MATH 8803: Nonlinear Dispersive Equations

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Contents

Lecture 1

Jan. 6 — Introduction to Dispersion

1.1 Introduction to Dispersion

Definition 1.1. An evolution equation is *dispersive* if when no boundary conditions are imposed (e.g. on $\mathbb{R}^n$), its wave solutions spread out in space as they evolve in time.

Example 1.1.1. Two classic examples of dispersive equations are:

- The *Schrödinger equation*: $iu_t + \Delta u = 0$.
- The *Airy (linearized KdV) equation*: $u_t + u_{xxx} = 0$.

Remark. Consider the equation $u_t + p(\partial_x)u = 0$, where p is a polynomial, and a plane-wave solution

$$u(t,x) = e^{i(kx-\omega t)} = e^{ik(x-(\omega/k)t)}.$$

Here k is the *wave number* or *space frequency*, and ω is the *(time) frequency*. Plugging the plane-wave solution into the equation, we obtain the relation $\omega(k) = -ip(ik)$, i.e.

$$\frac{\omega(k)}{k} = \frac{1}{ik}p(ik).$$

The above equation is known as the *dispersive relation*. This gives the traveling speed of the plane-wave solution with wave number k, which is called the *phase velocity*.

Example 1.1.2. The following are some examples of dispersive relations:

- For the *linear advection equation* $u_t + cu_x = 0$ with $c \in \mathbb{R}$, one can compute that $\omega/k = c$.
- For the Schrödinger equation $iu_t + \frac{1}{2}\Delta u = 0$, we have $\omega/k = k/2 \in \mathbb{R}$.

  In this case of the Schrödinger equation, plane waves with large wave number (large space frequency) travel faster than low-frequency waves.

Remark. In general, dispersion means that different frequency plane waves travel at different speeds.

Remark. Given initial data u_0, we can write using the Fourier transform that

$$u_0 = \int \widehat{u}_0(k)e^{ikx}\,dk.$$

Then we get the solution u as

$$u(t,x) = \int \widehat{u}_0(k)e^{ik(x-(\omega(k)/k)t)}\,dk.$$

Example 1.1.3. In the case of the linear advection equation, we obtain the solution as

$$u(t,x) = \int \widehat{u}_0(k) e^{ik(x-ct)}\, dk = u_0(x-ct).$$

For the Schrödinger equation, we instead have the solution

$$u(t,x) = \int \widehat{u}_0(k) e^{ik(x-(k/2)t)}\, dk.$$

Since different k travels at different speeds, the original profile quickly spreads out.

Exercise 1.1. Calculate the dispersive relation ω/k for the linearized KdV equation $u_t + u_{xxx} = 0$.

Example 1.1.4. The *KdV equation* is given by

$$\partial_t u + \partial_{xxx} u + 6u\partial_x u = 0.$$

This equation is used to model shallow water surfaces, and is a nonlinear dispersive equation. Russell observed a great bump of water in a channel that traveled for a long time and kept its shape. This is due to the nonlinear effects in the KdV equation, and these effects are called *solitons*.

Definition 1.2. A *soliton* is a self-reinforcing solitary wave (a wave packet or pulse) that maintains its shape while traveling at a constant speed.

1.2 Fourier Transform and the Free Schrödinger Equation

Consider the following free Schrödinger equation:

$$\begin{cases} i\partial_t \psi + \frac{1}{2}\Delta\psi = 0, \\ \psi|_{t=0} = \psi_0. \end{cases}$$

We will solve this equation using the *Fourier transform*

$$\widehat{f}(\xi) = \int_{\mathbb{R}^d} e^{-ix\cdot\xi} f(x)\, dx.$$

Note that one can recover f from its Fourier transform via the *inversion formula*

$$f(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{ix\cdot\xi} \widehat{f}(\xi)\, d\xi.$$

Exercise 1.2. Check that $(\partial_{x_j} f)^\wedge = i\xi_j \widehat{f}$.

Applying the Fourier transform to the free Schrödinger equation, one has

$$i\partial_t\psi + \frac{1}{2}\Delta\psi = 0 \quad \xrightarrow{\text{F.T.}} \quad i\partial_t\widehat{\psi} - \frac{1}{2}|\xi|^2\widehat{\psi} = 0$$

and initial condition $\widehat{\psi}(0,\xi) = \widehat{\psi}_0(\xi)$. So for fixed ξ, we have an ODE, so we can solve the equation via

$$\widehat{\psi}(t,\xi) = e^{-i|\xi|^2 t/2}\widehat{\psi}_0(\xi).$$

Now by applying the inverse Fourier transform, we obtain the solution

$$\psi(t,x) = (2\pi)^{-d} \int e^{ix\xi}\widehat{\psi}(t,\xi)\,d\xi = (2\pi)^{-d} \int e^{ix\xi} e^{-i|\xi|^2 t/2}\widehat{\psi}_0(\xi)\,d\xi.$$

Recalling Plancherel's theorem that $\|f\|_{L^2} = C\|\widehat{f}\|_{L^2}$ (for a constant C independent of f), we obtain

$$\|\psi(t,x)\|_{L^2} = C\|\widehat{\psi}(t,\xi)\|_{L^2} = C\|\widehat{\psi}(0,\xi)\|_{L^2} = \|\psi(0,x)\|_{L^2} = \|\psi_0(x)\|_{L^2},$$

where the second equality follows by noticing that $e^{-i|\xi|^2 t/2}$ has modulus 1. This shows that the linear Schrödinger evolution preserves the L^2 norm of the solution.

