MATH 8803: Nonlinear Dispersive Equations

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Contents

Lecture 1

Jan. 6 — Introduction to Dispersion

1.1 Introduction to Dispersion

Definition 1.1. An evolution equation is *dispersive* if when no boundary conditions are imposed (e.g. on $\mathbb{R}^n$), its wave solutions spread out in space as they evolve in time.

Example 1.1.1. Two classic examples of dispersive equations are:

- The *Schrödinger equation*: $iu_t + \Delta u = 0$.
- The *Airy (linearized KdV) equation*: $u_t + u_{xxx} = 0$.

Remark. Consider the equation $u_t + p(\partial_x)u = 0$, where p is a polynomial, and a plane-wave solution

$$u(t,x) = e^{i(kx-\omega t)} = e^{ik(x-(\omega/k)t)}.$$

Here k is the *wave number* or *space frequency*, and ω is the *(time) frequency*. Plugging the plane-wave solution into the equation, we obtain the relation $\omega(k) = -ip(ik)$, i.e.

$$\frac{\omega(k)}{k} = \frac{1}{ik}p(ik).$$

The above equation is known as the *dispersive relation*. This gives the traveling speed of the plane-wave solution with wave number k, which is called the *phase velocity*.

Example 1.1.2. The following are some examples of dispersive relations:

- For the *linear advection equation* $u_t + cu_x = 0$ with $c \in \mathbb{R}$, one can compute that $\omega/k = c$.
- For the Schrödinger equation $iu_t + \frac{1}{2}\Delta u = 0$, we have $\omega/k = k/2 \in \mathbb{R}$.

  In this case of the Schrödinger equation, plane waves with large wave number (large space frequency) travel faster than low-frequency waves.

Remark. In general, dispersion means that different frequency plane waves travel at different speeds.

Remark. Given initial data u_0, we can write using the Fourier transform that

$$u_0 = \int \widehat{u}_0(k)e^{ikx}\,dk.$$

Then we get the solution u as

$$u(t,x) = \int \widehat{u}_0(k)e^{ik(x-(\omega(k)/k)t)}\,dk.$$

Example 1.1.3. In the case of the linear advection equation, we obtain the solution as

$$u(t,x) = \int \widehat{u}_0(k) e^{ik(x-ct)}\, dk = u_0(x-ct).$$

For the Schrödinger equation, we instead have the solution

$$u(t,x) = \int \widehat{u}_0(k) e^{ik(x-(k/2)t)}\, dk.$$

Since different k travels at different speeds, the original profile quickly spreads out.

Exercise 1.1. Calculate the dispersive relation ω/k for the linearized KdV equation $u_t + u_{xxx} = 0$.

Example 1.1.4. The *KdV equation* is given by

$$\partial_t u + \partial_{xxx} u + 6u\partial_x u = 0.$$

This equation is used to model shallow water surfaces, and is a nonlinear dispersive equation. Russell observed a great bump of water in a channel that traveled for a long time and kept its shape. This is due to the nonlinear effects in the KdV equation, and these effects are called *solitons*.

Definition 1.2. A *soliton* is a self-reinforcing solitary wave (a wave packet or pulse) that maintains its shape while traveling at a constant speed.

1.2 Fourier Transform and the Free Schrödinger Equation

Consider the following free Schrödinger equation:

$$\begin{cases} i\partial_t \psi + \frac{1}{2}\Delta\psi = 0, \\ \psi|_{t=0} = \psi_0. \end{cases}$$

We will solve this equation using the *Fourier transform*

$$\widehat{f}(\xi) = \int_{\mathbb{R}^d} e^{-ix\cdot\xi} f(x)\, dx.$$

Note that one can recover f from its Fourier transform via the *inversion formula*

$$f(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{ix\cdot\xi} \widehat{f}(\xi)\, d\xi.$$

Exercise 1.2. Check that $(\partial_{x_j} f)^\wedge = i\xi_j \widehat{f}$.

Applying the Fourier transform to the free Schrödinger equation, one has

$$i\partial_t \psi + \frac{1}{2}\Delta\psi = 0 \quad \xrightarrow{\text{F.T.}} \quad i\partial_t \widehat{\psi} - \frac{1}{2}|\xi|^2 \widehat{\psi} = 0$$

and initial condition $\widehat{\psi}(0,\xi) = \widehat{\psi}_0(\xi)$. So for fixed ξ, we have an ODE, so we can solve the equation via

$$\widehat{\psi}(t,\xi) = e^{-i|\xi|^2 t/2} \widehat{\psi}_0(\xi).$$

Now by applying the inverse Fourier transform, we obtain the solution

$$\psi(t,x) = (2\pi)^{-d} \int e^{ix\xi} \widehat{\psi}(t,\xi)\, d\xi = (2\pi)^{-d} \int e^{ix\xi} e^{-i|\xi|^2 t/2} \widehat{\psi}_0(\xi)\, d\xi.$$

Recalling Plancherel's theorem that $\|f\|_{L^2} = C\|\widehat{f}\|_{L^2}$ (for a constant C independent of f), we obtain

$$\|\psi(t,x)\|_{L^2} = C\|\widehat{\psi}(t,\xi)\|_{L^2} = C\|\widehat{\psi}(0,\xi)\|_{L^2} = \|\psi(0,x)\|_{L^2} = \|\psi_0(x)\|_{L^2},$$

where the second equality follows by noticing that $e^{-i|\xi|^2 t/2}$ has modulus 1. This shows that the linear Schrödinger evolution preserves the L^2 norm of the solution.

Exercise 1.3. Compute that

$$\frac{d}{dt} \int_{\mathbb{R}^d} |\psi(t,x)|^2\, dx = 0.$$

This is an alternative way to show that the L^2 norm of the solution is preserved.

1.3 Sobolev Spaces

Definition 1.3. The *Sobolev spaces* $H^\gamma = W^{\gamma,2}$ for $\gamma \in \mathbb{R}$ are defined via the norm

$$\|f\|_{H^\gamma} = \left(\int_{\mathbb{R}^d} (1+|\xi|^2)^\gamma |\widehat{f}(\xi)|^2\, d\xi \right)^{1/2}.$$

The *homogeneous Sobolev spaces* $\dot{H}^\gamma$ are defined by the norm

$$\|f\|_{\dot{H}^\gamma} = \left(\int_{\mathbb{R}^d} |\xi|^{2\gamma} |\widehat{f}(\xi)|^2\, d\xi \right)^{1/2}.$$

Remark. If $\gamma \in \mathbb{N}$ and $d = 1$, then

$$\|f\|_{H^\gamma} \sim \sum_{m=0}^{\gamma} \|\partial_x^m f\|_{L^2}.$$

In particular, this means that $f \in H^\gamma$ if and only if $\partial_x^m f \in L^2$ for all $m \le \gamma$.

Exercise 1.4. Check that if $f_\lambda(x) = f(\lambda x)$, then $\widehat{f_\lambda}(\xi) = \lambda^{-d} \widehat{f}(\xi/\lambda)$.

Remark. In the Sobolev spaces, this means that (change variables $\eta = \xi/\lambda$ for the last equality)

$$\|f_\lambda\|_{\dot{H}^\gamma} = \left(\int_{\mathbb{R}^d} |\xi|^{2\gamma} |\widehat{f_\lambda}(\xi)|^2\, d\xi \right)^{1/2} = \left(\int_{\mathbb{R}^d} |\xi|^{2\gamma} |\lambda^{-d} \widehat{f}(\xi/\lambda)|^2\, d\xi \right)^{1/2} = \lambda^{\gamma - d/2} \|f\|_{\dot{H}^\gamma}.$$

Lemma 1.1. *In the Schrödinger equation, $\|\psi(t)\|_{H^\gamma} = \|\psi_0\|_{H^\gamma}$ and $\|\psi(t)\|_{\dot{H}^\gamma} = \|\psi_0\|_{\dot{H}^\gamma}$ for all t and γ.*

Proof. We can compute that

$$\|\psi(t)\|_{\dot{H}^\gamma} = \int_{\mathbb{R}^d} |\xi|^{2\gamma} |\widehat{\psi}(t,\xi)|^2\, d\xi = \int_{\mathbb{R}^d} |\xi|^{2\gamma} |e^{-i|\xi|^2/2} \widehat{\psi}_0(\xi)|^2\, d\xi = \int_{\mathbb{R}^d} |\xi|^{2\gamma} |\widehat{\psi}_0(\xi)|^2\, d\xi = \|\psi_0\|_{\dot{H}^\gamma}.$$

The same argument works for the H^γ case after replacing $|\xi|^{2\gamma}$ with $(1+|\xi|^2)^\gamma$. ☐

Lecture 2

Jan. 8 — Special Solutions

2.1 Special Solutions

Example 2.0.1. The following are special solutions to the Schrödinger equation:

1. Gaussian: $\psi_0 = e^{-|x|^2/2}$. One can compute the Fourier transform and get

$$\widehat{\psi}_0(\xi) = \int_{\mathbb{R}^d} e^{-ix\cdot\xi} e^{-|x|^2/2}\, dx = \int_{\mathbb{R}^d} e^{-|x+i\xi|^2/2} e^{-|\xi|^2/2}\, dx = e^{-|\xi|^2/2} \int_{\mathbb{R}^d} e^{-|x+i\xi|^2/2}\, dx.$$

The last integral is a contour integral in the complex plane along $\Im z = \xi$, and we can deform the contour via Cauchy's theorem to the real axis to obtain (the integrand is analytic on $0 \le \Im z \le \xi$)

$$\widehat{\psi}_0(\xi) = e^{-|\xi|^2/2} \int_{\mathbb{R}^d} e^{-|x|^2/2}\, dx = (2\pi)^{d/2} e^{-|\xi|^2/2}.$$

Then taking inverse Fourier transforms, we obtain the solution

$$\begin{aligned}\psi(t,x) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{i(x\cdot\xi - |\xi|^2 t/2)} \widehat{\psi}_0(\xi)\, d\xi &= (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{i(x\cdot\xi - |\xi|^2 t/2)} e^{-|\xi|^2/2}\, d\xi \\ &= (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{-\frac{1}{2}(1+it)|\xi|^2} e^{ix\cdot\xi}\, d\xi.\end{aligned}$$

Now formally put $\eta = (1+it)^{1/2}\xi$ to get

$$\psi(t,x) = (2\pi)^{-d/2}(1+it)^{-d/2} \int_{\mathbb{R}^d} e^{-\frac{1}{2}|\eta|^2} e^{ix\eta/(1+it)^{1/2}}\, d\eta.$$

Fill in the details of the above change of variables as an exercise (e.g. one has to worry about choosing a branch cut when taking the square root). Computing the integral explicitly, one obtains

$$\psi(t,x) = (1+it)^{-d/2} e^{-|x|^2/(2(1+it))}.$$

One can from this that ψ has decay in time. Furthermore, one can see that

$$|\psi(t,x)|^2 = (1+t^2)^{-d/2} e^{-|x|^2/(1+t^2)}.$$

From this we can observe an L^∞ decay of ψ like $t^{-d/2}$, and that the influence region of the solution grows like order t. We can also see again from this explicit computation that $\|\psi(t)\|_{L^2} = C$.

2. Modulated Gaussian: $\psi_0 = e^{-|x|^2/2}e^{ix\cdot v}$. The Fourier transform of this initial data is

$$\widehat{\psi}_0(\xi) = (2\pi)^{d/2}e^{-i|\xi-v|^2/2}.$$

So the solution corresponding to this initial data is

$$\begin{aligned}\psi(t,x) = (2\pi)^{-d}\int_{\mathbb{R}^d} e^{i(x\cdot\xi-|\xi|^2t/2)}\widehat{\psi}_0(\xi)\,d\xi &= (2\pi)^{-d/2}\int_{\mathbb{R}^d} e^{i(x\cdot\xi-|\xi|^2t/2)}e^{-|\xi-v|^2}\,d\xi\\ &= e^{ix\cdot v}e^{-|v|^2t/2}(2\pi)^{-d/2}\int_{\mathbb{R}^d} e^{i(x-vt)\cdot\xi}e^{-(1+it)|\xi|^2/2}\,d\xi\\ &= e^{ix\cdot v}e^{-|v|^2t/2}(1+it)^{-d/2}\exp\left(-\frac{|x-vt|^2}{2(1+it)}\right).\end{aligned}$$

From this we can see that the influence region of the solution moves with velocity v.

3. Fundamental solution: We want a *fundamental solution* K such that K solves

$$i\partial_t K + \frac{1}{2}\Delta K = 0 \quad\text{and}\quad K|_{t=0} = \delta_0.$$

We will find K by scaling arguments. Suppose such a K exists. Then we must have

$$\psi(t,x) = \int_{\mathbb{R}^d} K(t,x-y)\psi_0(y)\,dy \tag{1}$$

since $K|_{t=0} = \delta_0$. Now define the scaling $\psi_\lambda(t,x) = \psi(\lambda^2 t, \lambda x)$. Then ψ_λ also solves

$$i\partial_t\psi_\lambda + \frac{1}{2}\Delta\psi_\lambda = 0$$

and we have the initial condition $\psi_\lambda(0,x) = \psi_0(\lambda x)$. Then

$$\psi_\lambda(t,x) = \int_{\mathbb{R}^d} K(t,x-y)\psi_0(\lambda y)\,dy = \psi(\lambda^2 t,\lambda x).$$

Setting $t' = \lambda^2 t$, $x' = \lambda x$, and $y' = \lambda y$, we get

$$\psi(t',x') = \frac{1}{\lambda^d}\int_{\mathbb{R}^d} K\left(\frac{t'}{\lambda^2},\frac{x'-y'}{\lambda}\right)\psi_0(y')\,dy'. \tag{2}$$

Comparing (1) and (2), we see that we must have

$$K(t,x-y) = \lambda^{-d}K\left(\frac{t}{\lambda^2},\frac{x-y}{\lambda}\right).$$

Setting $u = x-y$, we get

$$K(t,u) = \lambda^{-d}K\left(\frac{t}{\lambda^2},\frac{u}{\lambda}\right).$$

Thus we expect $K(t,x) = t^{-d/2}\Phi(|x|^2/t)$ for some Φ. Now we use the fact that $i\partial_t K + \frac{1}{2}\Delta K = 0$. Setting $m = |x|^2/t$, one can plug in the above guess for K to obtain (note that $\Delta = \nabla\cdot\nabla$)

$$-\frac{id}{2}t^{-d/2-1}\Phi(m) - it^{-d/2}\Phi'(m)\frac{m}{t} + \frac{1}{2}t^{-d/2}\nabla\cdot\left(\frac{2x}{t}\Phi'(m)\right) = 0.$$

Then we get

$$-i\frac{d}{2}\Phi(m) - im\Phi'(m) + d\Phi'(m) + 2m\Phi''(m) = 0,$$

which gives

$$d\left(\Phi'(m) - \frac{i}{2}\Phi(m)\right) + 2m\frac{d}{dm}\left(\Phi'(m) - \frac{i}{2}\Phi(m)\right) = 0.$$

Now observe that $\Phi(m) = e^{im/2}$ solves the above equation. Since $\Phi(m)$ solves the equation, $c\Phi(m)$ also solves the equation for any $c \in \mathbb{C}$, and thus we have

$$K(t,x) = ct^{-d/2}\Phi(|x|^2/t) = ct^{-d/2}e^{i|x|^2/2t}.$$

To determine c, we use $K|_{t=0} = \delta_0$, from which one can obtain $c = (2\pi i)^{-d/2}$. Thus

$$K(t,x) = (2\pi it)^{-d/2}e^{i|x|^2/2t}.$$

The rough computation is that since $\widehat{K}(0,\xi) = 1$, we have

$$K = (2\pi)^{-d}\int_{\mathbb{R}^d} e^{i(x\cdot\xi - |\xi|^2t/2)}\widehat{K}(0,\xi)\,d\xi = (2\pi)^{-d}\int_{\mathbb{R}^d} e^{i(x\cdot\xi-|\xi|^2t/2)}\,d\xi.$$

This is not necessarily integrable a priori, but one can take limits and obtain

$$\begin{aligned} K = (2\pi)^{-d}\lim_{\epsilon\to 0^+}\int_{\mathbb{R}^d} e^{ix\cdot\xi}e^{-(\epsilon+it)|\xi|^2/2}\,d\xi &= \lim_{\epsilon\to 0^+}(\epsilon+it)^{-d/2}(2\pi)^{-d/2}e^{-|x|^2/(2(\epsilon+it))} \\ &= (2\pi it)^{-d/2}e^{-|x|^2/2it}. \end{aligned}$$

Note that this computation matches the result of the previous scaling argument.

Theorem 2.1. *Let $\psi_0 \in \mathcal{S}(\mathbb{R}^d)$.*[1] *Then there exists a solution to*

$$\begin{cases} i\partial_t\psi + \frac{1}{2}\Delta\psi = 0, \\ \psi|_{t=0} = \psi_0, \end{cases}$$

which is unique and given by

$$\psi(t,x) = \int_{\mathbb{R}^d} K(t,x-y)\psi_0(y)\,dy = (2\pi it)^{-d/2}\int_{\mathbb{R}^d} e^{-|x-y|^2/2it}\psi_0(y)\,dy.$$

Proof. This theorem is a summary of the results of the previous explicit computations. □

Remark. Recall that the Schrödinger evolution preserves the L^2 norm of a solution, i.e.

$$\|\psi(t)\|_{L^2} = \|\psi(0)\|_{L^2} = \|\psi_0\|_{L^2}.$$

The above theorem also gives an L^∞ bound (a so-called *dispersive estimate*)

$$\|\psi(t)\|_{L^\infty} \le |2\pi t|^{-d/2}\int_{\mathbb{R}^d}|\psi_0(y)|\,dy = |2\pi t|^{-d/2}\|\psi_0\|_{L^1}.$$

[1] Here $\mathcal{S}(\mathbb{R}^d)$ is the space of *Schwartz functions*.

