

# MATH 8803: Nonlinear Dispersive Equations

Frank Qiang  
Instructor: Gong Chen

Georgia Institute of Technology  
Spring 2025

# Contents

<b>1</b>	<b>Jan. 6 — Introduction to Dispersion</b>	<b>2</b>
1.1	Introduction to Dispersion . . . . .	2
1.2	Fourier Transform and the Free Schrödinger Equation . . . . .	3
1.3	Sobolev Spaces . . . . .	4
<b>2</b>	<b>Jan. 8 — Special Solutions</b>	<b>5</b>
2.1	Special Solutions . . . . .	5
<b>3</b>	<b>Jan. 15 — Strichartz Estimates</b>	<b>8</b>
3.1	Interpolation Results . . . . .	8
3.2	Strichartz Estimates . . . . .	8
<b>4</b>	<b>Jan. 22 — Strichartz Estimates, Part 2</b>	<b>11</b>
4.1	Proof of Strichartz Estimates . . . . .	11
4.2	Strichartz Estimates and Harmonic Analysis . . . . .	12
<b>5</b>	<b>Jan. 27 — Kato Smoothing</b>	<b>14</b>
5.1	Kato Smoothing . . . . .	14
5.2	Coarea Formula . . . . .	15
<b>6</b>	<b>Jan. 29 — Kato Smoothing, Part 2</b>	<b>17</b>
6.1	Proof of Kato Smoothing . . . . .	17
<b>7</b>	<b>Feb. 3 — Kato Smoothing, Part 3</b>	<b>19</b>
7.1	Sharpness of Kato Smoothing . . . . .	19
7.2	Schrödinger Equation with Potential . . . . .	20
<b>8</b>	<b>Feb. 5 — Schrödinger Equation with Potential</b>	<b>22</b>
8.1	Schrödinger Equation with Potential, Continued . . . . .	22
<b>9</b>	<b>Feb. 10 — Spectral Theory</b>	<b>25</b>
9.1	Review of Functional Analysis . . . . .	25
9.2	Spectral Theory . . . . .	27
<b>10</b>	<b>Feb. 12 — Spectral Theory, Part 2</b>	<b>28</b>
10.1	Examples for Spectral Theory . . . . .	28
10.2	More Spectral Theory . . . . .	30

# Lecture 1

## Jan. 6 — Introduction to Dispersion

### 1.1 Introduction to Dispersion

**Definition 1.1.** An evolution equation is *dispersive* if when no boundary conditions are imposed (e.g. on  $\mathbb{R}^n$ ), its wave solutions spread out in space as they evolve in time.

**Example 1.1.1.** Two classic examples of dispersive equations are:

- The *Schrödinger equation*:  $iu_t + \Delta u = 0$ .
- The *Airy (linearized KdV) equation*:  $u_t + u_{xxx} = 0$ .

**Remark.** Consider the equation  $u_t + p(\partial_x)u = 0$ , where  $p$  is a polynomial, and a plane-wave solution

$$u(t, x) = e^{i(kx - \omega t)} = e^{ik(x - (\omega/k)t)}.$$

Here  $k$  is the *wave number* or *space frequency*, and  $\omega$  is the *(time) frequency*. Plugging the plane-wave solution into the equation, we obtain the relation  $\omega(k) = -ip(ik)$ , i.e.

$$\frac{\omega(k)}{k} = \frac{1}{ik}p(ik).$$

The above equation is known as the *dispersive relation*. This gives the traveling speed of the plane-wave solution with wave number  $k$ , which is called the *phase velocity*.

**Example 1.1.2.** The following are some examples of dispersive relations:

- For the *linear advection equation*  $u_t + cu_x = 0$  with  $c \in \mathbb{R}$ , one can compute that  $\omega/k = c$ .
- For the Schrödinger equation  $iu_t + \frac{1}{2}\Delta u = 0$ , we have  $\omega/k = k/2 \in \mathbb{R}$ .

In this case of the Schrödinger equation, plane waves with large wave number (large space frequency) travel faster than low-frequency waves.

**Remark.** In general, dispersion means that different frequency plane waves travel at different speeds.

**Remark.** Given initial data  $u_0$ , we can write using the Fourier transform that

$$u_0 = \int \widehat{u}_0(k) e^{ikx} dk.$$

Then we get the solution  $u$  as

$$u(t, x) = \int \widehat{u}_0(k) e^{ik(x - (\omega(k)/k)t)} dk.$$

**Example 1.1.3.** In the case of the linear advection equation, we obtain the solution as

$$u(t, x) = \int \widehat{u}_0(k) e^{ik(x-ct)} dk = u_0(x - ct).$$

For the Schrödinger equation, we instead have the solution

$$u(t, x) = \int \widehat{u}_0(k) e^{ik(x-(k/2)t)} dk.$$

Since different  $k$  travels at different speeds, the original profile quickly spreads out.

**Exercise 1.1.** Calculate the dispersive relation  $\omega/k$  for the linearized KdV equation  $u_t + u_{xxx} = 0$ .

**Example 1.1.4.** The *KdV equation* is given by

$$\partial_t u + \partial_{xxx} u + 6u \partial_x u = 0.$$

This equation is used to model shallow water surfaces, and is a nonlinear dispersive equation. Russell observed a great bump of water in a channel that traveled for a long time and kept its shape. This is due to the nonlinear effects in the KdV equation, and these effects are called *solitons*.

**Definition 1.2.** A *soliton* is a self-reinforcing solitary wave (a wave packet or pulse) that maintains its shape while traveling at a constant speed.

## 1.2 Fourier Transform and the Free Schrödinger Equation

Consider the following free Schrödinger equation:

$$\begin{cases} i\partial_t \psi + \frac{1}{2}\Delta \psi = 0, \\ \psi|_{t=0} = \psi_0. \end{cases}$$

We will solve this equation using the *Fourier transform*

$$\widehat{f}(\xi) = \int_{\mathbb{R}^d} e^{-ix \cdot \xi} f(x) dx.$$

Note that one can recover  $f$  from its Fourier transform via the *inversion formula*

$$f(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{ix \cdot \xi} \widehat{f}(\xi) d\xi.$$

**Exercise 1.2.** Check that  $(\partial_{x_j} f)^\wedge = i\xi_j \widehat{f}$ .

Applying the Fourier transform to the free Schrödinger equation, one has

$$i\partial_t \psi + \frac{1}{2}\Delta \psi = 0 \quad \xrightarrow{\text{F.T.}} \quad i\partial_t \widehat{\psi} - \frac{1}{2}|\xi|^2 \widehat{\psi} = 0$$

and initial condition  $\widehat{\psi}(0, \xi) = \widehat{\psi}_0(\xi)$ . So for fixed  $\xi$ , we have an ODE, so we can solve the equation via

$$\widehat{\psi}(t, \xi) = e^{-i|\xi|^2 t/2} \widehat{\psi}_0(\xi).$$

Now by applying the inverse Fourier transform, we obtain the solution

$$\psi(t, x) = (2\pi)^{-d} \int e^{ix\xi} \widehat{\psi}(t, \xi) d\xi = (2\pi)^{-d} \int e^{ix\xi} e^{-i|\xi|^2 t/2} \widehat{\psi}_0(\xi) d\xi.$$

Recalling Plancherel's theorem that  $\|f\|_{L^2} = C\|\widehat{f}\|_{L^2}$  (for a constant  $C$  independent of  $f$ ), we obtain

$$\|\psi(t, x)\|_{L^2} = C\|\widehat{\psi}(t, \xi)\|_{L^2} = C\|\widehat{\psi}(0, \xi)\|_{L^2} = \|\psi(0, x)\|_{L^2} = \|\psi_0(x)\|_{L^2},$$

where the second equality follows by noticing that  $e^{-i|\xi|^2 t/2}$  has modulus 1. This shows that the linear Schrödinger evolution preserves the  $L^2$  norm of the solution.

**Exercise 1.3.** Compute that

$$\frac{d}{dt} \int_{\mathbb{R}^d} |\psi(t, x)|^2 dx = 0.$$

This is an alternative way to show that the  $L^2$  norm of the solution is preserved.

## 1.3 Sobolev Spaces

**Definition 1.3.** The *Sobolev spaces*  $H^\gamma = W^{\gamma,2}$  for  $\gamma \in \mathbb{R}$  are defined via the norm

$$\|f\|_{H^\gamma} = \left( \int_{\mathbb{R}^d} (1 + |\xi|^2)^\gamma |\widehat{f}(\xi)|^2 d\xi \right)^{1/2}.$$

The *homogeneous Sobolev spaces*  $\dot{H}^\gamma$  are defined by the norm

$$\|f\|_{\dot{H}^\gamma} = \left( \int_{\mathbb{R}^d} |\xi|^{2\gamma} |\widehat{f}(\xi)|^2 d\xi \right)^{1/2}.$$

**Remark.** If  $\gamma \in \mathbb{N}$  and  $d = 1$ , then

$$\|f\|_{H^\gamma} \sim \sum_{m=0}^{\gamma} \|\partial_x^m f\|_{L^2}.$$

In particular, this means that  $f \in H^\gamma$  if and only if  $\partial_x^m f \in L^2$  for all  $m \leq \gamma$ .

**Exercise 1.4.** Check that if  $f_\lambda(x) = f(\lambda x)$ , then  $\widehat{f}_\lambda(\xi) = \lambda^{-d} \widehat{f}(\xi/\lambda)$ .

**Remark.** In the Sobolev spaces, this means that (change variables  $\eta = \xi/\lambda$  for the last equality)

$$\|f_\lambda\|_{\dot{H}^\gamma} = \left( \int_{\mathbb{R}^d} |\xi|^{2\gamma} |\widehat{f}_\lambda(\xi)|^2 d\xi \right)^{1/2} = \left( \int_{\mathbb{R}^d} |\xi|^{2\gamma} |\lambda^{-d} \widehat{f}(\xi/\lambda)|^2 d\xi \right)^{1/2} = \lambda^{\gamma-d/2} \|f\|_{\dot{H}^\gamma}.$$

**Lemma 1.1.** In the Schrödinger equation,  $\|\psi(t)\|_{H^\gamma} = \|\psi_0\|_{H^\gamma}$  and  $\|\psi(t)\|_{\dot{H}^\gamma} = \|\psi_0\|_{\dot{H}^\gamma}$  for all  $t$  and  $\gamma$ .

*Proof.* We can compute that

$$\|\psi(t)\|_{\dot{H}^\gamma}^2 = \int_{\mathbb{R}^d} |\xi|^{2\gamma} |\widehat{\psi}(t, \xi)|^2 d\xi = \int_{\mathbb{R}^d} |\xi|^{2\gamma} |e^{-i|\xi|^2 t/2} \widehat{\psi}_0(\xi)|^2 d\xi = \int_{\mathbb{R}^d} |\xi|^{2\gamma} |\widehat{\psi}_0(\xi)|^2 d\xi = \|\psi_0\|_{\dot{H}^\gamma}^2.$$

The same argument works for the  $H^\gamma$  case after replacing  $|\xi|^{2\gamma}$  with  $(1 + |\xi|^2)^\gamma$ . □

# Lecture 2

## Jan. 8 — Special Solutions

### 2.1 Special Solutions

**Example 2.0.1.** The following are special solutions to the Schrödinger equation:

1. Gaussian:  $\psi_0 = e^{-|x|^2/2}$ . One can compute the Fourier transform and get

$$\widehat{\psi}_0(\xi) = \int_{\mathbb{R}^d} e^{-ix \cdot \xi} e^{-|x|^2/2} dx = \int_{\mathbb{R}^d} e^{-|x+i\xi|^2/2} e^{-|\xi|^2/2} dx = e^{-|\xi|^2/2} \int_{\mathbb{R}^d} e^{-|x+i\xi|^2/2} dx.$$

The last integral is a contour integral in the complex plane along  $\Im z = \xi$ , and we can deform the contour via Cauchy's theorem to the real axis to obtain (the integrand is analytic on  $0 \leq \Im z \leq \xi$ )

$$\widehat{\psi}_0(\xi) = e^{-|\xi|^2/2} \int_{\mathbb{R}^d} e^{-|x|^2/2} dx = (2\pi)^{d/2} e^{-|\xi|^2/2}.$$

Then taking inverse Fourier transforms, we obtain the solution

$$\begin{aligned} \psi(t, x) &= (2\pi)^{-d} \int_{\mathbb{R}^d} e^{i(x \cdot \xi - |\xi|^2 t/2)} \widehat{\psi}_0(\xi) d\xi = (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{i(x \cdot \xi - |\xi|^2 t/2)} e^{-|\xi|^2/2} d\xi \\ &= (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{-\frac{1}{2}(1+it)|\xi|^2} e^{ix \cdot \xi} d\xi. \end{aligned}$$

Now formally put  $\eta = (1+it)^{1/2} \xi$  to get

$$\psi(t, x) = (2\pi)^{-d/2} (1+it)^{-d/2} \int_{\mathbb{R}^d} e^{-\frac{1}{2}|\eta|^2} e^{ix\eta/(1+it)^{1/2}} d\eta.$$

Fill in the details of the above change of variables as an exercise (e.g. one has to worry about choosing a branch cut when taking the square root). Computing the integral explicitly, one obtains

$$\psi(t, x) = (1+it)^{-d/2} e^{-|x|^2/(2(1+it))}.$$

One can from this that  $\psi$  has decay in time. Furthermore, one can see that

$$|\psi(t, x)|^2 = (1+t^2)^{-d/2} e^{-|x|^2/(1+t^2)}.$$

From this we can observe an  $L^\infty$  decay of  $\psi$  like  $t^{-d/2}$ , and that the influence region of the solution grows like order  $t$ . We can also see again from this explicit computation that  $\|\psi(t)\|_{L^2} = C$ .

