

# MATH 7337: Harmonic Analysis

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# Lecture 1

## Aug. 19 — The Fourier Transform

### 1.1 The Fourier Transform on $L^1(\mathbb{R})$

All integrals will be taken over  $\mathbb{R}$  unless otherwise specified.

**Definition 1.1.** The *Fourier transform* of  $f \in L^1(\mathbb{R})$  is

$$\widehat{f}(\xi) = \int f(x) e^{-2\pi i \xi x} dx, \quad \xi \in \mathbb{R}.$$

**Remark.** Note that by the triangle inequality,

$$|\widehat{f}(\xi)| \leq \int |f(x) e^{-2\pi i \xi x}| dx = \int |f(x)| dx = \|f\|_1 < \infty,$$

so  $\widehat{f}(\xi)$  exists for all  $\xi \in \mathbb{R}$  (in fact,  $\widehat{f}$  is continuous).

**Remark.** The Fourier transform is an operator  $\mathcal{F} : L^1(\mathbb{R}) \rightarrow L^\infty(\mathbb{R})$  as  $\|\widehat{f}\|_\infty = \text{ess sup}_{\xi \in \mathbb{R}} |\widehat{f}(\xi)| \leq \|f\|_1$ . This is linear in  $f$ . The *operator norm* of  $\mathcal{F}$  is

$$\|\mathcal{F}\| = \|\mathcal{F}\|_{L^1 \rightarrow L^\infty} = \sup_{\|f\|_1=1} \|\widehat{f}\|_\infty \leq \sup_{\|f\|_1=1} \|f\|_1 = 1,$$

so  $\mathcal{F}$  is a bounded linear operator. However,  $\mathcal{F}$  is not isometric (norm-preserving) in general.

**Remark.** Observe that

$$\widehat{f}(0) = \int f(x) e^{-2\pi i \cdot 0 \cdot x} dx = \int f(x) dx.$$

So if  $f \geq 0$  and we normalize  $f$  so that  $\widehat{f}(0) = 1$ , then we have

$$|\widehat{f}(\xi)| \leq \int f(x) dx = \widehat{f}(0),$$

and so  $\|\widehat{f}\|_\infty = \text{ess sup}_{\xi \in \mathbb{R}} |\widehat{f}(\xi)| \leq 1$ . This is one particular case where  $\mathcal{F}$  does preserve the norm.

**Definition 1.2.** For  $r \neq 0$ , *dilation* of  $f$  by  $r$  is  $f_r(x) = rf(rx)$ . Note that  $\|f_r\|_1 = \|f\|_1$ .

**Example 1.2.1.** The *Dirichlet function* is  $d(\xi) = \sin(\xi)/\pi\xi \in C_0(\mathbb{R})$ .<sup>1</sup> Note that  $d \notin L^1(\mathbb{R})$ . We can also define the *sinc* function as  $\text{sinc } \xi = \sin(\pi\xi)/(\pi\xi) = d\pi(x)$ .

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<sup>1</sup>Recall that  $C_0(\mathbb{R})$  is the space of continuous functions  $f : \mathbb{R} \rightarrow \mathbb{C}$  such that  $\lim_{x \rightarrow \pm\infty} f(x) = 0$ .

However,  $d$  is the Fourier transform of a function in  $L^1(\mathbb{R})$ . Consider the

$$\chi_{-[T,T]}(x) = \begin{cases} 1 & \text{if } |x| \leq T, \\ 0 & |x| > T. \end{cases}$$

Note that  $\chi_{-[T,T]} \in L^1(\mathbb{R})$ . Then we have

$$\widehat{\chi}_{-[T,T]}(\xi) = \int_{-T}^T e^{-2\pi i \xi x} dx = \frac{e^{-2\pi i \xi x}}{-2\pi i \xi} \Big|_{-T}^T = \frac{\sin(2\pi T \xi)}{\pi \xi} = d_{2\pi T}(\xi),$$

so we see that  $\widehat{\chi}_{-[T,T]} \in C_0(\mathbb{R}) \subsetneq L^\infty(\mathbb{R})$ .

