}_2|_V,$$

so $V \subseteq A$, which shows that A is open. A similar argument shows that $Y \setminus A$ is open, so A is closed. Since Y is connected and A is open, closed, nonempty, we must have $A = Y$, hence $\widetilde{f}_1 = \widetilde{f}_2$. □

10.2 Covering Space Isomorphisms

Definition 10.2. We say that two covering spaces $(\widetilde{X}_1, p_1)$ and $(\widetilde{X}_2, p_2)$ of X are *isomorphic* if there exists a homeomorphism $h : \widetilde{X}_1 \to \widetilde{X}_2$ such that $p_2 \circ h = p_1$, i.e. the following diagram commutes:

$$\begin{tikzcd} \widetilde{X}_1 \arrow[r, dashed, "h"] \arrow[dr, "p_1"'] & \widetilde{X}_2 \arrow[d, "p_2"] \\ & X \end{tikzcd}$$

Corollary 10.1.1. *Suppose $(\widetilde{X}_1, p_1)$ and $(\widetilde{X}_2, p_2)$ are path connected, locally path connected covering spaces of X and $x_0 \in X$, $\widetilde{x}_i \in p_i^{-1}(x_0)$, then*

(a) if $(p_1)_*(\pi_1(\widetilde{X}_1, \widetilde{x}_1)) \subseteq (p_2)_*(\pi_1(\widetilde{X}_2, \widetilde{x}_2))$, then p_1 lifts to a covering map $p : \widetilde{X}_1 \to \widetilde{X}_2$ which takes the base point $\widetilde{x}_1$ to the base point $\widetilde{x}_2$;

(b) $(\widetilde{X}_1, \widetilde{x}_1)$ and $(\widetilde{X}_2, \widetilde{x}_2)$ are isomorphic covering spaces taking $\widetilde{x}_1$ to $\widetilde{x}_2$ if and only if

$$(p_1)_*(\pi_1(\widetilde{X}_1, \widetilde{x}_1)) = (p_2)_*(\pi_1(\widetilde{X}_2, \widetilde{x}_2));$$

(c) $(\widetilde{X}_1, \widetilde{x}_1)$ and $(\widetilde{X}_2, \widetilde{x}_2)$ are isomorphic covering spaces of X if and only if $(p_1)_*(\pi_1(\widetilde{X}_1, \widetilde{x}_1))$ is conjugate to $(p_2)_*(\pi_1(\widetilde{X}_2, \widetilde{x}_2))$.

Proof. (a) By Theorem 10.1, we get a lift $p : \widetilde{X}_1 \to \widetilde{X}_2$ of p_1 taking $\widetilde{x}_1$ to $\widetilde{x}_2$. We need to show that p is a covering map. Let $x \in \widetilde{X}_2$, and we show that x has an evenly covered neighborhood. Let U be a neighborhood of $p_2(x)$ in X which is evenly covered by both p_1 and p_2 (e.g. take the intersection of the two neighborhoods evenly covered by p_1 and p_2), so there exists a unique $\widetilde{U} \subseteq \widetilde{X}_2$ such that $x \in \widetilde{U}$ and $p_2|_{\widetilde{U}} : \widetilde{U} \to U$ is a homeomorphism. Write $p^{-1}(\widetilde{U}) = \bigcup_\alpha U_\alpha$. Clearly $\bigcup_\alpha U_\alpha \subseteq p^{-1}(U)$, so $p_1|_{\widetilde{U_\alpha}} : \widetilde{U_\alpha} \to U$ is a homeomorphism. Thus $p|_{U_\alpha} : p_2^{-1}|_{\widetilde{U}} \circ p_1|_{U_\alpha}$ is a homeomorphism $U_\alpha \to \widetilde{U}$, so $\widetilde{U}$ is evenly covered.

(b) $(\Rightarrow)$ This is clear. $(\Leftarrow)$ Let $\widetilde{p}_1 : \widetilde{X}_1 \to \widetilde{X}_2$ and $\widetilde{p}_2 : \widetilde{X}_2 \to \widetilde{X}_1$ be the lifts from part (a). We have:

$$\begin{tikzcd} \widetilde{X}_2 \arrow[r, "\widetilde{p}_2"] \arrow[dr, "p_2"'] & \widetilde{X}_1 \arrow[r, "\widetilde{p}_1"] \arrow[d, "p_1"] & \widetilde{X}_2 \arrow[dl, "p_2"] \\ & X & \end{tikzcd}$$

Note that $\widetilde{p}_1 \circ \widetilde{p}_2 : \widetilde{X}_2 \to \widetilde{X}_2$ takes $\widetilde{x}_2$ to $\widetilde{x}_2$ and is a lift of p_2 to $\widetilde{X}_2$. But $\mathrm{id}_{\widetilde{X}_2}$ is also a lift of p_2 and agrees with $\widetilde{p}_1 \circ \widetilde{p}_2$ on $\widetilde{x}_2$, so by Lemma 10.2, $\widetilde{p}_1 \circ \widetilde{p}_2 = \mathrm{id}_{\widetilde{X}_2}$. Similarly, we can show that $\widetilde{p}_2 \circ \widetilde{p}_1 = \mathrm{id}_{\widetilde{X}_1}$, so $\widetilde{p}_2$ is a homeomorphism and $\widetilde{X}_1, \widetilde{X}_2$ are isomorphic.

(c) This follows from Lemma 10.1 and part (b). $\square$

Lecture 11

Feb. 12 — Covering Spaces, Part 3

11.1 Galois Correspondence

Definition 11.1. A space X is *semi-locally simply connected* if each point $x \in X$ has a neighborhood U such that $\pi_1(U, x) \to \pi_1(X, x)$ is trivial.

Example 11.1.1. Let $X = \bigcup_{n=1}^{\infty} S_{1/n}(1/n, 0)$, where $S_{1/n}(1/n, 0)$ is a circle of radius $1/n$ about the point $(1/n, 0)$. Note that any neighorhood of $(0, 0)$ has non-trivial fundamental group that injects into $\pi_1(X, (0, 0))$. So X is not semi-locally simply connected. On the other hand, let X' be the cone on X:

$$X' = (X \times [0, 1])/(X \times \{0\}).$$

Then X' is semi-locally simply connected but not locally simply connected.

Remark. All CW-complexes and manifolds are semi-locally simply connected.

Theorem 11.1. *Let X be a path connected, locally path connected, semi-locally simply connected space, and fix $x_0 \in X$. Then there is a one-to-one correspondence*

$$\left\{\begin{matrix}\textit{basepoint preserving isomorphism classes}\\ \textit{of covering spaces } (\widetilde{X}, p, \widetilde{x_0})\end{matrix}\right\} \longleftrightarrow \{\textit{subgroups of } \pi_1(X, x_0)\}$$

given by $(\widetilde{X}, p, \widetilde{x_0}) \mapsto p_*(\pi_1(\widetilde{X}, \widetilde{x_0}))$ with inverse $H \mapsto (\widetilde{X}_H, p_H, \widetilde{x_0})$, which satisfies:

1. *If $H \leq K$, then $(\widetilde{X}_H, p_H, \widetilde{x}_H)$ is also a covering space of $(\widetilde{X}_K, p_K, \widetilde{x}_K)$.*
2. *If p_1 in $(\widetilde{X}_1, p_1, \widetilde{x}_1)$ lifts to a cover of $(\widetilde{X}_2, p_2, \widetilde{x}_2)$, then $p_{1*}(\pi_1(\widetilde{X}_1, \widetilde{x}_1)) \leq p_{2*}(\pi_1(\widetilde{X}_2, \widetilde{x}_2))$.*
3. *$[\pi_1(X, x_0) : H] = n$ if and only if $(\widetilde{X}_H, p_H, \widetilde{x}_H)$ is degree n.*

In addition, there is a one-to-one correspondence

$$\{\textit{covering spaces } (\widetilde{X}, p) \textit{ of } X\} \longleftrightarrow \{\textit{conjugacy classes of subgroups of } \pi_1(X, x_0)\}.$$

Proof. Note that (1) and (2) follow from Corollary 10.1.1 once we have the one-to-one correspondence, and (3) follows from Lemma 9.3. Also, once we have a well-defined map $H \mapsto (\widetilde{X}_H, p_H, \widetilde{x}_H)$ such that $(p_H)_*(\pi_1(\widetilde{X}_H, \widetilde{x}_H)) = H$, then the first one-to-one correspondence follows from Corollary 10.1.1(b). The second one-to-one correspondence follows from Corollary 10.1.1(c).

So we are left to construct $(\widetilde{X}_H, p_H, \widetilde{x}_H)$ given $H \leq \pi_1(X, x_0)$. Given H, we define an equivalence relation on paths based at x_0: Call two based paths $\gamma, \eta : [0, 1] \to X$ (i.e. $\gamma(0) = \eta(0) = x_0$) H-equivalent if:

1. $\gamma(1) = \eta(1)$, and
2. $[\gamma * \overline{\eta}] \in H$.

Denote H-equivalence by $\gamma \sim_H \eta$ and its equivalence classes by $\langle\gamma\rangle$. Show as an exercise that this is an equivalence relation, and if $H = \{e\}$, then $\gamma \sim_H \eta$ if and only if γ, η are homotopic rel endpoints. Let

1. $\widetilde{X}_H = \{\langle\gamma\rangle : \gamma \text{ if a path in } X \text{ based at } x_0\}$.

   Note that this is a set: We will need to put a topology on $\widetilde{X}_H$ later.
2. $p_H : \widetilde{X}_H \to X : \langle\gamma\rangle \mapsto \gamma(1)$.
3. $\widetilde{x}_H = \langle e_{x_0}\rangle$, where e_{x_0} is the constant path $x \mapsto x_0$. Clearly $p_H(\widetilde{x}_H) = x_0$.

Note that p_H is onto, since X is path connected. In order to put a topology on $\widetilde{X}_H$, we need to study the topology of X in terms of paths. We claim that

$$\mathcal{U} = \left\{ U \subseteq X : \begin{array}{c} U \text{ is open, path connected, and} \\ \pi_1(U, x) \to \pi_1(X, x) \text{ is trivial for some } x \in U \end{array} \right\}$$

is a basis for the topology on X.[1] To see this, note that if $\pi_1(U, x) \to \pi_1(X, x)$ is trivial for some $x \in U$, then this is true for all $y \in U$ since U is path-connected:

$$\begin{tikzcd} \pi_1(U,x) \arrow[r, "\text{triv}"] \arrow[d, "\phi_h"] & \pi_1(X,x) \arrow[d, leftrightarrow, "\phi_h"] \\ \pi_1(U,y) \arrow[r, dashed, "i_*"] & \pi_1(X,y) \end{tikzcd}$$

Also if $U \in \mathcal{U}$ and $V \subseteq U$ is open and path connected, then:

$$\begin{tikzcd} \pi_1(V,x) \arrow[r] \arrow[rr, dashed, bend right, "\text{triv}"'] & \pi_1(U,x) \arrow[r, "\text{triv}"] & \pi_1(X,x) \end{tikzcd}$$

so $V \in \mathcal{U}$ too. Now let $x \in X$ and U be any open set in X containing x. Since X is semi-locally simply connected, there exists an open set V containing x such that $\pi_1(V, x) \to \pi_1(X, x)$ is trivial. Then $U \cap V$ is an open set containing x and $\pi_1(U \cap V, x) \to \pi_1(X, x)$ is trivial. Now X being locally path connected implies that there exists an open set W such that $x \in W \subseteq U \cap V$ and W is path connected. Clearly $\pi_1(W, x) \to \pi_1(X, x)$ is trivial, so $W \in \mathcal{U}$. Thus $\mathcal{U}$ is a basis for the topology on X.

Now for each $U \in \mathcal{U}$ and a path γ from x_0 to a point in U, set

$$U_\gamma = \{\langle\gamma * \eta\rangle : \eta \text{ a path in } U \text{ such that } \eta(0) = \gamma(1)\}.$$

Note that U_γ is a subset of $\widetilde{X}_H$. We claim that $\{U_\gamma\}_{U \in \mathcal{U}, \gamma \text{ a path from } x_0 \text{ to } x \in U}$ forms a basis for a topology on $\widetilde{X}_H$.[2] To see this, note the following:

[1] Recall that a collection of open sets in X is a *basis for the topology on* X if for every $x \in X$ and open set U containing X, there exists a set O in the collection such that $x \in O \subseteq U$.

[2] Recall that a collection of subsets of a set X form a *basis for a topology* on X if for each pair of sets U, V in the collection and $x \in U \cap V$, there is another set W in the collection such that $x \in W \subseteq U \cap V$, and X is the union of the sets in the collection.

1. If $\langle\gamma\rangle = \langle\delta\rangle$, then $U_\gamma = U_\delta$ (so we can write $U_{\langle\gamma\rangle}$).

   This is because if $\gamma \sim_H \delta$, then $\gamma * \eta \sim_H \delta * \eta$ for all paths η in U with $\eta(0) = \eta(1)$:
$$\delta * \eta * \overline{(\gamma * \eta)} = \gamma * \eta * \overline{\eta} * \overline{\delta} \sim \gamma * \overline{\delta}.$$

2. $p_H : U_{\langle\gamma\rangle} \to U$ is onto since U is path connected.

3. $p_H : U_{\langle\gamma\rangle} \to U$ is injective.

   This is because if $p_H(\langle\gamma * \eta\rangle) = p_H(\langle\gamma * \eta'\rangle)$, then $\gamma * \eta(1) = \gamma * \eta'(1)$, so $\eta * \overline{\eta'}$ is a loop in U based at $x = \gamma(1)$. So $\eta * \overline{\eta'}$ is homotopic to a constant loop in X. Thus $\gamma * \eta \sim \gamma * \eta'$ rel endpoints, so
$$[(\langle * \eta) * \overline{(\gamma * \eta')}] = [e_{x_0}] \in H.$$
   This gives $\langle\gamma * \eta\rangle = \langle\gamma * \eta'\rangle$, i.e. p_H is injective.

4. If $\langle\gamma'\rangle \in U_{\langle\gamma\rangle}$, then $U_{\langle\gamma'\rangle} = U_{\langle\gamma\rangle}$.

   This is because by hypothesis, there exists a path η in U such that $\langle\gamma'\rangle = \langle\gamma * \eta\rangle$, so we can take $\gamma * \eta$ to represent γ' by note (1). So if $\langle\delta\rangle \in U_{\langle\gamma'\rangle}$, then
$$\delta = (\gamma * \eta) * \eta' \sim \gamma * (\eta * \eta'),$$
   so $\langle\delta\rangle \in U_{\langle\gamma\rangle}$. This shows that $U_{\langle\gamma'\rangle} \subseteq U_{\langle\gamma\rangle}$, and similarly one can prove that $U_{\langle\gamma\rangle} \subseteq U_{\langle\gamma'\rangle}$.

Now if $\langle\delta\rangle \in U_{\langle\gamma\rangle} \cap V_{\langle\gamma'\rangle}$, then $U_{\langle\gamma\rangle} = U_{\langle\delta\rangle}$ and $V_{\langle\gamma'\rangle} = V_{\langle\delta\rangle}$. by (4). So if W is any element of $\mathcal{U}$ such that $\delta(1) \in W \subseteq U \cap V$, then
$$\langle\delta\rangle = W_{\langle\delta\rangle} \subseteq U_{\langle\delta\rangle} \cap V_{\langle\delta\rangle} = U_\gamma \cap U_{\gamma'}.$$
Clearly $\widetilde{X}_H$ is the union of the sets in $\{U_\gamma\}$, so we have a basis for a basis on $\widetilde{X}_H$.

We claim that with this topology, $(\widetilde{X}_H, p_H)$ is a covering space of (X, x_0). For any $U \in \mathcal{U}$ and a path γ from x_0 to a point in U, $p|_{U_{\langle\gamma\rangle}} : U_{\langle\gamma\rangle} \to U$ is a homeomorphism. To see this, first note p_H is a bijection. Next, $p_H|_{U_{\langle\gamma\rangle}}$ is continuous since for any basic open set $V \subseteq U$ and path δ from x_0 to a point in V,
$$(p_H|_{U_{\langle\gamma\rangle}})^{-1}(V) = V_{\langle\delta\rangle}$$
which is open. This also shows that the image of a basic open set $V_{\langle\delta\rangle}$ is an open set V, so $p_H|_{U_{\langle\gamma\rangle}}$ is a homeomorphism. Note that this implies $p_H : \widetilde{X}_H \to X$ is continuous. Now if $x \in X$ is any point, let $U \in \mathcal{U}$ be an open set containing x. Then we have
$$p^{-1}(U) = \bigcup_{\substack{\gamma \text{ a path from } x_0 \\ \text{to a point in } U}} U_{\langle\gamma\rangle},$$
and $p_H|_{U_{\langle\gamma\rangle}} : U_{\langle\gamma\rangle} \to U$ is a homeomorphism, so U is evenly covered, i.e. $(\widetilde{X}_H, p_H)$ is a covering space.

It only remains to show $(p_H)_*(\pi_1(\widetilde{X}_H, \widetilde{x}_H)) = H$ in $\pi_1(X, x_0)$. If $[\gamma] \in H$, let $\gamma_t(s)$ be the path
$$\gamma_t : [0,1] \to X : s \mapsto \gamma(ts).$$
Note that $\widetilde{\gamma} : [0,1] \to \widetilde{X}_H$ given by $t \mapsto \langle\gamma_t\rangle$ is a loop in $\widetilde{X}_H$ since $\widetilde{\gamma}_0 = \gamma_0$ (the constant path in $\widetilde{X}_H$) and $\widetilde{\gamma}(1) = \langle\gamma\rangle = \langle e_{x_0}\rangle$ since $[\gamma] \in H$. We also have $(p_H \circ \widetilde{\gamma})(t) = \gamma_t(1) = \gamma(t)$, so $p_H \circ \widetilde{\gamma} = \gamma$. So $\widetilde{\gamma}$ is a lift of γ and a loop, therefore $[\gamma] \in \operatorname{im}((p_H)_*)$ by Lemma 9.3(b). Now if $\gamma \notin H$, then the path $\widetilde{t}$ is not a loop since $\widetilde{\gamma}(1) = \langle\gamma\rangle \notin H$, so $\widetilde{\gamma}(1) \neq \langle e_{x_0}\rangle$. So $[\gamma] \notin \operatorname{im}((p_H)_*)$ by Lemma 9.3(b). These two properties imply that $(p_H)_*(\pi_1(\widetilde{X}_H, \widetilde{x}_H)) = H$, which completes the proof of the theorem. $\square$

Remark. There is a "lattice" of subgroups of $\pi_1(X, x_0)$ and a "lattice" of covering spaces of X. The above theorem says that these are the same (similar as in Galois theory).

Remark. By the above correspondence, there is a unique covering space of X corresponding to the trivial subgroup $\{e\} \subseteq \pi_1(X, x_0)$. This is called the *universal cover* of X.

Lecture 12

Feb. 17 — Covering Spaces, Part 4

12.1 Deck Transformations

Definition 12.1. Let $p : \widetilde{X} \to X$ be a covering space. A *deck transformation*, or a *covering transformation*, is a covering space isomorphism, i.e. a homeomorphism $f : \widetilde{X} \to \widetilde{X}$ such that $p \circ f = p$. The set $G(\widetilde{X})$ of deck transformations is clearly a group under composition.

Example 12.1.1. Consider the covering map $p_4 : S^1 \to S^1$ given by $\theta \mapsto 4\theta$. Then each $\phi_k : S^1 \to S^1$ given by $\theta \mapsto \theta + (2\pi k)/4$ satisfies $p_4 \circ \phi_k = p_4$, so the ϕ_k are deck transformations. Now suppose $f : S^1 \to S^1$ is another deck transformation, so $f(\widetilde{x}_1) = \widetilde{x}_i$ for some i (let the $\widetilde{x}_i$ be the four preimages of $\widetilde{x}_1$ under p_4). But there exists k such that $\phi_k(\widetilde{x}_1) = \widetilde{x}_i$, and these are both lifts of p which agree at $\widetilde{x}_1$, so $f = \phi_k$ by Lemma 10.2. Thus $G(S^1 \xrightarrow{p_4} S^1) = \{\phi_0, \phi_1, \phi_2, \phi_3\} \cong \mathbb{Z}/4\mathbb{Z}$ (a generator is ϕ_1).

