MATH 6441: Algebraic Topology I

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Contents

Lecture 1

Jan. 6 — CW-Complexes

1.1 Introduction and Motivation

Algebraic topology builds "functions" (actually *functors*)

$$\{\text{topological spaces, continuous maps}\} \longrightarrow \{\text{algebraic things, algebraic maps}\},$$

where "algebraic things" can be groups, vector spaces, etc. The main objective of algebraic topology is to *distinguish topological spaces*, e.g. showing that $\mathbb{R}^n \not\cong \mathbb{R}^m$ for $n \neq m$. More applications are:

1. Studying maps between spaces.[1]
   - Does a given space M embed in N? For instance, for what m does $\mathbb{R}P^n$ embed in $\mathbb{R}^m$? (This is still not known in general.) Here $\mathbb{R}P^n$ is the real projective space.
   - Lifting maps, i.e. given $f : A \to B$ and $g : E \to B$, does there exist a map $\widetilde{f} : A \to E$ such that $g \circ \widetilde{f} = f$? In other words, is there a map $\widetilde{f}$ such that the following diagram commutes:

$$\begin{tikzcd} & E \arrow[d, "g"] \\ A \arrow[ur, dashed, "\widetilde{f}"] \arrow[r, "f"'] & B \end{tikzcd}$$

   - Fixed point problems: Given $f : X \to X$, does f have a fixed point, i.e. $x \in X$ such that $f(x) = x$? Such theorems are used to prove the existence of solutions to ordinary differential equations, for instance.
2. Group actions, e.g. which finite groups act freely on S^n?
3. Group theory, e.g. showing that every subgroup of a free group is free. Another example is that if F_n is the free group on n generators, then its *commutator* $[F_n, F_n]$ is not finitely generated.
4. Algebra, e.g. proving the fundamental theorem of algebra.

This course will cover the following topics:

1. The *fundamental group* $\pi_1(X, x_0)$ of a space X for $x_0 \in X$, and *covering spaces*.
2. The *homology groups* $H_k(X)$ for $k = 0, 1, 2, \ldots$. These groups are abelian.
3. The *cohomology ring* $H^*(X) = \bigoplus_{k=0}^{\infty} H^k(X)$.

But before getting to this, we need to develop some important ideas.

[1] All maps and functions in this class are continuous unless otherwise specified.

1.2 CW-Complexes

Definition 1.1. Let $D^n \subseteq \mathbb{R}^n$ be the unit disk and $S^{n-1} = \partial D^n$. Given a topological space Y and a continuous map $a : S^{n-1} \to Y$, the space obtained from Y by *attaching* an *n-cell* (*via a*) is

$$Y \cup_a D^n = (Y \sqcup D^n)/\sim,$$

where the equivalence relation $\sim$ is given by $x \sim a(x)$ for $x \in \partial D^n$. Here $\sqcup$ denotes disjoint union.

Definition 1.2. An *n-complex* or an *n-dimensional CW-complex* is defined inductively by:

- A (-1)-complex is the empty set $\varnothing$.
- An n-complex X^n is a space obtained from an $(n-1)$-complex by attaching n-cells.

An n-complex is *finite* if it involves only a finite number of cells. The *k-skeleton* of X is the union of all n-cells in X with $n \leq k$.

Remark. Any CW-complex is Hausdorff. See Hatcher for a proof.

Example 1.2.1. Here are some examples of CW-complexes:

- A 0-complex is a union of points. This is because $D^0 = \{\text{pt}\}$ and $\partial D^0 = \varnothing$.
- A 1-complex is a graph (points and lines connecting them).
- The torus T (a square with opposite sides identified) is a 2-complex. Here the 0-skeleton $T^{(0)}$ is the common corner on the square and the 1-skeleton $T^{(1)}$ is two sides of the square after taking the quotient. The 2-skeleton $T = T^{(2)}$ is the entire torus.
- Another example of a 2-complex is the two-holed torus, which is obtained by identifying the edges of an octagon (pairs of every other edge identified with opposite orientation).[2]
- A third example of a 2-complex is $X^{(1)} \cup_a D^2$ given an attaching map $D^2 \to X^{(1)}$.
- Consider the unit sphere $S^n \subseteq \mathbb{R}^{n+1}$. One way to give S^2 a CW-complex structure is to see the sphere as two disks D^2 glued together, resulting in one 0-cell, one 1-cell, and two 2-cells. Another way is to start with two points, attach two 1-cells to get a circle, and then attaching two disks to get S^2. This results in two 0-cells, two 1-cells, and two 2-cells.

  The second idea generalizes to S^n. We can write

  $$S^n = S^{n-1} \cup_{a_1} D^n \cup_{a_2} D^n,$$

  where S^{n-1} inductively has a CW-complex structure. This yields two k-cells for each $k \leq n$.

  Another way to put a CW-complex structure on S^n is to attach D^n to a point with $\partial D^n \to \{\text{pt}\}$. In particular, notice that a space can in general have several different CW-complex structures.
- Consider the *n-dimensional real projective space*

  $$\mathbb{R}P^n = \{\text{lines through the origin in } \mathbb{R}^{n+1}\}.$$

  Since each line through the origin passes through S^n twice, we can equivalently think of $\mathbb{R}P^n$ as the unit sphere S^n with antipodal points identified.

[2] This CW-decomposition of the two-holed torus results in one 0-cell, four 1-cells, and one 2-cell.

We can also think of this as D^n with antipodal points on ∂D^n identified. Since $\partial D^n = S^{n-1}$, this is simply $\mathbb{R}P^{n-1} \cup_a D^n$, where $a : \partial D^n \to \mathbb{R}P^{n-1}$ is the quotient map. This gives $\mathbb{R}P^n$ a CW-complex structure with one k-cell for each $k \leq n$.

- The complex projective space $\mathbb{C}P^n$ has a similar CW-complex structure with one k-cell for each even $k \leq 2n$. One can verify this as an exercise.
- Any smooth manifold has a CW-complex structure. See Hatcher.

Exercise 1.1. Show the product of CW-complexes is a CW-complex.

Definition 1.3. A *subcomplex* of a CW-complex X is a closed subset $A \subseteq X$ that is a union of cells in X. In particular, A is also a CW-complex and (X, A) is called a *CW-pair*.

1.3 Homotopy

Definition 1.4. Let X and Y be topological spaces. Two maps $f, g : X \to Y$ are *homotopic*, denoted $f \sim g$, if there exists a continuous map $\Phi : X \times [0, 1] \to Y$ such that

$$\Phi(x, 0) = f(x) \quad \text{and} \quad \Phi(x, 1) = g(x)$$

for all $x \in X$. In this case, Φ is called a *homotopy* between f and g.

Remark. We note the following:

- A homotopy Φ gives a family of maps $\phi_{t_0} : X \to Y$ given by $x \mapsto \phi(x, t_0)$ which is continuous in t_0. So maps are homotopic if there is a continuous family of maps between them.
- If $A \subseteq X$ then we say that f is *homotopic to g rel A*, denoted $f \sim_A g$, if there exists Φ as above with the additional property that $\Phi(x, t) = f(x)$ for all $x \in A$, i.e. points in A are fixed.
- If $A \subseteq X$ and $B \subseteq Y$, then the notation $f : (X, A) \to (X, B)$ means that $f : X \to Y$ and $f(a) \in B$ for each $a \in A$. We say that f is a *map of pairs*. If $f, g : (X, A) \to (Y, B)$, then f, g are *homotopic as maps of pairs* if each $\phi_t : (X, A) \to (X, B)$ is a map of pairs.

Lecture 2

Jan. 8 — Homotopy

2.1 More on Homotopy

Example 2.0.1. For any space X, any map $f : X \to [0,1]$ is homotopic to the map $g : X \to [0,1]$ given by $x \mapsto 0$. To see this, we have the homotopy $\Phi : X \times [0,1] \times [0,1]$ defined by

$$(x,t) \mapsto (1-t)f(x).$$

We can see that $\Phi(x,0) = f(x)$ and $\Phi(x,1) = 0 = g(x)$.

Exercise 2.1. Show that homotopy is an equivalence relation on maps $X \to Y$.

Definition 2.1. Let $C(X,Y) = \{\text{continuous maps from } X \text{ to } Y\}$. Let $[X,Y] = C(X,Y)/{\sim}$, i.e. homotopic maps are identified with each other.

Example 2.1.1. We have the following:

1. $[X,[0,1]] = \{g\}$ for any space X, where g is the map $x \mapsto 0$ as above.
2. $[\{*\},X] = \{\text{path components of } X\}$.

2.2 Homotopy Groups

Definition 2.2. We call a space X *pointed* if there is a designated "base point" $x_0 \in X$. Given two pointed spaces (X,x_0) and (Y,y_0), we define

$$[X,Y]_0 = \{\text{homotopy classes of maps of pairs } (X,\{x_0\}) \to (Y,\{y_0\})\}.$$

Definition 2.3. Let y_0 be the north pole in S^n, i.e. $S^n \subseteq \mathbb{R}^{n+1}$ is the unit sphere and $y_0 = (0,\dots,0,1)$. The *nth homotopy group* of a pointed space (X,x_0) is $\pi_n(X,x_0) = [S^n,X]_0$.

Remark. The homotopy group $\pi_n(X,x_0)$ is in fact a group. We will study $\pi_1(X,x_0)$ next and it is called the *fundamental group* of (X,x_0).

Remark. For which (Y,y_0) is $[Y,X]_0$ "naturally" a group for all (X,x_0)? Similarly, for which (Y,y_0) is $[X,Y]_0$ a group for all (X,x_0)? Here, given a map $f : (X_1,x_1) \to (X_2,x_2)$, there is an obvious *induced map* $f_* : [Y,X_1]_0 \to [Y,X_2]_0$ given by $[g] \mapsto [f \circ g]$. Similarly, there is a map $f^* : [X_2,Y]_0 \to [X_1,Y]_0$ given by $[g] \mapsto [g \circ f]$. In the questions above, "naturally" means that f_* and f^* are homomorphisms. for any (X_1,x_1) and (X_2,x_2). The (perhaps unsatisfying) answer is that a space satisfying the first condition is called an *H-space*, and a space satisfying the second is called an *H'-space*.

2.3 Homotopy Equivalence

Definition 2.4. We say that $f : X \to Y$ is the *homotopy inverse* to a function $g : Y \to X$ if $f \circ g \sim \mathrm{id}_Y$ and $g \circ f \sim \mathrm{id}_X$, where id_X and id_Y are the identity maps on X and Y. If g has a homotopy inverse, then we call g a *homotopy equivalence* from Y to X and we call X, Y *homotopy equivalent*.[1]

Exercise 2.2. Show that homotopy equivalence is an equivalence relation.

Lemma 2.1. *The following are equivalent:*

1. *X and Y are homotopy equivalent.*
2. *For any space Z, there is a one-to-one correspondence $\phi_Z : [X, Z] \to [Y, Z]$ such that for all continuous maps $h : Z \to Z'$, the following diagram commutes:*

$$\begin{array}{ccc} [X,Z] & \xrightarrow{\phi_Z} & [Y,Z] \\ \downarrow{\scriptstyle h_*} & & \downarrow{\scriptstyle h_*} \\ [X,Z'] & \xrightarrow{\phi_{Z'}} & [Y,Z'] \end{array}$$

3. *For any space Z, there is a one-to-one correspondence $\phi^Z : [Z, X] \to [Z, Y]$ such for that all continuous maps $h : Z \to Z'$, the following diagram commutes:*

$$\begin{array}{ccc} [Z',X] & \xrightarrow{\phi^{Z'}} & [Z',Y] \\ \downarrow{\scriptstyle h^*} & & \downarrow{\scriptstyle h^*} \\ [Z,X] & \xrightarrow{\phi^{Z}} & [Z,Y] \end{array}$$

Proof. This is left as an exercise. □

Remark. Based on the previous lemma, two spaces are homotopy equivalent if and only if homotopy classes of maps to and from the space are "naturally equivalent."