**Remark.** We will see in general that  $\mathcal{F} : L^1(\mathbb{R}) \rightarrow C_0(\mathbb{R})$ , this is the Riemann-Lebesgue lemma. The image of  $\mathcal{F}$  is a proper dense subspace of  $C_0(\mathbb{R})$ , which implies that  $\mathcal{F}^{-1}$  must be unbounded as a linear operator by Banach space theory.

**Proposition 1.1.** *If  $f \in L^1(\mathbb{R})$ , then  $\widehat{f}$  is uniformly continuous on  $\mathbb{R}$ , i.e.*

$$\|\widehat{f} - T_\eta \widehat{f}\|_\infty = \sup_{\xi \in \mathbb{R}} |\widehat{f}(\xi) - \widehat{f}(\xi - \eta)| \xrightarrow{\eta \rightarrow 0} 0,$$

where  $T_\eta \widehat{f}(\xi) = \widehat{f}(\xi - \eta)$ .

*Proof.* We can write the difference as

$$|\widehat{f}(\xi) - \widehat{f}(\xi - \eta)| = \left| \int f(x)(e^{-2\pi i \xi x} - e^{-2\pi i (\xi - \eta)x}) dx \right| \leq \int |f(x)| |e^{-2\pi i \xi x} - e^{-2\pi i (\xi - \eta)x}| dx.$$

Note that  $f \in L^1(\mathbb{R})$  and  $|e^{-2\pi i \xi x} - e^{-2\pi i (\xi - \eta)x}| = |1 - e^{2\pi i \eta x}| \rightarrow 0$  as  $\eta \rightarrow 0$  independent of  $\xi$ , so the statement follows from the dominated convergence theorem (the integrand is dominated by  $2|f|$ ).  $\square$

## 1.2 Motivation for the Fourier Transform

**Remark.** We will define the *inverse Fourier transform* of  $f \in L^1(\mathbb{R})$  as

$$\check{f}(x) = \int f(\xi) e^{2\pi i \xi x} d\xi.$$

Note that  $\check{f}(\xi) = \widehat{f}(-\xi)$ . With enough assumptions, this is an inverse to the Fourier transform.

**Proposition 1.2** (Fourier inversion formula). *If  $f, \widehat{f} \in L^1(\mathbb{R})$ , then*

$$f(x) = (\widehat{f})^\vee(x) = \int \widehat{f}(\xi) e^{2\pi i \xi x} d\xi.$$

**Remark.** Note that  $e_\xi(x) = e^{2\pi i \xi x} = \cos 2\pi \xi x + i \sin 2\pi \xi x$  and  $e_\xi : \mathbb{R} \rightarrow S^1 = \{z \in \mathbb{C} : |z| = 1\}$ . We have  $e_\xi(x + y) = e_\xi(x)e_\xi(y)$ , so  $e_\xi$  is a homomorphism, and it is also continuous. Thus  $e_\xi$  is a *character* on  $\mathbb{R}$  (in fact, every character on  $\mathbb{R}$  is of the form  $e_\xi$  for some  $\xi$ ). One can use this idea to define Fourier transforms in much more general settings.

**Remark.** The Fourier transform decomposes a function  $f$  into the pure harmonics  $e_\xi$ , and the inversion formula says that we can recover  $f$  as a “sum” of these pure harmonics.

# Lecture 2

## Aug. 21 — The Riemann-Lebesgue Lemma

### 2.1 Properties of the Fourier Transform

**Definition 2.1.** Define the following operators:

1. *Translation:*  $T_a f(x) = f(x - a)$  for  $a \in \mathbb{R}$ ;
2. *Modulation:*  $M_b f(x) = e^{2\pi i b x} f(x)$  for  $b \in \mathbb{R}$ ;
3. *Dilation:*  $f_\lambda(x) = \lambda f(\lambda x)$  for  $\lambda > 0$ ;
4. *Involution:*  $\tilde{f}(x) = \overline{f(-x)}$ .

**Remark.** Translation and modulation are isometries on  $L^p(\mathbb{R})$  for any  $p$ . Dilation as defined above is  $L^1$ -normalized, so it is only an isometry on  $L^1(\mathbb{R})$ .