Exercise 1.3. Compute that

$$\frac{d}{dt}\int_{\mathbb{R}^d} |\psi(t,x)|^2\,dx = 0.$$

This is an alternative way to show that the L^2 norm of the solution is preserved.

1.3 Sobolev Spaces

Definition 1.3. The *Sobolev spaces* $H^\gamma = W^{\gamma,2}$ for $\gamma \in \mathbb{R}$ are defined via the norm

$$\|f\|_{H^\gamma} = \left(\int_{\mathbb{R}^d} (1+|\xi|^2)^\gamma |\widehat{f}(\xi)|^2\,d\xi\right)^{1/2}.$$

The *homogeneous Sobolev spaces* $\dot{H}^\gamma$ are defined by the norm

$$\|f\|_{\dot{H}^\gamma} = \left(\int_{\mathbb{R}^d} |\xi|^{2\gamma} |\widehat{f}(\xi)|^2\,d\xi\right)^{1/2}.$$

Remark. If $\gamma \in \mathbb{N}$ and $d = 1$, then

$$\|f\|_{H^\gamma} \sim \sum_{m=0}^{\gamma} \|\partial_x^m f\|_{L^2}.$$

In particular, this means that $f \in H^\gamma$ if and only if $\partial_x^m f \in L^2$ for all $m \le \gamma$.

Exercise 1.4. Check that if $f_\lambda(x) = f(\lambda x)$, then $\widehat{f_\lambda}(\xi) = \lambda^{-d}\widehat{f}(\xi/\lambda)$.

Remark. In the Sobolev spaces, this means that (change variables $\eta = \xi/\lambda$ for the last equality)

$$\|f_\lambda\|_{\dot{H}^\gamma} = \left(\int_{\mathbb{R}^d} |\xi|^{2\gamma} |\widehat{f_\lambda}(\xi)|^2\,d\xi\right)^{1/2} = \left(\int_{\mathbb{R}^d} |\xi|^{2\gamma} |\lambda^{-d}\widehat{f}(\xi/\lambda)|^2\,d\xi\right)^{1/2} = \lambda^{\gamma-d/2}\|f\|_{\dot{H}^\gamma}.$$

Lemma 1.1. *In the Schrödinger equation, $\|\psi(t)\|_{H^\gamma} = \|\psi_0\|_{H^\gamma}$ and $\|\psi(t)\|_{\dot{H}^\gamma} = \|\psi_0\|_{\dot{H}^\gamma}$ for all t and γ.*

Proof. We can compute that

$$\|\psi(t)\|_{\dot{H}^\gamma} = \int_{\mathbb{R}^d} |\xi|^{2\gamma}|\widehat{\psi}(t,\xi)|^2\,d\xi = \int_{\mathbb{R}^d} |\xi|^{2\gamma}|e^{-i|\xi|^2/2}\widehat{\psi}_0(\xi)|^2\,d\xi = \int_{\mathbb{R}^d} |\xi|^{2\gamma}|\widehat{\psi}_0(\xi)|^2\,d\xi = \|\psi_0\|_{\dot{H}^\gamma}.$$

The same argument works for the H^γ case after replacing $|\xi|^{2\gamma}$ with $(1+|\xi|^2)^\gamma$. $\square$

Lecture 2

Jan. 8 — Special Solutions

2.1 Special Solutions

Example 2.0.1. The following are special solutions to the Schrödinger equation:

1. Gaussian: $\psi_0 = e^{-|x|^2/2}$. One can compute the Fourier transform and get

$$\widehat{\psi}_0(\xi) = \int_{\mathbb{R}^d} e^{-ix\cdot\xi} e^{-|x|^2/2}\,dx = \int_{\mathbb{R}^d} e^{-|x+i\xi|^2/2} e^{-|\xi|^2/2}\,dx = e^{-|\xi|^2/2}\int_{\mathbb{R}^d} e^{-|x+i\xi|^2/2}\,dx.$$

   The last integral is a contour integral in the complex plane along $\Im z = \xi$, and we can deform the contour via Cauchy's theorem to the real axis to obtain (the integrand is analytic on $0 \le \Im z \le \xi$)

$$\widehat{\psi}_0(\xi) = e^{-|\xi|^2/2}\int_{\mathbb{R}^d} e^{-|x|^2/2}\,dx = (2\pi)^{d/2} e^{-|\xi|^2/2}.$$

   Then taking inverse Fourier transforms, we obtain the solution

$$\begin{aligned}\psi(t,x) = (2\pi)^{-d}\int_{\mathbb{R}^d} e^{i(x\cdot\xi - |\xi|^2 t/2)}\widehat{\psi}_0(\xi)\,d\xi &= (2\pi)^{-d/2}\int_{\mathbb{R}^d} e^{i(x\cdot\xi-|\xi|^2t/2)} e^{-|\xi|^2/2}\,d\xi \\ &= (2\pi)^{-d/2}\int_{\mathbb{R}^d} e^{-\frac{1}{2}(1+it)|\xi|^2} e^{ix\cdot\xi}\,d\xi.\end{aligned}$$

   Now formally put $\eta = (1+it)^{1/2}\xi$ to get

$$\psi(t,x) = (2\pi)^{-d/2}(1+it)^{-d/2}\int_{\mathbb{R}^d} e^{-\frac{1}{2}|\eta|^2} e^{ix\eta/(1+it)^{1/2}}\,d\eta.$$

   Fill in the details of the above change of variables as an exercise (e.g. one has to worry about choosing a branch cut when taking the square root). Computing the integral explicitly, one obtains

$$\psi(t,x) = (1+it)^{-d/2} e^{-|x|^2/(2(1+it))}.$$

   One can from this that ψ has decay in time. Furthermore, one can see that

$$|\psi(t,x)|^2 = (1+t^2)^{-d/2} e^{-|x|^2/(1+t^2)}.$$

   From this we can observe an L^∞ decay of ψ like $t^{-d/2}$, and that the influence region of the solution grows like order t. We can also see again from this explicit computation that $\|\psi(t)\|_{L^2} = C$.