2. Modulated Gaussian:  $\psi_0 = e^{-|x|^2/2} e^{ix \cdot v}$ . The Fourier transform of this initial data is

$$\widehat{\psi}_0(\xi) = (2\pi)^{d/2} e^{-i|\xi-v|^2/2}.$$

So the solution corresponding to this initial data is

$$\begin{aligned} \psi(t, x) &= (2\pi)^{-d} \int_{\mathbb{R}^d} e^{i(x \cdot \xi - |\xi|^2 t/2)} \widehat{\psi}_0(\xi) d\xi = (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{i(x \cdot \xi - |\xi|^2 t/2)} e^{-|\xi-v|^2/2} d\xi \\ &= e^{ix \cdot v} e^{-|v|^2 t/2} (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{i(x-vt) \cdot \xi} e^{-(1+it)|\xi|^2/2} d\xi \\ &= e^{ix \cdot v} e^{-|v|^2 t/2} (1+it)^{-d/2} \exp\left(-\frac{|x-vt|^2}{2(1+it)}\right). \end{aligned}$$

From this we can see that the influence region of the solution moves with velocity  $v$ .

3. Fundamental solution: We want a *fundamental solution*  $K$  such that  $K$  solves

$$i\partial_t K + \frac{1}{2}\Delta K = 0 \quad \text{and} \quad K|_{t=0} = \delta_0.$$

We will find  $K$  by scaling arguments. Suppose such a  $K$  exists. Then we must have

$$\psi(t, x) = \int_{\mathbb{R}^d} K(t, x-y) \psi_0(y) dy \tag{1}$$

since  $K|_{t=0} = \delta_0$ . Now define the scaling  $\psi_\lambda(t, x) = \psi(\lambda^2 t, \lambda x)$ . Then  $\psi_\lambda$  also solves

$$i\partial_t \psi_\lambda + \frac{1}{2}\Delta \psi_\lambda = 0$$

and we have the initial condition  $\psi_\lambda(0, x) = \psi_0(\lambda x)$ . Then

$$\psi_\lambda(t, x) = \int_{\mathbb{R}^d} K(t, x-y) \psi_0(\lambda y) dy = \psi(\lambda^2 t, \lambda x).$$

Setting  $t' = \lambda^2 t$ ,  $x' = \lambda x$ , and  $y' = \lambda y$ , we get

$$\psi(t', x') = \frac{1}{\lambda^d} \int_{\mathbb{R}^d} K\left(\frac{t'}{\lambda^2}, \frac{x' - y'}{\lambda}\right) \psi_0(y') dy'. \tag{2}$$

Comparing (1) and (2), we see that we must have

$$K(t, x-y) = \lambda^{-d} K\left(\frac{t}{\lambda^2}, \frac{x-y}{\lambda}\right).$$

Setting  $u = x - y$ , we get

$$K(t, u) = \lambda^{-d} K\left(\frac{t}{\lambda^2}, \frac{u}{\lambda}\right).$$

Thus we expect  $K(t, x) = t^{-d/2} \Phi(|x|^2/t)$  for some  $\Phi$ . Now we use the fact that  $i\partial_t K + \frac{1}{2}\Delta K = 0$ . Setting  $m = |x|^2/t$ , one can plug in the above guess for  $K$  to obtain (note that  $\Delta = \nabla \cdot \nabla$ )

$$-\frac{id}{2} t^{-d/2-1} \Phi(m) - it^{-d/2} \Phi'(m) \frac{m}{t} + \frac{1}{2} t^{-d/2} \nabla \cdot \left( \frac{2x}{t} \Phi'(m) \right) = 0.$$

Then we get

$$-i\frac{d}{2}\Phi(m) - im\Phi'(m) + d\Phi'(m) + 2m\Phi''(m) = 0,$$

which gives

$$d\left(\Phi'(m) - \frac{i}{2}\Phi(m)\right) + 2m\frac{d}{dm}\left(\Phi'(m) - \frac{i}{2}\Phi(m)\right) = 0.$$

Now observe that  $\Phi(m) = e^{im/2}$  solves the above equation. Since  $\Phi(m)$  solves the equation,  $c\Phi(m)$  also solves the equation for any  $c \in \mathbb{C}$ , and thus we have

$$K(t, x) = ct^{-d/2}\Phi(|x|^2/t) = ct^{-d/2}e^{i|x|^2/2t}.$$

To determine  $c$ , we use  $K|_{t=0} = \delta_0$ , from which one can obtain  $c = (2\pi i)^{-d/2}$ . Thus

$$K(t, x) = (2\pi it)^{-d/2}e^{i|x|^2/2t}.$$

The rough computation is that since  $\widehat{K}(0, \xi) = 1$ , we have

$$K = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{ix \cdot \xi - |\xi|^2 t/2} \widehat{K}(0, \xi) d\xi = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{i(x \cdot \xi - |\xi|^2 t/2)} d\xi.$$

This is not necessarily integrable a priori, but one can take limits and obtain

$$\begin{aligned} K &= (2\pi)^{-d} \lim_{\epsilon \rightarrow 0^+} \int_{\mathbb{R}^d} e^{ix \cdot \xi} e^{-(\epsilon + it)|\xi|^2/2} d\xi = \lim_{\epsilon \rightarrow 0^+} (\epsilon + it)^{-d/2} (2\pi)^{-d/2} e^{-|x|^2/(2(\epsilon + it))} \\ &= (2\pi it)^{-d/2} e^{-|x|^2/2it}. \end{aligned}$$

Note that this computation matches the result of the previous scaling argument.

**Theorem 2.1.** *Let  $\psi_0 \in \mathcal{S}(\mathbb{R}^d)$ .<sup>1</sup> Then there exists a solution to*

$$\begin{cases} i\partial_t \psi + \frac{1}{2}\Delta \psi = 0, \\ \psi|_{t=0} = \psi_0, \end{cases}$$

*which is unique and given by*

$$\psi(t, x) = \int_{\mathbb{R}^d} K(t, x - y) \psi_0(y) dy = (2\pi it)^{-d/2} \int_{\mathbb{R}^d} e^{-|x-y|^2/2it} \psi_0(y) dy.$$

*Proof.* This theorem is a summary of the results of the previous explicit computations. □

**Remark.** Recall that the Schrödinger evolution preserves the  $L^2$  norm of a solution, i.e.

$$\|\psi(t)\|_{L^2} = \|\psi(0)\|_{L^2} = \|\psi_0\|_{L^2}.$$

The above theorem also gives an  $L^\infty$  bound (a so-called *dispersive estimate*)

$$\|\psi(t)\|_{L^\infty} \leq |2\pi t|^{-d/2} \int_{\mathbb{R}^d} |\psi_0(y)| dy = |2\pi t|^{-d/2} \|\psi_0\|_{L^1}.$$

---

<sup>1</sup>Here  $\mathcal{S}(\mathbb{R}^d)$  is the space of *Schwartz functions*.



# Lecture 3

## Jan. 15 — Strichartz Estimates

### 3.1 Interpolation Results

**Remark** (Interpolation). Consider a linear operator  $T$  which maps  $T : L^{p_1} \rightarrow L^{q_1}$  and  $T : L^{p_2} \rightarrow L^{q_2}$ , where  $1 \leq p_1 \leq p_2 \leq \infty$ . Then  $T$  also maps  $T : L^p \rightarrow L^q$  for any  $p, q$  such that

$$\frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2} \quad \text{and} \quad \frac{1}{q} = \frac{\theta}{q_1} + \frac{1-\theta}{q_2}$$

for some  $0 \leq \theta \leq 1$ . More specifically, if  $\|Tf\|_{L^{q_1}} \leq C_1\|f\|_{L^{p_1}}$  and  $\|Tf\|_{L^{q_2}} \leq C_2\|f\|_{L^{p_2}}$ , then

$$\|Tf\|_{L^q} \leq C_1^\theta C_2^{1-\theta} \|f\|_{L^p}.$$

This  $L^p$  interpolation is a result from real and functional analysis. Note that by interpolation, we have

$$\|\psi\|_{L^{p'}(\mathbb{R}^d)} \leq C|t|^{-d(1/p-1/2)} \|\psi_0\|_{L^p(\mathbb{R}^d)}$$

for  $1 \leq p \leq 2$ , where  $p'$  is the *Hölder conjugate* of  $p$ , i.e.  $1/p' + 1/p = 1$ .

### 3.2 Strichartz Estimates

**Remark.** We will now consider the inhomogeneous Schrödinger equation:

$$\begin{cases} i\psi_t + \frac{1}{2}\Delta\psi = F, & F \in \mathcal{S}_{x,t} \\ \psi(0) = \psi_0, & \psi_0 \in \mathcal{S}, \end{cases}$$

where  $F \in \mathcal{S}_{x,t}$  means that  $F$  is Schwartz in both  $x$  and  $t$ . We can solve this via the *Duhamel formula*:

$$\psi(t) = e^{it\Delta/2}\psi_0 - i \int_0^t e^{i(t-s)\Delta/2} F(s) ds,$$

where  $e^{it\Delta/2}$  is the *linear propagator* given by

$$e^{it\Delta/2}\psi_0 = (e^{-it|\xi|^2/2}\widehat{\psi_0})^\vee = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{ix\cdot\xi} e^{-it|\xi|^2/2} \widehat{\psi_0}(\xi) d\xi.$$

**Theorem 3.1** (Strichartz estimates). *For  $p' = 2 + 4/d$  and  $1/p' + 1/p = 1$ , we have the estimate<sup>1</sup>*

$$\|\psi\|_{L_{t,x}^{p'}(\mathbb{R} \times \mathbb{R}^d)} \lesssim \|\psi_0\|_{L_x^2(\mathbb{R}^d)} + \|F\|_{L_{t,x}^p(\mathbb{R} \times \mathbb{R}^d)}.$$

---

<sup>1</sup>Here  $A \lesssim B$  means that  $A \leq CB$  for some prescribed constant  $C$ .

**Remark.** If  $F = 0$ , this is the bound

$$\|\psi\|_{L_{t,x}^{p'}} \lesssim \|\psi_0\|_{L^2}$$

for  $p' > 2$ . Formally, this means that we gain integrability in  $x$ . Note that this gain in integrability is not pointwise in time, i.e. we do *not* have  $\|\psi(t)\|_{L_t^\infty L_x^{p'}} \lesssim \|\psi_0\|_{L_x^2}$ . We must instead average over  $t$ .