**Exercise 2.1.** If  $f \in L^1(\mathbb{R})$ , then

1.  $(T_a f)^\wedge(\xi) = (M_{-a} \hat{f})(\xi) = e^{-2\pi i \xi a} \hat{f}(\xi)$ ;
2.  $(M_b f)^\wedge(\xi) = (T_b \hat{f})(\xi) = \hat{f}(\xi - b)$ ;
3.  $(f_\lambda)^\wedge(\xi) = \lambda (f_{1/\lambda})^\wedge(\xi) = \hat{f}(\xi/\lambda)$ ;<sup>1</sup>
4.  $(\bar{f})^\wedge(\xi) = (\hat{f})^\sim(\xi) = \overline{\hat{f}(-\xi)}$ ;
5.  $(\tilde{f})^\wedge(\xi) = \overline{\hat{f}(\xi)}$ .

### 2.2 The Riemann-Lebesgue Lemma

**Definition 2.2.** Let  $C_c(\mathbb{R})$  be the space of continuous functions with compact support. For a continuous function, the *support* of  $f$ , denoted  $\text{supp}(f) = \overline{\{x \in \mathbb{R} : f(x) \neq 0\}}$ . So for a continuous function  $f$ ,  $\text{supp}(f)$  is compact if and only if  $f = 0$  outside some finite interval.

**Theorem 2.1.**  $C_c(\mathbb{R})$  is dense in  $L^p(\mathbb{R})$  for  $1 \leq p < \infty$ . In other words,

1. the closure of  $C_c(\mathbb{R})$  in  $L^p(\mathbb{R})$  is all of  $L^p(\mathbb{R})$ ;
2. for any  $f \in L^p(\mathbb{R})$  and  $\epsilon > 0$ , there exists  $g \in C_c(\mathbb{R})$  such that  $\|f - g\|_p < \epsilon$ ;
3. if  $f \in L^p(\mathbb{R})$ , then there exists  $g_n \in C_c(\mathbb{R})$  such that  $g_n \rightarrow f$  in  $L^p$ -norm, i.e.  $\|g_n - f\|_p \rightarrow 0$ .

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<sup>1</sup>Note that the result is an  $L^\infty$ -normalized dilation.

For  $p = \infty$ ,  $C_c(\mathbb{R})$  is dense in  $C_0(\mathbb{R})$  with respect to the  $L^\infty$ -norm (this is the same as the uniform norm for continuous functions).

*Proof.* We sketch the proof. First approximate  $f \in L^p(\mathbb{R})$  by a simple function (one that takes only finitely many distinct values)  $\phi = \sum_{k=1}^N c_k \chi_{E_k}$ , e.g. by rounding down to the nearest integer multiple of  $2^{-n}$ . Then use Urysohn's lemma to approximate  $\chi_{E_k}$  by a continuous function.  $\square$

**Exercise 2.2.** Fix  $1 \leq p < \infty$ . Prove that if  $f \in L^p(\mathbb{R})$ , then  $\lim_{a \rightarrow 0} \|f - T_a f\|_p = 0$ . We say that translation is *strongly continuous* on  $L^p(\mathbb{R})$ . For  $p = \infty$ , use  $C_0(\mathbb{R})$  and the uniform norm instead.

**Lemma 2.1** (Riemann-Lebesgue lemma). *If  $f \in L^1(\mathbb{R})$ , then  $\widehat{f} \in C_0(\mathbb{R})$ ,*

*Proof.* We have already seen that  $\widehat{f}$  is continuous. So it suffices to show decay at  $\infty$ . Write

$$\widehat{f}(\xi) = - \int f(x) e^{-2\pi i \xi x} e^{-2\pi i \xi (1/2\xi)} dx = - \int_{-\infty}^{\infty} f(x) e^{-2\pi i \xi (x+1/2\xi)} dx.$$

Now make the change of variables  $x \mapsto x - 1/2\xi$ , so we get

$$\widehat{f}(\xi) = - \int_{-\infty}^{\infty} f\left(x - \frac{1}{2\xi}\right) e^{-2\pi i \xi x} dx = - \int T_{1/2\xi} f(x) e^{-2\pi i \xi x} dx.$$

Taking an average with the usual expression for  $\widehat{f}(\xi)$ , we have

$$\widehat{f}(\xi) = \frac{1}{2} \int (f(x) - T_{1/2\xi} f(x)) e^{-2\pi i \xi x} dx.$$