2. Modulated Gaussian: $\psi_0 = e^{-|x|^2/2}e^{ix\cdot v}$. The Fourier transform of this initial data is
$$\widehat{\psi}_0(\xi) = (2\pi)^{d/2}e^{-i|\xi-v|^2/2}.$$
So the solution corresponding to this initial data is
$$\begin{aligned}\psi(t,x) = (2\pi)^{-d}\int_{\mathbb{R}^d} e^{i(x\cdot\xi-|\xi|^2t/2)}\widehat{\psi}_0(\xi)\,d\xi &= (2\pi)^{-d/2}\int_{\mathbb{R}^d} e^{i(x\cdot\xi-|\xi|^2t/2)}e^{-|\xi-v|^2}\,d\xi \\ &= e^{ix\cdot v}e^{-|v|^2t/2}(2\pi)^{-d/2}\int_{\mathbb{R}^d} e^{i(x-vt)\cdot\xi}e^{-(1+it)|\xi|^2/2}\,d\xi \\ &= e^{ix\cdot v}e^{-|v|^2t/2}(1+it)^{-d/2}\exp\left(-\frac{|x-vt|^2}{2(1+it)}\right).\end{aligned}$$
From this we can see that the influence region of the solution moves with velocity v.

3. Fundamental solution: We want a *fundamental solution* K such that K solves
$$i\partial_t K + \frac{1}{2}\Delta K = 0 \quad \text{and} \quad K|_{t=0} = \delta_0.$$
We will find K by scaling arguments. Suppose such a K exists. Then we must have
$$\psi(t,x) = \int_{\mathbb{R}^d} K(t,x-y)\psi_0(y)\,dy \tag{1}$$
since $K|_{t=0} = \delta_0$. Now define the scaling $\psi_\lambda(t,x) = \psi(\lambda^2 t, \lambda x)$. Then ψ_λ also solves
$$i\partial_t\psi_\lambda + \frac{1}{2}\Delta\psi_\lambda = 0$$
and we have the initial condition $\psi_\lambda(0,x) = \psi_0(\lambda x)$. Then
$$\psi_\lambda(t,x) = \int_{\mathbb{R}^d} K(t,x-y)\psi_0(\lambda y)\,dy = \psi(\lambda^2 t, \lambda x).$$
Setting $t' = \lambda^2 t$, $x' = \lambda x$, and $y' = \lambda y$, we get
$$\psi(t',x') = \frac{1}{\lambda^d}\int_{\mathbb{R}^d} K\left(\frac{t'}{\lambda^2}, \frac{x'-y'}{\lambda}\right)\psi_0(y')\,dy'. \tag{2}$$
Comparing (1) and (2), we see that we must have
$$K(t,x-y) = \lambda^{-d}K\left(\frac{t}{\lambda^2}, \frac{x-y}{\lambda}\right).$$
Setting $u = x-y$, we get
$$K(t,u) = \lambda^{-d}K\left(\frac{t}{\lambda^2}, \frac{u}{\lambda}\right).$$
Thus we expect $K(t,x) = t^{-d/2}\Phi(|x|^2/t)$ for some Φ. Now we use the fact that $i\partial_t K + \frac{1}{2}\Delta K = 0$. Setting $m = |x|^2/t$, one can plug in the above guess for K to obtain (note that $\Delta = \nabla\cdot\nabla$)
$$-\frac{id}{2}t^{-d/2-1}\Phi(m) - it^{-d/2}\Phi'(m)\frac{m}{t} + \frac{1}{2}t^{-d/2}\nabla\cdot\left(\frac{2x}{t}\Phi'(m)\right) = 0.$$

Then we get

$$-i\frac{d}{2}\Phi(m) - im\Phi'(m) + d\Phi'(m) + 2m\Phi''(m) = 0,$$

which gives

$$d\left(\Phi'(m) - \frac{i}{2}\Phi(m)\right) + 2m\frac{d}{dm}\left(\Phi'(m) - \frac{i}{2}\Phi(m)\right) = 0.$$

Now observe that $\Phi(m) = e^{im/2}$ solves the above equation. Since $\Phi(m)$ solves the equation, $c\Phi(m)$ also solves the equation for any $c \in \mathbb{C}$, and thus we have

$$K(t,x) = ct^{-d/2}\Phi(|x|^2/t) = ct^{-d/2}e^{i|x|^2/2t}.$$

To determine c, we use $K|_{t=0} = \delta_0$, from which one can obtain $c = (2\pi i)^{-d/2}$. Thus

$$K(t,x) = (2\pi i t)^{-d/2}e^{i|x|^2/2t}.$$

The rough computation is that since $\widehat{K}(0,\xi) = 1$, we have

$$K = (2\pi)^{-d}\int_{\mathbb{R}^d} e^{i(x\cdot\xi - |\xi|^2t/2)}\widehat{K}(0,\xi)\,d\xi = (2\pi)^{-d}\int_{\mathbb{R}^d} e^{i(x\cdot\xi - |\xi|^2t/2)}\,d\xi.$$

