MATH 6441: Algebraic Topology I

Frank Qiang
Instructor: John Etnyre

Georgia Institute of Technology
Spring 2025

Contents

Lecture 1

Jan. 6 — CW-Complexes

1.1 Introduction and Motivation

Algebraic topology builds "functions" (actually *functors*)

$$\{\text{topological spaces, continuous maps}\} \longrightarrow \{\text{algebraic things, algebraic maps}\},$$

where "algebraic things" can be groups, vector spaces, etc. The main objective of algebraic topology is to *distinguish topological spaces*, e.g. showing that $\mathbb{R}^n \not\cong \mathbb{R}^m$ for $n \neq m$. More applications are:

1. Studying maps between spaces.[1]
   - Does a given space M embed in N? For instance, for what m does $\mathbb{R}P^n$ embed in $\mathbb{R}^m$? (This is still not known in general.) Here $\mathbb{R}P^n$ is the real projective space.
   - Lifting maps, i.e. given $f : A \to B$ and $g : E \to B$, does there exist a map $\widetilde{f} : A \to E$ such that $g \circ \widetilde{f} = f$? In other words, is there a map $\widetilde{f}$ such that the following diagram commutes:

$$\begin{tikzcd} & E \arrow[d, "g"] \\ A \arrow[ur, dashed, "\widetilde{f}"] \arrow[r, "f"'] & B \end{tikzcd}$$

   - Fixed point problems: Given $f : X \to X$, does f have a fixed point, i.e. $x \in X$ such that $f(x) = x$? Such theorems are used to prove the existence of solutions to ordinary differential equations, for instance.
2. Group actions, e.g. which finite groups act freely on S^n?
3. Group theory, e.g. showing that every subgroup of a free group is free. Another example is that if F_n is the free group on n generators, then its *commutator* $[F_n, F_n]$ is not finitely generated.
4. Algebra, e.g. proving the fundamental theorem of algebra.

This course will cover the following topics:

1. The *fundamental group* $\pi_1(X, x_0)$ of a space X for $x_0 \in X$, and *covering spaces*.
2. The *homology groups* $H_k(X)$ for $k = 0, 1, 2, \ldots$. These groups are abelian.
3. The *cohomology ring* $H^*(X) = \bigoplus_{k=0}^{\infty} H^k(X)$.

But before getting to this, we need to develop some important ideas.

[1] All maps and functions in this class are continuous unless otherwise specified.

1.2 CW-Complexes

Definition 1.1. Let $D^n \subseteq \mathbb{R}^n$ be the unit disk and $S^{n-1} = \partial D^n$. Given a topological space Y and a continuous map $a : S^{n-1} \to Y$, the space obtained from Y by *attaching* an *n-cell* (*via* a) is

$$Y \cup_a D^n = (Y \sqcup D^n)/\sim,$$

where the equivalence relation $\sim$ is given by $x \sim a(x)$ for $x \in \partial D^n$.[2]

Definition 1.2. An *n-complex* or an *n-dimensional CW-complex* is defined inductively by:

- A (-1)-complex is the empty set $\varnothing$.
- An n-complex X^n is a space obtained from an $(n-1)$-complex by attaching n-cells.

An n-complex is *finite* if it involves only a finite number of cells. The *k-skeleton* of X is the union of all n-cells in X with $n \leq k$.

Remark. Any CW-complex is Hausdorff. See Hatcher for a proof.

Example 1.2.1. Here are some examples of CW-complexes:

- A 0-complex is a union of points. This is because $D^0 = \{\text{pt}\}$ and $\partial D^0 = \varnothing$.
- A 1-complex is a graph (points and lines connecting them).
- The torus T (a square with opposite sides identified) is a 2-complex. Here the 0-skeleton $T^{(0)}$ is the common corner on the square and the 1-skeleton $T^{(1)}$ is two sides of the square after taking the quotient. The 2-skeleton $T = T^{(2)}$ is the entire torus.
- Another example of a 2-complex is the two-holed torus, which is obtained by identifying the edges of an octagon (pairs of every other edge identified with opposite orientation).
- A third example of a 2-complex is $X^{(1)} \cup_a D^2$ given an attaching map $D^2 \to X^{(1)}$.
- Consider the unit sphere $S^n \subseteq \mathbb{R}^{n+1}$. One way to give S^2 a CW-complex structure is to see the sphere as two disks D^2 glued together, resulting in one 0-cell, one 1-cell, and two 2-cells. Another way is to start with two points, attach two 1-cells to get a circle, and then attaching two disks to get S^2. This results in two 0-cells, two 1-cells, and two 2-cells.

  The second idea generalizes to S^n. We can write

  $$S^n = S^{n-1} \cup_{a_1} D^n \cup_{a_2} D^n,$$

  where S^{n-1} inductively has a CW-complex structure. This yields two k-cells for each $k \leq n$.

  Another way to put a CW-complex structure on S^n is to attach D^n to a point with $\partial D^n \to \{\text{pt}\}$. In particular, notice that a space can in general have several different CW-complex structures.

- Consider the *n-dimensional real projective space*

  $$\mathbb{R}P^n = \{\text{lines through the origin in } \mathbb{R}^{n+1}\}.$$

  Since each line through the origin passes through S^n twice, we can equivalently think of $\mathbb{R}P^n$ as the unit sphere S^n with antipodal points identified.

[2] Here $A \sqcup B$ denotes the *disjoint union* of A and B.

We can also think of this as D^n with antipodal points on ∂D^n identified. Since $\partial D^n = S^{n-1}$, this is simply $\mathbb{R}P^{n-1} \cup_a D^n$, where $a : \partial D^n \to \mathbb{R}P^{n-1}$ is the quotient map. This gives $\mathbb{R}P^n$ a CW-complex structure with one k-cell for each $k \leq n$.

- The complex projective space $\mathbb{C}P^n$ has a similar CW-complex structure with one k-cell for each even $k \leq 2n$. One can verify this as an exercise.
- Any smooth manifold has a CW-complex structure. See Hatcher.

Exercise 1.1. Show the product of CW-complexes is a CW-complex.

Definition 1.3. A *subcomplex* of a CW-complex X is a closed subset $A \subseteq X$ that is a union of cells in X. In particular, A is also a CW-complex and (X, A) is called a *CW-pair*.

1.3 Homotopy

Definition 1.4. Let X and Y be topological spaces. Two maps $f, g : X \to Y$ are *homotopic*, denoted $f \sim g$, if there exists a continuous map $\Phi : X \times [0,1] \to Y$ such that

$$\Phi(x, 0) = f(x) \quad \text{and} \quad \Phi(x, 1) = g(x).$$

In this case, Φ is called a *homotopy* between f and g.

Remark. We note the following:

- A homotopy Φ gives a family of maps $\Phi_{t_0} : X \to Y$ given by $x \mapsto \Phi(x, t_0)$ which is continuous in t_0. So maps are homotopic if there is a continuous family of maps between them.
- If $A \subseteq X$ then we say that f is *homotopic to g rel A*, denoted $f \sim_A g$, if there exists Φ as above with the additional property that $\Phi(x, t) = f(x)$ for all $x \in A$, i.e. points in A are fixed.
- If $A \subseteq X$ and $B \subseteq Y$, then the notation $f : (X, A) \to (X, B)$ means that $f : X \to Y$ and $f(a) \in B$ for each $a \in A$. We say that f is a *map of pairs*. If $f, g : (X, A) \to (Y, B)$, then f, g are homotopic as maps of pairs if each Φ_t is a map of pairs.