MATH 8803: Nonlinear Dispersive Equations

Frank Qiang
Instructor: Gong Chen

Georgia Institute of Technology

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Contents

Lecture 1

Jan. 6 — Introduction to Dispersion

1.1 Introduction to Dispersion

Definition 1.1. An evolution equation is *dispersive* if when no boundary conditions are imposed (e.g. on $\mathbb{R}^n$), its wave solutions spread out in space as they evolve in time.

Example 1.1.1. Two classic examples of dispersive equations are:

- The *Schrödinger equation*: $iu_t + \Delta u = 0$.
- The *Airy (linearized KdV) equation*: $u_t + u_{xxx} = 0$.

Remark. Consider the equation $u_t + p(\partial_x)u = 0$, where p is a polynomial, and a plane-wave solution

$$u(t,x) = e^{i(kx-\omega t)} = e^{ik(x-(\omega/k)t)}.$$

Here k is the *wave number* or *space frequency*, and ω is the *(time) frequency*. Plugging the plane-wave solution into the equation, we obtain the relation $\omega(k) = -ip(ik)$, i.e.

$$\frac{\omega(k)}{k} = \frac{1}{ik}p(ik).$$

The above equation is known as the *dispersive relation*. This gives the traveling speed of the plane-wave solution with wave number k, which is called the *phase velocity*.

Example 1.1.2. The following are some examples of dispersive relations:

- For the *linear advection equation* $u_t + cu_x = 0$ with $c \in \mathbb{R}$, one can compute that $\omega/k = c$.
- For the Schrödinger equation $iu_t + \frac{1}{2}\Delta u = 0$, we have $\omega/k = k/2 \in \mathbb{R}$.

  In this case of the Schrödinger equation, plane waves with large wave number (large space frequency) travel faster than low-frequency waves.

Remark. In general, dispersion means that different frequency plane waves travel at different speeds.

Remark. Given initial data u_0, we can write using the Fourier transform that

$$u_0 = \int \widehat{u}_0(k)e^{ikx}\,dk.$$

Then we get the solution u as

$$u(t,x) = \int \widehat{u}_0(k)e^{ik(x-(\omega(k)/k)t)}\,dk.$$

Example 1.1.3. In the case of the linear advection equation, we obtain the solution as

$$u(t,x) = \int \widehat{u}_0(k) e^{ik(x-ct)}\,dk = u_0(x-ct).$$

For the Schrödinger equation, we instead have the solution

$$u(t,x) = \int \widehat{u}_0(k) e^{ik(x-(k/2)t)}\,dk.$$

Since different k travels at different speeds, the original profile quickly spreads out.

Exercise 1.1. Calculate the dispersive relation ω/k for the linearized KdV equation $u_t + u_{xxx} = 0$.

Example 1.1.4. The *KdV equation* is given by

$$\partial_t u + \partial_{xxx} u + 6u\partial_x u = 0.$$

This equation is used to model shallow water surfaces, and is a nonlinear dispersive equation. Russell observed a great bump of water in a channel that traveled for a long time and kept its shape. This is due to the nonlinear effects in the KdV equation, and these effects are called *solitons*.

Definition 1.2. A *soliton* is a self-reinforcing solitary wave (a wave packet or pulse) that maintains its shape while traveling at a constant speed.

1.2 Fourier Transform and the Free Schrödinger Equation

Consider the following free Schrödinger equation:

$$\begin{cases} i\partial_t \psi + \frac{1}{2}\Delta\psi = 0, \\ \psi|_{t=0} = \psi_0. \end{cases}$$

We will solve this equation using the *Fourier transform*

$$\widehat{f}(\xi) = \int_{\mathbb{R}^d} e^{-ix\cdot\xi} f(x)\,dx.$$

Note that one can recover f from its Fourier transform via the *inversion formula*

$$f(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{ix\cdot\xi} \widehat{f}(\xi)\,d\xi.$$

Exercise 1.2. Check that $(\partial_{x_j} f)^\wedge = i\xi_j \widehat{f}$.

