MATH 7337: Harmonic Analysis

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Contents

Lecture 1

Aug. 19 — The Fourier Transform

1.1 The Fourier Transform on $L^1(\mathbb{R})$

All integrals will be taken over $\mathbb{R}$ unless otherwise specified.

Definition 1.1. The *Fourier transform* of $f \in L^1(\mathbb{R})$ is

$$\widehat{f}(\xi) = \int f(x)e^{-2\pi i\xi x}\,dx, \quad \xi \in \mathbb{R}.$$

Remark. Note that by the triangle inequality,

$$|\widehat{f}(\xi)| \le \int |f(x)e^{-2\pi i\xi x}|\,dx = \int |f(x)|\,dx = \|f\|_1 < \infty,$$

so $\widehat{f}(\xi)$ exists for all $\xi \in \mathbb{R}$ (in fact, $\widehat{f}$ is continuous).

Remark. The Fourier transform is an operator $\mathcal{F} : L^1(\mathbb{R}) \to L^\infty(\mathbb{R})$ as $\|\widehat{f}\|_\infty = \operatorname{ess\,sup}_{\xi\in\mathbb{R}} |\widehat{f}(\xi)| \le \|f\|_1$. This is linear in f. The *operator norm* of $\mathcal{F}$ is

$$\|\mathcal{F}\| = \|\mathcal{F}\|_{L^1\to L^\infty} = \sup_{\|f\|_1=1} \|\widehat{f}\|_\infty \le \sup_{\|f\|_1=1} \|f\|_1 = 1,$$

so $\mathcal{F}$ is a bounded linear operator. However, $\mathcal{F}$ is not isometric (norm-preserving) in general.

Remark. Observe that

$$\widehat{f}(0) = \int f(x)e^{-2\pi i\cdot 0\cdot x}\,dx = \int f(x)\,dx.$$

So if $f \ge 0$ and we normalize f so that $\widehat{f}(0) = 1$, then we have

$$|\widehat{f}(\xi)| \le \int f(x)\,dx = \widehat{f}(0),$$

and so $\|\widehat{f}\|_\infty = \operatorname{ess\,sup}_{\xi\in\mathbb{R}} |\widehat{f}(\xi)| \le 1$. This is one particular case where $\mathcal{F}$ does preserve the norm.

Definition 1.2. For $r \ne 0$, *dilation* of f by r is $f_r(x) = rf(rx)$. Note that $\|f_r\|_1 = \|f\|_1$.

Example 1.2.1. The *Dirichlet function* is $d(\xi) = \sin(\xi)/\pi\xi \in C_0(\mathbb{R})$.[1] Note that $d \notin L^1(\mathbb{R})$. We can also define the *sinc* function as $\operatorname{sinc}\xi = \sin(\pi\xi)/(\pi\xi) = d\pi(x)$.

[1]Recall that $C_0(\mathbb{R})$ is the space of continuous functions $f : \mathbb{R} \to \mathbb{C}$ such that $\lim_{x\to\pm\infty} f(x) = 0$.

However, d is the Fourier transform of a function in $L^1(\mathbb{R})$. Consider the

$$\chi_{-[T,T]}(x) = \begin{cases} 1 & \text{if } |x| \leq T, \\ 0 & |x| > T. \end{cases}$$

Note that $\chi_{-[T,T]} \in L^1(\mathbb{R})$. Then we have

$$\widehat{\chi}_{-[T,T]}(\xi) = \int_{-T}^{T} e^{-2\pi i \xi x}\, dx = \frac{e^{-2\pi i \xi x}}{-2\pi i \xi}\bigg|_{-T}^{T} = \frac{\sin(2\pi T \xi)}{\pi \xi} = d_{2\pi T}(\xi),$$

so we see that $\widehat{\chi}_{-[T,T]} \in C_0(\mathbb{R}) \subsetneq L^\infty(\mathbb{R})$.

Remark. We will see in general that $\mathcal{F} : L^1(\mathbb{R}) \to C_0(\mathbb{R})$, this is the Riemann-Lebesgue lemma. The image of $\mathcal{F}$ is a proper dense subspace of $C_0(\mathbb{R})$, which implies that $\mathcal{F}^{-1}$ must be unbounded as a linear operator by Banach space theory.