Example 12.1.2. Let $p : \widetilde{X} \to X$ be as in Example 9.0.1(5). Let $\widetilde{x}_1$ be the center intersection point in $\widetilde{X}$ and $\widetilde{x}_2, \widetilde{x}_3$ be the two intersection points at the sides. Let x_0 be the intersection point in X. Note that if we lift a to $\widetilde{X}$ based at $\widetilde{x}_1$, then we get a path, but if we lift a to $\widetilde{X}$ based at $\widetilde{x}_3$, then we get a loop. So if $\phi : \widetilde{X} \to \widetilde{X}$ is a deck transformation, then it cannot take $\widetilde{x}_1$ to $\widetilde{x}_3$, since ϕ applied to a lift is a lift. Similarly, there are no deck transformations taking $\widetilde{x}_1$ to $\widetilde{x}_2$. So any deck transformation takes $\widetilde{x}_1$ to $\widetilde{x}_1$. Thus Lemma 10.2 says that any deck transformation is the identity, i.e. $G(\widetilde{X}) = \{1\}$.

Remark. In the first example, a covering map of degree 4 gave a group of order 4, whereas in the second example, a covering map of degree 3 gave a group of order 1.

Definition 12.2. A covering space $p : \widetilde{X} \to X$ is called *normal* if for every $x \in X$ and $\widetilde{x}, \widetilde{x}' \in p^{-1}(x)$, there exists $\phi \in G(\widetilde{X})$ such that $\phi(\widetilde{x}) = \widetilde{x}'$.

Remark. The first example is normal, while the second one is not.

Theorem 12.1. *Let $p : (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ be a path connected, locally path connected covering space of X. Let $H = p_*(\pi_1(\widetilde{X}, \widetilde{x}_0)) \leq \pi_1(X, x_0)$. Then*

1. *$(\widetilde{X}, p)$ is normal if and only if H is a normal subgroup of $\pi_1(X, x_0)$;*
2. *$G(\widetilde{X}) \cong N(H)/H$, where $N(H)$ is the normalizer of H, i.e. the largest subgroup of $\pi_1(X, x_0)$ in which H is normal.*

Proof. (1) ($\Rightarrow$) Let $[\gamma] \in \pi_1(X, x_0)$ and let $\widetilde{\gamma}$ be a lift of γ based at $\widetilde{x}_0$. Set $\widetilde{x}_1 = \widetilde{\gamma}(1)$. From the proof of Lemma 10.1, we have

$$[\gamma] p_*(\pi_1(\widetilde{X}, \widetilde{x}_0))[\gamma]^{-1} = p_*(\pi_1(\widetilde{X}, \widetilde{x}_1)).$$

Now $\widetilde{X}$ normal implies that there exists $\phi \in G(\widetilde{X})$ such that $\phi_1(\widetilde{x}_0) = \widetilde{x}_1$, so we get an isomorphism

$$\phi_* : \pi_1(\widetilde{X}, \widetilde{x}_1) \to \pi_1(\widetilde{X}, \widetilde{x}_0).$$

Then since $p_* \circ \phi_*$ (as ϕ is a deck transformation), we have

$$p_*(\pi_1(\widetilde{X}, \widetilde{x}_0)) = p_* \circ \phi_*(\pi_1(\widetilde{X}, \widetilde{x}_1)) = p_*(\pi_1(\widetilde{X}, \widetilde{x}_1)),$$

so that $[\gamma]p_*(\pi_1(\widetilde{X}, \widetilde{x}_0))[\gamma]^{-1} = p_*(\pi_1(\widetilde{X}, \widetilde{x}_0))$, i.e. $p_*(\pi_1(\widetilde{X}, \widetilde{x}_0))$ is normal.

($\Leftarrow$) Let $\widetilde{x}_0, \widetilde{x}_1$ be two points in $p^{-1}(x_0)$, and set $H_1 = p_*(\pi_1(\widetilde{X}, \widetilde{x}_1))$ Let h be a path from $\widetilde{x}_0$ to $\widetilde{x}_1$, and let $\gamma = p \circ h$. By Lemma 10.1, we have $[\gamma]H_1[\gamma]^{-1} = H$. Since H is normal, we have $H = H_1$, i.e.

$$p_*(\pi_1(\widetilde{X}, \widetilde{x}_1)) = p_*(\pi_1(\widetilde{X}, \widetilde{x}_0)).$$

Now by Theorem 10.1, there are lifts of p to ϕ_1, ϕ_2 such that

$$\begin{tikzcd} (\widetilde{X}, \widetilde{x}_1) \arrow[r, "\phi_1"] \arrow[dr, "p"'] & (\widetilde{X}, \widetilde{x}_0) \arrow[r, "\phi_2"] \arrow[d, "p"] & (\widetilde{X}, \widetilde{x}_0) \arrow[dl, "p"] \\ & (X, x_0) & \end{tikzcd}$$

so ϕ_1, ϕ_2 are deck transformations taking $\widetilde{x}_0$ to $\widetilde{x}_1$ or the reverse.

(2) From above, if $[\gamma]H[\gamma]^{-1} = H$, there exists $\phi \in G(\widetilde{X})$ such that $\phi(\widetilde{x}_0) = \widetilde{\gamma}(1)$, where $\widetilde{\gamma}$ is the lift of γ based at $\widetilde{x}_0$. So we get a map $\Phi : N(H) \to G(\widetilde{X})$. First, we claim that Φ is a homomorphism. Suppose $\phi_i = \Phi([\gamma_i])$ for $i = 1, 2$ and $[\gamma_i] \in N(H)$. Then $\phi_i(\widetilde{x}_0) = \widetilde{x}_i$, so $\widetilde{\phi}_i$ is a path from $\widetilde{x}_0$ to $\widetilde{x}_i$. Note $\phi_1(\widetilde{x}_2)$ is a lift of γ based at $\widetilde{x}_1$, and $\widetilde{\gamma}_1 * \phi_1(\widetilde{\gamma}_2)$ is a path from $\widetilde{x}_0$ to $\phi_1(\widetilde{x}_2)$. Then

$$[p \circ (\widetilde{\gamma}_1 * (\phi_1 \circ \widetilde{\gamma}_2))] = [\gamma_1 * \gamma_2] = [\gamma_1] \cdot [\gamma_2],$$

so $\phi_1 \circ \phi_2 = \Phi([\gamma_1] \cdot [\gamma_2])$, i.e. Φ is a homomorphism. It only remains to show (as an exercise) that Φ is surjective and that $\ker \Phi = H$, so $G(\widetilde{X}) \cong N(H)/H$ by the first isomorphism theorem. $\square$

Remark. If H is normal in $\pi_1(X, x_0)$, then $G(\widetilde{X}) = \pi_1(X, x_0)/H$. In particular, if $\widetilde{X}$ is the universal cover (so $\pi_1(\widetilde{X})$ is trivial), then $G(\widetilde{X}) = \pi_1(X, x_0)$.

Lecture 13

Feb. 19 — Singular Homology

13.1 Covering Spaces and Group Actions

Definition 13.1. A *group action* on a topological space X is a pair (G, ρ) where G is a group and $\rho : G \to \text{Homeo}(X)$ is a group homomorphism.[1]

Remark. If G acts on X, then we can form the quotient space X/G where two points $x_1, x_2 \in X$ are identified if there exists $g \in G$ such that $\rho(g)(x_1) = x_2$. Then X/G is called the *orbit space* of the action.

Theorem 13.1. *Let G be a group acting on X such that $(*)$ for all $x \in X$, there is a neighborhood U of x such that $g_1U \cap g_2U \neq \varnothing$ implies $g_1 = g_2$. Then:*

1. *$\rho : X \to X/G$ is a normal covering space;*
2. *$G \cong G(X \to X/G)$ if X is path-connected ($G(X \to X/G)$ is the group of deck transformations);*
3. *$G \cong \pi_1(X/G)/\rho_*(\pi_1(X))$ if X is path connected and locally path connected.*

Proof. This is left as an exercise (or see Hatcher). Note that the hypothesis implies each point in X/G has an evenly covered neighborhood, so (X, ρ) is a covering space. Also, any two preimages of a point in X/G are related by the group action, i.e. is a deck transformation of X, so (X, ρ) is normal. □

Exercise 13.1. Show that if G is a finite group acting freely on a Hausdorff space X (i.e. no non-identity element of G has a fixed point), then the action satisfies $(*)$.

Example 13.1.1. Let X be the four-holed torus, arranged in a triangular shape. Then rotation by $2\pi/3$ is an action of $G = \mathbb{Z}/3\mathbb{Z}$ on X with no fixed points. Then X/G is a two-holed torus, and by the above theorem X is a three-fold cover of X/G.

Example 13.1.2. The group $G = \mathbb{Z}/2\mathbb{Z}$ acts on S^n by sending $x \mapsto -x$, so S^n is a covering space of $S^n/G = \mathbb{R}P^n$. Note that we have already seen this before without appealing to the above theorem.

13.2 Introduction to Homology

Remark. There are many types of (ordinary) homology, i.e. singular, simplicial, cubical, cellular, etc. All of these are isomorphic on CW-complexes. We will define the most general homology theory, called *singular homology*, and then derive *cellular homology*, which is easier to compute.

[1] Here $\text{Homeo}(X)$ is the group of homeomorphisms of X.

There are also *generalized homology* theories, such as *bordism theory*, *K-theory*, etc. These are in many ways similar to ordinary homology, but they do not satisfy the dimension axiom, which can give them wildly different behavior. We will briefly mention bordism theory later on in the class.

Underlying these homology theories is *homological algebra*, which is the study of *chain complexes*. Such objects show up in many places and they give rise to more "homology theories," such as *Floer homology*, *Heegaard-Floer homology*. These are *not* "real homology theories" in the sense of algebraic topology.

13.3 Singular Homology

Definition 13.2. The *standard p-simplex* is

$$\Delta^p = \left\{ \sum_{i=0}^{p} t_i e_i \in \mathbb{R}^{p+1} : \sum_{i=0}^{p} t_i = 1, t_i \geq 0 \right\},$$

where $\{e_0, \dots, e_p\}$ are the standard basis vectors for $\mathbb{R}^{p+1}$.

Example 13.2.1. The following are examples of p-simplices:

1. For $p = 0$, we have $\Delta^0 = \{t_1 e_1 \in \mathbb{R} : t_1 = 1, t_1 \geq 0\} = e_1$, which is a point.
2. For $p = 1$, we have $\Delta^1 = \{t_0 e_0 + t_1 e_1 \in \mathbb{R}^2 : t_0 + t_1, t_i \geq 0\}$ is the convex hull of the points e_0 and e_1 in $\mathbb{R}^2$, which is a line segment.
3. For $p = 2$, the standard 2-simplex Δ^2 is the convex hull of $e_0, e_1, e_2 \in \mathbb{R}^3$, which is a triangle.

Definition 13.3. Given any vectors $v_0, \dots, v_p \in \mathbb{R}^n$, by $[v_0, \dots, v_p]$ we denote the map

$$\begin{aligned} \Delta^p &\longrightarrow \mathbb{R}^n \\ (t_0, \dots, t_p) &\longmapsto \sum_{i=0}^{p} t_i v_i \end{aligned}$$

Example 13.3.1. Consider the following examples:

1. $[e_0, \dots, e_p]$ parametrizes the standard p-simplex.
2. $[e_0, \dots, \widehat{e}_i, \dots, e_p]$ (where the hat means to remove e_i) is the ith *face map*, which parametrizes the ith *face* of Δ^p. Denote this map by $F_i^p = [e_0, \dots, \widehat{e}_i, \dots, e_p]$.

Remark. If $i > j$, then

$$F_i^p \circ F_j^{p-1} = [e_0, \dots, \widehat{e}_j, \dots, \widehat{e}_i, \dots, e_p],$$

so $F_2^2 \circ F_1^1$ identifies $[e_0]$ with $[e_0]$ in Δ^2. On the other hand, if $j \geq i$, then

$$F_i^p \circ F_j^{p-1} = [e_0, \dots, \widehat{e}_i, \dots, \widehat{e}_{j+1}, \dots, e_p].$$

For instance, $F_0^3 \circ F_1^2$ identifies $[e_0, e_1]$ with $[e_1, e_3]$ in Δ^3.

Definition 13.4. A *singular p-simplex* in a space X is a continuous map $\sigma : \Delta^p \to X$. The *singular group of p-chains* in X, denoted $C_p(X)$, is the free abelian group generated by singular p-simplices.

Remark. An element in $C_p(X)$ is a finite formal sum $\sum_{i=1}^{k} n_i \sigma_i$, where $n_i \in \mathbb{Z}$ and σ_i is a singular p-simplex. There is an obvious way to add elements in $C(X)$, and this makes $C(X)$ into a group

Definition 13.5. An element in $C_p(X)$ is called a *singular p-chain.* Given a singular p-simplex σ, the ith *face* of σ is $\sigma^{(i)} = \sigma \circ F_i^p$, and the *boundary* of σ is

$$\partial\sigma = \sum_{i=0}^{p} (-1)^i \sigma^{(i)}.$$

Remark. If σ is a singular p-simplex, then $\partial\sigma \in C_{p-1}(X)$.

Example 13.5.1. Let $\Delta^2 = [e_0, e_1, e_2]$ and $\sigma : \Delta^2 \to X$. Then $\partial\sigma = \sigma^{(0)} - \sigma^{(1)} + \sigma^{(2)}$ traces the edges $[\sigma(e_0), \sigma(e_1)]$, $[\sigma(e_1), \sigma(e_2)]$, and $[\sigma(e_2), \sigma(e_0)]$, which forms the boundary of $\sigma(\Delta^2) \subseteq X$.

Definition 13.6. Define the pth *boundary map* to be

$$\begin{aligned} \partial_p : C_p(X) &\longrightarrow C_{p-1}(X) \\ \sum_{i=1}^{k} n_i\sigma_i &\longmapsto \sum_{i=1}^{k} n_i(\partial\sigma_i). \end{aligned}$$

Lemma 13.1. *We have $\partial_{p-1} \circ \partial_p = 0$, i.e. $\partial^2 = 0$.*

Proof. We can calculate

$$\begin{aligned} \partial_{p-1} \circ \partial_p\sigma &= \partial_{p-1}\left(\sum_{i=0}^{p} (-1)^i(-1)^i \sigma \circ F_i^p\right) \\ &= \sum_{i=0}^{p} (-1)^i \left(\sum_{j=0}^{p-1} (-1)^j \sigma \circ F_i^p \circ F_j^{p-1}\right) \\ &= \sum_{0\le j<i\le p} (-1)^{i+j} \sigma \circ F_i^p \circ F_j^{p-1} + \sum_{0\le i\le j\le p-1} (-1)^{i+j} \sigma \circ F_i^p \circ F_j^{p-1} \\ &= \sum_{0\le j<i\le p} (-1)^{i+j} \sigma \circ [e_0, \ldots, \widehat{e}_j, \ldots, \widehat{e}_i, \ldots, e_p] \\ &\qquad + \sum_{0\le i\le j\le p-1} (-1)^{i+j} \sigma \circ [e_0, \ldots, \widehat{e}_i, \ldots, \widehat{e}_{j+1}, \ldots, e_p]. \end{aligned}$$

Relabel $k = j + 1$ in the second sum, we find that

$$\begin{aligned} \partial_{p-1} \circ \partial_p\sigma = &\sum_{0\le j<i\le p} (-1)^{i+j} \sigma \circ [e_0, \ldots, \widehat{e}_j, \ldots, \widehat{e}_i, \ldots, e_p] \\ &+ \sum_{0\le i<k\le p-1} (-1)^{i+k+1} \sigma \circ [e_0, \ldots, \widehat{e}_i, \ldots, \widehat{e}_k, \ldots, e_p] = 0 \end{aligned}$$

since these are the same but with an extra -1 on the second sum, so they cancel out to 0. □

Remark. The lemma implies that $\operatorname{im}\partial_{p+1} \subseteq \ker\partial_p$.

Definition 13.7. We define the *pth (singular) homology group* of X to be

$$H_p(X) = \frac{\ker\partial_p}{\operatorname{im}\partial_{p+1}}.$$

An element of $\ker\partial_p$ is called a *p-cycle*, and an element of $\operatorname{im}\partial_{p+1}$ is called a *p-boundary.* We say that two chains $c_1, c_2 \in C(X)$ are *homologous* if there exists some $d \in C_{p+1}(X)$ such that $c_1 - c_2 = \partial d$.

Remark. Writing $c_1 \sim c_2$ if c_1, c_2 are homologous and letting the equivalence classes be denoted by $[c]$, we can also write H_p as

$$H_p(X) = \{[c] : c \text{ is a } p\text{-cycle}\}.$$

Remark. There are no singular p-simplices for $p < 0$, so $C_p(X) = \{0\}$ for $p < 0$.

Lecture 14

Feb. 24 — Singular Homology, Part 2

14.1 Simple Computations of Homology

Lemma 14.1. *If X is a one-point space, then we have*

$$H_p(X) = \begin{cases} \mathbb{Z} & \text{if } p = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Since X is a one-point space, for each $p \geq 0$ there is a unique map $\sigma_p : \Delta^p \to X$. So $C_p(X) \cong \mathbb{Z}$, which is generated by σ_p. We can calculate the boundary of σ_p:

$$\partial_p \sigma_p \sum_{i=1}^{p} (-1)^i \sigma_p^{(i)} = \sum_{i=0}^{p} (-1)^i \sigma_{p-1} = \begin{cases} 0 & \text{if } p \text{ is odd or } p = 0, \\ \sigma_{p-1} & \text{if } p \text{ is even and } p > 0. \end{cases}$$

So if p is odd, then $H_p(X) = \ker \partial_p / \operatorname{im} \partial_{p+1} \cong \mathbb{Z}/\mathbb{Z} \cong \{0\}$. If p is even and $p > 0$, then

$$H_p(X) = \frac{\ker \partial_p}{\operatorname{im} \partial_{p+1}} \cong \frac{\{0\}}{\{0\}} \cong \{0\}.$$

Finally, we have $H_0(X) = \ker \partial_0 / \operatorname{im} \partial_1 \cong \mathbb{Z}/\{0\} \cong \mathbb{Z}$, which proves the claim. □

Theorem 14.1. *We have $H_0(X) \cong \bigoplus_n \mathbb{Z}$, where n is the number of path components of X.*

Proof. A singular 0-simplex $\sigma_0 : \Delta^0 = \{e_0\} \to X$ is determined by its image, i.e. by a point in X. So an element $\sigma \in C_0(X)$ is $c = \sum_{i=0}^k n_i x_i$, where $n_i \in \mathbb{Z}$ and $x_i \in X$. Define the map

$$\begin{aligned} \varepsilon : C_0(X) &\longrightarrow \mathbb{Z} \\ \sum_{i=0}^{k} n_i x_i &\longmapsto \sum_{i=0}^{k} n_i. \end{aligned}$$

It is easy to see that ε is a homomorphism. If σ is a singular 1-chain (with image a curve), then

$$\partial \sigma = \sum_{i=0}^{1} (-1)^i \sigma^{(i)} = \sigma^{(0)} - \sigma^{(1)} = \sigma(1) - \sigma(0).$$

Thus $\varepsilon(\partial \sigma) = 0$. More generally, if $c \in C_1(X)$ with $c = \sum_{i=0}^k n_i \sigma_i$, then

$$\partial c = \sum_{i=0}^{k} n_i \partial \sigma_i = \sum_{i=0}^{k} n_i (\sigma_i(1) - \sigma_i(0)),$$

so $\varepsilon(\partial c) = 0$. This shows that $\operatorname{im}\partial_1 \subseteq \ker \varepsilon$. Recalling that

$$H_0(X) = \frac{\ker \partial_0}{\operatorname{im}\partial_1} = \frac{C_0(X)}{\operatorname{im}\partial_1},$$

we see that ε induces a map $\varepsilon_* : H_0(X) \to \mathbb{Z}$ by $[c] \mapsto \varepsilon(c)$, which is called an *augmentation*.