Applying the Fourier transform to the free Schrödinger equation, one has

$$i\partial_t\psi + \frac{1}{2}\Delta\psi = 0 \quad \xrightarrow{\text{F.T.}} \quad i\partial_t\widehat{\psi} - \frac{1}{2}|\xi|^2\widehat{\psi} = 0$$

and initial condition $\widehat{\psi}(0,\xi) = \widehat{\psi}_0(\xi)$. So for fixed ξ, we have an ODE, so we can solve the equation via

$$\widehat{\psi}(t,\xi) = e^{-i|\xi|^2 t/2}\widehat{\psi}_0(\xi).$$

Now by applying the inverse Fourier transform, we obtain the solution

$$\psi(t,x) = (2\pi)^{-d} \int e^{ix\xi} \widehat{\psi}(t,\xi)\, d\xi = (2\pi)^{-d} \int e^{ix\xi} e^{-i|\xi|^2 t/2} \widehat{\psi}_0(\xi)\, d\xi.$$

Recalling Plancherel's theorem that $\|f\|_{L^2} = C\|\widehat{f}\|_{L^2}$ (for a constant C independent of f), we obtain

$$\|\psi(t,x)\|_{L^2} = C\|\widehat{\psi}(t,\xi)\|_{L^2} = C\|\widehat{\psi}(0,\xi)\|_{L^2} = \|\psi(0,x)\|_{L^2} = \|\psi_0(x)\|_{L^2},$$

where the second equality follows by noticing that $e^{-i|\xi|^2 t/2}$ has modulus 1. This shows that the linear Schrödinger evolution preserves the L^2 norm of the solution.

Exercise 1.3. Compute that

$$\frac{d}{dt} \int_{\mathbb{R}^d} |\psi(t,x)|^2\, dx = 0.$$

This is an alternative way to show that the L^2 norm of the solution is preserved.

1.3 Sobolev Spaces

Definition 1.3. The *Sobolev spaces* $H^\gamma = W^{\gamma,2}$ for $\gamma \in \mathbb{R}$ are defined via the norm

$$\|f\|_{H^\gamma} = \left(\int_{\mathbb{R}^d} (1+|\xi|^2)^\gamma |\widehat{f}(\xi)|^2\, d\xi \right)^{1/2}.$$

The *homogeneous Sobolev spaces* $\dot{H}^\gamma$ are defined by the norm

$$\|f\|_{\dot{H}^\gamma} = \left(\int_{\mathbb{R}^d} |\xi|^{2\gamma} |\widehat{f}(\xi)|^2\, d\xi \right)^{1/2}.$$

Remark. If $\gamma \in \mathbb{N}$ and $d = 1$, then

$$\|f\|_{H^\gamma} \sim \sum_{m=0}^{\gamma} \|\partial_x^m f\|_{L^2}.$$

In particular, this means that $f \in H^\gamma$ if and only if $\partial_x^m f \in L^2$ for all $m \le \gamma$.

Exercise 1.4. Check that if $f_\lambda(x) = f(\lambda x)$, then $\widehat{f_\lambda}(\xi) = \lambda^{-d} \widehat{f}(\xi/\lambda)$.

Remark. In the Sobolev spaces, this means that (change variables $\eta = \xi/\lambda$ for the last equality)

$$\|f_\lambda\|_{\dot{H}^\gamma} = \left(\int_{\mathbb{R}^d} |\xi|^{2\gamma} |\widehat{f_\lambda}(\xi)|^2\, d\xi \right)^{1/2} = \left(\int_{\mathbb{R}^d} |\xi|^{2\gamma} |\lambda^{-d} \widehat{f}(\xi/\lambda)|^2\, d\xi \right)^{1/2} = \lambda^{\gamma - d/2} \|f\|_{\dot{H}^\gamma}.$$