**Remark.** Why  $p'$  and why do we pick  $p'$  in the time integration? Actually,  $p'$  is the only possible choice for the above result. This follows by a scaling argument: Set

$$\psi_\lambda(t, x) = \psi(\lambda^2 t, \lambda x), \quad (\psi_\lambda)_0(x) = \psi_0(\lambda x), \quad F_\lambda(t, x) = \lambda^2 F(\lambda^2 t, \lambda x).$$

Then  $\psi_\lambda$  solves the equation

$$\begin{cases} i\partial_t \psi_\lambda + \frac{1}{2}\Delta \psi_\lambda = F_\lambda, \\ \psi_\lambda(0) = (\psi_\lambda)_0. \end{cases}$$

If the above theorem makes sense, then it must hold for both  $\psi_\lambda$  and  $\psi$ . Now

$$\|\psi_\lambda\|_{L_{t,x}^{p'}} = \lambda^{-d/p'} \lambda^{-2/p'} \|\psi\|_{L_{t,x}^{p'}}$$

by a change of variables, and

$$\|(\psi_\lambda)_0\|_{L_x^2} = \lambda^{-d/2} \|\psi_0\|_{L_x^2}.$$

Now if  $F = 0$ , then we have the estimates

$$\|\psi\|_{L_{t,x}^{p'}} \lesssim \|\psi_0\|_{L_x^2} \quad \text{and} \quad \|\psi_\lambda\|_{L_{t,x}^{p'}} \lesssim \|(\psi_\lambda)_0\|_{L_x^2}, \quad (*)$$

Using the scaling computations in the second estimate in  $(*)$  implies that

$$\|\psi\|_{L_{t,x}^{p'}} \lambda^{-d/p'} \lambda^{-2/p'} \lesssim \lambda^{-d/2} \|\psi_0\|_{L_x^2}.$$

This inequality should hold independent of  $\lambda$ , since otherwise taking  $\lambda \rightarrow \infty$  or  $\lambda \rightarrow 0$  yields a contradiction with the first inequality in  $(*)$ . Thus the powers in  $\lambda$  should match:

$$-\frac{d}{p'} - \frac{2}{p'} = -\frac{d}{2},$$

so we find that  $p'$  must be

$$p' = \frac{d+2}{d/2} + \frac{2d+4}{d} = 2 + \frac{4}{d}.$$

This uniquely determines  $p'$ . Now consider  $F \neq 0$ . Using a similar computation as before, we have

$$\|F_\lambda\|_{L_{t,x}^q} = \lambda^2 \lambda^{-d/q} \lambda^{-2/q} \|F\|_{L_{t,x}^q}.$$

Then the theorem says that  $\|\psi_\lambda\|_{L_{t,x}^{p'}} \lesssim \|\psi_0\|_{L_x^2} + \|F\|_{L_{t,x}^q}$ , so we have

$$\|\psi\|_{L_{t,x}^{p'}} \lambda^{-d/p'} \lambda^{-2/p'} \lesssim \lambda^{-d/2} \|\psi_0\|_{L_x^2} + \lambda^2 \lambda^{-d/q} \lambda^{-2/q} \|F\|_{L_{t,x}^q}.$$

Again the estimate should hold independent of  $\lambda$ , so the powers in  $\lambda$  must match:

$$-\frac{d}{p'} - \frac{2}{p'} = 2 - \frac{d}{q} - \frac{2}{q} = -\frac{d}{2},$$

which then gives  $p$  as

$$p = \left(1 - \frac{1}{p'}\right)^{-1} = \left(1 - \frac{d}{2d+4}\right)^{-1} = \frac{2d+4}{d+4}.$$

**Lemma 3.1.** *Let  $\psi(t) = e^{it\Delta/2}\psi_0$ . Then for  $1 \leq p \leq 2$ ,*

$$\|\psi(t)\|_{L_x^{p'}(\mathbb{R}^d)} \lesssim |t|^{-d(1/p-1/2)} \|\psi_0\|_{L_x^p(\mathbb{R}^d)}.$$

*Proof.* This is the interpolation result from the beginning of class. □

**Lemma 3.2** (Hardy-Littlewood-Sobolev inequality). *Let  $0 < \alpha < 1$  and  $g \in \mathcal{S}(\mathbb{R})$ . Let*

$$(T_\alpha g)(t) = \int_{-\infty}^{\infty} |t-s|^{-\alpha} g(s) ds.$$

*Then we have  $\|T_\alpha g\|_{L^q(\mathbb{R})} \lesssim \|g\|_{L^p(\mathbb{R})}$ , where  $1 < p < q < \infty$  such that  $1 + 1/q = \alpha + 1/p$ .*

*Proof.* One approach is via harmonic analysis and maximal functions. An alternative approach can be found in Theorem 4.3 of Analysis by Lieb and Loss. □

**Remark.** Recall *Young's inequality* that for

$$h(t) = \int f(t-s)g(s) dx,$$

we have  $\|h\|_{L^r} \leq \|f\|_{L^p} \|g\|_{L^q}$ , where  $1/r + 1 = 1/q + 1/p$ . The Hardy-Littlewood-Sobolev inequality can be seen as a generalized Young's inequality: If  $f(s) = |s|^{-\alpha}$ , then  $f$  barely fails to be in  $L^{1/\alpha}$ . Informally, we can think of " $f \in L^{1/\alpha}$ ," and the standard Young's inequality would imply Hardy-Littlewood-Sobolev.

**Remark.** We have  $q > p$  in the Hardy-Littlewood-Sobolev inequality, so we gain some integrability via fractional integration for  $p > 1$  (the type of integral defining  $T_\alpha g$  is known as *fractional integration*).

# Lecture 4

## Jan. 22 — Strichartz Estimates, Part 2

### 4.1 Proof of Strichartz Estimates

*Proof of Theorem 3.1.* The first step is a  $TT^*$  argument. Define the operator  $T$  by  $Tf = e^{it\Delta/2}f$ . We know that  $T : L_x^2(\mathbb{R}^d) \rightarrow L_t^\infty(\mathbb{R} \rightarrow L_x^2(\mathbb{R}^d))$ . The adjoint  $T^* : L_t^1 L_x^2 \rightarrow L_x^2$  is defined via the relation

$$\langle f, T^*G \rangle_{L_x^2} = \langle Tf, G \rangle_{L_{t,x}^2} = \iint (e^{it\Delta/2}f)(x) \overline{G}(t, x) dt dx = \int f(x) \int \overline{(e^{-it\Delta/2}G(t, \cdot))}(x) dt dx,$$

and so we have the formula

$$T^*G = \int (e^{-it\Delta/2}G(t, \cdot))(x) dt.$$

Then we can see that

$$(TT^*G)(t, x) = \int (e^{i(t-s)\Delta/2}G(s, \cdot))(x) ds = \left( e^{it\Delta/2} \int e^{-is\Delta/2}G(s, \cdot) ds \right)(x).$$

Note that there is a convolution structure in the time variable. Clearly  $TT^* : L_t^1 L_x^2 \rightarrow L_x^\infty L_x^2$ . Then the goal for now will be to show that

$$\|TT^*G\|_{L_{t,x}^{p'}} \leq C\|G\|_{L_{t,x}^p}.$$

To do this, observe that by the above expression for  $TT^*G$  and Minkowski's inequality, we have

$$\|TT^*G\|_{L_x^{p'}} \leq \int \|e^{i(t-s)\Delta/2}G(s)\|_{L_x^{p'}} ds \leq C \int |t-s|^{-d(1/p-1/2)} \|G(s)\|_{L_x^p} ds,$$

where the second inequality follows by Lemma 3.1. Now by Lemma 3.2,

$$\|TT^*G\|_{L_t^q L_x^{p'}} \leq C\|G(s)\|_{L_{t,x}^p}$$

if  $1/q + 1 = d(1/p - 1/2) + 1/p$  (also check that  $0 < \alpha = d(1/p - 1/2) < 1$ , where  $p = (2d+4)/(d+4)$ ). From this relation, we find that we must have  $q = 2 + 4/d = p'$ , so we have shown the goal.

Thus we have proved that  $TT^* : L_{t,x}^p \rightarrow L_{t,x}^{p'}$ , where  $p = (2d+4)/(d+4)$ . Now

$$\|T^*G\|_{L_x^2}^2 = \langle T^*G, T^*G \rangle_{L_x^2} = \langle TT^*G, G \rangle_{L_{t,x}^2} \leq \|TT^*G\|_{L_{t,x}^{p'}} \|G\|_{L_{t,x}^p} \leq C\|G\|_{L_{t,x}^p}^2,$$

where the first inequality follows by Hölder's inequality and the second follows from the goal we just proved. Thus we conclude that  $T^* : L_{t,x}^p \rightarrow L_x^2$ , and that  $T : L_x^2 \rightarrow L_{t,x}^{p'}$  by duality. Therefore,

$$\|T\psi_0\|_{L_{t,x}^{p'}} = \|e^{it\Delta/2}\psi_0\|_{L_{t,x}^{p'}} \leq C\|\psi_0\|_{L_x^2},$$

so we have proved the Strichartz estimates when  $F = 0$ .

In the case when  $F \neq 0$ , by Duhamel's formula we have

$$\psi(t) = e^{it\Delta/2}\psi_0 - i \int_0^t e^{i(t-s)\Delta/2} F(s) ds,$$

so by the triangle inequality we find that

$$\|\psi\|_{L_{t,x}^{p'}} \leq \|e^{it\Delta/2}\psi_0\|_{L_{t,x}^{p'}} + \left\| \int_0^t e^{i(t-s)\Delta/2} F(s) ds \right\|_{L_t^{p'}} \leq C\|\psi_0\|_{L_x^2} + \left\| \int_0^t \|e^{i(t-s)\Delta/2} F(s)\|_{L_x^{p'}} ds \right\|_{L_t^{p'}},$$

where the last inequality on the second term follows by Minkowski's inequality. Then using Lemma 3.1 and Lemma 3.2 in the same fashion as before, we can bound the latter term by

$$\left\| \int_0^t \|e^{i(t-s)\Delta/2} F(s)\|_{L_x^{p'}} ds \right\|_{L_t^{p'}} \leq C \left\| \int_{-\infty}^{\infty} |t-s|^{-d(1/p-1/2)} \|F(s)\|_{L_x^{p'}} ds \right\|_{L_t^{p'}} \leq C\|F\|_{L_{t,x}^p}.$$

Plugging this bound back in gives the desired inequality  $\|\psi\|_{L_{t,x}^{p'}} \leq C\|\psi_0\|_{L_x^2} + C\|F\|_{L_{t,x}^p}$ .  $\square$

**Remark.** Note that the term

$$TT^*G = \int_{-\infty}^{\infty} e^{i(t-s)\Delta/2} G(s) ds$$

looks similar to the term from the Duhamel formula

$$\int_0^t e^{i(t-s)\Delta/2} F(s) ds.$$

However, it is possible that these two have different estimates, which is why we had to argue separately.

## 4.2 Strichartz Estimates and Harmonic Analysis

**Remark.** The original intention of Strichartz for these estimates was for use in harmonic analysis. The Strichartz estimates can actually be derived from the *Stein-Tomas restriction theorem*.

Let  $S \subseteq \mathbb{R}^n$  with  $n = d + 1$ , where  $S$  is a hypersurface. If  $f \in L^1$ , then one can show (e.g. using the Riemann-Lebesgue lemma) that  $\widehat{f} \in L^\infty$ . So we can conclude that  $\widehat{f}$  has pointwise meaning, i.e.  $\widehat{f}(\xi)$  makes sense pointwise. In particular, we can make sense of  $\widehat{f}(\xi)$  on the hypersurface  $S$ .

On the other hand, if  $f \in L^2$ , then by Plancherel's theorem,  $\widehat{f} \in L^2$  as well. But an  $L^2$  function has no pointwise interpretation, i.e. we can modify it on a set of measure zero without changing the function. In particular, it is meaningless to restrict the function  $\widehat{f}$  to  $S$ , since  $S$  is a set of measure zero in  $\mathbb{R}^n$ .

In general, what about  $f \in L^p$  for  $1 < p < 2$ ? This is the topic of the *restriction theorems* in harmonic analysis. It turns out that the choice of which  $p$  work depends on the “curvature” of  $S$ .

**Theorem 4.1** (Stein-Tomas restriction theorem). *Let  $n = d + 1$  and  $S \subseteq \mathbb{R}^n$  be a hypersurface with non-vanishing Gaussian curvature. Let  $\sigma_S$  be the corresponding surface measure, and let  $\phi$  be a compactly supported on  $S$ . Then we have*

$$\|(\phi\sigma_S)^\vee\|_{L^r(\mathbb{R}^n)} \leq C\|\phi\|_{L^2(\sigma_S)},$$

where  $r = (2n + 2)/(n - 1)$ .