Lecture 3

Jan. 15 — Strichartz Estimates

3.1 Interpolation Results

Remark (Interpolation)**.** Consider a linear operator T which maps $T : L^{p_1} \to L^{q_1}$ and $T : L^{p_2} \to L^{q_2}$, where $1 \le p_1 \le p_2 \le \infty$. Then T also maps $T : L^p \to L^q$ for any p, q such that

$$\frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2} \quad \text{and} \quad \frac{1}{q} = \frac{\theta}{q_1} + \frac{1-\theta}{q_2}$$

for some $0 \le \theta \le 1$. More specifically, if $\|Tf\|_{L^{q_1}} \le C_1 \|f\|_{L^{p_1}}$ and $\|Tf\|_{L^{q_2}} \le C_2 \|f\|_{L^{p_2}}$, then

$$\|Tf\|_{L^q} \le C_1^\theta C_2^{1-\theta} \|f\|_{L^p}.$$

This L^p interpolation is a result from real and functional analysis. Note that by interpolation, we have

$$\|\psi\|_{L^{p'}(\mathbb{R}^d)} \le C|t|^{-d(1/p-1/2)} \|\psi_0\|_{L^p(\mathbb{R}^d)}$$

for $1 \le p \le 2$, where p' is the *Hölder conjugate* of p, i.e. $1/p' + 1/p = 1$.

3.2 Strichartz Estimates

Remark. We will now consider the inhomogeneous Schrödinger equation:

$$\begin{cases} i\psi_t + \frac{1}{2}\Delta\psi = F, & F \in \mathcal{S}_{x,t} \\ \psi(0) = \psi_0, & \psi_0 \in \mathcal{S}, \end{cases}$$

where $F \in \mathcal{S}_{x,t}$ means that F is Schwartz in both x and t. We can solve this via the *Duhamel formula*:

$$\psi(t) = e^{it\Delta/2}\psi_0 - i\int_0^t e^{i(t-s)\Delta/2} F(s)\, ds,$$

where $e^{it\Delta/2}$ is the *linear propagator* given by

$$e^{it\Delta/2}\psi_0 = (e^{-it|\xi|^2/2}\widehat{\psi}_0)^\vee = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{ix\cdot\xi} e^{-it|\xi|^2/2}\widehat{\psi}_0(\xi)\, d\xi.$$

Theorem 3.1 (Strichartz estimates)**.** *For $p' = 2 + 4/d$ and $1/p' + 1/p = 1$, we have the estimate*[1]

$$\|\psi\|_{L^{p'}_{t,x}(\mathbb{R}\times\mathbb{R}^d)} \lesssim \|\psi_0\|_{L^2_x(\mathbb{R}^d)} + \|F\|_{L^p_{t,x}(\mathbb{R}\times\mathbb{R}^d)}.$$

[1]Here $A \lesssim B$ means that $A \le CB$ for some prescribed constant C.

Remark. If $F = 0$, this is the bound

$$\|\psi\|_{L^{p'}_{t,x}} \lesssim \|\psi_0\|_{L^2}$$

for $p' > 2$. Formally, this means that we gain integrability in x. Note that this gain in integrability is not pointwise in time, i.e. we do *not* have $\|\psi(t)\|_{L^\infty_t L^{p'}_x} \lesssim \|\psi_0\|_{L^2_x}$. We must instead average over t.

Remark. Why p' and why do we pick p' in the time integration? Actually, p' is the only possible choice for the above result. This follows by a scaling argument: Set

$$\psi_\lambda(t,x) = \psi(\lambda^2 t, \lambda x), \quad (\psi_\lambda)_0(x) = \psi_0(\lambda x), \quad F_\lambda(t,x) = \lambda^2 F(\lambda^2 t, \lambda x).$$

Then ψ_λ solves the equation

$$\begin{cases} i\partial_t \psi_\lambda + \frac{1}{2}\Delta\psi_\lambda = F_\lambda, \\ \psi_\lambda(0) = (\psi_\lambda)_0. \end{cases}$$

If the above theorem makes sense, then it must hold for both ψ_λ and ψ. Now

$$\|\psi_\lambda\|_{L^{p'}_{t,x}} = \lambda^{-d/p'}\lambda^{-2/p'}\|\psi\|_{L^{p'}_{t,x}}$$

by a change of variables, and

$$\|(\psi_\lambda)_0\|_{L^2_x} = \lambda^{-d/2}\|\psi_0\|_{L^2_x}.$$

Now if $F = 0$, then we have the estimates

$$\|\psi\|_{L^{p'}_{t,x}} \lesssim \|\psi_0\|_{L^2_x} \quad \text{and} \quad \|\psi_\lambda\|_{L^{p'}_{t,x}} \lesssim \|(\psi_\lambda)_0\|_{L^2_x}, \tag{$*$}$$

Using the scaling computations in the second estimate in $(*)$ implies that

$$\|\psi\|_{L^{p'}_{t,x}}\lambda^{-d/p'}\lambda^{-2/p'} \lesssim \lambda^{-d/2}\|\psi_0\|_{L^2_x}.$$

This inequality should hold independent of λ, since otherwise taking $\lambda \to \infty$ or $\lambda \to 0$ yields a contradiction with the first inequality in $(*)$. Thus the powers in λ should match:

$$-\frac{d}{p'} - \frac{2}{p'} = -\frac{d}{2},$$

so we find that p' must be

$$p' = \frac{d+2}{d/2} + \frac{2d+4}{d} = 2 + \frac{4}{d}.$$

This uniquely determines p'. Now consider $F \neq 0$. Using a similar computation as before, we have

$$\|F_\lambda\|_{L^q_{t,x}} = \lambda^2\lambda^{-d/q}\lambda^{-2/q}\|F\|_{L^q_{t,x}}.$$

Then the theorem says that $\|\psi_\lambda\|_{L^{p'}_{t,x}} \lesssim \|\psi_0\|_{L^2_x} + \|F\|_{L^q_{t,x}}$, so we have

$$\|\psi\|_{L^{p'}_{t,x}}\lambda^{-d/p'}\lambda^{-2/p'} \lesssim \lambda^{-d/2}\|\psi_0\|_{L^2_x} + \lambda^2\lambda^{-d/q}\lambda^{-2/q}\|F\|_{L^q_{t,x}}.$$

Again the estimate should hold independent of λ, so the powers in λ must match:

$$-\frac{d}{p'} - \frac{2}{p'} = 2 - \frac{d}{q} - \frac{2}{q} = -\frac{d}{2},$$

which then gives p as

$$p = \left(1 - \frac{1}{p'}\right)^{-1} = \left(1 - \frac{d}{2d+4}\right)^{-1} = \frac{2d+4}{d+4}.$$

Lemma 3.1. *Let* $\psi(t) = e^{it\Delta/2}\psi_0$. *Then for* $1 \leq p \leq 2$,

$$\|\psi(t)\|_{L_x^{p'}(\mathbb{R}^d)} \lesssim |t|^{-d(1/p-1/2)} \|\psi_0\|_{L_x^p(\mathbb{R}^d)}.$$

Proof. This is the interpolation result from the beginning of class. □

Lemma 3.2 (Hardy-Littlewood-Sobolev inequality)**.** *Let* $0 < \alpha < 1$ *and* $g \in \mathcal{S}(\mathbb{R})$. *Let*

$$(T_\alpha g)(t) = \int_{-\infty}^{\infty} |t-s|^{-\alpha} g(s)\, ds.$$

Then we have $\|T_\alpha g\|_{L^q(\mathbb{R})} \lesssim \|g\|_{L^p(\mathbb{R})}$, *where* $1 < p < q < \infty$ *such that* $1 + 1/q = \alpha + 1/p$.

Proof. One approach is via harmonic analysis and maximal functions. An alternative approach can be found in Theorem 4.3 of Analysis by Lieb and Loss. □

Remark. Recall *Young's inequality* that for

$$h(t) = \int f(t-s) g(s)\, dx,$$

we have $\|h\|_{L^r} \leq \|f\|_{L^p} \|g\|_{L^q}$, where $1/r + 1 = 1/q + 1/p$. The Hardy-Littlewood-Sobolev inequality can be seen as a generalized Young's inequality: If $f(s) = |s|^{-\alpha}$, then f barely fails to be in $L^{1/\alpha}$. Informally, we can think of "$f \in L^{1/\alpha}$," and the standard Young's inequality would imply Hardy-Littlewood-Sobolev.

Remark. We have $q > p$ in the Hardy-Littlewood-Sobolev inequality, so we gain some integrability via fractional integration for $p > 1$ (the type of integral defining $T_\alpha g$ is known as *fractional integration*).

Lecture 4

Jan. 22 — Strichartz Estimates, Part 2

4.1 Proof of Strichartz Estimates

Proof of Theorem 3.1. The first step is a TT^* argument. Define the operator T by $Tf = e^{it\Delta/2}f$. We know that $T : L^2_x(\mathbb{R}^d) \to L^\infty_t(\mathbb{R} \to L^2_x(\mathbb{R}^d))$. The adjoint $T^* : L^1_t L^2_x \to L^2_x$ is defined via the relation

$$\langle f, T^*G\rangle_{L^2_x} = \langle Tf, G\rangle_{L^2_{t,x}} = \iint (e^{it\Delta/2}f)(x)\overline{G}(t,x)\,dtdx = \int f(x) \int \overline{(e^{-it\Delta/2}G(t,\cdot))}(x)\,dtdx,$$

and so we have the formula

$$T^*G = \int (e^{-it\Delta/2}G(t,\cdot))(x)\,dt.$$

Then we can see that

$$(TT^*G)(t,x) = \int (e^{i(t-s)\Delta/2}G(s,\cdot))(x)\,ds = \left(e^{it\Delta/2}\int e^{-is\Delta/2}G(s,\cdot)\,ds\right)(x).$$

Note that there is a convolution structure in the time variable. Clearly $TT^* : L^1_t L^2_x \to L^\infty_x L^2_x$. Then the goal for now will be to show that

$$\|TT^*G\|_{L^{p'}_{t,x}} \le C\|G\|_{L^p_{t,x}}.$$

To do this, observe that by the above expression for TT^*G and Minkowski's inequality, we have

$$\|TT^*G\|_{L^{p'}_x} \le \int \|e^{i(t-s)\Delta/2}G(s)\|_{L^{p'}_x}\,ds \le C\int |t-s|^{-d(1/p-1/2)}\|G(s)\|_{L^p_x}\,ds,$$

where the second inequality follows by Lemma 3.1. Now by Lemma 3.2,

$$\|TT^*G\|_{L^q_t L^{p'}_x} \le C\|G(s)\|_{L^p_{t,x}}$$

if $1/q + 1 = d(1/p - 1/2) + 1/p$ (also check that $0 < \alpha = d(1/p-1/2) < 1$, where $p = (2d+4)/(d+4)$). From this relation, we find that we must have $q = 2 + 4/d = p'$, so we have shown the goal.

Thus we have proved that $TT^* : L^p_{t,x} \to L^{p'}_{t,x}$, where $p = (2d+4)/(d+4)$. Now

$$\|T^*G\|^2_{L^2_x} = \langle T^*G, T^*G\rangle_{L^2_x} = \langle TT^*G, G\rangle_{L^2_{t,x}} \le \|TT^*G\|_{L^{p'}_{t,x}}\|G\|_{L^p_{t,x}} \le C\|G\|^2_{L^p_{t,x}},$$

where the first inequality follows by Hölder's inequality and the second follows from the goal we just proved. Thus we conclude that $T^* : L^p_{t,x} \to L^2_x$, and that $T : L^2_x \to L^{p'}_{t,x}$ by duality. Therefore,

$$\|T\psi_0\|_{L^{p'}_{t,x}} = \|e^{it\Delta/2}\psi_0\|_{L^{p'}_{t,x}} \le C\|\psi_0\|_{L^2_x},$$

so we have proved the Strichartz estimates when $F = 0$.

In the case when $F \neq 0$, by Duhamel's formula we have

$$\psi(t) = e^{it\Delta/2}\psi_0 - i\int_0^t e^{i(t-s)\Delta/2}F(s)\,ds,$$

so by the triangle inequality we find that

$$\|\psi\|_{L^{p'}_{t,x}} \leq \|e^{it\Delta/2}\psi_0\|_{L^{p'}_{t,x}} + \left\|\int_0^t e^{i(t-s)\Delta/2}F(s)\,ds\right\| \leq C\|\psi_0\|_{L^2_x} + \left\|\int_0^t \|e^{i(t-s)\Delta/2}F(s)\|_{L^{p'}_x}\,ds\right\|_{L^{p'}_t},$$

where the last inequality on the second term follows by Minkowski's inequality. Then using Lemma 3.1 and Lemma 3.2 in the same fashion as before, we can bound the latter term by

$$\left\|\int_0^t \|e^{i(t-s)\Delta/2}F(s)\|_{L^{p'}_x}\,ds\right\|_{L^{p'}_t} \leq C\left\|\int_{-\infty}^{\infty} |t-s|^{-d(1/p-1/2)}\|F(s)\|_{L^{p'}_x}\,ds\right\|_{L^{p'}_t} \leq C\|F\|_{L^p_{t,x}}.$$

Plugging this bound back in gives the desired inequality $\|\psi\|_{L^{p'}_{t,x}} \leq C\|\psi_0\|_{L^2_x} + C\|F\|_{L^p_{t,x}}$. □

Remark. Note that the term

$$TT^*G = \int_{-\infty}^{\infty} e^{i(t-s)\Delta/2}G(s)\,ds$$

looks similar to the term from the Duhamel formula

$$\int_0^t e^{i(t-s)\Delta/2}F(s)\,ds.$$

However, it is possible that these two have different estimates, which is why we had to argue separately.

4.2 Strichartz Estimates and Harmonic Analysis

Remark. The original intention of Strichartz for these estimates was for use in harmonic analysis. The Strichartz estimates can actually be derived from the *Stein-Tomas restriction theorem*.

Let $S \subseteq \mathbb{R}^n$ with $n = d+1$, where S is a hypersurface. If $f \in L^1$, then one can show (e.g. using the Riemann-Lebesgue lemma) that $\widehat{f} \in L^\infty$. So we can conclude that $\widehat{f}$ has pointwise meaning, i.e. $\widehat{f}(\xi)$ makes sense pointwise. In particular, we can make sense of $\widehat{f}(\xi)$ on the hypersurface S.

On the other hand, if $f \in L^2$, then by Plancherel's theorem, $\widehat{f} \in L^2$ as well. But an L^2 function has no pointwise interpretation, i.e. we can modify it on a set of measure zero without changing the function. In particular, it is meaningless to restrict the function $\widehat{f}$ to S, since S is a set of measure zero in $\mathbb{R}^n$.

In general, what about $f \in L^p$ for $1 < p < 2$? This is the topic of the *restriction theorems* in harmonic analysis. It turns out that the choice of which p work depends on the "curvature" of S.

Theorem 4.1 (Stein-Tomas restriction theorem)**.** *Let $n = d+1$ and $S \subseteq \mathbb{R}^n$ be a hypersurface with non-vanishing Gaussian curvature. Let σ_S be the corresponding surface measure, and let ϕ be a compactly supported on S. Then we have*

$$\|(\phi\sigma_S)^\vee\|_{L^r(\mathbb{R}^n)} \leq C\|\phi\|_{L^2(\sigma_S)},$$

where $r = (2n+2)/(n-1)$.

Remark. Now recall the explicit formula for a solution ψ to the Schrödinger equation:

$$\psi(t,s) = \int e^{i(x\cdot\xi - t|\xi|^2/2)}\widehat{\psi}_0(\xi)\,d\xi.$$

Also define the hypersurface $S = \{(\xi,\tau) : \tau = -|\xi|^2/2, \xi \in \mathbb{R}^d\}$.[1] Then

$$\psi(t,x) = (\phi\sigma_S)^\vee(t,x), \quad \phi(\xi,\tau) = \widehat{\psi}_0(\xi), \quad \sigma_S(d\xi,d\tau) = (2\pi)^d d\xi.$$

Indeed, we can see that

$$\psi(t,x) = (2\pi)^{-d}\int e^{i(x\cdot\xi + t\tau)}\phi(\xi,\tau)\,\sigma_S(d\xi,d\tau) = \int e^{i(x\cdot\xi - t|\xi|^2/2)}\widehat{\psi}_0(\xi)\,d\xi.$$

Then, the Stein-Tomas restriction theorem tells us that $\|(\phi\sigma_S)^\vee\|_{L^r(\mathbb{R}^n)} \le C\|\phi\|_{L^2(\sigma_S)}$, which implies

$$\|\psi\|_{L^{2+4/d}_{t,x}(\mathbb{R}\times\mathbb{R}^d)} \le C\|\widehat{\psi}_0\|_{L^2_\xi} \le C\|\psi_0\|_{L^2_x},$$

where the last inequality follows by Plancherel's theorem. The estimate above technically only holds for those ψ where $\widehat{\psi}_0$ is compactly supported, but one can extend this by a density argument to $\psi_0 \in L^2_x$.

See Chapter 11 of Muscalu-Schlag (Volume I) for more details.

[1] This particular hypersurface is called the *characteristic surface* of the Schrödinger equation.

Lecture 5

Jan. 27 — Kato Smoothing

5.1 Kato Smoothing

Theorem 5.1 (Kato 1/2 smoothing estimate)**.** *Let $d \geq 2$ and $\chi \geq 0$ be a smooth cutoff function such that $\widehat{\chi}$ is compactly supported. Then we have the estimate*

$$\|\chi(x)(-\Delta)^{1/4} e^{it\Delta/2} f\|_{L^2_{x,t}} \leq C\|f\|_{L^2}.$$

Remark. The above theorem says that when we localize, we gain a smoothing effect (1/2 derivatives). Also note that this effect is not pointwise, but rather after integrating in time.