Proposition 1.1. *If $f \in L^1(\mathbb{R})$, then $\widehat{f}$ is uniformly continuous on $\mathbb{R}$, i.e.*

$$\|\widehat{f} - T_\eta \widehat{f}\|_\infty = \sup_{\xi \in \mathbb{R}} |\widehat{f}(\xi) - \widehat{f}(\xi - \eta)| \xrightarrow{\eta \to 0} 0,$$

where $T_\eta \widehat{f}(\xi) = \widehat{f}(\xi - \eta)$.

Proof. We can write the difference as

$$|\widehat{f}(\xi) - \widehat{f}(\xi - \eta)| = \left| \int f(x)(e^{-2\pi i \xi x} - e^{-2\pi i (\xi - \eta) x})\, dx \right| \leq \int |f(x)| |e^{-2\pi i \xi x} - e^{-2\pi i (\xi - \eta) x}|\, dx.$$

Note that $f \in L^1(\mathbb{R})$ and $|e^{-2\pi i \xi x} - e^{-2\pi i (\xi - \eta) x}| = |1 - e^{2\pi i \eta x}| \to 0$ as $\eta \to 0$ independent of ξ, so the statement follows from the dominated convergence theorem (the integrand is dominated by $2f$). $\square$

1.2 Motivation for the Fourier Transform

Remark. We will define the *inverse Fourier transform* of $f \in L^1(\mathbb{R})$ as

$$\check{f}(x) = \int f(x) e^{2\pi i \xi x}\, d\xi.$$

Note that $\check{f}(\xi) = \widehat{f}(-\xi)$. With enough assumptions, this is an inverse to the Fourier transform.

Proposition 1.2 (Fourier inversion formula)**.** *If $f, \widehat{f} \in L^1(\mathbb{R})$, then*

$$f(x) = (\widehat{f})^\vee(x) = \int \widehat{f}(\xi) e^{2\pi i \xi x}\, d\xi.$$

Remark. Note that $e_\xi(x) = e^{2\pi i \xi x} = \cos 2\pi \xi x + i \sin 2\pi \xi x$ and $e_\xi : \mathbb{R} \to S^1 = \{z \in \mathbb{C} : |z| = 1\}$. We have $e_\xi(x + y) = e_\xi(x) e_\xi(y)$, so e_ξ is a homomorphism, and it is also continuous. Thus e_ξ is a *character* on $\mathbb{R}$ (in fact, every character on $\mathbb{R}$ is of the form e_ξ for some ξ). One can use this idea to define Fourier transforms in much more general settings.

Remark. The Fourier transform decomposes a function f into the pure harmonics e_ξ, and the inversion formula says that we can recover f as a "sum" of these pure harmonics.

Lecture 2

Aug. 21 — The Riemann-Lebesgue Lemma

2.1 Properties of the Fourier Transform

Definition 2.1. Define the following operators:

1. *Translation*: $T_a f(x) = f(x-a)$ for $a \in \mathbb{R}$;
2. *Modulation*: $M_b f(x) = e^{2\pi i b x} f(x)$ for $b \in \mathbb{R}$;
3. *Dilation*: $f_\lambda(x) = \lambda f(\lambda x)$ for $\lambda > 0$;
4. *Involution*: $\widetilde{f}(x) = \overline{f(-x)}$.

Remark. Translation and modulation are isometries on $L^p(\mathbb{R})$ for any p. Dilation as defined above is L^1-normalized, so it is only an isometry on $L^1(\mathbb{R})$.

Exercise 2.1. If $f \in L^1(\mathbb{R})$, then

1. $(T_a f)^\wedge(\xi) = (M_{-a}\widehat{f})(\xi) = e^{-2\pi i \xi a}\widehat{f}(\xi)$;
2. $(M_b f)^\wedge(\xi) = (T_b \widehat{f})(\xi) = \widehat{f}(\xi - b)$;
3. $(f_\lambda)^\wedge(\xi) = \lambda (f_{1/\lambda})^\wedge(\xi) = \widehat{f}(\xi/\lambda)$;[1]
4. $(\overline{f})^\wedge(\xi) = (\widehat{f})^\sim(\xi) = \overline{\widehat{f}(-\xi)}$;
5. $(\widetilde{f})^\wedge(\xi) = \overline{\widehat{f}(\xi)}$.

