MATH 8803: Representation Theory

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Contents

Lecture 1

Aug. 18 — Historical Perspective

1.1 Origin of Representation Theory

One motivation for representation theory is symmetries in physics. From a mathematical perspective, we consider *groups* and *algebras* (a vector space with a bilinear operation). In this course, we will study two types of groups:

1. *finite groups*, e.g. the symmetric group;
2. *Lie groups*, e.g. the rotation group.

Definition 1.1. A *representation* of a group G is a homomorphism $G \to \mathrm{End}(V)$, where V is some finite-dimensional vector space.

The history of representation theory is as follows:

1. In the late 19th century, people were interested in *crystallography*, in particular crystallographic groups and their classification. There are related objects called *Bieberbach groups* (e.g. $O(n)$ with translations, i.e. $\mathbb{R}^n \rtimes O(n)$).

   Sophus Lie discovered *Lie groups* in his main manuscript "Transformation groups." From Lie groups, one then derives *Lie algebras*.

2. In the early 20th century (1905), *special relativity* was discovered, which involves the *Lorentz group* $SO(1,3)$ (the transformations preserving the form $-t^2 + x^2 + y^2 + z^2$). This is a Lie group.

   Around the same time, E. Cartan developed the modern theory of *semisimple Lie groups* and *Lie algebras*, and H. Weyl studied their representations.

3. In the period 1920–1930, quantum ("matrix") mechanics was discovered. Here one has a Hilbert space $\mathcal{H}$ and a self-adjoint Hamiltonian (energy) operator H on $\mathcal{H}$. The symmetry operator A satisfies the commutator relation $[H, A] = 0$, and if we set $U = e^{iA}$, we have $UHU^\dagger = H$.

4. After the discovery of *spin* by W. Pauli, E. Wigner realized that spin was directly related to the representation theory of the universal cover $\pi : SU(2) \to SO(3)$.

   In the 1960s, there was a "zoo" of elementary particles. M. Gell-Mann and Y. Neeman realized that all of these can be described by representations of $SU(3)$. The led to the discovery of *quarks* and the later notion of grand unified theories and string theory in the 1970s.

   There are also connections to condensed matter theory and quantum information.

This course will cover the following topics:

1. basics about associative algebras and their representations, finite groups and their representations in general, the symmetric group and its representations, Young tableaux;
2. Lie groups and Lie algebras;
3. the structure of semisimple Lie algebras;
4. representations of $SL(n)$.

1.2 Introduction to Lie Groups and Lie Algebras

In general, groups are complicated, whereas algebras are less complicated. We begin with finite groups.

Definition 1.2. Let G be a finite group and $\mathbb{F}$ a field. The *group algebra* $\mathbb{F}G$ is

$$\mathbb{F}G = \left\{ \sum_g a_g g : a_g \in \mathbb{F} \right\}.$$

This forms an algebra over $\mathbb{F}$ with the obvious multiplication operation.

Example 1.2.1. Consider the rotation group, generated by the matrices

$$R_z(\theta) = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad R_x(\phi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{pmatrix}, \quad R_y(\psi) = \begin{pmatrix} \cos\psi & 0 & -\sin\psi \\ 0 & 1 & 0 \\ \sin\psi & 0 & \cos\psi \end{pmatrix}.$$

Letting δ be an infinitesimal value and using a Taylor expansion, we can write

$$R_z(\delta\theta) = 1 + \delta\theta \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 1 + \delta\theta M_z,$$

$$R_x(\delta\phi) = 1 + \delta\phi \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} = 1 + \delta\phi M_x,$$

$$R_y(\delta\psi) = 1 + \delta\psi \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} = 1 + \delta\psi M_y.$$

We can measure the commutativity of these matrices via

$$\begin{aligned} R_x(\delta\phi)R_y(\delta\psi)R_x^{-1}(\delta\phi)R_y^{-1}(\delta\psi) &= (1 + M_x\delta\phi)(1 + M_y\delta\psi)(1 - M_x\delta\phi)(1 - M_y\delta\psi) \\ &= 1 + \delta\phi\delta\psi(M_xM_y - M_yM_x). \end{aligned}$$

Exercise 1.1. Show that $[M_x, M_y] = -M_z$.

Remark. Thus we have a vector space spanned by M_x, M_y, M_z with an operation $[\cdot,\cdot]$ satisfying the identity $[M_x, M_y] = -M_z$. Note that this property is satisfied by the cross product on $\mathbb{R}^3$. The cross product also satisfies the following *Jacobi identity*:

$$[A, [B, C]] = [[A, B], C] + [B, [A, C]].$$

The above properties define a *Lie algebra*.

Definition 1.3. Let $\{e_k\}$ be a basis of a Lie algebra and $[e_i, e_j] = \sum_k c_{ij}^k e_k$. The *universal enveloping algebra* of the Lie algebra is the free associative algebra on $\{e_k\}$, modulo the relations $[e_i, e_j] = \sum_k c_{ij}^k e_k$.

Remark. One way to return to the Lie group from the Lie algebra is exponentiation, e.g. $R_z(\theta) = e^{\theta M_z}$.

1.3 Algebras and Modules

Let k be a commutative ring (most of the time $k = \mathbb{C}$). All rings will be associative and unital.

Definition 1.4. A *(associative and unital) k-algebra* is a unital ring A with a homomorphism $i : k \to A$ such that $i(r) \cdot a = a \cdot i(r)$, i.e. the image of i commutes with A.

Example 1.4.1. Any ring is a $\mathbb{Z}$-algebra.

Definition 1.5. A *homomorphism* of k-algebras is a k-linear homomorphism of unital rings.

Definition 1.6. Let A, B be unital rings, and M an abelian group. Then

1. a *left A-module structure* on M is a $\mathbb{Z}$-bilinear map $A \times M \to M$, associative in the sense that
$$a_1(a_2m) = (a_1a_2)m, \quad \text{for all } a_1, a_2 \in A,\ m \in M,$$
and such that $1_Am = m$ for all $m \in M$;
2. a *right A-module structure* on M is a $\mathbb{Z}$-bilinear map $M \times B \to M$, associative in the sense that
$$(mb_1)b_2 = m(b_1b_2), \quad \text{for all } b_1, b_2 \in B,\ m \in M,$$
and such that $m1_B = m$ for all $m \in M$;
3. an *A-B-bimodule structure* on M is a left A-module and right B-module structure on M, along with the condition that $(am)b = a(mb)$ for all $a \in A$, $b \in B$, and $m \in M$.

Remark. In general, an A-module will mean a left A-module by default.

Definition 1.7. Let M, N be left A-modules. An *A-module homomorphism* is a map $\varphi : M \to N$ such that $\varphi(am) = a\varphi(m)$ for all $a \in A$ and $m \in M$.

Example 1.7.1. A ring A is both a left/right A-module and an A-A-bimodule (the *regular bimodule*).

Definition 1.8. The *direct sum* $\bigoplus_{i\in I} M_i$ of left A-modules M_i is the collection of $(m_i)_{i\in I}$ with finitely many nonzero entries, with component-wise addition and scalar multiplication.

Example 1.8.1. Let I be an index set. Then $A^{\oplus I}$ is the *coordinate A-module.*

Definition 1.9. A *submodule* of M is a nontrivial subgroup closed under addition and invariant under the action of A.

Example 1.9.1. Submodules of the regular left/right A-module are the left/right ideals of A.

Definition 1.10. Let M be a left A-module and M_0 a submodule of M. The *quotient module* M/M_0 is the set of equivalence classes $m + M_0$, where the action of A is given by $a(m + M_0) = am + M_0$.

Lemma 1.1. *Let M, N be A-modules and $M_0 \subseteq M$ a submodule. Let $\varphi : M \to N$ be A-linear such that $\varphi(M_0) = \{0\}$. Then there exists a unique A-linear map $\underline{\varphi} : M/M_0 \to N$ such that $\varphi = \underline{\varphi} \circ \pi$, where $\pi : M \to M/M_0$ is the canonical projection.*

Lecture 2

Aug. 20 — Algebras and Modules

2.1 More on Algebras and Modules

Definition 2.1. A *free* module is a module which has a basis.

Example 2.1.1. Consider the coordinate module $A^{\oplus I}$. Then a basis is given by $e_i = \{\delta_{ij}\}_{j\in I}$ for $i \in I$.

Proposition 2.1. *Let M be a left A-module. Let I be an index set and let $m_i \in M$ for $i \in I$. Then*

1. *There exists a unique A-linear map $A^{\oplus I} \to M$ which sends $e_i \mapsto m_i$.*
2. *This map is surjective if and only if the elements m_i span M. In particular, every M is isomorphic to a quotient of a free module.*
3. *This map is an isomorphism if and only if $\{m_i\}$ form a basis of M. In particular, every coordinate module is a free module.*

Proof. Left as an exercise. □

Example 2.1.2. Suppose M is spanned by a single element m. Then $M \cong A/I$, where I is the left ideal

$$I = \{a \in A : am = 0\}.$$

Example 2.1.3. We can now construct the following examples of algebras:

1. Let $\mathrm{Mat}_n(A)$ be the set of $n \times n$ matrices with entries in A. If A is a k-algebra, then $\mathrm{Mat}_n(A)$ is also a k-algebra.
2. If G is a group, then the group algebra kG (for a ring k) given by

$$kG = \left\{ \sum_{g\in G} a_g g : a_g \in k \right\}$$

is a free module with basis identified with the elements of G.

The importance of of this object is as follows: Let G be a group and B an algebra. Consider the set of maps satisfying $1_G \mapsto 1_B$ and respecting the group multiplication. This set is in bijection with maps $kG \to B$ (they extend by linearity). If V is a vector space and $B = \mathrm{End}(V)$, then this statement says that there is a bijection between the representations of the group G and the representations of the group algebra kG.

3. If I is a two-sided ideal, then A/I has a natural algebra structure.
4. If A_1, A_2 are k-algebras, then the direct sum $A_1 \oplus A_2$ is again a k-algebra (with component-wise multiplication). One can extend this by induction to a finite direct sum, but note that we lose the multiplicative identity in an infinite direct sum (so we do not get an algebra in the infinite case).

2.2 Module of Homomorphisms

Definition 2.2. Let k be a commutative ring and A a k-algebra. Let M, N be left A-modules. Denote by $\mathrm{Hom}_A(M, N)$ the set of all A-module homomorphisms $M \to N$. Give $\mathrm{Hom}_A(M, N)$ a k-module structure via

$$[\varphi_1 + \varphi_2](m) = \varphi_1(m) + \varphi_2(m), \quad [r\varphi](m) = r\varphi(m)$$

for $\varphi_1, \varphi_2 \in \mathrm{Hom}_A(M, N)$, $r \in k$, and $m \in M$.