Remark. Let $f(t) = e^{it\Delta/2} f_0$, where $f_0 = e^{-|x|^2/2} e^{-ixv}$. Think of this as a quantum particle at the origin with an initial velocity v, so that the particle will stay in $B(0,1)$ for a period of order $O(1/|v|)$. Then

$$\left(\int_0^{1/|v|} \int_{B(0,1)} |f|^2 \, dx dt \right)^{1/2} \sim \left(\frac{1}{|v|} \right)^{1/2} = \frac{1}{|v|^{1/2}}.$$

Now $(-\Delta)^{1/2} f \sim |v|^{1/2}$, so these factors cancel each other out. This matches the above estimate.

Theorem 5.2 (1-D Kato smoothing estimate)**.** *For $d = 1$, we have*

$$\sup_x \|(-\Delta)^{1/4} e^{it\Delta/2} f\|_{L^2_t} \leq C\|f\|_{L^2}.$$

Proof. For $d = 1$, by the Fourier transform we have

$$\begin{aligned}
(-\Delta)^{1/4} e^{it\Delta/2} f &= \frac{1}{2\pi} \int_{\mathbb{R}} e^{ix\xi} |\xi|^{1/2} e^{-it\xi^2/2} \widehat{f}(\xi) \, d\xi \\
&= \frac{1}{2\pi} \int_{-\infty}^0 e^{ix\xi} |\xi|^{1/2} e^{-it\xi^2/2} \widehat{f}(\xi) \, d\xi + \frac{1}{2\pi} \int_0^\infty e^{ix\xi} |\xi|^{1/2} e^{-it\xi^2/2} \widehat{f}(\xi) \, d\xi.
\end{aligned}$$

Since we want an L^2_t bound, it indicates that we should apply Plancherel's theorem in time. We will prove the estimate for the latter integral, and the former integral is left as an exercise. Set $\eta = \xi^2$ with $d\eta = 2\xi \, d\xi = 2\sqrt{\eta} \, d\xi$. Applying this change of variables, we obtain

$$(-\Delta)^{1/4} e^{it\Delta/2} f = \frac{1}{2\pi} \int_0^\infty e^{ix\sqrt{\eta}} |\eta|^{1/4} e^{-it\eta/2} \widehat{f}(\sqrt{\eta}) \frac{1}{2\sqrt{\eta}} \, d\eta = \frac{1}{4\pi} \int_0^\infty e^{ix\sqrt{\eta}} |\eta|^{-1/4} e^{-it\eta/2} \widehat{f}(\sqrt{\eta}) \, d\eta.$$

Fix x and denote $h(\eta) = e^{ix\sqrt{\eta}}|\eta|^{-1/4}\widehat{f}(\sqrt{\eta})$. Then we have

$$(-\Delta)^{1/4}e^{it\Delta/2}f = \frac{1}{4\pi}\int_0^\infty e^{-it\eta/2}h(\eta)\,d\eta = (\star).$$

By Plancherel's theorem in time, we see that

$$\|(\star)\|_{L_t^2} \le \|h(\eta)\|_{L_\eta^2}.$$

Then we can estimate (setting $z = \sqrt{\eta}$ with $dz = d\eta/(2\sqrt{\eta})$)

$$\int_0^\infty |h(\eta)|^2\,d\eta = \int_0^\infty |\eta|^{-1/2}|\widehat{f}(\sqrt{\eta})|^2\,d\eta = 2\int_0^\infty |\widehat{f}(z)|^2\,dz = C\|f\|_{L^2}^2,$$

where the last inequality follows by Plancherel's theorem in z. This yields $\|(\star)\|_{L_t^2} \le C\|f\|_{L^2}$, and since these estimates are all independent of x, we get that $\sup_x \|(\star)\|_{L_t^2} \le C\|f\|_{L^2}$. Putting this together with an identical estimate for the other integral, we obtain the desired bound. □

Remark. The above 1-D version is stronger and implies the statement

$$\|\chi(-\Delta)^{1/4}e^{it\Delta/2}f\|_{L_{x,t}^2} \le C\|f\|_{L^2}$$

for the 1-D case. Check this as an exercise.

5.2 Coarea Formula

Remark. In dimension d, suppose we have two nice functions g and u such that $u^{-1}(t)$ is a $(d-1)$-dimensional hypersurface. Then the *coarea formula* says that

$$\int_{\mathbb{R}^d} g(x)|\nabla u(x)|\,dx = \int_{\mathbb{R}}\int_{\{u(x)=t\}} g(x)\,d\sigma(x)dt,$$

where $\sigma(x)$ is the surface measure on $\{u(x) = t\}$.

Note that for $d = 1$, this says that

$$\int g(x)|\partial_x u|\,dx = \int_{\mathbb{R}}\left(\int_{\{u(x)=t\}} g(x)\,dx\right)dt = \int_{\mathbb{R}} g(u^{-1}(t))\,dt.$$

In particular, this is the change of variables formula where $\eta = u^{-1}(t)$ (so $u(\eta) = t$ and $dt = \partial_\eta u\,d\eta$).

Lemma 5.1. *Let $F \in C_0^\infty$ and ϕ be smooth. Then one has*

$$\int_{\mathbb{R}}\int_{\mathbb{R}^d} e^{i\lambda\phi(x)}F(x)\,dxd\lambda = (2\pi)^d\int_{\{\phi=0\}}\frac{F(x)}{|\nabla\phi(x)|}\,d\sigma(x).$$

Proof. By the coarea formula (using $g(x) = e^{i\lambda\phi(x)}F(x)/|\nabla\phi(x)|$ and $u = \phi$), we have

$$\int_{\mathbb{R}}\int_{\mathbb{R}^d} e^{i\lambda\phi(x)}F(x)\,dxd\lambda = \int_{\mathbb{R}}\int_{\mathbb{R}} e^{i\lambda y}\int_{\{\phi=y\}}\frac{F(x)}{|\nabla\phi(x)|}\,d\sigma(x)dyd\lambda.$$

Denote by $h(y)$ the integral

$$h(y) = \int_{\{\phi=y\}} \frac{F(x)}{|\nabla\phi(x)|}\, d\sigma(x).$$

Then we can see that

$$\int_{\mathbb{R}}\int_{\mathbb{R}} e^{i\lambda y} \int_{\{\phi=y\}} \frac{F(x)}{|\nabla\phi(x)|}\, d\sigma(x) dy d\lambda = \int_{\mathbb{R}}\int_{\mathbb{R}} e^{i\lambda y} h(y)\, dy d\lambda = \int_{\mathbb{R}} \widehat{h}(\lambda)\, d\lambda = (2\pi)^d h(0).$$

This gives the desired equality after plugging in the definition of $h(0)$. □

Lecture 6

Jan. 29 — Kato Smoothing, Part 2

6.1 Proof of Kato Smoothing

Proof of Theorem 5.1. Note that we have (let $G = \chi^2$)

$$(*) \int_{\mathbb{R}} \int_{\mathbb{R}^d} \chi^2 |(-\Delta)^{1/4} e^{it\Delta/2} f|^2 \, dx dt = \int_{\mathbb{R}} \langle (-\Delta)^{1/4} e^{it\Delta/2} f, G(-\Delta)^{1/4} e^{it\Delta/2} f \rangle_{L^2_x} \, dt,$$

so by Plancherel's theorem in $x \mapsto \xi$, we have

$$(*) = \int_{\mathbb{R}} \langle |\xi|^{1/2} e^{-i|\xi|^2/2} \widehat{f}(\xi), H(\xi) \rangle_{L^2_\xi} \, dt,$$

where H is defined via (since the Fourier transform turns multiplication into convolution)

$$H(\xi) = \int_{\mathbb{R}} \widehat{G}(\xi - \eta) |\eta|^{1/2} e^{-it|\eta|^2/2} \widehat{f}(\eta) \, d\eta.$$

Thus we have

$$(*) = \int_{\mathbb{R}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |\xi|^{1/2} |\eta|^{1/2} \widehat{G}(\xi - \eta) \widehat{f}(\xi) \overline{\widehat{f}}(\eta) e^{-it(|\xi|^2 - |\eta|^2)/2} \, d\xi d\eta dt.$$

Applying Lemma 5.1 with ($\lambda = t$)

$$\phi = -\frac{|\xi|^2 + |\eta|^2}{2}, \quad F = |\xi|^{1/2} |\eta|^{1/2} \widehat{G}(\xi - \eta) \widehat{f}(\xi) \overline{\widehat{f}}(\eta),$$

we have (since $\phi = 0$ implies $|\xi| = |\eta|$ and $|\nabla \phi| = \sqrt{|\xi|^2 + |\eta|^2}$)

$$(*) = \int_{\{\phi = 0\}} \frac{F}{|\nabla \phi|} \, d\sigma = \int_{|\xi| = |\eta|} \frac{|\xi|^{1/2} |\eta|^{1/2}}{\sqrt{|\xi|^2 + |\eta|^2}} \widehat{G}(\xi - \eta) \widehat{f}(\xi) \overline{\widehat{f}}(\eta) \, d\sigma.$$

Since $|\xi| = |\eta|$ on the region of integration, we have

$$\begin{aligned} (*) = \int_{|\xi| = |\eta|} \frac{|\xi|}{\sqrt{2}\xi} \widehat{G}(\xi - \eta) \widehat{f}(\xi) \overline{\widehat{f}}(\eta) \, d\sigma &\sim \int_{|\xi| = |\eta|} \widehat{G}(\xi - \eta) \widehat{f}(\xi) \overline{\widehat{f}}(\eta) \, d\sigma \\ &\sim \int_{\mathbb{R}^d} |\widehat{G}(\xi - \eta)| (|\widehat{f}(\xi)|^2 + |\widehat{f}(\eta)|^2) \, d\sigma \\ &\leq \int_{\mathbb{R}^d} |\widehat{f}(\xi)|^2 \int_{|\eta| = |\xi} |\widehat{G}(\xi - \eta)| \, d\sigma(\eta) \, d\xi \lesssim \int_{\mathbb{R}^d} |\widehat{f}(\xi)|^2 \, d\xi, \end{aligned}$$

where the last inequality is since (we claim the following)

$$K(\xi) = \int_{|\eta|=|\xi|} |\widehat{G}(\xi - \eta)|\, d\sigma(\eta)$$

satisfies $|K(\xi)| \le C$ for some constant C. For the claim, note that $\widehat{G}(z)$ decays faster than z^{-N} as $z \to \infty$ for every N. Then setting $z = \xi - \eta$, we have

$$K(\xi) = \int_{|\eta|=|\xi|} |\widehat{G}(\xi - \eta)|\, d\sigma(\eta) = \int_{|\xi - z|=|\xi|} \widehat{G}(z)\, d\sigma(z).$$

Note that this is an integral on a sphere. The integral will be bounded on a compact subset of the sphere and decays rapidly on the rest of the sphere, so we have $|K(\xi)| \le C$ for all $\xi \in \mathbb{R}^d$. This gives

$$\|\chi^2(-\Delta)^{1/4} e^{it\Delta/2} f\|_{L^2_{x,t}}^2 = \int_{\mathbb{R}} \int_{\mathbb{R}^d} \chi^2 |(-\Delta)^{1/4} e^{it\Delta/2} f|^2\, dxdt \lesssim \int_{\mathbb{R}^d} |\widehat{f}(\xi)|^2\, d\xi \sim \|f\|_{L^2_x}^2,$$

which is the desired estimate. □

Remark. An alternative approach to prove the Kato smoothing estimate is the following: Take η a Schwartz function such that $\operatorname{supp} \widehat{\eta} \in (-1, 1)$. Then it suffices to show estimates for

$$(*) = \int_{\mathbb{R}} \int_{\mathbb{R}^d} |\eta(\epsilon t) \chi(x) (-\Delta)^{1/4} e^{it\Delta/2} f|^2\, dxdt$$

and then take $\epsilon \to 0$. By Plancherel's theorem in $x \mapsto \xi$, we have

$$(*) = \int_{\mathbb{R}} \int_{\mathbb{R}^d} \left| \int \eta(\epsilon t) \widehat{\chi}(\xi - \xi') |\xi'|^{1/2} e^{it|\xi'|^2/2} \widehat{f}(\xi')\, d\xi' \right|^2 d\xi dt$$

Next Plancherel's theorem in $t \mapsto \tau$ implies that

$$(*) = \int_{\mathbb{R}} \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} \widehat{\chi}(\xi - \xi') \frac{1}{\epsilon} \widehat{\eta}\left(\frac{\tau - |\xi'|^2/2}{\epsilon} \right) |\xi'|^{1/2} \widehat{f}(\xi')\, d\xi' \right|^2 d\xi d\tau.$$

Cauchy-Schwarz in ξ' implies that

$$(*) \le \int_{\mathbb{R}} \int_{\mathbb{R}^d} \left[\left(\int \widehat{\chi}(\xi - \xi') \frac{1}{\epsilon} \widehat{\eta}\left(\frac{\tau - |\xi'|^2/2}{\epsilon} \right) d\xi' \right) \left(\int \widehat{\chi}(\xi - \xi') \frac{1}{\epsilon} \widehat{\eta}\left(\frac{\tau - |\xi'|^2}{\epsilon} \right) |\xi'| |\widehat{f}(\xi)|^2\, d\xi' \right) \right] d\xi d\tau.$$

Now we count the area of the domain of integration:

$$\begin{cases} -\epsilon < \tau - |\xi'|^2/2 < \epsilon & (1) \\ |\xi - \xi'| \le 1 & (2) \end{cases}$$

Then (2) implies that $||\xi| - |\xi'|| \le 1 \sim O(1)$, so

$$||\xi|^2 - |\xi'|^2| = ||\xi| - |\xi'|| \cdot ||\xi| + |\xi'|| \sim O(1 + |\xi|),$$

and thus $\tau \sim |\xi'|^2$. Fix ξ, then the size of the interval in which τ can vary is of order $O(1 + |\xi|)$. For ξ', notice that (1) implies

$$|\sqrt{2\tau} - |\xi'| \sim O\left(\frac{\epsilon}{\sqrt{2\tau} + |\xi'|} \right) \sim O\left(\frac{\epsilon}{|\xi| + 1} \right).$$

So the size of the interval in which $|\xi'|$ can vary is of order $O(\epsilon/(|\xi| + 1))$. Then we have

$$(*) \lesssim \iint \left(\frac{1}{\epsilon} \frac{\epsilon}{|\xi| + 1} \right) \left(\frac{1}{\epsilon} \frac{\epsilon}{|\xi| + 1} |\xi| |\widehat{f}(\xi')|^2 \right) d\tau d\xi \lesssim \int \frac{1 + |\xi|}{|\xi| + 1} \frac{|\xi|}{|\xi| + 1} |\widehat{f}(\xi)|^2\, d\xi \lesssim \int |\widehat{f}(\xi)|^2\, d\xi.$$

Lecture 7

Feb. 3 — Kato Smoothing, Part 3

7.1 Sharpness of Kato Smoothing

Lemma 7.1 (Kato smoothing is sharp). *We have*

$$\|\chi(x)(-\Delta)^{1/4}e^{it\Delta/2}(e^{-|x|^2/2}e^{ixv})\|_{L^2_{x,t}} \sim 1 \quad \textit{and} \quad \sup_v \|\chi(-\Delta)^{1/4+\delta}e^{it\Delta/2}(e^{-|x|^2/2}e^{ixv})\|_{L^2_{x,t}} = \infty.$$

for any $\delta > 0$.

Proof. Let $\operatorname{supp}\widehat{\chi} \subseteq B(0,1)$ and $f_0 = e^{-|x|^2/2}e^{ixv}$. Choose η smooth with $\widehat{\eta}$ compactly supported. Then

$$\begin{aligned}\int_{\mathbb{R}}\int_{\mathbb{R}^d} |\chi(x)\eta(\epsilon t)(-\Delta)^{1/4}e^{it\Delta/2}f_0|^2\,dxdt &= \iint \left| \int \widehat{\chi}(\xi-\xi')\eta(\epsilon t)|\epsilon|^{1/2}e^{it|\xi|^2/2}\widehat{f}_0(\xi)\,d\xi \right|^2 d\xi dt \\ &= \iint \left| \int \widehat{\chi}(\xi-\xi')\frac{1}{\epsilon}\widehat{\eta}\left(\frac{\tau-|\xi|^2/2}{\epsilon}\right)|\xi|^{1/2}e^{-|\xi-v|^2/2}\,d\xi \right|^2 d\xi' d\tau\end{aligned}$$

by Plancherel's theorem first in $x \mapsto \xi$ and then in $t \mapsto \tau$, since $\widehat{\eta}$ and $\widehat{\chi}$ are compactly supported. Now, $-\epsilon < \tau - |\xi|^2/2 < \epsilon$ implies

$$||\xi| - \sqrt{2\tau}| < \frac{2\epsilon}{|\xi| + \sqrt{2\tau}}.$$

So the region of integration for $|\xi|$ is an annulus-shaped area between $\sqrt{2\tau}$ and $2\epsilon/(|\xi|+\sqrt{2\tau})$ (plot the above inequality to see the region). Thus for fixed τ, as $\epsilon \to 0$, we have

$$\begin{aligned}&\iint \left| \int \widehat{\chi}(\xi-\xi')\frac{1}{\epsilon}\widehat{\eta}\left(\frac{\tau-|\xi|^2/2}{\epsilon}\right)|\xi|^{1/2}e^{-|\xi-v|^2/2}\,d\xi \right|^2 d\xi' d\tau \\ &\sim \int_{\mathbb{R}}\int_{\mathbb{R}^d} \left| \int_{\sqrt{2\tau}\mathbb{S}^{d-1}} \widehat{\chi}(\xi-\xi')e^{-|\xi-v|^2/2}|\xi|^{1/2}\frac{2\epsilon}{|\xi|+\sqrt{2\tau}}\frac{1}{\epsilon}\,d\sigma(\xi) \right|^2 d\xi' d\tau \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}^d} \left| \int_{\sqrt{2\tau}\mathbb{S}^{d-1}} \widehat{\chi}(\xi-\xi')e^{-|\xi-v|^2/2}|\xi|^{1/2}\frac{2}{|\xi|+\sqrt{2\tau}}\,d\sigma(\xi) \right|^2 d\xi' d\tau.\end{aligned}$$