This is not necessarily integrable a priori, but one can take limits and obtain

$$\begin{aligned} K = (2\pi)^{-d}\lim_{\epsilon\to 0^+}\int_{\mathbb{R}^d} e^{ix\cdot\xi}e^{-(\epsilon+it)|\xi|^2/2}\,d\xi &= \lim_{\epsilon\to 0^+}(\epsilon+it)^{-d/2}(2\pi)^{-d/2}e^{-|x|^2/(2(\epsilon+it))} \\ &= (2\pi i t)^{-d/2}e^{-|x|^2/2it}. \end{aligned}$$

Note that this computation matches the result of the previous scaling argument.

Theorem 2.1. *Let $\psi_0 \in \mathcal{S}(\mathbb{R}^d)$.*[1] *Then there exists a solution to*

$$\begin{cases} i\partial_t\psi + \frac{1}{2}\Delta\psi = 0, \\ \psi|_{t=0} = \psi_0, \end{cases}$$

which is unique and given by

$$\psi(t,x) = \int_{\mathbb{R}^d} K(t,x-y)\psi_0(y)\,dy = (2\pi i t)^{-d/2}\int_{\mathbb{R}^d} e^{-|x-y|^2/2it}\psi_0(y)\,dy.$$

Proof. This theorem is a summary of the results of the previous explicit computations. □

Remark. Recall that the Schrödinger evolution preserves the L^2 norm of a solution, i.e.

$$\|\psi(t)\|_{L^2} = \|\psi(0)\|_{L^2} = \|\psi_0\|_{L^2}.$$

The above theorem also gives an L^∞ bound (a so-called *dispersive estimate*)

$$\|\psi(t)\|_{L^\infty} \le |2\pi t|^{-d/2}\int_{\mathbb{R}^d}|\psi_0(y)|\,dy = |2\pi t|^{-d/2}\|\psi_0\|_{L^1}.$$

[1] Here $\mathcal{S}(\mathbb{R}^d)$ is the space of *Schwartz functions*.

Lecture 3

Jan. 15 — Strichartz Estimates

3.1 Interpolation Results

Remark (Interpolation)**.** Consider a linear operator T which maps $T : L^{p_1} \to L^{q_1}$ and $T : L^{p_2} \to L^{q_2}$, where $1 \le p_1 \le p_2 \le \infty$. Then T also maps $T : L^p \to L^q$ for any p, q such that

$$\frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2} \quad \text{and} \quad \frac{1}{q} = \frac{\theta}{q_1} + \frac{1-\theta}{q_2}$$

for some $0 \le \theta \le 1$. More specifically, if $\|Tf\|_{L^{q_1}} \le C_1 \|f\|_{L^{p_1}}$ and $\|Tf\|_{L^{q_2}} \le C_2 \|f\|_{L^{p_2}}$, then

$$\|Tf\|_{L^q} \le C_1^\theta C_2^{1-\theta} \|f\|_{L^p}.$$

This L^p interpolation is a result from real and functional analysis. Note that by interpolation, we have

$$\|\psi\|_{L^{p'}(\mathbb{R}^d)} \le C|t|^{-d(1/p-1/2)} \|\psi_0\|_{L^p(\mathbb{R}^d)}$$

for $1 \le p \le 2$, where p' is the *Hölder conjugate* of p, i.e. $1/p' + 1/p = 1$.

3.2 Strichartz Estimates

Remark. We will now consider the inhomogeneous Schrödinger equation:

$$\begin{cases} i\psi_t + \frac{1}{2}\Delta\psi = F, & F \in \mathcal{S}_{x,t} \\ \psi(0) = \psi_0, & \psi_0 \in \mathcal{S}, \end{cases}$$

where $F \in \mathcal{S}_{x,t}$ means that F is Schwartz in both x and t. We can solve this via the *Duhamel formula*:

$$\psi(t) = e^{it\Delta/2}\psi_0 - i\int_0^t e^{i(t-s)\Delta/2} F(s)\, ds,$$

where $e^{it\Delta/2}$ is the *linear propagator* given by

$$e^{it\Delta/2}\psi_0 = (e^{-it|\xi|^2/2}\widehat{\psi}_0)^\vee = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{ix\cdot\xi} e^{-it|\xi|^2/2}\widehat{\psi}_0(\xi)\, d\xi.$$

Theorem 3.1 (Strichartz estimates)**.** *For $p' = 2 + 4/d$ and $1/p' + 1/p = 1$, we have the estimate*[1]

$$\|\psi\|_{L^{p'}_{t,x}(\mathbb{R}\times\mathbb{R}^d)} \lesssim \|\psi_0\|_{L^2_x(\mathbb{R}^d)} + \|F\|_{L^p_{t,x}(\mathbb{R}\times\mathbb{R}^d)}.$$

[1] Here $A \lesssim B$ means that $A \le CB$ for some prescribed constant C.

Remark. If $F = 0$, this is the bound
$$\|\psi\|_{L^{p'}_{t,x}} \lesssim \|\psi_0\|_{L^2}$$
for $p' > 2$. Formally, this means that we gain integrability in x. Note that this gain in integrability is not pointwise in time, i.e. we do *not* have $\|\psi(t)\|_{L^\infty_t L^{p'}_x} \lesssim \|\psi_0\|_{L^2_x}$. We must instead average over t.