Lemma 1.1. *In the Schrödinger equation, $\|\psi(t)\|_{H^\gamma} = \|\psi_0\|_{H^\gamma}$ and $\|\psi(t)\|_{\dot{H}^\gamma} = \|\psi_0\|_{\dot{H}^\gamma}$ for all t and γ.*

Proof. We can compute that

$$\|\psi(t)\|_{\dot{H}^\gamma} = \int_{\mathbb{R}^d} |\xi|^{2\gamma} |\widehat{\psi}(t,\xi)|^2\, d\xi = \int_{\mathbb{R}^d} |\xi|^{2\gamma} |e^{-i|\xi|^2/2} \widehat{\psi}_0(\xi)|^2\, d\xi = \int_{\mathbb{R}^d} |\xi|^{2\gamma} |\widehat{\psi}_0(\xi)|^2\, d\xi = \|\psi_0\|_{\dot{H}^\gamma}.$$

The same argument works for the H^γ case after replacing $|\xi|^{2\gamma}$ with $(1+|\xi|^2)^\gamma$. □

Lecture 2

Jan. 8 — Special Solutions

2.1 Special Solutions

Example 2.0.1. The following are special solutions to the Schrödinger equation:

1. Gaussian: $\psi_0 = e^{-|x|^2/2}$. One can compute the Fourier transform and get

$$\widehat{\psi}_0(\xi) = \int_{\mathbb{R}^d} e^{-ix\cdot\xi} e^{-|x|^2/2}\, dx = \int_{\mathbb{R}^d} e^{-|x+i\xi|^2/2} e^{-|\xi|^2/2}\, dx = e^{-|\xi|^2/2} \int_{\mathbb{R}^d} e^{-|x+i\xi|^2/2}\, dx.$$

The last integral is a contour integral in the complex plane along $\Im z = \xi$, and we can deform the contour via Cauchy's theorem to the real axis to obtain (the integrand is analytic on $0 \le \Im z \le \xi$)

$$\widehat{\psi}_0(\xi) = e^{-|\xi|^2/2} \int_{\mathbb{R}^d} e^{-|x|^2/2}\, dx = (2\pi)^{d/2} e^{-|\xi|^2/2}.$$

Then taking inverse Fourier transforms, we obtain the solution

$$\begin{aligned} \psi(t,x) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{i(x\cdot\xi - |\xi|^2 t/2)} \widehat{\psi}_0(\xi)\, d\xi &= (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{i(x\cdot\xi - |\xi|^2 t/2)} e^{-|\xi|^2/2}\, d\xi \\ &= (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{-\frac{1}{2}(1+it)|\xi|^2} e^{ix\cdot\xi}\, d\xi. \end{aligned}$$

Now formally put $\eta = (1+it)^{1/2}\xi$ to get

$$\psi(t,x) = (2\pi)^{-d/2} (1+it)^{-d/2} \int_{\mathbb{R}^d} e^{-\frac{1}{2}|\eta|^2} e^{ix\eta/(1+it)^{1/2}}\, d\eta.$$

Fill in the details of the above change of variables as an exercise (e.g. one has to worry about choosing a branch cut when taking the square root). Computing the integral explicitly, one obtains

$$\psi(t,x) = (1+it)^{-d/2} e^{-|x|^2/(2(1+it))}.$$

One can from this that ψ has decay in time. Furthermore, one can see that

$$|\psi(t,x)|^2 = (1+t^2)^{-d/2} e^{-|x|^2/(1+t^2)}.$$

From this we can observe an L^∞ decay of ψ like $t^{-d/2}$, and that the influence region of the solution grows like order t. We can also see again from this explicit computation that $\|\psi(t)\|_{L^2} = C$.