Since $\widehat{\chi}$ is supported in $B(0,1)$, for fixed ξ the range of values for ξ' is $O(1)$ (as $|\xi-\xi'| \le 1$). Since

$$-\epsilon < \tau - \frac{1}{2}|\xi|^2 < \epsilon \implies -\epsilon + \frac{1}{2}|\xi|^2 < \tau < \epsilon + \frac{1}{2}|\xi|^2$$

and $|\xi - v| \le O(1)$, we get

$$-\epsilon + \frac{1}{2}||v|-1|^2 < \tau < \epsilon + \frac{1}{2}||v|+1|^2.$$

Expanding $||v|-1|^2$ and $||v|+1|^2$, we see that as $\epsilon \to 0$, we have $|2\tau - |v|^2| \lesssim |v|+1$. Then

$$\begin{aligned}
&\int_{\mathbb{R}} \int_{\mathbb{R}^d} \left| \int_{\sqrt{2\tau}\mathbb{S}^{d-1}} \widehat{\chi}(\xi-\xi') e^{-|\xi-v|^2/2} |\xi|^{1/2} \frac{1}{|\xi|+\sqrt{2\tau}}\, d\sigma(\xi) \right|^2 d\xi' d\tau \\
&\quad \lesssim \int_{\mathbb{R}} \left| \int_{\sqrt{2\tau}\mathbb{S}^{d-1}} e^{-|\xi-v|^2/2} |\xi|^{1/2} \frac{1}{|\xi|+\sqrt{2\tau}}\, d\sigma(\xi) \right|^2 d\tau \\
&\quad \lesssim \int_{\mathbb{R}} \left| \frac{|\sqrt{\tau}|^{1/2}}{\sqrt{2\tau}} \right|^2 d\tau \approx \int_{|2\tau-|v|^2|\lesssim |v|+1} \left| \frac{|\sqrt{\tau}|^{1/2}}{\sqrt{2\tau}} \right|^2 d\tau \approx \frac{1}{v} \int_{|2\tau-|v|^2|\lesssim |v|+1} d\tau \approx \frac{1}{v}\cdot v \approx O(1)
\end{aligned}$$

uniformly in v. We can also see that if $|\xi|^{1/2}$ is replaced by $|\xi|^{1/2+\delta}$, then $1/|v|$ will become $1/|v|^{1-4\delta}$, so the above becomes $O(|v|^{4\delta})$, which goes to ∞ as $v \to \infty$. $\square$

Corollary 7.0.1. *When $d \geq 2$, for $\epsilon > 0$, we have*

$$\|(1+|x|)^{-1/2-\epsilon}(-\Delta)^{1/4} e^{it\Delta/2} f\|_{L^2_{x,t}} \leq C_\epsilon \|f\|_{L^2_x}.$$

This is the sharp Kato smoothing estimate for $d \geq 2$.

7.2 Schrödinger Equation with Potential

Remark. Now we will consider the Schrödinger equation with a potential:

$$\begin{cases} i\psi_t + \frac{1}{2}\Delta\psi + V(t,x)\psi = 0, \\ \psi|_{t=0} = \psi_0 \in H^\gamma(\mathbb{R}^d). \end{cases}$$

Lemma 7.2. *If $\psi_0 \in H^\gamma(\mathbb{R}^d)$, then $e^{it\Delta/2}\psi_0 \in C^0(\mathbb{R}, H^\gamma(\mathbb{R}^d)) \cap C^1(\mathbb{R}, H^{\gamma-2}(\mathbb{R}^d))$.*

Proof. For each $t, s \in \mathbb{R}$, we can write (since $e^{is\Delta/2}$ preserves each H^γ norm)

$$\|e^{it\Delta/2}\psi_0 - e^{is\Delta/2}\psi_0\|^2_{H^\gamma} = \|e^{i(t-s)\Delta}\psi_0 - \psi_0\|^2_{H^\gamma} = \int (1+|\xi|^2)^\gamma |e^{i(t-s)|\xi|^2/2} - 1| |\widehat{\psi}_0(\xi)|^2\, d\xi.$$

Since $\psi_0 \in H^\gamma(\mathbb{R}^d)$, as $t \to s$, the above quantity goes to 0 by the dominated convergence theorem. This shows that $e^{it\Delta/2\psi_0} \in C^0(\mathbb{R}; H^\gamma(\mathbb{R}^d)$. To check that $e^{it\Delta/2}\psi_0 \in C^1(\mathbb{R}; H^{\gamma-2}(\mathbb{R}^d))$, note

$$\frac{d}{dt} e^{it\Delta/2}\psi_0 = \frac{i}{2}\Delta e^{it\Delta/2}\psi_0.$$

Thus it suffices to see that $\Delta e^{it\Delta/2}\psi_0$ is continuous, i.e.

$$\|\Delta e^{it/2}\psi_0 - \Delta e^{is/2}\psi_0\|^2_{H^{\gamma-2}} \to 0$$

as $t \to s$, which follows by the dominated convergence theorem. $\square$

Theorem 7.1. *Assume that V is real with*

$$\sup_t \|\nabla_x^\alpha V(t,x)\|_{L^2_x} \leq C_\alpha \quad \textit{and} \quad \|\nabla_x^\alpha V(t,x) - \nabla_x^\alpha V(s,x)\|_{L^\infty_x} \to 0$$

as $t \to s$ for every multi-index α. Then for all $k \geq 0$, if $\psi_0 \in H^k(\mathbb{R}^d)$, there exists a unique solution $\psi \in C^0(\mathbb{R}; H^k(\mathbb{R}^d)) \cap C^1(\mathbb{R}, H^{k-2}(\mathbb{R}^d))$ to the Schrödinger equation with potential V, given by

$$\psi(t) = e^{it\Delta/2}\psi_0 + i\int_0^t e^{i(t-s)\Delta/2} V(s)\psi(s)\, ds.$$

Moreover, $\|\psi\|_{H^k(\mathbb{R}^d)} \leq C_{k,V}(1+|t|)^k$.

Proof. Define the operator A by the formula

$$A(\phi) = e^{it\Delta/2}\psi_0 - i\int_0^t e^{i(t-s)\Delta/2} V(s)\phi(s)\, ds.$$

Let $X = (C^0[0,T], H^k)$, and we want to find T such that A is a contraction in X (do this as an exercise). Repeating this on $[T, 2T]$, $[2T, 3T]$, ..., gives existence. Uniqueness follows by Gronwall's inequality.

Now we prove the upper bound for the growth. Gronwall's inequality gives us

$$\|\psi(t)\|_{H^k} \leq c_1 e^{c_2|t|}\|\psi_0\|_{H^k}.$$

Since V is real (the argument before this step works even when V is not real),

$$\begin{aligned}\frac{d}{dt}\|\psi(t)\|_{L^2}^2 = \frac{d}{dt}\langle\psi(t),\psi(t)\rangle = \frac{d}{dt} &= \langle\frac{d}{dt}\psi,\psi\rangle + \langle\psi,\frac{d}{dt}\psi\rangle = 2\operatorname{Re}\langle\frac{d}{dt}\psi,\psi\rangle \\ &= 2\operatorname{Re}\langle\frac{i}{2}\Delta\psi + iV(t,x)\psi,\psi\rangle = 2\operatorname{Re} i\int\left(-\frac{1}{2}|\nabla\phi|^2 + V(t,x)|\psi|^2\right) dx = 0\end{aligned}$$

since the integral is real, so that i times the integral is purely imaginary. So $\|\psi\|_{L^2}$ is conserved. Now let $\phi = \partial_{x_j}\psi(t,x)$, so that

$$i\phi_t + \frac{1}{2}\Delta\phi + V\phi = -\partial_{x_j}V\psi. \tag{$*$}$$

From the existence and uniqueness of the solution, we can use $V(t,0)\psi_0$ to denote the solution to

$$\begin{cases} i\partial_t + \frac{1}{2}\Delta\psi + V\psi = 0, \\ \psi|_{t=0} = \psi_0, \end{cases}$$

and use $V(t,s)\psi_0$ denote the solution for initial condition $\psi(s) = \psi_0$. Then $(*)$ can be solved as

$$\phi(t) = V(t,0)\partial_{x_j}\psi_0 + \int_0^t V(t,s)\partial_{x_j}V(s)\psi(s)\, ds.$$

This gives

$$\|\phi(t)\|_{L^2} \leq \|V(t,0)\partial_{x_j}\psi_0\|_{L^2} + \int_0^t \|V(t,s)\partial_{x_j}V(s)\psi(s)\|_{L^2}\, ds$$

by Minkowski's inequality. By the conservation of the L^2 norm, $\|V(t,0)\partial_{x_j}\psi_0\|_{L^2} = \|\partial_{x_j}\psi_0\|_{L^2}$, so

$$\|V(t,s)\partial_{x_j}V(s)\psi(s)\|_{L^2} = \|\partial_{x_j}V(s)\psi(s)\|_{L^2} \lesssim \|\psi(s)\|_{L^2} = \|\psi(0)\|_{L^2}.$$

Then we obtain

$$\|\phi\|_{L^2} \lesssim \|\partial_{x_j}\psi_0\|_{L^2} + \int_0^t \|\psi(s)\|_{L^2}\, ds \leq C_{1,V}(1+|t|)\|\psi(0)\|_{H^1},$$

and inducting on k gives the result $\|\psi(t)\|_{H^k} \leq C_{k,V}(1+|t|)^k\|\psi(0)\|_{H^k}$. □

Remark. Note that in the above theorem, we need a suitable integral (e.g. the Bochner integral) to be able to interpret the Duhamel formula in a Banach space.

Lecture 8

Feb. 5 — Schrödinger Equation with Potential

8.1 Schrödinger Equation with Potential, Continued

Theorem 8.1 (Bourgain). *Assume $d \geq 3$ and that V is real. Suppose that*

$$\sup_t \|\nabla_x^\alpha V(t,x)\|_{L_x^\infty} \leq C_\alpha$$

for every multi-index α and that $\sup_{|t-t_0|\leq 1} |V(t,x)|$ is compactly supported in x with the diameter of the support independent of t_0 (e.g. $V(t,x) = V(x-ct)$). Then

$$\|\psi(t)\|_{H^k} \leq C_\epsilon (1+|t|)^\epsilon \|\psi_0\|_{H^k}$$

for every $\epsilon > 0$.

Proof. Define (note that $|||f||| = \|f\|_{L^\infty + L^2}$, the dual space of $\|g\|_{L^1\cap L^2}$)

$$|||f||| = \inf_{f=f_1+f_2} (\|f_1\|_{L^2} + \|f_2\|_{L^\infty}).$$

Then we can write

$$\psi(t) = e^{it\Delta/2}\psi_0 + i\int_0^t e^{i(t-s)\Delta/2} V(s)\psi(s)\, ds.$$

Taking the derivative and splitting the integral, we obtain

$$\nabla^\alpha \psi(t) = e^{it\Delta/2}\nabla^\alpha\psi_0 + i\int_0^{t-A} e^{i(t-s)\Delta/2}\nabla^\alpha(V\psi)\, ds + i\int_{t-A}^t e^{i(t-s)\Delta/2}\nabla^\alpha(V\psi)\, ds,$$

where $A > 0$ is a constant to be chosen later. We will measure the first integral in L^∞ and the second integral in L^2. Taking norms, we find that

$$|||\nabla^\alpha\psi(t)||| \leq \left|\left|\left| e^{it\Delta/2}\nabla^\alpha\psi_0 \right|\right|\right| + \left\| \int_0^{t-A} e^{i(t-s)\Delta/2}\nabla^\alpha(V\psi)\, ds \right\|_{L^\infty} + \left\| \int_{t-A}^t e^{i(t-s)\Delta/2}\nabla^\alpha(V\psi)\, ds \right\|_{L^2}.$$

We can estimate the first integral by

$$\begin{aligned}\left\| \int_0^{t-A} e^{i(t-s)\Delta/2}\nabla^\alpha(V\psi)\, ds \right\|_{L^\infty} &\leq \int_0^{t-A} \left\| e^{i(t-s)\Delta/2}\nabla^\alpha(V\psi) \right\|_{L^\infty} ds \\ &\lesssim \int_0^{t-A} |t-s|^{-d/2} \|\nabla^\alpha(V\psi)\|_{L^1}\, ds \lesssim A^{-d/2+1} \sup_{0\leq s\leq t} \|\nabla^\alpha(V\psi)\|_{L^1},\end{aligned}$$

where the first inequality is by Minkowski's inequality, the second inequality is by the dispersive estimate, and the third is by explicit integration. Note that we needed the cutoff at $t-A$ here since $|t-s|^{-d/2}$ is not integrable at $t=s$. Then by the product rule and Cauchy-Schwarz, we have

$$\left\|\int_0^{t-A} e^{i(t-s)\Delta/2}\nabla^\alpha(V\psi)\,ds\right\|_{L^\infty} \lesssim A^{-d/2+1}\sup_{\substack{0\le\beta\le\alpha\\0\le s\le t}}\|\nabla^\beta V\nabla^{\alpha-\beta}\psi\|_{L^1} \lesssim A^{-d/2+1}\sup_{\substack{0\le\beta\le\alpha\\0\le s\le t}}\|\nabla^\beta\psi\|_{L^2}.$$

Note here that $d\ge 3$ implies $-d/2+1<0$, so if A is large, then $A^{-d/2+1}$ is small. The above implies

$$\left\|\int_0^{t-A} e^{i(t-s)\Delta/2}\nabla^\alpha(V\psi)\,ds\right\|_{L^\infty} \lesssim A^{-d/2+1}\sup_{\substack{0\le\beta\le\alpha\\0\le s\le t}}\left|\left|\left|\nabla^\beta\psi\right|\right|\right|.$$

For the second integral, we can write

$$\begin{aligned}\left\|\int_{t-A}^{t} e^{i(t-s)\Delta/2}\nabla^\alpha(V\psi)\,ds\right\|_{L^2} &= \sup_{\|\phi\|_{L^2}=1}\int_{t-A}^t\langle\phi, e^{i(t-s)\Delta/2}\nabla^\alpha(V\psi)\rangle\,ds\\ &= \sup_{\|\phi\|_{L^2=1}}\int_{t-A}^t\langle\phi e^{i(t-s)\Delta/2}\chi\nabla^\alpha(V\psi)\rangle\,ds,\end{aligned}$$

where χ is a smooth cutoff function such that $\chi\equiv 1$ in the support of V. Then

$$\left\|\int_{t-A}^{t} e^{i(t-s)\Delta/2}\nabla^\alpha(V\psi)\,ds\right\|_{L^2} = \sup_{\|\phi\|_{L^2=1}}\int_{t-A}^t\langle(1-\Delta)^{1/4}\chi e^{-i(t-s)\Delta/2}\phi, (1-\Delta)^{-1/4}\nabla^\alpha(V\psi)\rangle\,ds.$$

By Cauchy-Schwarz in x and t and Kato 1/2 smoothing, we obtain

$$\begin{aligned}&\left\|\int_{t-A}^{t} e^{i(t-s)\Delta/2}\nabla^\alpha(V\psi)\,ds\right\|_{L^2}\\ &\quad\le\sup_{\|\phi\|_{L^2=1}}\left(\int_{t-A}^t\|(1-\Delta)^{1/4}\chi e^{-i(t-s)\Delta/2}\phi\|_{L_x^2}^2\,ds\right)^{1/2}\left(\int_{t-A}^t\|(1-\Delta)^{-1/4}\nabla^\alpha(V\psi)\|_{L_x^2}^2\,ds\right)^{1/2}\\ &\quad\lesssim\|\phi\|_{L^2}+\sqrt{A}\sup_{t-A\le s\le t}\|(1-\Delta)^{-1/4}\nabla^\alpha(V\psi)\|_{L_x^2}.\end{aligned}$$

To deal with the last term, note that by Sobolev's inequality, we have

$$\|f\|_{H^{k-1/2}}\le C_k\|f\|_{H^k}^{1-\delta}\|f\|_{L^2}^\delta$$

for some $\delta=\delta(k)$. From this we get

$$\sup_{|\alpha|=k}\|(1-\Delta)^{-1/4}\nabla^\alpha(V\psi)\|_{L^2}\le C_k\|V\psi\|_{L^2}^\delta\sup_{|\beta|\le k}\|\nabla^\beta(V\psi)\|_{L^2}^{1-\delta}\le C_k\|\psi_0\|_{L^2}^\delta\sup_{|\beta|\le k}\left|\left|\left|\nabla^\beta(V\psi)\right|\right|\right|^{1-\delta},$$

where the second inequality is since the L^2 norm is conserved. This means that

$$|||\nabla^\alpha\psi|||\lesssim\|\nabla^\alpha\psi_0\|_{L^2}+A^{1-d/2}\sup_{\substack{0\le|\beta|\le|\alpha|\\0\le s\le t}}\left|\left|\left|\nabla^\beta\psi\right|\right|\right|+\sqrt{A}\cdot\|\psi_0\|_{L^2}^\delta\sup_{\substack{0\le|\beta|\le|\alpha|\\0\le s\le t}}\left|\left|\left|\nabla^\beta\psi\right|\right|\right|^{1-\delta}.$$

For the last term, observe that

$$x^{1-\delta}y^\delta\le\frac{1}{1-\delta}x+\frac{1}{\delta}y$$

by Young's inequality, and we can also use that for $\eta > 0$,

$$\left(\eta^{1/(1-\delta)}x\right)^{1-\delta}\left(n^{-1/\delta}y\right)^{\delta} \le \frac{\eta^{1/(1-\delta)}}{1-\delta}x + \frac{n^{-1/\delta}}{\delta}y.$$

Applying this to the last term, we obtain

$$\|\psi_0\|_{L^2}^{\delta} \sup_{\substack{0\le|\beta|\le|\alpha| \\ 0\le s\le t}} \left|\!\left|\!\left|\nabla^\beta\psi\right|\!\right|\!\right|^{1-\delta} \le \frac{\eta^{-1/\delta}}{\delta}\|\psi_0\|_{L^2} + \frac{\eta^{1/(1-\delta)}}{1-\delta}\sup_{\substack{0\le|\beta|\le|\alpha| \\ 0\le s\le t}} \left|\!\left|\!\left|\nabla^\beta\psi\right|\!\right|\!\right|.$$