Example 2.4.1. We have the following:

- Homeomorphic spaces are homotopy equivalent.
- Let $X = S^1$ and $Y = S^1 \times [0, 1]$. We claim that X is homotopy equivalent to Y.

  Define the maps $f : S^1 \to S^1 \times [0, 1]$ by $x \mapsto (x, 0)$ and $g : S^1 \times [0, 1] \to S^1$ by $(x, t) \mapsto x$. Then we can see that $g \circ f : S^1 \to S^1$ maps $x \mapsto x$, so $g \circ f = \mathrm{id}_{S^1}$. On the other hand, the composition $f \circ g : S^1 \times [0, 1] \to S^1 \times [0, 1]$ maps $(x, t) \mapsto (x, 0)$. Now $f \circ g \sim \mathrm{id}_{S^1 \times [0,1]}$ by homotopy. For instance, define $\Phi : (S^1 \times [0, 1]) \times [0, 1] \to (S^1 \times [0, 1])$ by $((x, t), s) \mapsto (x, st)$, so

$$\Phi((x,t),1) = (x,t) = \mathrm{id}_{S^1\times[0,1]}(x,t) \quad \text{and} \quad \Phi((x,t),0) = (x,0) = f \circ g.$$

  Thus f is a homotopy equivalence from S^1 to $S^1 \times [0, 1]$. Note that $S^1 \times [0, 1]$ is the annulus.

Definition 2.5. A space is called *contractible* if it is homotopy equivalent to a point.

Example 2.5.1. The spaces $[0, 1]$ and $\mathbb{R}^n$ are contractible (exercise).

[1] We will denote homotopy equivalence by $X \simeq Y$ or simply $X \sim Y$.

Definition 2.6. If $A \subseteq X$, then a *retraction of* X *to* A is a map $r : X \to A$ such that $r(a) = a$ for every $a \in A$. A *deformation retraction* of X to A is a retraction $r : X \to A$ that is homotopic rel A to the identity map id_X, i.e. we can find $\phi_t : X \to X$ for $t \in [0,1]$ such that $\phi_0(x) = x$ and $\phi_1(X) \subseteq A$ and $\phi_t(x) = x$ for all $x \in A$ and $t \in [0,1]$.

Remark. If X deformation retracts to A, then X is homotopy equivalent to A. To see this, suppose we have a homotopy $\phi_t : X \to X$ as above, and let $i : A \to X$ be the inclusion map. Then $\phi_1 \circ i = \mathrm{id}_A$ and $i \circ \phi_1 = \phi_1 \sim \phi_0 = \mathrm{id}_X$, so ϕ_1 is a homotopy equivalence from X to A.

Definition 2.7. Given two spaces X, Y and a map $f : X \to Y$, the *mapping cylinder* of f is the space

$$M_f = ((X \times [0,1]) \cup Y)/\sim,$$

where the equivalence relation $\sim$ is given by by $(x,1) \sim f(x)$ for $x \in X$.

Remark. The mapping cylinder M_f deformation retracts to Y. To see this, consider the map $\widetilde{\phi}_t$ given by $(x,s) \mapsto (x, (1-t)s + t)$ on $X \times [0,1]$ and $y \mapsto y$ on Y. Since $\widetilde{\phi}_t$ respects the equivalence relation, it descends to a map $\phi_t : M_f \to M_f$ on the quotient space. Note that $\phi_0 = \mathrm{id}_{M_f}$ and $\phi_1(M_f) = Y \subseteq M_f$, and $\phi_t = \mathrm{id}_Y$ for all t. Thus ϕ_1 is a deformation retraction. In particular, this means that $M_f \simeq Y$.

Remark. There are obvious inclusions $i : X \to M_f$ given by $x \mapsto (x,0)$ and $j : Y \to M_f$ given by $y \mapsto y$. Note that ϕ_1 defined above is the homotopy inverse to j. Now we have the diagram

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[dr, "i"'] & Y \arrow[d, "j"] \\
& M_f
\end{tikzcd}$$

where j is a homotopy equivalence and $j \circ f \sim i$ (exercise).

Remark. The above remark shows the following "slogan" of algebraic topology:

Any map is an inclusion up to homotopy.

Example 2.7.1. Let X be three circles with two enclosed in a third bigger one, and let Y be two circles enclosing the inner two circles of X connected by a line segment. Let Z be the region inside by the outer circle of X and outside the inner two circles of X.

Define $f : X \to Y$ to be the map which sends $x \in X$ to the point in Y at the other end of an interval (points on the inner circles of X are mapped by radial lines to the circles in Y, and points on the outer circle of X are mapped radially to either the circles or the line segment in Y). One can write an explicit formula for f as an exercise.

Then Z is homeomorphic to M_f, and in particular $Z \simeq Y$. Similarly, M_f is homotopy equivalent to two circles joined at a point, or a circle with a diameter. Thus by transitivity, these two spaces and Z are all homotopy equivalent to each other.

Lecture 3

Jan. 13 — Homotopy, Part 2

3.1 More on Homotopy Equivalence

Lemma 3.1. *If (X, A) is a CW pair and A is contractible, then $X/A \simeq X$.*[1]

Exercise 3.1. The following are some applications of this lemma:

1. Let X be a connected graph (i.e. a 1-complex), and let A be an edge in X connecting distinct vertices. Then A is contractible, so $X/A \simeq X$. Continuing this process, let A be a maximal tree in X, which will also be contractible. Then $X \simeq X/A$, so any connected graph is homotopy equivalent to a *wedge of circles* (with number of circles equal to the number of self-loops in the graph).[2]

2. Consider the space X obtained by attaching a 1-cell A connecting the north and south poles on a sphere. Let B be half of a great circle connecting the endpoints of A. Clearly A and B are both contractible. After collapsing A, we see that $X \simeq X/A$, which is S^2 with the north and south poles identified. On the other hand, by contracting B instead we see that $X \simeq X/B$, which is $S^2 \vee S^1$.

   Note that after contracting A, the subset B is actually no longer contractible.

3. Let X be a torus with attached disks A_1, A_2, A_3 in the tube of the torus. Then

$$X \simeq ((X/A_1)/A_2)/A_3,$$

   which is three spheres lying in a circle, each attached to the next one at a single point.

   We can also obtain this space by considering the space Y of three spheres attached in a line with an extra 1-cell B attached at the ends of the chain of circles. Also let A be the union of halves of great circles going through the chain of circles, with the same endpoints as B. Then we can see that this creates the same space as before, so that $X \simeq Y/B \simeq Y$. On the other hand, by contracting A, we see that $Y \simeq Y/A = S^2 \vee S^2 \vee S^2 \vee S^1$.

   Of course, all of these spaces are then homotopy equivalent to each other by transitivity.

Lemma 3.2. *Let (X, A) be a CW pair and $f, g : A \to Y$ be homotopic maps. Then $X \cup_f Y \simeq X \cup_g Y$.*

Example 3.0.1. Let $Y = S^2$, and $X = D^2$, and $A = \partial D^2$. Let $g : A \to Y$ map A to a great circle, and let $f : A \to Y$ map A to the north pole. One can show as an exercise that $f \sim g$ (e.g. by pulling the equator towards the north pole). So the lemma says that $X \cup_g Y$, which is a sphere with a disk glued along its equator, is homotopy equivalent to $X \cup_f Y = S^2 \vee S^2$.

[1]Here X/A denotes the quotient of X obtained by collapsing all of A to a single point.
[2]A *wedge* of pointed spaces $(X, x_0) \vee (Y, y_0)$ is the space obtained from $X \sqcup Y$ by identifying x_0 and y_0.

3.2 Homotopy Extension Property

Remark. To prove both of these lemmas, we need the *homotopy extension property* (HEP).

Definition 3.1. A space X and a subspace $A \subseteq X$ have the *homotopy extension property* if given $F_0 : X \to Y$ (for any Y and F_0) and a homotopy $f_t : A \to Y$ such that $f_0 = F_0|_A$, then there is a homotopy $F_t : X \to Y$ such that $F_t|_A = f_t$ for every t.

Lemma 3.3. *A pair (X, A) has the homotopy extension property if and only if*

$$(X \times \{0\}) \cup (A \times [0,1])$$

is a retract of $X \times [0,1]$.

Proof. ($\Leftarrow$) We will assume that A is closed (not necessarily but makes the proof easier, and almost all examples satisfy this). By assumption, we have a retraction

$$r : (X \times [0,1]) \to (X \times \{0\}) \cup (A \times [0,1]).$$

Given $F_0 : X \to Y$ and $f_t : A \to Y$ such that $f_0 = F_0|_A$, we can define a map

$$\widetilde{F} : (X \times \{0\}) \cup (A \times [0,1]) \to Y$$

by $x \mapsto F_0(x)$ on $X \times \{0\}$ and $(a,t) \mapsto f_t(a)$ on $A \times [0,1]$. This map $\widetilde{F}$ is continuous since the definitions of $\widetilde{F}$ agree on the intersection and the intersection $A \times \{0\}$ is closed. Now define

$$F : X \times [0,1] \to Y$$

by $F = \widetilde{F} \circ r$, which is a homotopy of F_0 that extends f_t.

($\Rightarrow$) Let $Y = (X \times [0,1]) \cup (A \times [0,1])$. Let

$$F_0 : X \to (X \times \{0\}) \cup (A \times [0,1])$$

be given by $x \mapsto (x, 0)$, and

$$f_t : A \mapsto (X \times \{0\}) \cup (A \times [0,1])$$

be given by $a \mapsto (a,t)$. Then the homotopy extension property yields an extension

$$F : X \times [0,1] \to (X \times \{0\}) \cup (A \times [0,1]),$$

which is a retraction, as desired. $\square$

Lemma 3.4. *If (X, A) is a CW pair, then $(X \times \{0\}) \cup (A \times [0,1])$ is a (deformation) retract of $X \times [0,1]$. In particular, (X, A) satisfies the homotopy extension property.*

Proof. The main idea is that for any disk D^n, the space $(D^n \times \{0\}) \cup (\partial D^n \times [0,1])$ is a deformation retract of $D^n \times [0,1]$. To see this, let $D^n \subseteq \mathbb{R}^n$ be the unit disk and $D^n \times [0,1] \subseteq \mathbb{R}^{n+1}$. Let

$$p = (0, \dots, 0, 2).$$

For any $x \in D^n \times [0,1]$, let ℓ_x be the line through p and x. Note that

$$\ell_x \cap ((D^n \times [0,1] \cup (\partial D^n \times [0,1]))$$

is a unique point. Define $\widetilde{r}(x)$ to be this point, which yields a map

$$\widetilde{r} : D^n \times [0,1] \to (D^n \times \{0\}) \cup (\partial D^n \times [0,1])$$

Note that for $x \in (D^n \times \{0\}) \cup (\partial D^n \times [0,1])$, then $\widetilde{r}(x) = x$ since the point of intersection is unique and x is already in the intersection. Show as an exercise that $\widetilde{r}$ is continuous. Then setting

$$\widetilde{r}_t = t\widetilde{r} + (1-t)\,\mathrm{id}_{D^n \times [0,1]}$$

gives a deformation retraction of $D^n \times [0,1]$ onto $(D^n \times \{0\}) \cup (\partial D^n \times [0,1])$.