**Remark.** Now recall the explicit formula for a solution  $\psi$  to the Schrödinger equation:

$$\psi(t, s) = \int e^{i(x \cdot \xi - t|\xi|^2/2)} \widehat{\psi}_0(\xi) d\xi.$$

Also define the hypersurface  $S = \{(\xi, \tau) : \tau = -|\xi|^2/2, \xi \in \mathbb{R}^d\}$ .<sup>1</sup> Then

$$\psi(t, x) = (\phi \sigma_S)^\vee(t, x), \quad \phi(\xi, \tau) = \widehat{\psi}_0(\xi), \quad \sigma_S(d\xi, d\tau) = (2\pi)^d d\xi.$$

Indeed, we can see that

$$\psi(t, x) = (2\pi)^{-d} \int e^{i(x \cdot \xi + t\tau)} \phi(\xi, \tau) \sigma_S(d\xi, d\tau) = \int e^{i(x \cdot \xi - t|\xi|^2/2)} \widehat{\psi}_0(\xi) d\xi.$$

Then, the Stein-Tomas restriction theorem tells us that  $\|(\phi \sigma_S)^\vee\|_{L^r(\mathbb{R}^n)} \leq C \|\phi\|_{L^2(\sigma_S)}$ , which implies

$$\|\psi\|_{L_{t,x}^{2+4/d}(\mathbb{R} \times \mathbb{R}^d)} \leq C \|\widehat{\psi}_0\|_{L_\xi^2} \leq C \|\psi_0\|_{L_x^2},$$

where the last inequality follows by Plancherel's theorem. The estimate above technically only holds for those  $\psi$  where  $\widehat{\psi}_0$  is compactly supported, but one can extend this by a density argument to  $\psi_0 \in L_x^2$ .

See Chapter 11 of Muscalu-Schlag (Volume I) for more details.

---

<sup>1</sup>This particular hypersurface is called the *characteristic surface* of the Schrödinger equation.

# Lecture 5

## Jan. 27 — Kato Smoothing

### 5.1 Kato Smoothing

**Theorem 5.1** (Kato 1/2 smoothing estimate). *Let  $d \geq 2$  and  $\chi \geq 0$  be a smooth cutoff function such that  $\widehat{\chi}$  is compactly supported. Then we have the estimate*

$$\|\chi(x)(-\Delta)^{1/4}e^{it\Delta/2}f\|_{L^2_{x,t}} \leq C\|f\|_{L^2}.$$

**Remark.** The above theorem says that when we localize, we gain a smoothing effect (1/2 derivatives). Also note that this effect is not pointwise, but rather after integrating in time.

**Remark.** Let  $f(t) = e^{it\Delta/2}f_0$ , where  $f_0 = e^{-|x|^2/2}e^{-ixv}$ . Think of this as a quantum particle at the origin with an initial velocity  $v$ , so that the particle will stay in  $B(0,1)$  for a period of order  $O(1/|v|)$ . Then

$$\left( \int_0^{1/|v|} \int_{B(0,1)} |f|^2 dx dt \right)^{1/2} \sim \left( \frac{1}{|v|} \right)^{1/2} = \frac{1}{|v|^{1/2}}.$$

Now  $(-\Delta)^{1/2}f \sim |v|^{1/2}$ , so these factors cancel each other out. This matches the above estimate.

**Theorem 5.2** (1-D Kato smoothing estimate). *For  $d = 1$ , we have*

$$\sup_x \|(-\Delta)^{1/4}e^{it\Delta/2}f\|_{L^2_t} \leq C\|f\|_{L^2}.$$

*Proof.* For  $d = 1$ , by the Fourier transform we have

$$\begin{aligned} (-\Delta)^{1/4}e^{it\Delta/2}f &= \frac{1}{2\pi} \int_{\mathbb{R}} e^{ix\xi} |\xi|^{1/2} e^{-it\xi^2/2} \widehat{f}(\xi) d\xi \\ &= \frac{1}{2\pi} \int_{-\infty}^0 e^{ix\xi} |\xi|^{1/2} e^{-it\xi^2/2} \widehat{f}(\xi) d\xi + \frac{1}{2\pi} \int_0^{\infty} e^{ix\xi} |\xi|^{1/2} e^{-it\xi^2/2} \widehat{f}(\xi) d\xi. \end{aligned}$$

Since we want an  $L^2_t$  bound, it indicates that we should apply Plancherel's theorem in time. We will prove the estimate for the latter integral, and the former integral is left as an exercise. Set  $\eta = \xi^2$  with  $d\eta = 2\xi d\xi = 2\sqrt{\eta} d\xi$ . Applying this change of variables, we obtain

$$(-\Delta)^{1/4}e^{it\Delta/2}f = \frac{1}{2\pi} \int_0^{\infty} e^{ix\sqrt{\eta}} |\eta|^{1/4} e^{-it\eta/2} \widehat{f}(\sqrt{\eta}) \frac{1}{2\sqrt{\eta}} d\eta = \frac{1}{4\pi} \int_0^{\infty} e^{ix\sqrt{\eta}} |\eta|^{-1/4} e^{-it\eta/2} \widehat{f}(\sqrt{\eta}) d\eta.$$

Fix  $x$  and denote  $h(\eta) = e^{ix\sqrt{\eta}}|\eta|^{-1/4}\widehat{f}(\sqrt{\eta})$ . Then we have

$$(-\Delta)^{1/4}e^{it\Delta/2}f = \frac{1}{4\pi} \int_0^\infty e^{-it\eta/2}h(\eta) d\eta = (\star).$$

By Plancherel's theorem in time, we see that

$$\|(\star)\|_{L_t^2} \leq \|h(\eta)\|_{L_\eta^2}.$$

Then we can estimate (setting  $z = \sqrt{\eta}$  with  $dz = d\eta/(2\sqrt{\eta})$ )

$$\int_0^\infty |h(\eta)|^2 d\eta = \int_0^\infty |\eta|^{-1/2}|\widehat{f}(\sqrt{\eta})|^2 d\eta = 2 \int_0^\infty |\widehat{f}(z)|^2 dz = C\|f\|_{L^2}^2,$$

where the last inequality follows by Plancherel's theorem in  $z$ . This yields  $\|(\star)\|_{L_t^2} \leq C\|f\|_{L^2}$ , and since these estimates are all independent of  $x$ , we get that  $\sup_x \|(\star)\|_{L_t^2} \leq C\|f\|_{L^2}$ . Putting this together with an identical estimate for the other integral, we obtain the desired bound.  $\square$

**Remark.** The above 1-D version is stronger and implies the statement

$$\|\chi(-\Delta)^{1/4}e^{it\Delta/2}f\|_{L_{x,t}^2} \leq C\|f\|_{L^2}$$

for the 1-D case. Check this as an exercise.

## 5.2 Coarea Formula

**Remark.** In dimension  $d$ , suppose we have two nice functions  $g$  and  $u$  such that  $u^{-1}(t)$  is a  $(d-1)$ -dimensional hypersurface. Then the *coarea formula* says that

$$\int_{\mathbb{R}^d} g(x)|\nabla u(x)| dx = \int_{\mathbb{R}} \int_{\{u(x)=t\}} g(x) d\sigma(x) dt,$$

where  $\sigma(x)$  is the surface measure on  $\{u(x) = t\}$ .

Note that for  $d = 1$ , this says that

$$\int g(x)|\partial_x u| dx = \int_{\mathbb{R}} \left( \int_{\{u(x)=t\}} g(x) dx \right) dt = \int_{\mathbb{R}} g(u^{-1}(t)) dt.$$

In particular, this is the change of variables formula where  $\eta = u^{-1}(t)$  (so  $u(\eta) = t$  and  $dt = \partial_\eta u d\eta$ ).

**Lemma 5.1.** *Let  $F \in C_0^\infty$  and  $\phi$  be smooth. Then one has*

$$\int_{\mathbb{R}} \int_{\mathbb{R}^d} e^{i\lambda\phi(x)} F(x) dx d\lambda = (2\pi)^d \int_{\{\phi=0\}} \frac{F(x)}{|\nabla\phi(x)|} d\sigma(x).$$

*Proof.* By the coarea formula (using  $g(x) = e^{i\lambda\phi(x)} F(x)/|\nabla\phi(x)|$  and  $u = \phi$ ), we have

$$\int_{\mathbb{R}} \int_{\mathbb{R}^d} e^{i\lambda\phi(x)} F(x) dx d\lambda = \int_{\mathbb{R}} \int_{\mathbb{R}} e^{i\lambda y} \int_{\{\phi=y\}} \frac{F(x)}{|\nabla\phi(x)|} d\sigma(x) dy d\lambda.$$



Denote by  $h(y)$  the integral

$$h(y) = \int_{\{\phi=y\}} \frac{F(x)}{|\nabla\phi(x)|} d\sigma(x).$$

Then we can see that

$$\int_{\mathbb{R}} \int_{\mathbb{R}} e^{i\lambda y} \int_{\{\phi=y\}} \frac{F(x)}{|\nabla\phi(x)|} d\sigma(x) dy d\lambda = \int_{\mathbb{R}} \int_{\mathbb{R}} e^{i\lambda y} h(y) dy d\lambda = \int_{\mathbb{R}} \widehat{h}(\lambda) d\lambda = (2\pi)^d h(0).$$

This gives the desired equality after plugging in the definition of  $h(0)$ . □

# Lecture 6

## Jan. 29 — Kato Smoothing, Part 2

### 6.1 Proof of Kato Smoothing

*Proof of Theorem 5.1.* Note that we have (let  $G = \chi^2$ )

$$(*) \int_{\mathbb{R}} \int_{\mathbb{R}^d} \chi^2 |(-\Delta)^{1/4} e^{it\Delta/2} f|^2 dx dt = \int_{\mathbb{R}} \langle (-\Delta)^{1/4} e^{it\Delta/2} f, G(-\Delta)^{1/4} e^{it\Delta/2} f \rangle_{L_x^2} dt,$$

so by Plancherel's theorem in  $x \mapsto \xi$ , we have

$$(*) = \int_{\mathbb{R}} \langle |\xi|^{1/2} e^{-i|\xi|^2/2} \widehat{f}(\xi), H(\xi) \rangle_{L_\xi^2} dt,$$

where  $H$  is defined via (since the Fourier transform turns multiplication into convolution)

$$H(\xi) = \int_{\mathbb{R}} \widehat{G}(\xi - \eta) |\eta|^{1/2} e^{-it|\eta|^2/2} \widehat{f}(\eta) d\eta.$$

Thus we have

$$(*) = \int_{\mathbb{R}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |\xi|^{1/2} |\eta|^{1/2} \widehat{G}(\xi - \eta) \widehat{f}(\xi) \overline{\widehat{f}(\eta)} e^{-it(|\xi|^2 - |\eta|^2)/2} d\xi d\eta dt.$$

Applying Lemma 5.1 with  $(\lambda = t)$

$$\phi = -\frac{|\xi|^2 + |\eta|^2}{2}, \quad F = |\xi|^{1/2} |\eta|^{1/2} \widehat{G}(\xi - \eta) \widehat{f}(\xi) \overline{\widehat{f}(\eta)},$$

we have (since  $\phi = 0$  implies  $|\xi| = |\eta|$  and  $|\nabla \phi| = \sqrt{|\xi|^2 + |\eta|^2}$ )

$$(*) = \int_{\{\phi=0\}} \frac{F}{|\nabla \phi|} d\sigma = \int_{|\xi|=|\eta|} \frac{|\xi|^{1/2} |\eta|^{1/2}}{\sqrt{|\xi|^2 + |\eta|^2}} \widehat{G}(\xi - \eta) \widehat{f}(\xi) \overline{\widehat{f}(\eta)} d\sigma.$$

Since  $|\xi| = |\eta|$  on the region of integration, we have

$$\begin{aligned} (*) &= \int_{|\xi|=|\eta|} \frac{|\xi|}{\sqrt{2}\xi} \widehat{G}(\xi - \eta) \widehat{f}(\xi) \overline{\widehat{f}(\eta)} d\sigma \sim \int_{|\xi|=|\eta|} \widehat{G}(\xi - \eta) \widehat{f}(\xi) \overline{\widehat{f}(\eta)} d\sigma \\ &\sim \int_{\mathbb{R}^d} |\widehat{G}(\xi - \eta)| (|\widehat{f}(\xi)|^2 + |\widehat{f}(\eta)|^2) d\sigma \\ &\leq \int_{\mathbb{R}^d} |\widehat{f}(\xi)|^2 \int_{|\eta|=|\xi|} |\widehat{G}(\xi - \eta)| d\sigma(\eta) d\xi \lesssim \int_{\mathbb{R}^d} |\widehat{f}(\xi)|^2 d\xi, \end{aligned}$$

where the last inequality is since (we claim the following)

$$K(\xi) = \int_{|\eta|=\xi} |\widehat{G}(\xi - \eta)| d\sigma(\eta)$$

satisfies  $|K(\xi)| \leq C$  for some constant  $C$ . For the claim, note that  $\widehat{G}(z)$  decays faster than  $z^{-N}$  as  $z \rightarrow \infty$  for every  $N$ . Then setting  $z = \xi - \eta$ , we have

$$K(\xi) = \int_{|\eta|=\xi} |\widehat{G}(\xi - \eta)| d\sigma(\eta) = \int_{|\xi-z|=\xi} \widehat{G}(z) d\sigma(z).$$