Putting everything together, we obtain

$$\sup_{\substack{|\alpha|=k \\ 0\le s\le t}} |||\nabla^\alpha\psi||| \le \|\psi_0\|_{H^k} + A^{1-d/2}\sup_{\substack{0\le s\le t \\ |\alpha|\le k}} |||\psi||| + \frac{\sqrt{A}\cdot\eta^{-1/\delta}}{\delta}\|\psi_0\|_{L^2} + \frac{\sqrt{A}\cdot\eta^{1/(1-\delta)}}{1-\delta}\sup_{\substack{0\le s\le t \\ |\alpha|\le k}} |||\psi|||.$$

Now take A large, so that $A^{1-d/2}$ is very small (note that since $A \le t$, we also take t to be large enough; for t small, apply Gronwall). Take η small, so that $\sqrt{A}\cdot\eta^{1/(1-\delta)}$ is very small. Then we have

$$\sup_{\substack{|\alpha|=k \\ 0\le s\le t}} |||\nabla^\alpha\psi||| \le C_{\alpha,\eta}\|\psi_0\|_{H^k}.$$

Now recall that (since the Schrödinger evolution conserves the H^k norms)

$$\|\psi(t)\|_{H^k} \le \|\psi_0\|_{H^k} + \int_0^t \|V(s)\psi(s)\|_{H^k}\,ds.$$

The above calculations show that

$$\|V\psi\|_{H^k} \lesssim \sup_{|\alpha|\le k}\|\nabla^\alpha(V\psi)\|_{L^2} \lesssim \sup_{|\alpha|\le k}|||\nabla^\alpha\psi||| \lesssim \|\psi_0\|_{H^k},$$

and so we have

$$\|\psi(t)\|_{H^k} \lesssim \|\psi_0\|_{H^k} + \int_0^t \|\psi_0\|_{H^k}\,ds \lesssim (1+|t|)\|\psi_0\|_{H^k}.$$

Note that the growth of the norm is independent of k. To get $(1+|t|)^\epsilon$, first note that the L^2 norm is conserved. So if $S(t)$ is the solution operator, then we have

$$\|S(t)\psi_0\|_{L^2} = \|\psi_0\|_{L^2} \quad \text{and} \quad \|S(t)\psi_0\|_{H^{k_1}} \le C_{k_1}(1+|t|)\|\psi_0\|_{H^{k_1}}$$

for every $k_1 \in \mathbb{N}$. Since $S(t)$ is a linear operator, by interpolation we have

$$\|S(t)\|_{\mathcal{L}(H^k,H^k)} \le \|S(t)\|_{\mathcal{L}(L^2,L^2)}^{1-k/k_1}\|S(t)\|_{\mathcal{L}(H^{k_1},H^{k_1})}^{k/k_1} \le C_{k,k_1}(1+|t|)^{k/k_1}$$

for every $0 \le k \le k_1$ (since $\|S(t)\|_{\mathcal{L}(L^2,L^2)}$ is bounded). For any $\epsilon > 0$, for fixed k take k_1 large enough so that $k/k_1 \le \epsilon$. Then

$$\|S(t)\|_{\mathcal{L}(H^k,H^k)} \le C_{k,k_1}(1+|t|)^{k/k_1} \le C_\epsilon(1+|t|)^\epsilon,$$

which implies the desired result. □

Remark. The compact support assumption could be replaced by rapid decay in the above theorem.

Lecture 9

Feb. 10 — Spectral Theory

9.1 Review of Functional Analysis

Remark. Our goal now is to study the decay of a solution ψ to

$$i\psi_t - \frac{1}{2}\Delta\psi + V(x)\psi = 0, \quad V \text{ real}.$$

We will study $e^{iHt}\psi_0$, where $H = -\Delta/2 + V$, in particular the spectral theory for H in $L^2(\mathbb{R}^d)$.

Definition 9.1. Let $\mathcal{H}$ be a Hilbert space, with inner product $\langle \cdot, \cdot \rangle$. We will use $\mathcal{H} = L^2(\mathbb{R}^d)$ with

$$\langle f, g \rangle = \int_{\mathbb{R}^d} f\overline{g}\, dx.$$

Let T be a linear operator which is densely defined on $\mathcal{H}$. Let $D(T)$ denote the domain of T. We say that T is *symmetric* if $\langle Tx, y \rangle = \langle x, Ty \rangle$. The *adjoint* T^* of T is defined as

$$D(T^*) = \{f \in \mathcal{H} : \text{there exists } h \in \mathcal{H} \text{ such that } \langle T\phi, f \rangle = \langle \phi, h \rangle \text{ for all } \phi \in D(T)\},$$

and we set $T^* f = h$. We say T is *closed* provided that the graph of T is a closed subset of $\mathcal{H} \times \mathcal{H}$, i.e. if $f_n \to f$ in $\mathcal{H}$ with $f_n \in D(T)$ and $Tf_n \to g$ in $\mathcal{H}$, then $f \in D(T)$ and $Tf = g$.

Definition 9.2. We say that a linear operator T is *self-adjoint* if $(T, D(T)) = (T^*, D(T^*))$.

Example 9.2.1. Let $T = i\partial_x$. For $\mathcal{H} = L^2(\mathbb{R})$ and $D(T) = C_c^1(\mathbb{R})$, one can check that T is self-adjoint. But if we take $\mathcal{H} = L^2([0,\infty))$ and $D(T) = C_c^1([0,\infty))$, then T is not self-adjoint (but T is symmetric).

Example 9.2.2. Let $T = -\partial_x^2$ on $L^2([0,1])$. Then for

$$D(T) = \{f \in H^2([0,1]) : f(0) = f(1) = f'(0) = f'(1) = 0\},$$

T is symmetric but not self-adjoint. One can check that $D(T^*) = H^2([0,1])$.

Lemma 9.1. *Let T be a densely defined linear operator on $\mathcal{H}$. Then:*

1. *If T is symmetric, then $D(T^*) \supseteq D(T)$ (so T^* is densely defined) and all eigenvalues of T are real.*
2. *For every T, its adjoint T^* is closed.*
3. *We have $(\ker T)^\perp = \overline{\operatorname{Ran}(T^*)}$ and $(\ker T^*)^\perp = \overline{\operatorname{Ran}(T)}$.*

4. If T is self-adjoint, i.e. $(T, D(T)) = (T^*, D(T^*))$, then

$$\operatorname{Spec} T = \mathbb{C} \setminus \{z \in \mathbb{C} : (T-z)^{-1} \textit{ exists as a bounded linear operator}\} \subseteq \mathbb{R}$$

and $\|(T-z)^{-1}\|_{\mathcal{L}(\mathcal{H},\mathcal{H})} \lesssim |\operatorname{Im} z|^{-1}$.

Proof. (1) One can check this by definition.

(2) Take a sequence $f_n \in D(T^*)$ with $f_n \to f$, and suppose that $T^* f_n \to g$. By definition,

$$\langle T\phi, f_n\rangle = \langle \phi, T^* f_n\rangle \quad \text{for every } \phi \in D(T).$$

Then $f_n \to f$ implies that $\langle T\phi, f_n\rangle \to \langle T\phi, f\rangle$, and $T^* f_n \to g$ implies that $\langle \phi, T^* f_n\rangle \to \langle \phi, g\rangle$. Then we have $\langle T\phi, f\rangle = \langle \phi, g\rangle$ for every $\phi \in D(T)$, so $f \in D(T^*)$ and $T^* f = g$ by definition.

(3) Note that in general, if $A \subseteq \mathcal{H}$, then $A^\perp$ is closed. To see this, let $f_n \in A^\perp$ and $f_n \to f$ in $\mathcal{H}$. Then $\langle f_n, a\rangle = 0$ for every $a \in A$ since $f_n \in A^\perp$. So letting $n \to \infty$, we have $\langle f, a\rangle = 0$, i.e. $f \in A^\perp$.

We first show that $\overline{\operatorname{Ran}(T)} \subseteq (\ker T^*)^\perp$. For any $h \in \operatorname{Ran}(T)$, there exists f such that $h = Tf$. Then for any $g \in \ker T^* \subseteq D(T^*)$, we have

$$\langle Tf, g\rangle = \langle f, T^* g\rangle = 0,$$

so $f \in (\ker T^*)^\perp$. Thus $\operatorname{Ran}(T) \subseteq (\ker T^*)^\perp$, and taking closures, $\overline{\operatorname{Ran}(T)} \subseteq \overline{(\ker T^*)^\perp} = (\ker T^*)^\perp$.

Now we show that $(\ker T^*)^\perp \subseteq \overline{\operatorname{Ran}(T)}$. Let $f \in (\overline{\operatorname{Ran}(T)})^\perp$, so $f \in (\operatorname{Ran}(T))^\perp$. So

$$\langle T^* f, g\rangle = \langle f, Tg\rangle = 0, \quad \text{for all } g \in D(T),$$

so $T^* f = 0$. This gives $f \in \ker(T^*)$. So $(\overline{\operatorname{Ran}(T)})^\perp \subseteq \ker(T^*)$. Then taking orthogonal complements,

$$(\ker T^*)^\perp \subseteq ((\overline{\operatorname{Ran}(T)})^\perp)^\perp = \overline{\operatorname{Ran}(T)}$$

since $\overline{\operatorname{Ran}(T)}$ is closed. This gives $(\ker T^*)^\perp = \overline{\operatorname{Ran}(T)}$, and the other equality follows by duality.

(4) Let $T = T^*$ and $z = x + iy$. Then we can write

$$|\langle (T-z)f, f\rangle| = |\langle (T-(x+iy))f, f\rangle| = |\langle (T-x)f, f\rangle - iy\langle f, f\rangle| \geq |y| \|f\|_{\mathcal{H}}^2,$$

since $\langle (T-x)f, f\rangle$ is real and $iy\langle f, f\rangle$ is purely imaginary. On the other hand,

$$\langle (T-z)f, f\rangle \leq \|f\|_{\mathcal{H}} \|(T-z)f\|_{\mathcal{H}},$$

by Cauchy-Schwarz, so $\|(T-z)f\|_{\mathcal{H}} \geq |y| \|f\|_{\mathcal{H}}$ where $y = \operatorname{Im} Z$. So $\ker(T-z) = 0$ if $|y| \neq 0$. Now

$$\overline{\operatorname{Ran}(T-z)} = (\ker(T^* - \overline{z}))^\perp = (\ker(T - \overline{z}))^\perp = \mathcal{H}$$

if $|y| \neq 0$ by (3), so we just need to show that $\operatorname{Ran}(T-z)$ is closed. Assume $(T-z)f_n \to g$. Note that

$$\|(T-z)f_n - (T-z)f_m\|_{\mathcal{H}} \geq |y| \|f_n - f_m\|_{\mathcal{H}},$$

so $(T-z)f_n$ being Cauchy implies that $\{f_n\}$ is also Cauchy. So $f_n \to f$ for some $f \in \mathcal{H}$, i.e. $T = T^*$ is closed. This then implies that $T - z$ is closed, from which it follows that $f \in D(T-z)$ and $(T-z)f = g$, so $\operatorname{Ran}(T-z)$ is closed if $|y| \neq 0$. So $\ker(T-z) = \{0\}$ and $\operatorname{Ran}(T-z) = \mathcal{H}$ if $|y| \neq 0$. So $T - z$ is invertible, hence $\operatorname{Spec} T \subseteq \mathbb{R}$ and $\|(T-z)^{-1}\|_{\mathcal{L}(\mathcal{H},\mathcal{H})} \lesssim |\operatorname{Im} z|^{-1}$. $\square$

9.2 Spectral Theory

Definition 9.3. The *resolvent* of T, denoted $\rho(T) \subseteq \mathbb{C}$ is defined as follows: We have $\lambda \in \rho(T)$ if we can find a bounded linear operator $R(\lambda)$ from $\mathcal{H}$ to $\mathcal{H}$ such that

$$(T - \lambda)R(\lambda) = R(\lambda)(T - \lambda) = I.$$

The *spectrum* of T is $\sigma(T) = \mathbb{C} \setminus \rho(T)$.

Definition 9.4. Let $\mathcal{B}$ be the Borel sets in $\mathbb{R}$. A *spectral measure* E on $\mathbb{R}$ is an orthogonal projection valued measure on $\mathcal{B}$, i.e. a map E such that $E(A)$ is an orthogonal projection for every $A \in \mathcal{B}$, and

1. $E(\varnothing) = 0$ and $E(\mathbb{R}) = I$;
2. for $A, B \in \mathcal{B}$, we have $E(A)E(B) = E(A \cap B)$;
3. for disjoint $A_j \in \mathcal{B}$, we have $E(\bigcup_{j=1}^{\infty} A_j) = \sum_{j=1}^{\infty} E(A_j)$.

Remark. Given a spectral measure E, for $x, y \in \mathcal{H}$, we can define

$$E_{x,y}(A) = \langle E(A)x, y\rangle \quad \text{for } A \in \mathcal{B}.$$

For fixed x, y, $E_{x,y}$ is a complex-valued Borel measure. In particular, $E_{x,x}$ is a positive Borel measure.

Theorem 9.1 (Spectral theorem)**.** *Let $(T, D(T))$ be self-adjoint on $\mathcal{H}$. Then there is a unique spectral measure E such that*

$$\langle Tx, y\rangle = \int \lambda \, dE_{x,y}(\lambda) \quad \textit{for all } x \in D(T),\ y \in \mathcal{H}.$$

Remark. We make the following remarks:

1. We have $\operatorname{supp} E_{x,y} \subseteq \sigma(T)$.
2. If f is a Borel measurable function, then we can interpret

   $$\langle f(T)x, y\rangle = \int f(\lambda)\, dE_{x,y}(\lambda).$$

   In particular, we have a map $f \mapsto f(T)$.

Lecture 10

Feb. 12 — Spectral Theory, Part 2

10.1 Examples for Spectral Theory

Example 10.0.1. Let $\mathcal{H} = \mathbb{R}^n$ with the standard inner product, and let T be a Hermitian matrix. Then T is self-adjoint, and the corresponding spectral measure E is given by

$$E(I) = \sum_{j:\lambda_j \in I} P_{L_j},$$

where $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k$ are the eigenvalues of T without multiplicity and $L_1, \ldots, L_k$ are eigenspaces.

Example 10.0.2. Let T be a self-adjoint operator. Then

$$e^{itA} f = \left(\int e^{it\lambda)} \, E(d\lambda) \right) f.$$

From this we have

$$\|e^{itA} f\|_{L^2}^2 = \langle \int e^{it\lambda} \, E(d\lambda) f, \int e^{it\lambda} \, E(d\lambda) f \rangle = \iint e^{it\lambda} e^{-it\mu} \, \langle E(d\lambda) E(d\mu) f, f \rangle.$$

Fixing λ and integrating over μ, the only point that survives is $\mu = \lambda$, so

$$\|e^{itA} f\|_{L^2}^2 = \int \langle E d(\lambda) f, f \rangle = \langle E(\mathbb{R}) f, f \rangle = \|f\|_{L^2}^2.$$

since $E(\mathbb{R}) = I$. Thus if T is self-adjoint, then $\|e^{itT} f\|_{L^2} = \|f\|_{L^2}$, i.e. e^{itT} is unitary. Also,

$$e^{itT} e^{isT} f = e^{i(t+s)T} f \quad \text{and} \quad e^{-itT} e^{itT} = I = e^{itT} e^{-itT}.$$

The above are group properties for e^{itT} as t varies.

Example 10.0.3. Let $\mathcal{H} = L^2(X, \mu)$, where μ is a positive probability measure. Define $T = M_\phi$ to be the multiplication operator by ϕ, i.e. $Tf = \phi f$ for $\phi : X \to \mathbb{R}$. We have

$$D(T) = \{f \in L^2 : \phi f \in L^2\},$$

which is dense in $\mathcal{H}$ since $\mu(|\phi| > M) \to 0$ as $M \to \infty$. So T is densely defined. Then

$$\langle Tf, g \rangle = \int \phi f \overline{g} \, \mu(dx) = \int f \overline{\phi g} \, \mu(dx) = \langle f, Tg \rangle$$

since ϕ is real. So T is symmetric, which implies $D(T) \subseteq D(T^*)$. To get the reverse inclusion, notice that for $g \in D(T^*)$, we have $\langle Tf, g\rangle = \langle f, h\rangle$ for some $h \in \mathcal{H}$, for all $f \in D(T)$. Then

$$\int f\overline{\phi g}\,\mu(dx) \int \phi f \overline{g}\,\mu(dx) = \langle Tf, g\rangle = \langle f, h\rangle = \int f\overline{h}\mu(dx).$$

Since $D(T)$ is dense, we get $\phi g = h \in \mathcal{H} = L^2(X, \mu)$, so $g \in D(T)$. This gives $D(T^*) \subseteq D(T)$, so we see that $D(T^*) = D(T)$ and T is self-adjoint. The spectral theorem then gives the spectral resolution E:

$$E(I)f = \chi_{\{\phi\in I\}}f,$$

where χ denotes an indicator function.