We now claim that if X is path connected, then ε_* is an isomorphism. Clearly ε_* is onto, since $\varepsilon_*([x]) = 1$ and 1 is a generator for $\mathbb{Z}$. Now fix a point $x_0 \in X$. Then for any $x \in X$ let $\lambda_x : [0,1] \to X$ be a path from x_0 to x, which exists since X is path connected. So λ_x is a singular 1-chain, and

$$\partial_1 \lambda_x = x - x_0.$$

Now given $[c] \in H_0(X)$ such that $\varepsilon_*([c]) = 0$, we can write $c = \sum_{i=0}^k n_i x_i$. Then

$$\partial_1\left(\sum_{i=0}^k n_i \lambda_{x_i}\right) = \sum_{i=0}^k n_i(x_i - x_0) = \sum_{i=0}^k n_i x_i - \left(\sum_{i=0}^k n_i\right) x_0 = \sum_{i=0}^k n_i x_i = c$$

since $\varepsilon_*([c]) = 0$, which means that $[c] = 0$ in $H_0(X)$. So ε_* is injective, and hence an isomorphism.

Finally, in the general case, check as an exercise that if the path components of X are X_α for $\alpha \in A$, then $C_p(X) = \bigoplus_{\alpha\in A} C_p(X_\alpha)$ and $H_p(X) = \bigoplus_{\alpha \in A} H_p(X_\alpha)$.[1] This suffices to prove the theorem. $\square$

Remark. If we set $\widetilde{\partial}_0 = \varepsilon$ and $\widetilde{\partial}_i = \partial_i$ for $i > 0$, then the proof above shows $\widetilde{\partial}_i \circ \widetilde{\partial}_{i+1} = 0$ for each i. So we can take the homology and define the *reduced homology groups* $\widetilde{H}_i(X)$ via

$$\widetilde{H}_i(X) = \frac{\ker \widetilde{\partial}_i}{\operatorname{im}\widetilde{\partial}_{i+1}}.$$

Clearly $\widetilde{H}_i(X) \cong H_i(X)$ for $i \geq 1$, but $H_0(X) = \widetilde{H}_0(X) \oplus \mathbb{Z}$. In this language, we can write

$$\widetilde{H}_i(\{\mathrm{pt}\}) = 0, \quad \text{for all } i.$$

14.2 Homology and the Fundamental Group

Remark. If $\gamma : [0,1] \to X$ is a loop based at x_0, then γ is a singular 1-chain with $\partial_1 \gamma = x_0 - x_0 = 0$, so that $[\gamma] \in H_1(X)$. This gives a map $\phi : \pi_1(X, x_0) \to H_1(X)$ called the *Hurewitz map*. We will see that it is well-defined and a homomorphism in the next theorem.

Definition 14.1. The *abelianization* G^{ab} of a group G is the largest abelian quotient of G. This means that if A is any abelian group and $f : G \to A$, then f factors through G^{ab}, i.e.

$$\begin{tikzcd} G \arrow[r, "f"] \arrow[dr, "q"'] & A \\ & G^{\mathrm{ab}} \arrow[u, dashed, "\widetilde{f}"'] \end{tikzcd}$$

where $q : G \to G^{\mathrm{ab}}$ is the quotient map.

[1] The main idea is that Δ^p is connected, so $\sigma(\Delta^p)$ must land in a single path component X_α, so $C_p(X)$ decomposes. Similarly, the ∂ sends an element of $C_{p+1}(X_\alpha)$ to an element of $C_p(X_\alpha)$, so $H_p(X)$ decomposes.

Exercise 14.1. Show that $G^{\mathrm{ab}} \cong G/[G,G]$, where $[G,G]$ is the *commutator subgroup* of G, i.e. the smallest normal subgroup of G containing $\{xyx^{-1}y^{-1} \mid x, y \in G\}$.

Theorem 14.2. *If X is path connected, then the Hurewitz map induces an isomorphism*

$$\phi_* : (\pi_1(X, x_0))^{\mathrm{ab}} \to H_1(X),$$

where $(\pi_1(X, x_0))^{\mathrm{ab}}$ is the abelianization of $\pi_1(X, x_0)$.

Proof. Denote an equivalence class in $\pi_1(X, x_0)$ by $[\gamma]$ and one in $H_1(X)$ by $[\![\gamma]\!]$. Note that:

1. If γ, η are paths in X with $\gamma(1) = \eta(0)$, then the 1-chain $\gamma * \eta - \gamma - \eta$ is a boundary.

   To see this, define $\sigma : \Delta^2 \to X$ on $[e_0, e_1, e_2]$ by γ on the edge $[e_0, e_1]$ and η on the edge $[e_1, e_2]$. Extend σ to be constant on parallel lines perpendicular to $[e_0, e_2]$, where σ takes the value of the intersection point of the line with one of the other edges $[e_0, e_1]$ or $[e_1, e_2]$. Note that this defines σ on $[e_0, e_1]$ to be $\gamma * \eta$. Thus we can compute that

   $$\partial(-\sigma) = -\sigma^{(0)} + \sigma^{(1)} - \sigma^{(2)} = -\eta + \gamma * \eta - \gamma.$$

2. If γ is a path in X, then $\gamma + \overline{\gamma}$ is a boundary, and a constant path is a boundary.

   First if γ is a constant path, then we can simply let σ be a singular 2-simplex that is the same constant. Then $\partial\sigma = \sigma^{(0)} - \sigma^{(1)} + \sigma^{(2)} = \gamma - \gamma + \gamma = \gamma$.

   Now given any γ, let σ be the singular 2-simplex which is defined on $[e_0, e_1]$ to be γ. Let σ be constant on lines perpendicular to $[e_1, e_2]$. Note that this defines σ to be some constant c on $[e_0, e_2]$ and $\overline{\gamma}$ on $[e_1, e_2]$. Let σ' be a singular 2-simplex that is constantly c, then

   $$\partial(\sigma + \sigma') = (\sigma^{(0)} - \sigma^{(1)} + \sigma^{(2)}) + c = (\overline{\gamma} - c + \gamma) + c = \overline{\gamma} + \gamma.$$

3. If γ and η are homotopic rel endpoints, then $\gamma - \eta$ is a boundary.

   Let $H : [0,1] \times [0,1] \to X$ be the homotopy from γ to η. View H on the unit square as γ on the bottom, η on the top, and constant on the sides. Since H is constant on the left edge, we can collapse the left edge to a point to get a 2-simplex, so H descends to a map $\sigma_H : \Delta^2 \to X$. Note that σ_H is γ on $[e_0, e_1]$, η on $[e_0, e_2]$, and a constant c on $[e_1, e_2]$. Then

   $$\partial\sigma_H = \sigma_H^{(0)} - \sigma_H^{(1)} + \sigma_H^{(2)} = c - \eta + \gamma.$$

   Since the constant path c is a boundary, the above implies that $\gamma - \eta$ also is.

Note that (3) implies $\phi : \pi_1(X, x_0) \to H_1(X)$ is well-defined, and (1) implies ϕ is a homomorphism since

$$\phi([\gamma] \cdot [\eta]) = \phi([\gamma * \eta]) = [\![\gamma * \eta]\!] \overset{(1)}{=} [\![\gamma]\!] + [\![\eta]\!] = \phi([\gamma]) + \phi([\eta]).$$

Since $H_1(X)$ is abelian, ϕ induces a homomorphism $\phi_* : (\pi_1(X, x_0))^{\mathrm{ab}} \to H_1(X)$.

To see that ϕ_* is a bijection, we construct an inverse to ϕ_*. For each $x \in X$, let γ_x be a path from x_0 to x (if $x = x_0$, we let γ_x be the constant path). Given a singular 1-simplex σ, let $\widehat{\sigma} = \gamma_{\sigma(0)} * \sigma * \overline{\gamma}_{\sigma(1)}$, and define $\psi(\sigma) = [\widehat{\sigma}]$. Since $(\pi_1(X, x_0))^{\mathrm{ab}}$ is abelian and $C_1(X)$ is a free abelian group, this defines a homomorphism $\psi : C_1(X) \to (\pi_1(X, x_0))^{\mathrm{ab}}$. Note that

$$\psi \circ \phi_*([\gamma]) = [e_{x_0} * \gamma * \overline{e_{x_0}}] = [\gamma],$$

where e_{x_0} is the constant path at x_0. If σ is a singular 2-simplex with vertices y_0, y_1, y_2, then

$$\begin{aligned}\psi(\partial_2\sigma) &= \psi(\sigma^{(0)} - \sigma^{(1)} + \sigma^{(2)}) = \psi(\sigma^{(2)})\psi(\sigma^{(0)})\psi(\sigma^{(1)})^{-1} \\ &= [\gamma_{y_0} * \sigma^{(2)} * \overline{\gamma}_{y_1} * \gamma_{y_1} * \overline{\sigma}^{(1)} * \overline{\gamma}_{y_2} * \gamma_{y_2} * \sigma^{(0)} * \overline{\gamma}_{y_0}] \\ &= [\gamma_{y_0} * \sigma^{(2)} * \sigma^{(1)} * \overline{\sigma}^{(0)} * \overline{\gamma}_{y_0}] = [e_{x_0}]\end{aligned}$$

since $\gamma_{y_0} * \sigma^{(2)} * \sigma^{(1)} * \overline{\sigma}^{(0)} * \overline{\gamma}_{y_0}$ bounds a disk. Thus $\operatorname{im}\partial_2 \subseteq \ker\psi$, and so ψ induces a homomorphism $\psi_* : H_1(X) \to (\pi_1(X, x_0))^{\mathrm{ab}}$. From above, we know that $\psi_* \circ \phi_* = \mathrm{id}$, and one can show as an exercise that $\phi_* \circ \psi_* = \mathrm{id}$. This shows that ϕ_* is an isomorphism. $\square$

Lecture 15

Mar. 3 — Homological Algebra

15.1 Introduction to Homological Algebra

Definition 15.1. A sequence of abelian groups C_* and group homomorphisms $\partial_n : C_n \to C_{n-1}$ is called a *chain complex* if $\partial_{n-1} \circ \partial_n = 0$ for all n. The *homology* of a chain complex is

$$H_n(C_*, \partial) = \frac{\ker \partial_n}{\operatorname{im} \partial_{n+1}}.$$

Example 15.1.1. The singular chain groups and the boundary map ∂ is a chain complex.

Definition 15.2. Given two chain complexes (C_*, ∂) and (C'_*, ∂'), a *chain map* is a sequence of group homomorphisms $f_n : C_n \to C'_n$ such that $\partial'_n \circ f_n = f_{n-1} \circ \partial_n$, i.e. we have

$$\begin{tikzcd} \cdots \arrow[r] & C_n \arrow[r, "\partial_n"] \arrow[d, "f_n"] & C_{n-1} \arrow[r] \arrow[d, "f_{n-1}"] & \cdots \\ \cdots \arrow[r] & C'_n \arrow[r, "\partial'_n"] & C'_{n-1} \arrow[r] & \cdots \end{tikzcd}$$

Lemma 15.1. *A chain map $\{f_n\} : (C_*, \partial) \to (C'_*, \partial')$ induces homomorphisms*

$$(f_n)_* : H_n(C_*, \partial) \to H_n(C'_*, \partial').$$

Proof. If $[h] \in H_n(C_*, \partial) = \ker \partial_n / \operatorname{im} \partial_{n+1}$, then $h \in C_n$ and $\partial_n h = 0$. Then

$$\partial'_n(f_n(h)) = f_{n-1}(\partial_n h) = f_{n-1}(0) = 0,$$

so we can define $(f_n)_*([h]) = [f_n(h)]$. To see that this is well-defined, suppose $[h] = [h'] \in H_n(C_*, \partial)$. Then there exists $k \in C_{n+1}$ such that $h' = h + \partial_{n+1} k$. We can write

$$f_n(h') = f_n(h + \partial_{n+1} k) = f_n(h) + f_n(\partial_{n+1} k) = f_n(h) + \partial'_{n+1}(f_{n+1}(k)),$$

i.e. $[f_n(h')] = [f_n(h)]$. Finally, check as an exercise that the $(f_n)_*$ are homomorphisms. $\square$

15.2 Induced Homomorphisms in Homology

Remark. If $f : X \to Y$ is a continuous map, then we can define maps

$$\begin{aligned} f_n : C_n(X) &\longrightarrow C_n(Y) \\ \sum n_i \sigma_i &\longmapsto \sum n_i (f \circ \sigma_i). \end{aligned}$$

Note that we have

$$f_{n-1}(\partial_n\sigma) = f_{n-1}\left(\sum_{i=0}^{n}(-1)^i\sigma^{(i)}\right) = \sum_{i=0}^{n}(-1)^i(f\circ\sigma^{(i)}) = \sum_{i=0}^{n}(-1)^i(f\circ\sigma)^{(i)} = \partial_n(f_n(\sigma)),$$

In particular, we see that $\{f_n\}$ is a chain map. So by Lemma 15.1, we get a map $f_* : H_n(X) \to H_n(Y)$ induced on singular homology for each n.

Exercise 15.1. Check the following:

1. $(f \circ g)_* = f_* \circ g_*$;
2. $(\mathrm{id}_X)_* = \mathrm{id}_{H_n(X)}$.

Definition 15.3. Given chain complexes (C_*, ∂) and (C'_*, ∂') and chain maps $\{f_n\}$ and $\{g_n\}$ from C_* to C'_*, we say there is a *chain homotopy* between them if there is a sequence of homomorphisms

$$p_n : C_n \to C'_{n+1} \quad \text{such that} \quad \partial'_{n+1}\circ p_n + p_{n-1}\circ\partial_n = f_n - g_n.$$

In other words, we have the diagram

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[d, shift right, bend right, "f_{n+1}"'] \arrow[d, shift left, "g_{n+1}"] & C_n \arrow[r, "\partial_n"] \arrow[dl, "p_n"] \arrow[d, shift right, "f_n"'] \arrow[d, shift left, "g_n"] & C_{n-1} \arrow[r] \arrow[dl, "p_n"] \arrow[d, shift right, "f_{n-1}"'] \arrow[d, shift left, bend left, "g_{n-1}"] & \cdots \\
\cdots \arrow[r] & C'_{n+1} \arrow[r, "\partial'_{n+1}"'] & C'_n \arrow[r, "\partial'_n"'] & C'_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

Lemma 15.2. *If there is a chain homotopy between $\{f_n\}$ and $\{g_n\}$, then*

$$(f_n)_* = (g_n)_* : H_n(C_*, \partial) \to H_n(C'_*, \partial').$$

Proof. Let $h \in C_n$ and $\partial_n h = 0$, so h represents an element of $H_n(C_*, \partial)$. Then

$$f_n(h) - g_n(h) = \partial'_{n+1}(p_n(h)) + p_{n-1}(\partial_n(h)) = \partial'_{n+1}(p_n(h)),$$

so $[f_n(h)] = [g_n(h)]$ in $H_n(C'_*, \partial')$. □

Theorem 15.1. *Let $f, g : X \to Y$ be homotopic maps. Then $\{f_n\}$ and $\{g_n\}$ are chain homotopic maps $(C_*(X), \partial) \to (C_*(Y), \partial')$ and thus $f_* = g_* : H_n(X) \to H_n(Y)$.*

Proof. Let $H : X \times [0,1] \to Y$ be the homotopy, so

$$H(x, 0) = f(x) \quad \text{and} \quad H(x, 1) = g(x)$$

Given a simplex $\sigma^n \to X$, define $p_n(\sigma)$ by

$$\begin{aligned} H\circ(\sigma\times\mathrm{id}_{[0,1]}) : \Delta^n \times [0,1] &\longmapsto Y \\ (x,t) &\longmapsto H(\sigma(x), t). \end{aligned}$$

To make sense of this, we need to see how to write $H \circ (\sigma \times \mathrm{id}_{[0,1]})$ as a sum of singular $(n+1)$-simplices in Y. For $n = 0$, note that the interval $\{e_0\} \times [e_0, e_1]$ is already Δ^1, so

$$\begin{aligned} p_0(\sigma) &= H\circ(\sigma\times\mathrm{id}_{[0,1]}) = H(e_0, t) \\ \partial_1(p_0(\sigma)) &= \partial(H(e_0,t)) = (H(e_0,t))^{(0)} - (H(e_0,t))^{(1)} \\ &= H(e_0,1) - H(e_0,0) = g(e_0) - f(e_0) = g\circ\sigma - f\circ\sigma. \end{aligned}$$

Letting $p_{-1} = 0$, we have $g_0 - f_0 = \partial_1 \circ p_0 + p_{-1} \circ \partial_0$ as desired. For $n = 1$, note that $[e_0, f_0] \times [e_0, e_1]$ (the last corner is f_1) is a square, but we can divide it into two 2-simplices by

$$\tau_1 = [e_0, e_1, f_1] \quad \text{and} \quad \tau_0 = [e_0, f_0, f_1].$$

Let $H_\sigma = H \circ (\sigma \times \mathrm{id}_{[0,1]})$, then we have

$$\begin{aligned}
p_1(\sigma) &= H_\sigma|_{\tau_0} - H_\sigma|_{\tau_1} \\
\partial_2(p_1(\sigma)) &= (H_\sigma|_{[f_0,f_1]} - H_\sigma|_{[e_0,f_1]} + H_\sigma|_{[e_0,f_0]}) - (H_\sigma|_{[e_1,f_1]} - H_\sigma|_{[e_0,f_1]} + H_\sigma|_{[e_0,e_1]}) \\
&= H_\sigma|_{[f_0,f_1]} - H_\sigma|_{[e_0,e_1]} + H_\sigma|_{[e_0,f_0]} - H_\sigma|_{[e_1,f_1]} \\
&= g(\sigma) - f(\sigma) - p_0(\partial\sigma)
\end{aligned}$$

In general, let $e_0, \ldots, e_n$ be the vertices of Δ^n in $\mathbb{R}^{n+1}$, where we think of $\mathbb{R}^{n+1} \subseteq \mathbb{R}^{n+2}$ with $x_{n+2} = 0$. Let $f_0, \ldots, f_n$ be the points in $\mathbb{R}^{n+2}$ above $e_0, \ldots, e_n$ with $x_{n+2} = 1$. So $[e_0, \ldots, e_n]$ describes $\Delta^n \times \{0\}$ and $[f_0, \ldots, f_n]$ describes $\Delta^n \times \{1\}$. Show as an exercise that $D^n \times [0, 1]$ is the union of $(n+1)$-simplices

$$[e_0, \ldots, e_i, f_i, \ldots, f_n].$$

Now we can define $p_n : C_n(X) \to C_{n+1}(Y)$ by

$$\begin{aligned}
p_n(\sigma) &= \sum_{i=0}^{n} (-1)^i (H \circ (\sigma \times \mathrm{id}_{[0,1]}))|_{[e_0,\ldots,e_i,f_i,\ldots,f_n]} \\
\partial_{n+1}(p_n(\sigma)) &= \sum_{j \le i} (-1)^i (-1)^j (H \circ (\sigma \times \mathrm{id}_{[0,1]}))|_{[e_0,\ldots,\widehat{e}_j,\ldots,e_i,f_i,\ldots,f_n]} \\
&\qquad + \sum_{i \le j} (-1)^i (-1)^{j+1} (H \circ (\sigma \times \mathrm{id}_{[0,1]}))|_{[e_0,\ldots,e_i,f_i,\ldots,\widehat{f}_j,\ldots,f_n]}.
\end{aligned}$$

The $i = j$ terms all cancel except for

$$[\widehat{e}_0, f_0, \ldots, f_n] = [f_0, \ldots, f_n] \quad \text{and} \quad [e_0, \ldots, e_n, \widehat{f}_n] = [e_0, \ldots, e_n],$$

which give $g \circ \sigma$ and $-(f \circ \sigma)$, respectively. Check as an exercise that many other terms cancel, and those which remain are precisely those which give $p_{n-1}(\partial_n \sigma)$. This would prove the result. □

Corollary 15.1.1. *If $f : X \to Y$ is a homotopy equivalence, then*

$$f_* : H_n(X) \to H_n(Y)$$

is an isomorphism for all n.