Remark. Why p' and why do we pick p' in the time integration? Actually, p' is the only possible choice for the above result. This follows by a scaling argument: Set
$$\psi_\lambda(t,x) = \psi(\lambda^2 t, \lambda x), \quad (\psi_\lambda)_0(x) = \psi_0(\lambda x), \quad F_\lambda(t,x) = \lambda^2 F(\lambda^2 t, \lambda x).$$
Then ψ_λ solves the equation
$$\begin{cases} i\partial_t \psi_\lambda + \frac{1}{2}\Delta\psi_\lambda = F_\lambda, \\ \psi_\lambda(0) = (\psi_\lambda)_0. \end{cases}$$
If the above theorem makes sense, then it must hold for both ψ_λ and ψ. Now
$$\|\psi_\lambda\|_{L^{p'}_{t,x}} = \lambda^{-d/p'}\lambda^{-2/p'}\|\psi\|_{L^{p'}_{t,x}}$$
by a change of variables, and
$$\|(\psi_\lambda)_0\|_{L^2_x} = \lambda^{-d/2}\|\psi_0\|_{L^2_x}.$$
Now if $F = 0$, then we have the estimates
$$\|\psi\|_{L^{p'}_{t,x}} \lesssim \|\psi_0\|_{L^2_x} \quad \text{and} \quad \|\psi_\lambda\|_{L^{p'}_{t,x}} \lesssim \|(\psi_\lambda)_0\|_{L^2_x}, \tag{$*$}$$
Using the scaling computations in the second estimate in $(*)$ implies that
$$\|\psi\|_{L^{p'}_{t,x}}\lambda^{-d/p'}\lambda^{-2/p'} \lesssim \lambda^{-d/2}\|\psi_0\|_{L^2_x}.$$
This inequality should hold independent of λ, since otherwise taking $\lambda \to \infty$ or $\lambda \to 0$ yields a contradiction with the first inequality in $(*)$. Thus the powers in λ should match:
$$-\frac{d}{p'} - \frac{2}{p'} = -\frac{d}{2},$$
so we find that p' must be
$$p' = \frac{d+2}{d/2} + \frac{2d+4}{d} = 2 + \frac{4}{d}.$$
This uniquely determines p'. Now consider $F \neq 0$. Using a similar computation as before, we have
$$\|F_\lambda\|_{L^q_{t,x}} = \lambda^2\lambda^{-d/q}\lambda^{-2/q}\|F\|_{L^q_{t,x}}.$$
Then the theorem says that $\|\psi_\lambda\|_{L^{p'}_{t,x}} \lesssim \|\psi_0\|_{L^2_x} + \|F\|_{L^q_{t,x}}$, so we have
$$\|\psi\|_{L^{p'}_{t,x}}\lambda^{-d/p'}\lambda^{-2/p'} \lesssim \lambda^{-d/2}\|\psi_0\|_{L^2_x} + \lambda^2\lambda^{-d/q}\lambda^{-2/q}\|F\|_{L^q_{t,x}}.$$
Again the estimate should hold independent of λ, so the powers in λ must match:
$$-\frac{d}{p'} - \frac{2}{p'} = 2 - \frac{d}{q} - \frac{2}{q} = -\frac{d}{2},$$
which then gives p as
$$p = \left(1 - \frac{1}{p'}\right)^{-1} = \left(1 - \frac{d}{2d+4}\right)^{-1} = \frac{2d+4}{d+4}.$$

Lemma 3.1. *Let $\psi(t) = e^{it\Delta/2}\psi_0$. Then for $1 \le p \le 2$,*

$$\|\psi(t)\|_{L_x^{p'}(\mathbb{R}^d)} \lesssim |t|^{-d(1/p-1/2)}\|\psi_0\|_{L_x^p(\mathbb{R}^d)}.$$

Proof. This is the interpolation result from the beginning of class. $\square$

Lemma 3.2 (Hardy-Littlewood-Sobolev inequality)**.** *Let $0 < \alpha < 1$ and $g \in \mathcal{S}(\mathbb{R})$. Let*

$$(T_\alpha g)(t) = \int_{-\infty}^{\infty} |t-s|^{-\alpha} g(s)\, ds.$$

Then we have $\|T_\alpha g\|_{L^q(\mathbb{R})} \lesssim \|g\|_{L^p(\mathbb{R})}$, where $1 < p < q < \infty$ such that $1 + 1/q = \alpha + 1/p$.

Proof. One approach is via harmonic analysis and maximal functions. An alternative approach can be found in Theorem 4.3 of Analysis by Lieb and Loss. $\square$

Remark. Recall *Young's inequality* that for

$$h(t) = \int f(t-s)g(s)\, dx,$$

we have $\|h\|_{L^r} \le \|f\|_{L^p}\|g\|_{L^q}$, where $1/r + 1 = 1/q + 1/p$. The Hardy-Littlewood-Sobolev inequality can be seen as a generalized Young's inequality: If $f(s) = |s|^{-\alpha}$, then f barely fails to be in $L^{1/\alpha}$. Informally, we can think of "$f \in L^{1/\alpha}$," and the standard Young's inequality would imply Hardy-Littlewood-Sobolev.

Remark. We have $q > p$ in the Hardy-Littlewood-Sobolev inequality, so we gain some integrability via fractional integration for $p > 1$ (the type of integral defining $T_\alpha g$ is known as *fractional integration*).