Example 10.0.4. Let $T = -\Delta$ on $L^2(\mathbb{R}^d)$. Clearly $D(T) = H^2(\mathbb{R}^d)$ is dense in $L^2(\mathbb{R}^d)$. Integration by parts shows that $-\Delta$ is symmetric. Now suppose $g \in D(T^*)$. Then

$$\langle -\Delta f, g\rangle = \langle f, h\rangle \quad \text{for all } f \in D(T) \text{ and some } h \in L^2.$$

Since $\langle -\Delta f, g\rangle = \langle f, -\Delta g\rangle = Lf$ (note that $-\Delta g \in \mathcal{S}'$, i.e. in the sense of distributions), we have

$$|Lf| \leq \|f\|_{L^2}\|h\|_{L^2} = C\|f\|_{L^2},$$

where $C = \|h\|_{L^2}$. So by the Hahn-Banach theorem, L can be extended to a continuous functional on L^2 such that $\|L\|_{B(L^2,L^2)} \leq \|h\|_{L^2}$. We can identify $Lf = \langle f, -\Delta g\rangle$, and thus

$$\|\Delta g\|_{L^2} \leq \|h\|_{L^2}.$$

This implies $\Delta g \in L^2$, so by the elliptic theory, $g \in H^2 = D(T)$. This gives $D(T^*) \subseteq D(T)$ and thus $D(T) = D(T^*)$ (since $D(T) \subseteq D(T^*)$ because T is symmetric), so T is self-adjoint. To understand the spectral resolution, we can use the Fourier transform: $\widehat{Tf} = \xi^2\widehat{f}$. Writing $\mathcal{F}(f) = \widehat{f}$, we have

$$\mathcal{F}T\mathcal{F}^{-1} = M_{\xi^2}.$$

Note that $\mathcal{F}$ is unitary on $L^2_x \to L^2_\xi$ by Plancherel's theorem. This says that $\mathcal{F}$ diagonalizes Δ. Letting E_T be the spectral resolution for T and E_{ξ^2} be the spectral resolutoin for M_{ξ^2}, we have

$$\mathcal{F}E_T\mathcal{F}^{-1} = E_{\xi^2}.$$

Since $E^2_\xi(I) = \chi_{\{\xi^2\in I\}}$, we can write

$$E_T(I)f = (\chi_{\{\xi^2\in I\}}\widehat{f})^\vee.$$

Example 10.0.5. Let $\mathcal{H} = L^2(\mathbb{R}^d)$ and $T = -\Delta/2 + V$, where $V \in L^\infty$ is real. We claim that T is self-adjoint. It is easy to check that T is symmetric, so $D(T) \subseteq D(T^*)$, where $D(T) = H^2(\mathbb{R}^d)$. For the reverse inclusion, take $g \in D(T^*)$, so $\langle Tf, g\rangle = \langle f, h\rangle$ for some $h \in \mathcal{H}$. Then

$$\langle -\Delta f/2, g\rangle = \langle f, h\rangle - \langle Vf, g\rangle,$$

so we have

$$|\langle -\Delta f/2, g\rangle| \leq (\|h\|_{L^2} + \|V\|_{L^\infty}\|g\|_{L^2})\|f\|_{L^2}.$$

As before, the Hahn-Banach theorem implies that $-\Delta g \in L^2$, and elliptic theory yields $g \in H^2 = D(T)$. Thus $D(T^*) \subseteq D(T)$, so we see that T is self-adjoint.

Lecture 11

Feb. 17 — Spectral Theory, Part 3

11.1 More Spectral Theory

Definition 11.1. Define the following parts of $\mathcal{H}$ (with respect to Lebesgue measure):

$$\begin{aligned}\mathcal{H}_{\mathrm{ac}} &= \{x \in \mathcal{H} : E_{x,x} \text{ is absolutely continuous}\},\\ \mathcal{H}_{\mathrm{sc}} &= \{x \in \mathcal{H} : E_{x,x} \text{ is singular continuous}\},\\ \mathcal{H}_{\mathrm{pp}} &= \{x \in \mathcal{H} : E_{x,x} \text{ is pure point}\}.\end{aligned}$$

Lemma 11.1. *Let T be self-adjoint and E its spectral resolution. Then $\mathcal{H} = \mathcal{H}_{\mathrm{ac}} \oplus \mathcal{H}_{\mathrm{sc}} \oplus \mathcal{H}_{\mathrm{pp}}$.*

Proof. We first show that (1) $\mathcal{H}_{\mathrm{ac}}, \mathcal{H}_{\mathrm{sc}}, \mathcal{H}_{\mathrm{pp}}$ are closed subspaces, then that (2) they are orthogonal to each other, and finally (3) the decomposition for an arbitrary $f \in \mathcal{H}$.

(1) Take $f, g \in \mathcal{H}_{\mathrm{ac}}$, we first show that $f + g \in \mathcal{H}_{\mathrm{ac}}$. If S is a Borel set such that $|S| = 0$ (Lebesgue measure), then we want to show that $E(S)(f + g) = 0$. Since $f, g \in \mathcal{H}_{\mathrm{ac}}$, we note that

$$E(S)f = E(S)g = 0.$$

Indeed, for $f \in \mathcal{H}_{\mathrm{ac}}$, we have $\langle E(S)f, f\rangle = E_{f,f}(S) = 0$. Then

$$0 = \langle E(S)f, f\rangle = \langle E(S)E(S)f, f\rangle = \langle E(S)f, E(S)f\rangle$$

since $E(S)$ is a projection and also self-adjoint. This shows that $E(S)f = 0$ by the definiteness of the inner product, and similarly, $E(S)g = 0$. To check $E_{f+g,f+g}(S) = 0$, we note that

$$E_{f+g,f+g}(S) = \langle E(S)(f+g), f+g\rangle = \langle E(S)f, f\rangle + \langle E(S)g, f\rangle + \langle E(S)f, g\rangle + \langle E(S)g, g\rangle = 0$$

since $E(S)f = E(S)g = 0$. This implies that $f + g \in \mathcal{H}_{\mathrm{ac}}$, which shows that $\mathcal{H}_{\mathrm{ac}}$ is a subspace. To see that $\mathcal{H}_{\mathrm{ac}}$ is closed, let $f_n \in \mathcal{H}_{\mathrm{ac}}$ with $f_n \to f$ in $\mathcal{H}$. Take any S with $|S| = 0$, so

$$\langle E(S)f, f\rangle = \lim_{n\to\infty} \langle E(S)f_n, f_n\rangle = \lim_{n\to\infty} 0 = 0.$$

Thus $E(S)f = 0$, so $f \in \mathcal{H}_{\mathrm{ac}}$. Thus $\mathcal{H}_{\mathrm{ac}}$ is a closed subspace.

Now let $f, g \in \mathcal{H}_{\mathrm{sc}}$. Let S_f, S_g be the support of $E_{f,f}, E_{g,g}$, respectively. Since $f, g \in \mathcal{H}_{\mathrm{sc}}$, we must have $|S_f| = |S_g| = 0$. We also know $E_{f,f}(S_f^c) = E_{g,g}(S_g^c) = 0$, since S_f^c, S_g^c are outside of the support of $E_{f,f}, E_{g,g}$. Let $S_{f+g} = S_f \cup S_g$, so $|S_{f+g}| = 0$ by the above. Furthermore,

$$E(S_{f+g}^c)(f+g) = E(S_f^c \cap S_g^c)(f+g) = E(S_g^c)E(S_f^c)f + E(S_f^c)E(S_g^c)g = 0$$

This implies that $\mathcal{H}_{\text{sc}}$ is a subspace. Now take $f_n \in \mathcal{H}_{\text{sc}}$ with $f_n \to f$ in $\mathcal{H}$. Similarly, let S_n be the support of the E_{f_n,f_n}. Let $S_f = \bigcup_{n=1}^{\infty} S_{f_n}$, so $|S_f| = 0$ by countable subadditivity. Then

$$E(S_f^c)f_n = \left(\prod_j \neq nE(S_{f_j}^c)\right) \circ E(S_{f_n}^c)f_n = 0,$$

and letting $n \to \infty$ implies that $E(S_f^c)f = 0$. Since the support of $E_{f,f}$ is of measure 0 in the Lebesgue sense, $f \in \mathcal{H}_{\text{sc}}$. Thus we see that $\mathcal{H}_{\text{sc}}$ is a closed subspace.

One can verify that $\mathcal{H}_{\text{pp}}$ is a closed subspace in a similar manner, which completes the proof for (1).

(2) We claim that $\mathcal{H}_{\text{ac}}$ is orthogonal to $\mathcal{H}_{\text{sc}}$ and $\mathcal{H}_{\text{pp}}$. Take $f \in \mathcal{H}_{\text{ac}}$ and $g \in \mathcal{H}_{\text{sc}} \cup \mathcal{H}_{\text{pp}}$. Let S_g be the support of $E_{g,g}$, so $|S_g| = 0$. Note that $g = E(S_g)g$. Then we can write

$$\langle f, g\rangle = \langle f, E(S_g)g\rangle = \langle E(S_g)f, g\rangle = 0$$

since f is absolutely continuous with respect to Lebesgue measure. Thus $f \perp g$.

A similar argument shows that $\mathcal{H}_{\text{sc}} \perp \mathcal{H}_{\text{pp}}$, so which proves (2).

(3) Given $f \in \mathcal{H}$, the measure $E_{f,f}$ can be decomposed (with respect to Lebesgue measure) into an absolutely continuous part E_1, a singular continuous part E_2, and a pure point part E_3. Denote the support of E_1, E_2, E_3 by S_1, S_2, S_3. Thus we can write

$$f = E(S_1)f + E(S_2)f + E(S_3)f.$$

We have $E(S_1)f \in \mathcal{H}_{\text{ac}}$, $E(S_2)f \in \mathcal{H}_{\text{sc}}$, and $E(S_3)f \in \mathcal{H}_{\text{pp}}$, so we get $\mathcal{H} = \mathcal{H}_{\text{ac}} \oplus \mathcal{H}_{\text{sc}} \oplus \mathcal{H}_{\text{pp}}$. □

Definition 11.2. Define $E_{\text{ac}} = E \circ P_{\text{ac}}$, where P_{ac} is the projection onto $\mathcal{H}_{\text{ac}}$.

Lemma 11.2. *Let T be a self-adjoint operator and E be its spectral resolution. Then for a.e. $\lambda \in \mathbb{R}$,*

$$\langle E_{\text{ac}}(d\lambda)f, g\rangle = \langle \frac{1}{2\pi i}(R(\lambda + i0) - R(\lambda - i0))f, g\rangle \, d\lambda,$$

where $R(\lambda \pm i0) = \lim_{\epsilon \to 0^+} R(\lambda \pm i\epsilon)$ and $R(\lambda \pm i\epsilon) = (T - (\lambda \pm i\epsilon))^{-1}$.

Proof. By spectral theory, we can write

$$\begin{aligned}&\langle \frac{1}{2\pi i}((T - (x + i\epsilon))^{-1} - (T - (x - i\epsilon))^{-1})f, g\rangle \\ &\quad = \frac{1}{2\pi i}\left(\frac{1}{\mu - (\lambda + i\epsilon)} - \frac{1}{\mu - (\lambda - i\epsilon)}\right)\langle E(d\mu)f, g\rangle = \int \frac{1}{\pi}\frac{\epsilon}{(\mu - \lambda)^2 + \epsilon^2}\langle E(d\mu)f, g\rangle.\end{aligned}$$

Note that this is the Poisson kernel in the upper half-plane. Letting $\epsilon \to 0^+$, we have

$$\int \frac{1}{\pi}\frac{\epsilon}{(\mu - \lambda)^2 + \epsilon^2}\langle E(d\mu)f, g\rangle \to \frac{\langle E_{\text{ac}}(d\lambda)f, g\rangle}{d\lambda}$$

for a.e. λ (in terms of Lebesgue measure) since the Poisson kernel is an approximate identity. □

Remark. The above lemma says that the spectral density can be written as the jump of the resolvent.

11.2 Computing Resolvents

Remark. What can we do with resolvents? We are interested in $-\Delta + V$. Take $V = 0$, then

$$R(\lambda + i\epsilon) = (-\Delta - (\lambda + i\epsilon))^{-1}.$$

Let $d = 3$, then we have

$$((-\Delta - (\lambda + i\epsilon))^{-1} f)^\wedge = \frac{\widehat{f}(\xi)}{\xi^2 - (\lambda + i\epsilon)}.$$

Applying Fourier inversion, we get

$$((-\Delta - (\lambda + i\epsilon))^{-1} f)(x) = \frac{1}{(2\pi)^3} \int \frac{e^{ix\xi} \widehat{f}(\xi)}{\xi^2 - (\lambda + i\epsilon)} \, d\xi.$$

Note that we can write

$$((-\Delta - (\lambda + i\epsilon))^{-1} f)(x) = \int K(x-y) f(x) \, dy,$$

where the kernel K satisfies

$$\widehat{K}(\xi) = \frac{1}{\xi^2 - (\lambda + i\epsilon)}.$$

By the inversion formula, this means that

$$K = \frac{1}{(2\pi)^3} \int \frac{e^{ix\xi}}{\xi^2 - (\lambda + i\epsilon)} \, d\xi.$$

Using polar coordinates, set $\xi = r\omega$, and we obtain

$$K = \int_0^\infty \int_{\mathbb{S}^2} e^{irx\cdot\omega} \, d\sigma_{\mathbb{S}^2}(\omega) \, \frac{r^2}{r^2 - (\lambda + i\epsilon)} \, dr. \qquad (*)$$

We can first compute that

$$\int_{\mathbb{S}^2} e^{ia\omega\cdot\vec{e}_3} \, d\sigma_{\mathbb{S}^2}(\omega) = 2\pi \int_0^\pi e^{ia\cos\theta} \sin\theta \, d\theta = 2\pi \int_{-1}^1 e^{iau} \, du = 4\pi \frac{\sin(au)}{a}.$$

Then we can calculate the inner integral in $(*)$ by symmetry (Δ is rotationally invariant),:

$$\int_{\mathbb{S}^2} e^{irx\cdot\omega} \, d\sigma_{\mathbb{S}^2}(\omega) = 4\pi \frac{\sin(r|x|)}{r|x|}.$$

Substituting this into $(*)$, we get

$$\begin{aligned} K &= \frac{1}{2\pi^2} \int_0^\infty \frac{\sin(r|x|)}{r|x|} \frac{r^2}{r^2 - (\lambda + i\epsilon)} \, dr = \frac{1}{2\pi^2 |x|} \int_0^\infty \frac{\sin(r|x|)}{r^2 - (\lambda + i\epsilon)} \, r \, dr \\ &= \frac{1}{4\pi^2 |x|} \int_{-\infty}^\infty \frac{\sin(r|x|)}{r^2 - (\lambda + i\epsilon)} \, r \, dr = \frac{1}{16\pi^2 |x|} \int_{-\infty}^\infty (e^{i|x|r} - e^{-ir|x|}) \left(\frac{1}{r - \sqrt{\lambda + i\epsilon}} - \frac{1}{r + \sqrt{\lambda + i\epsilon}} \right) dr. \end{aligned}$$

By some complex analysis (residue computations), we get $K = e^{i|x|\sqrt{\lambda + i\epsilon}}/4\pi|x|$, so that

$$R(\lambda + i0) = \lim_{\epsilon \to 0^+} R(\lambda + i\epsilon) = \frac{1}{4\pi|x|} e^{i|x|\sqrt{\lambda}}, \quad -\infty < \lambda < \infty,$$

where we take the principal branch of the square root.

Lecture 12

Feb. 19 — Dispersive Decay with Potential

12.1 More on Computing Resolvents

Example 12.0.1. We will compute the resolvent for $-\partial_x^2$. For $\epsilon > 0$, we would like to study

$$R(\lambda \pm i\epsilon) = (-\partial_x^2 - (\lambda \pm i\epsilon))^{-1}.$$

It is reduced to compute the Green's function in 1 dimension: $G_{\lambda+i\epsilon}(x, y)$ such that

$$(-\partial_x^2 - (\lambda + i\epsilon))G_{\lambda+i\epsilon}(x, y) = \delta(x - y)$$

and is normalized by $G_{\lambda+i\epsilon}(-\infty, y) = G_{\lambda+i\epsilon}(\infty, y) = 0$. Note that the function

$$g_1 = e^{i\sqrt{\lambda+i\epsilon}x}$$

solves the ODE $(-\partial_x^2 - (\lambda + i\epsilon))g_1 = 0$ and satisfies $g_1(x) \to 0$ as $x \to \infty$. The function

$$g_2 = e^{-i\sqrt{\lambda+i\epsilon}x}$$

also solves $(-\partial_x^2 - (\lambda + i\epsilon))g_2 = 0$ but satisfies $g_2(x) \to 0$ as $x \to -\infty$. If $x < y$, then by the uniqueness of the ODE solution we should expect that $G_{\lambda+i\epsilon}$ is a multiple of g_2. Similarly, if $x > y$ we should expect that $G_{\lambda+i\epsilon}$ is a multiple of g_1. Stitching the solutions together, we find that

$$G_{\lambda+i\epsilon}(x, y) = \frac{1}{W[g_1, g_2]} \begin{cases} g_2(x)g_1(y) & -\infty < x < y, \\ g_1(x)g_2(y) & y < x < \infty, \end{cases}$$

where $W[g_1, g_2] = g_1g_2' - g_1'g_2 = -2i\sqrt{\lambda + i\epsilon}$ is the Wronskian. Passing $\epsilon \to 0$,

$$R(\lambda + i0) = \lim_{\epsilon\to0^+} G_{\lambda+i\epsilon}(x, y) = -\frac{e^{i\sqrt{\lambda}|x-y|}}{2i\sqrt{\lambda}}.$$

We can also note that

$$R(\lambda - i0) = \overline{R(\lambda + i0)} = \frac{e^{-i\sqrt{\lambda}|x-y|}}{2i\sqrt{\lambda}}.$$

Exercise 12.1. By spectral theory, we know that we can write

$$e^{t\partial_x^2 t}f = \frac{1}{2\pi i}\int_0^\infty e^{i\lambda t}[R(\lambda + i0) - R(\lambda - i0)]f\, d\lambda.$$

We have an explicit formula for $R(\lambda \pm i0)$ from the above computation, e.g.

$$R(\lambda + i0)f = -\int_{\mathbb{R}} \frac{e^{i\sqrt{\lambda}|x-y|}}{2i\sqrt{\lambda}} f(y)\, dy.$$

Check that this is the same formula for $e^{i\partial_x^2 t} f$ that we obtained via Fourier transforms.

Remark. The Fourier transform only works well for standard differential operators and not for perturbed versions such as $-\partial_x^2 + V$. Spectral theory will allow us to better understand such operators.

