

# MATH 8803: Nonlinear Dispersive Equations

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# Lecture 1

## Jan. 6 — Introduction to Dispersion

### 1.1 Introduction to Dispersion

**Definition 1.1.** An evolution equation is *dispersive* if when no boundary conditions are imposed (e.g. on  $\mathbb{R}^n$ ), its wave solutions spread out in space as they evolve in time.

**Example 1.1.1.** Two classic examples of dispersive equations are:

- The *Schrödinger equation*:  $iu_t + \Delta u = 0$ .
- The *Airy (linearized KdV) equation*:  $u_t + u_{xxx} = 0$ .

**Remark.** Consider the equation  $u_t + p(\partial_x)u = 0$ , where  $p$  is a polynomial, and a plane-wave solution

$$u(t, x) = e^{i(kx - \omega t)} = e^{ik(x - (\omega/k)t)}.$$

Here  $k$  is the *wave number* or *space frequency*, and  $\omega$  is the *(time) frequency*. Plugging the plane-wave solution into the equation, we obtain the relation  $\omega(k) = -ip(ik)$ , i.e.

$$\frac{\omega(k)}{k} = \frac{1}{ik}p(ik).$$

The above equation is known as the *dispersive relation*. This gives the traveling speed of the plane-wave solution with wave number  $k$ , which is called the *phase velocity*.

**Example 1.1.2.** The following are some examples of dispersive relations:

- For the *linear advection equation*  $u_t + cu_x = 0$  with  $c \in \mathbb{R}$ , one can compute that  $\omega/k = c$ .
- For the Schrödinger equation  $iu_t + \frac{1}{2}\Delta u = 0$ , we have  $\omega/k = k/2 \in \mathbb{R}$ .

In this case of the Schrödinger equation, plane waves with large wave number (large space frequency) travel faster than low-frequency waves.

**Remark.** In general, dispersion means that different frequency plane waves travel at different speeds.

**Remark.** Given initial data  $u_0$ , we can write using the Fourier transform that

$$u_0 = \int \widehat{u}_0(k) e^{ikx} dk.$$

Then we get the solution  $u$  as

$$u(t, x) = \int \widehat{u}_0(k) e^{ik(x - (\omega(k)/k)t)} dk.$$

**Example 1.1.3.** In the case of the linear advection equation, we obtain the solution as

$$u(t, x) = \int \widehat{u}_0(k) e^{ik(x-ct)} dk = u_0(x - ct).$$

For the Schrödinger equation, we instead have the solution

$$u(t, x) = \int \widehat{u}_0(k) e^{ik(x-(k/2)t)} dk.$$

Since different  $k$  travels at different speeds, the original profile quickly spreads out.

**Exercise 1.1.** Calculate the dispersive relation  $\omega/k$  for the linearized KdV equation  $u_t + u_{xxx} = 0$ .

**Example 1.1.4.** The *KdV equation* is given by

$$\partial_t u + \partial_{xxx} u + 6u \partial_x u = 0.$$

This equation is used to model shallow water surfaces, and is a nonlinear dispersive equation. Russell observed a great bump of water in a channel that traveled for a long time and kept its shape. This is due to the nonlinear effects in the KdV equation, and these effects are called *solitons*.

**Definition 1.2.** A *soliton* is a self-reinforcing solitary wave (a wave packet or pulse) that maintains its shape while traveling at a constant speed.

## 1.2 Fourier Transform and the Free Schrödinger Equation

Consider the following free Schrödinger equation:

$$\begin{cases} i\partial_t \psi + \frac{1}{2}\Delta \psi = 0, \\ \psi|_{t=0} = \psi_0. \end{cases}$$

We will solve this equation using the *Fourier transform*

$$\widehat{f}(\xi) = \int_{\mathbb{R}^d} e^{-ix \cdot \xi} f(x) dx.$$

Note that one can recover  $f$  from its Fourier transform via the *inversion formula*

$$f(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{ix \cdot \xi} \widehat{f}(\xi) d\xi.$$

**Exercise 1.2.** Check that  $(\partial_{x_j} f)^\wedge = i\xi_j \widehat{f}$ .

Applying the Fourier transform to the free Schrödinger equation, one has

$$i\partial_t \psi + \frac{1}{2}\Delta \psi = 0 \quad \xrightarrow{\text{F.T.}} \quad i\partial_t \widehat{\psi} - \frac{1}{2}|\xi|^2 \widehat{\psi} = 0$$

and initial condition  $\widehat{\psi}(0, \xi) = \widehat{\psi}_0(\xi)$ . So for fixed  $\xi$ , we have an ODE, so we can solve the equation via

$$\widehat{\psi}(t, \xi) = e^{-i|\xi|^2 t/2} \widehat{\psi}_0(\xi).$$

Now by applying the inverse Fourier transform, we obtain the solution

$$\psi(t, x) = (2\pi)^{-d} \int e^{ix\xi} \widehat{\psi}(t, \xi) d\xi = (2\pi)^{-d} \int e^{ix\xi} e^{-i|\xi|^2 t/2} \widehat{\psi}_0(\xi) d\xi.$$