To show the general case that $X \times [0,1]$ retracts to $(X \times \{0\}) \cup (A \times [0,1])$, we induct on the dimension of cells. Define r on $X^{(0)} \times [0,1] \to (X \times \{0\}) \cup (A \times [0,1])$ by the following:

- if a 0-cell $D^0 \subseteq A$, then let r be the identity on $D^0 \times [0,1]$;
- if a 0-cell D^0 is not in A, then send $D^0 \times [0,1]$ to $D^0 \subseteq X \times \{0\}$.

Now inductively assume r is defined on $X^{(k-1)} \times [0,1] \to (X \times \{0\}) \cup (A \times [0,1])$. For each k-cell D^k,

- if $D^k \subseteq A$, then let r be the identity on $D^k \times [0,1]$;
- if D^k is not in A, then note that $\partial D^k \times [0,1] \to X^{(k-1)} \times [0,1]$ is defined by induction, and we have an “inclusion” (here $a : \partial D^k \to X^{(k-1)}$ is the attaching map for D^k)

$$\begin{tikzcd} D^k \arrow[r, "i"] \arrow[rr, bend right, "j"'] & X^{(k-1)} \cup D^k \arrow[r, "q"] & (X^{(k-1)} \cup D^k)/\{(x \in \partial D^k) \sim (a(x) \in X^{k-1})\} \end{tikzcd}$$

  So we let $D^k \times \{0\} \to (X \times \{0\}) \cup (A \times [0,1])$ be the map j into $X \times \{0\}$. This defines r on

$$(D^k \times [0,1]) \cup (\partial D^k \times [0,1]),$$

  which extends to $D^k \times [0,1]$ by composing with the map $\widetilde{r}$ from above.

This inductively defines the retraction r on all of $X \times [0,1]$. The last claim follows by Lemma 3.3. □

Remark. Now we can prove the first two lemmas from the beginning of the day.

Proof of Lemma 3.1. We prove that the result holds for any (X, A) which satisfies the homotopy extension property. We show that the quotient map $q : X \to X/A$ has a homotopy inverse. Since A is contractible, we know there exists a homotopy $f_t : A \to A \subseteq X$, such that $f_0 = \mathrm{id}_A$ and f_1 is constant. Let $F_0 : X \to X$ be the identity (note that $F_0|_A = \mathrm{id}_A$), so that the homotopy extension property gives a homotopy $F_t : X \to X$ extending f_t. Since $F_t(A) \subseteq A$ for every t, we get maps

$$\widetilde{F}_t : X/A \to X/A,$$

which are well defined since points in A are mapped to points in A. Furthermore, $F_1(A) = \{\mathrm{pt}\}$, so F_1 factors through X/A to give a map $h : X/A \to X$ satisfying $F_1 = h \circ q$. This gives the diagram:

$$\begin{tikzcd} X \arrow[r, "F_1"] \arrow[d, "q"'] & X \arrow[d, "q"] \\ X/A \arrow[ur, "h"] \arrow[r, "\overline{F}_1"'] & X/A \end{tikzcd}$$

It is easy to check that $h \circ q = F_1$ and $q \circ h = \overline{F}_1$, so that the diagram commutes. But then

$$h \circ q = F_1 \sim F_0 = \mathrm{id}_X \quad \text{and} \quad q \circ h = \overline{F}_1 \sim \overline{F}_0 = \mathrm{id}_{X/A},$$

so q is a homotopy equivalence. □

Proof of Lemma 3.2. Let $F : A \times [0,1] \to Y$ be a homotopy, which extends to $F : X \times [0,1] \to Y$ by the homotopy extension property. Consider the mapping cylinder

$$M_F = (X \times [0,1]) \cup_F Y,$$

and one can show that $M_F \simeq X \cup_g Y \simeq X \cup_f Y$. □

Lecture 4

Jan. 15 — Fundamental Group

4.1 Fundamental Group

Remark. The basic idea of the *fundamental group* is to study the topology of a space with loops mapped into the space. For instance, intuitively, any loop in S^2 can be "pulled back" to (i.e. is homotopic to) a constant loop. On the other hand, a loop wrapping around the hole in T^2 gets stuck and cannot be "pulled back" to a constant loop. The same issue happens for a loop in T^2 around the cylindrical part. The fundamental group is a way to make this intuition precise and measure the "holes" in a space.

Definition 4.1. The *fundamental group* of a based space (X, x_0) is

$$\pi_1(X, x_0) = [(S^1, n), (X, x_0)]_0,$$

i.e. the homotopy classes of loops in X based at x_0. Here $n = (0, 1)$ is the north pole of S^1.

Exercise 4.1. Let $S^1 \subseteq \mathbb{R}^2$ be the unit circle and let $p : [0, 1] \to S^1$ be given by

$$t \mapsto (\cos 2\pi t, \sin 2\pi t).$$

Show that p is a quotient map, so we can think of S^1 as $[0, 1]$ with $0, 1$ identified. Moreover, show that there is a one-to-one correspondence between maps of the form[1]

$$\gamma : ([0, 1], \{0, 1\}) \to (X, x_0) \quad \text{and} \quad \widetilde{\gamma} : (S^1, \{(1, 0)\}) \to (X, x_0)$$

given by $\widetilde{\gamma} \mapsto \widetilde{\gamma} \circ p = \gamma$, and that homotopies of $\widetilde{\gamma}$ rel $\{0, 1\}$ correspond to homotopies of S^1 rel $\{(1, 0)\}$.

Remark. Using the above exercise, we can think of $\pi_1(X, x_0) = [S^1, X]_0$ instead as

$$\pi_1(X, x_0) = [([0, 1], \{0, 1\}), (X, x_0)]_0.$$

Given a based loop $\gamma : [0, 1] \to X$, we denote its equivalence class in $\pi_1(X, x_0)$ by $[\gamma]$.

Definition 4.2. If γ_1, γ_2 are loops in X based at $x_0 \in X$, their *concatenation* $\gamma_1 * \gamma_2 : [0, 1] \to X$ is

$$t \mapsto \begin{cases} \gamma_1(2t) & \text{if } 0 \leq t \leq 1/2, \\ \gamma_2(2t - 1) & \text{if } 1/2 \leq t \leq 1. \end{cases}$$

Remark. Concatenation of loops indeed yields another loop since $\gamma_1 * \gamma_2(0) = \gamma_1 * \gamma_2(1) = x_0$ and $\gamma_1 * \gamma_2$ is continuous since the definitions agree on the closed set $\{1/2\}$.

[1] Such a loop γ is called a *based loop*.

Remark. We can clearly see that $\gamma_1 * \gamma_2$ is well-defined given γ_1 and γ_2, but can we define $[\gamma_1] * [\gamma_2]$ for homotopy classes of loops in a well-defined manner? We need to check that if $\gamma_1 \sim \gamma_2$ and $\delta_1 \sim \delta_2$ (i.e. $\gamma_1, \gamma_2 \in [\gamma_1]$ and $\delta_1, \delta_2 \in [\delta_1]$), then we also have $\gamma_1 * \delta_1 \sim \gamma_2 * \delta_2$.

To do this, let $H : [0,1] \times [0,1] \to X$ be the homotopy from γ_1 to γ_2 and $G : [0,1] \times [0,1] \to X$ be the homotopy from δ_1 to δ_2. We need to construct a homotopy $\widetilde{H} : [0,1] \times [0,1] \to X$ from $\gamma_1 * \delta_1$ to $\gamma_2 * \delta_2$.

Note that if we think of $[0,1] \times [0,1]$ as the unit square, then $\widetilde{H}$ is already defined on the boundary: the left and right sides are constantly x_0, the top side is $\gamma_2 * \delta_2$, and the bottom side is $\gamma_1 * \delta_1$. So we only need to define it on the interior. For this, note that the vertical line in the middle of the square is also constantly x_0 by construction: This creates two rectangles on each half, which we can fill with H and G.

More formally, we can construct the homotopy $\widetilde{H} : [0,1] \times [0,1] \to X$ explicitly via

$$(t,s) \mapsto \begin{cases} H(2t, s) & \text{if } 0 \le t \le 1/2, \\ G(2t-1, s) & \text{if } 1/2 \le t \le 1. \end{cases}$$

This is continuous since the definitions agree on the closed set $\{t = 1/2\}$. Thus, setting

$$[\gamma_1] * [\delta_1] = [\gamma_1 * \delta_1]$$

gives a well-defined binary operation by the above discussion.

Lemma 4.1. *The pair $(\pi_1(X, x_0), *)$ is a group.*

Proof. For the identity, let $e : [0,1] \to X$ be the constant loop $t \mapsto x_0$. We will show that

$$[e] * [\gamma] = [\gamma] = [\gamma] * [e].$$

The picture is that $[0,1] \times [0,1]$ has γ on the top side and $\gamma * e$ on the bottom. By drawing a line from the midpoint of the bottom side and the top-right corner, we see that we can fill the right portion with just x_0 and the left portion with γ. The equation of this line is $s = 2t - 1$, so $t = (s+1)/2$. Thus from the picture, we can write the explicit homotopy $H : [0,1] \times [0,1] \to X$ via

$$H(t,s) = \begin{cases} \gamma(2/(s+1), t) & \text{if } 0 \le t \le (s+1)/2, \\ x_0 & \text{if } (s+1)/2 \le t \le 1. \end{cases}$$

One can use a similar construction to show that $[e] * [\gamma] = [\gamma]$, so that $[e]$ is an identity element.

Now we show the existence of inverses. Given a loop γ, define $\overline{\gamma}$ via $\overline{\gamma}(t) = \gamma(1-t)$, i.e. γ backwards. Set $\gamma_s(t) = \gamma(st)$. Note that as t goes from 0 to 1, γ_s goes from $\gamma(0)$ to $\gamma(s)$, and also that $\overline{\gamma}_s(t) = \gamma(s - st)$. So we can write the homotopy $H : [0,1] \times [0,1] \to X$ between $\gamma * \overline{\gamma}$ and e by

$$H(t,s) = \begin{cases} \gamma_s(2t) & \text{if } 0 \le t \le 1/2, \\ \overline{\gamma}_s(2t-1) & \text{if } 1/2 \le t \le 1 \end{cases} = \begin{cases} \gamma(2st) & \text{if } 0 \le t \le 1/2, \\ \gamma(s - s(2t-1)) & \text{if } 1/2 \le t \le 1. \end{cases}$$

Thus setting $[\gamma]^{-1} = [\overline{\gamma}]$ gives us inverses.