Remark. Let L, M, N be left A-modules. Then we can define a k-bilinear map

$$\begin{aligned} \mathrm{Hom}_A(M, N) \times \mathrm{Hom}_A(L, M) &\longrightarrow \mathrm{Hom}_A(L, N) \\ (\varphi, \psi) &\longmapsto \varphi \circ \psi. \end{aligned}$$

Exercise 2.1. Let N_2 be an A-module, $N_1 \subseteq N_2$ an A-submodule, and $N_3 = N_2/N_1$. Let $i : N_1 \hookrightarrow N_2$ be the inclusion and $\pi : N_2 \to N_3$ the projection. Define the maps

$$\begin{aligned} \widetilde{\iota} : \mathrm{Hom}(M, N_1) &\to \mathrm{Hom}(M, N_2) \\ \varphi_1 &\longmapsto i \circ \varphi_1 \\ \widetilde{\pi} : \mathrm{Hom}(M, N_2) &\to \mathrm{Hom}(M, N_3) \\ \varphi_2 &\longmapsto \pi \circ \varphi_2. \end{aligned}$$

Then show that $\widetilde{\iota}$ is injective and $\mathrm{Im}\,\widetilde{\iota} = \ker \widetilde{\pi}$.

Remark. Let B be a k-algebra and M and A-B-bimodule. Then for all A-modules N, we have that $\mathrm{Hom}_A(M, N)$ is a left B-module via

$$[b\varphi](m) = \varphi(mb).$$

Similarly, if N is an A-C-bimodule, then $\mathrm{Hom}_A(M, N)$ is a right C-module via

$$[\varphi c](m) = \varphi(m)c.$$

So if M is an A-B-bimodule and N an A-C-bimodule, then $\mathrm{Hom}_A(M, N)$ is a B-C-bimodule.

Remark. Let M be a left A-module. We write $\mathrm{End}_A(M)$ in place of $\mathrm{Hom}_A(M, M)$, and composition gives $\mathrm{End}_A(M)$ the structure of a k-algebra. If $M = A^{\oplus n}$, then we can identify

$$\mathrm{End}_A(M) = \mathrm{Mat}_n(A^{\mathrm{opp}}),$$

where the opposite algebra exchanges the order of multiplication in the original algebra (this is because $\mathrm{End}_A(M)$ must respect the action by A). Then M becomes an A-$(\mathrm{Mat}_n(A))^{\mathrm{opp}}$-bimodule.

Remark. If M, N are two left A-modules, then $\mathrm{Hom}_A(M, N)$ is an $\mathrm{End}_A(N)$-$\mathrm{End}_A(M)$-bimodule (by taking into account compositions).

2.3 Tensor Product of Modules

Remark. Let A be a k-algebra, M a right A-module, and N a left A-module. We want to produce a k-module $M \otimes_A N$, which will be the *tensor product* of M and N over A.

Definition 2.3. Let L be a k-module. We say that a map $\varphi : M \times N \to L$ is *A-bilinear* if it is k-linear in both arguments and satisfies

$$\varphi(ma, n) = \varphi(m, an)$$

for any $a \in A$, $m \in M$, and $n \in N$.

Definition 2.4 (Universal property of the tensor product)**.** There is an A-bilinear map

$$\begin{aligned} M \times N &\longrightarrow M \otimes_A N \\ (m,n) &\longmapsto m \otimes n \end{aligned}$$

such that for any A-bilinear map $\varphi : M \times N \to L$, there exists a unique k-linear map $\psi : M \otimes_A N \to L$ such that $\varphi(m, n) = \psi(m \otimes n)$. As a diagram, this says that

$$\begin{tikzcd} M \times N \arrow[rr, "(m,n)\mapsto m\otimes n"] \arrow[dr, "\varphi"'] & & M \otimes_A N \arrow[dl, "\psi"] \\ & L & \end{tikzcd}$$

Exercise 2.2. If we choose $M \otimes'_A N$ with bilinear map $(m, n) \mapsto m \otimes' n$, then there exists a unique isomorphism $i : M \otimes_A N \to M \otimes'_A N$ given by $i(m \otimes n) = m \otimes' n$.

Corollary 2.0.1. *Assume $M \otimes_A N$ satisfies the universal property. Then $\{m \otimes n\}$ span $M \otimes_A N$.*

Theorem 2.1. *The tensor product $M \otimes_A N$ exists for all right A-modules M and left A-modules N.*

Proof. We sketch the proof. First take M to be free. Then we can define $M \otimes_A N$ as $N^{\oplus I}$, where we have $(e_i a_i) \otimes n = (a_i n)_{i \in I}$. The universal property is easy to check for this case, and the general case can be done by writing M as a quotient of a free module. □

Example 2.4.1. If M, N are both free and $\{e_i\}_{i \in I}$, $\{f_j\}_{j \in J}$ are bases of M, N, respectively, then $M \otimes_A N$ is a free k-module with basis vectors $\{e_i \otimes f_j\}_{i \in I, j \in J}$.

Exercise 2.3. Let $M = A/I$, where I is a right ideal. Show that $M \otimes_A N = N/IN$. Find out what happens when $N = A/J$, where J is a left ideal, what can you say about $M \otimes_A N$ in terms of A, I, J?

Proposition 2.2. *Assume B is a k-algebra and M a B-A-module. Then $M \otimes_A N$ is a left B-module.*

Proof. Define $\varphi_b : M \times N \to M \otimes_A N$ by $(m, n) \mapsto bm \otimes n$. This is bilinear, so by the universal property, there exists $\psi_b : M \otimes_A N \to M \otimes_A N$ such that $\psi_b(m \otimes n) = bm \otimes n$, which gives the B-action. □

Definition 2.5. Let L be a B-module. A map $\varphi : M \times N \to L$ is called *B-A-linear* if it is k-linear in both arguments and

$$\varphi(ma, n) = \varphi(m, an), \quad \varphi(bm, n) = b\varphi(m, n)$$

for all $m \in M$, $n \in N$, $b \in B$, and $a \in A$.

Proposition 2.3. *The left B-module $M \otimes_A N$ has the following universal property:*

Let L be any left B-module and $\varphi : M \times N \to L$ a B-A-linear map. Then there exists a unique B-linear map $\psi : M \otimes_A N \to L$ such that $\psi(m \otimes n) = \varphi(m, n)$.

Example 2.5.1. Let A_1, A_2 be k-algebras. Then

1. $A_1 \otimes_k A_2$ has the structure of a k-algebra via

$$(a_1 \otimes a_2)(b_1 \otimes b_2) = (a_1 b_1) \otimes (a_2 b_2),$$

   where $1 \otimes 1$ is a unit element.

2. Let M_i be a left A_i-module for $i = 1, 2$. Then $M_1 \otimes_k M_2$ is a module for $A_1 \otimes_k A_2$.

2.4 Tensor-Hom Adjunction

Proposition 2.4 (Tensor-Hom adjunction)**.** *Let A, B be associative algebras, N a B-module, M an A-module, and L an A-B-bimodule. Then*

1. $L \otimes_B N$ is an A-module;

2. $\mathrm{Hom}_A(L, M)$ is a B-module.

Moreover, there is a natural k-linear isomorphism

$$\mathrm{Hom}_A(L \otimes_B N, M) \xrightarrow{\cong} \mathrm{Hom}_B(N, \mathrm{Hom}_A(L, M)).$$

Proof. By the universal property, there is a natural map

$$\mathrm{Hom}_A(L \otimes_B N, M) \xrightarrow{\cong} \mathrm{Bilin}_{A,B}(L \times N, M).$$

So it suffices to find

$$\begin{aligned} \mathrm{Hom}_B(N, \mathrm{Hom}_A(L, M)) &\xrightarrow{\cong} \mathrm{Bilin}_{A,B}(L \times N, M) \\ f &\longmapsto \varphi_f. \end{aligned}$$

Construct this map by $\psi_f(e, n) = [f(n)](e)$, with inverse $h \mapsto \psi(\cdot, h)$ for $\psi \in \mathrm{Bilin}_{A,B}(L \times N, M)$. $\square$

Example 2.5.2. If we have an algebra homomorphism $B \to A$, where A is a an A-B-bimodule. One can show as an exercise that $\mathrm{Hom}_A(A, M)$ is naturally identified with M as an A-module and B-module. Thus by the Tensor-Hom adjunction, we have a natural isomorphism

$$\mathrm{Hom}_A(A \otimes_B N, M) \xrightarrow{\cong} \mathrm{Hom}_B(N, M).$$

Definition 2.6. The A-module $A \otimes_B N$ is said to be *induced* from N.

Remark. Assume there is Hom from $A \to B$. Then B is an A-B-bimodule. Take it as L in the Tensor-Hom adjunction. Note that $B \otimes_B N \cong N$ as A-modules, and we have a natural isomorphism

$$\mathrm{Hom}_A(N, M) \xrightarrow{\cong} \mathrm{Hom}_B(N, \mathrm{Hom}_A(B, M)).$$

Definition 2.7. The B-module $\mathrm{Hom}_A(B, M)$ is said to be *coinduced* from M.

Lecture 3

Aug. 25 — Complete Reducibility

3.1 Reducibility of Modules

Remark. Consider an associative algebra A over a field $\mathbb{F}$. We proceed to study completely reducible representations of A. Let U be an A-module.

Definition 3.1. An A-module U is *irreducible* if it only has two distinct submodules ($\{0\}$ and U).

Remark. With this definition, $\{0\}$ is not irreducible.

Definition 3.2. An A-module U is *completely reducible* if for any submodule $U' \subseteq U$, there exists an A-submodule U'' such that $U = U' \oplus U''$.

Exercise 3.1. Show that any submodule and any quotient module of a completely reducible A-module is also completely reducible.

Example 3.2.1. Consider $A = \mathrm{End}_{\mathbb{F}}(U)$. Then U is an A-module and is irreducible (there is a linear operator $\alpha : U \to U$ taking $u \mapsto v$ for any $u, v \in U$, so there are no nontrivial invariant subspaces).

Proposition 3.1. *Let U_1, U_2 be completely reducible A-modules. Then $U_1 \oplus U_2$ is completely reducible.*

Proof. Left as an exercise. □

Corollary 3.0.1. *Let U be a finite-dimensional A-module. Then the following are equivalent:*

1. *U is completely reducible;*
2. *U is isomorphic to a direct sum of irreducible submodules.*

Exercise 3.2. Show that every irreducible A-module is isomorphic to a quotient module for a regular module (i.e. one isomorphic to A). In particular, every irreducible module over a finite-dimensional associative $\mathbb{F}$-algebra is finite-dimensional.

3.2 Schur's Lemma

Theorem 3.1 (Schur's lemma)**.** *Let A be an associative $\mathbb{F}$-algebra and U, V irreducible A-modules. Then*

1. *if U, V are not isomorphic, then $\mathrm{Hom}_A(U, V) = 0$;*
2. *$\mathrm{End}_A(U)$ is a skew field (i.e. a division ring). Furthermore, if U is finite-dimensional and $\mathbb{F}$ is algebraically closed, then $\dim \mathrm{End}_A(U) = 1$.*

Proof. (1) Assume we have a nonzero homomorphism $\varphi : U \to V$. Then $\ker\varphi \subsetneq U$, and $\operatorname{Im}\varphi \subseteq V$ is nontrivial, so by irreducibility φ must be an isomorphism.