Lecture 4

Jan. 22 — Strichartz Estimates, Part 2

4.1 Proof of Strichartz Estimates

Proof of Theorem 3.1. The first step is a TT^* argument. Define the operator T by $Tf = e^{it\Delta/2}f$. We know that $T : L^2_x(\mathbb{R}^d) \to L^\infty_t(\mathbb{R} \to L^2_x(\mathbb{R}^d))$. The adjoint $T^* : L^1_t L^2_x \to L^2_x$ is defined via the relation

$$\langle f, T^*G\rangle_{L^2_x} = \langle Tf, G\rangle_{L^2_{t,x}} = \iint (e^{it\Delta/2}f)(x)\overline{G}(t,x)\,dtdx = \int f(x)\int \overline{(e^{-it\Delta/2}G(t,\cdot))}(x)\,dtdx,$$

and so we have the formula

$$T^*G = \int (e^{-it\Delta/2}G(t,\cdot))(x)\,dt.$$

Then we can see that

$$(TT^*G)(t,x) = \int (e^{i(t-s)\Delta/2}G(s,\cdot))(x)\,ds = \left(e^{it\Delta/2}\int e^{-is\Delta/2}G(s,\cdot)\,ds\right)(x).$$

Note that there is a convolution structure in the time variable. Clearly $TT^* : L^1_t L^2_x \to L^\infty_x L^2_x$. Then the goal for now will be to show that

$$\|TT^*G\|_{L^{p'}_{t,x}} \le C\|G\|_{L^p_{t,x}}.$$

To do this, observe that by the above expression for TT^*G and Minkowski's inequality, we have

$$\|TT^*G\|_{L^{p'}_x} \le \int \|e^{i(t-s)\Delta/2}G(s)\|_{L^{p'}_x}\,ds \le C\int |t-s|^{-d(1/p-1/2)}\|G(s)\|_{L^p_x}\,ds,$$

where the second inequality follows by Lemma 3.1. Now by Lemma 3.2,

$$\|TT^*G\|_{L^q_t L^{p'}_x} \le C\|G(s)\|_{L^p_{t,x}}$$

if $1/q + 1 = d(1/p-1/2) + 1/p$ (also check that $0 < \alpha = d(1/p-1/2) < 1$, where $p = (2d+4)/(d+4)$). From this relation, we find that we must have $q = 2 + 4/d = p'$, so we have shown the goal.

Thus we have proved that $TT^* : L^p_{t,x} \to L^{p'}_{t,x}$, where $p = (2d+4)/(d+4)$. Now

$$\|T^*G\|^2_{L^2_x} = \langle T^*G, T^*G\rangle_{L^2_x} = \langle TT^*G, G\rangle_{L^2_{t,x}} \le \|TT^*G\|_{L^{p'}_{t,x}}\|G\|_{L^p_{t,x}} \le C\|G\|^2_{L^p_{t,x}},$$

where the first inequality follows by Hölder's inequality and the second follows from the goal we just proved. Thus we conclude that $T^* : L^p_{t,x} \to L^2_x$, and that $T : L^2_x \to L^{p'}_{t,x}$ by duality. Therefore,

$$\|T\psi_0\|_{L^{p'}_{t,x}} = \|e^{it\Delta/2}\psi_0\|_{L^{p'}_{t,x}} \le C\|\psi_0\|_{L^2_x},$$

so we have proved the Strichartz estimates when $F = 0$.

In the case when $F \neq 0$, by Duhamel's formula we have

$$\psi(t) = e^{it\Delta/2}\psi_0 - i\int_0^t e^{i(t-s)\Delta/2}F(s)\,ds,$$

so by the triangle inequality we find that

$$\|\psi\|_{L^{p'}_{t,x}} \leq \|e^{it\Delta/2}\psi_0\|_{L^{p'}_{t,x}} + \left\|\int_0^t e^{i(t-s)\Delta/2}F(s)\,ds\right\| \leq C\|\psi_0\|_{L^2_x} + \left\|\int_0^t \|e^{i(t-s)\Delta/2}F(s)\|_{L^{p'}_x}\,ds\right\|_{L^{p'}_t},$$

where the last inequality on the second term follows by Minkowski's inequality. Then using Lemma 3.1 and Lemma 3.2 in the same fashion as before, we can bound the latter term by

$$\left\|\int_0^t \|e^{i(t-s)\Delta/2}F(s)\|_{L^{p'}_x}\,ds\right\|_{L^{p'}_t} \leq C\left\|\int_{-\infty}^{\infty} |t-s|^{-d(1/p-1/2)}\|F(s)\|_{L^{p'}_x}\,ds\right\|_{L^{p'}_t} \leq C\|F\|_{L^p_{t,x}}.$$

Plugging this bound back in gives the desired inequality $\|\psi\|_{L^{p'}_{t,x}} \leq C\|\psi_0\|_{L^2_x} + C\|F\|_{L^p_{t,x}}$. □

Remark. Note that the term

$$TT^*G = \int_{-\infty}^{\infty} e^{i(t-s)\Delta/2}G(s)\,ds$$

looks similar to the term from the Duhamel formula

$$\int_0^t e^{i(t-s)\Delta/2}F(s)\,ds.$$

However, it is possible that these two have different estimates, which is why we had to argue separately.