Proof. There is a homotopy inverse $g : Y \to X$ such that $(f \circ g) \simeq \mathrm{id}_Y$ and $(g \circ f) \simeq \mathrm{id}_X$. Then

$$f_* \circ g_* = (f \circ g)_* = (\mathrm{id}_Y)_* = \mathrm{id}_{H_n(Y)} \quad \text{and} \quad g_* \circ f_* = (g \circ f)_* = (\mathrm{id}_X)_* = \mathrm{id}_{H_n(X)}.$$

The first says that f_* is surjective and the second says f_* is injective, so f_* is an isomorphism. □

Remark. The corollary shows that if X is contractible, then

$$H_n(X) \cong \begin{cases} \mathbb{Z} & \text{if } n = 0, \\ 0 & \text{if } n \neq 0. \end{cases}$$

Lecture 16

Mar. 5 — Relative Homology

16.1 Relative Homology

Remark. Let A be a subspace of X. Then $C_n(A) \leq C_n(X)$, and note that $C_n(A)$ lies in the kernel of

$$C_n(X) \xrightarrow{\partial_n} C_{n-1}(X) \xrightarrow{\text{quotient}} C_{n-1}(X)/C_{n-1}(A)$$

So ∂_n induces a homomorphism $\partial_n : C_n(X)/C_n(A) \to C_{n-1}(X)/C_{n-1}(A)$.

Definition 16.1. Define $C_n(X, A) = C_n(X)/C_n(A)$ and $\partial_n : C_n(X, A) \to C_{n-1}(X, A)$ We still have $\partial_{n-1} \circ \partial_n = 0$ for all n, so this is a chain complex. The *relative homology* of (X, A) is

$$H_n(X, A) = H_n(C_*(X, A), \partial) = \frac{\ker(\partial_n : C_n(X, A) \to C_{n-1}(X, A))}{\operatorname{im}(\partial_{n+1} : C_{n+1}(X, A) \to C_n(X, A))}.$$

An element in $H_n(X, A)$ is represented by a *relative cycle*, i.e. $\alpha \in C_n(X)$ such that $\partial\alpha \in C_{n-1}(A)$. A relative cycle $\alpha \in C_n(X)$ trivial in $H_n(X, A)$ if it is a *relative boundary*, i.e. there exists $\beta \in C_{n+1}(X)$ and $\gamma \in C_n(A)$ such that $\alpha = \partial\beta + \gamma$.

Definition 16.2. Recall that a sequence of homomorphisms

$$A \xrightarrow{\phi} B \xrightarrow{\psi} C$$

is called *exact at B* if $\operatorname{im}\phi = \ker\psi$. A longer sequence is called *exact* if it is exact at each term.[1]

Lemma 16.1. *If (A_n, ∂_A), (B_n, ∂_B), (C_n, ∂_C) are chain complexes and*

$$\{\phi_n\} : (A_*, \partial_A) \to (B_*, \partial_B) \quad \textit{and} \quad \{\psi_n\} : (B_*, \partial_B) \to (C_*, \partial_C)$$

are chain maps such that the sequence

$$0 \longrightarrow A_n \xrightarrow{\phi} B_n \xrightarrow{\psi} C_n \longrightarrow 0$$

is exact for all n, then there exists a long exact sequence

$$\cdots \longrightarrow H_n(A_*, \partial_A) \xrightarrow{(\phi_n)_*} H_n(B_*, \partial_B) \xrightarrow{(\psi_n)_*} H_n(C_*, \partial_C) \xrightarrow{\partial_n} H_{n-1}(A_*, \partial_A) \longrightarrow \cdots$$

Corollary 16.0.1. *Let A be a subspace of X. Let $i : A \to X$ be the inclusion and $q_n : C_n(X) \to C_n(X, A)$ be the quotient map. Then the sequence*

[1]Note that a sequence is exact precisely when it has trivial homology.

$$0 \longrightarrow C_n(A) \xrightarrow{i} C_n(X) \xrightarrow{q_n} C_n(X,A) \longrightarrow 0$$

is exact, so there exists an exact sequence

$$\cdots \longrightarrow H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{q_*} H_n(X,A) \xrightarrow{\partial_*} H_{n-1}(A) \longrightarrow \cdots$$

The above is called the long exact sequence of a pair (X, A).

Proof. The second part follows from Lemma 16.1. For the first part, note that $i : C_n(A) \to C_n(X)$ is clearly injective and $q_n : C_n(X) \to C_n(X,A) = C_n(X)/C_n(A)$ is clearly surjective. Also, we have

$$\operatorname{im} i = C_n(A) = \ker q_n,$$

so the sequence is exact, as claimed. $\square$

Exercise 16.1. Verify that ∂_* in the exact sequence is the following map: For $h \in H_n(X,A)$, choose $\alpha \in C_n(X)$ such that $\partial_n \alpha \in C_{n-1}(A)$ and $h = [\alpha]$, then define $\partial_* h = [\partial_n \alpha] \in H_{n-1}(A)$.

Proof of Lemma 16.1. The main difficulty lies in defining ∂_*.

Consider the following diagram (which commutes since the ϕ_n, ψ_n are chain maps):

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\
A \arrow[r, "\partial_A"] \arrow[d, "\phi_n"] & A_{n-1} \arrow[r, "\partial_A"] \arrow[d, "\phi_{n-1}"] & A_{n-2} \arrow[d, "\phi_{n-2}"] \\
B \arrow[r, "\partial_B"] \arrow[d, "\psi_n"] & B_{n-1} \arrow[r, "\partial_B"] \arrow[d, "\psi_{n-1}"] & B_{n-2} \arrow[d, "\psi_{n-2}"] \\
C \arrow[r, "\partial_C"] \arrow[d] & C_{n-1} \arrow[r, "\partial_C"] \arrow[d] & A_{n-2} \arrow[d] \\
0 & 0 & 0
\end{tikzcd}$$

Given $[c] \in H_n(C_n, \partial_C)$, we know that $\partial_C c = 0$. Since ψ_n is surjective, there exists some $b \in B_n$ such that $\psi(b) = c$. Note that

$$\psi_{n-1}(\partial_B b) = \partial_C(\psi_n(b)) = \partial_C c = 0,$$

so $\partial_B b \in \ker \psi_{n-1} = \operatorname{im} \phi_{n-1}$. So there is a unique $a \in A_{n-1}$ such that $\phi_{n-1}(a) = \partial_B b$. Note that

$$\phi_{n-2}(\partial_A a) = \partial_B(\psi_{n-1}(a)) = \partial_B(\partial_B b) = 0.$$

Since ϕ_{n-2} is injective, we have $\partial_A a = 0$ and $[a] \in H_{n-1}(A)$. So we can define $\partial_*[c] = [a]$.

To see that ∂_* is well-defined, note that we made two choices: the representative of $[c]$ and the choice of b such that $\psi_n(b) = c$. We need to see that ∂_* is independent of these choices. Let c, c' represent $[c]$. Since $c, c' \in [c]$, there exists $\overline{c}$ such that $\partial \overline{c} = c - c'$. Choose $b, b' \in B_n$ such that $\phi_n(b) = c$ and $\phi(b') = c'$ (this will also show the independence of b by taking $c' = c$). There exists $\overline{b}$ such that $\psi_{n+1}(\overline{b}) = \overline{c}$. Then

$$\psi_n(\partial_B \overline{b} - b + b') = \partial_C(\psi_{n+1}(\overline{b})) - \psi_n(b) + \psi_n(b') = (c - c') - c + c' = 0.$$

Thus there exists $\overline{a} \in A_n$ such that $\phi_n(\overline{a}) = \partial_B \overline{b} - b + b'$. Let a, a' be such that $\phi_{n-1}(a) = \partial_B b$ and $\phi_{n-1}(a') = \partial_B b'$, as we did above. We can then see that

$$\begin{aligned}\phi_{n-1}(\partial_A \overline{a} + a - a') &= \partial_B(\phi_n(\overline{a})) + \phi_{n-1}(a) - \phi_{n-1}(a') = \partial_B(\partial_B \overline{b} - b + b') + \partial_B b - \partial_B b' \\ &= (0 - \partial_B b + \partial_B b') + \partial_B b - \partial_B b' = 0.\end{aligned}$$

Therefore, $a' = a + \partial_A \overline{a}$ and $[a] = [a']$, so ∂_* is well-defined.

Complete the remainder of the proof as an exercise (i.e. check that ∂_* is a homomorphism and that the sequence is in fact exact at each term). □

Example 16.2.1. Consider (X, x_0). The long exact sequence of (X, x_0) is (for $n \geq 2$)

$$\begin{array}{ccccccccc} \cdots \longrightarrow & H_n(x_0) & \longrightarrow & H_n(X) & \longrightarrow & H_n(X, x_0) & \longrightarrow & H_{n-1}(x_0) & \longrightarrow \cdots \\ & \| & & & & & & \| & \\ & 0 & & & & & & 0 & \end{array}$$

In particular, this means that $H_n(X) \cong H_n(X, x_0)$. For $n = 0, 1$, we have

$$\begin{array}{ccccccccccccc} H_1(x_0) & \longrightarrow & H_1(X) & \longrightarrow & H_1(X, x_0) & \longrightarrow & H_0(x_0) & \xrightarrow{i_*} & H_0(X) & \longrightarrow & H_0(X, x_0) & \longrightarrow & 0 \\ \| & & & & & & \| & & \| & & & & \\ 0 & & & & & & \mathbb{Z} & & \bigoplus_n \mathbb{Z} & & & & \end{array}$$

Since i_* is injective, we must have $\operatorname{im}(H_1(X, x_0) \to H_0(x_0)) = 0$, and so $H_1(X) \cong H_1(X, x_0)$. Now i_* maps $\mathbb{Z}$ to the $\mathbb{Z}$ factor of $\bigoplus_n \mathbb{Z}$ corresponding to the path component containing x_0. Since the last map $H_0(X) \to H_0(X, x_0)$ is surjective, we have

$$H_0(X, x_0) \cong \frac{H_0(X)}{\ker(H_0(X) \to H_0(X, x_0))} \cong \frac{H_0(X)}{\operatorname{im} i_*} \cong \frac{\bigoplus_n \mathbb{Z}}{\mathbb{Z}} \cong \bigoplus_{n-1} \mathbb{Z}.$$

Note that we are able to conclude the last step since we know i_* maps $\mathbb{Z}$ to an entire $\mathbb{Z}$ factor (in general, we may end up with a direct sum of $\bigoplus_{n-1} \mathbb{Z}$ with a finite group). This shows that $H_n(X, x_0) = \widetilde{H}_n(X)$.

Exercise 16.2. Show the following:

1. If $f : (X, A) \to (Y, B)$ is continuous, then $f_* : H_n(X, A) \to H_n(Y, B)$.
2. If $f, g : (X, A) \to (Y, B)$ are homotopic as maps of pairs, then $f_* = g_*$.
3. If $A \subseteq B \subseteq X$, then one gets a long exact sequence

$$\cdots \longrightarrow H_n(B, A) \longrightarrow H_n(X, A) \longrightarrow H_n(X, B) \longrightarrow H_{n-1}(B, A) \longrightarrow \cdots$$

Hint: For the last part, note that one can write

$$C_n(X, A) = \frac{C_n(X)}{C_n(A)} \cong \frac{C_n(X)/C_n(B)}{C_n(B)/C_n(A)} \cong \frac{C_n(X, B)}{C_n(B, A)}.$$

16.2 Excision

Theorem 16.1 (Excision)**.** *Let $Z \subseteq A \subseteq X$ and $\overline{Z} \subseteq \operatorname{int} A$. Then the inclusion map*

$$i : (X \setminus Z, A \setminus Z) \to (X, A)$$

induces an isomorphism

$$i_* : H_n(X \setminus Z, A \setminus Z) \to H_n(X, A).$$

Remark. We will prove this theorem in a later lecture.

Definition 16.3. A pair $A \subseteq X$ is called *good* if A is nonempty, closed, and has a neighborhood U such that A is a deformation retract of U, i.e. there exists a homotopy $H : U \times [0,1] \to U$ such that $H(x,0) = x$ for all $x \in U$, and $H(x,1) \in A$, $H(x,t) = x$ for all $x \in A$ and $t \in [0,1]$.

Exercise 16.3. If A is a submanifold of a manifold X, then show that $A \subseteq X$ is good. If X is obtained from A by attaching a cell, then show that $A \subseteq X$ is good.

Theorem 16.2. *If (X, A) is a good pair, then the quotient map*

$$q : (X, A) \to (X/A, A/A) \cong (X/A, \{\text{pt}\})$$

induces an isomorphism

$$q_* : H_n(X, A) \to H_n(X/A, A/A) \cong \widetilde{H}_n(X/A).$$

Proof. Let U be the neighborhood of A that deformation retracts to A and let $H : U \times [0,1] \to U$ be the deformation retraction. We have the diagram:

$$\begin{tikzcd}
H_n(X,A) \arrow[r, "i"] \arrow[d, "q_*"] & H_n(X,U), \arrow[d, "q_*"] & H_n(X-A, U-A) \arrow[l, "\cong"'] \arrow[d, "q_*"] \\
H_n(X/A, A/A) \arrow[r, "i"] & H_n(X/A, U/A) & H_n(X/A - A/A, U/A - A/A) \arrow[l, "\cong"']
\end{tikzcd}$$

The isomorphism arrows are by excision. Now note that

$$q : (X - A, U - A) \to (X/A - A/A, U/A - A/A)$$

is a homeomorphism (actually the identity), so the last quotient map on the right is an isomorphism. Thus the middle quotient map is also an isomorphism by excision.

Let $h_t(x) = H(x,t)$, and consider the composition

$$(A, A) \xrightarrow{i} (U, A) \xrightarrow{h_1} (A, A)$$

Note that $h_1 \circ i = \mathrm{id}_{(A,A)}$ and $i \circ h_1 \simeq \mathrm{id}_{(U,A)}$ via H, so $i_* : H_n(A,A) \to H_n(U,A)$ is an isomorphism. Now consider the long exact sequence of $A \subseteq U \subseteq X$:

$$\begin{tikzcd}
H_n(U,A) \arrow[r] \arrow[d, equal] & H_n(X,A) \arrow[r, "i_*"] & H_n(X,U) \arrow[r, "\partial"] & H_{n-1}(U,A) \arrow[d, equal] \\
H_n(A,A) = 0 & & & H_{n-1}(A,A) = 0
\end{tikzcd}$$

This tells us that $i_* : H_n(X,A) \to H_n(X,U)$ is an isomorphism, and a similar argument shows that $i_* : H_n(X/A, A/A) \to H_n(X/A, U/A)$ is as well, so $q_* : H_n(X,A) \to H_n(X/A, A/A)$ also is. $\square$

Lecture 17

Mar. 10 — Applications of Excision

17.1 Homology Computations Using Excision

Proposition 17.1. *We have*

$$H_k(S^n) = \begin{cases} \mathbb{Z} & \text{if } k = 0, n, \\ 0 & \text{if } k \neq 0, n, \end{cases} \quad \text{and} \quad H_k(D, \partial D^n) = \begin{cases} \mathbb{Z} & \text{if } k = n, \\ 0 & \text{if } k \neq n. \end{cases}$$

Proof. First note that (interpret $k = n = 0$ as two copies of $\mathbb{Z}$)

$$H_k(S^0) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & \text{if } k = 0, \end{cases} \quad \text{and} \quad H_k(D^n) = \begin{cases} \mathbb{Z} & \text{if } k = 0, \\ 0 & \text{if } k \neq 0. \end{cases}$$

For $n = 1$, consider the long exact sequence of (D^1, S^0):

$$\begin{array}{ccccccc} H_k(S^0) & \longrightarrow & H_k(D^1) & \xrightarrow{\cong} & H_k(D^1, S^0) & \longrightarrow & H_{k-1}(S^0) \\ \| & & \| & & & & \| \\ 0 & & 0 & & & & 0 \end{array}$$

so $H_k(D^1, S^0) = 0$ for $k \geq 2$ (note that $D^1/S^0 \cong S^1$, thus $H_k(S^1) = 0$ also for $k \geq 2$). Now consider

$$\begin{array}{ccccccccccccc} H_1(S^0) & \longrightarrow & H_1(D^1) & \longrightarrow & H_1(D^1, S^0) & \xrightarrow{\partial} & H_0(S^0) & \xrightarrow{i_*} & H_0(D^1) & \longrightarrow & H_0(D^1, S^0) & \longrightarrow & 0 \\ \| & & \| & & \| & & \| & & \| & & \| & & \\ 0 & & 0 & & H_1(S^1) & & \mathbb{Z} \oplus \mathbb{Z} & & \mathbb{Z} & & \widetilde{H}_0(S^1) = 0 & & \end{array}$$

By the exactness of the above sequence at $H_0(S^0)$, we have

$$\mathbb{Z} \cong \frac{\mathbb{Z} \oplus \mathbb{Z}}{\ker i_*} \cong \frac{\mathbb{Z} \oplus \mathbb{Z}}{\operatorname{im} \partial} \cong \frac{\mathbb{Z} \oplus \mathbb{Z}}{H_1(S^1)},$$

so we must have $H_1(S^1) \cong \mathbb{Z}$. Now by induction, assume the proposition is true for S^{n-1} (with $n \geq 1$). Consider the long exact sequence of (D^n, S^{n-1}): If $n \geq 2$,

$$\begin{array}{ccccccc} H_n(D^n) & \longrightarrow & H_n(D^n, S^{n-1}) & \longrightarrow & H_{n-1}(S^{n-1}) & \longrightarrow & H_{n-1}(D^n) \\ \| & & & & \| & & \| \\ 0 & & & & \mathbb{Z} & & 0 \end{array}$$

so we have $H_n(S^n) \cong H_n(D^n, S^{n-1}) \cong \mathbb{Z}$. Now notice that

$$\begin{array}{ccccccc} H_1(D^n) & \longrightarrow & H_1(D^n, S^{n-1}) & \xrightarrow{\phi} & H_0(S^{n-1}) & \xrightarrow{i_*} & H_0(D^n) \\ \| & & & & \| & & \| \\ 0 & & & & \mathbb{Z} & & \mathbb{Z} \end{array}$$

where $H_0(D^n) \cong \mathbb{Z}$ since i_* maps a generator of $H_0(S^{n-1})$ to a generator of $H_0(D^n)$. Thus $\operatorname{im}\phi = \ker i_* = 0$, so we have $H_1(S^n) \cong H_1(D^n, S^{n-1}) \cong 0$. Finally, for $k \neq n, 1, 0$, we get

$$\begin{array}{ccccccc} H_k(D^n) & \longrightarrow & H_k(D^n, S^{n-1}) & \longrightarrow & H_{k-1}(S^{n-1}) & \longrightarrow & H_{k-1}(D^n) \\ \| & & & & \| & & \| \\ 0 & & & & 0 & & 0 \end{array}$$

so we have $H_k(S^n) \cong H_k(D^n, S^{n-1}) \cong 0$. □

Corollary 17.0.1. *∂D^n is not a retract of D^n, and any map $f : D^n \to D^n$ has a fixed point.*

Proof. If $r : D^n \to \partial D^n$ is a retract, then $r \circ i = \mathrm{id}_{\partial D^n}$, so

$$r_* \circ i_* : H_{n-1}(\partial D^n) \to H_{n-1}(D^n)$$

is the identity map. In particular, $r_* \circ i_*$ is an isomorphism, but

$$r_* : H_{n-1}(D^n) \to H_{n-1}(S^{n-1})$$

is the zero map (since $H_{n-1}(D^n) = 0$ and $H_{n-1}(S^{n-1}) = \mathbb{Z}$), a contradiction.