Taking absolute values, we obtain

$$|\widehat{f}(\xi)| \leq \frac{1}{2} \int |f(x) - T_{1/2\xi} f(x)| dx = \frac{1}{2} \|f - T_{1/2\xi} f\|_1 \xrightarrow{\xi \rightarrow \pm\infty} 0$$

by the strong continuity of translation on  $L^1(\mathbb{R})$ .  $\square$

**Exercise 2.3.** The following is an alternative proof of the Riemann-Lebesgue lemma. Recall that we have  $\widehat{\chi}_{-T,T} = d_{2\pi T} \in C_0(\mathbb{R})$ . By taking translations and dilations, we see that  $\widehat{\chi}_{[a,b]} \in C_0(\mathbb{R})$ . Consider *really simple functions*  $\phi = \sum_{k=1}^N c_k \chi_{[a_k, b_k]}$ , and by linearity we can write

$$\widehat{\phi} = \sum_{k=1}^N c_k \widehat{\chi}_{[a_k, b_k]} \in C_0(\mathbb{R}).$$

Note that really simple functions are also dense in  $L^p(\mathbb{R})$  for  $1 \leq p < \infty$ . So if  $f \in L^1(\mathbb{R})$ , there exist really simple  $\phi_n \rightarrow f$  in  $L^1$ -norm. On the Fourier side, we have

$$\|\widehat{f} - \widehat{\phi}_n\|_\infty \leq \|f - \phi_n\|_1 \rightarrow 0.$$

Since  $\phi_n \rightarrow f$  uniformly and  $C_0(\mathbb{R})$  is a Banach space, we conclude  $\widehat{f} \in C_0(\mathbb{R})$ . Fill in the details.

## 2.3 Position and Momentum Operators

**Definition 2.3.** The *position operator*  $P : L^1(\mathbb{R}) \rightarrow L^1(\mathbb{R})$  is given by  $Pf(x) = xf(x)$ . Note that  $P$  is unbounded on  $L^1(\mathbb{R})$  (in fact,  $P$  is not defined on all of  $L^1(\mathbb{R})$ ). Restrict  $P$  to the domain

$$D_P = \{f \in L^1(\mathbb{R}) : xf(x) \in L^1(\mathbb{R})\},$$

which is dense in  $L^1(\mathbb{R})$ . Note that  $D_P$  cannot be bounded as it does not admit an extension to  $L^1(\mathbb{R})$ .

**Exercise 2.4.** Show that  $\sup_{\|f\|_1=1, f \in D_P} \|Pf\|_1 = \infty$ .

**Definition 2.4.** The *momentum operator*  $M : L^1(\mathbb{R}) \rightarrow L^1(\mathbb{R})$  is given by  $Mf = f'/2\pi i$ . Similarly,  $M$  is unbounded and defined only on a dense subset of  $L^1(\mathbb{R})$ .

**Remark.** We have the relation  $(Mf)^\wedge(\xi) = \xi P\hat{f}(\xi)$ , whenever the statement makes sense.

## 2.4 The HRT Conjecture

**Conjecture 2.1** (HRT conjecture). Assume  $g$  is not zero a.e.,  $a_k, b_k$  are distinct, and consider finite linear combinations of translations and modulations of  $g \in L^2(\mathbb{R})$  of the following form:

$$\sum_{k=1}^N c_k e^{2\pi i b_k x} g(x - a_k). \quad (*)$$

If  $(*) = 0$ , then must it be that  $c_1 = \dots = c_N = 0$ ? In other words, are these linearly independent?

**Remark.** Consider the special case  $b_k = 0$  for every  $k$ , so  $\sum c_k T_{a_k} g = \sum c_k g(x - a_k) = 0$  a.e. Then

$$\left( \sum c_k T_{a_k} g \right)^\wedge = \sum c_k M_{-a_k} \hat{g} = \left( \sum_{k=1}^N c_k e^{-2\pi i a_k \xi} \right) \hat{g}(\xi) = 0.$$

Since  $\hat{g}$  is not zero a.e., we must have  $\sum_{k=1}^N c_k e^{-2\pi i a_k \xi} = 0$ , which implies  $c_k = 0$  for all  $k$ . In particular, this means that translations alone are linearly independent (the same is true for modulations alone).