12.2 Dispersive Decay with Potential

Remark. Our next goal is to show the dispersive decay for $e^{iHt} f$, where $H = -\Delta + V$, in $d = 1, 3$. In particular, for V sufficiently nice and $P_{\mathrm{c}} = P_{\mathrm{ac}}$, we will show that

$$\|e^{iHt} P_{\mathrm{c}} f\|_{L^\infty} \lesssim t^{-d/2} \|f\|_{L^1}, \quad d = 1, 3.$$

If V is nice, then $\mathcal{H}_{\mathrm{sc}} = \varnothing$, so $\mathcal{H} = \mathcal{H}_{\mathrm{ac}} \oplus \mathcal{H}_{\mathrm{pp}}$ and we may simply write $P_{\mathrm{c}} = P_{\mathrm{ac}}$.

Theorem 12.1. *Let $d = 1$ and $H = -\partial_x^2 + V$, where V is sufficiently nice (e.g. enough decay). Then*

$$\|e^{iHt} P_{\mathrm{c}} f\|_{L^\infty} \lesssim t^{-1/2} \|f\|_{L^1}.$$

Remark. Why do we need P_{c}? This is because if $f \in \mathrm{Ran}(P_{\mathrm{pp}})$, then $Hf = \lambda f$ for some $\lambda \in \mathbb{R}$ (since H is self-adjoint). Then $e^{iHt} f = e^{i\lambda t} f$, which is oscillatory in t and does not decay.

Lemma 12.1 (High energy part). *Let $\chi \in C^\infty(\mathbb{R})$ such that $\chi(\lambda) \equiv 1$ for $\lambda \geq 2\lambda_0$ and $\chi(\lambda) \equiv 0$ for $\lambda \leq \lambda_0$ for some large but fixed λ_0. Then we have*

$$\|e^{iHt} \chi(H) f\|_{L^\infty} \lesssim t^{-1/2} \|f\|_{L^1}.$$

Proof. This proof is due to Goldberg–Schlag (2004), we will give an alternative (more direct but less general) proof later on. Note that $\sigma(-\partial_x^2) = [0, \infty)$, so $\langle -\partial_x^2 f, f \rangle \geq 0$. So one can always invert $-\partial_x^2 - \lambda$ for $\lambda < 0$. Then the absolutely continuous spectrum of $-\partial_x^2 + V$ will also be $[0, \infty)$ by Weyl's theorem provided that V decays sufficiently fast. Then by spectral theory, one can write

$$\langle e^{iHt} \chi(H) P_{\mathrm{c}} f, g \rangle = \int_0^\infty e^{i\lambda t} \chi(\lambda) \langle (R(\lambda + i0) - R(\lambda - i0)) f, g \rangle \, d\lambda.$$

The resolvent in this case is

$$R(\lambda \pm i0) = \lim_{\epsilon \to 0^+} (-\partial_x^2 + V - (\lambda \pm i\epsilon))^{-1} = \lim_{\epsilon \to 0^+} (H_0 + V - (\lambda \pm i\epsilon))^{-1}$$

where $H_0 = -\partial_x^2$. Then one can write

$$R(\lambda \pm i\epsilon) = (H_0 + V - (\lambda \pm i\epsilon))^{-1} = R_0(\lambda \pm i\epsilon) - R_0(\lambda \pm i\epsilon) V R(\lambda \pm i\epsilon).$$

The above is the *second resolvent identity*, which follows from

$$(H - (\lambda \pm i\epsilon)) R(\lambda \pm i\epsilon) = I$$

and

$$-(H-(\lambda \pm i\epsilon))R_0(\lambda \pm i\epsilon) = (H_0 + V - (\lambda \pm i\epsilon))R_0(\lambda \pm i\epsilon) = VR_0(\lambda \pm i\epsilon) + I,$$

which gives

$$-(H-(\lambda \pm i\epsilon))(R(\lambda \pm i\epsilon)VR_0(\lambda \pm i\epsilon)) = VR_0(\lambda \pm i\epsilon).$$

Adding the above equations gives the desired identity. Thus we can formally write

$$R(\lambda \pm i\epsilon) = R_0(\lambda \pm i\epsilon) - R_0(\lambda \pm i\epsilon)VR(\lambda \pm i\epsilon) + \cdots = \sum_{n=0}^{\infty} R_0(\lambda \pm i\epsilon)(-VR_0(\lambda \pm i\epsilon))^n,$$

which is known as the *Born series.*[1] Now we claim that for $\lambda > \lambda_0$ with λ_0 large enough,

$$\langle R(\lambda + i0)f, g\rangle = \sum_{n=0}^{\infty} \langle R_0(\lambda + i0)(-VR_0(\lambda + i0))^n f, g\rangle$$

for Schwartz functions $f, g \in \mathcal{S}$. To see this, we can write

$$\langle R_0(\lambda + i\epsilon)f, g\rangle = -\iint \frac{1}{2i\sqrt{\lambda + i\epsilon}} e^{i\sqrt{\lambda+i\epsilon}|x-y|} f(y)\, dy\, \overline{g(x)}\, dx.$$

Thus we have the bound

$$|\langle R_0(\lambda + i\epsilon)f, g\rangle| \le \frac{1}{2\sqrt{\lambda}} \|f\|_{L^1} \|g\|_{L^1}.$$

In general, we can estimate

$$\begin{aligned}
&|\langle R_0(\lambda + i\epsilon)(VR_0(\lambda + i\epsilon))^n f, g\rangle| \\
&\quad \le \int \cdots \int \frac{1}{|2\sqrt{\lambda + \epsilon}|^{n+1}} |V(x_1) \dots V(x_n)| |f(x_0)| |g(x_{n+1})|\, dx_0 \dots dx_{n+1} \\
&\quad \le \frac{1}{(2\sqrt{\lambda + i\epsilon})^{n+1}} \|V\|_{L^1}^n \|f\|_{L^1} \|g\|_{L^1}.
\end{aligned}$$

Now take λ_0 such that $2\sqrt{\lambda_0} > \|V\|_{L^1}$, so that the series

$$\sum_{n=0}^{\infty} \langle R_0(\lambda + i0)(-VR_0(\lambda + i0))^n f, g\rangle$$

converges uniformly in λ. We will finish the proof next time. $\square$

[1]Formally, one can see this as the *Neumann series* for $(H - (\lambda \pm i\epsilon))^{-1} = (H_0 - (\lambda \pm i\epsilon))^{-1}(1 + V/(H_0 - (\lambda + i\epsilon)))^{-1}$ and apply the geometric series formula.

Lecture 13

Feb. 24 — Dispersive Decay with Potential, Part 2

13.1 High Energy Estimates, Continued

Proposition 13.1. *We have the estimate*

$$|\langle e^{iHt}\chi(H)P_{\mathrm{ac}}f, g\rangle| \lesssim t^{-1/2}\|f\|_{L^1}\|g\|_{L^1}.$$

Proof. Using the Born series, we have

$$\begin{aligned}
&|\langle e^{iHt}\chi(H)P_{\mathrm{ac}}f, g\rangle| \\
&\le \sum_{n=0}^{\infty}\left|\frac{(-1)^n}{2\pi i}\int_0^\infty e^{it\lambda}\chi(\lambda)\left(\langle R_0(\lambda+i0)(VR_0(\lambda+i0))^n f, g\rangle - \langle R_0(\lambda-i0)(VR_0(\lambda-i0))^n f, g\rangle\right)d\lambda\right|.
\end{aligned}$$

The general term in the above series is (the minus term is similar)

$$\int_0^\infty e^{it\lambda}\chi(\lambda)\langle R_0(\lambda+i0)(VR_0(\lambda+i0))^n f, g\rangle\,d\lambda.$$

Changing variables using $\lambda = \eta^2$, we have

$$\int_0^\infty e^{it\eta^2}\chi(\eta^2)\cdot 2\eta\langle R_0(\eta^2+i0)(VR_0(\eta^2+i0))^n f, g\rangle\,d\eta.$$

Using the calculation that $R_0(\lambda \pm i0) = \pm i e^{\pm i|x|\sqrt{\lambda}}/2\sqrt{\lambda}$, we have $R_0(\eta^2+i0) = ie^{i|x|\eta}/2\eta$, and we get

$$\iint f(x_0)g(x_{n+1})\int\cdots\int V(x_1)\ldots V(x_n)\int_0^\infty e^{it\eta^2}2\eta\cdot\chi(\eta^2)\frac{e^{i\eta\sum_{j=1}^{n+1}|x_j-x_{j-1}|}}{(-2\eta i)^{n+1}}\,d\eta\,dx_1\ldots dx_n\,dx_0dx_{n+1}.$$

Note the above integral by $(*)$, and by Minkowski's inequality we have

$$\begin{aligned}
&|(*)| \\
&\le \int\cdots\int |f(x_0)||g(x_{n+1})|V(x_1)\ldots V(x_n)\sup_{x_0,\ldots,x_{n+1}}\left|\int_0^\infty e^{it\eta^2}2\eta\cdot\chi(\eta^2)\frac{e^{i\eta\sum_{j=1}^{n+1}|x_j-x_{j-1}|}}{(-2\eta i)^{n+1}}\,d\eta\right|dx_0\ldots dx_{n+1}.
\end{aligned}$$

Setting $a = \sum_{j=1}^{n+1}|x_j - x_{j-1}|$, the inner integral satisfies

$$\sup_{x_0,\ldots,x_{n+1}}\left|\int_0^\infty e^{it\eta^2}2\eta\cdot\chi(\eta^2)\frac{e^{i\eta\sum_{j=1}^{n+1}|x_j-x_{j-1}|}}{(-2\eta i)^{n+1}}\,d\eta\right| = \sup_a\left|\int_0^\infty e^{it\eta^2}2\eta\cdot\chi(\eta^2)\frac{e^{i\eta a}}{(-2\eta i)^{n+1}}\,d\eta\right|.$$

By a suitable change of variables, it suffices to study

$$\sup_a \left| \int_{-\infty}^{\infty} e^{it\eta^2} 2\eta \cdot \chi(\eta^2) \frac{e^{i\eta a}}{(-2\eta i)^{n+1}} \, d\eta \right| = \sup_a \left| \int_{-\infty}^{\infty} e^{it\eta^2 + i\eta a} \widehat{\mu}_n(\eta) \, d\eta \right|,$$

where we have

$$\widehat{\mu}_n(\eta) = \frac{2\eta \cdot \chi(\eta^2)}{(-2\eta i)^{n+1}} = \|e^{i\partial_x^2 t} \mu_n\|_{L^\infty} \leq t^{-1/2} \|\mu_n\|_{L^1}.$$

The key idea is to reduce the computation to the 1-D free Schrödinger equation. Writing

$$\mu_n(x) = \int_{-\infty}^{\infty} \frac{\chi(\lambda^2)}{\lambda^n} e^{-i\lambda x} \, d\lambda,$$

we want to estimate $\|\mu_n\|_{L^1}$. For $n = 0$, we have

$$\mu_0(x) = \int_{-\infty}^{\infty} \chi(\lambda^2) e^{-i\lambda x} \, d\lambda = \int_{-\infty}^{\infty} (1 + (1 - \chi(\lambda^2))) e^{-i\lambda x} \, d\lambda = \delta_0 + \int_{-\infty}^{\infty} (1 - \chi(\lambda))^2 e^{-i\lambda x} \, d\lambda.$$

The latter integral is a Schwartz function since $1 - \chi(\lambda)$ is compactly supported. So $\|\mu_0\|_{L^1} < \infty$. For general n, we would like to estimate $\|x^2 \mu_n(x)\|_{L^\infty}$:

$$\|x^2 \mu_n(x)\|_{L^\infty} = \|(\widehat{\mu}_n''(x))^\vee\|_{L^\infty} \leq \|\widehat{\mu}_n''\|_{L^1} = \left\| \left(\frac{\chi(\lambda^2)}{\lambda^n} \right)'' \right\| \lesssim \lambda_0^{-n/2},$$

where the second inequality is by Young's inequality and the last inequality is because

$$\left(\frac{\chi(\lambda^2)}{\lambda^n} \right)'' \lesssim \left| \frac{\chi'(\lambda^2)}{\lambda^{n+2}} \right| + \left| \frac{\lambda \chi'(\lambda^2)}{\lambda^{n+1}} \right| + \left| \frac{\lambda^2 \chi''(\lambda^2)}{\lambda^n} \right|$$

and the observation that χ', χ'' are compactly supported functions. Thus we have

$$|\mu_n(x)| \lesssim (\lambda_0)^{-n/2} |x|^{-2}.$$

This gives us decay as $x \to \infty$, but we also need to estimate $\|\mu_n\|_{L^\infty}$: For $n \geq 2$,

$$\|\mu_n\|_{L^\infty} \lesssim \|\widehat{\mu}_n\|_{L^1} = \left\| \frac{\chi(\lambda^2)}{\lambda^n} \right\| \lesssim (\lambda_0)^{-n/2}$$

since $\widehat{\mu}_n$ is integrable for $n \geq 2$. When $n = 1$, we have

$$\|\mu_1(x)\|_{L^\infty} = \left\| \left(\frac{\chi(\lambda^2)}{\lambda} \right)^\vee \right\|_{L^\infty} = \|(\chi(\lambda^2))^\vee * (1/\lambda)^\vee\|_{L^\infty} = \|\chi(\lambda^2)\|_{L^1} \|(1/\lambda)^\vee\|_{L^\infty} < \infty$$

since $\|\chi(\lambda^2)\|_{L^1}$ is finite and $(1/\lambda)^\vee = -i \, \mathrm{sign}(x)$, so $\|(1/\lambda)^\vee\|_{L^\infty} \leq 1$. Thus for $n \geq 1$,

$$\|\mu_n\|_{L^\infty} \lesssim (\lambda_0)^{-n/2} \quad \text{and} \quad \|\mu_n(x)\| \lesssim (\lambda_0)^{-n/2} |x|^{-2},$$

which gives $\|\mu_n\|_{L^1} \lesssim (\lambda_0)^{-n/2}$. When $n = 0$, we already know $\|\mu_0\|_{L^1} < \infty$. This gives

$$\sup_a \left| \int_0^{\infty} e^{it\eta^2} 2\eta \cdot \chi(\eta^2) \frac{e^{i\eta a}}{(-2\eta i)^{n+1}} \, d\eta \right| \lesssim |t|^{-1/2} (\lambda_0)^{-n/2},$$

and so we have
$$(*) \lesssim \|f\|_{L^1}\|g\|_{L^1}\|V\|_{L^1}^n(\lambda_0)^{-n/2}|t|^{-1/2}.$$
As long as $\lambda_0^{-1/2}\|V\|_{L^1} < 1$, we can sum the Born series and get
$$\langle e^{iHt}\chi(H)P_{\mathrm{ac}}f, g\rangle \leq |t|^{-1/2}\|f\|_{L^1}\|g\|_{L^1}\sum_{n=0}^{\infty}(\lambda_0^{-1/2}\|V\|_{L^1})^n \lesssim |t|^{-1/2}\|f\|_{L^1}\|g\|_{L^1},$$
which is the desired estimate. □

Remark. The takeaway from this proof is that for the high frequency part, we use a Born series, the free resolvent, and explicit computations.

13.2 Low Energy Estimates

Remark. In this setting, the influence of the potential becomes more serious.