(2) Let $\varphi \in \operatorname{End}_A(U)$. From (1), we know that φ is an isomorphism, so φ has an inverse, i.e. $\operatorname{End}_A(U)$ is a skew field. For the second part, since $\mathbb{F}$ is algebraically closed, we can find an eigenvalue z for φ. Then $\varphi - z\operatorname{Id}_U$ is not invertible, so we have $\varphi - z\operatorname{Id}_U = 0$ by (1). $\square$

Exercise 3.3. Consider $1, i, j, k$, where $i^2 = j^2 = k^2 = -1$ and $ij = -ji = k$. The *quaternion algebra* over $\mathbb{R}$ is given by

$$\mathbb{H}_{\mathbb{R}} = \{q = w + xi + yj + zk : w, x, y, z \in \mathbb{R}\}$$

Note that $\overline{q} = w - xi - yj - zk$ satisfies $q\overline{q} = w^2 + x^2 + y^2 + z^2$, so $q^{-1} = \overline{q}/(w^2 + x^2 + y^2 + z^2)$, i.e. $\mathbb{H}_{\mathbb{R}}$ is a skew field. Show that $\operatorname{End}_{\mathbb{H}_{\mathbb{R}}}(\mathbb{H}_{\mathbb{R}}) \cong \mathbb{H}_{\mathbb{R}}^{\mathrm{opp}}$.

Remark. We have an embedding $\mathbb{H}_{\mathbb{R}} \hookrightarrow \operatorname{Mat}_2(\mathbb{C})$ given by

$$q \longmapsto \begin{pmatrix} w + xi & y + zi \\ -y + zi & w - xi \end{pmatrix}.$$

If we replace $\mathbb{R}$ with $\mathbb{C}$, then $\mathbb{H}_{\mathbb{C}} \cong \operatorname{Mat}_2(\mathbb{C})$, which is reducible (consider the sum of column spaces).

Definition 3.3. Let U be an A-module. We say that U is *endotrivial* if $\operatorname{End}_A(U)$ consists only of scalar maps, i.e. maps of the form $z\operatorname{Id}$.

Remark. Suppose $\mathbb{F}$ is algebraically closed and uncountable (e.g. $\mathbb{C}$), A has countable dimension over $\mathbb{F}$, and U an irreducible A-module. Then U is endotrivial.

Definition 3.4. Define the *center* of A to be

$$\mathcal{Z}(A) = \{z \in A : za = az \text{ for all } a \in A\}.$$

Note that this is a commutative algebra.

Exercise 3.4. Schur's lemma gives a description of the center of A. Let U be an endotrivial A-module (e.g. a finite-dimensional module over if $\mathbb{F}$ is algebraically closed). Show that $z \in \mathcal{Z}(A)$ acts as a scalar on U. We call the algebra homomorphism $\mathcal{Z}(A) \to \mathbb{F}$ the *central character* of U.

3.3 Completely Reducible Modules

Remark. Consider finite direct sums of endotrivial irreducible modules:

$$\bigoplus_{i=1}^{k} U_i \otimes M_i,$$

where the U_i are endotrivial modules and the M_i are vector spaces known as *multiplicity spaces*. Note that $U_1^{\oplus i} = U_1 \otimes \mathbb{F}^i$. The A-action on the direct sum for $a \in A$ is given by

$$a(u_1 \otimes m_1, \ldots, u_k \otimes m_k) = (au_1 \otimes m_1, \ldots, au_k \otimes m_k).$$

We will use Schur's lemma to understand homomorphisms between such modules.

Write $U^j = \bigoplus_{i=1}^k U_i \otimes M_i^j$ for $j = 1, 2$. We can produce a linear map

$$\bigoplus_{i=1}^k \mathrm{Hom}_{\mathbb{F}}(M_i^1, M_i^2) \longrightarrow \mathrm{Hom}_A(U^1, U^2)$$

in the following manner: For $\underline{\varphi} = (\varphi_1, \ldots, \varphi_k) \in \bigoplus_{i=1}^k \mathrm{Hom}_{\mathbb{F}}(M_i^1, M_i^2)$, we can define

$$\psi_{\underline{\varphi}}\left(\sum_{i=1}^k u_i \otimes m_i^1\right) = \sum_{i=1}^k u_i \otimes \varphi_i(m_i^1).$$

Theorem 3.2. *We have the following:*

1. *The map $\underline{\varphi} \mapsto \psi_{\underline{\varphi}}$ defines a vector space isomorphism*

$$\bigoplus_{i=1}^k \mathrm{Hom}_{\mathbb{F}}(M_i^1, M_i^2) \xrightarrow{\cong} \mathrm{Hom}_A(U^1, U^2).$$

2. *Every A-module homomorphism $U_1 \to U_2$ sends $U_i \otimes M_i^1$ to $U_i \otimes M_i^2$ for any i.*

Proof. Left as an exercise (use Schur's lemma). □

Corollary 3.2.1. *We have the following:*

1. *there is an isomorphism $\mathrm{Hom}_A(U_i, U) \xrightarrow{\cong} M_i$;*
2. *there is an isomorphism $\bigoplus_{i=1}^k U_i \otimes \mathrm{Hom}_A(U_i, U) \cong U$ given by*

$$\sum_{i=1}^k u_i \otimes \varphi_i \longmapsto \sum_{i=1}^k \varphi_i(u_i).$$

Proposition 3.2. *For any A-submodule $U' \subseteq U$, there exists a unique collection of determined subspaces $M_i' \subseteq M_i$ such that $U' = \bigoplus_{i=1}^k U_i \otimes M_i'$ as submodules of U.*

Proof. Note that $\mathrm{Hom}_A(U_i, U') \subseteq \mathrm{Hom}_A(U_i, U)$, set $M_i' = \mathrm{Hom}_A(U_i, U')$, and use Corollary 3.2.1. □

Theorem 3.3. *Let U_i be irreducible modules for A and consider maps $\beta_i : A \to \mathrm{End}_{\mathbb{F}}(U_i)$. Set*

$$\beta = \beta_1 \oplus \cdots \oplus \beta_k : A \longrightarrow \bigoplus_{i=1}^k \mathrm{End}_{\mathbb{F}}(U_i),$$

where the U_i are pairwise non-isomorphic. Then the homomorphism β is surjective.

Proof. Replace A by $A/\ker\beta$, so that β is injective. Then β equips $\bigoplus_{i=1}^k \mathrm{End}(U_i)$ with an A-bimodule structure, and there is a natural isomorphism $\mathrm{End}_{\mathbb{F}}(U_i) \cong U_i \otimes U_i^*$. View U_i as the multiplicity space for the right A-module and U_i^* as the multiplicity space for the left A-module. By Proposition 3.2,

$$A = \bigoplus_{i=1}^k U_i \otimes V_i$$

as a left A-module for some $V_i \subseteq U_i^*$. Similarly for the right A-module, we have

$$A = \bigoplus_{i=1}^{k} W_i \otimes U_i^*$$

for some $W_i \subseteq U_i$. Then we must have $U_i \oplus V_i = W_i \oplus U_i^*$, so $U_i \cong W_i$ and $V_i \cong U_i^*$ (the identity $1 \in A$ guarantees that no component is zero). Thus β is surjective. □

Corollary 3.3.1. *Let $\mathbb{F}$ be algebraically closed and A a finite-dimensional $\mathbb{F}$-algebra. Then the set of isomorphism classes of irreducible A-modules is finite and non-empty.*

Proof. First this set is nonempty since A is nonzero, so it has an irreducible subrepresentation. To see that it is finite, note that for all collections $U_1, \ldots, U_k$, the map $A \to \bigoplus_{i=1}^k \operatorname{End}_{\mathbb{F}}(U_i)$ is surjective, so

$$\dim A \geq \sum_{i=1}^{k} (\dim U_i)^2.$$

This proves the desired result, since A is finite-dimensional. □

3.4 Simple Algebras

Definition 3.5. An algebra A is *simple* if the only two-sided ideals are $\{0\}$ and A (i.e. A is irreducible as a bimodule over itself).

Theorem 3.4. *Let $\mathbb{F}$ be an algebraically closed field and A a finite-dimensional $\mathbb{F}$-algebra. Then the following are equivalent:*

1. *A is simple;*
2. *$A \cong \operatorname{End}_{\mathbb{F}}(U)$ for some finite-dimensional vector space U.*

Proof. $(1 \Rightarrow 2)$: The algebra A has an irreducible representation U, i.e. we have a map $A \to \operatorname{End}_{\mathbb{F}}(U)$. Since A is simple, this map must have trivial kernel, i.e. it is injective. We also already know that it is surjective, so this map is an isomorphism.

$(2 \Rightarrow 1)$: Assume I is a two-sided ideal in $\operatorname{End}_{\mathbb{F}}(U) \cong U \otimes U^*$ and view $I \subseteq U \otimes U^*$. Show as an exercise that we must have $I = \{0\}$. □

Theorem 3.5. *Every finite-dimensional module V for $A = \operatorname{End}_{\mathbb{F}}(U)$ is isomorphic to a direct sum of several copies of U.*

Proof. Recall that every finitely generated module V is a quotient of $A^{\oplus \ell}$ for some $\ell \in \mathbb{N}$. We can write $A = U \otimes U^*$. Let $A^{\oplus \ell} = U \otimes M$ and consider the quotient map $\pi : U \otimes M \to V$. Then $\ker \pi \subseteq U \otimes M$ must be of the form $U \oplus M_0$, so we have $V \cong (U \otimes M)/(U \otimes M_0) = U \otimes (M/M_0)$. □

Lecture 4

Aug. 27 — Semisimple Algebras

4.1 Semisimple Algebras

Definition 4.1. A finite-dimensional $\mathbb{F}$-algebra A is called *semisimple* if it is isomorphic to a direct sum of simple algebras.

Remark. If $\mathbb{F}$ is algebraically closed, then A is a direct sum of matrix algebras, i.e. $\bigoplus_{i=1}^k \operatorname{End}_{\mathbb{F}}(U_i)$.

Theorem 4.1. *Let $U_1, \ldots, U_k$ be finite-dimensional vector spaces over $\mathbb{F}$. Let $A = \bigoplus_{i=1}^k \operatorname{End}_{\mathbb{F}}(U_i)$, so that U_i is an irreducible A-module. Then every finite-dimensional A-module V is isomorphic to a direct sum of several copies of $U_1, \ldots, U_k$.*

Proof. Left as an exercise. □

Corollary 4.1.1. *Let $\mathbb{F}$ be algebraically closed, and A be semisimple and finite-dimensional. Then*

1. *The number of isomorphism classes of irreducible A-modules is equal to* $\dim \mathcal{Z}(A)$.
2. *Different irreducible modules have different central characters.*

Proof. (1) Let $A = \bigoplus_{i=1}^k \operatorname{End}_{\mathbb{F}}(U_i)$. By Theorem 4.1, the number of irreducible representations is k. We can also write

$$\mathcal{Z}\left(\bigoplus_{i=1}^k A_k\right) = \bigoplus_{i=1}^k \mathcal{Z}(A_i),$$

where $A_i = \operatorname{End}_{\mathbb{F}}(U_i)$. Since $\dim \mathcal{Z}(A_i) = 1$, we have $\dim \mathcal{Z}(\bigoplus_{i=1}^k A_k) = k$ as well.

(2) Use the projections $\mathcal{Z} \to \mathcal{Z}(A_i) \to \mathbb{F}$, which correspond to the central characteres. □

4.2 Characterizations of Semisimple Algebras

Definition 4.2. Let A be a finite-dimensional algebra. We say that a two-sided ideal $I \subseteq A$ is *nilpotent* if $I^n = \{0\}$ for some n.

Exercise 4.1. If I, J are nilpotent, then show that $I + J$ is also nilpotent.