Finally, for associativity, we need to see that $(\gamma_1 * \gamma_2) * \gamma_3 \sim \gamma_1 * (\gamma_2 * \gamma_3)$. Again by drawing a picture, we see that we can draw two diagonal lines connecting the starting points of γ_2 on top and bottom and the ending points of γ_1 on top and bottom. Write an explicit formula for the homotopy as an exercise. $\square$

4.2 Induced Homomorphisms

Remark. If $f : X \to Y$ and $x_0 \in X$, let $y_0 = f(x_0)$. Then given a based loop $\gamma : [0,1] \to X$, note that the composition $f \circ \gamma : [0,1] \to Y$ is a based loop in Y. Also, if $\gamma \sim \delta$, then $f \circ \gamma \sim f \circ \delta$ (if H is a homotopy from $\gamma \sim \delta$, then $f \circ H$ is a homotopy from $f \circ \gamma$ to $f \circ \delta$). In particular, f induces a map

$$f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0).$$

Lemma 4.2. *The induced map $f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ is a homomorphism.*

Proof. Note that

$$\gamma_1 * \gamma_2(t) = \begin{cases} \gamma_1(2t) & \text{if } 0 \le t \le 1/2, \\ \gamma_2(2t-1) & \text{if } 1/2 \le t \le 1 \end{cases}$$

and that

$$(f \circ \gamma_1) * (f \circ \gamma_2)(t) = \begin{cases} f(\gamma_1(2t)) & \text{if } 0 \le t \le 1/2, \\ f(\gamma_2(2t-1)) & \text{if } 1/2 \le t \le 1, \end{cases}$$

so $f \circ (\gamma_1 * \gamma_2) = (f \circ \gamma_1) * (f \circ \gamma_2)$, which implies $f_*([\gamma_1 * \gamma_2]) = f_*([\gamma_1]) * f_*([\gamma_2])$. □

Exercise 4.2. Check the following as an exercise:

1. $(f \circ g)_* = f_* \circ g_*$;
2. if $f : X \to Y$ is homotopic rel base point to $g : X \to Y$, then $f_* = g_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$.

Remark. How does π_1 depend on the base point? Let $x_0, x_1 \in X$, and suppose that there exists a path $h : [0,1] \to X$ with $h(0) = x_0$ and $h(1) = x_1$. Then if γ is a loop based at x_1, we can get a loop based at x_0 by going from x_0 to x_1 along h, taking γ, and then going back to x_0. More explicitly, this is

$$h * \gamma * \overline{h}(t) = \begin{cases} h(3t) & \text{if } 0 \le t \le 1/3, \\ \gamma(3t-1) & \text{if } 1/3 \le t \le 2/3, \\ \overline{h}(3t-2) & \text{if } 2/3 \le t \le 1. \end{cases}$$

Lemma 4.3. *The path h induces an isomorphism $\phi_h : \pi_1(X, x_1) \to \pi_1(X, x_0)$ by $[\gamma] \mapsto [h * \gamma * \overline{h}]$.*

Proof. Check as an exercise that ϕ_h is a well-defined homomorphism. To show that ϕ_h is an isomorphism, we claim that $\phi_{\overline{h}}$ is an inverse of ϕ_h. To see this, let $[\gamma] \in \pi_1(X, x_0)$. Then

$$\phi_h \circ \phi_{\overline{h}}([\gamma]) = [h * \overline{h} * \gamma * h * \overline{h}] = [h * \overline{h}] * [\gamma] * [h * \overline{h}] = [e] * [\gamma] * [e] = [\gamma],$$

where the second equality follows by the same proof for associativity of $*$. This proves the result. □

Remark. Note the following based on the above lemma:

1. The isomorphism class of $\pi_1(X, x_0)$ only depends on the path component of X containing x_0.
2. The isomorphism *depends on h*. One needs to be careful about using the correct identification.

Lecture 5

Jan. 22 — Simple Computations

5.1 Fundamental Groups and Homotopy Equivalence

Lemma 5.1. *Suppose $f_0, f_1 : X \to Y$ are homotopic by the homotopy $H : X \times [0,1] \to Y$. Let $x_0 \in X$ be a basepoint and define $h : [0,1] \to Y$ by $t \mapsto H(x_0, t)$. Then the following diagram commutes:*

$$\begin{tikzcd} \pi_1(X, x_0) \arrow[r, "(f_0)_*"] \arrow[dr, "(f_1)_*"'] & \pi_1(Y, f_0(x_0)) \\ & \pi_1(Y, f_1(x_0)) \arrow[u, "\phi_h"'] \end{tikzcd}$$

Proof. Fix an arbitrary $[\gamma] \in \pi_1(X, x_0)$, and we construct a homotopy from $h * (f_1 \circ \gamma) * \overline{h}$ to $f_0 \circ \gamma$. The picture is the following: we have $h * (f_1 \circ \gamma) * \overline{h}$ at the top and $f_0 \circ \gamma$ at the bottom, where t parametrizes the horizontal direction and s parametrizes the vertical direction. Draw a trapezoidal shape by connecting the middle two points on the top edge to the two bottom corners.

Define the homotopy as follows: For a fixed s, define $H'(t, s)$ first by $h_s(t)$ in the first third, $H(\gamma(t), s)$ in the second third, and then $\overline{h}_s(t)$ in the last third. Construct the explicit homotopy as an exercise. □

Theorem 5.1. *If $f : X \to Y$ is a homotopy equivalence, then the induced map*

$$f_* : \pi_1(X, x_0) \to \pi_1(Y, f(x_0))$$

on fundamental groups is an isomorphism.

Proof. Let g be a homotopy inverse to f, so we have the composition:

$$\pi_1(X, x_0) \xrightarrow{f_*} \pi_1(Y, f(x_0)) \xrightarrow{g_*} \pi_1(X, g(f(x_0)))$$

But we know $g \circ f = \mathrm{id}_X$, so by the lemma there exists a path $h : X \to X$ from x_0 to $g(f(x_0))$ such that $g_* \circ f_* = \phi_h$ is an isomorphism. So f_* is injective. Similarly, $f_* \circ g_*$ is an isomorphism, and therefore f_* is surjective. Since we already know f_* is a homomorphism, this shows that f_* is an isomorphism. □

Remark. Recall that we have defined a "functor"

$$\{\text{pointed topological spaces, pointed maps}\} \to \{\text{groups, homomorphisms}\},$$

where homotopy equivalent spaces are mapped to isomorphic groups and homotopic maps give rise to the "same" homomorphism. We will finally make some computations of fundamental groups next.

5.2 Simple Computations of Fundamental Groups

Lemma 5.2. *If X is contractible, then $\pi_1(X, x_0) = \{1\}$ for all $x_0 \in X$, where $\{1\}$ is the trivial group.*

Proof. If $Y = \{y_0\}$ is a one-point space, then there exists a unique loop $\gamma : [0,1] \to X$ given by $t \mapsto y_0$. So $\pi_1(Y, y_0) = \{1\}$. Since X is contractible, it is homotopy equivalent to Y and so $\pi_1(X, x_0) = \{1\}$. □

Definition 5.1. We say that a space X is *simply connected* if

1. X is path-connected, and
2. $\pi_1(X, x_0) = \{1\}$ for some $x_0 \in X$.

Remark. Simply connected means that "points in X are connected in a very simple way."

Lemma 5.3. *A space X is simply connected if and only if every two points in X are connected by a unique homotopy class of paths in X.*

Proof. ($\Leftarrow$) Clearly X is path-connected. Furthermore, any loop based at x_0 is a path from x_0 to itself, and the constant is as well. Thus any loop is homotopic to the constant loop, i.e. $\pi_1(X, x_0) = \{1\}$.

($\Rightarrow$) For any $a, b \in X$, there exists a path from a to b (since X is path-connected). Now suppose $\gamma, \delta : [0,1] \to X$ are paths from a to b. By $\pi_1(X, a) = \{1\}$ we know that $\gamma * \overline{\delta} \sim e_a$, so

$$\gamma \sim \gamma * (\overline{\delta} * \delta) \sim (\gamma * \overline{\delta}) * \delta \sim e_a * \delta \sim \delta,$$

i.e. γ and δ are in the same homotopy class. □

Lemma 5.4. *Let $X = A \cup B$, where $A, B, A \cap B$ are open and path-connected. Let $x_0 \in A \cap B$. Then any loop $\gamma : [0,1] \to X$ based at x_0 can be written as*

$$\gamma \sim \gamma_1 * \gamma_2 * \cdots * \gamma_n,$$

where each γ_i is a loop in A or B based at x_0.

Proof. We first claim that there exist $0 = t_0 < t_1 < \cdots < t_n = 1$ such that $\operatorname{im} \gamma|_{[t_{i-1}, t_i]} \subseteq A$ or B, and $\gamma(t_i) \in A \cap B$ for every i. The proof of this will use the following topology fact:

> **Lemma** (Lebesgue number lemma)**.** Let X be a compact metric space and $\{U_\alpha\}_{\alpha \in A}$ be an open cover. Then there exists a *Lebesgue number* $\delta > 0$ such that for all sets S with $\operatorname{diam}(S) = \sup_{x,y \in S} d(x,y) < \delta$, there exists $\alpha \in A$ such that $S \subseteq U_\alpha$.

To prove the claim, let $U_1 = \gamma^{-1}(A)$ and $U_2 = \gamma^{-1}(B)$, which is an open cover of $[0,1]$. So there exists $\delta > 0$ such that if $|b - a| < \delta$, then $[a,b] \subseteq U_i$ for $i = 1$ or 2. Thus $\gamma([a,b]) \subseteq A$ or B. Now let n be a positive integer such that $1/n < \delta$, so that for each i we have

$$\operatorname{im} \gamma|_{[i/n, (i+1)/n]} \subseteq A \text{ or } B.$$

So we start with $t_i = 1/n$ for $i = 0, \ldots, n$. Now if $\gamma|_{[t_{i-1}, t_i]}$ and $\gamma|_{[t_i, t_{i+1}]}$ both have image in A (or both in B), then throw out t_i. Then $\gamma|_{[t_{i-1}, t_i]}$ will have image in A or B and $\gamma(t_i) \in A \cap B$, as desired.

Now given the claim, let $\delta_i : [0,1] \to A \cap B$ connect x_0 to $\gamma(t_1)$, and set $\gamma_i = \gamma|_{[t_{i-1}, t_i]}$. Then

$$\gamma \sim \gamma_1 * \gamma_2 * \cdots * \gamma_n \sim (\gamma_1 * \overline{\delta}_1) * (\delta_1 * \gamma_2 * \overline{\delta}_2) * \cdots * (\delta_{n-1} * \gamma_n),$$

where each of the above loops is either in A or B. □

Theorem 5.2. *We have $\pi_1(S^n, x_0) = \{1\}$ for all $n \geq 2$.*

Proof. We have $\pi_1(S^n, x_0) = \{1\}$ for all $n \geq 2$. Let $A = S^n \setminus \{(0, \ldots, 0, 1)\}$ and $B = S^n \setminus \{(0, \ldots, 0, -1)\}$. Note that $A \cong B \cong \mathbb{R}^n$, so they are path-connected. Furthermore, $A \cap B = S^{n-1} \times \mathbb{R}$, which is also path-connected if $n \geq 2$.[1] Now take any $x_0 \in A \cap B$. Then any $[\gamma] \in \pi_1(S^n, x_0)$ can be written as

$$[\gamma] = [\gamma_1] * [\gamma_2] * \cdots * [\gamma_n],$$

where $[\gamma_i] \in \pi_1(A, x_0)$ or $\pi_1(B, x_0)$ by the lemma. But we have

$$\pi_1(A, x_0) = \pi_1(B, x_0) = \{1\},$$

so $[\gamma] = [e_{x_0}]$ and hence $\pi_1(S^n, x_0) = \{1\}$. $\square$

Theorem 5.3. *Given two spaces X and Y, we have $\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0)$.*

Proof. The map $\Phi : \pi_1(X, x_0) \times \pi_1(Y, y_0) \to \pi_1(X \times Y, (x_0, y_0))$ given by

$$([\gamma], [\delta]) \mapsto [\gamma \times \delta],$$

where $(\gamma \times \delta)(t) = (\gamma(t), \delta(t))$, is an isomorphism. Check as an exercise that Φ is well-defined and a bijection (for the second part, consider the projections $p_X : X \times Y \to X$ and $p_Y : X \times Y \to Y$). $\square$

5.3 Fundamental Group of the Circle

Our next objective is the following computation:

Theorem 5.4. *We have $\pi_1(S^1, (1, 0)) \cong \mathbb{Z}$. In particular, the map sending $n \in \mathbb{Z}$ to*

$$\gamma_n : [0, 1] \to S^1 : t \mapsto (\cos 2\pi nt, \sin 2\pi nt)$$

is an isomorphism $\mathbb{Z} \to \pi_1(S^1, (1, 0))$.

