MATH 7337: Harmonic Analysis

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Fall 2025

Contents

Lecture 1

Aug. 19 — The Fourier Transform

1.1 The Fourier Transform on $L^1(\mathbb{R})$

All integrals will be taken over $\mathbb{R}$ unless otherwise specified.

Definition 1.1. The *Fourier transform* of $f \in L^1(\mathbb{R})$ is

$$\widehat{f}(\xi) = \int f(x) e^{-2\pi i \xi x}\, dx, \quad \xi \in \mathbb{R}.$$

Remark. Note that by the triangle inequality,

$$|\widehat{f}(\xi)| \leq \int |f(x) e^{-2\pi i \xi x}|\, dx = \int |f(x)|\, dx = \|f\|_1 < \infty,$$

so $\widehat{f}(\xi)$ exists for all $\xi \in \mathbb{R}$ (in fact, $\widehat{f}$ is continuous).

Remark. The Fourier transform is an operator $\mathcal{F} : L^1(\mathbb{R}) \to L^\infty(\mathbb{R})$ as $\|\widehat{f}\|_\infty = \operatorname{ess\,sup}_{\xi \in \mathbb{R}} |\widehat{f}(\xi)| \leq \|f\|_1$. This is linear in f. The *operator norm* of $\mathcal{F}$ is

$$\|\mathcal{F}\| = \|\mathcal{F}\|_{L^1 \to L^\infty} = \sup_{\|f\|_1 = 1} \|\widehat{f}\|_\infty \leq \sup_{\|f\|_1 = 1} \|f\|_1 = 1,$$

so $\mathcal{F}$ is a bounded linear operator. However, $\mathcal{F}$ is not isometric (norm-preserving) in general.

Remark. Observe that

$$\widehat{f}(0) = \int f(x) e^{-2\pi i \cdot 0 \cdot x}\, dx = \int f(x)\, dx.$$

So if $f \geq 0$ and we normalize f so that $\widehat{f}(0) = 1$, then we have

$$|\widehat{f}(\xi)| \leq \int f(x)\, dx = \widehat{f}(0),$$

and so $\|\widehat{f}\|_\infty = \operatorname{ess\,sup}_{\xi \in \mathbb{R}} |\widehat{f}(\xi)| \leq 1$. This is one particular case where $\mathcal{F}$ does preserve the norm.

Definition 1.2. For $r \neq 0$, *dilation* of f by r is $f_r(x) = r f(rx)$. Note that $\|f_r\|_1 = \|f\|_1$.

Example 1.2.1. The *Dirichlet function* is $d(\xi) = \sin(\xi)/\pi\xi \in C_0(\mathbb{R})$.[1] Note that $d \notin L^1(\mathbb{R})$. We can also define the *sinc* function as $\operatorname{sinc} \xi = \sin(\pi\xi)/(\pi\xi) = d\pi(x)$.

[1] Recall that $C_0(\mathbb{R})$ is the space of continuous functions $f : \mathbb{R} \to \mathbb{C}$ such that $\lim_{x \to \pm\infty} f(x) = 0$.

However, d is the Fourier transform of a function in $L^1(\mathbb{R})$. Consider the

$$\chi_{-[T,T]}(x) = \begin{cases} 1 & \text{if } |x| \le T, \\ 0 & |x| > T. \end{cases}$$

Note that $\chi_{-[T,T]} \in L^1(\mathbb{R})$. Then we have

$$\widehat{\chi}_{-[T,T]}(\xi) = \int_{-T}^{T} e^{-2\pi i \xi x}\, dx = \left. \frac{e^{-2\pi i \xi x}}{-2\pi i \xi} \right|_{-T}^{T} = \frac{\sin(2\pi T\xi)}{\pi \xi} = d_{2\pi T}(\xi),$$

so we see that $\widehat{\chi}_{-[T,T]} \in C_0(\mathbb{R}) \subsetneq L^\infty(\mathbb{R})$.

Remark. We will see in general that $\mathcal{F} : L^1(\mathbb{R}) \to C_0(\mathbb{R})$, this is the Riemann-Lebesgue lemma. The image of $\mathcal{F}$ is a proper dense subspace of $C_0(\mathbb{R})$, which implies that $\mathcal{F}^{-1}$ must be unbounded as a linear operator by Banach space theory.

Proposition 1.1. *If $f \in L^1(\mathbb{R})$, then $\widehat{f}$ is uniformly continuous on $\mathbb{R}$, i.e.*

$$\|\widehat{f} - T_\eta \widehat{f}\|_\infty = \sup_{\xi \in \mathbb{R}} |\widehat{f}(\xi) - \widehat{f}(\xi - \eta)| \xrightarrow{\eta \to 0} 0,$$

where $T_\eta \widehat{f}(\xi) = \widehat{f}(\xi - \eta)$.

Proof. We can write the difference as

$$|\widehat{f}(\xi) - \widehat{f}(\xi - \eta)| = \left| \int f(x)(e^{-2\pi i \xi x} - e^{-2\pi i (\xi - \eta) x})\, dx \right| \le \int |f(x)||e^{-2\pi i \xi x} - e^{-2\pi i(\xi-\eta)x}|\, dx.$$

Note that $f \in L^1(\mathbb{R})$ and $|e^{-2\pi i \xi x} - e^{-2\pi i(\xi - \eta)x}| = |1 - e^{2\pi i \eta x}| \to 0$ as $\eta \to 0$ independent of ξ, so the statement follows from the dominated convergence theorem (the integrand is dominated by $2f$). $\square$

1.2 Motivation for the Fourier Transform

Remark. We will define the *inverse Fourier transform* of $f \in L^1(\mathbb{R})$ as

$$\check{f}(x) = \int f(x) e^{2\pi i \xi x}\, d\xi.$$

Note that $\check{f}(\xi) = \widehat{f}(-\xi)$. With enough assumptions, this is an inverse to the Fourier transform.

Proposition 1.2 (Fourier inversion formula)**.** *If $f, \widehat{f} \in L^1(\mathbb{R})$, then*

$$f(x) = (\widehat{f})^\vee(x) = \int \widehat{f}(\xi) e^{2\pi i \xi x}\, d\xi.$$

Remark. Note that $e_\xi(x) = e^{2\pi i \xi x} = \cos 2\pi \xi x + i \sin 2\pi \xi x$ and $e_\xi : \mathbb{R} \to S^1 = \{z \in \mathbb{C} : |z| = 1\}$. We have $e_\xi(x + y) = e_\xi(x) e_\xi(y)$, so e_ξ is a homomorphism, and it is also continuous. Thus e_ξ is a *character* on $\mathbb{R}$ (in fact, every character on $\mathbb{R}$ is of the form e_ξ for some ξ). One can use this idea to define Fourier transforms in much more general settings.

Remark. The Fourier transform decomposes a function f into the pure harmonics e_ξ, and the inversion formula says that we can recover f as a "sum" of these pure harmonics.