Definition 4.3. The maximal nilpotent ideal of A, denoted $\operatorname{rad}(A)$, is called the *radical* of A.

Theorem 4.2. *Let $\mathbb{F}$ be algebraically closed and A a finite-dimensional algebra. Then the following are equivalent:*

1. *A is semisimple;*
2. *all finite-dimensional representations of A are completely reducible;*
3. $\mathrm{rad}(A) = \{0\}$.

Proof. ($1 \Rightarrow 2$) We have already shown this.

($2 \Rightarrow 3$) Let $I = \mathrm{rad}(A)$, so $I^n = \{0\}$ for some $n \in \mathbb{N}$. Let N be a finite-dimensional A-module. Then $I^\ell N$ is an A-submodule for $\ell = 0, \dots, n$. Since N is completely reducible and $I^{\ell+1}N \subseteq I^\ell N$, we have

$$I^\ell N = N_\ell \oplus I^{\ell+1}.$$

Acting on both sides by I, we get $IN_\ell \subseteq I^{\ell+1}$, so $IN_\ell = \{0\}$. Continuing, we get $IN = 0$, so $A = N$.

($3 \Rightarrow 1$) Take $N_1, \dots, N_k$ to be pairwise non-isomorphic irreducible A-modules. We have an epimorphism $A \to \bigoplus_{i=1}^k \mathrm{End}_{\mathbb{F}}(N_i)$. Let I be the kernel, so I acts trivially on every irreducible A-module. We claim that I is nilpotent. Take A to be the regular module. Take a filtration

$$A = A_0 \supseteq A_1 \supseteq \cdots \supseteq A_n = \{0\},$$

where A_i/A_{i+1} is irreducible. Now I acts trivially on A_i/A_{i+1}, so $IA_i \subseteq A_{i+1}$ for all i, thus $I^n = \{0\}$. □

Remark. Assume $\mathrm{char}(\mathbb{F}) = 0$. Consider the following bilinear form on A:

$$(a, b)_U = \mathrm{tr}_U(ab),$$

where U is any A-module. Note that U could be A.

Theorem 4.3. *Let $\mathrm{char}(\mathbb{F}) = 0$, and let A be a finite-dimensional $\mathbb{F}$-algebra. Then A is semisimple if and only if $(a, b)_A$ is nondegenerate.*

Proof. ($\Rightarrow$) Assume A is semisimple, so $A = \bigoplus_{i=1}^k \mathrm{End}(U_i)$. Note that the restriction of $(\cdot,\cdot)_A$ to the direct summand $\mathrm{End}_{\mathbb{F}}(U_i)$ coincides with $(\cdot,\cdot)_{\mathrm{End}_{\mathbb{F}}(U_i)}$. Let $E_{j\ell}$ denote the matrix with all 0s except a single 1 in the (j, ℓ) entry. Then we can compute that

$$(E_{j\ell}, E_{j'\ell'})_{\mathrm{End}_{\mathbb{F}}(U_i)} = \delta_{ej'} \,\mathrm{tr}_{\mathrm{End}_{\mathbb{F}}(U_i)}(E_{j\ell'}) = \delta_{e\ell}\delta_{j\ell'} \dim U_i.$$

So if $\{E_{j\ell}\}$ is a basis, then $\{(\dim U_i)^{-1} E_{j\ell}\}$ is the dual basis. This is nondenegerate if $\mathrm{char}(\mathbb{F}) = 0$.

($\Leftarrow$) Suppose $(\cdot,\cdot)_A$ is nondegenerate. If I is a nilpotent ideal, then for any $a \in I$ such that $a^n = 0$. Then $\mathrm{tr}_A(a) = 0$ for any $a \in I$, so $I \in \ker(\cdot,\cdot) = 0$. Since $(\cdot,\cdot)$ is nondegenerate, we have $I = \{0\}$. □

4.3 Double Centralizer Theorem

Theorem 4.4 (Double centralizer theorem)**.** *Let V be a finite-dimensional vector space over $\mathbb{F}$. Let $A \subseteq \mathrm{End}_{\mathbb{F}}(V)$ be a semisimple algebra, and set $B = \mathrm{End}_A(V)$. Then $A = \mathrm{End}_B(V)$.*

Proof. Let $A = \bigoplus_{i=1}^k \mathrm{End}(U_i)$ and V be a faithful representation of A, so V is completely reducible:

$$V \cong \bigoplus_{i=1}^k U_i \oplus M_i,$$

where the M_i are multiplicity spaces. Let $a = (\varphi_1, \dots, \varphi_k) \in A$ (for $\varphi_i \in \mathrm{End}(U_i)$) act on $\mathrm{End}_{\mathbb{F}}(V)$ by

$$(\varphi_1, \dots, \varphi_k) \longmapsto \sum_{i=1}^{k} \varphi_i \otimes \mathrm{Id}_{M_i} .$$

Note that the M_i are nonzero since V is faithful. Then $B = \bigoplus_{i=1}^{n} \mathrm{End}(M_i)$ embeds into $\mathrm{End}_{\mathbb{F}}(V)$ via

$$(\psi_1, \dots, \psi_k) \longmapsto \sum_{i=1}^{k} \mathrm{Id}_{U_i} \otimes \psi_i,$$

which completes the proof. □

4.4 Representations of Finite Groups

Remark. Recall that to any group G we can associate the group algebra $\mathbb{F}G$. For any representation of G, there is a representation of $\mathbb{F}G$ and vice versa.

Remark. Consider the following operations with representations. Let U, V be representations of G.

1. the *tensor product* $U \otimes_{\mathbb{F}} V$, where $g(U \otimes V) = (gU \otimes gV)$;
2. the *dual* U^* defined by $\langle g\alpha, u\rangle = \langle \alpha, g^{-1}u\rangle$ for $u \in U$, $\alpha \in U^*$, $g \in G$;
3. $\mathrm{Hom}_{\mathbb{F}}(U, V)$, with action given by $[g\varphi](h) = g[\varphi(g^{-1}u)]$ for $\varphi \in \mathrm{Hom}_{\mathbb{F}}(U, V)$.

Exercise 4.2. Show the following:

1. The tensor product of representations satisfies associativity, distributivity, and commutativity.
2. There is an isomorphism of representations $U^* \otimes V \to \mathrm{Hom}(U, V)$.
3. $\mathrm{Hom}_G(U, V) \subseteq \mathrm{Hom}(U, V)$ coincides with the space of G-invariant elements.

Remark. For the rest of this section, assume $\mathbb{F}$ is algebraically closed and $\mathrm{char}\,\mathbb{F} = 0$.

Theorem 4.5. *The group algebra $\mathbb{F}G$ is semisimple.*

Proof. It suffices to show that $(\cdot, \cdot)_{\mathbb{F}G}$ is nondegenerate. Take $g, g' \in G$, and note that $gg' : h \mapsto gg'h$, so

$$(g, g')_{\mathbb{F}G} = \mathrm{tr}_{\mathbb{F}G}(gg') = \delta_{1,gg'}|G|,$$

which is nondegenerate. Moreover, the basis $\{g\}$ in $\mathbb{F}G$ corresponds to the dual basis $\{|G|^{-1}g^{-1}\}$. □

Corollary 4.5.1. *(Let $\mathbb{F}$ be algebraically closed and* $\mathrm{char}\,\mathbb{F} = 0$*.)*

1. *Every finite-dimensional representation of G is completely reducible.*
2. *The number of isomorphism classes of irreducible representations is equal to the number of conjugacy classes of G.*
3. *If $U_1, \dots, U_k$ are all of the pairwise non-isomorphic irreducible representations of G, then*

$$|G| = \sum_{i=1}^{k} (\dim U_i)^2.$$

Proof. (1) This follows from the semisimplicity of $\mathbb{F}G$.

(2) It suffices to show that $\dim \mathcal{Z}(\mathbb{F}G)$ equals the number of conjugacy classes of G. We have

$$\mathcal{Z}(\mathbb{F}G) = \left\{ \sum_{g \in G} a_g g : a_g \text{ is constant on conjugacy classes} \right\},$$

i.e. we must have $a_{hgh^{-1}} = a_g$ for any $h \in G$. So the dimension is the number of conjugacy classes.

(3) This automatically follows from looking at the dimension of $\mathbb{F}G$. $\square$

Lecture 5

Sept. 3 — Representations of Finite Groups

5.1 Representations of S_4

Remark. We will write *irrep* for "irreducible representation."

Example 5.0.1. Consider the symmetric group S_4, with $|S_4| = 24$. The conjugacy classes of S_4 are parametrized by partitions of 4: If we have a partition

$$\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k), \quad \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq 1, \quad \lambda_1 + \lambda_2 + \cdots + \lambda_k = 4,$$

then the corresponding conjugacy class has cycle type λ. For example, the conjugacy classes are given by

1. $\lambda_1 = 4$: $[4]$;
2. $\lambda_1 = 3$, $\lambda_2 = 1$: $[3, 1]$;
3. $\lambda_1 = 2$, $\lambda_2 = 2$: $[2, 2]$;
4. $\lambda_1 = 2$, $\lambda_2 = 1$, $\lambda_3 = 1$: $[2, 1, 1]$;
5. $\lambda_1 = 1$, $\lambda_2 = 1$, $\lambda_3 = 1$, $\lambda_4 = 1$: $[1, 1, 1, 1]$.

In particular, this means that S_4 has 5 irreps. We can enumerate them as follows:

1. We have the 1-dimensional representations: the trivial representation and the sign sign_4.
2. Let S_4 act on $\mathbb{C}^4$ by permuting the basis vectors. The span of (x, x, x, x) gives a 1-dimensional subrepresentation, but it has a unique 3-dimensional complement refl_4.
3. We can take a tensor product $\text{refl}_4 \otimes \text{sign}_4$, which is also 3-dimensional. One can check that this is different from refl_4 by looking at the determinant.
4. We have found two 1-dimensional and two 3-dimensional irreps, which account for $1+1+9+9 = 20$ of the 24 dimensions. Thus there is a missing 2-dimensional representation.

   Note that there is a projection $\pi : S_4 \to S_3$ which is a homomorphism with kernel $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Figure this out and find the last irrep as an exercise.

Exercise 5.1. Let G be a finite abelian group. Prove that all irreps of G are 1-dimensional.

5.2 Characters

Definition 5.1. Let G be a group, and let U a finite-dimensional representation of G. The *character* $\chi_U : G \to \mathbb{F}$ is defined by $\chi_U(g) = \text{tr}_U(g)$.

Exercise 5.2. Prove the following:

1. χ_U is constant on conjugacy classes of G.
2. $\chi_{U\oplus V} = \chi_U \oplus \chi_V$.
3. $\chi_{U\otimes V} = \chi_U \chi_V$.

Remark. For the rest of this section, assume G is finite and $\mathbb{F} = \mathbb{C}$. So we know every representation of G is completely reducible. Denote by $\mathbb{C}[G]$ the algebra of complex-valued functions on G, and $\mathbb{C}[G]^G$ the subalgebra of functions constant on conjugacy classes (i.e. the G-invariant functions). Clearly the character χ_U lies in $\mathbb{C}[G]^G$ for any finite-dimensional representation U of G.