**Remark.** The general case of the HRT conjecture is still open. Note that after taking a Fourier transform, we end up with the same problem, just for  $\hat{g}$  instead of  $g$ .

# Lecture 3

## Aug. 3 — Convolution

### 3.1 Convolution

**Definition 3.1.** If  $f, g$  are measurable on  $\mathbb{R}$ , their *convolution* is (formally)

$$(f * g)(x) = \int_{-\infty}^{\infty} f(y)g(x - y) dy.$$

**Remark.** When it exists, we have

$$(f * g)(x) = \int_{-\infty}^{\infty} f(y)g(x - y) dy = \int_{-\infty}^{\infty} f(x - y)g(y) dy = (g * f)(x)$$

by the change of variables  $y \mapsto x - y$ . So  $f * g = g * f$ , if it exists. Similarly,  $f * (g * h) = (f * g) * h$  if each of these convolutions exist.

**Remark.** If we take  $g_T = \chi_{-T,T}/2T$  (note that  $\|g_T\|_1 = 1$ ), then

$$(f * g_T)(x) = \int_{-\infty}^{\infty} f(y)g_T(x - y) dy = \frac{1}{2T} \int_{x-T}^{x+T} f(y) dy = \text{Avg}_{[-T,T]} f(x),$$

so we can see convolution as a averaging or smoothing operation (also known as *mollification*).

**Remark.** We would like to show  $f, g \in L^1(\mathbb{R})$  implies  $f * g \in L^1(\mathbb{R})$ . Note that  $(f * g)^\wedge = \widehat{f}\widehat{g} \in C_0(\mathbb{R})$ , since  $C_0(\mathbb{R})$  is closed under multiplication, even though  $L^1(\mathbb{R})$  is not.

**Remark.** The *Lebesgue differentiation theorem* says that if  $f \in L^1_{\text{loc}}(\mathbb{R})$ , then  $(f * g_T)(x) \rightarrow f(x)$  a.e.

### 3.2 Properties of Convolution

**Remark.** Use the notation

$$\langle f, g \rangle = \int_{-\infty}^{\infty} f(x)\overline{g(x)} dx,$$

whenever this integral exists. Then *Hölder's inequality* says that if  $1/p + 1/p' = 1$  with  $1 \leq p \leq \infty$  and  $f \in L^p(\mathbb{R})$ ,  $g \in L^{p'}(\mathbb{R})$ , then  $fg \in L^1(\mathbb{R})$  and we have

$$|\langle f, g \rangle| \leq \int |f(x)||g(x)| dx \leq \|f\|_p \|g\|_{p'}.$$



**Theorem 3.1.** For  $1 \leq p \leq \infty$ , if  $f \in L^p(\mathbb{R})$  and  $g \in L^1(\mathbb{R})$ , then  $f * g \in L^\infty(\mathbb{R})$ .

*Proof.* By Hölder's inequality, we can write

$$\int |f(y)g(x-y)| dy \leq \|f\|_p \|g(x)\|_{p'} < \infty,$$

so  $(f * g)(x)$  exists for every  $x \in \mathbb{R}$ . □

**Exercise 3.1.** Show that  $f * g \in C_b(\mathbb{R}) = \{h : \mathbb{R} \rightarrow \mathbb{C} : h \text{ is continuous and bounded}\}$ .

**Remark.** Denote  $g^*(y) = \overline{g(-y)}$ . Then we have

$$(f * g)(x) = \int f(y)g(x-y) dy = \int f(y)\overline{g^*(y-x)} dy = \langle f, T_x g^* \rangle.$$

**Theorem 3.2.** Let  $f, g \in L^1(\mathbb{R})$ . Then

1.  $f(y)g(x-y)$  is measurable and integrable on  $\mathbb{R}^2$ ;
2. for a.e.  $x \in \mathbb{R}$ ,  $f(y)g(x-y)$  is measurable and integrable on  $\mathbb{R}$  as a function of  $y$ ;
3.  $f * g \in L^1(\mathbb{R})$  and  $\|f * g\|_1 \leq \|f\|_1 \|g\|_1$ , i.e. convolution is submultiplicative on  $L^1(\mathbb{R})$ ;
4.  $(f * g)^\wedge(\xi) = \widehat{f}(\xi)\widehat{g}(\xi)$  for every  $\xi \in \mathbb{R}$ .