4.2 Strichartz Estimates and Harmonic Analysis

Remark. The original intention of Strichartz for these estimates was for use in harmonic analysis. The Strichartz estimates can actually be derived from the *Stein-Tomas restriction theorem*.

Let $S \subseteq \mathbb{R}^n$ with $n = d+1$, where S is a hypersurface. If $f \in L^1$, then one can show (e.g. using the Riemann-Lebesgue lemma) that $\widehat{f} \in L^\infty$. So we can conclude that $\widehat{f}$ has pointwise meaning, i.e. $\widehat{f}(\xi)$ makes sense pointwise. In particular, we can make sense of $\widehat{f}(\xi)$ on the hypersurface S.

On the other hand, if $f \in L^2$, then by Plancherel's theorem, $\widehat{f} \in L^2$ as well. But an L^2 function has no pointwise interpretation, i.e. we can modify it on a set of measure zero without changing the function. In particular, it is meaningless to restrict the function $\widehat{f}$ to S, since S is a set of measure zero in $\mathbb{R}^n$.

In general, what about $f \in L^p$ for $1 < p < 2$? This is the topic of the *restriction theorems* in harmonic analysis. It turns out that the choice of which p work depends on the "curvature" of S.

Theorem 4.1 (Stein-Tomas restriction theorem)**.** *Let $n = d+1$ and $S \subseteq \mathbb{R}^n$ be a hypersurface with non-vanishing Gaussian curvature. Let σ_S be the corresponding surface measure, and let ϕ be a compactly supported on S. Then we have*

$$\|(\phi\sigma_S)^\vee\|_{L^r(\mathbb{R}^n)} \leq C\|\phi\|_{L^2(\sigma_S)},$$

where $r = (2n+2)/(n-1)$.

Remark. Now recall the explicit formula for a solution ψ to the Schrödinger equation:

$$\psi(t,s) = \int e^{i(x\cdot\xi - t|\xi|^2/2)}\widehat{\psi}_0(\xi)\,d\xi.$$

Also define the hypersurface $S = \{(\xi,\tau) : \tau = -|\xi|^2/2, \xi \in \mathbb{R}^d\}$.[1] Then

$$\psi(t,x) = (\phi\sigma_S)^\vee(t,x), \quad \phi(\xi,\tau) = \widehat{\psi}_0(\xi), \quad \sigma_S(d\xi,d\tau) = (2\pi)^d d\xi.$$

Indeed, we can see that

$$\psi(t,x) = (2\pi)^{-d}\int e^{i(x\cdot\xi + t\tau)}\phi(\xi,\tau)\,\sigma_S(d\xi,d\tau) = \int e^{i(x\cdot\xi - t|\xi|^2/2)}\widehat{\psi}_0(\xi)\,d\xi.$$

Then, the Stein-Tomas restriction theorem tells us that $\|(\phi\sigma_S)^\vee\|_{L^r(\mathbb{R}^n)} \le C\|\phi\|_{L^2(\sigma_S)}$, which implies

$$\|\psi\|_{L^{2+4/d}_{t,x}(\mathbb{R}\times\mathbb{R}^d)} \le C\|\widehat{\psi}_0\|_{L^2_\xi} \le C\|\psi_0\|_{L^2_x},$$

where the last inequality follows by Plancherel's theorem. The estimate above technically only holds for those ψ where $\widehat{\psi}_0$ is compactly supported, but one can extend this by a density argument to $\psi_0 \in L^2_x$.

See Chapter 11 of Muscalu-Schlag (Volume I) for more details.

[1]This particular hypersurface is called the *characteristic surface* of the Schrödinger equation.

Lecture 5

Jan. 27 — Kato Smoothing

5.1 Kato Smoothing

Theorem 5.1 (Kato 1/2 smoothing estimate)**.** *Let $d \geq 2$ and $\chi \geq 0$ be a smooth cutoff function such that $\widehat{\chi}$ is compactly supported. Then we have the estimate*

$$\|\chi(x)(-\Delta)^{1/4}e^{it\Delta/2}f\|_{L^2_{x,t}} \leq C\|f\|_{L^2}.$$

Remark. The above theorem says that when we localize, we gain a smoothing effect (1/2 derivatives). Also note that this effect is not pointwise, but rather after integrating in time.

Remark. Let $f(t) = e^{it\Delta/2}f_0$, where $f_0 = e^{-|x|^2/2}e^{-ixv}$. Think of this as a quantum particle at the origin with an initial velocity v, so that the particle will stay in $B(0,1)$ for a period of order $O(1/|v|)$. Then

$$\left(\int_0^{1/|v|}\int_{B(0,1)}|f|^2\,dxdt\right)^{1/2} \sim \left(\frac{1}{|v|}\right)^{1/2} = \frac{1}{|v|^{1/2}}.$$

Now $(-\Delta)^{1/2}f \sim |v|^{1/2}$, so these factors cancel each other out. This matches the above estimate.