Definition 5.2. Define a Hermitian scalar product on $\mathbb{C}[G]^G$ (a priori only on the characters) by

$$(\chi_1, \chi_2) = \frac{1}{|G|}\sum_{g\in G} \chi_1(g)\overline{\chi_2(g)}.$$

Proposition 5.1. *Let U, V be finite-dimensional representations of G. Then*

$$(\chi_U, \chi_V) = \dim \mathrm{Hom}_G(U, V).$$

Proof. We first note that $\chi_{U^*} = \overline{\chi}_U$. To see this, observe that since G is finite, we have $g^n = 1$ for some n. In particular, the eigenvalues $\lambda_i(g)$ of g have $|\lambda_i(g)| = 1$. Thus $\lambda_i(g^{-1}) = \overline{\lambda_i(g)}$, so we see the result after taking traces. Another way to see this is the following: For a representation $\rho : G \to U$, we can make each $\rho(g)$ into a unitary operator as follows. Begin with a pairing $\langle \cdot, \cdot \rangle_0$ on U and define

$$\langle v, w\rangle = \frac{1}{|G|}\sum_{g\in G} \langle \rho(g)v, \rho(g)w\rangle_0, \quad v, w \in U.$$

Then $\rho(g)$ is unitary with respect to $\langle \cdot, \cdot \rangle$, and we get the result.

Continuing, we have $V \otimes U^* = \mathrm{Hom}_{\mathbb{C}}(U, V)$, so $\chi_{\mathrm{Hom}(U,V)} = \chi_V \overline{\chi}_U$. Consider the averaging element

$$\epsilon = |G|^{-1}\sum_{g\in G} g \in \mathbb{C}[G].$$

This is a projector on G-invariants (W^G) in any representation W. Thus $\mathrm{tr}_W(\epsilon) = \dim W^G$. Applying this to $W = \mathrm{Hom}(U, V)$ and noting that $\mathrm{Hom}_G(U, V) = \mathrm{Hom}(U, V)^G$, we get

$$\dim \mathrm{Hom}_G(U, V) = \mathrm{tr}_{\mathrm{Hom}(U,V)}(\epsilon) = |G|^{-1}\sum_{g\in G} \chi_{\mathrm{Hom}(U,V)}(g) = |G|^{-1}\sum_{g\in G} \chi_V(g)\overline{\chi_U(g)} = (\chi_V, \chi_U),$$

which proves the desired claim. $\square$

Corollary 5.0.1. *The characters of irreps form an orthonormal basis in $\mathbb{C}[G]^G$.*

Proof. Schur's lemma implies orthonormality. Since the number of irreps equals the number of conjugacy classes, the characters must form a basis. $\square$

5.3 Induced Representations

Remark. In this section, we only assume k is a commutative ring.

Let $H \subseteq G$, where H, G are finite groups, let kH, kG be the corresponding group algebras, and let U be a representation of H. Treating kG as a kG-kH-bimodule, we can construct the tensor product

$$kG \otimes_{kH} U.$$

Similarly, treating kG as a kH-kG-bimodule, we can construct the representation

$$\operatorname{Hom}_{kH}(kG, U).$$

In fact, these two representations are isomorphic, we call it the *induced representation*, denoted $\operatorname{Ind}_H^G U$.

Proposition 5.2. *There is a natural isomorphism $kG \otimes_{kH} U \cong \operatorname{Hom}_{kH}(kG, U)$.*

Proof. First treat kG as a kH-kG-bimodule, so we can consider $\operatorname{Hom}_{kH}(kG, kH)$ since kG, kH are both left kH-modules. So for any element $\varphi : kG \to kH$, we have

$$\varphi(hg) = h\varphi(g), \quad h \in H, g \in G.$$

with a left G-action and right H-action given by

$$[g\varphi](g') = \varphi(g'g) \quad \text{and} \quad [\varphi h](g') = \varphi(hg').$$

Note that kG is a free left kH-module with basis given by the orbits of H. Show as an exercise that

$$\begin{aligned} \operatorname{Hom}_{kH}(kG, kH) \otimes_{kH} U &\xrightarrow{\cong} \operatorname{Hom}_{kH}(kG, U) \\ \alpha \otimes u &\longmapsto (x \mapsto \alpha(x)u) \end{aligned}$$

is an isomorphism. From here it suffices to show that

$$kG \xrightarrow{\cong} \operatorname{Hom}_{kH}(kG, kH)$$

as kG-kH-bimodules. Define this map via $g \mapsto \varphi_g \in \operatorname{Hom}_{kH}(kG, kH)$, where

$$\varphi_g(g') = \begin{cases} g'g & \text{if } g'g \in H, \\ 0 & \text{otherwise.} \end{cases}$$

We need to show that φ is H-equivariant, G-equivariant, and an isomorphism of k-modules.

To see H-equivariance, note that $\varphi_{gh}(g')$ and $\varphi_g(g')h$ are nonzero and equal if and only if $gg' \in H$. For the G-equivariance, note that $\varphi_{g_1 g}(g')$ and $[g_1\varphi_g](g')$ are given by

$$\begin{aligned} \varphi_{g_1 g}(g') &= g'g_1g \quad \text{if } g'g_1g \in H, \\ [g_1\varphi_g](g') &= g'g_1g \quad \text{if } g'g_1g \in H \end{aligned}$$

and zero otherwise, so they coincide. To prove that φ is an isomorphism of k-modules, we need to check that the φ_g form a basis in $\operatorname{Hom}_{kH}(kG, kH)$. Let $g_1, \ldots, g_\ell$ be representatives of the left H-orbits in G. Then we claim that the following map is an isomorphism of k-modules:

$$\begin{aligned} \operatorname{Hom}_{kH}(kG, kH) &\xrightarrow{\cong} (kH)^{\oplus \ell} \\ \varphi &\mapsto \{\varphi(g_i)\}_{i=1}^{\ell}. \end{aligned}$$

This follows since for any $g \in G$ and $i \in \{1, \ldots, \ell\}$, there is a unique element $h \in H$ such that $hg_i = g^{-1}$, so φ_g is sent to the corresponding summand. $\square$

Corollary 5.0.2 (Frobenius reciprocity)**.** *Let U, V be representations of H, G, respectively. Then*

1. $\mathrm{Hom}_G(\mathrm{Ind}_H^G(U), V) \cong \mathrm{Hom}_H(U, V)$*;*
2. $\mathrm{Hom}_G(V, \mathrm{Ind}_H^G(U)) \cong \mathrm{Hom}_H(V, U)$*.*

Proof. This follows from the Tensor-Hom adjunction, check it as an exercise. □

Remark. What really is $\mathrm{Ind}_H^G U$? Consider the set of maps (of sets) $G \to U$, denote it by $\mathrm{Fun}(G, U)$. The action of G on itself gives $\mathrm{Fun}(G, U)$ the structure of a kG-module. Then we can define

$$\mathrm{Fun}_H(G, U) = \{f \in \mathrm{Fun}(G, U) : f(hg) = hf(g)\} \subseteq \mathrm{Fun}(G, U),$$

which we can identify with the induced representation $\mathrm{Hom}_{kH}(kG, U)$.

Lecture 6

Sept. 8 — Representations of S_n

6.1 Motivation for Studying S_n and Summary

Remark. The finite *simple* groups (those with no nontrivial normal subgroups) are classified as follows:

1. abelian groups: cyclic groups of finite order;
2. alternating groups: $U_n \subseteq S_n$ (the subgroup of even permutations) for $n \geq 5$;
3. 26 exceptional finite simple groups;
4. finite simple groups of *Lie type* (analogues of Lie groups for finite fields).

The final parts of the classification were done by Gorenstein (1960–1980s) and Aschbacher-Smith (2004).

Remark. We study S_n because it is easier to work with than directly studying U_n, and we can recover representations of U_n from those of S_n via Frobenius reciprocity.

Remark. We have previously seen the following using our abstract theory:

1. Representations of S_n are the same as representations of $\mathbb{C}S_n$.
2. The algebra $\mathbb{C}S_n$ is semisimple: $\mathbb{C}S_n \cong \bigoplus_V \operatorname{End}_{\mathbb{C}}(V)$, where V runs over the isomorphism classes of irreps of S_n.
3. The number of irreps of S_n (up to isomorphism) coincides with the number of conjugacy classes.

Remark. In the case of S_n, the conjugacy classes are enumerated by partitions of n:

$$(n_1, n_2, \ldots, n_k), \quad n_1 \geq n_2 \geq \cdots \geq n_k.$$

We can write repeated parts via $(m_1^{d_1}, \ldots, m_e^{d_e})$, where $m_1 > m_2 > \cdots > m_e$. So for S_6, we have

$$(2, 2, 1, 1) \longleftrightarrow (2^2, 1^2).$$

6.2 The Inductive Approach: Background

Remark. We will follow the *inductive approach*, due to Okounkov-Vershik. Consider the inclusions

$$\{1\} = S_1 \subseteq S_2 \subseteq \cdots \subseteq S_{n-1} \subseteq S_n.$$

Note that if $H \subseteq G$ are finite groups, then an irrep of $\mathbb{C}G$ decomposes into irreps of $\mathbb{C}H$.

In general, if $B \subseteq A$ are finite-dimensional associative algebras and $\tau : B \to A$ is a homomorphism, then any A-module is also a B-module by the homomorphism τ. We have isomorphisms

$$A \xrightarrow{\cong} \bigoplus_{V \in \mathrm{Irr}(A)} \mathrm{End}_{\mathbb{C}}(V),$$
$$B \xrightarrow{\cong} \bigoplus_{U \in \mathrm{Irr}(B)} \mathrm{End}_{\mathbb{C}}(U).$$

Let $M_{V,U} = \mathrm{Hom}_B(U, V)$ be multiplicity spaces. Then there is a B-linear isomorphism

$$\begin{aligned} \bigoplus_i U_i \otimes M_{V,U_i} &\xrightarrow{\cong} V \\ \sum_i u_i \otimes \varphi_i &\longmapsto \sum_i \varphi_i(u_i). \end{aligned}$$

We can compute $M_{V,U}$ from an algebraic perspective.

Definition 6.1. Define the *centralizer* of B in A to be

$$\mathcal{Z}_B(A) = \{a \in A : a\tau(b) = \tau(b)a \text{ for all } b \in B\}.$$

Exercise 6.1. Prove the following:

1. $\mathcal{Z}_A(A) = \mathcal{Z}(A)$.
2. $\mathcal{Z}_B(A)$ is a subalgebra of A.