The second statement follows from the first as in the $n = 2$ case we have already seen. □

Corollary 17.0.2 (Invariance of domain)**.** *Let $U \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^m$. If U is homeomorphic to V, then we have $n = m$.*

Proof. For any $x \in U$, we have $\mathbb{R}^n - U \subseteq \mathbb{R}^n - \{x\} \subseteq \mathbb{R}^n$, so excision says

$$H_k(\mathbb{R}^n, \mathbb{R}^n - \{x\}) \cong H_k(\mathbb{R}^n - (\mathbb{R}^n - U), (\mathbb{R}^n - \{x\}) - (\mathbb{R}^n - U)) \cong H_k(U, U - \{x\}).$$

Consider the long exact sequence of $(\mathbb{R}^n, \mathbb{R}^n - \{x\})$:

$$\begin{array}{ccccccc} H_k(\mathbb{R}^n) & \longrightarrow & H_k(\mathbb{R}^n, \mathbb{R}^n - \{x\}) & \longrightarrow & H_{k-1}(\mathbb{R}^{n-1}) & \longrightarrow & H_{k-1}(\mathbb{R}^n) \\ \| & & & & & & \| \\ 0 & & & & & & 0 \end{array}$$

so

$$H_k(U, U - \{x\}) \cong H_k(\mathbb{R}^n, \mathbb{R}^n - \{x\}) \cong H_k(\mathbb{R}^n - \{x\}) \cong H_{k-1}(S^{n-1}) \cong \begin{cases} \mathbb{Z} & \text{if } k = n, 0, \\ 0 & \text{if } k \neq n, 0 \end{cases}$$

since $\mathbb{R}^n - \{x\} \simeq S^{n-1}$. By the same argument, we can also get

$$H_k(V, V - \{y\}) \cong \begin{cases} \mathbb{Z} & \text{if } k = m, 0, \\ 0 & \text{if } k \neq m, 0. \end{cases}$$

Now if $\phi : U \to V$ is a homeomorphism, then ϕ induces a homeomorphism of pairs

$$\phi : (U, U - \{x\}) \to (V, V - \{\phi(x)\}).$$

Then ϕ induces an isomorphism on homology, so H_k must be $\mathbb{Z}$ at the same place, i.e. $m = n$. □

Remark. The above says that if $M \neq \varnothing$ is an n-manifold, then it is not an m-manifold for any $m \neq n$.

Proposition 17.2. *We have the following:*

1. *We can identify $(D^n, \partial D^n)$ with $(\Delta^n, \partial D^n)$, and under this identification, $H_n(D^n, \partial D^n) \cong \mathbb{Z}$ is generated by the identity map $\Delta^n \to \Delta^n$.*
2. *We can identify S^n with $\Delta_1^n \cup \Delta_2^n$, where $\partial \Delta_1^n$ is glued to $\partial \Delta_2^n$ by the identity map. Then $H_n(S^n) \cong \mathbb{Z}$ is generated by $f_1 - f_2$, where $f_i : \Delta_i^n \to S^n$ is the inclusion.*

Proof. See `https://etnyre.math.gatech.edu/class/6441Spring21/Section%20IIA-C.pdf`. □

Remark. When making computations, it is sometimes useful to know that the long exact sequences we are working with "respect" maps between spaces. This is called *naturality.*

17.2 Naturality

Lemma 17.1. *If we have chain complex and chain maps such that the following diagram:*

$$\begin{tikzcd}
0 \arrow[r] & A_* \arrow[r, "i"] \arrow[d, "\alpha"] & B_* \arrow[r, "j"] \arrow[d, "\beta"] & C_* \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & A'_* \arrow[r, "i'"] & B'_* \arrow[r, "j'"] & C'_* \arrow[r] & 0
\end{tikzcd}$$

commutes, then the maps induced on the long exact sequence also commute.

Proof. Since $\beta \circ i = i' \circ \alpha$, we have $\beta_* \circ i_* = i'_* \circ \alpha_*$, and similarly for $\gamma_* \circ j_* = j'_* \circ \beta_*$. Now recall that we defined $\partial[c] = [a]$, where $a \in A_{n-1}$ such that $i(a) = \partial b$ for some $b \in B_n$ such that $j(b) = c$. Now note that $\partial(\gamma(c)) = [\alpha(a)]$ because $\gamma(c) = \gamma(j(b)) = j(\beta(b))$ and

$$i'(\alpha(a)) = \beta(i(a)) = \beta(\partial b) = \partial(\beta(b))$$

as β is a chain map. This proves the claim. □

Theorem 17.1 (Naturality)**.** *If $f : (X, A) \to (Y, B)$ is a map of pairs, then this diagram commutes:*

$$\begin{tikzcd}
\cdots \arrow[r] & H_k(A) \arrow[r, "i_*"] \arrow[d, "f_*"] & H_k(X) \arrow[r, "j_*"] \arrow[d, "f_*"] & H_k(X, A) \arrow[r, "\partial"] \arrow[d, "f_*"] & H_{k-1}(A) \arrow[r] \arrow[d, "f_*"] & \cdots \\
\cdots \arrow[r] & H_k(B) \arrow[r, "i_*"] & H_k(Y) \arrow[r, "j_*"] & H_k(Y, A) \arrow[r, "\partial"] & H_{k-1}(B) \arrow[r] & \cdots
\end{tikzcd}$$

and similarly for the long exact sequence of a triple.

Proof. Note that the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & C_k(A) \arrow[r] \arrow[d, "f_*"] & C_k(X) \arrow[r] \arrow[d, "f_*"] & C_k(X,A) \arrow[r] \arrow[d, "f_*"] & 0 \\
0 \arrow[r] & C_k(B) \arrow[r] & C_k(Y) \arrow[r] & C_k(Y,B) \arrow[r] & 0
\end{tikzcd}$$

clearly commutes, so the result follows from Lemma 17.1. $\square$

Lecture 18

Mar. 12 — Mayer-Vietoris Theorem

18.1 Mayer-Vietoris Theorem

Theorem 18.1 (Mayer-Vietoris)**.** *Let $A, B \subseteq X$ be subspaces such that $X = (\operatorname{int} A) \cup (\operatorname{int} B)$, and let*

$$\begin{tikzcd} A \cap B \arrow[r, "i_A"] \arrow[dr, "i_B"'] & A \arrow[dr, "j_A"] & \\ & B \arrow[r, "j_B"'] & X \end{tikzcd}$$

be the inclusions. Then the sequence

$$\cdots \longrightarrow H_n(A \cap B) \xrightarrow{\phi} H_n(A) \oplus H_n(B) \xrightarrow{\psi} H_n(X) \xrightarrow{\partial} H_{n-1}(A \cap B) \longrightarrow \cdots$$

is exact, where $\phi = (i_A)_* \oplus (i_B)_*$, $\psi([a],[b]) = (j_A)_*([a]) - (j_B)_*([b])$, and

$$\partial[z] = [\partial a] \quad \text{where } z = a + b \text{ for } a \in C_*(A), b \in C_*(B).$$

Remark. This is like an analogue of the Seifert-van Kampen theorem for homology.

18.2 Applications of the Mayer-Vietoris Theorem

Example 18.0.1. Consider $T^2 = S^1 \times S^1$. Let c (e.g. $\{\pi/4\} \times S^1$) be a circle around the center of the torus. Let A be a small neighborhood of c (e.g. $(0, \pi/2) \times S^1$) and $B = T^2 - c \cong S^1 \times I$. Note that $A \cap B$ is the union of two annuli. We have $A \simeq S^1$, $B \simeq S^1$, and $A \cap B \simeq S^1 \cup S^1$. Hence

$$H_k(A) \cong H_k(B) \cong \begin{cases} \mathbb{Z} & \text{if } k = 0, 1, \\ 0 & \text{if } k \neq 0, 1, \end{cases} \quad \text{and} \quad H_k(A \cap B) \cong \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & \text{if } k = 0, 1, \\ 0 & \text{if } k \neq 0, 1. \end{cases}$$

By Mayer-Vietoris, we have the exact sequence:

$$\begin{tikzcd} 0 \arrow[d, equal] & & \mathbb{Z} \oplus \mathbb{Z} \arrow[d, equal] & \mathbb{Z} \oplus \mathbb{Z} \arrow[d, equal] & & \\ H_2(A) \oplus H_2(B) \arrow[r] & H_2(X) \arrow[r, "\partial_2"] & H_1(A \cap B) \arrow[r, "\phi_1"] & H_1(A) \oplus H_1(B) & & \\ \arrow[r, "\psi_1"] & H_1(X) \arrow[r, "\partial_1"] & H_0(A \cap B) \arrow[r, "\phi_0"] \arrow[d, equal] & H_0(A) \oplus H_0(B) \arrow[r, "\psi_0"] \arrow[d, equal] & H_0(X) \arrow[r] \arrow[d, equal] & 0 \\ & & \mathbb{Z} \oplus \mathbb{Z} & \mathbb{Z} \oplus \mathbb{Z} & \mathbb{Z} & \end{tikzcd}$$

Note that ϕ_0 is induced by the inclusion, so $\phi_0(1,0) = \phi_0(0,1) = (1,1)$, and similarly one can also see that $\phi_1(1,0) = \phi_1(0,1) = (1,1)$. Since $H_2(A) \cong H_2(B) \cong 0$, we have $\ker \partial_2 = 0$ and so

$$\langle (1,-1) \rangle \cong \mathbb{Z} \cong \ker \phi_1 = \operatorname{im} \partial_2 = \frac{H_2(X)}{\ker \partial_2} = H_2(X).$$

Next, we have $\ker \psi_1 = \operatorname{im} \phi_1 \cong \mathbb{Z}$, which is generated by $(1,1)$. Then

$$\operatorname{im} \psi_1 \cong \frac{\mathbb{Z} \oplus \mathbb{Z}}{\ker \psi_1} \cong \frac{\mathbb{Z} \oplus \mathbb{Z}}{\mathbb{Z}} \cong \frac{\mathbb{Z} \oplus \mathbb{Z}}{\langle (1,1) \rangle} \cong \mathbb{Z}.$$

We also have $\operatorname{im} \partial_1 = \ker \phi_) \cong \mathbb{Z}$ (generated by $(1,-1)$, as before), so

$$\mathbb{Z} \cong \operatorname{im} \partial_1 \cong \frac{H_1(X)}{\ker \partial_1} \cong \frac{H_1(X)}{\operatorname{im} \psi_1} \cong \frac{H_1(X)}{\mathbb{Z}}.$$

Check as an exercise that this implies $H_1(X) \cong \mathbb{Z} \oplus \mathbb{Z}$. Also verify that $H_k(X) \cong 0$ for $k \geq 3$, so

$$H_k(T^2) = \begin{cases} \mathbb{Z} & \text{if } k = 0, 2, \\ \mathbb{Z} \oplus \mathbb{Z} & \text{if } k = 1, \\ 0 & \text{if } k \geq 3. \end{cases}$$

18.3 Proof of the Mayer-Vietoris Theorem

Lemma 18.1. *Consider two long exact sequences and maps between them as below:*

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1} \arrow[r] \arrow[d, "\gamma_{n+1}"] & A_n \arrow[r, "f_n"] \arrow[d, "\alpha_n"] & B_n \arrow[r, "g_n"] \arrow[d, "\beta_n"] & C_n \arrow[r, "h_n"] \arrow[d, "\gamma_n"] & A_{n-1} \arrow[r] \arrow[d, "\alpha_{n-1}"] & \cdots \\
\cdots \arrow[r] & C'_{n+1} \arrow[r] & A'_n \arrow[r, "f'_n"] & B'_n \arrow[r, "g'_n"] & C'_n \arrow[r, "h'_n"] & A'_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

so that the diagram commutes. If γ_n is an isomorphism, then the sequence

$$\cdots \longrightarrow A_n \xrightarrow{\Phi_n} A'_n \oplus B_n \xrightarrow{\Psi_n} B'_n \xrightarrow{\Gamma_n} A_{n-1} \longrightarrow \cdots$$

is exact, where Φ, Ψ, Γ are defined by

$$\Phi_n(a) = (\alpha_n(a), f_n(a)) \quad \Psi_n(a', b) = \beta_n(b) - f'_n(a) \quad \Gamma_n(b') = h_n \circ \gamma_n^{-1} \circ g'_n(b').$$

Proof. We check exactness at B'_n (i.e. $\operatorname{im} \Psi_n = \ker \Gamma_n$), the rest is left as an exercise. We can see that

$$\Gamma_n \circ \Psi_n(a', b) = \Gamma_n(\beta_n(b) - f'_n(a)) = h_n \circ \gamma_n^{-1} \circ g'_n(\beta_n(b) - f'_n(a)) = h_n(\gamma_n^{-1}(g'_n(\beta_n(b))))$$

since $g'_n \circ f'_n = 0$ by assumption. We can then write $(\beta_n \circ \gamma'_n = \gamma_n \circ g_n)$

$$\Gamma_n \circ \Psi_n(a', b) = h_n(\gamma_n^{-1}(\gamma_n(g_n(b)))) = h_n(g_n(b)) = 0$$

since $h_n \circ g_n$ by assumption. So $\operatorname{im} \Psi_n \subseteq \ker \Gamma_n$. For the reverse inclusion, let $b' \in \ker \Gamma_n$, so $\gamma_n^{-1} \circ g'_n(b')$ lies in $\ker h_n$. Then there exists $b \in B_n$ such that $g_n(b) = \gamma_n^{-1} \circ g'_n(b')$, so (write $g'_n(b') = \gamma_n \circ g_n(b)$)

$$g'_n(\beta_n(b) - b') = g'_n(\beta_n(b)) - g'_n(b') = \gamma_n(g_n(b)) - g'_n(b') = 0.$$

So there exists $a' \in A'_n$ such that $f'_n(a') = \beta_n(b) - b'$, and thus $b' = \beta_n(b) - f'_n(a') = \Psi_n(a', b)$. $\square$

Proof Theorem 18.1. Consider the long exact sequence of the pairs $(A, A\cap B)$ and (X, B):

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & H_n(A\cap B) & \xrightarrow{(i_A)_*} & H_n(A) & \xrightarrow{(j_A)_*} & H_n(A, A\cap B) & \longrightarrow & H_{n-1}(A\cap B) & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle (i_B)_*} & & \downarrow{\scriptstyle (j_A)_*} & & \downarrow{\scriptstyle I_*} & & \downarrow{\scriptstyle (i_B)_*} & & \\
\cdots & \longrightarrow & H_n(B) & \xrightarrow{(j_B)_*} & H_n(X) & \longrightarrow & H_n(X, B) & \longrightarrow & H_{n-1}(B) & \longrightarrow & \cdots
\end{array}$$

Verify as an exercise that $X/B \cong A/(A\cap B)$. So if $(A, A\cap B)$ and (X, B) are good pairs, then I_* is an isomorphism, since excision allows us to write

$$H_n(A, A\cap B) \cong H_n(A/(A\cap B)) \cong H_n(X/B) \cong H_n(X, B).$$

From here, Mayer-Vietoris (in the case that $(A, A\cap B)$ and (X, B) are good pairs) follows from the lemma. In general, I_* is still an isomorphism even when $(A, A\cap B)$ and (X, B) are not good pairs, but the argument is much more difficult (see Hatcher for the details). □

18.4 Proof of Excision

Proof of Theorem 16.1. The main idea is the following: For any n and $\alpha = \sum m_i\sigma_i \in C_n(X, A)$, we can find $\beta \in C_{n+1}(X, A)$ such that $\alpha + \partial\beta = \sum_{i=1}^{\ell} m_i\tau_i$, where $\operatorname{im}\tau_1 \subseteq X - Z$ or $\operatorname{im}\tau_1 \subseteq A$.

Call the above statement $(*)$, and we will prove it later. We will first prove excision assuming $(*)$.

To see that i_* is onto, let $[\alpha] \in H_n(X, A)$. Then $(*)$ implies that there is $\alpha' \in [\alpha]$ such that $\alpha' = \sum_i m_i\tau_i$, where the τ_i are as in $(*)$. Define $\alpha'' = \sum_{i, \operatorname{im}\tau_i \not\subseteq A} m_i\tau_i$. Note the following:

1. $\alpha'' = \alpha$ in $C_n(X)/C_n(A) = C_n(X, A)$.
2. $\partial\alpha'' \in C_{n-1}(A)$ (in fact is contained in $C_{n-1}(A - Z)$).

   To see this, note that $-\partial\alpha'' = \partial(a' - \alpha'') - \partial\alpha'$, where

   $$\alpha'' \subseteq X - Z \quad \text{and} \quad \alpha' - \alpha'', \alpha' \subseteq A,$$

   so $\partial\alpha'' \subseteq X - Z$ and $\partial(a' - \alpha''), \partial\alpha' \subseteq A$, and thus $\partial\alpha'' \subseteq A - Z$.
3. $\alpha'' \in C_n(X - Z)/C_n(A - Z)$

Thus α'' defines an element $[\alpha''] \in H_n(X - Z, A - Z)$. Clearly $i_*([\alpha'']) = [\alpha'] = [\alpha]$, so i_* is onto.

To see that i_* is injective, suppose that $[\alpha] \in H_n(X - Z, A - Z)$ and $i_*([\alpha]) = 0$, so

$$i \circ \alpha = \alpha = \partial\beta, \quad \text{for some } \beta \in C_{n+1}(X, A).$$

Then $(*)$ implies that there exists $\gamma \in C_{n+2}(X, A)$ such that $\beta + \partial\gamma = \sum m_i\tau_i$, where the τ_i are as in $(*)$. Clearly $\alpha = \partial(\beta + \partial\gamma)$. Again define $\beta' = \sum_{i, \operatorname{im}\tau_i \not\subseteq A} m_i\tau_i$, and note:

1. $\beta' \in C_n(X - Z)$.
2. $\partial\beta' = \partial\beta +$ terms with image in $A \cap (X - A) = A - Z$, so $\partial\beta' = \alpha$ in $C_n(X - Z, A - Z)$.

Thus $[\alpha] = 0$ in $H_n(X - Z, A - Z)$, so i_* is injective. This proves excision given $(*)$

We now proceed to prove $(*)$. We will need *barycentric subdivision*, defined as follows:

0. subdivision of 0-simplex: do nothing;
1. subdivision of 1-simplex: divide into two equal pieces;
2. subdivision of 2-simplex: subdivide the edges $[e_0, e_1]$, $[e_1, e_2]$, $[e_2, e_0]$ of the triangle to get vertices f_0, f_1, f_2, add a vertex e at the center of the triangle, and add (6) edges between e and the e_i, f_i:
$$[e_0, e_1, e_2] = [e_0, f_0, e] \cup [f_0, e_1, e] \cup \dots;$$
3. subdivision of n-simplex ($n \geq 3$): subdivide all faces, add center point, add all edges, add all 2-simplices, ..., add all $(n-1)$-simplices.

Show as an exercise that "up to boundaries, we can subdivide simplices," i.e. if $\sigma : \Delta^n \to X$ is a singular n-simplex and $\Delta^n = \Delta^n_1 \cup \cdots \cup \Delta^n_n$ is its barycentric subdivision, then there is an $(n+1)$-chain τ with
$$\sigma + \partial\tau = \sum_{i=1}^{k} (\pm\sigma|_{\Delta^n_i}).$$

Note that if we continually barycentrically subdivide, the subsimplices have diameter $d \to 0$. Now given $\sigma : \Delta^n \to X$, note that $\{\sigma^{-1}(\operatorname{int} A), \sigma^{-1}(X - \overline{Z})\}$ is an open cover of Δ^n. Since Δ^n is a compact metric space, there exists a Lebesgue number $\delta > 0$ for the cover. So if we subdivide enough, all subsimplices are either in $\sigma^{-1}(\operatorname{int} A)$ or $\sigma^{-1}(X - \overline{Z})$, which proves $(*)$. $\square$

Lecture 19

Mar. 24 — Degree

19.1 Degree

Definition 19.1. Given a map $f : S^n \to S^n$, we get an induced map $f_* : \widetilde{H}_n(S^n) \to \widetilde{H}_n(S^n)$ (recall that $\widetilde{H}_n(S^n) \cong \mathbb{Z}$). Then the *degree* of f, denoted $\deg f$, is $f_*(1) \in \mathbb{Z}$.

Remark. Note the following:

1. $\deg(\mathrm{id}_{S^n}) = 1$.
2. $\deg f$ only depends on f up to homotopy (since $f_* = g_*$ if $f \simeq g$).
3. If f is not surjective, then $\deg f = 0$.

