MATH 8803: Nonlinear Dispersive Equations

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Contents

Lecture 1

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1.1 Introduction to Dispersion

Definition 1.1. An evolution equation is *dispersive* if when no boundary conditions are imposed (e.g. on $\mathbb{R}^n$), its wave solutions spread out in space as they evolve in time.

Example 1.1.1. Two classic examples of dispersive equations are:

- The *Schrödinger equation*: $iu_t + \Delta u = 0$.
- The *Airy (linearized KdV) equation*: $u_t + u_{xxx} = 0$.

Remark. Consider the equation $u_t + p(\partial_x)u = 0$, where p is a polynomial, and a plane-wave solution

$$u(t,x) = e^{i(kx-\omega t)} = e^{ik(x-(\omega/k)t)}.$$

Here k is the *wave number* or *space frequency*, and ω is the *(time) frequency*. Plugging the plane-wave solution into the equation, we obtain the relation $\omega(k) = -ip(ik)$, i.e.

$$\frac{\omega(k)}{k} = \frac{1}{ik}p(ik).$$

The above equation is known as the *dispersive relation*. This gives the traveling speed of the plane-wave solution with wave number k, which is called the *phase velocity*.

Example 1.1.2. The following are some examples of dispersive relations:

- For the *linear advection equation* $u_t + cu_x = 0$ with $c \in \mathbb{R}$, one can compute that $\omega/k = c$.
- For the Schrödinger equation $iu_t + \frac{1}{2}\Delta u = 0$, we have $\omega/k = k/2 \in \mathbb{R}$.

  In this case of the Schrödinger equation, plane waves with large wave number (large space frequency) travel faster than low-frequency waves.

Remark. In general, dispersion means that different frequency plane waves travel at different speeds.

Remark. Given initial data u_0, we can write using the Fourier transform that

$$u_0 = \int \widehat{u}_0(k)e^{ikx}\,dk.$$

Then we get the solution u as

$$u(t,x) = \int \widehat{u}_0(k)e^{ik(x-(\omega(k)/k)t)}\,dk.$$

Example 1.1.3. In the case of the linear advection equation, we obtain the solution as

$$u(t,x) = \int \widehat{u}_0(k)e^{ik(x-ct)}\,dk = u_0(x-ct).$$

For the Schrödinger equation, we instead have the solution

$$u(t,x) = \int \widehat{u}_0(k)e^{ik(x-(k/2)t)}\,dk.$$

Since different k travels at different speeds, the original profile quickly spreads out.

Exercise 1.1. Calculate the dispersive relation ω/k for the linearized KdV equation $u_t + u_{xxx} = 0$.

Example 1.1.4. The *KdV equation* is given by

$$\partial_t u + \partial_{xxx}u + 6u\partial_x u = 0.$$

This equation is used to model shallow water surfaces, and is a nonlinear dispersive equation. Russell observed a great bump of water in a channel that traveled for a long time and kept its shape. This is due to the nonlinear effects in the KdV equation, and these effects are called *solitons*.

Definition 1.2. A *soliton* is a self-reinforcing solitary wave (a wave packet or pulse) that maintains its shape while traveling at a constant speed.

1.2 Fourier Transform and the Free Schrödinger Equation

Consider the following free Schrödinger equation:

$$\begin{cases} i\partial_t\psi + \frac{1}{2}\Delta\psi = 0, \\ \psi|_{t=0} = \psi_0. \end{cases}$$

We will solve this equation using the *Fourier transform*

$$\widehat{f}(\xi) = \int_{\mathbb{R}^d} e^{-ix\cdot\xi} f(x)\,dx.$$

Note that one can recover f from its Fourier transform via the *inversion formula*

$$f(x) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{ix\cdot\xi}\widehat{f}(\xi)\,d\xi.$$

Exercise 1.2. Check that $(\partial_{x_j} f)^\wedge = i\xi_j\widehat{f}$.