Remark. The proof is an example of a very important technique that we will see again soon. The proof involves studying the map

$$p : \mathbb{R} \to S^1 : t \mapsto (\cos 2\pi t, \sin 2\pi t).$$

Note that $p^{-1}((1, 0)) = \mathbb{Z}$. This is a particular example of a *covering map*, which we will study later.

Definition 5.2. If $\gamma : [0, 1] \to S^1$ is a path based at the point $(1, 0)$, then a *lift of γ based at $n \in \mathbb{Z}$* is a map $\widetilde{\gamma}_n : [0, 1] \to \mathbb{R}$ such that $\widetilde{\gamma}_n(0) = n$ and $p \circ \widetilde{\gamma}_n = \gamma$.

Lemma 5.5. *We have the following:*

(a) *For each $n \in \mathbb{Z}$, each loop $\gamma : [0, 1] \to S^1$ based at $(0, 1)$ lifts to a unique path $\widetilde{\gamma}_n$ based at n.*

(b) *If $\gamma \sim \gamma'$ are loops in S^1 based at $(0, 1)$ and $\widetilde{\gamma}_n, \widetilde{\gamma}'_n$ are their lifts based at n, then $\widetilde{\gamma}_n \sim \widetilde{\gamma}'_n$ rel $\{0, 1\}$.*

The above properties are called path lifting *and* homotopy lifting.

Proof. We will prove this next class. $\square$

[1] Note that this does not work for $n = 1$: the intersection $S^0 \times \mathbb{R} = \{\pm 1\} \times \mathbb{R}$ is not path-connected.

Proof of Theorem 5.4. Given $[\gamma] \in \pi_1(S^1, (1,0))$, part (a) of Lemma 5.5 says that there is a unique lift $\widetilde{\gamma}_0 : [0,1] \to \mathbb{R}$. Since $\widetilde{\gamma}_0(1) \in p^{-1}((1,0)) = \mathbb{Z}$, we can define $\Phi : \pi_1(S^1, (1,0)) \to \mathbb{Z}$ by $[\gamma] \mapsto \widetilde{\gamma}_0(1)$. Part (b) of Lemma 5.5 says that Φ is well-defined. We need to show the following:

1. Φ is surjective:

   Let $\widetilde{\delta}^n(t) = nt$ for $t \in [0,1]$ and $\delta^n = p \circ \widetilde{\delta}$. Then $\widetilde{\delta}^n$ is the lift of δ^n based at 0, and $\Phi([\delta^n]) = n$.

2. Φ is injective:

   Suppose γ, γ' are loops in S^1 such that
   $$\Phi([\gamma]) = \widetilde{\gamma}_0(1) = \widetilde{\gamma}'_0(1) = \Phi([\gamma']).$$
   Set $\widetilde{H}(s,t) = (1-t)\widetilde{\gamma}_0(s) + t\widetilde{\gamma}'_0(s)$ and $H = p \circ \widetilde{H}$. Then H is a homotopy from γ to γ' (exercise).

3. Φ is a homomorphism:

   Let $[\gamma], [\gamma'] \in \pi_1(S^1, (1,0))$ and let $\widetilde{\gamma}_0, \widetilde{\gamma}'_0$ be their lifts based at 0. Then
   $$\Phi([\gamma]) = \widetilde{\gamma}_0(1) = n \in \mathbb{Z} \quad \text{and} \quad \Phi([\gamma']) = \widetilde{\gamma}'_0(1) = m \in \mathbb{Z}.$$
   Then note the following:

   (a) $t \mapsto n + \widetilde{\gamma}'_0(t)$ is a lift of γ' and starts at n, so by uniqueness it is $\widetilde{\gamma}'_n$, and

   (b) $\widetilde{\gamma}_0 * \widetilde{\gamma}'_n$ is a lift of $\gamma * \gamma'$.

   So $\Phi([\gamma] * [\gamma']) = \widetilde{\gamma} * \widetilde{\gamma}'_0(1) = n + m = \Phi([\gamma]) + \Phi([\gamma'])$, i.e. Φ is a homomorphism.

Thus Φ is an isomorphism, and we have $\pi_1(S^1, (1,0)) \cong \mathbb{Z}$. □

Lecture 6

Jan. 27 — Fundamental Group of the Circle

6.1 Path Lifting

Proof of Lemma 5.5. (a) Let $A = S^1 \setminus \{(1,0)\}$, and note that

$$p^{-1}(A) = \bigcup_{i \in \mathbb{Z}} (i, i+1) = \bigcup_{i \in \mathbb{Z}} A_i.$$

Notice that each restriction $p|_{A_i} : A_i \to A$ is a homeomorphism. Now let $B = S^1 \setminus \{(-1,0)\}$, so

$$p^{-1}(B) = \bigcup_{i \in \mathbb{Z}} \left(i - \frac{1}{2}, i + \frac{1}{2} \right) = \bigcup_{i \in \mathbb{Z}} B_i.$$

Similarly, each $p|_{B_i} : B_i \to B$ is a homeomorphism. Now if $\gamma : [0,1] \to S^1$ is contained in A (or B), we can choose any $i \in \mathbb{Z}$ and let $\widetilde{\gamma} = (p|_{A_i})^{-1} \circ \gamma$, giving a lift of γ. Then for a general $\gamma : [0,1] \to S^1$ with $\gamma(0) = (1,0)$, the set $\{\gamma^{-1}(A), \gamma^{-1}(B)\}$ is an open cover of the compact metric space $[0,1]$, so there exists a Lebesgue number $\delta > 0$ such that any interval $[a,b]$ with $b - a < \delta$ lies in either $\gamma^{-1}(A)$ or $\gamma^{-1}(B)$. Choose n such that $1/n < \delta$. If $t_n = i/n$ for $i = 0, \dots, n$, then

$$\gamma([t_i, t_{i+1}]) \subseteq A \text{ or } B$$

for every i. Again for convenience, if $[t_{i-1}, t_i]$ and $[t_i, t_{i+1}]$ are both in $\gamma^{-1}(A)$ or $f^{-1}(B)$, then discard t_i. So we have a partition $0 = t_0 < t_1 < \dots < t_k = 1$ such that (note that γ starts at $(1,0) \notin A$)

$$\gamma([t_i, t_{i+1}]) \subseteq \begin{cases} A & \text{if } i \text{ is odd,} \\ B & \text{if } i \text{ is even.} \end{cases}$$

Then we want to build $\widetilde{\gamma}_n$. Define $\widetilde{\gamma}_n$ on $[t_0, t_1]$ to be $(p|_{B_n})^{-1} \circ \gamma|_{[t_0,t_1]}$. Now $\widetilde{\gamma}_n(t_1) \in A_i$ for a unique i, so define $\widetilde{\gamma}_n$ on $[t_1, t_2]$ by $(p|_{A_i})^{-1} \circ \gamma|_{[t_1,t_2]}$. Note that $\widetilde{\gamma}_n$ is continuous on $[t_0, t_2]$ since the two definitions agree at $t = t_1$. Inductively continue to define the lift $\widetilde{\gamma}_n$ on all of $[0,1]$.

(b) The proof is very similar to path lifting. Given a homotopy $H : [0,1] \times [0,1] \to S^1$, we can find a Lebesgue number $\delta > 0$ for $\{H^{-1}(A), H^{-1}(B)\}$. Pick n such that $\sqrt{2}/n < \delta$ and break $[0,1] \times [0,1]$ into n^2 squares of side length $1/n$. The diameter of each square is at most $\sqrt{2}/n$, so each square can be lifted as above. Finish the construction as an exercise to lift H on all of $[0,1] \times [0,1]$. □

6.2 Applications of the Fundamental Group of S^1

Corollary 6.0.1. *There is no retraction $D^2 \to \partial D^2$.*

Proof. Suppose there was a retraction $r : D^2 \to \partial D^2$, and let $i : S^1 \to D^2$ be the inclusion of S^1 as the boundary of D^2. Then we have the composition:

$$S^1 \xrightarrow{i} D^2 \xrightarrow{r} S^1$$

Noting that $r \circ i = S^1 \to S^1$ is the identity, so $(r \circ i)_* : \pi_1(S^1, (1,0)) \to \pi_1(S^1, (1,0))$ is the identity map. In particular, $r_* \circ i_* = (r \circ i)_*$ is the identity map, hence i_* must be injective. But

$$i_* : \pi_1(S^1, (1,0)) \to \pi_1(D^2, (1,0))$$

where $\pi_1(S^1, (1,0)) \cong \mathbb{Z}$ and $\pi_1(D^2, (1,0)) = \{1\}$, so i_* cannot be injective. Contradiction. □

Corollary 6.0.2. *Any map $f : D^2 \to D^2$ has a fixed point, i.e. $x \in D^2$ such that $f(x) = x$.*

Proof. Suppose otherwise that $f : D^2 \to D^2$ has no fixed points. Then for each $x \in D^2$, there is a unique ray R_x starting at $f(x)$ and going through x. Note that $R_x \cap \partial D^2$ in a unique point (on the interior of R_x). Define $r : D^2 \to S^1$ by $x \mapsto R_x \cap \partial D^2$. Show that r is continuous as an exercise (e.g. parametrize the line). But then r is a retraction $D^2 \to \partial D^2$, a contradiction. □

Remark. There are more applications such as the fundamental theorem of algebra, the ham sandwich theorem, and the Borsuk-Ulam theorem. See Hatcher for more details.

6.3 Free Products of Groups

Definition 6.1. Let G_1 and G_2 be groups. A *word* in $G_1 \sqcup G_2$ is a finite sequence

$$x = (x_1, x_2, \ldots, x_n)$$

for some n, where each x_i is in G_1 or G_2. Define an equivalence relation on words in $G_1 \sqcup G_2$ which is generated by (show as an exercise that this is in fact an equivalence relation):

1. replace a, b in a sequence by ab if a, b are in the same group (or the reverse of this), and
2. if e_i (the identity in G_i) is in a sequence, then remove it (or add it in any place in a sequence).

Denote the equivalence class of a word x by $[x]$. Call a word $x = (x_1, \ldots, x_n)$ *reduced* if

1. $x_j \neq e_i$ for any j or i, and
2. x_i and x_{i+1} are from different groups.