   To see this, note that if f is not surjective, then there exists $p \in S^n$ such that $p \notin \operatorname{im} f$. Then

   $$\begin{tikzcd} S^n \arrow[r, "f"] \arrow[dr, "\widetilde{f}"'] & S^n \\ & S^n \setminus \{f(p)\} \arrow[u, hook, "i"'] \end{tikzcd} \qquad \begin{tikzcd} \widetilde{H}_n(S^n) \arrow[r, "f_*"] \arrow[dr, "\widetilde{f}_*"'] & \widetilde{H}_n(S^n) \\ & \widetilde{H}_n(S^n \setminus \{f(p)\}) \arrow[u, hook, "i_*"'] \end{tikzcd}$$

   Since $\widetilde{H}_n(S^n \setminus \{f(p)\}) = 0$, we have $f_*(1) = i_*(\widetilde{f}_*(1)) = 0$.
4. If f is a reflection (e.g. $f(x_0, \ldots, x_n) = (-x_0, x_1, \ldots, x_n)$), then $\deg f = -1$.

   To see this, we induct on n. For $n = 0$, we have $S^0 = \{\pm 1\}$ and

   $$H_0(S^0) = H_0(\{1\}) \oplus H_0(\{-1\}) = \mathbb{Z} \oplus \mathbb{Z},$$

   where $f_*(a, b) = (b, a)$. Recall that the reduced homology is computed with

   $$C_1(S^0) \xrightarrow{\partial_0} C_0(S^0) \xrightarrow{\varepsilon} \mathbb{Z}$$

   where ε is given by $\sum m_i x_i \mapsto \sum m_i$. Then we defined $\widetilde{H}_0(S^0) = \ker \varepsilon / \operatorname{im} \partial_0$. Note that we have

   $$\begin{aligned} \varepsilon : \mathbb{Z} \oplus \mathbb{Z} &\longrightarrow \mathbb{Z} \\ (a, b) &\longmapsto a + b \end{aligned}$$

   so $\ker \varepsilon$ (hence also $\widetilde{H}_0(S^0) \cong \mathbb{Z}$) is generated by $(1, -1)$. Thus $f_*(1, -1) = (-1, 1) = -(1, -1)$, so $\deg f = -1$. Now assume the result is true for S^k with $k < n$. Let

   $$D^n_\pm = \{(x_0, \ldots, x_n) \in S^n : \pm x_n \geq 0\}$$

and note that f preserves $D^n_\pm$. Since (S^n, D^n_+) is a good pair, $\widetilde{H}_n(S^n, D^n_+) \cong \widetilde{H}_n(S^n/D^n_+)$, and

$$\begin{tikzcd}
\widetilde{H}_n(S^n) \arrow[r, "\cong"] \arrow[d, "f_*"] & H_n(S^n, D^n_+) \arrow[r, "\cong"] \arrow[d, "f_*"] & H_n(D^n_-, \partial D^n_-) \arrow[r, dashed, "\cong"] \arrow[d, "f_*"] & H_{n-1}(\partial D^n_-) \arrow[d, "f_*"] \\
\widetilde{H}_n(S^n) \arrow[r, "\cong"] & H_n(S^n, D^n_+) \arrow[r, "\cong"] & H_n(D^n_-, \partial D^n_-) \arrow[r, dashed, "\cong"] & H_{n-1}(\partial D^n_-)
\end{tikzcd}$$

where $\partial D^n_- \cong S^{n-1}$. To see that $H_n(D^n_-, \partial D^n_-) \cong H_{n-1}(\partial D^n_-)$, note that for $n > 1$,

$$\begin{tikzcd}
H_n(D^n_-) \arrow[r] \arrow[d, equal] & H_n(D^n_-, \partial D^n_-) \arrow[r, "\partial"] & H_{n-1}(\partial D^n_-) \arrow[r] & H_{n-1}(D^n_-) \arrow[d, equal] \\
0 & & & 0
\end{tikzcd}$$

is exact, so ∂ is an isomorphism. Check the $n = 1$ case as an exercise.

This completes the proof by induction since we already know the result for $\partial D^n_- \cong S^{n-1}$.

5. If f is the antipodal map (i.e. $f(x_0, \ldots, x_n) = (-x_0, \ldots, -x_n)$), then $\deg f = (-1)^{n+1}$.

   This follows from the previous result, since the antipodal map is a composition of reflections.

Lemma 19.1. *Let $f, g : X \to S^n \subseteq \mathbb{R}^{n+1}$. If $f(x) \neq -g(x)$ for all$x \in X$, then $f \simeq g$.*

Proof. Define the homotopy $H : X \times [0,1] \to S^n$ by

$$(x,t) \longmapsto \frac{tf(x) + (1-t)g(x)}{\|tf(x) + (1-t)g(x)\|},$$

which is well-defined since $g(x) \neq -f(x)$. □

Corollary 19.0.1. *Let $f : S^n \to S^n$. Then*

1. *if f has no fixed points, then* $\deg f = (-1)^{n+1}$*;*
2. *if there is no point $x \in S^n$ such that $f(x) = -x$, then* $\deg f = 1$.

Proof. Apply the lemma to f and the antipodal map (resp. identity map). □

Corollary 19.0.2. *If n is even, then any map $f : S^n \to S^n$ has a fixed point or an antipodal point (i.e. $f(x) = -x$).*

Proof. If f has neither, then $1 = \deg f = -1$, a contradiction. □

Corollary 19.0.3. *The sphere S^n has a non-zero vector field if and only if n is odd.*

Proof. ($\Rightarrow$) We prove the contrapositive. If n is even, consider a vector field v, so $v(x)$ is a vector in $\langle x \rangle^\perp$ for each $x \in S^n$. If $v(x) \neq 0$ for all x, then $f : S^n \to S^n$ given by

$$v(x) \mapsto \frac{v(x)}{\|v(x)\|}$$

is a function with no fixed points or antipodal points, a contradiction.

($\Leftarrow$) If n is odd, then $v(x_0, x_1, \ldots, x_{n-1}, x_n) = (x_1, -x_0, \ldots, x_n, -x_{n-1})$ is a non-zero vector field. □

Remark. The case $n = 2$ is the *hairy ball theorem.*

Remark. It is true that $f, g : S^n \to S^n$ are homotopic if and only if $\deg f = \deg g$.

19.2 Computing the Degree

Definition 19.2. Let $n > 0$ and $f : S^n \to S^n$. Take a point $y \in S^n$ such that $f^{-1}(\{y\})$ consists of finitely many points $x_1, \ldots, x_k$. Then for $n \neq 1$ (check $n = 1$ as an exercise), we have the exact sequence

$$\begin{tikzcd} H_n(S^n - \{y\}) \arrow[r] \arrow[d, equal] & H_n(S^n) \arrow[r, "i_*"] & H_n(S^n, S^n - \{y\}) \arrow[r] & H_{n-1}(S^n - \{y\}) \arrow[d, equal] \\ 0 & & & 0 \end{tikzcd}$$

So i_* is an isomorphism $H_n(S^n) \to H_n(S^n, S^n - \{y\})$, and similarly $j_* : H_n(S^n) \to H_n(S^n, S^n - \{x_i\})$ is also an isomorphism. Let V be a neighborhood of y and U_i be a neighborhood of x_i such that $f(U_i) \subseteq V$ and $x_j \notin U_i$ for $i \neq j$. Then by excision, we have an isomorphism

$$H_n(S^n) \cong H_n(S^n, S^n - \{y\}) \cong H_n(S^n - (S^n - V), (S^n - \{y\}) - (S^n - V)) \cong H_n(V, V - \{y\})$$

and similarly for $H_n(U_i, U_i - \{x_i\})$. Now f gives a map $f_* : H_n(U_i, U_i - \{x_i\}) \to H_n(V, V - \{y\})$, which we can view as a map $f_* : \mathbb{Z} \to \mathbb{Z}$. Define the *local degree of f at x_i* to be $\deg(f, x_i) = f_*(1) \in \mathbb{Z}$ (choose the generators using the isomorphism with $H_n(S^n)$).

Remark. Note the following:

- If we change V or U_i, we get the same number (as long as $f(U_i) \subseteq V$).
- If $f|_{U_i} : U_i \to f(U_i)$ is a homeomorphism, then take $V = f(U_i)$ and we see that

$$(f|_{U_i})_* : H_n(U_i, U_i - \{x_i\}) \to H_n(V, V - \{y\})$$

  is an isomorphism, so if we view $(f|_{U_i})_*$ as a map $\mathbb{Z} \to \mathbb{Z}$, then $1 \mapsto \pm 1$. So $\deg(f, x_i) = \pm 1$.

Lemma 19.2. *With $f : S^n \to S^n$, y, $x_1, \ldots, x_k$ as above,* $\deg f = \sum_{i=1}^k \deg(f, x_i)$.

Proof. Choose the U_i to be disjoint and set $Z = S^n - \bigcup_{i=1}^k U_i$. Then by excision,

$$\begin{aligned} H_n(S^n, S^n - f^{-1}(\{y\})) &= H_n(S^n, S^n - \{x_1, \ldots, x_k\}) \cong H_n(S^n - Z, (S^n - \{x_1, \ldots, x_k\}) - Z) \\ &= H_n\left(\bigcup_{i=1}^k U_i, \bigcup_{i=1}^k (U_i - \{x_i\})\right) \cong \bigoplus_{i=1}^k H_n(U_i, U_i - \{x_i\}). \end{aligned}$$

So we have the diagram (let g be the composition map $H_n(S^n) \to \bigoplus_{i=1}^k H_n(U_i, U_i - \{x_i\})$)

$$\begin{tikzcd} H_n(S^n) \arrow[rr, "f_*"] \arrow[d, "i_*"] & & H_n(S^n) \arrow[d, "i_*", "\cong"'] \\ H_n(S^n, S^n - f^{-1}(\{y\})) \arrow[d, "\cong"] & & H_n(S^n, S^n - \{y\}) \arrow[d, "\cong"] \\ \bigoplus_{i=1}^k H_n(U_i, U_i - \{x_i\}) \arrow[rr, "\bigoplus (f|_{U_i})_*"'] & & H_n(V, V - \{y\}) \end{tikzcd}$$

Note that $H_n(S^n) \to H_n(U_i, U_i - \{x_i\})$ (viewed as $\mathbb{Z} \to \mathbb{Z}$) sends $1 \mapsto 1$, so $g(1) = (1, 1, \ldots, 1)$ and

$$\deg f = f_*(1) = \Big(\bigoplus (f|_{U_i})_*\Big) \circ g_*(1) = \bigoplus (f|_{U_i})_*(1) = \sum_{i=1}^{k} \deg(f, x_i),$$

which is the desired result. □

Example 19.2.1. Consider the following:

1. Let $n > 0$ and $f_n : S^1 \to S^1$ be the map $z \mapsto z^n$, where S^1 is viewed as the unit circle in $\mathbb{C}$. Let $y = 1$ and pick $x_1, \ldots, x_n$ evenly spaced around the circle with $x_1 = y$. Choose the U_i such that $f|_{U_i} : U_i \to V$ is a homeomorphism, so $\deg(f_n, x_i) = \pm 1$. We can extend $f|_{U_i} : U_i \to V$ to a homeomorphism $g_i : S^1 \to S^1$. One can shows as an exercise that g_i must be homotopic to id_{S^1}. Since $n > 0$, we have
$$1 = \deg g_i = \deg(g_i, x_i) = \deg(f, x_i),$$
so all local degrees are 1 and $\deg f = n$. Now if $n < 0$, let $f_n = f_{-n} \circ r$ for some reflection r, so
$$\deg f_n = (\deg f_{-n})(\deg r) = (-n)(-1) = n.$$
Thus we see that $\deg f_n = n$ for all $n \neq 0$.

2. Now let $D_1, \ldots, D_k$ be disjoint disks in S^n for $n > 1$. Let $C = S^n - \bigcup_{i=1}^k D_i$, then
$$X = S^n / C \cong \bigvee_{i=1}^{k} S^n.$$
Let $q : S^n \to X$ be the quotient map. Let $U = X - \{\text{wedge point}\}$ and V be a neighborhood of the wedge point, and note that $U \cap V \cong \coprod_{i=1}^n (S^{n-1} \times (0, 1))$. Mayer-Vietoris gives the exact sequence
$$\begin{array}{ccccccc} H_n(U) \oplus H_n(V) & \longrightarrow & H_n(X) & \longrightarrow & H_{n-1}(U \cap V) & \longrightarrow & H_{n-1}(U) \oplus H_{n-1}(V) \\ \| & & & & \| & & \| \\ 0 & & & & \bigoplus_{i=1}^k \mathbb{Z} & & 0 \end{array}$$
so we see that $H_n(X) \cong \bigoplus_{i=1}^k \mathbb{Z}$. Now let $f_i : X \to S^n$ collapse all but the ith copy of S^n to a point. Then we claim that $(f_i)_* : H_n(X) \to H_n(S^n)$ maps $(m_1, \ldots, m_k) \mapsto m_i$. To see this, consider the map $j_i : S^n \to X$ which includes S^n into the ith S^n in X. Then
$$(f_\ell \circ j_i)_*(1) = 0 \quad \text{if } i \neq \ell,$$
whereas $f_i \circ j_i$ is the identity on S^n, so $(f_i \circ j_i)_*(1) = 1$. So we see that
$$(j_i)_*(1) = (0, \ldots, 0, 1, 0, \ldots, 1) = 0 \quad \text{and} \quad (f_i)_*(0, \ldots, 0, 1, 0, \ldots, 0) = 1,$$
where the 1 appears in the ith slot. Then let $f : X \to S^n$ be f_n on the ith sphere, so $f(1, \ldots, 1) = k$ and we see that $(f_* \circ q_*)(1) = k$, i.e. the degree is k.

Lecture 20

Mar. 26 — Cellular Homology

20.1 Cellular Homology

Definition 20.1. Let X be a CW complex. Define the *cellular chain groups*

$$C_n^{\mathrm{CW}}(X) = \text{free abelian group generated by the } n\text{-cells } e_1^n, \ldots, e_{\ell_n}^n.$$

Let $f_i^n : \partial e_i^n \to X^{(n-1)}$ be the attaching map for e_i^n. Let e_i^n be an n-cell and e_j^{n-1} be an $(n-1)$-cell. Let $g_{ij} : S^{n-1} \to S^{n-1}$ be the composition of the following maps:

$$S^{n-1} \xrightarrow{\cong} \partial e_i^n \xrightarrow{f_i^n} X^{(n-1)} \xrightarrow{q} X^{(n-1)}/X^{(n-2)} \xrightarrow{\cong} \bigvee\nolimits_{\ell_{n-1}} S^{n-1} \xrightarrow{p_j} S^{n-1}$$

Define the boundary map as follows: For $n > 1$, set

$$\begin{aligned} \partial_n^{\mathrm{CW}} : C_n^{\mathrm{CW}}(X) &\longrightarrow C_{n-1}^{\mathrm{CW}}(X) \\ e_i^n &\longmapsto \sum_{j=1}^{\ell_{n-1}} (\deg g_{ij}) e_j^{n-1} \end{aligned}$$

and for $n = 1$, define $\partial_1^{\mathrm{CW}} e_i^1 = \partial e_i^1$ where ∂ is the singular boundary map.

Remark. If X has only one 0-cell, then $\partial_1^{\mathrm{CW}} = 0$.

Theorem 20.1. *We have $\partial_n^{\mathrm{CW}} \circ \partial_{n+1}^{\mathrm{CW}} = 0$ and moreover,*

$$H_n(X) \cong \frac{\ker \partial_n^{\mathrm{CW}}}{\operatorname{im} \partial_{n+1}^{\mathrm{CW}}}.$$

Definition 20.2. Define the *cellular homology groups* to be

$$H_n^{\mathrm{CW}}(X) = \frac{\ker \partial_n^{\mathrm{CW}}}{\operatorname{im} \partial_{n+1}^{\mathrm{CW}}}.$$

The previous theorem says that $H_n(X) \cong H_n^{\mathrm{CW}}(X)$ if X is a CW complex.

20.2 Computations of Cellular Homology

Example 20.2.1. Give the torus T^2 the CW complex structure with one 0-cell e_1^0, two 1-cells e_1^1, e_2^1, and one 2-cell e_1^2. Viewing T^2 as a square with opposite sides identified (and pointing to the right and up), let e_1^0 be the bottom-left corner, e_1^1 and e_1^2 be the bottom and left sides, and e_1^2 be the interior of the square. So can we write the cellular chain complex as follows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C_2^{\mathrm{CW}}(T^2) & \xrightarrow{\partial_2^{\mathrm{CW}}} & C_1^{\mathrm{CW}}(T^2) & \xrightarrow{\partial_1^{\mathrm{CW}}} & C_0^{\mathrm{CW}}(T^2) & \longrightarrow & 0 \\ & & \| & & \| & & \| & & \\ & & \mathbb{Z} = \langle e_1^2 \rangle & & \mathbb{Z} \oplus \mathbb{Z} = \langle e_1^1, e_2^1 \rangle & & \mathbb{Z} = \langle e_1^0 \rangle & & \end{array}$$

Since T^2 has one 0-cell, we have $\partial_1^{\mathrm{CW}} = 0$. To compute $\partial_2^{\mathrm{CW}} e_1^2$, note that the map g_{11} is given by

$$S^1 \xrightarrow{\cong} \partial e_1^2 \xrightarrow{f_i^n} X^{(1)} \xrightarrow{q} X^{(1)}/X^{(0)} \xrightarrow{\cong} e_1^2 \vee e_1^1 \xrightarrow{p_j} e_1 \xrightarrow{\cong} S^1$$

Consider the points $x_1 = (1, 1/4)$ and $x_2 = (1, 3/4)$ on the boundary of the square, viewed as $[0,1]^2 \subseteq \mathbb{R}^2$. These points are both mapped to the same point y under g_{11}, and note that at x_1 the direction on ∂e_1^2 and $S^1 = e_1^1/\partial e_1^1$ agree while at x_2 they disagree. Thus from our previous computations, we see $\deg(g_{11}, x_1) = 1 = -\deg(g_{11}, x_2)$, so $\deg(g_{11}) = 0$. Similarly, $\deg g_{12} = 0$, and so $\partial_2^{\mathrm{CW}} = 0$. So

$$H_n^{\mathrm{CW}}(T^2) = C_n^{\mathrm{CW}}(T^2) = \begin{cases} \mathbb{Z} & \text{if } n = 0, 2, \\ \mathbb{Z} \oplus \mathbb{Z} & \text{if } n = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Exercise 20.1. Let Σ_g be the surface of genus g. Use cellular homology to verify that

$$H_n(\Sigma_g) = \begin{cases} \mathbb{Z} & \text{if } n = 0, 2, \\ \bigoplus_{2g} \mathbb{Z} & \text{if } n = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Remark. Consider the following:

1. $H_k(X)$ has at most ℓ_k generators, in particular $H_k(X) = 0$ if X has no k-cells.
2. If X has cells only in even dimensions, then $H_n(X) = C_n^{\mathrm{CW}}(X)$.

Example 20.2.2. Recall that $\mathbb{C}P^n$ has a CW decomposition with one cell in each even dimension between 0 and $2n$, so

$$H_n(\mathbb{C}P^n) = \begin{cases} \mathbb{Z} & \text{if } n = 0, 2, \ldots, 2n, \\ 0 & \text{otherwise.} \end{cases}$$

Example 20.2.3. Let a, b be two circles and $X = (a \vee b) \cup (\text{two 2-cells } e_1^2, e_2^2)$, where e_1^2 is attached along $a^5 b^{-3}$ and e_2^2 is attached along $b^3(ab)^{-2}$. As above, we get

$$\begin{array}{ccccc} C_2^{\mathrm{CW}}(X) & \xrightarrow{\partial_2^{\mathrm{CW}}} & C_1^{\mathrm{CW}}(X) & \xrightarrow{\partial_1^{\mathrm{CW}}} & C_0^{\mathrm{CW}}(X) \\ \| & & \| & & \| \\ \mathbb{Z} \oplus \mathbb{Z} = \langle e_1^2, e_2^2 \rangle & & \mathbb{Z} \oplus \mathbb{Z} = \langle a, b \rangle & & \mathbb{Z} \end{array}$$

We can compute that $\partial_2^{\mathrm{CW}} e_1^2 = 5a - 3b$ and $\partial_2^{\mathrm{CW}} e_2^2 = -2a + b$, and so

$$\partial_2^{\mathrm{CW}} = \begin{pmatrix} 5 & -3 \\ -2 & 1 \end{pmatrix}.$$

This matrix is invertible over $\mathbb{Z}$, so ∂_2^{CW} is an isomorphism. We already know $\partial_1^{\mathrm{CW}} = 0$, so

$$H_k(X) = \begin{cases} \mathbb{Z} & \text{if } k = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Remark. Note that in the previous example, X has the homology of a point. But X is not contractible: One can use the van Kampen theorem to see that

$$\pi_1(X, x_0) \cong \langle a, b | a^5 b^{-3}, b^3 (ab)^{-2} \rangle,$$

which one can show is a group of order 120. This shows that π_1 "sees" things that H_k does not for any k (of course, H_k also "sees" things π_1 cannot, e.g. S^k is not contractible since $H_k(S^k) \neq 0$).