Applying the Fourier transform to the free Schrödinger equation, one has

$$i\partial_t\psi + \frac{1}{2}\Delta\psi = 0 \quad \xrightarrow{\text{F.T.}} \quad i\partial_t\widehat{\psi} - \frac{1}{2}|\xi|^2\widehat{\psi} = 0$$

and initial condition $\widehat{\psi}(0,\xi) = \widehat{\psi}_0(\xi)$. So for fixed ξ, we have an ODE, so we can solve the equation via

$$\widehat{\psi}(t,\xi) = e^{-i|\xi|^2 t/2}\widehat{\psi}_0(\xi).$$

Now by applying the inverse Fourier transform, we obtain the solution

$$\psi(t,x) = \frac{1}{(2\pi)^d}\int e^{ix\xi}\widehat{\psi}(t,\xi)\,d\xi = \frac{1}{(2\pi)^d}\int e^{ix\xi}e^{-i|\xi|^2t/2}\widehat{\psi}_0(\xi)\,d\xi.$$

Recalling Plancherel's theorem that $\|f\|_{L^2} = C\|\widehat{f}\|_{L^2}$ (for a constant C independent of f), we obtain

$$\|\psi(t,x)\|_{L^2} = C\|\widehat{\psi}(t,\xi)\|_{L^2} = C\|\widehat{\psi}(0,\xi)\|_{L^2} = \|\psi(0,x)\|_{L^2} = \|\psi_0(x)\|_{L^2},$$

where the second equality is noticing that $e^{-i|\xi|^2t/2}$ is purely imaginary. This is a rigorous justification that the linear Schrödinger evolution preserves the L^2 norm of the solution.

Exercise 1.3. Compute that

$$\frac{d}{dt}\int_{\mathbb{R}^d}|\psi(t,x)|^2\,dx = 0.$$

This is an alternative way to show that the L^2 norm of the solution is preserved.

1.3 Sobolev Spaces

Definition 1.3. The *Sobolev spaces* $H^\gamma = W^{\gamma,2}$ for $\gamma \in \mathbb{R}$ are defined via the norm

$$\|f\|_{H^\gamma} = \left(\int_{\mathbb{R}^d}(1+|\xi|^2)^\gamma|\widehat{f}(\xi)|^2\,d\xi\right)^{1/2}.$$

The *homogeneous Sobolev spaces* $\dot{H}^\gamma$ are defined by the norm

$$\|f\|_{\dot{H}^\gamma} = \left(\int_{\mathbb{R}^d}|\xi|^{2\gamma}|\widehat{f}(\xi)|^2\,d\xi\right)^{1/2}.$$

Remark. If $\gamma \in \mathbb{N}$ and $d = 1$, then

$$\|f\|_{H^\gamma} \sim \sum_{m=0}^{\gamma}\|\partial_x^m f\|_{L^2}.$$

In particular, this means that $f \in H^\gamma$ if and only if $\partial_x^m f \in L^2$ for all $m \le \gamma$.

Exercise 1.4. Check that if $f_\lambda(x) = f(\lambda x)$, then $\widehat{f_\lambda}(\xi) = \lambda^{-d}\widehat{f}(\xi/\lambda)$.

Remark. In the Sobolev spaces, this means that (change variables $\eta = \xi/\lambda$ for the last equality)

$$\|f_\lambda\|_{\dot{H}^\gamma} = \left(\int_{\mathbb{R}^d}|\xi|^{2\gamma}|\widehat{f_\lambda}(\xi)|^2\,d\xi\right)^{1/2} = \left(\int_{\mathbb{R}^d}|\xi|^{2\gamma}|\lambda^{-d}\widehat{f}(\xi/\lambda)|^2\,d\xi\right)^{1/2} = \lambda^{\gamma-d/2}\|f\|_{\dot{H}^\gamma}.$$

Lemma 1.1. *In the Schrödinger equation, $\|\psi(t)\|_{H^\gamma} = \|\psi_0\|_{H^\gamma}$ and $\|\psi(t)\|_{\dot{H}^\gamma} = \|\psi_0\|_{\dot{H}^\gamma}$ for all t and γ.*

Proof. We can compute that

$$\|\psi(t)\|_{\dot{H}^\gamma} = \int_{\mathbb{R}^d}|\xi|^{2\gamma}|\widehat{\psi}(t,\xi)|^2\,d\xi = \int_{\mathbb{R}^d}|\xi|^{2\gamma}|e^{-i|\xi|^2/2}\widehat{\psi}_0(\xi)|^2\,d\xi = \int_{\mathbb{R}^d}|\xi|^{2\gamma}|\widehat{\psi}_0(\xi)|^2\,d\xi = \|\psi_0\|_{\dot{H}^\gamma}.$$

The same argument works for the H^γ case after replacing $|\xi|^{2\gamma}$ with $(1+|\xi|^2)^\gamma$. □