Lemma 6.1. *There is an isomorphism $\mathcal{Z}_B(A) \cong \bigoplus_{U,V} \mathrm{End}(M_{V,U})$, with U, V such that $M_{V,U} \neq 0$.*

Proof. We have the isomorphism

$$A \xrightarrow{\cong} \bigoplus_V \mathrm{End}(V),$$

and we can view $\tau : B \to A$ as $(\tau_V)_{V \in \mathrm{Irr}(A)}$, where $\tau_V : B \to \mathrm{End}(V)$. Similarly, we can view an element $a \in A$ as $(a_V) \in \bigoplus_V \mathrm{End}(V)$. Then $a \in \mathcal{Z}_B(A)$ if and only if $a_V \in \mathcal{Z}_B(\mathrm{End}(V))$ for all V, so

$$\mathcal{Z}_B(A) = \bigoplus_V \mathcal{Z}_B(\mathrm{End}(V)).$$

Then $\mathcal{Z}_B(\mathrm{End}(V)) \cong \mathrm{End}_B(V) \cong \bigoplus_U \mathrm{End}(M_{V,U})$, which completes the proof. □

Remark. Show that the following actions of $\mathcal{Z}_B(A)$ on $\mathrm{End}(M_{V,U}) = \mathrm{Hom}_B(U, V)$ are the same:

1. $\mathrm{End}_B(V)$ acts on $\mathrm{Hom}_B(U, V)$ via

$$\begin{aligned} \mathrm{End}_B(V) \times \mathrm{Hom}_B(U, V) &\longrightarrow \mathrm{Hom}_B(U, V) \\ (\alpha, \varphi) &\longmapsto \alpha \circ \varphi; \end{aligned}$$

2. for $z \in \mathcal{Z}_B(A)$, $\varphi \in \mathrm{Hom}_B(U, V)$, we can define $z\varphi \in \mathrm{Hom}_B(U, V)$ by

$$[z\varphi](u) = z\varphi(u),$$

where the right-hand side is the A-action on V.

Corollary 6.0.1. *The following conditions are equivalent:*

1. *for all $U \in \mathrm{Irr}(B)$ and $V \in \mathrm{Irr}(A)$, we have $\dim \mathrm{Hom}_B(U, V) \leq 1$;*
2. *$\mathcal{Z}_B(A)$ is commutative.*

Proof. $\mathcal{Z}_B(A) = \bigoplus_{U,V} \mathrm{End}(M_{V,U})$ is commutative if and only if $\mathrm{End}(M_{V,U})$ has dimension 1 or 0. □

Example 6.1.1. Let $A = \mathrm{Mat}_4(\mathbb{C}) \oplus \mathrm{Mat}_3(\mathbb{C})$ and $B = \mathrm{Mat}_2(\mathbb{C}) \oplus \mathbb{C}^{\oplus 2}$. Define $\tau : B \to A$ by

$$\tau(x_1, x_2, x_3) = (\mathrm{diag}(x_1, x_2, x_2), \mathrm{diag}(x_1, x_3)), \quad x_1 \in \mathrm{Mat}_2(\mathbb{C}), x_2, x_3 \in \mathbb{C}.$$

We have B-modules U_1, U_2, U_3 of dimensions $2, 1, 1$ and A-modules V_1, V_2 of dimensions $4, 3$. Note that M_{V_1,U_2} is 2-dimensional, and M_{V_1,U_1}, M_{V_2,U_1}, M_{V_2,U_3} are 1-dimensional. So far, we have

$$\mathcal{Z}_B(A) \cong \mathrm{Mat}_2(\mathbb{C}) \oplus \mathbb{C}^{\oplus 3}.$$

To verify this directly, we know that $\mathcal{Z}_B(A)$ consists of pairs $(y_1, y_2) \in \mathrm{Mat}_4(\mathbb{C}) \oplus \mathrm{Mat}_3(\mathbb{C})$ such that y_1 commutes with $\mathrm{diag}(x_1, x_2, x_2)$ and y_2 commutes with $\mathrm{diag}(x_1, x_3)$. So

$$y_1 = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & b & c \\ 0 & 0 & d & e \end{pmatrix}, \quad y_2 = \begin{pmatrix} f & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & g \end{pmatrix}.$$

So $\mathcal{Z}_B(A)$ is parametrized by the 2×2 matrix and the 3 scalars a, f, g.

6.3 The Inductive Approach: Properties of $\mathbb{C}S_n$

Remark. Let $S_m \subseteq S_n$ for $m < n$, and let $\mathcal{Z}_m(n)$ be the corresponding centralizer for group algebras.

Lemma 6.2. *Let $H \subseteq G$ be finite groups. Then $\mathbb{Z}_{\mathbb{C}H}(\mathbb{C}G) \subseteq \mathbb{C}G$ consists of elements of the form $\sum_{g \in G} a_g g$ such that $a_{hgh^{-1}} = a_g$ for all $h \in H$. In particular, $\mathcal{Z}_{\mathbb{C}H}(\mathbb{C}G)$ has a basis indexed by the H-conjugacy classes in G, given by (for a conjugacy class C)*

$$C \longmapsto b_C = \sum_{g \in C} g \in \mathcal{Z}_{\mathbb{C}H}(\mathbb{C}G).$$

Example 6.1.2. Note that for $\mathbb{C}S_m \subseteq \mathbb{C}S_n$, conjugation permutes the first m elements. For example, for $S_3 \subseteq S_6$, we can write a conjugacy class as $(* * 4)(5 *)(6)$, which contains elements like $(1\ 2\ 4)(5\ 3)$ and $(2\ 3\ 4)(5\ 1)$. For $m = n - 1$, consider the conjugacy class $(*\ n)$, which consists of

$$(1\ n), \quad (2\ n), \quad \ldots, \quad (n-1\ n).$$

Then the basis element $b_{(*\,n)}$ (called the nth *Jucys-Murphy element*) is given by

$$b_{(*\,n)} = \sum_{i=1}^{n-1} (i\ n).$$

Lecture 7

Sept. 10 — Representations of S_n, Part 2

7.1 Properties of $\mathbb{C}S_n$, Continued

Remark. We will now determine algebra generators of $\mathcal{Z}_m(n)$. It contains

1. $\mathcal{Z}_m(m)$: the center of $\mathbb{C}S_m$;
2. $S_{[m+1,n]}$: the subgroup of S_n containing permutations fixing $1, \ldots, m$;
3. $J_k = \sum_{i=1}^{k-1} (i\ k)$ for $k = m+1, \ldots, n$.

Note that $J_{m+1}, \ldots, J_n$ pairwise commute (check this as an exercise).

Theorem 7.1. *The algebra $\mathcal{Z}_m(n)$ is generated by the subalgebras $\mathcal{Z}_m(m)$, $\mathbb{C}S_{[m+1,n]}$, and the elements $J_{m+1}, \ldots, J_n$.*

Proof. Let C be an S_m-conjugacy class in S_n. Define $\deg(C)$ to be the number of elements in $\{1, \ldots, n\}$ which are moved by the corresponding permutations (for instance, $(*\ n) = (*\ *)$ has degree 2). Note that we either have $\deg(C) = 0$ or $\deg(C) \geq 2$.

Let A be the subalgebra of $\mathcal{Z}_m(n)$ generated by $\mathcal{Z}_m(m)$, $\mathbb{C}S_{n-1}$, and $J_{m+1}, \ldots, J_n$. We need to show that $b_C \in A$ for every C. Assume it is not true, and pick C of minimal degree such that $b_C \notin A$. First we show that $\deg(C) > 2$. If $\deg(C) = 2$, then we have two possibilities:

1. $C = (*\ k)$ for $k > m$. Then
$$b_{(*\,k)} = \sum_{i=1}^{m} (i\ k) = J_k - \sum_{i=m+1}^{k-1} (i\ k).$$
Then $J_k \in A$ and $\sum_{i=m+1}^{k-1} (i\ k) \in \mathbb{C}S_{[m+1,n]} \subseteq A$, so we are good.
2. $C = (k\ \ell)$ for $m < k < \ell \leq n$. Then $b_{(k\,\ell)} \in \mathbb{C}S_{[m+1,n]} \subseteq A$.
3. $C = (*\ *)$. Then $b_{(*\,*)} \in \mathcal{Z}_m(m) \subseteq A$.

So $\deg(C) > 2$. Now assume that C has more than 1 cycle of degree ≥ 2. Write $C = C'C''$, then
$$b_{C'} b_{C''} = \alpha b_C + \sum_{C_0, \deg C_0 < \deg C} \alpha_{C_0} b_{C_0}.$$

Since $b_{C'}, b_{C''}, b_{C_0} \in A$ by minimality of C, we also get $\alpha b_C \in A$, so $b_C \in A$ since $\alpha \neq 0$ (note that we may have characteristic issues here if we are not working over $\mathbb{C}$).

So we may assume C is a single cycle. Pick a cycle $(i_1\ i_2\ \cdots\ i_k) \in S_n$. Then if $j \notin \{i_1, \dots, i_k\}$,

$$(i_1\ i_2\ \cdots\ i_k)(i_s\ j) = (i_1\ i_2\ \cdots\ i_{s-1}\ j\ i_{s+1}\ \cdots\ i_k).$$

If $j \in \{i_1, \dots, i_k\}$, then $(i_1\ i_2\ \cdots\ i_k)(i_s\ j)$ either splits into two cycles or reduces the degree by 1.

So suppose a cycle in C has elements from $\{1, \dots, m\}$ and $k \in \{m+1, \dots, n\}$. We can assume that k is next to $*$. Denote by C' the cycle obtained after eliminating $*$. Then

$$b_{C'} b_{(*\,k)} = \alpha b_C + \sum_{C_0} \alpha_{C_0} b_{C_0},$$

where C_0 either contains disjoint cycles or cycles of smaller degree. Thus we get $b_{C'}, b_{(*\,k)}, b_{C_0} \in A$ by the minimality of C, so $b_C \in A$ as well.

Thus we may assume the elements in our 1-cycle C sit in either $\{1, \dots, m\}$ or $\{m+1, \dots, n\}$. In the first case, $b_C \in \mathcal{Z}_m(m) \subseteq A$, and in the second case, $b_C \in \mathbb{C}S_{[m+1,n]} \subseteq A$. $\square$

Corollary 7.1.1. *We have the following:*

1. *$\mathcal{Z}_{n-1}(n)$ is commutative;*
2. *for all $U \in \mathrm{Irr}(\mathbb{C}S_{n-1})$ and $V \in \mathrm{Irr}(\mathbb{C}S_n)$, the multiplicity of U in V is either 0 or 1;*
3. *the element J_n acts on each irreducible $\mathbb{C}S_{n-1}$-submodule of $V \in \mathrm{Irr}(\mathbb{C}S_n)$ by a scalar.*

Proof. (1) $\mathcal{Z}_{n-1}(n)$ is generated by $\mathcal{Z}(n-1)$ and J_n, which commute.

(2) This follows from the statement about abelian centralizers for algebras.

(3) This follows from Schur's lemma. $\square$

Example 7.0.1. We will determine how J_n acts on various modules and how they decompose:

1. $V = \mathrm{refl}_n$, which is a $\mathbb{C}S_n$-module and is given by

   $$\mathrm{refl}_n = \{(x_1, \dots, x_n) \in \mathbb{C}^n : x_1 + \cdots + x_n = 0\}.$$

   As a $\mathbb{C}S_{n-1}$-module, refl_n decomposes as follows

   - $U_1 = \{(x_1, \dots, x_{n-1}, 0) \in \mathbb{C}^n : x_1 + \cdots + x_{n-1} = 0\}$. This is refl_{n-1}.
   - $U_0 = \{(-x, \dots, -x, (n-1)x) \in \mathbb{C}^n\}$. This is the trivial representation.

   Note that $J_n = \sum_{i=1}^{n-1} (i\ n)$ acts on $(x_1, \dots, x_n)$ by

   $$(x_1, \dots, x_n) \longmapsto ((n-2)x_1 + x_n, \dots, (n-2)x_{n-2} + x_n, x_1 + \cdots + x_n).$$

   On refl_{n-1}, the eigenvalue is $n-2$, and on the trivial subrepresentation, the eigenvalue is -1.