20.3 Proof of Cellular Homology

Lemma 20.1. *If X is a CW complex, then*

1. *If ℓ_k is the number of k-cells in X, then*

$$H_k(X^{(n)}, X^{(n-1)}) = \begin{cases} \bigoplus_{\ell_k} \mathbb{Z} & \text{if } k = n, \\ 0 & \text{otherwise.} \end{cases}$$

2. *$H_k(X^{(n)}) = 0$ for all $k > n$.*
3. *If $i : X^{(n)} \to X$ is the inclusion map, then $i_* : H_k(X^{(n)}) \to H_k(X)$ is an isomorphism for $k < n$.*

Proof. (1) Note that $(X^{(n)}, X^{(n-1)})$ is a good pair, so

$$H_k(X^{(n)}, X^{(n-1)}) \cong \widetilde{H}_k(X^{(n)}/X^{(n-1)}) = H_k\left(\bigvee_{\ell_n} S^n\right),$$

so the statement holds for $n \geq 1$. One can check the $n = 0$ separately as an exercise.

(2) We have the long exact sequence of a pair: For $k \neq n, n-1$,

$$\begin{array}{ccccccc} H_k(X^{(n)}, X^{(n-1)}) & \longrightarrow & H_k(X^{(n-1)}) & \longrightarrow & H_K(X^{(n)}) & \longrightarrow & H_{k-1}(X^{(n)}, X^{(n-1)}) \\ \| & & & & \| & & \\ 0 & & & & 0 & & \end{array}$$

so $H_k(X^{(n)}) \cong H_k(X^{(n-1)})$ for $k \neq n, n-1$. So for $k > n$, we have

$$H_k(X^{(n)}) \cong H_k(X^{(n-1)}) \cong \ldots \cong H_k(X^{(0)}) = 0.$$

(3) If $k < n$, then

$$H_k(X^{(n)}) \cong H_k(X^{(n+1)}) \cong \ldots \cong H_k(X^{(n+\ell)})$$

for any ℓ. So if X is a finite-dimensional CW complex, then clearly $H_k(X^{(n)}) \cong H_k(X)$ (if instead X is infinite-dimensional, then we need the fact that homology commutes with *direct limits*). $\square$

Proof of Theorem 20.1. By the lemma, we know that $C_n^{\mathrm{CW}}(X) \cong H_n(X^{(n)}, X^{(n-1)})$. Consider the long exact sequence of the triple $(X^{(n+1)}, X^{(n)}, X^{(n-1)})$:

$$\ldots \longrightarrow H_{n+1}(X^{(n+1)}, X^{(n-1)}) \longrightarrow H_{n+1}(X^{(n+1)}, X^{(n)}) \xrightarrow{d_{n+1}} H_n(X^{(n)}, X^{(n-1)}) \longrightarrow \ldots$$

We claim that $\partial_n^{\mathrm{CW}} = d_n$, which we will check later. We first see that the theorem follows from this claim: Consider the long exact sequence of $(X^{(n+1)}, X^{(n)})$ and $(X^{(n)}, X^{(n-1)})$:

$$\begin{tikzcd}
 & 0 \arrow[d, equal] & & \\
C_{n+1}^{\mathrm{CW}}(X) \arrow[d, equal] & H_n(X^{(n-1)}) \arrow[d] & H_n(X) \arrow[d, equal] & 0 \arrow[d, equal] \\
H_{n+1}(X^{(n+1)}, X^{(n)}) \arrow[r, "\partial_{n+1}"] \arrow[dr, bend right, "j_n \circ \partial_{n+1}"'] & H_n(X^{(n)}) \arrow[r] \arrow[d, "j_n"] & H_n(X^{(n+1)}) \arrow[r] & H_n(X^{(n+1)}, X^{(n)}) \\
 & H_n(X^{(n)}, X^{(n-1)}) \arrow[r, equal] \arrow[d, "\partial_n"] & C_n^{\mathrm{CW}}(X) & \\
 & H_{n-1}(X^{(n-1)}) & &
\end{tikzcd}$$

Show as an exercise that $j_n \circ \partial_{n+1} = d_{n+1}$. (Hint: This is just a diagram chase. One can see that the choices used to define d_{n+1} and ∂_{n+1} are the same.) Then we have

$$d_n \circ d_{n+1} = j_{n-1} \circ \partial_n \circ j_n \circ \partial_{n+1} = j_{n-1} \circ 0 \circ \partial_{n+1} = 0$$

since ∂_n, j_n are adjacent terms in the long exact sequence. So we can consider $\ker d_n / \operatorname{im} d_{n+1}$. From the diagram above, we see that $H_n(X) = H_n(X^{(n)})/\operatorname{im} \partial_{n+1}$. Note that j_n is injective, so we have

$$\operatorname{im} \partial_{n+1} \cong j_n(\operatorname{im} \partial_{n+1}) = \operatorname{im}(j_n \circ \partial_{n+1}) = \operatorname{im} d_{n+1}.$$

Since j_{n-1} is also injective, we also see that

$$H_n(X^{(n)}) \cong \operatorname{im} j_n = \ker \partial_n = \ker(j_{n-1} \circ \partial_n) = \ker d_n.$$

Then by the claim that $\partial_n^{\mathrm{CW}} = d_n$, we have

$$H_n(X) \cong \frac{H_n(X^{(n)})}{\operatorname{im} \partial_{n+1}} \cong \frac{\ker d_n}{\operatorname{im} d_{n+1}} = \frac{\ker \partial_n^{\mathrm{CW}}}{\operatorname{im} \partial_{n+1}^{\mathrm{CW}}},$$

which proves the theorem. So it suffices to prove the claim.

To prove the claim, note that $i : (e_i^n, \partial e_i^n) \to (X^{(n)}, X^{(n-1)})$ is given by an "inclusion" (technically the inclusion is into $X^{(n-1)} \cup e_i^n$, which only becomes $X^{(n)}$ after taking a quotient). The induced map

$$\begin{aligned} i_* : H_k(e_i^n, \partial e_i^n) &\longrightarrow H_k(X^{(n)}, X^{(n-1)}) \\ \mathbb{Z} &\longrightarrow \bigoplus_{\ell_n} \mathbb{Z} \end{aligned}$$

is injective and maps $\mathbb{Z}$ to the factor corresponding to e_i^n. So we have the commutative diagram:

$$\begin{tikzcd}
H_n(e_i^n, \partial e_i^n) \arrow[r, "\partial"] \arrow[d, "i_*"] & H_{n-1}(\partial e_i^n) \arrow[d, "(f_i^n)_*"] \\
H_n(X^{(n)}, X^{(n-1)}) \arrow[r, "\partial_n"] \arrow[dr, bend right, "d_n"'] & H_{n-1}(X^{(n-1)}) \arrow[d, "j_{n-1}"] \\
 & H_{n-1}(X^{(n-1)}, X^{(n-2)})
\end{tikzcd}$$

So the generator of $C_n^{\mathrm{CW}}(X) \cong H_n(X^{(n)}, X^{(n-1)})$ corresponding to e_i^n (i.e. $i_*(1)$) maps under d_n to

$$j_{n-1} \circ (f_i^n)_*(0, \dots, 1, \dots, 0) \in H_{n-1}(X^{(n-1)}, X^{(n-2)}),$$

(the 1 is in the ith factor). Then (the tuple denotes the coefficients of $e_1^{n-1}, \dots, e_{\ell_{n-1}}^{n-1}$)

$$\begin{aligned} d_n(\text{generator corresponding to } e_i) &= d_n \circ i(1) = j_{n-1} \circ (f_i^n)_* \circ \partial(1) \\ &= (\deg g_{i1}, \deg g_{i2}, \dots, \deg g_{i\ell_{n-1}}) = \partial_n^{\mathrm{CW}}(e_i^n), \end{aligned}$$

which proves the desired claim. $\square$

Lecture 21

Mar. 31 — Cohomology

21.1 Homology with Coefficients

Definition 21.1. Given any abelian group G and a space X, let

$$C_n(X;G) = \left\{ \sum_{i=1}^{k} g_i\sigma_i : g_i \in G, \sigma_i \text{ are singular } n\text{-simplices} \right\}$$

and define

$$\partial_n \left(\sum_{i=1}^{k} g_i\sigma_i \right) = \sum_{i=1}^{k} g_i \partial_n \sigma_i = \sum_{i=1}^{k} \sum_{j=0}^{n} g_i (-1)^j \sigma_i^{(j)}.$$

Just like before, $\partial_n \circ \partial_{n+1} = 0$, so we can define the *homology of X with coefficients in G* to be

$$H_n(X;G) = \frac{\ker \partial_n}{\operatorname{im} \partial_{n+1}}.$$

If $G = \mathbb{Z}$, then this is the usual singular homology.

Remark. All theorems about the original definition of singular homology work in this setting as well. We can also define the *cellular homology with coefficients in G* in a similar way.

Example 21.1.1. Consider $\mathbb{R}P^2$. Using $\mathbb{Z}$-coefficients, we get

$$\begin{tikzcd} 0 \arrow[r] & C_2^{\mathrm{CW}}(\mathbb{R}P^2, \mathbb{Z}) \arrow[r] \arrow[d, equal] & C_1^{\mathrm{CW}}(\mathbb{R}P^2, \mathbb{Z}) \arrow[r] \arrow[d, equal] & C_0^{\mathrm{CW}}(\mathbb{R}P^2, \mathbb{Z}) \arrow[r] \arrow[d, equal] & 0 \\ & \mathbb{Z} \arrow[r, "\times 2"] & \mathbb{Z} \arrow[r, "0"] & \mathbb{Z} & \end{tikzcd}$$

so we find that

$$H_k(\mathbb{R}P^2;\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{if } k = 0, \\ \mathbb{Z}/2\mathbb{Z} & \text{if } k = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Using $\mathbb{Z}/2\mathbb{Z}$-coefficients, however, we get

$$\begin{tikzcd} 0 \arrow[r] & C_2^{\mathrm{CW}}(\mathbb{R}P^2, \mathbb{Z}/2\mathbb{Z}) \arrow[r] \arrow[d, equal] & C_1^{\mathrm{CW}}(\mathbb{R}P^2, \mathbb{Z}/2\mathbb{Z}) \arrow[r] \arrow[d, equal] & C_0^{\mathrm{CW}}(\mathbb{R}P^2, \mathbb{Z}/2\mathbb{Z}) \arrow[r] \arrow[d, equal] & 0 \\ & \mathbb{Z}/2\mathbb{Z} \arrow[r, "0"] & \mathbb{Z}/2\mathbb{Z} \arrow[r, "0"] & \mathbb{Z}/2\mathbb{Z} & \end{tikzcd}$$

since multiplication by 2 in $\mathbb{Z}/2\mathbb{Z}$ is the zero map. Thus we have

$$H_k(\mathbb{R}P^2; \mathbb{Z}/2\mathbb{Z}) \cong \begin{cases} \mathbb{Z}/2\mathbb{Z} & \text{if } k = 0, 1, 2, \\ 0 & \text{otherwise.} \end{cases}$$

Remark. We will see later that $H_k(X; \mathbb{Z})$ determines $H_k(X; G)$ for any other abelian group G. However, $H_k(X; G)$ may be sufficient for some purposes and may be easier to work with.

21.2 Cohomology

Definition 21.2. A sequence of abelian groups C^* and maps $\delta_n : C^n \to C^{n+1}$ is called a *cochain complex* if $\delta_{n+1} \circ \delta_n = 0$. for all n. The "homology" of this complex is called the *cohomology* of (C^*, δ):

$$H^n(C^*, \delta) = \frac{\ker \delta_n}{\operatorname{im} \delta_{n-1}}.$$

Definition 21.3. If (C_*, ∂) is a chain complex and G is any abelian group, then let

$$C^n = \operatorname{Hom}(C_n, G) = \{\text{homomorphisms } C_n \to G\}$$

and set $\delta_n = \partial_{n+1}^* : C^n \to C^{n+1}$, i.e. $\delta_n(\tau) = \tau \circ \partial_{n+1}$ (this is the same dual map from linear algebra).

Remark. Note that we have

$$(\partial_{n+1} \circ \delta_n(\tau))(\sigma) = (\delta_n(\tau))(\partial_{n+1}\sigma) = \tau(\partial_{n+1} \circ \partial_{n+1}(\sigma)) = 0,$$

so $\delta_{n+1} \circ \delta_n = 0$ and (C^*, δ) is a cochain complex.

Definition 21.4. Call $H^n(C_*; G) = \ker \delta_n / \operatorname{im} \delta_{n-1}$ the *cohomology of* (C_*, ∂) *with coefficients in* G. In the case that $G = \mathbb{Z}$, we usually abbreviate this via $H^n(C_*) = H^n(C_*; \mathbb{Z})$.

Remark. As groups, $H^n(C_*; G)$ is determined by (and also determines) $H_n(C_*; G)$. However, we will see that we can put a ring structure on $\bigoplus_{n=0}^{\infty} H^n(C_*; G)$, which is a stronger invariant.

Definition 21.5. If (A_*, ∂), (B_*, ∂') are two chain complexes and $\alpha : (A_*, \partial) \to (B_*, \partial')$ is a chain map, then $\alpha^* : B^* \to A^*$ given by $\beta \to \beta \circ \alpha$ is a *cochain map*, i.e. $\delta \circ \alpha^* = \alpha^* \circ \delta'$. Hence, the map α induces a homomorphism $\alpha^* : H^n(B_*; G) \to H^n(A_*; G)$.

Exercise 21.1. Show the following:

1. If $\alpha : (A_*, \partial) \to (B_*, \partial')$ and $\beta : (B_*, \partial') \to (C_*, \partial'')$ are chain maps, then $(\beta \circ \alpha)^* = \alpha^* \circ \beta^*$.
2. $\mathrm{id}^* = \mathrm{id}$ and $0^* = 0$, where id is the identity map and 0 is the zero map.

Remark. As mentioned above, $H_n(C_*; \partial)$ determines $H^n(C_*, \partial)$, but this is not obvious.

Example 21.5.1. Consider the chain complex (C_*, ∂):

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{0} & \mathbb{Z} & \xrightarrow{\times 2} & \mathbb{Z} & \xrightarrow{0} & \mathbb{Z} & \longrightarrow & 0 \\ & & \| & & \| & & \| & & \| & & \\ & & C_3 & & C_2 & & C_1 & & C_0 & & \end{array}$$

so we have $H_3(C_*) \cong H_0(C_*) \cong \mathbb{Z}$, $H_2(C_*) \cong 0$ and $H_1(C_*) \cong \mathbb{Z}/2\mathbb{Z}$. After dualizing, we have

$$\begin{array}{ccccccccccc} & & C_3 & & C_2 & & C_1 & & C_0 & & \\ & & \| & & \| & & \| & & \| & & \\ 0 & \longleftarrow & \mathbb{Z} & \xleftarrow[0]{} & \mathbb{Z} & \xleftarrow[\times 2]{} & \mathbb{Z} & \xleftarrow[0]{} & \mathbb{Z} & \longleftarrow & 0 \end{array}$$

so we get $H^3(C_*) \cong H^0(C_*) \cong \mathbb{Z}$, $H_2(C_*) \cong \mathbb{Z}/2\mathbb{Z}$ and $H_1(C_*) \cong 0$. In particular, we see that H^n is not something simple like $\operatorname{Hom}(H_n, \mathbb{Z})$ in general.

Remark. There is a natural map from the cohomology and homology groups to G:

$$\begin{aligned} H^n(C_*; G) \times H_n(C_*; G) &\longrightarrow G, \\ ([\alpha], [\beta]) &\longmapsto \alpha(\beta). \end{aligned}$$

(Check that this is well-defined.) So there is a "natural" map $\Phi : H^n(C_*; G) \to \operatorname{Hom}(H_n(C_*, \partial), G)$ by

$$\begin{aligned} [\alpha] \longmapsto (\phi_{[\alpha]} : H_n(C_*, \partial) &\longrightarrow G), \\ [\beta] &\longmapsto \alpha(\beta). \end{aligned}$$

We want to understand Φ better: If A is an abelian group, then there exist free abelian groups F, R and homomorphisms such that the following sequence:

$$0 \longrightarrow R \xrightarrow{f} F \xrightarrow{g} A \longrightarrow 0$$

is exact (this follows by the first isomorphism theorem, since a subgroup of a free group is free).

Exercise 21.2. Show that functor $\operatorname{Hom}(\cdot, G)$ is *left exact*, i.e. given an exact sequence of the form

$$G_1 \xrightarrow{\alpha} G_2 \xrightarrow{\beta} G_3 \longrightarrow 0$$

then the following sequence:

$$0 \longrightarrow \operatorname{Hom}(G_3, G) \xrightarrow{\beta^*} \operatorname{Hom}(G_2, G) \xrightarrow{\alpha^*} \operatorname{Hom}(G_1, G)$$

is also exact, but given an exact sequence of the form

$$0 \longrightarrow G_1 \xrightarrow{\alpha} G_2$$

then it is not always true that α^* is surjective, i.e. we can lose exactness on the right.

Definition 21.6. Define the functor $\operatorname{Ext}(A, G) = \operatorname{Hom}(R, G)/\operatorname{im} f^*$.[1]

Remark. Note that the sequence

$$0 \longrightarrow \operatorname{Hom}(A, G) \xrightarrow{g^*} \operatorname{Hom}(F, G) \xrightarrow{f^*} \operatorname{Hom}(R, G) \longrightarrow \operatorname{Ext}(A, G) \longrightarrow 0$$

is exact, so $\operatorname{Ext}(A, G)$ is exactly the obstruction to f^* being surjective.

Example 21.6.1. If $A = \mathbb{Z}$, then we have

$$\begin{array}{ccccccccc} & & R & & F & & G & & \\ & & \| & & \| & & \| & & \\ 0 & \longrightarrow & 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\mathrm{id}} & \mathbb{Z} & \longrightarrow & 0 \end{array}$$

so $\operatorname{Ext}(\mathbb{Z}, G) = \operatorname{Hom}(R, G)/\operatorname{im} f^* = 0/\operatorname{im} f^* = 0$. For $\mathbb{Z}/n\mathbb{Z}$, This is

[1] The name "Ext" comes from "extension".

$$\begin{array}{ccccccccc} & & R & & F & & G & & \\ & & \| & & \| & & \| & & \\ 0 & \longrightarrow & \mathbb{Z} & \xrightarrow[f]{\times n} & \mathbb{Z} & \xrightarrow[g]{} & \mathbb{Z}/n\mathbb{Z} & \longrightarrow & 0 \end{array}$$

so we have $\operatorname{Ext}(\mathbb{Z}/n\mathbb{Z}, G) = \operatorname{Hom}(\mathbb{Z}, G)/\operatorname{im} f^* \cong G/nG$.

Exercise 21.3. Show that $\operatorname{Ext}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/m\mathbb{Z}) = \mathbb{Z}/d\mathbb{Z}$, where $d = \gcd(m, n)$.

Exercise 21.4. Show the following:

1. $\operatorname{Ext}(A, G)$ is independent of F, R, f, g.
2. $\operatorname{Ext}(H \oplus H', G) \cong \operatorname{Ext}(H, G) \oplus \operatorname{Ext}(H', G)$.
3. $\operatorname{Ext}(H, G) = 0$ if H is free.
4. $\operatorname{Ext}(Z/n\mathbb{Z}, G) = G/nG$
5. From above, we can compute $\operatorname{Ext}(H, G)$ for any finitely generated abelian groups H, G.
6. $\operatorname{Ext}(G, \mathbb{Q}) = 0$ for any G.