2. Modulated Gaussian: $\psi_0 = e^{-|x|^2/2}e^{ix\cdot v}$. The Fourier transform of this initial data is
$$\widehat{\psi}_0(\xi) = (2\pi)^{d/2}e^{-i|\xi-v|^2/2}.$$
So the solution corresponding to this initial data is
$$\begin{aligned}\psi(t,x) = (2\pi)^{-d}\int_{\mathbb{R}^d} e^{i(x\cdot\xi-|\xi|^2t/2)}\widehat{\psi}_0(\xi)\,d\xi &= (2\pi)^{-d/2}\int_{\mathbb{R}^d} e^{i(x\cdot\xi-|\xi|^2t/2)}e^{-|\xi-v|^2}\,d\xi \\ &= e^{ix\cdot v}e^{-|v|^2t/2}(2\pi)^{-d/2}\int_{\mathbb{R}^d} e^{i(x-vt)\cdot\xi}e^{-(1+it)|\xi|^2/2}\,d\xi \\ &= e^{ix\cdot v}e^{-|v|^2t/2}(1+it)^{-d/2}\exp\left(-\frac{|x-vt|^2}{2(1+it)}\right).\end{aligned}$$
From this we can see that the influence region of the solution moves with velocity v.

3. Fundamental solution: We want a *fundamental solution* K such that K solves
$$i\partial_t K + \frac{1}{2}\Delta K = 0 \quad \text{and} \quad K|_{t=0} = \delta_0.$$
We will find K by scaling arguments. Suppose such a K exists. Then we must have
$$\psi(t,x) = \int_{\mathbb{R}^d} K(t,x-y)\psi_0(y)\,dy \tag{1}$$
since $K|_{t=0} = \delta_0$. Now define the scaling $\psi_\lambda(t,x) = \psi(\lambda^2 t, \lambda x)$. Then ψ_λ also solves
$$i\partial_t\psi_\lambda + \frac{1}{2}\Delta\psi_\lambda = 0$$
and we have the initial condition $\psi_\lambda(0,x) = \psi_0(\lambda x)$. Then
$$\psi_\lambda(t,x) = \int_{\mathbb{R}^d} K(t,x-y)\psi_0(\lambda y)\,dy = \psi(\lambda^2 t, \lambda x).$$
Setting $t' = \lambda^2 t$, $x' = \lambda x$, and $y' = \lambda y$, we get
$$\psi(t',x') = \frac{1}{\lambda^d}\int_{\mathbb{R}^d} K\left(\frac{t'}{\lambda^2}, \frac{x'-y'}{\lambda}\right)\psi_0(y')\,dy'. \tag{2}$$
Comparing (1) and (2), we see that we must have
$$K(t,x-y) = \lambda^{-d}K\left(\frac{t}{\lambda^2}, \frac{x-y}{\lambda}\right).$$
Setting $u = x-y$, we get
$$K(t,u) = \lambda^{-d}K\left(\frac{t}{\lambda^2}, \frac{u}{\lambda}\right).$$
Thus we expect $K(t,x) = t^{-d/2}\Phi(|x|^2/t)$ for some Φ. Now we use the fact that $i\partial_t K + \frac{1}{2}\Delta K = 0$. Setting $m = |x|^2/t$, one can plug in the above guess for K to obtain (note that $\Delta = \nabla\cdot\nabla$)
$$-\frac{id}{2}t^{-d/2-1}\Phi(m) - it^{-d/2}\Phi'(m)\frac{m}{t} + \frac{1}{2}t^{-d/2}\nabla\cdot\left(\frac{2x}{t}\Phi'(m)\right) = 0.$$

Then we get

$$-i\frac{d}{2}\Phi(m) - im\Phi'(m) + d\Phi'(m) + 2m\Phi''(m) = 0,$$

which gives

$$d\left(\Phi'(m) - \frac{i}{2}\Phi(m)\right) + 2m\frac{d}{dm}\left(\Phi'(m) - \frac{i}{2}\Phi(m)\right) = 0.$$