Theorem 5.2 (1-D Kato smoothing estimate)**.** *For $d = 1$, we have*

$$\sup_x \|(-\Delta)^{1/4}e^{it\Delta/2}f\|_{L^2_t} \leq C\|f\|_{L^2}.$$

Proof. For $d = 1$, by the Fourier transform we have

$$\begin{aligned}
(-\Delta)^{1/4}e^{it\Delta/2}f &= \frac{1}{2\pi}\int_{\mathbb{R}} e^{ix\xi}|\xi|^{1/2}e^{-it\xi^2/2}\widehat{f}(\xi)\,d\xi \\
&= \frac{1}{2\pi}\int_{-\infty}^0 e^{ix\xi}|\xi|^{1/2}e^{-it\xi^2/2}\widehat{f}(\xi)\,d\xi + \frac{1}{2\pi}\int_0^\infty e^{ix\xi}|\xi|^{1/2}e^{-it\xi^2/2}\widehat{f}(\xi)\,d\xi.
\end{aligned}$$

Since we want an L^2_t bound, it indicates that we should apply Plancherel's theorem in time. We will prove the estimate for the latter integral, and the former integral is left as an exercise. Set $\eta = \xi^2$ with $d\eta = 2\xi\,d\xi = 2\sqrt{\eta}\,d\xi$. Applying this change of variables, we obtain

$$(-\Delta)^{1/4}e^{it\Delta/2}f = \frac{1}{2\pi}\int_0^\infty e^{ix\sqrt{\eta}}|\eta|^{1/4}e^{-it\eta/2}\widehat{f}(\sqrt{\eta})\frac{1}{2\sqrt{\eta}}\,d\eta = \frac{1}{4\pi}\int_0^\infty e^{ix\sqrt{\eta}}|\eta|^{-1/4}e^{-it\eta/2}\widehat{f}(\sqrt{\eta})\,d\eta.$$

Fix x and denote $h(\eta) = e^{ix\sqrt{\eta}}|\eta|^{-1/4}\widehat{f}(\sqrt{\eta})$. Then we have

$$(-\Delta)^{1/4}e^{it\Delta/2}f = \frac{1}{4\pi}\int_0^\infty e^{-it\eta/2}h(\eta)\,d\eta = (\star).$$

By Plancherel's theorem in time, we see that

$$\|(\star)\|_{L^2_t} \le \|h(\eta)\|_{L^2_\eta}.$$

Then we can estimate (setting $z = \sqrt{\eta}$ with $dz = d\eta/(2\sqrt{\eta})$)

$$\int_0^\infty |h(\eta)|^2\,d\eta = \int_0^\infty |\eta|^{-1/2}|\widehat{f}(\sqrt{\eta})|^2\,d\eta = 2\int_0^\infty |\widehat{f}(z)|^2\,dz = C\|f\|_{L^2}^2,$$

where the last inequality follows by Plancherel's theorem in z. This yields $\|(\star)\|_{L^2_t} \le C\|f\|_{L^2}$, and since these estimates are all independent of x, we get that $\sup_x \|(\star)\|_{L^2_t} \le C\|f\|_{L^2}$. Putting this together with an identical estimate for the other integral, we obtain the desired bound. □

Remark. The above 1-D version is stronger and implies the statement

$$\|\chi(-\Delta)^{1/4}e^{it\Delta/2}f\|_{L^2_{x,t}} \le C\|f\|_{L^2}$$

for the 1-D case. Check this as an exercise.

5.2 Coarea Formula

Remark. In dimension d, suppose we have two nice functions g and u such that $u^{-1}(t)$ is a $(d-1)$-dimensional hypersurface. Then the *coarea formula* says that

$$\int_{\mathbb{R}^d} g(x)|\nabla u(x)|\,dx = \int_{\mathbb{R}}\int_{\{u(x)=t\}} g(x)\,d\sigma(x)dt,$$

where $\sigma(x)$ is the surface measure on $\{u(x) = t\}$.

Note that for $d = 1$, this says that

$$\int g(x)|\partial_x u|\,dx = \int_{\mathbb{R}}\left(\int_{\{u(x)=t\}} g(x)\,dx\right)dt = \int_{\mathbb{R}} g(u^{-1}(t))\,dt.$$

In particular, this is the change of variables formula where $\eta = u^{-1}(t)$ (so $u(\eta) = t$ and $dt = \partial_\eta u\,d\eta$).

Lemma 5.1. *Let $F \in C_0^\infty$ and ϕ be smooth. Then one has*

$$\int_{\mathbb{R}}\int_{\mathbb{R}^d} e^{i\lambda\phi(x)}F(x)\,dxd\lambda = (2\pi)^d\int_{\{\phi=0\}}\frac{F(x)}{|\nabla\phi(x)|}\,d\sigma(x).$$

Proof. By the coarea formula (using $g(x) = e^{i\lambda\phi(x)}F(x)/|\nabla\phi(x)|$ and $u = \phi$), we have

$$\int_{\mathbb{R}}\int_{\mathbb{R}^d} e^{i\lambda\phi(x)}F(x)\,dxd\lambda = \int_{\mathbb{R}}\int_{\mathbb{R}} e^{i\lambda y}\int_{\{\phi=y\}}\frac{F(x)}{|\nabla\phi(x)|}\,d\sigma(x)dyd\lambda.$$

Denote by $h(y)$ the integral

$$h(y) = \int_{\{\phi=y\}} \frac{F(x)}{|\nabla\phi(x)|}\, d\sigma(x).$$

Then we can see that

$$\int_{\mathbb{R}}\int_{\mathbb{R}} e^{i\lambda y}\int_{\{\phi=y\}} \frac{F(x)}{|\nabla\phi(x)|}\, d\sigma(x)dyd\lambda = \int_{\mathbb{R}}\int_{\mathbb{R}} e^{i\lambda y} h(y)\, dyd\lambda = \int_{\mathbb{R}} \widehat{h}(\lambda)\, d\lambda = (2\pi)^d h(0).$$

This gives the desired equality after plugging in the definition of $h(0)$. $\square$