21.3 Universal Coefficient Theorem

Theorem 21.1 (Universal coefficient theorem)**.** *The following sequence:*

$$0 \longrightarrow \operatorname{Ext}(H_{n-1}(C_*), G) \longrightarrow H^n(C_*; G) \xrightarrow{\Phi} \operatorname{Hom}(H_n(C_*), G) \longrightarrow 0$$

is exact, splits (i.e. $H^n(C_*; G) = \operatorname{Ext}(H_{n-1}(C_*), G) \oplus \operatorname{Hom}(H_n(C_*), G)$ but not in a natural way), and is natural with respect to chain maps. In particular, the cohomology groups are determined by homology.

Proof. This is purely a fact of algebra, see Hatcher. □

Corollary 21.1.1. *If F_n is the free part of $H_n(C_*)$ and T_n is the torsion part of $H_n(C_*)$, then*

$$H^n(C_*; \mathbb{Z}) \cong F_n \oplus T_{n-1}.$$

Proof. This follows from the universal coefficient theorem and the exercises. □

Example 21.6.2. Suppose that we have

$$H_n(C_n, \partial) = \begin{cases} \mathbb{Z} & \text{if } n = 0, \\ \mathbb{Z}/2\mathbb{Z} & \text{if } n \text{ is odd}, \\ 0 & \text{otherwise.} \end{cases}$$

Then we can compute that

$$H^n(C_*; \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } n = 0, \\ 0 & \text{if } n \text{ is odd}, \\ \mathbb{Z}/2\mathbb{Z} & \text{otherwise.} \end{cases}$$

This follows directly from the corollary.

For $\mathbb{Z}/2\mathbb{Z}$-coefficients however, we need to use the universal coefficient theorem:

$$\begin{aligned} H^n(C_*;\mathbb{Z}/2\mathbb{Z}) &= \begin{cases} \mathrm{Hom}(\mathbb{Z};\mathbb{Z}/2\mathbb{Z}) & \text{if } n=0, \\ \mathrm{Hom}(\mathbb{Z}/2\mathbb{Z};\mathbb{Z}/2\mathbb{Z}) \oplus \mathrm{Ext}(0;\mathbb{Z}/2\mathbb{Z}) & \text{if } n \text{ is odd}, \\ \mathrm{Hom}(0;\mathbb{Z}/2\mathbb{Z}) \oplus \mathrm{Ext}(\mathbb{Z}/2\mathbb{Z};\mathbb{Z}/2\mathbb{Z}) & \text{otherwise} \end{cases} \\ &= \mathbb{Z}/2\mathbb{Z} \quad \text{for all } n \geq 0 \end{aligned}$$

since $\mathrm{Ext}(0;\mathbb{Z}/2\mathbb{Z}) = \mathrm{Hom}(0;\mathbb{Z}/2\mathbb{Z}) = 0$.

Corollary 21.1.2. *If a chain map induces an isomorphism on all homology groups, then it induces an isomorphism on all cohomology groups.*

Proof. If $\alpha : (C_*, \partial) \to (C'_*, \partial')$ induces isomorphisms on all homology groups, then

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Ext}(H_{n-1}(C_*), G) \arrow[r] & H^n(C_*; G) \arrow[r] & \mathrm{Hom}(H_n(C_*), G) \arrow[r] & 0 \\ 0 \arrow[r] & \mathrm{Ext}(H_{n-1}(C'_*), G) \arrow[r] \arrow[u, "\alpha_*"] & H^n(C'_*; G) \arrow[r] \arrow[u, "\alpha_*"] & \mathrm{Hom}(H_n(C'_*), G) \arrow[r] \arrow[u, "\alpha_*"] & 0 \end{tikzcd}$$

The α^* on the ends are isomorphisms, so the α^* in the middle is also an isomorphism. $\square$

Lecture 22

Apr. 2 — Cohomology, Part 2

22.1 Cohomology of Spaces

Definition 22.1. Let X be a topological space and $(C_n(X), \partial)$ be the singular chain groups of X. The cohomology of this complex, denoted $H^n(X; G)$, is called the *cohomology of X (with coefficients in G)*. Similarly for a pair (X, A), we get $H^n(X, A; G)$ from the chain complex $(C_n(X, A), \partial)$.

Remark. From Corollary 21.1.2, we know that if X is a CW complex, then we get the same cohomology groups if we use $(C_*^{\mathrm{CW}}(X), \partial^{\mathrm{CW}})$ in place of $(C_*(X), \partial)$.

Remark. We state the following facts that carry over from homology:

1. If $f : X \to Y$ is a map, then we get a chain map $f_* : C_n(X) \to C_n(Y)$, and thus a homomorphism
$$f^* : H^n(Y; G) \to H^n(X; G).$$

2. If $f, g : X \to Y$ are homotopic, then f_*, g_* are chain homotopic, i.e. there exists a map $p_n : C_n(X) \to C_{n+1}(Y)$ such that $\partial_{n+1} \circ p_n + p_{n-1} \circ \partial_n = f_n - g_n$. Dualizing, this becomes
$$p^* \circ \delta + \delta \circ p^* = f^* - g^*.$$
This implies that if $f \simeq g$, then $f^* = g^*$ on $H^n(Y; G)$.

Remark. Exactly as we did for homology, we can also prove:

1. *Exact sequence of a pair:* We have the long exact sequence
$$\cdots \longrightarrow H^n(X, A) \xrightarrow{j^*} H^n(X) \xrightarrow{i^*} H^n(A) \xrightarrow{\delta} H^{n+1}(X, A) \longrightarrow \cdots$$
and if $f : (X, A) \to (Y, B)$, then the following diagram commutes:
$$\begin{tikzcd} H^n(A) \arrow[r, "\delta"] & H^{n+1}(X, A) \\ H^n(B) \arrow[u, "f^*"] \arrow[r, "\delta"] & H^{N+1}(Y, B) \arrow[u, "f^*"] \end{tikzcd}$$

2. *Excision*: If $Z \subseteq \overline{Z} \subseteq \operatorname{int} A \subseteq A \subseteq X$, then the inclusion map
$$(X - Z, A - Z) \longrightarrow (X, A)$$
induces an isomorphism
$$H^n(X, A) \longrightarrow H^n(X - Z, A - Z).$$

3. For a point, we have

$$H^n(\{\mathrm{pt}\}; G) \cong \begin{cases} G & \text{if } n = 0, \\ 0 & \text{otherwise.} \end{cases}$$

4. *Mayer-Vietoris*: If $X = A \cup B$ with A, B open, then

$$\cdots \longrightarrow H^n(X) \longrightarrow H^n(A) \oplus H^n(B) \longrightarrow H^n(A \cap B) \longrightarrow H^{n+1}(X) \longrightarrow \cdots$$

Exercise 22.1. Show directly (i.e. without appealing to the universal coefficient theorem) that

$$H^k(D^n; G) = \begin{cases} G & \text{if } k = 0, \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad H^k(S^n; G) \cong H^k(D^n, \partial D^n; G) \cong \begin{cases} G & \text{if } k = 0, n, \\ 0 & \text{otherwise.} \end{cases}$$

22.2 Products

Remark. We will define two products on cohomology: The first is the *cross product*

$$\begin{aligned} H^p(X) \times H^q(Y) &\longrightarrow H^{p+q}(X \times Y) \\ (\alpha, \beta) &\longmapsto \alpha \times \beta, \end{aligned}$$

which satisfies the following properties: It is *bilinear*:

$$\begin{aligned} (\alpha_1 + \alpha_2) \times \beta &= (\alpha_1 \times \beta) + (\alpha_2 \times \beta), \\ \alpha \times (\beta_1 + \beta_2) &= (\alpha \times \beta_1) + (\alpha \times \beta_2), \end{aligned}$$

and *natural*: If $f : X' \to X$ and $g : Y' \to Y$, then

$$(f^*\alpha) \times (g^*\beta) = (f \times g)^*(\alpha \times \beta).$$

The second is the *cup product*:

$$\begin{aligned} H^p(X) \times H^q(X) &\longrightarrow H^{p+q}(X) \\ (\alpha, \beta) &\longmapsto \alpha \cup \beta, \end{aligned}$$

which is also *bilinear*:

$$\begin{aligned} (\alpha_1 + \alpha_2) \cup \beta &= (\alpha_1 \cup \beta) + (\alpha_2 \cup \beta), \\ \alpha \cup (\beta_1 + \beta_2) &= (\alpha \cup \beta_1) + (\alpha \cup \beta_2), \end{aligned}$$

and *natural*: If $f : X' \to X$, then

$$f^*(\alpha \cup \beta) = (f^*\alpha) \cup (f^*\beta).$$

Remark. We will see that the cup product is more useful, but the cross product is usually easier to compute. However, they are *logically equivalent.* To see this, let $p_1 : X \times Y \to X$ and $p_2 : X \times Y \to Y$ be the projection maps and $\Delta : X \to X \times X : p \mapsto (p, p)$ be the diagonal map. Suppose we have a cup product $\cup$, then we can define a cross product via

$$\begin{aligned} \times_\cup : H^p(X) \times H^q(Y) &\longrightarrow H^{p+q}(X \times Y) \\ (\alpha, \beta) &\longmapsto (p_1^*\alpha) \cup (p_2^*\beta). \end{aligned}$$

Check as an exercise that $x_\times$ is bilinear and natural. Conversely, given a cross product $\times$, we can define

$$\begin{aligned} \cup_\times : H^p(X) \times H^q(X) &\longrightarrow H^{p+q}(X) \\ (\alpha, \beta) &\longmapsto \Delta^*(\alpha \times \beta), \end{aligned}$$

which one can check is a cup product. Note that $\Delta^* : H^*(X \times X) \to H^*(X)$ is the reason we need to work with cohomology for this construction.

Remark. Note that given $\cup$, we have $\cup_{X_\cup} = \cup$. Indeed, we have

$$\begin{aligned} \alpha \cup_{\times_\cup} \beta = \Delta^*(\alpha \times_\cup \beta) = \Delta^*(p_1^*(\alpha) \cup p_2^*(\beta)) &= (\Delta^* \circ p_1^*)(\alpha) \cup (\Delta^* \circ p_2^*)(\beta) \\ &= (p_1 \circ \Delta)^*(\alpha) \cup (p_2 \circ \Delta)^*(\beta) = \alpha \cup \beta. \end{aligned}$$

Similarly, one can verify that given $\times$, we have $\times_{\cup_\times} = \times$.

22.3 Tensor Products

Definition 22.2. Let G, H be two abelian groups, and let $F(G \times H)$ be the free abelian group generated by $G \times H$, i.e. finite formal sums $\sum_i n_i(g_i, h_i)$ with $n_i, \mathbb{Z}$, $g_i \in G$, and $h_i \in H$. Let

$$\begin{aligned} S = \text{subgroup of } F(G \times H) \text{ generated by } &(g + g', h) - (g, h) - (g', h), \\ &(g, h + h') - (g, h) - (g, h'), \\ &(ng, h) - n(g, h), \\ &(g, nh) - n(g, h), \\ \text{for all } g, g' \in G,\ h, h', &\in H, \text{ and } n \in \mathbb{Z}. \end{aligned}$$

The *tensor product* of G and H is the group $G \otimes H = F(G \times H)/S$. Denote the coset of (g, h) by $g \otimes h$, so that elements of $G \otimes H$ are of the form $\sum_{i=1}^k n_i(g_i \otimes h_i)$.

Remark. In $G \otimes H$, we have the following properties:

$$\begin{aligned} (g + g') \otimes h &= g \otimes h + g' \otimes h, \\ g \otimes (h + h') &= g \otimes h + g \otimes h', \\ (ng) \otimes h &= n(g \otimes h) = g \otimes (nh). \end{aligned}$$

Exercise 22.2. Verify the following properties of the tensor product:

1. $G \otimes H \cong H \otimes G$.
2. $(\bigoplus_i G_i) \otimes H \cong \bigoplus_i (G_i \otimes H)$.
3. $(G \otimes H) \otimes K \cong G \otimes (H \otimes K)$.
4. $\mathbb{Z} \otimes G \cong G$.
5. $\mathbb{Z}/n \otimes G \cong G/nG$.
6. Given homomorphisms $f : G \to G'$ and $g : H \to H'$, the map

$$\begin{aligned} f \otimes g : G \otimes H &\longrightarrow G' \otimes H' \\ x \otimes y &\longmapsto f(x) \otimes g(y) \end{aligned}$$

is a well-defined homomorphism.

7. A bilinear map $\phi : G \times H \to K$ induces a homomorphism $G \otimes H \to K$ by $g \otimes h \mapsto \phi(g, h)$.

Remark. Part (7) of the previous exercise is part of the reason why we define the tensor product: It turns a bilinear map into a linear map, which is easier to work with.

Remark. We can define tensor products more generally for R-modules, where R is a commutative ring with unit. We will not need this, however.

22.4 Cross and Cup Products

Definition 22.3. Define the *tensor product* of chain complexes (C_*, ∂) and (C'_*, δ) to be the chain complex $C \otimes C'$ with

$$(C \otimes C')_n = \bigoplus_{i+j=n} (C_i \otimes C_j)$$

and boundary map $\partial^{\otimes} : C \otimes C' \to C \otimes C'$ given by

$$\partial^{\otimes}(a \otimes b) = (\partial a) \otimes b + (-1)^i a \otimes (\partial' b).$$

Exercise 22.3. Verify that $(\partial^{\otimes})^2 = 0$.

Definition 22.4. Define the *algebraic cross product* by

$$\begin{aligned} \times_{\text{alg}} : H_p(C) \times H_q(C') &\longrightarrow H_{p+q}(C \otimes C') \\ [z] \otimes [w] &\longmapsto [z \otimes w]. \end{aligned}$$

Remark. This is well-defined, e.g. if $z = \overline{z} + \partial\tau$ (i.e. $z, \overline{z} \in [z]$), then

$$z \oplus w = (\overline{z} + \partial\tau) \otimes w = \overline{z} \otimes w + \partial(\tau \otimes w) = \overline{z} \otimes w \otimes \partial^{\otimes}(\tau \otimes w)$$

since $\partial' w = 0$, so $[z \otimes w] = [\overline{z} \otimes w]$. One can also check $z \otimes w$ is independent of the choice of $w \in [w]$.

Exercise 22.4. Show that $\times_{\text{alg}}$ is natural with respect to chain maps.

Theorem 22.1 (1/2 Künneth Sequence)**.** *The following sequence is exact:*

$$0 \longrightarrow \bigoplus\nolimits_{p+q=n} (H_p(C) \otimes H_q(C')) \xrightarrow{\times_{\text{alg}}} H_n(C \otimes C')$$

Proof. This is purely a fact of algebra (we will also not need it), see Hatcher. □

Remark. If X, Y are CW complexes, we get a CW structure on $X \times Y$ by taking products of cells: If e^i_j is an i-cell of X and $\widehat{e}^{i'}_{j'}$ is an i'-cell of Y, then $e^i_j \times \widehat{e}^{i'}_{j'}$ is an $(i + i')$-cell of $X \times Y$.

$$\alpha^i_j : \partial e^i_j \to X^{i-1} \quad \text{and} \quad \widehat{\alpha}^{i'}_{j'} : \partial \widehat{e}^{i'}_{j'} \to Y^{i'-1}$$

are the attaching maps for e^i_j and $\widehat{e}^{i'}_{j'}$, then the attaching map of $e^i_j \times \widehat{e}^{i'}_{j'}$ is

$$\begin{aligned} \partial(e^i_j \times \widehat{e}^{i'}_{j'}) = [\partial e^i_j \times \widehat{e}^{i'}_{j'}] \cup [e^i_j \times \partial \widehat{e}^{i'}_{j'}] &\longrightarrow (X^{i-1} \times Y^i) \cup (X^i \times Y^{i-1}) \subseteq (X \times Y)^{i+i'-1} \\ [\partial e^i_j \times \widehat{e}^{i'}_{j'}] \ni (x, y) &\longmapsto (\alpha^i_j(x), y) \\ [e^i_j \times \partial \widehat{e}^{i'}_{j'}] \ni (x, y) &\longmapsto (x, \widehat{\alpha}^{i'}_{j'}(y)). \end{aligned}$$

Remark. For $a = \sum \alpha^k e^i_k \in C^{\mathrm{CW}}_i(X)$ and $b = \sum \beta^\ell \widehat{e}^{i'}_\ell \in C^{\mathrm{CW}}_{i'}(Y)$ (with $\alpha^k, \beta^\ell \in \mathbb{Z}$), define

$$a \otimes b = \sum \alpha^k \beta^\ell (e^i_k \otimes \widehat{e}^{i'}_\ell).$$

From above, the boundary map is given by

$$\begin{aligned} \partial^{\mathrm{CW}}(a \times b) &= \sum \alpha^k \beta^\ell (\partial^{\mathrm{CW}} e^i_k \times \widehat{e}^{i'}_\ell + (-1)^i e^i_k \times \partial^{\mathrm{CW}} \widehat{e}^{i'}_\ell) \\ &= \partial^{\mathrm{CW}} a \times b + (-1)^i a \times \partial^{\mathrm{CW}} b. \end{aligned}$$

Thus we get chain maps

$$\bigoplus_{p+q=n} (C^{\mathrm{CW}}_p(X) \otimes C^{\mathrm{CW}}_q(Y)) \xrightarrow{B} C^{\mathrm{CW}}_n(X \times Y)$$
$$C^{\mathrm{CW}}(X \times Y) \xrightarrow{A} \bigoplus_{p+q=n} (C^{\mathrm{CW}}_p(X) \otimes C^{\mathrm{CW}}_q(Y)),$$

where A is given by

$$A\left(\sum_{p_i+q_j=n} a^{ij}(e^{p_i}_i \times e^{q_j}_j) \right) = \sum_{p_i+q_j=n} a^{ij}(e^{p_i}_i \otimes e^{q_j}_j).$$

One can easily check $A \circ B(a \otimes b) = a \otimes b$ and $B \circ A(a) = a$, so we get an isomorphism on homology:

$$H^{\mathrm{CW}}_n(X \times Y) \xrightarrow{\cong} H_n(C^{\mathrm{CW}}(X) \otimes C^{\mathrm{CW}}(Y))$$

Remark. In singular homology, do the following:

$$\begin{aligned} B : C_p(X) \otimes C_q(Y) &\longrightarrow C_{p+q}(X \times Y) \\ (\sigma : \Delta^p \to X, \tau : \Delta^q \to Y) &\longmapsto (\sigma \times \tau : \Delta^p \times \Delta^q \to X \times Y), \end{aligned}$$

where we can break $\Delta^p \times \Delta^q$ into several $(p+q)$-simplices. Now given $\sigma : \Delta^n \to X$, set

$$\begin{aligned} {}_p\sigma : \Delta^p \to X : (t_0, \dots, t_p) &\longmapsto \sigma(t_0, \dots, t_p, 0, \dots, 0) \\ \sigma_q : \Delta^q \to X : (t_0, \dots, t_q) &\longmapsto \sigma(0, \dots, 0, t_0, \dots, t_q). \end{aligned}$$

We can then define

$$\begin{aligned} A : C_n(X \times Y) &\longrightarrow \bigoplus_{p+q=n} (C_p(X) \otimes C_q(Y)) \\ \sigma &\longmapsto \sum_{p+q=n} {}_p(p_X \circ \sigma) \otimes (p_Y \circ \sigma)_q, \end{aligned}$$

where $p_X : X \times Y \to X$ and $p_Y : X \times Y \to Y$ are the projection maps.

Theorem 22.2 (Eilenberg-Zilber)**.** *The maps A, B induce isomorphisms on singular homology.*

Definition 22.5. The *homological cross product* $\times$ is given by the composition

$$H_p(X) \otimes H_q(Y) \xrightarrow{\times_{\mathrm{alg}}} H_{p+q}(C(X) \otimes C(Y)) \xrightarrow{B} H_{p+q}(X \times Y)$$