Now observe that $\Phi(m) = e^{im/2}$ solves the above equation. Since $\Phi(m)$ solves the equation, $c\Phi(m)$ also solves the equation for any $c \in \mathbb{C}$, and thus we have

$$K(t,x) = ct^{-d/2}\Phi(|x|^2/t) = ct^{-d/2}e^{i|x|^2/2t}.$$

To determine c, we use $K|_{t=0} = \delta_0$, from which one can obtain $c = (2\pi i)^{-d/2}$. Thus

$$K(t,x) = (2\pi it)^{-d/2}e^{i|x|^2/2t}.$$

The rough computation is that since $\widehat{K}(0,\xi) = 1$, we have

$$K = (2\pi)^{-d}\int_{\mathbb{R}^d} e^{i(x\cdot\xi - |\xi|^2t/2)}\widehat{K}(0,\xi)\,d\xi = (2\pi)^{-d}\int_{\mathbb{R}^d} e^{i(x\cdot\xi-|\xi|^2t/2)}\,d\xi.$$

This is not necessarily integrable a priori, but one can take limits and obtain

$$\begin{aligned} K = (2\pi)^{-d}\lim_{\epsilon\to0^+}\int_{\mathbb{R}^d} e^{ix\cdot\xi}e^{-(\epsilon+it)|\xi|^2/2}\,d\xi &= \lim_{\epsilon\to0^+}(\epsilon+it)^{-d/2}(2\pi)^{-d/2}e^{-|x|^2/(2(\epsilon+it))} \\ &= (2\pi it)^{-d/2}e^{-|x|^2/2it}. \end{aligned}$$

Note that this computation matches the result of the previous scaling argument.

Theorem 2.1. *Let $\psi_0 \in \mathcal{S}(\mathbb{R}^d)$.*[1] *Then there exists a solution to*

$$\begin{cases} i\partial_t\psi + \frac{1}{2}\Delta\psi = 0, \\ \psi|_{t=0} = \psi_0, \end{cases}$$

which is unique and given by

$$\psi(t,x) = \int_{\mathbb{R}^d} K(t,x-y)\psi_0(y)\,dy = (2\pi it)^{-d/2}\int_{\mathbb{R}^d} e^{-|x-y|^2/2it}\psi_0(y)\,dy.$$

Proof. This theorem is a summary of the results of the previous explicit computations. □

Remark. Recall that the Schrödinger evolution preserves the L^2 norm of a solution, i.e.

$$\|\psi(t)\|_{L^2} = \|\psi(0)\|_{L^2} = \|\psi_0\|_{L^2}.$$

The above theorem also gives an L^∞ bound (a so-called *dispersive estimate*)

$$\|\psi(t)\|_{L^\infty} \le |2\pi t|^{-d/2}\int_{\mathbb{R}^d}|\psi_0(y)|\,dy = |2\pi t|^{-d/2}\|\psi_0\|_{L^1}.$$

[1] Here $\mathcal{S}(\mathbb{R}^d)$ is the space of *Schwartz functions*.

Lecture 3

Jan. 15 — Strichartz Estimates

3.1 Interpolation Results

Remark (Interpolation)**.** Consider a linear operator T which maps $T : L^{p_1} \to L^{q_1}$ and $T : L^{p_2} \to L^{q_2}$, where $1 \leq p_1 \leq p_2 \leq \infty$. Then T also maps $T : L^p \to L^q$ for any p, q such that

$$\frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2} \quad \text{and} \quad \frac{1}{q} = \frac{\theta}{q_1} + \frac{1-\theta}{q_2}$$

for some $0 \leq \theta \leq 1$. More specifically, if $\|Tf\|_{L^{q_1}} \leq C_1 \|f\|_{L^{p_1}}$ and $\|Tf\|_{L^{q_2}} \leq C_2 \|f\|_{L^{p_2}}$, then

$$\|Tf\|_{L^q} \leq C_1^\theta C_2^{1-\theta} \|f\|_{L^p}.$$

This L^p interpolation is a result from real and functional analysis. Note that by interpolation, we have

$$\|\psi\|_{L^{p'}(\mathbb{R}^d)} \leq C|t|^{-d(1/p-1/2)} \|\psi_0\|_{L^p(\mathbb{R}^d)}$$

for $1 \leq p \leq 2$, where p' is the *Hölder conjugate* of p, i.e. $1/p' + 1/p = 1$.