Recalling Plancherel's theorem that  $\|f\|_{L^2} = C\|\widehat{f}\|_{L^2}$  (for a constant  $C$  independent of  $f$ ), we obtain

$$\|\psi(t, x)\|_{L^2} = C\|\widehat{\psi}(t, \xi)\|_{L^2} = C\|\widehat{\psi}(0, \xi)\|_{L^2} = \|\psi(0, x)\|_{L^2} = \|\psi_0(x)\|_{L^2},$$

where the second equality follows by noticing that  $e^{-i|\xi|^2 t/2}$  has modulus 1. This shows that the linear Schrödinger evolution preserves the  $L^2$  norm of the solution.

**Exercise 1.3.** Compute that

$$\frac{d}{dt} \int_{\mathbb{R}^d} |\psi(t, x)|^2 dx = 0.$$

This is an alternative way to show that the  $L^2$  norm of the solution is preserved.

## 1.3 Sobolev Spaces

**Definition 1.3.** The *Sobolev spaces*  $H^\gamma = W^{\gamma,2}$  for  $\gamma \in \mathbb{R}$  are defined via the norm

$$\|f\|_{H^\gamma} = \left( \int_{\mathbb{R}^d} (1 + |\xi|^2)^\gamma |\widehat{f}(\xi)|^2 d\xi \right)^{1/2}.$$

The *homogeneous Sobolev spaces*  $\dot{H}^\gamma$  are defined by the norm

$$\|f\|_{\dot{H}^\gamma} = \left( \int_{\mathbb{R}^d} |\xi|^{2\gamma} |\widehat{f}(\xi)|^2 d\xi \right)^{1/2}.$$

**Remark.** If  $\gamma \in \mathbb{N}$  and  $d = 1$ , then

$$\|f\|_{H^\gamma} \sim \sum_{m=0}^{\gamma} \|\partial_x^m f\|_{L^2}.$$

In particular, this means that  $f \in H^\gamma$  if and only if  $\partial_x^m f \in L^2$  for all  $m \leq \gamma$ .

**Exercise 1.4.** Check that if  $f_\lambda(x) = f(\lambda x)$ , then  $\widehat{f}_\lambda(\xi) = \lambda^{-d} \widehat{f}(\xi/\lambda)$ .

**Remark.** In the Sobolev spaces, this means that (change variables  $\eta = \xi/\lambda$  for the last equality)

$$\|f_\lambda\|_{\dot{H}^\gamma} = \left( \int_{\mathbb{R}^d} |\xi|^{2\gamma} |\widehat{f}_\lambda(\xi)|^2 d\xi \right)^{1/2} = \left( \int_{\mathbb{R}^d} |\xi|^{2\gamma} |\lambda^{-d} \widehat{f}(\xi/\lambda)|^2 d\xi \right)^{1/2} = \lambda^{\gamma-d/2} \|f\|_{\dot{H}^\gamma}.$$

**Lemma 1.1.** In the Schrödinger equation,  $\|\psi(t)\|_{H^\gamma} = \|\psi_0\|_{H^\gamma}$  and  $\|\psi(t)\|_{\dot{H}^\gamma} = \|\psi_0\|_{\dot{H}^\gamma}$  for all  $t$  and  $\gamma$ .

*Proof.* We can compute that

$$\|\psi(t)\|_{\dot{H}^\gamma}^2 = \int_{\mathbb{R}^d} |\xi|^{2\gamma} |\widehat{\psi}(t, \xi)|^2 d\xi = \int_{\mathbb{R}^d} |\xi|^{2\gamma} |e^{-i|\xi|^2 t/2} \widehat{\psi}_0(\xi)|^2 d\xi = \int_{\mathbb{R}^d} |\xi|^{2\gamma} |\widehat{\psi}_0(\xi)|^2 d\xi = \|\psi_0\|_{\dot{H}^\gamma}^2.$$

The same argument works for the  $H^\gamma$  case after replacing  $|\xi|^{2\gamma}$  with  $(1 + |\xi|^2)^\gamma$ . □