2.2 The Riemann-Lebesgue Lemma

Definition 2.2. Let $C_c(\mathbb{R})$ be the space of continuous functions with compact support. For a continuous function, the *support* of f, denoted $\operatorname{supp}(f) = \overline{\{x \in \mathbb{R} : f(x) \neq 0\}}$. So for a continuous function f, $\operatorname{supp}(f)$ is compact if and only if $f = 0$ outside some finite interval.

Theorem 2.1. *$C_c(\mathbb{R})$ is dense in $L^p(\mathbb{R})$ for $1 \le p < \infty$. In other words,*

1. *the closure of $C_c(\mathbb{R})$ in $L^p(\mathbb{R})$ is all of $L^p(\mathbb{R})$;*
2. *for any $f \in L^p(\mathbb{R})$ and $\epsilon > 0$, there exists $g \in C_c(\mathbb{R})$ such that $\|f - g\|_p < \epsilon$;*
3. *if $f \in L^p(\mathbb{R})$, then there exists $g_n \in C_c(\mathbb{R})$ such that $g_n \to f$ in L^p-norm, i.e. $\|g_n - f\|_p \to 0$.*

[1] Note that the result is an L^∞-normalized dilation.

For $p = \infty$, $C_c(\mathbb{R})$ is dense in $C_0(\mathbb{R})$ with respect to the L^∞-norm (this is the same as the uniform norm for continuous functions).

Proof. We sketch the proof. First approximate $f \in L^p(\mathbb{R})$ by a simple function (one that takes only finitely many distinct values) $\phi = \sum_{k=1}^N c_k \chi_{E_k}$, e.g. by rounding down to the nearest integer multiple of 2^{-n}. Then use Urysohn's lemma to approximate χ_{E_k} by a continuous function. □

Exercise 2.2. Fix $1 \le p < \infty$. Prove that if $f \in L^p(\mathbb{R})$, then $\lim_{a\to 0} \|f - T_a f\|_p = 0$. We say that translation is *strongly continuous* on $L^p(\mathbb{R})$. For $p = \infty$, use $C_0(\mathbb{R})$ and the uniform norm instead.

Lemma 2.1 (Riemann-Lebesgue lemma)**.** *If $f \in L^1(\mathbb{R})$, then $\widehat{f} \in C_0(\mathbb{R})$,*

Proof. We have already seen that $\widehat{f}$ is continuous. So it suffices to show decay at ∞. Write

$$\widehat{f}(\xi) = -\int f(x) e^{-2\pi i \xi x} e^{-2\pi i \xi (1/2\xi)}\, dx = -\int_{-\infty}^{\infty} f(x) e^{-2\pi i \xi (x + 1/2\xi)}\, dx.$$

Now make the change of variables $x \mapsto x - 1/2\xi$, so we get

$$\widehat{f}(\xi) = -\int_{-\infty}^{\infty} f\left(x - \frac{1}{2\xi}\right) e^{-2\pi i \xi x}\, dx = -\int T_{1/2\xi} f(x) e^{-2\pi i \xi x}\, dx.$$

Taking an average with the usual expression for $\widehat{f}(\xi)$, we have

$$\widehat{f}(\xi) = \frac{1}{2} \int (f(x) - T_{1/2\xi} f(x)) e^{-2\pi i \xi x}\, dx.$$

Taking absolute values, we obtain

$$|\widehat{f}(\xi)| \le \frac{1}{2} \int |f(x) - T_{1/2\xi} f(x)|\, dx = \frac{1}{2} \|f - T_{1/2\xi} f\|_1 \xrightarrow[\xi \to \pm\infty]{} 0$$

by the strong continuity of translation on $L^1(\mathbb{R})$. □

Exercise 2.3. The following is an alternative proof of the Riemann-Lebesgue lemma. Recall that we have $\widehat{\chi}_{-T,T} = d_{2\pi T} \in C_0(\mathbb{R})$. By taking translations and dilations, we see that $\widehat{\chi}_{[a,b]} \in C_0(\mathbb{R})$. Consider *really simple functions* $\phi = \sum_{k=1}^N c_k \chi_{[a_k, b_k]}$, and by linearity we can write

$$\widehat{\phi} = \sum_{k=1}^N c_k \widehat{\chi}_{[a_k, b_k]} \in C_0(\mathbb{R}).$$