Note that this is an integral on a sphere. The integral will be bounded on a compact subset of the sphere and decays rapidly on the rest of the sphere, so we have  $|K(\xi)| \leq C$  for all  $\xi \in \mathbb{R}^d$ . This gives

$$\|\chi^2(-\Delta)^{1/4} e^{it\Delta/2} f\|_{L_{x,t}^2}^2 = \int_{\mathbb{R}} \int_{\mathbb{R}^d} \chi^2 |(-\Delta)^{1/4} e^{it\Delta/2} f|^2 dx dt \lesssim \int_{\mathbb{R}^d} |\widehat{f}(\xi)|^2 d\xi \sim \|f\|_{L_x^2}^2,$$

which is the desired estimate.  $\square$

**Remark.** An alternative approach to prove the Kato smoothing estimate is the following: Take  $\eta$  a Schwartz function such that  $\text{supp } \widehat{\eta} \in (-1, 1)$ . Then it suffices to show estimates for

$$(*) = \int_{\mathbb{R}} \int_{\mathbb{R}^d} |\eta(\epsilon t) \chi(x) (-\Delta)^{1/4} e^{it\Delta/2} f|^2 dx dt$$

and then take  $\epsilon \rightarrow 0$ . By Plancherel's theorem in  $x \mapsto \xi$ , we have

$$(*) = \int_{\mathbb{R}} \int_{\mathbb{R}^d} \left| \int \eta(\epsilon t) \widehat{\chi}(\xi - \xi') |\xi'|^{1/2} e^{it|\xi'|^2/2} \widehat{f}(\xi') d\xi' \right|^2 d\xi dt$$

Next Plancherel's theorem in  $t \mapsto \tau$  implies that

$$(*) = \int_{\mathbb{R}} \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} \widehat{\chi}(\xi - \xi') \frac{1}{\epsilon} \widehat{\eta} \left( \frac{\tau - |\xi'|^2/2}{\epsilon} \right) |\xi'|^{1/2} \widehat{f}(\xi') d\xi' \right|^2 d\xi d\tau.$$

Cauchy-Schwarz in  $\xi'$  implies that

$$(*) \leq \int_{\mathbb{R}} \int_{\mathbb{R}^d} \left[ \left( \int \widehat{\chi}(\xi - \xi') \frac{1}{\epsilon} \widehat{\eta} \left( \frac{\tau - |\xi'|^2/2}{\epsilon} \right) d\xi' \right) \left( \int \widehat{\chi}(\xi - \xi') \frac{1}{\epsilon} \widehat{\eta} \left( \frac{\tau - |\xi'|^2}{\epsilon} \right) |\xi'| |\widehat{f}(\xi)|^2 d\xi' \right) \right] d\xi d\tau.$$

Now we count the area of the domain of integration:

$$\begin{cases} -\epsilon < \tau - |\xi'|^2/2 < \epsilon & (1) \\ |\xi - \xi'| \leq 1 & (2) \end{cases}$$

Then (2) implies that  $||\xi| - |\xi'|| \leq 1 \sim O(1)$ , so

$$||\xi|^2 - |\xi'|^2| = ||\xi| - |\xi'|| \cdot (|\xi| + |\xi'|) \sim O(1 + |\xi|),$$

and thus  $\tau \sim |\xi'|^2$ . Fix  $\xi$ , then the size of the interval in which  $\tau$  can vary is of order  $O(1 + |\xi|)$ . For  $\xi'$ , notice that (1) implies

$$|\sqrt{2\tau} - |\xi'|| \sim O\left(\frac{\epsilon}{\sqrt{2\tau} + |\xi'|}\right) \sim O\left(\frac{\epsilon}{|\xi| + 1}\right).$$

So the size of the interval in which  $|\xi'|$  can vary is of order  $O(\epsilon/(|\xi| + 1))$ . Then we have

$$(*) \lesssim \iint \left( \frac{1}{\epsilon} \frac{\epsilon}{|\xi| + 1} \right) \left( \frac{1}{\epsilon} \frac{\epsilon}{|\xi| + 1} |\xi| |\widehat{f}(\xi')|^2 \right) d\tau d\xi \lesssim \int \frac{1 + |\xi|}{|\xi| + 1} \frac{|\xi|}{|\xi| + 1} |\widehat{f}(\xi)|^2 d\xi \lesssim \int |\widehat{f}(\xi)|^2 d\xi.$$

# Lecture 7

## Feb. 3 — Kato Smoothing, Part 3

### 7.1 Sharpness of Kato Smoothing

**Lemma 7.1** (Kato smoothing is sharp). *We have*

$$\|\chi(x)(-\Delta)^{1/4}e^{it\Delta/2}(e^{-|x|^2/2}e^{ixv})\|_{L^2_{x,t}} \sim 1 \quad \text{and} \quad \sup_v \|\chi(-\Delta)^{1/4+\delta}e^{it\Delta/2}(e^{-|x|^2/2}e^{ixv})\|_{L^2_{x,t}} = \infty.$$

for any  $\delta > 0$ .

*Proof.* Let  $\text{supp } \widehat{\chi} \subseteq B(0, 1)$  and  $f_0 = e^{-|x|^2/2}e^{ixv}$ . Choose  $\eta$  smooth with  $\widehat{\eta}$  compactly supported. Then

$$\begin{aligned} \int_{\mathbb{R}} \int_{\mathbb{R}^d} |\chi(x)\eta(\epsilon t)(-\Delta)^{1/4}e^{it\Delta/2}f_0|^2 dx dt &= \iint \left| \int \widehat{\chi}(\xi - \xi')\eta(\epsilon t)|\epsilon|^{1/2}e^{it|\xi|^2/2}\widehat{f_0}(\xi) d\xi \right|^2 d\xi dt \\ &= \iint \left| \int \widehat{\chi}(\xi - \xi')\frac{1}{\epsilon}\widehat{\eta}\left(\frac{\tau - |\xi|^2/2}{\epsilon}\right)|\xi|^{1/2}e^{-|\xi-v|^2/2} d\xi \right|^2 d\xi' d\tau \end{aligned}$$

by Plancherel's theorem first in  $x \mapsto \xi$  and then in  $t \mapsto \tau$ , since  $\widehat{\eta}$  and  $\widehat{\chi}$  are compactly supported. Now,  $-\epsilon < \tau - |\xi|^2/2 < \epsilon$  implies

$$||\xi| - \sqrt{2\tau}| < \frac{2\epsilon}{|\xi| + \sqrt{2\tau}}.$$

So the region of integration for  $|\xi|$  is an annulus-shaped area between  $\sqrt{2\tau}$  and  $2\epsilon/(|\xi| + \sqrt{2\tau})$  (plot the above inequality to see the region). Thus for fixed  $\tau$ , as  $\epsilon \rightarrow 0$ , we have

$$\begin{aligned} &\iint \left| \int \widehat{\chi}(\xi - \xi')\frac{1}{\epsilon}\widehat{\eta}\left(\frac{\tau - |\xi|^2/2}{\epsilon}\right)|\xi|^{1/2}e^{-|\xi-v|^2/2} d\xi \right|^2 d\xi' d\tau \\ &\sim \int_{\mathbb{R}} \int_{\mathbb{R}^d} \left| \int_{\sqrt{2\tau}\mathbb{S}^{d-1}} \widehat{\chi}(\xi - \xi')e^{-|\xi-v|^2/2}|\xi|^{1/2}\frac{2\epsilon}{|\xi| + \sqrt{2\tau}}\frac{1}{\epsilon} d\sigma(\xi) \right|^2 d\xi' d\tau \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}^d} \left| \int_{\sqrt{2\tau}\mathbb{S}^{d-1}} \widehat{\chi}(\xi - \xi')e^{-|\xi-v|^2/2}|\xi|^{1/2}\frac{2}{|\xi| + \sqrt{2\tau}} d\sigma(\xi) \right|^2 d\xi' d\tau. \end{aligned}$$

Since  $\widehat{\chi}$  is supported in  $B(0, 1)$ , for fixed  $\xi$  the range of values for  $\xi'$  is  $O(1)$  (as  $|\xi - \xi'| \leq 1$ ). Since

$$-\epsilon < \tau - \frac{1}{2}|\xi|^2 < \epsilon \implies -\epsilon + \frac{1}{2}|\xi|^2 < \tau < \epsilon + \frac{1}{2}|\xi|^2$$

and  $|\xi - v| \leq O(1)$ , we get

$$-\epsilon + \frac{1}{2}||v| - 1|^2 < \tau < \epsilon + \frac{1}{2}||v| + 1|^2.$$

Expanding  $||v| - 1|^2$  and  $||v| + 1|^2$ , we see that as  $\epsilon \rightarrow 0$ , we have  $|2\tau - |v|^2| \lesssim |v| + 1$ . Then

$$\begin{aligned} & \int_{\mathbb{R}} \int_{\mathbb{R}^d} \left| \int_{\sqrt{2\tau}\mathbb{S}^{d-1}} \widehat{\chi}(\xi - \xi') e^{-|\xi-v|^2/2} |\xi|^{1/2} \frac{1}{|\xi| + \sqrt{2\tau}} d\sigma(\xi) \right|^2 d\xi' d\tau \\ & \lesssim \int_{\mathbb{R}} \left| \int_{\sqrt{2\tau}\mathbb{S}^{d-1}} e^{-|\xi-v|^2/2} |\xi|^{1/2} \frac{1}{|\xi| + \sqrt{2\tau}} d\sigma(\xi) \right|^2 d\tau \\ & \lesssim \int_{\mathbb{R}} \left| \frac{|\sqrt{\tau}|^{1/2}}{\sqrt{2\tau}} \right|^2 d\tau \approx \int_{|2\tau - |v|^2| \lesssim |v| + 1} \left| \frac{|\sqrt{\tau}|^{1/2}}{\sqrt{2\tau}} \right|^2 d\tau \approx \frac{1}{v} \int_{|2\tau - |v|^2| \lesssim |v| + 1} d\tau \approx \frac{1}{v} \cdot v \approx O(1) \end{aligned}$$

uniformly in  $v$ . We can also see that if  $|\xi|^{1/2}$  is replaced by  $|\xi|^{1/2+\delta}$ , then  $1/|v|$  will become  $1/|v|^{1-4\delta}$ , so the above becomes  $O(|v|^{4\delta})$ , which goes to  $\infty$  as  $v \rightarrow \infty$ .  $\square$

**Corollary 7.0.1.** *When  $d \geq 2$ , for  $\epsilon > 0$ , we have*

$$\|(1 + |x|)^{-1/2-\epsilon} (-\Delta)^{1/4} e^{it\Delta/2} f\|_{L_{x,t}^2} \leq C_\epsilon \|f\|_{L_x^2}.$$

*This is the sharp Kato smoothing estimate for  $d \geq 2$ .*

## 7.2 Schrödinger Equation with Potential

**Remark.** Now we will consider the Schrödinger equation with a potential:

$$\begin{cases} i\psi_t + \frac{1}{2}\Delta\psi + V(t, x)\psi = 0, \\ \psi|_{t=0} = \psi_0 \in H^\gamma(\mathbb{R}^d). \end{cases}$$

**Lemma 7.2.** *If  $\psi_0 \in H^\gamma(\mathbb{R}^d)$ , then  $e^{it\Delta/2}\psi_0 \in C^0(\mathbb{R}, H^\gamma(\mathbb{R}^d)) \cap C^1(\mathbb{R}, H^{\gamma-2}(\mathbb{R}^d))$ .*

*Proof.* For each  $t, s \in \mathbb{R}$ , we can write (since  $e^{is\Delta/2}$  preserves each  $H^\gamma$  norm)

$$\|e^{it\Delta/2}\psi_0 - e^{is\Delta/2}\psi_0\|_{H^\gamma}^2 = \|e^{i(t-s)\Delta}\psi_0 - \psi_0\|_{H^\gamma}^2 = \int (1 + |\xi|^2)^\gamma |e^{i(t-s)|\xi|^2/2} - 1| |\widehat{\psi_0}(\xi)|^2 d\xi.$$

Since  $\psi_0 \in H^\gamma(\mathbb{R}^d)$ , as  $t \rightarrow s$ , the above quantity goes to 0 by the dominated convergence theorem. This shows that  $e^{it\Delta/2}\psi_0 \in C^0(\mathbb{R}, H^\gamma(\mathbb{R}^d))$ . To check that  $e^{it\Delta/2}\psi_0 \in C^1(\mathbb{R}, H^{\gamma-2}(\mathbb{R}^d))$ , note

$$\frac{d}{dt} e^{it\Delta/2}\psi_0 = \frac{i}{2} \Delta e^{it\Delta/2}\psi_0.$$

Thus it suffices to see that  $\Delta e^{it\Delta/2}\psi_0$  is continuous, i.e.

$$\|\Delta e^{it/2}\psi_0 - \Delta e^{is/2}\psi_0\|_{H^{\gamma-2}}^2 \rightarrow 0$$

as  $t \rightarrow s$ , which follows by the dominated convergence theorem.  $\square$

**Theorem 7.1.** *Assume that  $V$  is real with*

$$\sup_t \|\nabla_x^\alpha V(t, x)\|_{L_x^2} \leq C_\alpha \quad \text{and} \quad \|\nabla_x^\alpha V(t, x) - \nabla_x^\alpha V(s, x)\|_{L_x^\infty} \rightarrow 0$$

as  $t \rightarrow s$  for every multi-index  $\alpha$ . Then for all  $k \geq 0$ , if  $\psi_0 \in H^k(\mathbb{R}^d)$ , there exists a unique solution  $\psi \in C^0(\mathbb{R}, H^k(\mathbb{R}^d)) \cap C^1(\mathbb{R}, H^{k-2}(\mathbb{R}^d))$  to the Schrödinger equation with potential  $V$ , given by

$$\psi(t) = e^{it\Delta/2}\psi_0 + i \int_0^t e^{i(t-s)\Delta/2}V(s)\psi(s) ds.$$

Moreover,  $\|\psi\|_{H^k(\mathbb{R}^d)} \leq C_{k,V}(1 + |t|)^k$ .