# Lecture 2

## Jan. 8 — Special Solutions

### 2.1 Special Solutions

**Example 2.0.1.** The following are special solutions to the Schrödinger equation:

1. Gaussian:  $\psi_0 = e^{-|x|^2/2}$ . One can compute the Fourier transform and get

$$\widehat{\psi}_0(\xi) = \int_{\mathbb{R}^d} e^{-ix \cdot \xi} e^{-|x|^2/2} dx = \int_{\mathbb{R}^d} e^{-|x+i\xi|^2/2} e^{-|\xi|^2/2} dx = e^{-|\xi|^2/2} \int_{\mathbb{R}^d} e^{-|x+i\xi|^2/2} dx.$$

The last integral is a contour integral in the complex plane along  $\Im z = \xi$ , and we can deform the contour via Cauchy's theorem to the real axis to obtain (the integrand is analytic on  $0 \leq \Im z \leq \xi$ )

$$\widehat{\psi}_0(\xi) = e^{-|\xi|^2/2} \int_{\mathbb{R}^d} e^{-|x|^2/2} dx = (2\pi)^{d/2} e^{-|\xi|^2/2}.$$

Then taking inverse Fourier transforms, we obtain the solution

$$\begin{aligned} \psi(t, x) &= (2\pi)^{-d} \int_{\mathbb{R}^d} e^{i(x \cdot \xi - |\xi|^2 t/2)} \widehat{\psi}_0(\xi) d\xi = (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{i(x \cdot \xi - |\xi|^2 t/2)} e^{-|\xi|^2/2} d\xi \\ &= (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{-\frac{1}{2}(1+it)|\xi|^2} e^{ix \cdot \xi} d\xi. \end{aligned}$$

Now formally put  $\eta = (1+it)^{1/2} \xi$  to get

$$\psi(t, x) = (2\pi)^{-d/2} (1+it)^{-d/2} \int_{\mathbb{R}^d} e^{-\frac{1}{2}|\eta|^2} e^{ix\eta/(1+it)^{1/2}} d\eta.$$

Fill in the details of the above change of variables as an exercise (e.g. one has to worry about choosing a branch cut when taking the square root). Computing the integral explicitly, one obtains

$$\psi(t, x) = (1+it)^{-d/2} e^{-|x|^2/(2(1+it))}.$$

One can from this that  $\psi$  has decay in time. Furthermore, one can see that

$$|\psi(t, x)|^2 = (1+t^2)^{-d/2} e^{-|x|^2/(1+t^2)}.$$

From this we can observe an  $L^\infty$  decay of  $\psi$  like  $t^{-d/2}$ , and that the influence region of the solution grows like order  $t$ . We can also see again from this explicit computation that  $\|\psi(t)\|_{L^2} = C$ .

2. Modulated Gaussian:  $\psi_0 = e^{-|x|^2/2} e^{ix \cdot v}$ . The Fourier transform of this initial data is

$$\widehat{\psi}_0(\xi) = (2\pi)^{d/2} e^{-i|\xi-v|^2/2}.$$

So the solution corresponding to this initial data is

$$\begin{aligned} \psi(t, x) &= (2\pi)^{-d} \int_{\mathbb{R}^d} e^{i(x \cdot \xi - |\xi|^2 t/2)} \widehat{\psi}_0(\xi) d\xi = (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{i(x \cdot \xi - |\xi|^2 t/2)} e^{-|\xi-v|^2/2} d\xi \\ &= e^{ix \cdot v} e^{-|v|^2 t/2} (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{i(x-vt) \cdot \xi} e^{-(1+it)|\xi|^2/2} d\xi \\ &= e^{ix \cdot v} e^{-|v|^2 t/2} (1+it)^{-d/2} \exp\left(-\frac{|x-vt|^2}{2(1+it)}\right). \end{aligned}$$

From this we can see that the influence region of the solution moves with velocity  $v$ .

3. Fundamental solution: We want a *fundamental solution*  $K$  such that  $K$  solves

$$i\partial_t K + \frac{1}{2}\Delta K = 0 \quad \text{and} \quad K|_{t=0} = \delta_0.$$

We will find  $K$  by scaling arguments. Suppose such a  $K$  exists. Then we must have

$$\psi(t, x) = \int_{\mathbb{R}^d} K(t, x-y) \psi_0(y) dy \tag{1}$$

since  $K|_{t=0} = \delta_0$ . Now define the scaling  $\psi_\lambda(t, x) = \psi(\lambda^2 t, \lambda x)$ . Then  $\psi_\lambda$  also solves

$$i\partial_t \psi_\lambda + \frac{1}{2}\Delta \psi_\lambda = 0$$

and we have the initial condition  $\psi_\lambda(0, x) = \psi_0(\lambda x)$ . Then

$$\psi_\lambda(t, x) = \int_{\mathbb{R}^d} K(t, x-y) \psi_0(\lambda y) dy = \psi(\lambda^2 t, \lambda x).$$