Show that each $[x]$ contains a unique reduced word (note: uniqueness is hard). The *free product* of G_1 and G_2 is the group $G_1 * G_2$ of all equivalence classes of words in $G_1 \sqcup G_2$, with multiplication

$$[x_1, \ldots, x_n] \cdot [y_1, \ldots, y_m] = [x_1, \ldots, x_n, y_1, \ldots, y_m].$$

Remark. Note that in $G_1 * G_2$, the identity e is the empty word and the inverse is given by

$$[x_1, \ldots, x_n]^{-1} = [x_n^{-1}, \ldots, x_1^{-1}].$$

Check as an exercise that $G_1 * G_2$ is in fact a group (really only need to check associativity).

Proposition 6.1. *Let $j_i : G_i \to G_1 * G_2$ be the inclusion of G_i into $G_1 * G_2$. Given any homomorphisms $\phi_i : G_i \to H$ where H is any group, there exists a unique homomorphism $\phi : G_1 * G_2 \to H$ such that $\phi \circ j_i = \phi_i$, i.e. the following diagram commutes:*

$$\begin{tikzcd}
 & G_1 \arrow[d, "j_1"] \arrow[ddr, bend left, "\phi_1"] & \\
G_2 \arrow[r, "j_2"] \arrow[drr, bend right, "\phi_2"'] & G_1 * G_2 \arrow[dr, dashed, "\phi"] & \\
 & & H
\end{tikzcd}$$

Proof. If the x_i are reduced and $x_1 \in G_1$, then define

$$\phi_1(x_1, x_2, \ldots, x_n) = \phi_1(x_1) \cdot \phi_2(x_2). \cdot \phi_1(x_3) \cdot \ldots.$$

Then check the following as an exercise:

1. Show such that ϕ exists and is unique.
2. Show this property *defines* the free product, i.e. if D is another group satisfying the property in the proposition, then $D \cong G_1 * G_2$.

The second part above says that this is the *universal property* of the free product. □

Example 6.1.1. Represent $\mathbb{Z}$ in product notation via $\{x^n\}$, where $x^n x^m = x^{n+m}$. Then

$$\mathbb{Z} * \mathbb{Z} = \{x^n\} * \{y^m\} = \{e, x^{n_1} y^{m_1} \ldots x^{n_k}, x^{n_1} y^{m_1} \ldots y^{m_k}, y^{m_1} x^{n_1} \ldots y^{m_k}, y^{m_1} x^{n_1} \ldots x^{n_k}\}.$$

This group is called the *free group on two generators*, and $\mathbb{Z}$ is the *free group on one generator*.

Lecture 7

Jan. 29 — Some Group Theory

7.1 Group Presentations

Definition 7.1. The *free group* on n generators, denoted F_n, is defined inductively via

$$F_n = F_{n-1} * \mathbb{Z},$$

where $F_1 = \mathbb{Z}$. (One can also consider F_∞.)

Remark. Note that any homomorphism $\phi : \mathbb{Z} \to G$ (for any group G) is determined by $\phi(1)$. Moreover, given any $g \in G$, there is a unique homomorphism which maps $1 \mapsto g$. Thus by the universal property of the free product, a homomorphism $F_n \to G$ is determined uniquely by a choice of $g_1, \dots, g_n \in G$.

Definition 7.2. A *group presentation* is a group $\langle X|R\rangle$ defined as follows:

- X is some set (of generators);
- R is a set of words (relations) in $X \cup X^{-1}$ (formally denote $x \in X$ as $x^{-1} \in X^{-1}$);
- let $n = |X|$ and F_n be the free group on n generators, so that we can think of $R \subseteq F_n$;
- let $\langle R\rangle$ be the smallest normal subgroup of F_n containing R;
- define the group $\langle X|R\rangle = F_n/\langle R\rangle$.

We say that $\langle X|R\rangle$ is a *presentation* of a group G if $G \cong \langle X|R\rangle$.

Example 7.2.1. The group $\langle g|g^n\rangle$ is all the words in g, g^{-1}:

$$\dots, g^{-2}, g^{-1}, e, g, g^2, g^3, \dots,$$

but $g^n = e$, so $g^{n+1} = g^n g = eg = g$ and thus we have $g^{-1} = eg^{-1} = g^n g^{-1} = g^{n-1} g g^{-1} = g^{n-1}$. So there is a one-to-one correspondence between elements of $\langle g|g^n\rangle$ and g^k for $k = 0, \dots, n-1$.

Exercise 7.1. Show that $\langle g|g^n\rangle \cong \mathbb{Z}/n$, so that $\langle g|g^n\rangle$ is a presentation of $\mathbb{Z}/n$.

Lemma 7.1. *Every group has a presentation.*

Proof. Let G be a group. Let $X \subseteq G$ be a collection of elements of G that generate G (e.g. take $X = G$ itself). Let $n = |X|$, so there exists a unique $\phi : F_n \to G$ sending the generators of F_n to the $g_i \in X$. Let $N = \ker \varphi$, so the first isomorphism theorem says that $G \cong F_n/N$ (note that ϕ is clearly surjective). Let R be a subset of N that generates N (e.g. take $R = N$). Then $G \cong \langle X|R\rangle$. $\square$

Remark. Using fewer generators to write G or N may give more less complicated presentations of G.

Definition 7.3. We say that G is *finitely generated* if $G \cong \langle X | R \rangle$ such that $|X| < \infty$. We say that G is *finitely presented* if both $|X|, |R| < \infty$.

Exercise 7.2. Show the following:

1. If $G = \langle g_1, \dots, g_n | r_1, \dots, r_m \rangle$, then for any group H and any map $h : \{g_1, \dots, g_n\} \to H$ satisfying $h(r_i) = e_H$ (the notation $h(r_i)$ means to replace any letters g_j in r_i by $h(g_j)$), there exists a unique homomorphism $\phi_h : G \to H$ such that $\phi_h(g_i) = h(g_i)$.

2. If $G_1 = \langle g_1, \dots, g_n | r_1, \dots, r_m \rangle$ and $G_2 = \langle h_1, \dots, h_k | s_1, \dots, s_\ell \rangle$, then
$$G_1 * G_2 = \langle g_1, \dots, g_n, h_1, \dots, h_k | r_1, \dots, r_m, s_1, \dots, s_\ell \rangle.$$

Definition 7.4. Given groups G_1, G_2 and K, and homomorphisms $\psi_i : K \to G_i$, the *free product with amalgamation* is
$$G_1 *_K G_2 = \frac{G_1 * G_2}{\langle \{\psi_1(k)\psi_2(k)^{-1}\}_{k \in K} \rangle},$$
where $\langle \{\psi_1(k)\psi_2(k)^{-1}\}_{k \in K} \rangle$ is the smallest normal subgroup of $G_1 * G_2$ containing $\{\psi_1(k)\psi_2(k)^{-1}\}_{k \in K}$.

Remark. The idea is that $G_1 *_K G_2$ is the set of all words in $G_1 \cup G_2$ but if we see $\psi_1(k)$ in a word, we can replace it by $\psi_2(k)$ and vice versa. In terms of group presentations, if
$$\begin{aligned} G_1 &= \langle g_1, \dots, g_n | r_1, \dots, r_m \rangle, \\ G_2 &= \langle g_1', \dots, g_{n'}' | r_1', \dots, r_{m'}' \rangle, \\ K &= \langle h_1, \dots, h_k | s_1, \dots, s_\ell \rangle, \end{aligned}$$
then we can write
$$G_1 *_K G_2 = \langle g_1, \dots, g_n, g_1', \dots, g_{n'}' | r_1, \dots, r_m, r_1', \dots, r_{m'}', \psi_1(h_1)(\psi_2(h_1))^{-1}, \dots, \psi_1(h_k)(\psi_2(h_k))^{-1} \rangle.$$
Note that the s_i show up in the above since ψ_1, ψ_2 are homomorphisms.

Exercise 7.3. Show the following:

1. Check that the above presentation for $G_1 *_K G_2$ is correct.

2. Let $\iota_i : G_i \to G_1 * G_2$ be the inclusions and $j_i : G_i \to G_1 *_K G_2$ be the induced maps. Then given any homomorphisms $\phi_i : G_i \to H$ (where H is any group) such that
$$\phi_1 \circ \psi_1(k) = \phi_2 \circ \psi_2(k) \quad \text{for all } k \in K,$$
then there exists a unique homomorphism $\Phi : G_1 *_K G_2 \to H$ such that $\Phi \circ j_i = \phi_i$, i.e.
$$\begin{tikzcd} & G_1 \arrow[dr, "j_1"] \arrow[drr, bend left, "\phi_1"] & & \\ K, \arrow[ur, "\psi_1"] \arrow[dr, "\psi_2"'] & & G_1 *_K G_2 \arrow[r, dashed, "\Phi"] & H \\ & G_2 \arrow[ur, "j_2"'] \arrow[urr, bend right, "\phi_2"'] & & \end{tikzcd}$$
Show that this is the *universal property* for the free product with amalgamation.

7.2 Seifert-van Kampen Theorem

Theorem 7.1 (Seifert-van Kampen)**.** *Let X be a topological space with base point x_0. Let $A, B \subseteq X$ be open sets with $X = A \cup B$ such that $A, B, A \cap B$ are path-connected and $x_0 \in A \cap B$. Let*

$$\psi_A : \pi_1(A, x_0) \to \pi_1(X, x_0) \quad \text{and} \quad \psi_B : \pi_1(B, x_0) \to \pi_1(X, x_0)$$

be the homomorphisms induced from the inclusions $A \cap B \to A$ and $A \cap B \to B$. Then

$$\pi_1(X, x_0) \cong \pi_1(A, x_0) *_{\pi_1(A\cap B, x_0)} \pi_1(B, x_0).$$

Remark. There is a more general version where $X = \bigcup_{\alpha \in A} U_\alpha$. See Hatcher for more details.

Example 7.4.1. Consider $W_2 = S^1 \vee S^1$, a wedge of two circles. Let x_0 be the point of intersection of the circles. Let A be an open neighborhood of the left circle and B be an open neighborhood of the right circle. Note that $A \simeq B \simeq S^1$ and $A \cap B \simeq \{\text{pt}\}$. So we see that

$$\pi_1(A, x_0) \cong \mathbb{Z} \cong \langle g_1 | \ \rangle, \quad \pi_1(B, x_0) \cong \mathbb{Z} \cong \langle g_2 | \ \rangle, \quad \pi_1(A \cap B, x_0) = \{e\},$$

and so $\psi_A : \pi_1(A \cap B, x_0) \to \pi_1(A, x_0)$ and $\psi_B : \pi_1(A \cap B, x_0) \to \pi_1(B, x_0)$ must both map $e \mapsto e$. Thus

$$\pi_1(W_2, x_0) \cong \pi_1(A, x_0) *_{\pi_1(A\cap B, x_0)} \pi_1(B, x_0) \cong \langle g_1, g_2 | \psi_A(e)(\psi_B(e))^{-1} \rangle \cong \langle g_1, g_2 | \ \rangle \cong F_2,$$

by the Seifert-van Kampen theorem.

Exercise 7.4. Show the following:

1. If W_n is a wedge of n circles, then $\pi_1(W_n, x_0) \cong F_n$.
2. We have $\pi_1(\text{any connected graph}) = F_n$ for some n.