*Proof.* Define the operator  $A$  by the formula

$$A(\phi) = e^{it\Delta/2}\psi_0 - i \int_0^t e^{i(t-s)\Delta/2}V(s)\phi(s) ds.$$

Let  $X = (C^0[0, T], H^k)$ , and we want to find  $T$  such that  $A$  is a contraction in  $X$  (do this as an exercise). Repeating this on  $[T, 2T]$ ,  $[2T, 3T]$ ,  $\dots$ , gives existence. Uniqueness follows by Gronwall's inequality.

Now we prove the upper bound for the growth. Gronwall's inequality gives us

$$\|\psi(t)\|_{H^k} \leq c_1 e^{c_2|t|} \|\psi_0\|_{H^k}.$$

Since  $V$  is real (the argument before this step works even when  $V$  is not real),

$$\begin{aligned} \frac{d}{dt} \|\psi(t)\|_{L^2}^2 &= \frac{d}{dt} \langle \psi(t), \psi(t) \rangle = \frac{d}{dt} = \left\langle \frac{d}{dt} \psi, \psi \right\rangle + \left\langle \psi, \frac{d}{dt} \psi \right\rangle = 2 \operatorname{Re} \left\langle \frac{d}{dt} \psi, \psi \right\rangle \\ &= 2 \operatorname{Re} \left\langle \frac{i}{2} \Delta \psi + iV(t, x)\psi, \psi \right\rangle = 2 \operatorname{Re} i \int \left( -\frac{1}{2} |\nabla \phi|^2 + V(t, x) |\psi|^2 \right) dx = 0 \end{aligned}$$

since the integral is real, so that  $i$  times the integral is purely imaginary. So  $\|\psi\|_{L^2}$  is conserved. Now let  $\phi = \partial_{x_j} \psi(t, x)$ , so that

$$i\phi_t + \frac{1}{2} \Delta \phi + V\phi = -\partial_{x_j} V\psi. \quad (*)$$

From the existence and uniqueness of the solution, we can use  $V(t, 0)\psi_0$  to denote the solution to

$$\begin{cases} i\partial_t + \frac{1}{2} \Delta \psi + V\psi = 0, \\ \psi|_{t=0} = \psi_0, \end{cases}$$

and use  $V(t, s)\psi_0$  denote the solution for initial condition  $\psi(s) = \psi_0$ . Then  $(*)$  can be solved as

$$\phi(t) = V(t, 0)\partial_{x_j}\psi_0 + \int_0^t V(t, s)\partial_{x_j}V(s)\psi(s) ds.$$

This gives

$$\|\phi(t)\|_{L^2} \leq \|V(t, 0)\partial_{x_j}\psi_0\|_{L^2} + \int_0^t \|V(t, s)\partial_{x_j}V(s)\psi(s)\|_{L^2} ds$$

by Minkowski's inequality. By the conservation of the  $L^2$  norm,  $\|V(t, 0)\partial_{x_j}\psi_0\|_{L^2} = \|\partial_{x_j}\psi_0\|_{L^2}$ , so

$$\|V(t, s)\partial_{x_j}V(s)\psi(s)\|_{L^2} = \|\partial_{x_j}V(s)\psi(s)\|_{L^2} \lesssim \|\psi(s)\|_{L^2} = \|\psi(0)\|_{L^2}.$$

Then we obtain

$$\|\phi\|_{L^2} \lesssim \|\partial_{x_j}\psi_0\|_{L^2} + \int_0^t \|\psi(s)\|_{L^2} ds \leq C_{1,V}(1 + |t|) \|\psi(0)\|_{H^1},$$

and inducting on  $k$  gives the result  $\|\psi(t)\|_{H^k} \leq C_{k,V}(1 + |t|)^k \|\psi(0)\|_{H^k}$ .  $\square$

**Remark.** Note that in the above theorem, we need a suitable integral (e.g. the Bochner integral) to be able to interpret the Duhamel formula in a Banach space.

# Lecture 8

## Feb. 5 — Schrödinger Equation with Potential

### 8.1 Schrödinger Equation with Potential, Continued

**Theorem 8.1** (Bourgain). Assume  $d \geq 3$  and that  $V$  is real. Suppose that

$$\sup_t \|\nabla_x^\alpha V(t, x)\|_{L_x^\infty} \leq C_\alpha$$

for every multi-index  $\alpha$  and that  $\sup_{|t-t_0| \leq 1} |V(t, x)|$  is compactly supported in  $x$  with the diameter of the support independent of  $t_0$  (e.g.  $V(t, x) = V(x - ct)$ ). Then

$$\|\psi(t)\|_{H^k} \leq C_\epsilon (1 + |t|)^\epsilon \|\psi_0\|_{H^k}$$

for every  $\epsilon > 0$ .

*Proof.* Define (note that  $\|f\| = \|f\|_{L^\infty + L^2}$ , the dual space of  $\|g\|_{L^1 \cap L^2}$ )

$$\|f\| = \inf_{f=f_1+f_2} (\|f_1\|_{L^2} + \|f_2\|_{L^\infty}).$$

Then we can write

$$\psi(t) = e^{it\Delta/2} \psi_0 + i \int_0^t e^{i(t-s)\Delta/2} V(s) \psi(s) ds.$$

Taking the derivative and splitting the integral, we obtain

$$\nabla^\alpha \psi(t) = e^{it\Delta/2} \nabla^\alpha \psi_0 + i \int_0^{t-A} e^{i(t-s)\Delta/2} \nabla^\alpha (V\psi) ds + i \int_{t-A}^t e^{i(t-s)\Delta/2} \nabla^\alpha (V\psi) ds,$$

where  $A > 0$  is a constant to be chosen later. We will measure the first integral in  $L^\infty$  and the second integral in  $L^2$ . Taking norms, we find that

$$\|\nabla^\alpha \psi(t)\| \leq \|e^{it\Delta/2} \nabla^\alpha \psi_0\| + \left\| \int_0^{t-A} e^{i(t-s)\Delta/2} \nabla^\alpha (V\psi) ds \right\|_{L^\infty} + \left\| \int_{t-A}^t e^{i(t-s)\Delta/2} \nabla^\alpha (V\psi) ds \right\|_{L^2}.$$

We can estimate the first integral by

$$\begin{aligned} \left\| \int_0^{t-A} e^{i(t-s)\Delta/2} \nabla^\alpha (V\psi) ds \right\|_{L^\infty} &\leq \int_0^{t-A} \|e^{i(t-s)\Delta/2} \nabla^\alpha (V\psi)\|_{L^\infty} ds \\ &\lesssim \int_0^{t-A} |t-s|^{-d/2} \|\nabla^\alpha (V\psi)\|_{L^1} ds \lesssim A^{-d/2+1} \sup_{0 \leq s \leq t} \|\nabla^\alpha (V\psi)\|_{L^1}, \end{aligned}$$

where the first inequality is by Minkowski's inequality, the second inequality is by the dispersive estimate, and the third is by explicit integration. Note that we needed the cutoff at  $t - A$  here since  $|t - s|^{-d/2}$  is not integrable at  $t = s$ . Then by the product rule and Cauchy-Schwarz, we have

$$\left\| \int_0^{t-A} e^{i(t-s)\Delta/2} \nabla^\alpha (V\psi) ds \right\|_{L^\infty} \lesssim A^{-d/2+1} \sup_{\substack{0 \leq \beta \leq \alpha \\ 0 \leq s \leq t}} \|\nabla^\beta V \nabla^{\alpha-\beta} \psi\|_{L^1} \lesssim A^{-d/2+1} \sup_{\substack{0 \leq \beta \leq \alpha \\ 0 \leq s \leq t}} \|\nabla^\beta \psi\|_{L^2}.$$

Note here that  $d \geq 3$  implies  $-d/2 + 1 < 0$ , so if  $A$  is large, then  $A^{-d/2+1}$  is small. The above implies

$$\left\| \int_0^{t-A} e^{i(t-s)\Delta/2} \nabla^\alpha (V\psi) ds \right\|_{L^\infty} \lesssim A^{-d/2+1} \sup_{\substack{0 \leq \beta \leq \alpha \\ 0 \leq s \leq t}} \|\nabla^\beta \psi\|.$$

For the second integral, we can write

$$\begin{aligned} \left\| \int_{t-A}^t e^{i(t-s)\Delta/2} \nabla^\alpha (V\psi) ds \right\|_{L^2} &= \sup_{\|\phi\|_{L^2}=1} \int_{t-A}^t \langle \phi, e^{i(t-s)\Delta/2} \nabla^\alpha (V\psi) \rangle ds \\ &= \sup_{\|\phi\|_{L^2}=1} \int_{t-A}^t \langle \phi e^{i(t-s)\Delta/2} \chi \nabla^\alpha (V\psi) \rangle ds, \end{aligned}$$

where  $\chi$  is a smooth cutoff function such that  $\chi \equiv 1$  in the support of  $V$ . Then

$$\left\| \int_{t-A}^t e^{i(t-s)\Delta/2} \nabla^\alpha (V\psi) ds \right\|_{L^2} = \sup_{\|\phi\|_{L^2}=1} \int_{t-A}^t \langle (1-\Delta)^{1/4} \chi e^{-i(t-s)\Delta/2} \phi, (1-\Delta)^{-1/4} \nabla^\alpha (V\psi) \rangle ds.$$

By Cauchy-Schwarz in  $x$  and  $t$  and Kato 1/2 smoothing, we obtain

$$\begin{aligned} &\left\| \int_{t-A}^t e^{i(t-s)\Delta/2} \nabla^\alpha (V\psi) ds \right\|_{L^2} \\ &\leq \sup_{\|\phi\|_{L^2}=1} \left( \int_{t-A}^t \|(1-\Delta)^{1/4} \chi e^{-i(t-s)\Delta/2} \phi\|_{L_x^2}^2 ds \right)^{1/2} \left( \int_{t-A}^t \|(1-\Delta)^{-1/4} \nabla^\alpha (V\psi)\|_{L_x^2}^2 ds \right)^{1/2} \\ &\lesssim \|\phi\|_{L^2} + \sqrt{A} \sup_{t-A \leq s \leq t} \|(1-\Delta)^{-1/4} \nabla^\alpha (V\psi)\|_{L_x^2}. \end{aligned}$$

To deal with the last term, note that by Sobolev's inequality, we have

$$\|f\|_{H^{k-1/2}} \leq C_k \|f\|_{H^k}^{1-\delta} \|f\|_{L^2}^\delta$$

for some  $\delta = \delta(k)$ . From this we get

$$\sup_{|\alpha|=k} \|(1-\Delta)^{-1/4} \nabla^\alpha (V\psi)\|_{L^2} \leq C_k \|V\psi\|_{L^2}^\delta \sup_{|\beta| \leq k} \|\nabla^\beta (V\psi)\|_{L^2}^{1-\delta} \leq C_k \|\psi_0\|_{L^2}^\delta \sup_{|\beta| \leq k} \|\nabla^\beta (V\psi)\|^{1-\delta},$$

where the second inequality is since the  $L^2$  norm is conserved. This means that

$$\|\nabla^\alpha \psi\| \lesssim \|\nabla^\alpha \psi_0\|_{L^2} + A^{1-d/2} \sup_{\substack{0 \leq |\beta| \leq |\alpha| \\ 0 \leq s \leq t}} \|\nabla^\beta \psi\| + \sqrt{A} \cdot \|\psi_0\|_{L^2}^\delta \sup_{\substack{0 \leq |\beta| \leq |\alpha| \\ 0 \leq s \leq t}} \|\nabla^\beta \psi\|^{1-\delta}.$$