*Proof.* (1) Let  $h(x, y) = f(y)g(x-y)$ . Then we have

$$\{h > a\} = h^{-1}((a, \infty)) = \{(x, y) : f(y)g(x-y) > a\} = \{f > a\} \times \mathbb{R},$$

which is measurable in  $\mathbb{R}^2$  since  $\{f > a\}$  and  $\mathbb{R}$  are measurable in  $\mathbb{R}$ . Similarly,  $g(y)$  is measurable on  $\mathbb{R}^2$ , so  $F(x, y) = f(y)g(x-y)$  is measurable on  $\mathbb{R}^2$ . Now make a linear change of variables  $T(x, y) = (y, x-y)$ , so  $H = F \circ T = f(y)g(x-y)$  is measurable (note that linear maps preserve measurability).

Now we can integrate by Tonelli's theorem and see that

$$\begin{aligned} \iint |f(y)g(x-y)| dx dy &= \int |f(y)| \left( \int |g(x-y)| dx \right) dy = \int |f(y)| \left( \int |g(z)| dz \right) dy \\ &= \int |f(y)| \|g\|_1 dy = \|f\|_1 \|g\|_1 < \infty, \end{aligned}$$

hence  $f(y)g(x-y)$  is integrable on  $\mathbb{R}^2$ .

(2) This follows by Fubini's theorem since  $f(y)g(x-y)$  is integrable.

(3) By (2),  $(f * g)(x)$  exists for a.e.  $x$ , and

$$\int |(f * g)(x)| dx = \int \left| \int f(y)g(x-y) dy \right| dx \leq \iint |f(y)g(x-y)| dy dx \leq \|f\|_1 \|g\|_1,$$

which is the desired inequality.

(4) Justify the following calculation as an exercise via Fubini/Tonelli's theorem:

$$\begin{aligned}(f * g)^\wedge(\xi) &= \int (f * g)(x) e^{-2\pi i \xi x} dx = \int \left( \int f(y) g(x - y) dy \right) e^{-2\pi i \xi x} dx \\ &= \iint f(y) e^{-2\pi i \xi y} g(x - y) e^{-2\pi i \xi (x - y)} dy dx.\end{aligned}$$

By Fubini's theorem, we can exchange orders and write

$$\begin{aligned}(f * g)^\wedge(\xi) &= \int f(y) e^{-2\pi i \xi y} \left( \int g(x - y) e^{-2\pi i \xi (x - y)} dx \right) dy \\ &= \int f(y) e^{-2\pi i \xi y} \left( \int g(z) e^{-2\pi i \xi z} dz \right) dy = \widehat{f}(\xi) \widehat{g}(\xi),\end{aligned}$$

which is the desired equality. □

**Corollary 3.2.1.**  $L^1(\mathbb{R})$  is closed under convolution.

**Definition 3.2.** An *algebra* is a vector space  $A$  with a product such that

- (a)  $(fg)h = f(gh)$ ,
- (b)  $f(g + h) = fg + fh$ ,
- (c)  $\alpha(fg) = (\alpha f)g = f(\alpha g)$ .

If  $fg = gf$  always, then we say that  $A$  is *commutative*. A Banach space which is also an algebra with a submultiplicative product is a *Banach algebra*.

**Example 3.2.1.** With convolution as a product,  $L^1(\mathbb{R})$  becomes a commutative Banach algebra without identity. Similarly,  $C_0(\mathbb{R})$  is also a commutative Banach algebra without identity (under pointwise products). The space  $\mathcal{B}(X)$  of bounded linear operators on a Banach space  $X$  is also a Banach space under the operator norm, and we have  $\|AB\| \leq \|A\|\|B\|$  with composition as a product. So  $\mathcal{B}(X)$  is a noncommutative Banach algebra, with identity.

### 3.3 Young's Inequality

**Theorem 3.3** (Young's inequality, special case). Fix  $1 \leq p \leq \infty$ . If  $f \in L^p(\mathbb{R})$  and  $g \in L^1(\mathbb{R})$ , then  $f * g \in L^p(\mathbb{R})$  and  $\|f * g\|_p \leq \|f\|_p \|g\|_1$ .

