MATH 8803: Nonlinear Dispersive Equations

Frank Qiang
Instructor: Gong Chen

Georgia Institute of Technology
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Contents

Lecture 1

Jan. 6 — Introduction to Dispersion

1.1 Introduction to Dispersion

Definition 1.1. An evolution equation is *dispersive* if when no boundary conditions are imposed (e.g. on $\mathbb{R}^n$), its wave solutions spread out in space as they evolve in time.

Example 1.1.1. Two classic examples of dispersive equations are:

- The *Schrödinger equation*: $iu_t + \Delta u = 0$.
- The *Airy (linearized KdV) equation*: $u_t + u_{xxx} = 0$.

Remark. Consider the equation $u_t + p(\partial_x)u = 0$, where p is a polynomial, and a plane-wave solution

$$u(t,x) = e^{i(kx-\omega t)} = e^{ik(x-(\omega/k)t)}.$$

Here k is the *wave number* or *space frequency*, and ω is the *(time) frequency*. Plugging the plane-wave solution into the equation, we obtain the relation $\omega(k) = -ip(ik)$, i.e.

$$\frac{\omega(k)}{k} = \frac{1}{ik}p(ik).$$

The above equation is known as the *dispersive relation*. This gives the traveling speed of the plane-wave solution with wave number k, which is called the *phase velocity*.

Example 1.1.2. The following are some examples of dispersive relations:

- For the *linear advection equation* $u_t + cu_x = 0$ with $c \in \mathbb{R}$, one can compute that $\omega/k = c$.
- For the Schrödinger equation $iu_t + \frac{1}{2}\Delta u = 0$, we have $\omega/k = k/2 \in \mathbb{R}$.

  In this case of the Schrödinger equation, plane waves with large wave number (large space frequency) travel faster than low-frequency waves.

Remark. In general, dispersion means that different frequency plane waves travel at different speeds.

Remark. Given initial data u_0, we can write using the Fourier transform that

$$u_0 = \int \widehat{u}_0(k)e^{ikx}\,dk.$$

Then we get the solution u as

$$u(t,x) = \int \widehat{u}_0(k)e^{ik(x-(\omega(k)/k)t)}\,dk.$$

Example 1.1.3. In the case of the linear advection equation, we obtain the solution as

$$u(t,x) = \int \widehat{u}_0(k) e^{ik(x-ct)}\, dk = u_0(x-ct).$$

For the Schrödinger equation, we instead have the solution

$$u(t,x) = \int \widehat{u}_0(k) e^{ik(x-(k/2)t)}\, dk.$$

Since different k travels at different speeds, the original profile quickly spreads out.

Exercise 1.1. Calculate the dispersive relation ω/k for the linearized KdV equation $u_t + u_{xxx} = 0$.

Example 1.1.4. The *KdV equation* is given by

$$\partial_t u + \partial_{xxx} u + 6u\partial_x u = 0.$$

This equation is used to model shallow water surfaces, and is a nonlinear dispersive equation. Russell observed a great bump of water in a channel that traveled for a long time and kept its shape. This is due to the nonlinear effects in the KdV equation, and these effects are called *solitons*.

Definition 1.2. A *soliton* is a self-reinforcing solitary wave (a wave packet or pulse) that maintains its shape while traveling at a constant speed.

1.2 Fourier Transform and the Free Schrödinger Equation

Consider the following free Schrödinger equation:

$$\begin{cases} i\partial_t \psi + \frac{1}{2}\Delta\psi = 0, \\ \psi|_{t=0} = \psi_0. \end{cases}$$

We will solve this equation using the *Fourier transform*

$$\widehat{f}(\xi) = \int_{\mathbb{R}^d} e^{-ix\cdot\xi} f(x)\, dx.$$

Note that one can recover f from its Fourier transform via the *inversion formula*

$$f(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{ix\cdot\xi} \widehat{f}(\xi)\, d\xi.$$

Exercise 1.2. Check that $(\partial_{x_j} f)^\wedge = i\xi_j \widehat{f}$.