For the last term, observe that

$$x^{1-\delta} y^\delta \leq \frac{1}{1-\delta} x + \frac{1}{\delta} y$$



by Young's inequality, and we can also use that for  $\eta > 0$ ,

$$(\eta^{1/(1-\delta)}x)^{1-\delta} (n^{-1/\delta}y)^\delta \leq \frac{\eta^{1/(1-\delta)}}{1-\delta}x + \frac{n^{-1/\delta}}{\delta}y.$$

Applying this to the last term, we obtain

$$\|\psi_0\|_{L^2}^\delta \sup_{\substack{0 \leq |\beta| \leq |\alpha| \\ 0 \leq s \leq t}} \|\nabla^\beta \psi\|^{1-\delta} \leq \frac{\eta^{-1/\delta}}{\delta} \|\psi_0\|_{L^2} + \frac{\eta^{1/(1-\delta)}}{1-\delta} \sup_{\substack{0 \leq |\beta| \leq |\alpha| \\ 0 \leq s \leq t}} \|\nabla^\beta \psi\|.$$

Putting everything together, we obtain

$$\sup_{\substack{|\alpha|=k \\ 0 \leq s \leq t}} \|\nabla^\alpha \psi\| \leq \|\psi_0\|_{H^k} + A^{1-d/2} \sup_{\substack{0 \leq s \leq t \\ |\alpha| \leq k}} \|\psi\| + \frac{\sqrt{A} \cdot \eta^{-1/\delta}}{\delta} \|\psi_0\|_{L^2} + \frac{\sqrt{A} \cdot \eta^{1/(1-\delta)}}{1-\delta} \sup_{\substack{0 \leq s \leq t \\ |\alpha| \leq k}} \|\psi\|.$$

Now take  $A$  large, so that  $A^{1-d/2}$  is very small (note that since  $A \leq t$ , we also take  $t$  to be large enough; for  $t$  small, apply Gronwall). Take  $\eta$  small, so that  $\sqrt{A} \cdot \eta^{1/(1-\delta)}$  is very small. Then we have

$$\sup_{\substack{|\alpha|=k \\ 0 \leq s \leq t}} \|\nabla^\alpha \psi\| \leq C_{\alpha,\eta} \|\psi_0\|_{H^k}.$$

Now recall that (since the Schrödinger evolution conserves the  $H^k$  norms)

$$\|\psi(t)\|_{H^k} \leq \|\psi_0\|_{H^k} + \int_0^t \|V(s)\psi(s)\|_{H^k} ds.$$

The above calculations show that

$$\|V\psi\|_{H^k} \lesssim \sup_{|\alpha| \leq k} \|\nabla^\alpha(V\psi)\|_{L^2} \lesssim \sup_{|\alpha| \leq k} \|\nabla^\alpha \psi\| \lesssim \|\psi_0\|_{H^k},$$

and so we have

$$\|\psi(t)\|_{H^k} \lesssim \|\psi_0\|_{H^k} + \int_0^t \|\psi_0\|_{H^k} ds \lesssim (1+|t|)\|\psi_0\|_{H^k}.$$

Note that the growth of the norm is independent of  $k$ . To get  $(1+|t|)^\epsilon$ , first note that the  $L^2$  norm is conserved. So if  $S(t)$  is the solution operator, then we have

$$\|S(t)\psi_0\|_{L^2} = \|\psi_0\|_{L^2} \quad \text{and} \quad \|S(t)\psi_0\|_{H^{k_1}} \leq C_{k_1}(1+|t|)\|\psi_0\|_{H^{k_1}}$$

for every  $k_1 \in \mathbb{N}$ . Since  $S(t)$  is a linear operator, by interpolation we have

$$\|S(t)\|_{\mathcal{L}(H^k, H^k)} \leq \|S(t)\|_{\mathcal{L}(L^2, L^2)}^{1-k/k_1} \|S(t)\|_{\mathcal{L}(H^{k_1}, H^{k_1})}^{k/k_1} \leq C_{k,k_1}(1+|t|)^{k/k_1}$$

for every  $0 \leq k \leq k_1$  (since  $\|S(t)\|_{\mathcal{L}(L^2, L^2)}$  is bounded). For any  $\epsilon > 0$ , for fixed  $k$  take  $k_1$  large enough so that  $k/k_1 \leq \epsilon$ . Then

$$\|S(t)\|_{\mathcal{L}(H^k, H^k)} \leq C_{k,k_1}(1+|t|)^{k/k_1} \leq C_\epsilon(1+|t|)^\epsilon,$$

which implies the desired result.  $\square$

**Remark.** The compact support assumption could be replaced by rapid decay in the above theorem.

# Lecture 9

## Feb. 10 — Spectral Theory

### 9.1 Review of Functional Analysis

**Remark.** Our goal now is to study the decay of a solution  $\psi$  to

$$i\psi_t - \frac{1}{2}\Delta\psi + V(x)\psi = 0, \quad V \text{ real.}$$

We will study  $e^{iHt}\psi_0$ , where  $H = -\Delta/2 + V$ , in particular the spectral theory for  $H$  in  $L^2(\mathbb{R}^d)$ .

**Definition 9.1.** Let  $\mathcal{H}$  be a Hilbert space, with inner product  $\langle \cdot, \cdot \rangle$ . We will use  $\mathcal{H} = L^2(\mathbb{R}^d)$  with

$$\langle f, g \rangle = \int_{\mathbb{R}^d} f \bar{g} \, dx.$$

Let  $T$  be a linear operator which is densely defined on  $\mathcal{H}$ . Let  $D(T)$  denote the domain of  $T$ . We say that  $T$  is *symmetric* if  $\langle Tx, y \rangle = \langle x, Ty \rangle$ . The *adjoint*  $T^*$  of  $T$  is defined as

$$D(T^*) = \{f \in \mathcal{H} : \text{there exists } h \in \mathcal{H} \text{ such that } \langle T\phi, f \rangle = \langle \phi, h \rangle \text{ for all } \phi \in D(T)\},$$

and we set  $T^*f = h$ . We say  $T$  is *closed* provided that the graph of  $T$  is a closed subset of  $\mathcal{H} \times \mathcal{H}$ , i.e. if  $f_n \rightarrow f$  in  $\mathcal{H}$  with  $f_n \in D(T)$  and  $Tf_n \rightarrow g$  in  $\mathcal{H}$ , then  $f \in D(T)$  and  $Tf = g$ .

**Definition 9.2.** We say that a linear operator  $T$  is *self-adjoint* if  $(T, D(T)) = (T^*, D(T^*))$ .

**Example 9.2.1.** Let  $T = i\partial_x$ . For  $\mathcal{H} = L^2(\mathbb{R})$  and  $D(T) = C_c^1(\mathbb{R})$ , one can check that  $T$  is self-adjoint. But if we take  $\mathcal{H} = L^2([0, \infty))$  and  $D(T) = C_c^1([0, \infty))$ , then  $T$  is not self-adjoint (but  $T$  is symmetric).

**Example 9.2.2.** Let  $T = -\partial_x^2$  on  $L^2([0, 1])$ . Then for

$$D(T) = \{f \in H^2([0, 1]) : f(0) = f(1) = f'(0) = f'(1) = 0\},$$

$T$  is symmetric but not self-adjoint. One can check that  $D(T^*) = H^2([0, 1])$ .

**Lemma 9.1.** Let  $T$  be a densely defined linear operator on  $\mathcal{H}$ . Then:

1. If  $T$  is symmetric, then  $D(T^*) \supseteq D(T)$  (so  $T^*$  is densely defined) and all eigenvalues of  $T$  are real.
2. For every  $T$ , its adjoint  $T^*$  is closed.
3. We have  $(\ker T)^\perp = \overline{\text{Ran}(T^*)}$  and  $(\ker T^*)^\perp = \overline{\text{Ran}(T)}$ .

4. If  $T$  is self-adjoint, i.e.  $(T, D(T)) = (T^*, D(T^*))$ , then

$$\text{Spec } T = \mathbb{C} \setminus \{z \in \mathbb{C} : (T - z)^{-1} \text{ exists as a bounded linear operator}\} \subseteq \mathbb{R}$$

$$\text{and } \|(T - z)^{-1}\|_{\mathcal{L}(\mathcal{H}, \mathcal{H})} \lesssim |\text{Im } z|^{-1}.$$

*Proof.* (1) One can check this by definition.

(2) Take a sequence  $f_n \in D(T^*)$  with  $f_n \rightarrow f$ , and suppose that  $T^*f_n \rightarrow g$ . By definition,

$$\langle T\phi, f_n \rangle = \langle \phi, T^*f_n \rangle \quad \text{for every } \phi \in D(T).$$

Then  $f_n \rightarrow f$  implies that  $\langle T\phi, f_n \rangle \rightarrow \langle T\phi, f \rangle$ , and  $T^*f_n \rightarrow g$  implies that  $\langle \phi, T^*f_n \rangle \rightarrow \langle \phi, g \rangle$ . Then we have  $\langle T\phi, f \rangle = \langle \phi, g \rangle$  for every  $\phi \in D(T)$ , so  $f \in D(T^*)$  and  $T^*f = g$  by definition.

(3) Note that in general, if  $A \subseteq \mathcal{H}$ , then  $A^\perp$  is closed. To see this, let  $f_n \in A^\perp$  and  $f_n \rightarrow f$  in  $\mathcal{H}$ . Then  $\langle f_n, a \rangle = 0$  for every  $a \in A$  since  $f_n \in A^\perp$ . So letting  $n \rightarrow \infty$ , we have  $\langle f, a \rangle = 0$ , i.e.  $f \in A^\perp$ .

We first show that  $\overline{\text{Ran}(T)} \subseteq (\ker T^*)^\perp$ . For any  $h \in \text{Ran}(T)$ , there exists  $f$  such that  $h = Tf$ . Then for any  $g \in \ker T^* \subseteq D(T^*)$ , we have

$$\langle Tf, g \rangle = \langle f, T^*g \rangle = 0,$$

so  $f \in (\ker T^*)^\perp$ . Thus  $\text{Ran}(T) \subseteq (\ker T^*)^\perp$ , and taking closures,  $\overline{\text{Ran}(T)} \subseteq \overline{(\ker T^*)^\perp} = (\ker T^*)^\perp$ .

Now we show that  $(\ker T^*)^\perp \subseteq \overline{\text{Ran}(T)}$ . Let  $f \in (\overline{\text{Ran}(T)})^\perp$ , so  $f \in (\text{Ran}(T))^\perp$ . So

$$\langle T^*f, g \rangle = \langle f, Tg \rangle = 0, \quad \text{for all } g \in D(T),$$

so  $T^*f = 0$ . This gives  $f \in \ker(T^*)$ . So  $(\overline{\text{Ran}(T)})^\perp \subseteq \ker(T^*)$ . Then taking orthogonal complements,

$$(\ker T^*)^\perp \subseteq ((\overline{\text{Ran}(T)})^\perp)^\perp = \overline{\text{Ran}(T)}$$

since  $\overline{\text{Ran}(T)}$  is closed. This gives  $(\ker T^*)^\perp = \overline{\text{Ran}(T)}$ , and the other equality follows by duality.