Note that really simple functions are also dense in $L^p(\mathbb{R})$ for $1 \le p < \infty$. So if $f \in L^1(\mathbb{R})$, there exist really simple $\phi_n \to f$ in L^1-norm. On the Fourier side, we have

$$\|\widehat{f} - \widehat{\phi}_n\|_\infty \le \|f - \phi_n\|_1 \longrightarrow 0.$$

Since $\phi_n \to \widehat{f}$ uniformly and $C_0(\mathbb{R})$ is a Banach space, we conclude $\widehat{f} \in C_0(\mathbb{R})$. Fill in the details.

2.3 Position and Momentum Operators

Definition 2.3. The *position operator* $P : L^1(\mathbb{R}) \to L^1(\mathbb{R})$ is given by $Pf(x) = xf(x)$. Note that P is unbounded on $L^1(\mathbb{R})$ (in fact, P is not defined on all of $L^1(\mathbb{R})$). Restrict P to the domain

$$D_P = \{f \in L^1(\mathbb{R}) : xf(x) \in L^1(\mathbb{R})\},$$

which is dense in $L^1(\mathbb{R})$. Note that D_P cannot be bounded as it does not admit an extension to $L^1(\mathbb{R})$.

Exercise 2.4. Show that $\sup_{\|f\|_1=1, f\in D_P} \|Pf\|_1 = \infty$.

Definition 2.4. The *momentum operator* $M : L^1(\mathbb{R}) \to L^1(\mathbb{R})$ is given by $Mf = f'/2\pi i$. Similarly, M is unbounded and defined only on a dense subset of $L^1(\mathbb{R})$.

Remark. We have the relation $(Mf)^\wedge(\xi) = \xi P\widehat{f}(\xi)$, whenever the statement makes sense.

2.4 An Open Problem

Conjecture 2.1 (HRT conjecture). *Assume g is not zero a.e., a_k, b_k are distinct, and consider finite linear combinations of translations and modulations of $g \in L^2(\mathbb{R})$ of the following form:*

$$\sum_{k=1}^{N} c_k e^{2\pi i b_k x} g(x - a_k). \qquad (*)$$

If $(*) = 0$, then must it be that $c_1 = \cdots = c_N = 0$? In other words, are these linearly independent?

Remark. Consider the special case $b_k = 0$ for every k, so $\sum c_k T_{a_k} g = \sum c_k g(x - a_k) = 0$ a.e. Then

$$\left(\sum c_k T_{a_k} g\right)^\wedge = \sum c_k M_{-a_k} \widehat{g} = \left(\sum_{k=1}^{N} c_k e^{-2\pi i a_k \xi}\right) \widehat{g}(\xi) = 0.$$

Since $\widehat{g}$ is not zero a.e., we must have $\sum_{k=1}^N c_k e^{-2\pi i a_k \xi} = 0$, which implies $c_k = 0$ for all k. In particular, this means that translations alone are linearly independent (the same is true for modulations alone).

Remark. The general case of the HRT conjecture is still open. Note that after taking a Fourier transform, we end up with the same problem, just for $\widehat{g}$ instead of g.

2.5 Convolution

Definition 2.5. If f, g are measurable on $\mathbb{R}$, their *convolution* is (formally)

$$(f * g)(x) = \int_{-\infty}^{\infty} f(y) g(x - y)\, dy.$$

Remark. Note that if $g = \chi_{-T,T}/2T$, then

$$\int_{-\infty}^{\infty} f(y) g(x - y)\, dy = \frac{1}{2T} \int_{x-T}^{x+T} f(y) dy = \mathrm{Avg}_{[-T,T]} f,$$

so we can see convolution as a averaging or smoothing operation (also known as *mollification*).

Remark. We would like to show $f, g \in L^1(\mathbb{R})$ implies $f * g \in L^1(\mathbb{R})$. Note that $(f * g)^\wedge = \widehat{f}\widehat{g} \in C_0(\mathbb{R})$, since $C_0(\mathbb{R})$ is closed under multiplication, even though $L^1(\mathbb{R})$ is not.