Setting  $t' = \lambda^2 t$ ,  $x' = \lambda x$ , and  $y' = \lambda y$ , we get

$$\psi(t', x') = \frac{1}{\lambda^d} \int_{\mathbb{R}^d} K\left(\frac{t'}{\lambda^2}, \frac{x' - y'}{\lambda}\right) \psi_0(y') dy'. \tag{2}$$

Comparing (1) and (2), we see that we must have

$$K(t, x-y) = \lambda^{-d} K\left(\frac{t}{\lambda^2}, \frac{x-y}{\lambda}\right).$$

Setting  $u = x - y$ , we get

$$K(t, u) = \lambda^{-d} K\left(\frac{t}{\lambda^2}, \frac{u}{\lambda}\right).$$

Thus we expect  $K(t, x) = t^{-d/2} \Phi(|x|^2/t)$  for some  $\Phi$ . Now we use the fact that  $i\partial_t K + \frac{1}{2}\Delta K = 0$ . Setting  $m = |x|^2/t$ , one can plug in the above guess for  $K$  to obtain (note that  $\Delta = \nabla \cdot \nabla$ )

$$-\frac{id}{2} t^{-d/2-1} \Phi(m) - it^{-d/2} \Phi'(m) \frac{m}{t} + \frac{1}{2} t^{-d/2} \nabla \cdot \left( \frac{2x}{t} \Phi'(m) \right) = 0.$$

Then we get

$$-i\frac{d}{2}\Phi(m) - im\Phi'(m) + d\Phi'(m) + 2m\Phi''(m) = 0,$$

which gives

$$d\left(\Phi'(m) - \frac{i}{2}\Phi(m)\right) + 2m\frac{d}{dm}\left(\Phi'(m) - \frac{i}{2}\Phi(m)\right) = 0.$$

Now observe that  $\Phi(m) = e^{im/2}$  solves the above equation. Since  $\Phi(m)$  solves the equation,  $c\Phi(m)$  also solves the equation for any  $c \in \mathbb{C}$ , and thus we have

$$K(t, x) = ct^{-d/2}\Phi(|x|^2/t) = ct^{-d/2}e^{i|x|^2/2t}.$$

To determine  $c$ , we use  $K|_{t=0} = \delta_0$ , from which one can obtain  $c = (2\pi i)^{-d/2}$ . Thus

$$K(t, x) = (2\pi it)^{-d/2}e^{i|x|^2/2t}.$$

The rough computation is that since  $\widehat{K}(0, \xi) = 1$ , we have

$$K = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{i(x \cdot \xi - |\xi|^2 t/2)} \widehat{K}(0, \xi) d\xi = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{i(x \cdot \xi - |\xi|^2 t/2)} d\xi$$

This is not necessarily integrable a priori, but one can take limits and obtain

$$\begin{aligned} K &= (2\pi)^{-d} \lim_{\epsilon \rightarrow 0^+} \int_{\mathbb{R}^d} e^{ix \cdot \xi} e^{-(\epsilon + it)|\xi|^2/2} d\xi = \lim_{\epsilon \rightarrow 0^+} (\epsilon + it)^{-d/2} (2\pi)^{-d/2} e^{-|x|^2/(2(\epsilon + it))} \\ &= (2\pi it)^{-d/2} e^{-|x|^2/2it}. \end{aligned}$$

Note that this computation matches the result of the previous scaling argument.

**Theorem 2.1.** *Let  $\psi_0 \in \mathcal{S}(\mathbb{R}^d)$ .<sup>1</sup> Then there exists a solution to*

$$\begin{cases} i\partial_t \psi + \frac{1}{2}\Delta \psi = 0, \\ \psi|_{t=0} = \psi_0, \end{cases}$$

*which is unique and given by*

$$\psi(t, x) = \int_{\mathbb{R}^d} K(t, x - y) \psi_0(y) dy = (2\pi it)^{-d/2} \int_{\mathbb{R}^d} e^{-|x-y|^2/2it} \psi_0(y) dy.$$

*Proof.* This theorem is a summary of the previous explicit computations. □

**Remark.** Recall that the Schrödinger evolution preserves the  $L^2$  norm of a solution, i.e.

$$\|\psi(t)\|_{L^2} = \|\psi(0)\|_{L^2} = \|\psi_0\|_{L^2}.$$

The above theorem also gives an  $L^\infty$  bound (a so-called *dispersive estimate*)

$$\|\psi(t)\|_{L^\infty} \leq |2\pi t|^{-d/2} \int_{\mathbb{R}^d} |\psi_0(y)| dy = |2\pi t|^{-d/2} \|\psi_0\|_{L^1}.$$

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<sup>1</sup>Here  $\mathcal{S}(\mathbb{R}^d)$  is the space of Schwartz functions.