2. When $n = 4$, there was a representation V of dimension 2, given by the pull-back of refl_3 under the projection $S_4 \to S_3$. The kernel of the projection is the normal subgroup

   $$\{e, (1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3)\},$$

where S_3 permutes $(1\ 2)(3\ 4)$, $(1\ 3)(2\ 4)$, and $(1\ 4)(2\ 3)$. Now

$$J_4 = (1\ 4) + (2\ 4) + (3\ 4),$$

and we are looking for an action of J_4 on V. We can take

$$J_4|_V = (2\ 3) + (1\ 3) + (1\ 2),$$

which is an element of $\mathbb{C}S_3$. Note that refl_3 is given by

$$\mathrm{refl}_3 = \{(x_1, x_2, x_3) \in \mathbb{C}^3 : x_1 + x_2 + x_3 = 0\}.$$

When $J_4|_V$ acts on refl_3, we get $x_1 + x_2 + x_3 = 0$ in every coordinate, for any $(x_1, x_2, x_3) \in \mathrm{refl}_3$, so the eigenvalue in this case is 0.

7.2 Branching Graphs

Remark. Let V^n be an irrep for $\mathbb{C}S_n$. We know that V^n decomposes into a direct sum of non-isomorphic $\mathbb{C}S_{n-1}$-modules. These then decompose into $\mathbb{C}S_{n-2}$-modules, and so on.

Definition 7.1. The *branching graph* is a directed graph, where the vertices are labeled by isomorphism classes of $\mathbb{C}S_n$-modules (for all n), and the edge $U \to V$ exists if V is an irreducible module for $\mathbb{C}S_n$ and U is an irreducible module for $\mathbb{C}S_{n-1}$ which occurs in the decomposition of V.

Example 7.1.1. The following is the branching graph up to S_4:

$$\begin{tikzcd}
\mathrm{triv}_4 & & \mathrm{refl}_4 & \mathbb{C}^2 & \mathrm{refl}_4 \otimes \mathrm{sign}_4 & & \mathrm{sign}_4 \\
& \mathrm{triv}_3 \arrow[ul] \arrow[ur] & & \mathrm{refl}_3 \arrow[ul] \arrow[u] \arrow[ur] & & \mathrm{sign}_3 \arrow[ul] \arrow[ur] & \\
& & \mathrm{triv}_2 \arrow[ul] \arrow[ur] & & \mathrm{sign}_2 \arrow[ul] \arrow[ur] & & \\
& & & \mathrm{triv}_1 \arrow[ul] \arrow[ur] & & &
\end{tikzcd}$$

Note that there is a left-right symmetry in the graph, which comes from tensoring with sign_n.

Definition 7.2. Let $V^m \in \mathrm{Irr}(\mathbb{C}S_m)$ and $V^n \in \mathrm{Irr}(\mathbb{C}S_n)$ for $m < n$. Define $\mathrm{Path}(V^m, V^n)$ to be the set of all paths from V^m to V^n in the branching graph. If $m = 1$, we write $\mathrm{Path}(V^n) = \mathrm{Path}(V^1, V^n)$, and we denote $\mathrm{Path}_n = \bigsqcup_{V^n \in \mathrm{Irr}(\mathbb{C}S_n)} \mathrm{Path}(V^n)$.

Remark. For $\overline{P} = (V^m \to V^{m+1} \to \cdots \to V^n) \in \mathrm{Path}(V^m, V^n)$, denote by $V^m(\overline{P})$ a copy of V^m in V^n according to the path $\overline{P}$. Then we can write the decomposition of V^n by

$$V^n = \bigoplus_{V^m \in \mathrm{Irr}(\mathbb{C}S_m)} \bigoplus_{\overline{P} \in \mathrm{Path}(V^m, V^n)} V^m(\overline{P}).$$

Definition 7.3. Denote by $\varphi_{\overline{P}} : V^m \to V^n$ the homomorphism sending V^m to its copy in V^n according to the path $\overline{P}$, which is defined uniquely up to rescaling, and define

$$w_{\overline{P}} = (w_{m+1}, \ldots, w_n) \in \mathbb{C}^{n-m}$$

where w_k is the scalar by which J_k acts on $V^{k-1} \subseteq V^k$. Call $w_{\overline{P}}$ the *weight* of $\overline{P}$.

Remark. Recall that $\mathrm{Hom}_{\mathbb{C}S_m}(V^m, V^n)$ is an irreducible $\mathcal{Z}_m(n)$-module from properties of centralizers.

Lemma 7.1. *We have the following:*

1. *The elements $\varphi_{\overline{P}}$ form a basis in $\mathrm{Hom}_{\mathbb{C}S_m}(V^m, V^n)$.*
2. *Each $\varphi_{\overline{P}}$ is an eigenvector for J_k with eigenvalue w_k, for each $k = m+1, \dots, n$, where*

$$(w_{m+1}, \dots, w_n) = w_{\overline{P}}.$$

Proof. (1) We can write

$$\begin{aligned}\mathrm{Hom}_{\mathbb{C}S_m}(V^m, V^n) &= \bigoplus_{V'^m \in \mathrm{Irr}(S_m)} \bigoplus_{\overline{P} \in \mathrm{Path}(V'^m, V^n)} \mathrm{Hom}(V^m, V'^m(\overline{P})) \\ &= \bigoplus_{\overline{P} \in \mathrm{Path}(V^m, V^n)} \mathrm{Hom}(V^m, V^m(\overline{P})),\end{aligned}$$

where the second equality is by Schur's lemma. By Schur's lemma again, $\mathrm{Hom}(V^m, V^m(\overline{P})) \cong \mathbb{C}$. Since the $\varphi_{\overline{P}}$ correspond to these summands, this proves (1).

(2) For any $u \in V^m$, we have $[J_k \varphi_{\overline{P}}](u) = J_k[\varphi_{\overline{P}}(u)]$. By construction, $V^m(\overline{P})$ lies in some copy of V^{k-1} in V^k for $k = m+1, \dots, n$, so $J_k \varphi_{\overline{P}} = w_k \varphi_{\overline{P}}$ implies (2). $\square$

Lecture 8

Sept. 15 — Representations of S_n, Part 3

8.1 More on Branching Graphs

Remark. Consider $\mathrm{Hom}_{\mathbb{C}S_m}(V^m, V^n)$. When $m = 1$, we may identify $\mathrm{Hom}_{\mathbb{C}S_1}(V^1, V^n) = \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}, V^n)$ with V^n itself. For $P \in \mathrm{Path}(V^n)$, we will write v_P for φ_P.

Corollary 8.0.1. *We have the following:*

1. *the vectors v_P for $P \in \mathrm{Path}(V^n)$ form a basis in V^n;*
2. *each v_P is an eigenvector for J_k with eigenvalue w_k for $k = 1, \ldots, n$. Note $w_1 = 0$ since $J_1 = 0$.*

Example 8.0.1. Consider the following:

1. $V^n = \mathrm{refl}_n$. We have $\mathrm{refl}_n \cong \mathrm{refl}_{n-1} \oplus \mathrm{triv}_{n-1}$. When $n = 2$, we have $\mathrm{refl}_2 = \mathrm{triv}_1$. Then any path $P \in \mathrm{Path}(V^n)$ must be of the form

$$P = \mathrm{triv}_1 \to \cdots \to \mathrm{triv}_i \to \mathrm{refl}_{i+1} \to \cdots \to \mathrm{refl}_n .$$

   The corresponding weights are $w_P = (0, 1, \ldots, i-1, -1, i, \ldots, n-2)$: Recall from before that J_k acts on $\mathrm{refl}_{k-1} \subseteq \mathrm{refl}_k$ by $k-2$ and $\mathrm{triv}_{k-1} \subseteq \mathrm{refl}_k$ by -1.

Exercise 8.1. Check that $v_P = (1, \ldots, 1, -i, 0, \ldots, 0)$ in Example 8.0.1 (there are i ones).

Exercise 8.2. Let $V = \mathbb{C}^2$ be a representation of S_4. Write down two elements in $\mathrm{Path}(\mathbb{C}^2)$ and find the corresponding weights.

Corollary 8.0.2. *Let $m < n$ and $V^m \in \mathrm{Irr}(\mathbb{C}S_m)$, $V^n \in \mathrm{Irr}(\mathbb{C}S_n)$, $\underline{P} \in \mathrm{Path}(V^m)$, $\overline{P} \in \mathrm{Path}(V^m, V^n)$. Let P be the path obtained by concatenating $\underline{P}$ and $\overline{P}$. Then v_P is proportional to $\varphi_{\overline{P}}(v_{\underline{P}})$.*

Proof. Both are clearly nonzero and lie in $V^1(P)$, which is one-dimensional. □

8.2 Properties of Weights

Theorem 8.1. *Let $P, P' \in \mathrm{Path}_n$. If $w_P = w_{P'}$, then $P = P'$.*

Proof. The proof is by induction. The $n = 1$ case is trivial. Now suppose the statement is true for $n - 1$. Let $\underline{P}, \underline{P'} \in \mathrm{Path}_{n-1}$ be truncations of $P, P' \in \mathrm{Path}_n$. Assume that

$$\begin{cases} w_P = (w_1, \ldots, w_n), \\ w_{P'} = (w'_1, \ldots, w'_n), \end{cases}$$

so $w_{\underline{P}} = (w_1, \dots, w_{n-1})$ and $w_{\underline{P'}} = (w'_1, \dots, w'_{n-1})$. If $w_P = w_{P'}$, then we have $w_{\underline{P}} = w_{\underline{P'}}$ and thus $\underline{P} = \underline{P'}$ by the inductive hypothesis.

Now assume V, V' are the endpoints of P, P', respectively, $V, V' \in \mathrm{Irr}(\mathbb{C}S_n)$. We need to show that $V \cong V'$. Let $U \in \mathrm{Irr}(\mathbb{C}S_{n-1})$ be the endpoint of $\underline{P} = \underline{P'}$. Note that each $z \in \mathcal{Z}_{n-1}(n)$ acts on $U \subseteq V$ and $U \subseteq V'$ as a scalar. Denote these scalars by $\chi(z)$ and $\chi'(z)$, and note that $\chi(z) = \chi'(z)$: We know that $\mathcal{Z}_{n-1}(n)$ is generated by $\mathcal{Z}_{n-1}$ and J_n, any $z \in \mathcal{Z}_{n-1}$ acts on U as a scalar with $\chi(z) = \chi'(z)$, and J_n acts on both U's embedded in V, V' by w_n, so $\chi(J_n) = \chi'(J_n) = w_n$.

Let $\mathcal{Z}_n(n)$ be the center of $\mathbb{C}S_n$, which is contained in $\mathcal{Z}_{n-1}(n)$. Every $z \in \mathcal{Z}_n(n)$ acts on V and V' as scalars $\chi_V(z)$ and $\chi_{V'}(z)$, which must be the same scalars by which z acts on U. Then χ_V and $\chi_{V'}$ are the same central characters, so we find that $V \cong V'$. $\square$

Definition 8.1. Define $\mathrm{Wt}_n = \{w_P : p \in \mathrm{Path}_n\}$. We say that two elements in Wt_n are *r-equivalent* (the r is for "representation") if the weights of the two paths are in the same irreducible module.