*Proof.* The case  $p = \infty$  is easy by Hölder's inequality and  $p = 1$  is done, so assume  $1 < p < \infty$ . Then

$$|(f * g)(x)| \leq \int |f(y)| |g(x - y)| dy = \int (|f(y)| |g(x - y)|^{1/p}) (|g(x - y)|^{1/p'}) dy,$$

By Hölder's inequality, we can write

$$\begin{aligned}|(f * g)(x)| &\leq \left( \int |f(y)|^p |g(x - y)| dy \right)^{1/p} \left( \int |g(x - y)| dy \right)^{1/p'} \\ &\leq \|g\|_1^{1/p'} \left( \int |f(y)|^p |g(x - y)| dy \right)^{1/p}.\end{aligned}$$

Now taking  $L^p$ -norms, we get

$$\|f * g\|_p^p = \int |(f * g)(x)|^p dx \leq \|g\|_1^{p/p'} \iint |f(y)|^p |g(x - y)| dy dx.$$

By Tonelli's theorem, we can exchange orders and write

$$\|f * g\|_p^p \leq \|g\|_1^{p/p'} \int |f(y)|^p \left( \int |g(x - y)| dx \right) dy \leq \|g\|_1^{1+p/p'} \|f\|_p^p = \|g\|_1^p \|f\|_p^p,$$

so we get the desired inequality  $\|f * g\|_p \leq \|f\|_p \|g\|_1$  after taking  $p$ th roots.  $\square$

**Exercise 3.2** (Young's inequality, general case). Let  $1 \leq p, q, r \leq \infty$  satisfy  $1/r = 1/p + 1/q - 1$ . If  $f \in L^p(\mathbb{R})$  and  $g \in L^q(\mathbb{R})$ , then

$$\|f * g\|_r \leq \|f\|_p \|g\|_q.$$

**Remark.** Recall *Minkowski's inequality* (the triangle inequality in  $L^p(\mathbb{R})$ ):

$$\left\| \sum f_k \right\|_p \leq \sum \|f_k\|_p.$$

*Minkowski's integral inequality* then says that for  $1 \leq p \leq \infty$ ,

$$\left\| \int f_x dx \right\|_p = \left( \int \left| \int f(x, y) dx \right|^p dy \right)^{1/p} \leq \int \left( \int |f(x, y)|^p dy \right)^{1/p} dx = \int \|f_x\|_p dx.$$

One can also use this to prove Young's inequality.

**Remark.** The *Babenko-Beckner constant* is the optimal constant in front of Hölder's inequality:

$$A_p = \left( \frac{p^{1/p}}{(p')^{1/p'}} \right)^{1/2}.$$

The optimal constant in Young's inequality is  $A_p A_q A_{r'}$ , i.e. we have

$$\|f * g\|_r \leq (A_p A_q A_{r'}) \|f\|_p \|g\|_q.$$

### 3.4 The Dirac Delta

**Remark.** Is there an identity for convolution? Suppose there was a function  $\delta \in L^1(\mathbb{R})$  (the *Dirac delta function*) such that  $f * \delta = f$  for all  $f \in L^1(\mathbb{R})$ . Then we have  $(f * \delta)^\wedge = \widehat{f}$ , so

$$\widehat{f}(\xi) \widehat{\delta}(\xi) = \widehat{f}(\xi) \quad \text{for all } f \in L^1(\mathbb{R}).$$

Take  $f(x) = e^{-x^2}$  with  $\widehat{f}(\xi) = e^{-\xi^2}$  and note that  $\widehat{f}(\xi)$  is everywhere nonzero. Then  $\widehat{\delta}(\xi) = 1$  for all  $\xi \in \mathbb{R}$ , which contradicts the Riemann-Lebesgue lemma.

The correct way to work with the Dirac delta is to use the measure

$$\delta(E) = \begin{cases} 1, & 0 \in E, \\ 0, & 0 \notin E. \end{cases}$$

One can then integrate against the measure  $\delta$  to achieve a similar effect.

# Lecture 4

Aug. 28 — Convolution, Part 2