Applying the Fourier transform to the free Schrödinger equation, one has

$$i\partial_t\psi + \frac{1}{2}\Delta\psi = 0 \quad \xrightarrow{\text{F.T.}} \quad i\partial_t\widehat{\psi} - \frac{1}{2}|\xi|^2\widehat{\psi} = 0$$

and initial condition $\widehat{\psi}(0,\xi) = \widehat{\psi}_0(\xi)$. So for fixed ξ, we have an ODE, so we can solve the equation via

$$\widehat{\psi}(t,\xi) = e^{-i|\xi|^2 t/2}\widehat{\psi}_0(\xi).$$

Now by applying the inverse Fourier transform, we obtain the solution

$$\psi(t,x) = (2\pi)^{-d} \int e^{ix\xi} \widehat{\psi}(t,\xi)\, d\xi = (2\pi)^{-d} \int e^{ix\xi} e^{-i|\xi|^2 t/2} \widehat{\psi}_0(\xi)\, d\xi.$$

Recalling Plancherel's theorem that $\|f\|_{L^2} = C\|\widehat{f}\|_{L^2}$ (for a constant C independent of f), we obtain

$$\|\psi(t,x)\|_{L^2} = C\|\widehat{\psi}(t,\xi)\|_{L^2} = C\|\widehat{\psi}(0,\xi)\|_{L^2} = \|\psi(0,x)\|_{L^2} = \|\psi_0(x)\|_{L^2},$$

where the second equality follows by noticing that $e^{-i|\xi|^2 t/2}$ has modulus 1. This shows that the linear Schrödinger evolution preserves the L^2 norm of the solution.

Exercise 1.3. Compute that

$$\frac{d}{dt} \int_{\mathbb{R}^d} |\psi(t,x)|^2\, dx = 0.$$

This is an alternative way to show that the L^2 norm of the solution is preserved.

1.3 Sobolev Spaces

Definition 1.3. The *Sobolev spaces* $H^\gamma = W^{\gamma,2}$ for $\gamma \in \mathbb{R}$ are defined via the norm

$$\|f\|_{H^\gamma} = \left(\int_{\mathbb{R}^d} (1+|\xi|^2)^\gamma |\widehat{f}(\xi)|^2\, d\xi \right)^{1/2}.$$

The *homogeneous Sobolev spaces* $\dot{H}^\gamma$ are defined by the norm

$$\|f\|_{\dot{H}^\gamma} = \left(\int_{\mathbb{R}^d} |\xi|^{2\gamma} |\widehat{f}(\xi)|^2\, d\xi \right)^{1/2}.$$

Remark. If $\gamma \in \mathbb{N}$ and $d = 1$, then

$$\|f\|_{H^\gamma} \sim \sum_{m=0}^{\gamma} \|\partial_x^m f\|_{L^2}.$$

In particular, this means that $f \in H^\gamma$ if and only if $\partial_x^m f \in L^2$ for all $m \le \gamma$.

Exercise 1.4. Check that if $f_\lambda(x) = f(\lambda x)$, then $\widehat{f_\lambda}(\xi) = \lambda^{-d} \widehat{f}(\xi/\lambda)$.

Remark. In the Sobolev spaces, this means that (change variables $\eta = \xi/\lambda$ for the last equality)

$$\|f_\lambda\|_{\dot{H}^\gamma} = \left(\int_{\mathbb{R}^d} |\xi|^{2\gamma} |\widehat{f_\lambda}(\xi)|^2\, d\xi \right)^{1/2} = \left(\int_{\mathbb{R}^d} |\xi|^{2\gamma} |\lambda^{-d} \widehat{f}(\xi/\lambda)|^2\, d\xi \right)^{1/2} = \lambda^{\gamma - d/2} \|f\|_{\dot{H}^\gamma}.$$

Lemma 1.1. *In the Schrödinger equation, $\|\psi(t)\|_{H^\gamma} = \|\psi_0\|_{H^\gamma}$ and $\|\psi(t)\|_{\dot{H}^\gamma} = \|\psi_0\|_{\dot{H}^\gamma}$ for all t and γ.*

Proof. We can compute that

$$\|\psi(t)\|_{\dot{H}^\gamma} = \int_{\mathbb{R}^d} |\xi|^{2\gamma} |\widehat{\psi}(t,\xi)|^2\, d\xi = \int_{\mathbb{R}^d} |\xi|^{2\gamma} |e^{-i|\xi|^2/2} \widehat{\psi}_0(\xi)|^2\, d\xi = \int_{\mathbb{R}^d} |\xi|^{2\gamma} |\widehat{\psi}_0(\xi)|^2\, d\xi = \|\psi_0\|_{\dot{H}^\gamma}.$$

The same argument works for the H^γ case after replacing $|\xi|^{2\gamma}$ with $(1+|\xi|^2)^\gamma$. ☐

Lecture 2

Jan. 8 — Special Solutions

2.1 Special Solutions

Example 2.0.1. The following are special solutions to the Schrödinger equation:

1. Gaussian: $\psi_0 = e^{-|x|^2/2}$. One can compute the Fourier transform and get

$$\widehat{\psi}_0(\xi) = \int_{\mathbb{R}^d} e^{-ix\cdot\xi} e^{-|x|^2/2}\,dx = \int_{\mathbb{R}^d} e^{-|x+i\xi|^2/2} e^{-|\xi|^2/2}\,dx = e^{-|\xi|^2/2} \int_{\mathbb{R}^d} e^{-|x+i\xi|^2/2}\,dx.$$

The last integral is a contour integral in the complex plane along $\Im z = \xi$, and we can deform the contour via Cauchy's theorem to the real axis to obtain (the integrand is analytic on $0 \le \Im z \le \xi$)

$$\widehat{\psi}_0(\xi) = e^{-|\xi|^2/2} \int_{\mathbb{R}^d} e^{-|x|^2/2}\,dx = (2\pi)^{d/2} e^{-|\xi|^2/2}.$$

Then taking inverse Fourier transforms, we obtain the solution

$$\begin{aligned} \psi(t,x) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{i(x\cdot\xi - |\xi|^2 t/2)} \widehat{\psi}_0(\xi)\,d\xi &= (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{i(x\cdot\xi-|\xi|^2 t/2)} e^{-|\xi|^2/2}\,d\xi \\ &= (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{-\frac{1}{2}(1+it)|\xi|^2} e^{ix\cdot\xi}\,d\xi. \end{aligned}$$

Now formally put $\eta = (1+it)^{1/2}\xi$ to get

$$\psi(t,x) = (2\pi)^{-d/2}(1+it)^{-d/2} \int_{\mathbb{R}^d} e^{-\frac{1}{2}|\eta|^2} e^{ix\eta/(1+it)^{1/2}}\,d\eta.$$

Fill in the details of the above change of variables as an exercise (e.g. one has to worry about choosing a branch cut when taking the square root). Computing the integral explicitly, one obtains

$$\psi(t,x) = (1+it)^{-d/2} e^{-|x|^2/(2(1+it))}.$$

One can from this that ψ has decay in time. Furthermore, one can see that

$$|\psi(t,x)|^2 = (1+t^2)^{-d/2} e^{-|x|^2/(1+t^2)}.$$

From this we can observe an L^∞ decay of ψ like $t^{-d/2}$, and that the influence region of the solution grows like order t. We can also see again from this explicit computation that $\|\psi(t)\|_{L^2} = C$.

2. Modulated Gaussian: $\psi_0 = e^{-|x|^2/2}e^{ix\cdot v}$. The Fourier transform of this initial data is

$$\widehat{\psi}_0(\xi) = (2\pi)^{d/2}e^{-i|\xi-v|^2/2}.$$

So the solution corresponding to this initial data is

$$\begin{aligned}\psi(t,x) = (2\pi)^{-d}\int_{\mathbb{R}^d} e^{i(x\cdot\xi-|\xi|^2t/2)}\widehat{\psi}_0(\xi)\,d\xi &= (2\pi)^{-d/2}\int_{\mathbb{R}^d} e^{i(x\cdot\xi-|\xi|^2t/2)}e^{-|\xi-v|^2}\,d\xi\\ &= e^{ix\cdot v}e^{-|v|^2t/2}(2\pi)^{-d/2}\int_{\mathbb{R}^d} e^{i(x-vt)\cdot\xi}e^{-(1+it)|\xi|^2/2}\,d\xi\\ &= e^{ix\cdot v}e^{-|v|^2t/2}(1+it)^{-d/2}\exp\left(-\frac{|x-vt|^2}{2(1+it)}\right).\end{aligned}$$

From this we can see that the influence region of the solution moves with velocity v.

3. Fundamental solution: We want a *fundamental solution* K such that K solves

$$i\partial_t K + \frac{1}{2}\Delta K = 0 \quad \text{and} \quad K|_{t=0} = \delta_0.$$

We will find K by scaling arguments. Suppose such a K exists. Then we must have

$$\psi(t,x) = \int_{\mathbb{R}^d} K(t,x-y)\psi_0(y)\,dy \tag{1}$$

since $K|_{t=0} = \delta_0$. Now define the scaling $\psi_\lambda(t,x) = \psi(\lambda^2 t, \lambda x)$. Then ψ_λ also solves

$$i\partial_t\psi_\lambda + \frac{1}{2}\Delta\psi_\lambda = 0$$

and we have the initial condition $\psi_\lambda(0,x) = \psi_0(\lambda x)$. Then

$$\psi_\lambda(t,x) = \int_{\mathbb{R}^d} K(t,x-y)\psi_0(\lambda y)\,dy = \psi(\lambda^2 t, \lambda x).$$