3.2 Strichartz Estimates

Remark. We will now consider the inhomogeneous Schrödinger equation:

$$\begin{cases} i\psi_t + \frac{1}{2}\Delta\psi = F, & F \in \mathcal{S}_{x,t} \\ \psi(0) = \psi_0, & \psi_0 \in \mathcal{S}, \end{cases}$$

where $F \in \mathcal{S}_{x,t}$ means that F is Schwartz in both x and t. We can solve this via the *Duhamel formula*:

$$\psi(t) = e^{it\Delta/2}\psi_0 - i\int_0^t e^{i(t-s)\Delta/2} F(s)\, ds,$$

where $e^{it\Delta/2}$ is the *linear propagator* given by

$$e^{it\Delta/2}\psi_0 = (e^{-it|\xi|^2/2}\widehat{\psi}_0)^\vee = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{ix\cdot\xi} e^{-it|\xi|^2/2}\widehat{\psi}_0(\xi)\, d\xi.$$

Theorem 3.1 (Strichartz estimates)**.** *For $p' = 2 + 4/d$, we have the estimate*[1]

$$\|\psi\|_{L^{p'}_{t,x}(\mathbb{R}\times\mathbb{R}^d)} \lesssim \|\psi_0\|_{L^2_x(\mathbb{R}^d)} + \|F\|_{L^p_{t,x}(\mathbb{R}\times\mathbb{R}^d)}.$$

[1] Here $A \lesssim B$ means that $A \leq CB$ for some prescribed constant C.

Remark. If $F = 0$, this is the bound
$$\|\psi\|_{L^{p'}_{t,x}} \lesssim \|\psi_0\|_{L^2}$$
for $p' > 2$. Formally, this means that we gain integrability in x. Note that this gain in integrability is not pointwise in time, i.e. we do *not* have $\|\psi(t)\|_{L^\infty_t L^{p'}_x} \lesssim \|\psi_0\|_{L^2_x}$. We must instead average over t.