Example 7.4.2. Consider the torus $T^2 = S^1 \times S^1$. Recall that for a product, we know

$$\pi_1(T^2) \cong \pi_1(S^1) \times \pi_1(S^1) \cong \mathbb{Z} \times \mathbb{Z}.$$

We can also use van Kampen's theorem to see this. Think of T^2 as a square with opposite sides identified. Let A be the square with a circle missing in the circle, which deformation retracts to the boundary of the square. Let B be a big disk in the middle, so that $A \cap B$ deformation retracts to a circle. So

$$A \simeq W_2, \quad B \simeq \{\text{pt}\}, \quad A \cap B \simeq S^1.$$

Thus by our previous computations, we have

$$\pi_1(A, x_0) \cong \langle g_1, g_2 | \ \rangle, \quad \pi_1(B, x_0) \cong \{e\}, \quad \pi_1(A \cap B, x_0) \cong \langle h | \ \rangle.$$

The inclusion $\psi_B : \pi_1(A \cap B, x_0) \to \pi_1(B, x_0)$ sends $h \to e$, and $\psi_A : \pi_1(A \cap B, x_0) \to \pi_1(A, x_0)$ sends $h \mapsto g_1 g_2 g_1^{-1} g_2^{-1}$. To see the last claim, push a loop in $A \cap B$ to the boundary of the square (note that under the homotopy equivalence of A and W_2, we can assume x_0 maps to a corner point), where it follows the four edges of the square. These are the loops $g_1, g_2, g_1^{-1}, g_2^{-1}$, where the last two loops are oriented in the opposite direction of the first two. Thus by van Kampen's theorem,

$$\pi_1(T^2, x_0) \cong \pi_1(A, x_0) *_{\pi_1(A\cap B, x_0)} \pi_1(B, x_0) \cong \langle g_1, g_2 | \psi_A(h)(\psi_B(h))^{-1} \rangle = \langle g_1, g_2 | g_1 g_2 g_1^{-1} g_2^{-1} \rangle.$$

Exercise 7.5. Check the following:

1. Show that $\mathbb{Z} \times \mathbb{Z} \cong \langle g_1, g_2 | g_1 g_2 g_1^{-1} g_2^{-1} \rangle$, so that $\pi_1(T^2) \cong \mathbb{Z} \times \mathbb{Z}$ again.
2. Compute $\pi_1(\Sigma_g, x_0)$, where Σ_g is the surface of genus g. Show that $\pi_1(\Sigma_g)$ is not abelian if $g > 1$.

Lecture 8

Feb. 3 — Seifert-van Kampen Theorem

8.1 Applications of the Seifert-van Kampen Theorem

Theorem 8.1. *Let X be a path-connected space $f : \partial D^n \to X$ be continuous with $x_0 \in \partial D^n$. Set*

$$Y = X \cup_f D^n = X \sqcup D^n / \{(x \in D^n) \sim (f(x) \in X)\}.$$

Then (for $n = 1$, we need X to have a base point $f(x_0)$ with an open neighborhood $U \simeq \{f(x_0)\}$)

$$\pi_1(Y, y_0) = \begin{cases} \pi_1(X, f(x_0)) * \mathbb{Z} & \text{if } n = 1, \\ \pi_1(X, f(x_0))/\langle r \rangle & \text{if } n = 2, \\ \pi_1(X, f(x_0)) & \text{if } n \geq 3, \end{cases}$$

where $r = f_*(g)$ where g generates $\pi_1(\partial D^2, x_0) \cong \mathbb{Z}$.

Proof. We proof this in the case $n = 2$. Let

$$A = X \cup_f (D^2 \setminus \{0\}) = X \cup_f (S^1 \times (0, 1]) \simeq X$$

and B be the interior of D^2 (so $B \simeq \{\text{pt}\}$. Then we can see that

$$A \cap B = (\operatorname{int} D^2) \setminus \{0\} = S^1 \times (0, 1) \simeq S^1.$$

Note that we can choose y_0 to be $f(x_0) \in X$ because the that is where it is sent under the deformation retraction from A to X. Now $\psi_A : \pi_1(A \cap B, y_0) \to \pi_1(A, y_0)$

$$\pi_1(A \cap B, y_0) \cong \langle g | \; \rangle \quad \text{and} \quad \pi_1(A, y_0) \cong \pi_1(X, f(x_0))$$

is given by $g \mapsto f_*(g)$, and $\psi_B(g) = e$. Thus the Seifert-van Kampen theorem implies

$$\pi_1(Y, y_0) \cong \pi_1(A, y_0) *_{\pi_1(A \cap B, y_0)} \pi_1(B, y_0) \cong \frac{\pi_1(A, y_0) * \{e\}}{\langle \psi_A(g)(\psi_B(g))^{-1} \rangle} \cong \frac{\pi_1(A, y_0)}{\langle r \rangle} \cong \frac{\pi_1(X, f(x_0))}{\langle r \rangle},$$

which is the desired result for $n = 2$. The $n \geq 3$ case is similar (except the intersection is now contractible in this case). Check the $n = 1$ case as an exercise. $\square$

Remark. This allows us to compute the fundamental group of any CW complex (hence any manifold).

Corollary 8.1.1. *Let G be a finitely presented group. Then there exists a topological space X (in fact, a compact CW complex) such that $\pi_1(X, x_0) \cong G$.*

Proof. Let $G = \langle g_1, \dots, g_n | r_1, \dots, r_m$ and W_n be the wedge of n circles, so that

$$\pi_1(W_n, x_0) \cong F_n \cong \langle g_1, \dots, g_n | \ \rangle.$$

Now for each r_i let $f_i : S^1 \to W_n$ be a map such that $(f_i)_*(g) = r_i$ (show as an exercise that such f_i exists; essentially for each word, take the loops corresponding to each letter in order). Let

$$X = W_n \cup_{f_i} \left(\bigsqcup_{i=1}^m D^2 \right),$$

and the previous theorem tells us that $\pi_1(X, x_0) \cong \langle g_1, \dots, g_n | r_1, \dots, r_m \rangle \cong G$. $\square$

Remark. The topological space realizing G as its fundamental group is not unique. For instance, we can take the above construction and add a 5-cell, which does not change the fundamental group.

8.2 Proof of the Seifert-van Kampen Theorem

Proof of Theorem 7.1. We have the inclusions $A \subseteq X$ and $B \subseteq X$, which induce maps

$$\phi_A : \pi_1(A, x_0) \to \pi_1(X, x_0) \quad \text{and} \quad \phi_B : \pi_1(B, x_0) \to \pi_1(X, x_0).$$

By the universal property of free products, we get a map $\Phi : \pi_1(A, x_0) * \pi_1(B, x_0) \to \pi_1(X, x_0)$ by

$$([\gamma_1], [\delta_1], [\gamma_2], \dots) \mapsto \phi_A([\gamma_1])\phi_B([\delta_1])\phi_A([\gamma_2]) \dots.$$

Note that if $[\gamma] \in \pi_1(A \cap B, x_0)$, then $\psi_A([\gamma]) = [\gamma] = \psi_B([\gamma])$, so

$$\phi_A \circ \psi_A([\gamma]) = [\gamma] = \phi_B \circ \psi_B([\gamma]).$$

This tells us that $\Phi(\psi_A([\gamma])(\psi_B([\gamma]))^{-1}) = e$, so we see that

$$K = \langle \psi_A([\gamma])(\psi_B([\gamma]))^{-1} \rangle_{[\gamma] \in \pi_1(A \cap B, x_0)}$$

lies in the kernel of Φ. This gives us an induced map (still called it Φ)

$$\Phi : \pi_1(A, x_0) *_{\pi_1(A \cap B, x_0)} \pi_1(B, x_0) \to \pi_1(X, x_0).$$

Lemma 5.4 says that Φ is surjective, so it suffices to check injectivity. To do this, let $[\gamma_i] \in \pi_1(A, x_0)$ and $[\eta_i] \in \pi_1(B, x_0)$ with

$$\Phi([\gamma_1][\eta_1] \dots, [\gamma_n], [\eta_n]) = [\gamma_1 * \eta_1 * \dots * \gamma_n * \eta_n] = e. \tag{$*$}$$

We need to see that we can get from the word $[\gamma_1][\eta_1] \dots [\gamma_n][\eta_n]$ to the empty word by a sequence of:

1. replace a, b by $a \cdot b$ if a, b are in the same group (and the reverse of this);
2. if we see $\psi_A(k)$ in the word, we can replace it with $\psi_B(k)$ (and the reverse of this).

We will prove the theorem only for $n = 2$. Now $(*)$ says that there exists a homotopy H between x_0 and $\gamma_1 * \eta_1$. As before, we can use the Lebesgue number lemma to find n such that squares of side length $1/n$ are mapped by H into either A or B (we can assume that the number of γ_i, η_i divides n). Check as an exercise that we can assume $H(i/n, j/n) = x_0$, i.e. that we can change H and γ_i, η_i by a homotopy such that the homotopy of γ_i is in A and the homotopy of η_i is in B (hint: consider radial lines around $(i/n, j/n)$). So we have an $n \times n$ grid where each corner point is x_0, and the bottom edges are

$$\gamma_1 \sim \gamma_1' * \gamma_1'' \quad \text{and} \quad \eta_1 \sim \eta_1' * \eta_1'',$$

where these homotopies take place in A or B, respectively. Let a_1, a_2, a_3, a_4 be the four edges lying above $\gamma_1', \gamma_1'', \eta_1', \eta_1''$ on the grid (recall that $n = 2$ in this case) We will show that we can go from $[\gamma_1][\eta_1]$ to $[a_1][a_2][a_3][a_4]$ using (1) and (2). Then we can inductively push this to the top, which is the empty word.

Let $\delta_1, \delta_2, \delta_3$ be the three interior edges which connect the bottom row and the second-to-last row. Since each square maps to A or B, the first three squares lie in A and the last one lies in B. Let

$$G = \pi_1(A, x_0) \quad \text{and} \quad H = \pi_1(B, x_0).$$

Note that we have

$$\delta_1 * \eta_1' \sim \delta_1 * \eta' * \delta_3 * \overline{\delta}_3 \text{ in } G, \quad \delta_1 * \eta_1' * \delta_3 \sim a_1 * a_2 * a_3 \text{ in } G, \quad \delta_3 * \eta_1'' \sim a_4 \text{ in } H,$$

and thus we can write

$$\begin{aligned}
[\gamma_1]^G[\eta_1]^H = [\gamma_1]^G([\eta_1'][\eta_1''])^H &\overset{(1)}{=} [\gamma_1]^G[\eta_1']^H[\eta_1'']^H \\
&\overset{(2)}{=} [\gamma_1]^G[\eta_1']^G[\eta_1'']^H \\
&\overset{(1)}{=} ([\gamma_1][\eta_1'])^G[\eta_1'']^H = ([\gamma_1][\eta_1'][\delta_3][\overline{\delta}_3])^G[\eta_1'']^H \\
&\overset{(2)}{=} ([\gamma_1][\eta_1'][\delta_3])^G[\overline{\delta}_3]^H[\eta_1'']^H \\
&\overset{(1)}{=} ([a_1][a_2][a_3])^G([\overline{\delta}_3][\eta_1''])^H = ([a_1][a_2][a_3])^G[a_4]^H \\
&\overset{(1)}{=} [a_1]^G[a_2]^G[a_3]^G[a_4]^H,
\end{aligned}$$

which proves the statement for $n = 2$. See Hatcher for the general case. □

8.3 Covering Spaces

Definition 8.1. A *covering space* of a space X is a pair $(\widetilde{X}, p)$ where $\widetilde{X}$ is a space and $p : \widetilde{X} \to X$ such that every point $x \in X$ has an *evenly covered neighborhood*. An open set U is *evenly covered* if

$$p^{-1}(U) = \text{disjoint union of open sets } \{U_\alpha\} \text{ in } \widetilde{X}$$

such that $p|_{U_\alpha} : U_\alpha \to U$ is a homeomorphism for every α.