Setting $t' = \lambda^2 t$, $x' = \lambda x$, and $y' = \lambda y$, we get

$$\psi(t',x') = \frac{1}{\lambda^d}\int_{\mathbb{R}^d} K\left(\frac{t'}{\lambda^2}, \frac{x'-y'}{\lambda}\right)\psi_0(y')\,dy'. \tag{2}$$

Comparing (1) and (2), we see that we must have

$$K(t,x-y) = \lambda^{-d}K\left(\frac{t}{\lambda^2}, \frac{x-y}{\lambda}\right).$$

Setting $u = x-y$, we get

$$K(t,u) = \lambda^{-d}K\left(\frac{t}{\lambda^2}, \frac{u}{\lambda}\right).$$

Thus we expect $K(t,x) = t^{-d/2}\Phi(|x|^2/t)$ for some Φ. Now we use the fact that $i\partial_t K + \frac{1}{2}\Delta K = 0$. Setting $m = |x|^2/t$, one can plug in the above guess for K to obtain (note that $\Delta = \nabla\cdot\nabla$)

$$-\frac{id}{2}t^{-d/2-1}\Phi(m) - it^{-d/2}\Phi'(m)\frac{m}{t} + \frac{1}{2}t^{-d/2}\nabla\cdot\left(\frac{2x}{t}\Phi'(m)\right) = 0.$$

Then we get

$$-i\frac{d}{2}\Phi(m) - im\Phi'(m) + d\Phi'(m) + 2m\Phi''(m) = 0,$$

which gives

$$d\left(\Phi'(m) - \frac{i}{2}\Phi(m)\right) + 2m\frac{d}{dm}\left(\Phi'(m) - \frac{i}{2}\Phi(m)\right) = 0.$$

Now observe that $\Phi(m) = e^{im/2}$ solves the above equation. Since $\Phi(m)$ solves the equation, $c\Phi(m)$ also solves the equation for any $c \in \mathbb{C}$, and thus we have

$$K(t,x) = ct^{-d/2}\Phi(|x|^2/t) = ct^{-d/2}e^{i|x|^2/2t}.$$

To determine c, we use $K|_{t=0} = \delta_0$, from which one can obtain $c = (2\pi i)^{-d/2}$. Thus

$$K(t,x) = (2\pi i t)^{-d/2}e^{i|x|^2/2t}.$$

The rough computation is that since $\widehat{K}(0,\xi) = 1$, we have

$$K = (2\pi)^{-d}\int_{\mathbb{R}^d} e^{i(x\cdot\xi - |\xi|^2 t/2)}\widehat{K}(0,\xi)\,d\xi = (2\pi)^{-d}\int_{\mathbb{R}^d} e^{i(x\cdot\xi - |\xi|^2 t/2)}\,d\xi$$

This is not necessarily integrable a priori, but one can take limits and obtain

$$\begin{aligned} K = (2\pi)^{-d}\lim_{\epsilon\to 0^+}\int_{\mathbb{R}^d} e^{ix\cdot\xi}e^{-(\epsilon+it)|\xi|^2/2}\,d\xi &= \lim_{\epsilon\to 0^+}(\epsilon+it)^{-d/2}(2\pi)^{-d/2}e^{-|x|^2/(2(\epsilon+it))} \\ &= (2\pi i t)^{-d/2}e^{-|x|^2/2it}. \end{aligned}$$

Note that this computation matches the result of the previous scaling argument.

Theorem 2.1. *Let $\psi_0 \in \mathcal{S}(\mathbb{R}^d)$.[1] Then there exists a solution to*

$$\begin{cases} i\partial_t\psi + \frac{1}{2}\Delta\psi = 0, \\ \psi|_{t=0} = \psi_0, \end{cases}$$

which is unique and given by

$$\psi(t,x) = \int_{\mathbb{R}^d} K(t,x-y)\psi_0(y)\,dy = (2\pi i t)^{-d/2}\int_{\mathbb{R}^d} e^{-|x-y|^2/2it}\psi_0(y)\,dy.$$

Proof. This theorem is a summary of the results of the previous explicit computations. □

Remark. Recall that the Schrödinger evolution preserves the L^2 norm of a solution, i.e.

$$\|\psi(t)\|_{L^2} = \|\psi(0)\|_{L^2} = \|\psi_0\|_{L^2}.$$

The above theorem also gives an L^∞ bound (a so-called *dispersive estimate*)

$$\|\psi(t)\|_{L^\infty} \le |2\pi t|^{-d/2}\int_{\mathbb{R}^d}|\psi_0(y)|\,dy = |2\pi t|^{-d/2}\|\psi_0\|_{L^1}.$$

[1] Here $\mathcal{S}(\mathbb{R}^d)$ is the space of Schwartz functions.