(4) Let  $T = T^*$  and  $z = x + iy$ . Then we can write

$$|\langle (T - z)f, f \rangle| = |\langle (T - (x + iy))f, f \rangle| = |\langle (T - x)f, f \rangle - iy\langle f, f \rangle| \geq |y|\|f\|_{\mathcal{H}}^2,$$

since  $\langle (T - x)f, f \rangle$  is real and  $iy\langle f, f \rangle$  is purely imaginary. On the other hand,

$$\langle (T - z)f, f \rangle \leq \|f\|_{\mathcal{H}}\|(T - z)f\|_{\mathcal{H}},$$

by Cauchy-Schwarz, so  $\|(T - z)f\|_{\mathcal{H}} \geq |y|\|f\|_{\mathcal{H}}$  where  $y = \text{Im } z$ . So  $\ker(T - z) = \{0\}$  if  $|y| \neq 0$ . Now

$$\overline{\text{Ran}(T - z)} = (\ker(T^* - \bar{z}))^\perp = (\ker(T - \bar{z}))^\perp = \mathcal{H}$$

if  $|y| \neq 0$  by (3), so we just need to show that  $\text{Ran}(T - z)$  is closed. Assume  $(T - z)f_n \rightarrow g$ . Note that

$$\|(T - z)f_n - (T - z)f_m\|_{\mathcal{H}} \geq |y|\|f_n - f_m\|_{\mathcal{H}},$$

so  $(T - z)f_n$  being Cauchy implies that  $\{f_n\}$  is also Cauchy. So  $f_n \rightarrow f$  for some  $f \in \mathcal{H}$ , i.e.  $T = T^*$  is closed. This then implies that  $T - z$  is closed, from which it follows that  $f \in D(T - z)$  and  $(T - z)f = g$ , so  $\text{Ran}(T - z)$  is closed if  $|y| \neq 0$ . So  $\ker(T - z) = \{0\}$  and  $\text{Ran}(T - z) = \mathcal{H}$  if  $|y| \neq 0$ . So  $T - z$  is invertible, hence  $\text{Spec } T \subseteq \mathbb{R}$  and  $\|(T - z)^{-1}\|_{\mathcal{L}(\mathcal{H}, \mathcal{H})} \lesssim |\text{Im } z|^{-1}$ .  $\square$

## 9.2 Spectral Theory

**Definition 9.3.** The *resolvent* of  $T$ , denoted  $\rho(T) \subseteq \mathbb{C}$  is defined as follows: We have  $\lambda \in \rho(T)$  if we can find a bounded linear operator  $R(\lambda)$  from  $\mathcal{H}$  to  $\mathcal{H}$  such that

$$(T - \lambda)R(\lambda) = R(\lambda)(T - \lambda) = I.$$

The *spectrum* of  $T$  is  $\sigma(T) = \mathbb{C} \setminus \rho(T)$ .

**Definition 9.4.** Let  $\mathcal{B}$  be the Borel sets in  $\mathbb{R}$ . A *spectral measure*  $E$  on  $\mathbb{R}$  is an orthogonal projection valued measure on  $\mathcal{B}$ , i.e. a map  $E$  such that  $E(A)$  is an orthogonal projection for every  $A \in \mathcal{B}$ , and

1.  $E(\emptyset) = 0$  and  $E(\mathbb{R}) = I$ ;
2. for  $A, B \in \mathcal{B}$ , we have  $E(A)E(B) = E(A \cap B)$ ;
3. for disjoint  $A_j \in \mathcal{B}$ , we have  $E(\bigcup_{j=1}^{\infty} A_j) = \sum_{j=1}^{\infty} E(A_j)$ .

**Remark.** Given a spectral measure  $E$ , for  $x, y \in \mathcal{H}$ , we can define

$$E_{x,y}(A) = \langle E(A)x, y \rangle \quad \text{for } A \in \mathcal{B}.$$

For fixed  $x, y$ ,  $E_{x,y}$  is a complex-valued Borel measure. In particular,  $E_{x,x}$  is a positive Borel measure.

**Theorem 9.1** (Spectral theorem). *Let  $(T, D(T))$  be self-adjoint on  $\mathcal{H}$ . Then there is a unique spectral measure  $E$  such that*

$$\langle Tx, y \rangle = \int \lambda dE_{x,y}(\lambda) \quad \text{for all } x \in D(T), y \in \mathcal{H}.$$

**Remark.** We make the following remarks:

1. We have  $\text{supp } E_{x,y} \subseteq \sigma(T)$ .
2. If  $f$  is a Borel measurable function, then we can interpret

$$\langle f(T)x, y \rangle = \int f(\lambda) dE_{x,y}(\lambda).$$

In particular, we have a map  $f \mapsto f(T)$ .

# Lecture 10

## Feb. 12 — Spectral Theory, Part 2

### 10.1 Examples for Spectral Theory

**Example 10.0.1.** Let  $\mathcal{H} = \mathbb{R}^n$  with the standard inner product, and let  $T$  be a Hermitian matrix. Then  $T$  is self-adjoint, and the corresponding spectral measure  $E$  is given by

$$E(I) = \sum_{j: \lambda_j \in I} P_{L_j},$$

where  $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_k$  are the eigenvalues of  $T$  without multiplicity and  $L_1, \dots, L_k$  are eigenspaces.

**Example 10.0.2.** Let  $T$  be a self-adjoint operator. Then

$$e^{itA}f = \left( \int e^{it\lambda} E(d\lambda) \right) f.$$

From this we have

$$\|e^{itA}f\|_{L^2}^2 = \left\langle \int e^{it\lambda} E(d\lambda)f, \int e^{it\lambda} E(d\lambda)f \right\rangle = \iint e^{it\lambda} e^{-it\mu} \langle E(d\lambda)E(d\mu)f, f \rangle.$$

Fixing  $\lambda$  and integrating over  $\mu$ , the only point that survives is  $\mu = \lambda$ , so

$$\|e^{itA}f\|_{L^2}^2 = \int \langle E(d\lambda)f, f \rangle = \langle E(\mathbb{R})f, f \rangle = \|f\|_{L^2}^2.$$

since  $E(\mathbb{R}) = I$ . Thus if  $T$  is self-adjoint, then  $\|e^{itT}f\|_{L^2} = \|f\|_{L^2}$ , i.e.  $e^{itT}$  is unitary. Also,

$$e^{itT}e^{isT}f = e^{i(t+s)T}f \quad \text{and} \quad e^{-itT}e^{itT} = I = e^{itT}e^{-itT}.$$

The above are group properties for  $e^{itT}$  as  $t$  varies.

**Example 10.0.3.** Let  $\mathcal{H} = L^2(X, \mu)$ , where  $\mu$  is a positive probability measure. Define  $T = M_\phi$  to be the multiplication operator by  $\phi$ , i.e.  $Tf = \phi f$  for  $\phi : X \rightarrow \mathbb{R}$ . We have

$$D(T) = \{f \in L^2 : \phi f \in L^2\},$$

which is dense in  $\mathcal{H}$  since  $\mu(|\phi| > M) \rightarrow 0$  as  $M \rightarrow \infty$ . So  $T$  is densely defined. Then

$$\langle Tf, g \rangle = \int \phi f \bar{g} \mu(dx) = \int f \overline{\phi g} \mu(dx) = \langle f, Tg \rangle$$

since  $\phi$  is real. So  $T$  is symmetric, which implies  $D(T) \subseteq D(T^*)$ . To get the reverse inclusion, notice that for  $g \in D(T^*)$ , we have  $\langle Tf, g \rangle = \langle f, h \rangle$  for some  $h \in \mathcal{H}$ , for all  $f \in D(T)$ . Then

$$\int f \overline{\phi g} \mu(dx) \int \phi f \bar{g} \mu(dx) = \langle Tf, g \rangle = \langle f, h \rangle = \int f \bar{h} \mu(dx).$$

Since  $D(T)$  is dense, we get  $\phi g = h \in \mathcal{H} = L^2(X, \mu)$ , so  $g \in D(T)$ . This gives  $D(T^*) \subseteq D(T)$ , so we see that  $D(T^*) = D(T)$  and  $T$  is self-adjoint. The spectral theorem then gives the spectral resolution  $E$ :

$$E(I)f = \chi_{\{\phi \in I\}} f,$$

where  $\chi$  denotes an indicator function.

**Example 10.0.4.** Let  $T = -\Delta$  on  $L^2(\mathbb{R}^d)$ . Clearly  $D(T) = H^2(\mathbb{R}^d)$  is dense in  $L^2(\mathbb{R}^d)$ . Integration by parts shows that  $-\Delta$  is symmetric. Now suppose  $g \in D(T^*)$ . Then

$$\langle -\Delta f, g \rangle = \langle f, h \rangle \quad \text{for all } f \in D(T) \text{ and some } h \in L^2.$$

Since  $\langle -\Delta f, g \rangle = \langle f, -\Delta g \rangle = Lf$  (note that  $-\Delta g \in \mathcal{S}'$ , i.e. in the sense of distributions), we have

$$|Lf| \leq \|f\|_{L^2} \|h\|_{L^2} = C\|f\|_{L^2},$$

where  $C = \|h\|_{L^2}$ . So by the Hahn-Banach theorem,  $L$  can be extended to a continuous functional on  $L^2$  such that  $\|L\|_{B(L^2, L^2)} \leq \|h\|_{L^2}$ . We can identify  $Lf = \langle f, -\Delta g \rangle$ , and thus

$$\|\Delta g\|_{L^2} \leq \|h\|_{L^2}.$$

This implies  $\Delta g \in L^2$ , so by the elliptic theory,  $g \in H^2 = D(T)$ . This gives  $D(T^*) \subseteq D(T)$  and thus  $D(T) = D(T^*)$  (since  $D(T) \subseteq D(T^*)$  because  $T$  is symmetric), so  $T$  is self-adjoint. To understand the spectral resolution, we can use the Fourier transform:  $\widehat{Tf} = \xi^2 \widehat{f}$ . Writing  $\mathcal{F}(f) = \widehat{f}$ , we have

$$\mathcal{F}T\mathcal{F}^{-1} = M_{\xi^2}.$$

Note that  $\mathcal{F}$  is unitary on  $L_x^2 \rightarrow L_\xi^2$  by Plancherel's theorem. This says that  $\mathcal{F}$  diagonalizes  $\Delta$ . Letting  $E_T$  be the spectral resolution for  $T$  and  $E_{\xi^2}$  be the spectral resolution for  $M_{\xi^2}$ , we have

$$\mathcal{F}E_T\mathcal{F}^{-1} = E_{\xi^2}.$$

Since  $E_\xi^2(I) = \chi_{\{\xi^2 \in I\}}$ , we can write

$$E_T(I)f = (\chi_{\{\xi^2 \in I\}} \widehat{f})^\vee.$$

**Example 10.0.5.** Let  $\mathcal{H} = L^2(\mathbb{R}^d)$  and  $T = -\Delta/2 + V$ , where  $V \in L^\infty$  is real. We claim that  $T$  is self-adjoint. It is easy to check that  $T$  is symmetric, so  $D(T) \subseteq D(T^*)$ , where  $D(T) = H^2(\mathbb{R}^d)$ . For the reverse inclusion, take  $g \in D(T^*)$ , so  $\langle Tf, g \rangle = \langle f, h \rangle$  for some  $h \in \mathcal{H}$ . Then

$$\langle -\Delta f/2, g \rangle = \langle f, h \rangle - \langle Vf, g \rangle,$$

so we have

$$|\langle -\Delta f/2, g \rangle| \leq (\|h\|_{L^2} + \|V\|_{L^\infty} \|g\|_{L^2}) \|f\|_{L^2}.$$

As before, the Hahn-Banach theorem implies that  $-\Delta g \in L^2$ , and elliptic theory yields  $g \in H^2 = D(T)$ . Thus  $D(T^*) \subseteq D(T)$ , so we see that  $T$  is self-adjoint.

## 10.2 More Spectral Theory

**Definition 10.1.** Define the following parts of  $\mathcal{H}$ :

$$\begin{aligned}\mathcal{H}_{\text{ac}} &= \{x \in \mathcal{H} : E_{x,x} \text{ is absolutely continuous}\}, \\ \mathcal{H}_{\text{s}} &= \{x \in \mathcal{H} : E_{x,x} \text{ is singular continuous}\}, \\ \mathcal{H}_{\text{pp}} &= \{x \in \mathcal{H} : E_{x,x} \text{ is pure point}\}.\end{aligned}$$

**Lemma 10.1.** *We have the decomposition  $\mathcal{H} = \mathcal{H}_{\text{pp}} \oplus \mathcal{H}_{\text{s}} \oplus \mathcal{H}_{\text{ac}}$ .*

*Proof.* We will show this next time. □

**Definition 10.2.** Define  $E_{\text{ac}} = E \circ P_{\text{ac}}$ , where  $P_{\text{ac}}$  is the projection onto  $\mathcal{H}_{\text{ac}}$ .

**Lemma 10.2.** *Let  $T$  be a self-adjoint operator and  $E$  be its spectral resolution. Then for a.e.  $\lambda$ ,*

$$\langle E_{\text{ac}}(d\lambda)f, g \rangle = \langle \frac{1}{2\pi i} (R(\lambda + i0) - R(\lambda - i0))f, g \rangle d\lambda,$$

where  $R(\lambda \pm i0) = \lim_{\epsilon \rightarrow 0} R(\lambda \pm i\epsilon)$ .