Remark. Why p' and why do we also pick p' in the time integration? Actually, p' is the only possible choice for the above result. This follows by a scaling argument: Set
$$\psi_\lambda(t,x) = \psi(\lambda^2 t, \lambda x), \quad (\psi_\lambda)_0(x) = \psi_0(\lambda x), \quad F_\lambda(t,x) = \lambda^2 F(\lambda^2 t, \lambda x).$$
Then ψ_λ solves the equation
$$\begin{cases} i\partial_t \psi_\lambda + \frac{1}{2}\Delta\psi_\lambda = F_\lambda, \\ \psi_\lambda(0) = (\psi_\lambda)_0. \end{cases}$$
If the above theorem makes sense, then it must hold for both ψ_λ and ψ. Now
$$\|\psi_\lambda\|_{L^{p'}_{t,x}} = \lambda^{-d/p'}\lambda^{-2/p'}\|\psi\|_{L^{p'}_{t,x}}$$
by a change of variables, and
$$\|(\psi_\lambda)_0\|_{L^2_x} = \lambda^{-d/2}\|\psi_0\|_{L^2_x}.$$
Now if $F = 0$, then we have the estimates
$$\|\psi\|_{L^{p'}_{t,x}} \lesssim \|\psi_0\|_{L^2_x} \quad \text{and} \quad \|\psi_\lambda\|_{L^{p'}_{t,x}} \lesssim \|(\psi_\lambda)_0\|_{L^2_x}, \tag{$*$}$$
Using the scaling computations in the second estimate in $(*)$ implies that
$$\|\psi\|_{L^{p'}_{t,x}}\lambda^{-d/p'}\lambda^{-2/p'} \lesssim \lambda^{-d/2}\|\psi_0\|_{L^2_x}.$$
This inequality should hold independent of λ, since otherwise taking $\lambda \to \infty$ or $\lambda \to 0$ yields a contradiction with the first inequality in $(*)$. Thus the powers in λ should match:
$$-\frac{d}{p'} - \frac{2}{p'} = -\frac{d}{2},$$
so we find that p' must be
$$p' = \frac{d+2}{d/2} + \frac{2d+4}{d} = 2 + \frac{4}{d}.$$
This uniquely determines p'. Now consider $F \neq 0$. Using a similar computation as before, we have
$$\|F_\lambda\|_{L^q_{t,x}} = \lambda^2\lambda^{-d/q}\lambda^{-2/q}\|F\|_{L^q_{t,x}}.$$
Then the theorem says that $\|\psi_\lambda\|_{L^{p'}_{t,x}} \lesssim \|\psi_0\|_{L^2_x} + \|F\|_{L^q_{t,x}}$, so we have
$$\|\psi\|_{L^{p'}_{t,x}}\lambda^{-d/p'}\lambda^{-2/p'} \lesssim \lambda^{-d/2}\|\psi_0\|_{L^2_x} + \lambda^2\lambda^{-d/q}\lambda^{-2/q}\|F\|_{L^q_{t,x}}.$$
Again the estimate should hold independent of λ, so the powers in λ must match:
$$-\frac{d}{p'} - \frac{2}{p'} = 2 - \frac{d}{q} - \frac{2}{q} = -\frac{d}{2},$$
which then gives p as
$$p = \left(1 - \frac{1}{p'}\right)^{-1} = \left(1 - \frac{d}{2d+4}\right)^{-1} = \frac{2d+4}{d+4}.$$

Lemma 3.1. *Let $\psi(t) = e^{it\Delta/2}\psi_0$. Then for $1 \le p \le 2$,*

$$\|\psi(t)\|_{L_x^{p'}(\mathbb{R}^d)} \lesssim |t|^{-d(1/p-1/2)}\|\psi_0\|_{L_x^p(\mathbb{R}^d)}.$$

Proof. This is the interpolation result from the beginning of class. □

Lemma 3.2 (Hardy-Littlewood-Sobolev inequality)**.** *Let $0 < \alpha < 1$ and $g \in \mathcal{S}(\mathbb{R})$. Let*

$$(T_\alpha g)(t) = \int_{-\infty}^{\infty} |t-s|^{-\alpha} g(s)\,ds.$$

Then we have $\|T_\alpha g\|_{L^q(\mathbb{R})} \lesssim \|g\|_{L^p(\mathbb{R})}$, where $1 < p < q < \infty$ such that $1 + 1/q = \alpha + 1/p$.

Proof. One approach is via harmonic analysis and maximal functions. An alternative approach can be found in Theorem 4.3 of Analysis by Lieb and Loss. □

Remark. Recall *Young's inequality* that for

$$h(t) = \int f(t-s)g(s)\,dx,$$

we have $\|h\|_{L^r} \le \|f\|_{L^p}\|g\|_{L^q}$, where $1/r + 1 = 1/q + 1/p$. The Hardy-Littlewood-Sobolev inequality can be seen as a generalized Young's inequality: If $f(s) = |s|^{-\alpha}$, then f barely fails to be in $L^{1/\alpha}$. Informally, we can think of "$f \in L^{1/\alpha}$," and the standard Young's inequality would imply Hardy-Littlewood-Sobolev.

Remark. We have $q > p$ in the Hardy-Littlewood-Sobolev inequality, so we gain some integrability via fractional integration for $p > 1$ (the type of integral defining $T_\alpha g$ is known as *fractional integration*).

Proof of Theorem 3.1. This proof is left for next class. □