Lemma 13.1 (Low energy). *Let V be "nice" and χ a cutoff function near 0. Then*
$$\|e^{iHt}\chi(H)P_{\mathrm{ac}}f\|_{L^\infty} \leq |t|^{-1/2}\|f\|_{L^1}.$$

Proof. We use spectral theory. Write
$$e^{iHt}\chi(H)P_{\mathrm{ac}}f = \frac{1}{2\pi i}\int_0^\infty \chi(\lambda)e^{-it\lambda}[R(\lambda+i0)-R(\lambda-i0)]f\,d\lambda.$$
We will use the convention (note that $(\eta+i\epsilon)^2 = \eta^2-\epsilon^2+2i\eta\epsilon$)
$$R(\eta^2+i0) = \lim_{\epsilon\to 0^+} R((\eta+i\epsilon)^2) = \lim_{\epsilon\to 0^+} R(\eta^2+i\,\mathrm{sign}(\eta)\epsilon).$$
Then we can write
$$e^{iHt}\chi(H)P_{\mathrm{ac}}f = \frac{1}{\pi i}\int_{-\infty}^\infty \eta\cdot\chi(\eta^2)e^{it\eta^2}R(\eta^2+i0)f\,d\eta.$$
Using a Green's function, we can write
$$R(\lambda^2+i0)(x,y) = \frac{f_+(\lambda,y)f_-(\lambda,x)}{W[f_+(\lambda,\cdot),f_-(\lambda,\cdot)]}\mathbb{1}_{\{x<y\}} + \frac{f_+(\lambda,x)f_-(\lambda,y)}{W[f_+(\lambda,\cdot),f_-(\lambda,\cdot)]}\mathbb{1}_{\{y<x\}},$$
where $f_\pm$ are the Jost functions solving
$$\begin{cases} -f_\pm'' + Vf_\pm = \lambda^2 f_\pm, \\ |f_+(\lambda,x) - e^{i\lambda x}| \xrightarrow[x\to\infty]{} 0, \\ |f_-(\lambda,x) - e^{-i\lambda x}| \xrightarrow[x\to\infty]{} 0. \end{cases}$$
The conclusion from this is that
$$\begin{aligned}\langle e^{iHt}\chi(H)P_{\mathrm{ac}}f, g\rangle &= \frac{1}{\pi i}\int_{-\infty}^\infty\int e^{it\lambda^2}\lambda\cdot\chi(\lambda^2)\frac{f_+(\lambda,y)f_-(\lambda,x)}{W(\lambda)}\,d\lambda\,f(x)\overline{g(y)}\mathbb{1}_{\{x<y\}}\,dx\,dy \\ &\quad + \frac{1}{\pi i}\int_{-\infty}^\infty\int e^{it\lambda^2}\lambda\cdot\chi(\lambda^2)\frac{f_+(\lambda,x)f_-(\lambda,y)}{W(\lambda)}\,d\lambda\,f(x)\overline{g(y)}\mathbb{1}_{\{y<x\}}\,dx\,dy.\end{aligned}$$
We will finish the proof next time. □

Lecture 14

Feb. 26 — Dispersive Decay with Potential, Part 3

14.1 Low Energy Estimates, Continued

Proof of Lemma 13.1, continued. Our goal is to understand $f_\pm$. Define $m_+(\lambda, x) = e^{i\lambda x} f_+(\lambda)$, so that

$$m_+(\lambda, x) \to 1 \quad \text{as } x \to \infty$$

"Volterra iteration" gives more precise estimates, but we will go with a rougher approach: m_+ satisfies

$$\partial_x^2 m_+(\lambda, x) + 2i\lambda \partial_x m_+(\lambda, x) + V(x) m_+(\lambda, x) = 0.$$

Then we have the *Volterra equation*

$$m_+(\lambda, x) = 1 + \int_x^\infty D_\lambda(x-y) V(y) m_+(\lambda, y)\, dy, \quad D_\lambda = \int_0^x e^{2i\lambda z}\, dz = \frac{e^{2i\lambda x} - 1}{2i\lambda}.$$

Denote $\dot{m}_+(\lambda, x) = \partial_\lambda m_+(\lambda, x)$ and $\ddot{m}_+(\lambda, x) = \partial_\lambda^2 m_+(\lambda, x)$, which satisfy

$$\dot{m}_+(\lambda, x) = \int_x^\infty D_\lambda(x-y) V(y) \dot{m}_+(\lambda, y)\, dy + \int_x^\infty \dot{D}_\lambda(x-y) V(y) m_+(\lambda, y)\, dy$$

and a similar integral equation for $\ddot{m}_+$. We want to show the existence of m_+, and we are interested in the behavior of m_+ as $x \to \infty$. We will solve the equation on $[x_0, \infty)$ for x_0 large enough. Set

$$z_+(\lambda, x) = \langle \lambda \rangle \frac{m_+(k, x) - 1}{W_+'(x)}, \quad W_+'(x) = \int_x^\infty \langle y \rangle |V(y)|\, dy.$$

where $\langle \lambda \rangle = \sqrt{1 + \lambda^2}$ is the *Japanese bracket*. Note that $W_+'(x) \to 0$ with the same rate, and if

$$\int \langle y \rangle^n |V(y)|\, dy < \infty,$$

then $|W_+'(x)| \le \langle x \rangle^{1-n}$. From the equation of m_+, we can see that

$$(I - L) z_+(k, x) = z_+(k, x) - L z_+ = \frac{\langle \lambda \rangle}{W_+'(x)} \int_x^\infty D_\lambda(y - x) V(y)\, dy,$$

where $L z_+$ is given by

$$L z_+ = \frac{1}{W_+'(x)} \int_x^\infty D_\lambda(y - x) V(y) W_+'(y) z_+(y)\, dy.$$

Our goal is to show that $(I-L)$ is invertible on $L^\infty([x_0,\infty))$. Note that we have $|D_\lambda(z)| \lesssim |z|/|\lambda|$ since $|e^{ix}-1| \le |x|$, and also $|D_\lambda(z)| \le |z|$. Then we can estimate

$$\left\| \langle\lambda\rangle W_+' \int_x^\infty D_\lambda(y-x)V(y)\,dy \right\|_{L^\infty([x_0,\infty))} \lesssim \left\| \langle\lambda\rangle W_+' \int_x^\infty \frac{|y-x|}{|\lambda|} V(y)\,dy \right\|_{L^\infty([x_0,\infty))} < \infty$$

So if $(I-L)z_+ = G$, then from the above computation $\|G\|_{L^\infty([x_0,\infty))} < \infty$. We also have

$$\begin{aligned}
\|Lz_+\|_{L^\infty([x_0,\infty))} &\lesssim \left\| \frac{1}{W_+'(x)} \int_x^\infty |y-x| V(y) W_+'(y) \|z_+\|_{L^\infty}\,dy \right\|_{L^\infty([x_0,\infty))} \\
&\lesssim W_+'(x_0)\|z_+\|_{L^\infty([x_0,\infty))} \left\| \frac{1}{W_+'(x)} \int_x^\infty |y-x||V(y)|\,dy \right\|_{L^\infty([x_0,\infty))} \\
&\lesssim W_+'(x_0)\|z_+\|_{L^\infty([x_0,\infty))}.
\end{aligned}$$

As $|x_0| \to \infty$, we have $\|Lz_+\|_{L^\infty([x_0,\infty))} \ll \|z_+\|_{L^\infty([x_0,\infty))}$, so $(I-L)$ is invertible on $L^\infty([x_0,\infty))$. Thus z_+ exists in $L^\infty([x_0,\infty))$, hence m_+ also exists in $L^\infty([x_0,\infty))$.

We still need to work on $(-\infty, x_0)$. For $-1 \le x \le x_0$, we have

$$|z_+(\lambda,x)| \lesssim 1 + \int_x^{x_0} \langle y\rangle |V(y)||z_+(\lambda,y)|\,dy,$$

so by Gronwall's inequality, $|z_+(\lambda,x)| \lesssim 1$. For $x \le 0$, we can notice that

$$\frac{|z_+(\lambda,x)|}{\langle x\rangle} \lesssim 1 + \frac{1}{\langle x\rangle}\int_x^0 \langle x-y\rangle |V(y)| \frac{|z_+(\lambda,y)|}{\langle y\rangle}\,dy \lesssim 1 + \int_x^0 \langle y\rangle |V(y)| \frac{|z_+(\lambda,y)|}{\langle y\rangle}\,dy$$

Applying Gronwall's inequality again, we obtain $|z_+(\lambda,x)/\langle x\rangle| \lesssim 1$.[1] So we know the existence of f_+ and m_+, and a similar argument shows the existence of f_- and m_-. We can also obtain estimates for $\dot{m}_\pm$ and $\ddot{m}_\pm$ in a similar fashion. We continue the proof from here next time. □

Remark. Inserting the formula for m_+ again in the Volterra equation yields

$$\begin{aligned}
m_+(\lambda,x) &= 1 + \int_x^\infty D_\lambda(x-y)V(y)m_+(\lambda,y)\,dy \\
&= 1 + \int_x^\infty D_\lambda(x-y)V(y)\,dy + \int_x^\infty D_\lambda(x-y)V(y)\int_y^\infty D_\lambda(y-z)V(z)m_+(\lambda,z)\,dz\,dy \\
&= \dots.
\end{aligned}$$

Iterating this process (Volterra iteration) and bounding each term not involving m_+ gives more precise estimates (argue that one ends up with a summable series and get estimates from there).

[1] Note that one can also apply Gronwall's inequality for $x > 0$, but the resulting estimate $|z_+(\lambda,x)| \lesssim \langle x\rangle$ does not give precise asymptotic information for z_+ as $x \to \infty$.

Lecture 15

Mar. 3 — Dispersive Decay with Potential, Part 4

15.1 Low Energy Estimates, Continued

Lemma 15.1. *Suppose $W(0) \neq 0$. Then we have*

$$\sup_{x<y} \left| \int_{-\infty}^{\infty} e^{it\lambda^2} \frac{\lambda\chi(\lambda^2)}{W(\lambda)} f_+(\lambda, y) f_-(\lambda, x)\, d\lambda \right| \leq |t|^{-1/2}.$$

Proof. We first consider the case $x < 0 < y$. Then $f_+(\lambda, y) \to e^{iy\lambda}$ as $y \to \infty$ and $f_-(\lambda, x) \to e^{-ix\lambda}$ as $x \to -\infty$. So we can write

$$\begin{aligned} \left| \int_{-\infty}^{\infty} e^{it\lambda^2} e^{i\lambda(y-x)} \frac{\lambda\chi(x^2)}{W(\lambda)} m_+(\lambda, y) m_-(\lambda, x)\, d\lambda \right| &= \left| e^{i\partial_x^2 t} \left(\frac{\lambda\chi(\lambda^2)}{W(\lambda)} m_+(\lambda, y) m_-(\lambda, x) \right)^{\vee} \right| \\ &\lesssim |t|^{-1/2} \left\| \left(\frac{\lambda\chi(\lambda^2)}{W(\lambda)} m_+(\lambda, y) m_-(\lambda, x) \right)^{\vee} \right\|_{L^1}. \end{aligned}$$

Let τ be the dual variable of λ, and write the inner function as $G(x, y; \tau)$, where x, y are fixed. Then we would like to bound $\|G(x, y; \tau)\|_{L^1_\tau}$, for which it suffices to show $|G(x, y; \tau)| \lesssim \langle \tau \rangle^{-2}$. By Fourier duality, it suffices to bound the derivatives of G in λ:

$$\partial_\lambda^2 \left(\frac{\lambda\chi(\lambda^2)}{W(\lambda)} m_+(\lambda, y) m_-(\lambda, x) \right), \quad \partial_\lambda \left(\frac{\lambda\chi(\lambda^2)}{W(\lambda)} m_+(\lambda, y) m_-(\lambda, x) \right), \quad \left(\frac{\lambda\chi(\lambda^2)}{W(\lambda)} m_+(\lambda, y) m_-(\lambda, x) \right).$$

Recalling that χ is compactly supported, it is sufficent to bound the L^∞ norms of the above quantities. So we just need to show that $\partial_\lambda^j W(\lambda)$ and $\partial_\lambda^j m_\pm$ are bounded for $j = 0, 1, 2$. Recall that

$$W(\lambda) = W[f_+(\lambda, x), f_-(\lambda, x)] = [e^{i\lambda x} m_+(\lambda, x), e^{-i\lambda x} m_-(\lambda, x)] = [m_+(\lambda, 0), m_-(\lambda, 0)],$$

where we assumed $W(0) \neq 0$. Note that the Wronskian is independent of the point at which we evaluate, so that $W(\lambda)$ does not depend on x. We previously estimated $\partial_\lambda^j m_\pm$ for $x \geq 0$ and $j = 0, 1, 2$, so we have L^∞ bounds for $\partial_\lambda^j W(\lambda)$ and $\partial_\lambda^j m_\pm$ for $j = 0, 1, 2$. This proves pointwise decay in this case.

Now we consider the case $0 \leq x < y$. We need to analyze $f_+(\lambda, y) f_-(\lambda, x)$ (note that $f_-(\lambda, x)$ behaves badly as $x \to \infty$, we only know that $f_-(\lambda, x)$ might grow linearly as $x \to \infty$). To deal with this, we use some 1-D scattering theory. Note that $f_\pm(\lambda, x)$ solve the ODE $-\partial_x^2 f_\pm + V f_\pm = \lambda^2 f$, so we can write

$$f_\pm(\lambda, x) = \alpha_\pm(\lambda) f_\mp f(\lambda, x) + \beta_\pm(\lambda) f_\mp(-\lambda, x)$$

for $\lambda \neq 0$. Since $f_-(\lambda, x)$ and $f_-(-\lambda, x)$ are linearly independent ($W[f_-(\lambda, x), f_-(-\lambda, x)] = 2i\lambda$ and $\lambda \neq 0$). Since $0 \leq x < y$, we can use this to write

$$\begin{aligned} f_+(\lambda, y) f_-(\lambda, x) &= f_+(\lambda, y)(\alpha_-(\lambda) f_+(\lambda, x) + \beta_-(\lambda) f_+(-\lambda, x)) \\ &= \alpha_-(\lambda) f_+(\lambda, y) f_+(\lambda, x) + \beta_-(\lambda) f_+(\lambda, y) f_+(-\lambda, x), \end{aligned}$$

and we can now make the same argument as in the previous case to get the decay estimate.

In the final case $x < y \leq 0$, the idea is the same as when $x_0 \leq x < y$: We can write $f_+(\lambda, y)$ as a linear combination of $f_-(-\lambda, y)$ and $f_-(\lambda, y)$ and proceed as before. □

Remark. This completes the proof of the low energy part.

Remark. Consider the two *scattering relations* from above

$$\begin{aligned} f_+(\lambda, x) &= \alpha_+(\lambda) f_-(\lambda, x) + \beta_+(\lambda) f_-(-\lambda, x) \\ f_-(\lambda, x) &= \alpha_-(\lambda) f_+(\lambda, x) + \beta_-(\lambda) f_+(-\lambda, x). \end{aligned}$$

We can rewrite the first equation as

$$\frac{1}{\beta_+(\lambda)} f_+(\lambda, x) = \frac{\alpha_+(\lambda)}{\beta_+(\lambda)} f_-(\lambda, x) + f_-(-\lambda, x),$$

and we find that

$$\begin{aligned} \frac{1}{\beta_+(\lambda)} f_+(\lambda, x) &\xrightarrow[x \to +\infty]{} \frac{1}{\beta_+(\lambda)} e^{i\lambda x} \\ \frac{\alpha_+(\lambda)}{\beta_+(\lambda)} f_-(\lambda, x) &\xrightarrow[x \to -\infty]{} \frac{\alpha_+(\lambda)}{\beta_+(\lambda)} e^{-i\lambda x} \\ f_-(-\lambda, x) &\xrightarrow[x \to -\infty]{} e^{i\lambda x}. \end{aligned}$$

We call $T(\lambda) = 1/\beta_+(\lambda)$ the *transmission coefficient* and $R_+(\lambda) = \alpha_+(\lambda)/\beta_+(\lambda)$ the *reflection coefficient*. One also has the relation $|T(\lambda)|^2 + |R_+(\lambda)|^2 = 1$.

15.2 Distorted Fourier Transforms

Remark. We give an alternate approach to proving the low energy estimates via "distorted Fourier transforms." Recall that the usual Fourier transform diagonalizes differentiation. The "distorted Fourier transform" $\widetilde{\mathcal{F}}$ will diagonalize $-\partial_x^2 + V$, i.e. it will satisfy

$$\widetilde{\mathcal{F}}((-\partial_x^2 + V)g)(\xi) = \xi^2 \widetilde{\mathcal{F}}(g).$$

In Fourier space, the scattering relations from before become:

$$\begin{aligned} T(\xi) f_+(x, \xi) &= f_-(x, -\xi) + R_-(\xi) f_-(x, \xi) \\ T(\xi) f_-(x, \xi) &= f_+(x, -\xi) + R_+(\xi) f_+(x, \xi) \end{aligned}$$

Recall that for $H = -\partial_x^2 + V$, we have the spectral resolution

$$(H - (\xi^2 + i0))^{-1}(x, y) = R(\xi^2 + i0)(x, y) = \frac{f_+(x, \xi) f_-(y, \xi) \mathbb{1}_{\{x \geq y\}} + f_-(x, \xi) f_+(y, \xi) \mathbb{1}_{\{x < y\}}}{W[f_+(\cdot, \xi), f_-(\cdot, \xi)]}$$

for $x, y \in \mathbb{R}$ and $\xi \in \mathbb{R}$.

Lemma 15.2. *The density of the spectral resolution $E(d\xi^2)$ on $[0,\infty)$ can be written as*

$$\frac{E(d\xi^2)}{d\xi}(x,y) = \frac{|T(\xi)|^2}{2\pi}[f_+(x,\xi)f_+(y,-\xi) + f_-(x,\xi)f_-(y,-\xi)].$$

Alternatively, it can also be written as

$$\frac{E(d\xi^2)}{d\xi}(x,y) = \frac{1}{\pi}\begin{cases} \mathrm{Re}[T(\xi)f_+(x,\xi)f_-(y,\xi)], & \text{if } x \geq y, \\ \mathrm{Re}[T(\xi)f_+(y,\xi)f_-(x,\xi)], & \text{if } x \geq y. \end{cases}$$

Proof. Recall that (make a change of variables $\lambda = \xi^2$ in the usual formula for $E(d\lambda)$)

$$\frac{E(d\xi^2)}{d\xi}(x,y) = \frac{\xi}{\pi i}[R(\xi^2 + i0) - R(\xi^2 - i0)](x,y).$$

Then we can write

$$\frac{E(d\xi^2)}{d\xi}(x,y) = \frac{\xi}{2\pi}[W(\xi)^{-1}f_+(x,\xi)f_-(y,\xi) - W(-\xi)^{-1}f_+(x,-\xi)f_-(y,-\xi)].$$

Using the scattering relations, we have $T(\xi)W(\xi) = -2\xi i$, so we have

$$\frac{E(d\xi^2)}{d\xi}(x,y) = \frac{1}{2\pi}(T(\xi)f_+(x,\xi)f_-(y,\xi) + T(-\xi)f_+(x,-\xi)f_-(y,-\xi)).$$

Write $f_-(y,\xi)$ in the first term using $f_+(y,\xi)$ and $f_-(y,-\xi)$, and write $f_+(x,-\xi)$ in the second term as $f_-(x,\xi)$ and $f_+(x,\xi)$ to obtain (note that $T(-\xi) = T(\xi)$ and $R_\pm(-\xi) = \overline{R_\pm(\xi)}$)

$$\begin{aligned}\frac{E(d\xi^2)}{d\xi}(x,y) &= \frac{1}{2\pi}[T(\xi)f_+(x,\xi)(T(-\xi)f_+(y,-\xi) - R_-(-\xi)f_-(y,\xi)) \\ &\quad + T(-\xi)(T(\xi)f_-(x,\xi) - R_+(\xi)f_+(x,\xi))f_-(y,-\xi)] \\ &= \frac{|T(\xi)|^2}{2\pi}[f_+(x,\xi)f_+(y,-\xi) + f_-(x,\xi)f_-(y,-\xi)].\end{aligned}$$

This proves the case $x > y$, and $x < y$ follows similarly. □

Definition 15.1. Define the *distorted Fourier basis*

$$e(x,\xi) = \frac{1}{\sqrt{2\pi}}\begin{cases} T(\xi)f_+(x,\xi) & \text{if } \xi \geq 0, \\ T(-\xi)f_-(x,-\xi) & \text{if } \xi < 0. \end{cases}$$