Example 8.1.1. The following are examples of covering maps:

1. If $p : \widetilde{X} \to X$ is a homeomorphism, then it is a covering map.
2. We saw that $p : \mathbb{R} \to S^1$ given by $t \mapsto (\cos 2\pi t, \sin 2\pi t)$ is a covering map.

Exercise 8.1. If $(\widetilde{X}, p)$ is a covering space of X and $(\widetilde{Y}, p)$ is a covering space of of Y, then show that

$$p \times p' : \widetilde{X} \times \widetilde{Y} \to X \times Y, \quad (x, y) \mapsto (p(x), p'(y))$$

is a covering map. This gives a covering map $\mathbb{R}^2 \to T^2$ by $(t, s) \mapsto (\cos 2\pi t, \sin 2\pi t, \cos 2\pi s, \sin 2\pi s)$.

Lecture 9

Feb. 5 — Covering Spaces

9.1 More on Covering Spaces

Example 9.0.1. The following are more examples of covering maps:

3. Define $p_n : S^1 \to S^1$ by $\theta \mapsto n\theta$. These are covering maps for each $n \in \mathbb{Z}$.
4. Let X be a wedge of two circles (corresponding to a, b), and let $\widetilde{X}$ be a circle with three outer circles attached, evenly spaced (the inner circle corresponding to a_1, a_2, a_3 and the three outer circles corresponding to b_1, b_2, b_3). Let p map a_i to a and b_i to b. This is a covering map.

   Write out a formula for p and really check that p is a covering map as an exercise.
5. Again let X be a wedge of two circles, labeled a, b. Let $\widetilde{X}$ be a wedge of two circles, with an extra circle attached on either side. Let a_3, b_3 be the extra circles on the left and right, let b_1, b_2 be the top and bottom halves of the circle on the left, and let a_1, a_2 be the top and bottom halves of the circles on the right. Let p map a_i to a and b_i to b. Then p is a covering map.
6. Consider the quotient map $p : S^2 \to \mathbb{R}P^2$. This map is a covering map. Note that each point in $\mathbb{R}P^2$ has a neighborhood whose preimage is two open sets on opposite sides of S^2.

Lemma 9.1. *Let $(\widetilde{X}, p)$ be a covering space of a connected space X. Then the cardinality $|p^{-1}(x)|$ is independent of $x \in X$.*

Proof. Fix $x_0 \in X$ and let $n = |p^{-1}(x_0)|$. Let $A = \{x \in X : |p^{-1}(X)| = n\}$, and note that $A \neq \varnothing$ since $x_0 \in A$. We will show that A is both open and closed, which implies that $A = X$ by connectedness.

To see that A is open, let $x \in A$. By the definition of a covering space, there exists an open set U in X such that $x \in U$ and $p^{-1}(U) = \{U_1, \ldots, U_n\}$. Thus for any $x' \in U$, we have

$$p^{-1}(x') \subseteq p^{-1}(U) = \{U_1, \ldots, U_n\},$$

and $p^{-1}(x') \cap U_i = \{\mathrm{pt}\}$ since $p|_{U_i} : U_i \to U$ is a homeomorphism. Thus $|p^{-1}(x')| = n$, so $x' \in A$. This holds for each $x' \in U$, so $U \subseteq A$, i.e. A is an open set.

One can make a similar argument to check that if $x \notin A$, then there exists a open set U about x such that $U \cap A = \varnothing$. This shows that $X \setminus A$ is open, i.e. A is closed, which completes the proof. ☐

Definition 9.1. We call $|p^{-1}(x)|$ the *degree* of the covering space.

Definition 9.2. If $(\widetilde{X}, p)$ is a covering space for X and $f : Y \to X$ is a continuous map, then a *lift* of f to $\widetilde{X}$ is a map $\widetilde{f} : Y \to \widetilde{X}$ such that $p \circ \widetilde{f} = f$, i.e. the following diagram commutes:

$$\begin{tikzcd} & \widetilde{X} \arrow[d, "p"] \\ Y \arrow[r, "f"] \arrow[ur, dashed, "\widetilde{f}"] & X \end{tikzcd}$$

If $f(y_0) = x_0$ and $\widetilde{x}_0 \in \widetilde{X}$ with $p(\widetilde{x}_0) = x_0$, then $\widetilde{f}$ is a *lift of f based at $\widetilde{x}_0$* if $\widetilde{f}$ is a lift and $\widetilde{f}(y_0) = \widetilde{x}_0$.

Lemma 9.2. *Let $(\widetilde{X}, p)$ be a covering space of X, $x_0 \in X$, and $\widetilde{x}_0 \in p^{-1}(x_0)$. Then*

(a) each path $\gamma : [0,1] \to X$ based at x_0 has a unique lift $\widetilde{\gamma} : [0,1] \to X$ based at $\widetilde{x}_0$.

(b) if $H : Y \times [0,1] \to X$ is a homotopy with $h_0(y) = H(y,0)$ and $\widetilde{h}_0 : Y \to \widetilde{X}$ a lift of h_0, then there is a unique lift $\widetilde{H} : Y \times [0,1] \to \widetilde{X}$ of H such that $\widetilde{H}(y,0) = h_0(y)$.

The above properties are called path lifting *and* homotopy lifting.

Proof. (a) The proof of this part is exactly the proof of part (a) in Lemma 5.5.

(b) The proof from Lemma 5.5 works if $Y = [0,1]$. For the general case, see the proof of Theorem 23. $\square$

Lemma 9.3. *If $(\widetilde{X}, p)$ is a path connected covering space of X and $x_0 \in X$, $\widetilde{x}_0 \in p^{-1}(x_0)$, then the map $p_* : \pi_1(\widetilde{X}, \widetilde{x}_0) \to \pi_1(X, x_0)$ satisfies the following:*

1. *p_* is injective;*
2. *the image of p_* is the set of loops in $\pi_1(X, x_0)$ that when lifted are loops in $\widetilde{X}$ based at $\widetilde{x}_0$;*
3. *the index $[\pi_1(X, x_0) : p_*(\pi_1(\widetilde{X}, \widetilde{x}_0))]$ is the degree of $(\widetilde{X}, p)$.*

Proof. (1) Suppose that $p_*([\gamma]) = [e]$, so there exists a homotopy H in X between x_0 and $p \circ \gamma$. Note that γ is a lift of $H(t, 0)$, so Lemma 9.2 says H lifts to a homotopy $\widetilde{H}$ starting at γ in $\widetilde{X}$. Note that $H|_{\{0\}\times[0,1]}$ is a constant loop and the loop $t \mapsto \widetilde{x}_0$ is a lift of $H|_{\{0\}\times[0,1]}$ based at $\widetilde{x}_0$, so by uniqueness we see that $\widetilde{H}|_{\{0\}\times[0,1]} = \widetilde{x}_0$. Similarly, we see that $\widetilde{H}_{\{1\}\times[0,1]} = \widetilde{x}_0$ and $\widetilde{H}_{[0,1]\times\{1\}} = \widetilde{x}_0$. Thus $\widetilde{H}$ is a homotopy of loops based at $\widetilde{x}_0$ from γ to the constant loop, i.e. $[\gamma] = [e_{x_0}]$. So p_* is injective.

(2) Clearly if $[\gamma] \in \pi_1(X, x_0)$ lifts to a loop $\widetilde{\gamma}$ based at $\widetilde{x}_0$, then $[\gamma] = p_*(\widetilde{\gamma})$, so $[\gamma]$ is in the image of p_*. Now if $[\eta] = p_*([\gamma])$, then $\eta \sim p \circ \gamma$ in X. Let $\widetilde{\eta}$ be the lift of η based at $\widetilde{x}_0$. By Lemma 9.2, the homotopy $\eta \sim p \circ \gamma$ lifts to a homotopy $\widetilde{\eta} \sim \gamma$ rel endpoints. But γ is a loop, so $\widetilde{\eta}$ must be a loop.

(3) Let $H = p_*(\pi_1(\widetilde{X}, \widetilde{x}_0)) \leq \pi_1(X, x_0)$. If $[\gamma] \in \pi_1(X, x_0)$ and $[\delta] \in H$, then note that by part (2), δ lifts to a loop $\widetilde{\delta}$ based at $\widetilde{x}_0$. Let $\widetilde{\delta * \gamma}$ be a lift of $\delta * \gamma$ based at $\widetilde{x}_0$, and note that $\widetilde{\gamma}(1) = \widetilde{\delta * \gamma}(1) = \widetilde{\delta} * \widetilde{\gamma}(1)$. This allows us to define

$$\phi : \{\text{right cosets of } H\} \to p^{-1}(x_0)$$

by $H[\gamma] \mapsto \widetilde{\gamma}(1)$, which is well-defined by the above arguments.

If $\widetilde{x}_1 \in p^{-1}(x_0)$, then let $\widetilde{\gamma}$ be a path in $\widetilde{X}$ from $\widetilde{x}_0$ to $\widetilde{x}_1$. Let $\gamma = p \circ \widetilde{\gamma}$, which is a loop in X based at x_0. Clearly $\phi(H[\gamma]) = \widetilde{\gamma}(1) = \widetilde{x}_1$, so ϕ is onto. Now suppose that $\phi(H[\gamma]) = \phi(H[\eta])$. If $\widetilde{\gamma}, \widetilde{\eta}$ are lifts of γ based at $\widetilde{x}_0$, then $\widetilde{\gamma}(1) = \widetilde{\eta}(1)$. Thus $\widetilde{\gamma} * \overline{\widetilde{\eta}}$ is a loop in $\widetilde{X}$, so

$$p_*([\widetilde{\gamma} * \overline{\widetilde{\eta}}]) = [\gamma] * [\overline{\eta}] = [\gamma] * [\eta]^{-1} \in H.$$

This gives $H[\gamma] = H[\eta]$, so ϕ is injective. Thus ϕ is a bijection, so $[\pi_1(X, x_0) : H] = |p^{-1}(x_0)|$. $\square$

Example 9.2.1. Recall the following examples from before:

1. For the covering map $p : \mathbb{R} \to S^1$, we have $p_* : \pi_1(\mathbb{R}, 0) \to \pi_1(S^1, (1, 0))$ which sends $e \mapsto 0$, if we view p_* as $p_* : \{e\} \to \mathbb{Z}$. One can see all three properties of the above lemma in this example, in particular that $[\pi_1(S^1) : \pi_1(\mathbb{R})] = \infty = |p^{-1}((1, 0))|$.
2. Let $p_n : S^1 \to S^1$ send $\theta \mapsto n\theta$. Then $(p_n)_* : \pi_1(S^1, (1, 0)) \to \pi_1(S^1, (1, 0))$, which we can view as a map $(p_n)_* : \mathbb{Z} \to \mathbb{Z}$, or $(p_n)_* : \langle g| \ \rangle \to \langle h| \ \rangle$. Then $(p_n)_*$ maps $g \mapsto h^n$, so the subgroup of $\mathbb{Z}$ corresponding to this covering space is $n\mathbb{Z}$. Again we can see all three properties of the above lemma, in particular that we have $[\mathbb{Z} : n\mathbb{Z}] = n = |p_n^{-1}((1, 0))|$.

Exercise 9.1. Check the properties of the above lemma explicitly for Example 9.0.1(4). Note that picking different base points can yield different images of p_*.