Remark. Theorem 8.1 states that there is a one-to-one correspondence $\mathrm{Path}_n \longleftrightarrow \mathrm{Wt}_n$. Moreover, r-equivalence is an equivalence relation and gives a one-to-one correspondence between equivalence classes and isomorphism classes of irreducible representations.

Theorem 8.1 also implies that the basis vectors v_P for $P \in \mathrm{Path}(V^n)$ are in bijection with weights in the corresponding equivalence class. Thus it suffices to study weights going forward.

Remark. We now see what happens when we vary paths. Consider a path

$$P = (V^1 \to \cdots \to V^n) \in \mathrm{Path}_n .$$

Pick $i \in \{1, \dots, n-1\}$, and consider the space of all paths of the form

$$P' = (V'^1 \to \cdots \to V'^n), \quad \text{where } V'^j = V^j \text{ for } j \neq i.$$

Denote this set by $\mathrm{Path}(P, i)$. We will prove the following theorem later:

Theorem 8.2. *Let $w_P = (w_1, \dots, w_n)$. Then the following are true:*

1. $w_i \neq w_{i+1}$;
2. *if* $w_{i+1} = w_i \pm 1$, *then* $\mathrm{Path}(P, i) = \{P\}$;
3. *if* $w_{i+1} \neq w_i \pm 1$, *then* $\mathrm{Path}(P, i)$ *consists of two elements* P, P' *and* $w_{P'}$ *is obtained from* w_P *by permuting* w_i, w_{i+1};
4. *if* $i < n-1$, *then* $w_i = w_{i+1} \pm 1$ *implies* $w_{i+2} \neq w_i$.

Remark. To simplify notation, denote $V = V^n$, $\mathcal{Z}_{i-1}(i+1) \subseteq \mathbb{C}S_n$, and

$$V_{P,i} = \mathrm{Span}\{v_{P'} : P' \in \mathrm{Path}(P, i)\}.$$

Note that the $v_{P'}$ actually form a basis of $V_{P,i}$.

Proposition 8.1. *The subspace $V_{P,i} \subseteq V$ is an irreducible $\mathcal{Z}_{i-1}(i+1)$-module.*

Proof. Let $P = P_0 P_1 P_2$, where $P_0 \in \mathrm{Path}(V^{i-1})$, $P_1 \in \mathrm{Path}(V^{i-1}, V^{i+1})$, and $P_2 \in \mathrm{Path}(V^{i+1}, V^n)$. Then $\mathrm{Path}(P, i)$ consists of paths of the form $P_0 P'_1 P_2$, where $P'_1 \in \mathrm{Path}(V^{i-1}, V^{i+1})$. We have

$$V_{P_0 P'_1 P_2} = \varphi_{P_2}(\varphi_{P'_1}(v_{P_0})).$$

Now consider the linear map

$$\begin{aligned} \mathrm{Hom}_{\mathbb{C}S_{i-1}}(V^{i-1}, V^{i+1}) &\longrightarrow V \\ \psi &\longmapsto \varphi_{P_2}(\psi(v_{P_0})). \end{aligned}$$

Note that we have $\varphi_{P_1'} \mapsto v_{P_0 P_1' P_2}$ in $V_{P,i}$, where the $v_{P_0 P_1' P_2}$ form a basis of $V_{P,i}$ and the $\varphi_{P_1'}$ form a basis in $\mathrm{Hom}_{\mathbb{C}S_{i-1}}(V^{i-1}, V^{i+1})$. In particular, this map is injective with image $V_{P,i}$.

It only remains to show that this map is $\mathcal{Z}_{i-1}(i+1)$-linear, which is left as an exercise. □

8.3 The Degenerate Affine Hecke Algebra

Remark. We want to study $\mathcal{Z}_{i-1}(i+1) \subseteq \mathbb{C}S_n$ better. We know $\mathcal{Z}_{i-1}(i+1)$ is generated by $\mathcal{Z}_{i-1}(i-1)$, J_i, J_{i+1}, and $(i, i+1)$, and we know that $V_{P,i}$ is an irreducible representation for $\mathcal{Z}_{i-1}(i+1)$. Note that the elements in $\mathcal{Z}_{i-1}(i-1)$ act as scalars, so we only need to worry about J_i, J_{i+1}, and $(i, i+1)$.

Lemma 8.1. *We have the following relations:*

1. $J_i J_{i+1} = J_{i+1} J_i$;
2. $(i, i+1)^2 = 1$;
3. $(i, i+1) J_i = J_{i+1}(i, i+1) - 1$.

Proof. We already know (1) and (2). For

$$(i, i+1) J_i (i, i+1) = \sum_{j=1}^{i-1} (j, i+1) = J_{i+1} - (i, i+1),$$

which becomes (3) after right-multiplying by $(i, i+1)$. □

Definition 8.2. Define the *degenerate affine Hecke algebra* $\mathcal{H}(2)$ to be the algebra with generators X_1, X_2, T and relations $X_1 X_2 = X_2 X_1$, $T^2 = 1$, and $T X_1 = X_2 T - 1$ (equivalently, $X_1 T = T X_2 - 1$).

Remark. There is a unique homomorphism $\mathcal{H}(2) \to \mathcal{Z}_{i-1}(i+1)$ given by

$$X_1 \mapsto J_i, \quad X_2 \mapsto J_{i+1}, \quad T \mapsto (i, i+1).$$

Corollary 8.2.1. *Let M be an irreducible module for $\mathcal{Z}_{i-1}(i+1)$. Then M stays irreducible as an $\mathcal{H}(2)$-module.*

Proof. Note that $\mathcal{Z}_{i-1}(i-1)$ is the central subalgebra of $\mathcal{Z}_{i-1}(i+1)$. Any element of the center acts as a scalar on an irreducible $\mathcal{Z}_{i-1}(i+1)$-module, so a subspace invariant under $\mathcal{Z}_{i-1}(i+1)$ is also invariant under $\mathcal{H}(2)$. This proves the claim. □

Remark. A basis of $\mathcal{H}(2)$ is given by $\{X_1^{d_1} X_2^{d_2} \sigma : \sigma \in \{1, T\}\}$.

Remark. One can generalize this construction to $\mathcal{Z}_i(d)$ to get $\mathcal{H}(d)$, with generators $X_1, \ldots, X_d$ and $T_1, \ldots, T_{d-1}$, with similar relations.

Example 8.2.1. We consider finite-dimensional irreps of $\mathcal{H}(2)$. Note that X_1, X_2 commute, so they have a common eigenvector $m \in M$. So $X_1 m = am$ and $X_2 m = bm$ for $a, b \in \mathbb{C}$. We have two cases:

1. $Tm \sim m$. Since $T^2 = 1$, we have two options:
   (a) $Tm = m$. Then $TX_1 m = am$, and applying $TX_1 = X_2 T - 1$ to m, we get
   $$(X_2 T - 1)m = (b-1)m.$$
   Thus we must have $b = a + 1$.
   (b) $Tm = -m$. Then one can check that $b = a - 1$ as an exercise.
2. m, Tm are linearly independent. Then
   $$\begin{aligned} X_1(Tm) &= (TX_2 - 1)m = b(Tm) - m, \\ X_2(Tm) &= (TX_1 + 1)m = a(Tm) + m. \end{aligned}$$
   In particular, $\mathrm{Span}\{m, Tm\}$ is stable under $\mathcal{H}(2)$. Since M is irreducible, $\{m, Tm\}$ is a basis of M. In this case, one can check that
   $$T \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad X_1 \mapsto \begin{pmatrix} a & 0 \\ -1 & b \end{pmatrix}, \quad X_2 \mapsto \begin{pmatrix} b & 0 \\ 1 & a \end{pmatrix}$$
   defines an $\mathcal{H}(2)$-module on $\mathbb{C}^2$, denoted as $M(a,b)$.

Lemma 8.2. *$M(a,b)$ is irreducible if and only if $a \neq b \pm 1$. If $a \neq b \pm 1$, then $M(a,b) \cong M(a',b')$ if and only if $(a,b) = (a',b')$ or $(b,a) = (a',b')$.*

Lecture 9

Sept. 17 — Combinatorial Weights

9.1 More on the Degenerate Affine Hecke Algebra

Proof of Lemma 8.2. Assume $a \neq b$. Then X_1, X_2 have two distinct eigenvalues, hence they are diagonalizable. Since $a \neq b$, every subspace in $M(a,b)$ stable under X_1 (or X_2) must be the sum of these eigenspaces. If one has a 1-dimensional submodule for $\mathcal{H}(2)$, then T must preserve it. If $a = b \pm 1$, then $m \pm Tm$ is an eigenvector for X_1, X_2, T, so $M(a,b)$ is not irreducible.

For the last part, we can simply switch the two eigenvalues. $\square$

Proposition 9.1. *The finite-dimensional irreps of $\mathcal{H}(2)$ are classified by pairs of complex numbers (a,b), $(a,b) \mapsto L(a,b)$, where $L(a,b) \cong L(b,a)$ if $b \neq a, a \pm 1$. Moreover, we have*

1. *If $b = a+1$, then $L(a,b) = \mathbb{C}$ with $T \mapsto 1$, $X_1 = a$, $X_2 = b$.*
2. *If $b = a-1$, then $L(a,b) = \mathbb{C}$ with $T \mapsto -1$, $X_1 = a$, $X_2 = b$.*
3. *If $b \neq a \pm 1$, then $L(a,b) \cong M(a,b)$.*
4. *The action of X_1, X_2 on $L(a,b)$ is diagonalizable if and only if $a \neq b$.*

Proof. This is Example 8.2.1 and Lemma 8.2. $\square$

Proof of Theorem 8.2. Let $w_P = (w_1, \dots, w_n)$, $P' \in \mathrm{Path}(V, i)$, and $w_{P'} = (w'_1, \dots, w'_n)$, where the w'_j depend only on V_j, V_{j-1}. Note that $V'_j = V_j$ for all $j \neq i$ implies $w'_j = w_j$ for all $j \neq i$. We have shown that $V_{P,i}$ is an irreducible $\mathcal{Z}_{i-1}(i+1)$-module and also an irreducible $\mathcal{H}(2)$-module, and that X_1, X_2 (J_i, J_{i+1}) are diagonalizable with eigenvalues (w_i, w_{i+1}) and (w'_i, w'_{i+1}). This proves (1)-(3).

(4) If $w_{i+1} = w_i \pm 1$, then $w_{i+2} \neq w_i$ (check this as an exercise). By (2), $w_{i+1} = w_i \neq 1$ implies that $V_{P,i+1}$ is also 1-dimensional, and $\mathbb{C}v_P$ is invariant under $(i, i+1)$, $(i+1, i+2)$. Now observe that

$$(i, i+1)(i+1, i+2)(i, i+1) = (i, i+2) = (i+1, i+2)(i, i+1)(i+1, i+2),$$

which is the same element. But $(i, i+1)$ and $(i+1, i+2)$ act on v_P by ± 1 and ∓ 1, respectively, so the above implies that $\mp 1 = \pm 1$, which is a contradiction. $\square$

9.2 Combinatorial Weights

Definition 9.1. We say two elements of $\mathbb{C}^n$ are *c-equivalent* (the c is for "combinatorial") if one can be obtained from the other through a sequence of *admissible* transpositions (those where the difference between two adjacent entries in the transposition is not ± 1).

Definition 9.2. A *combinatorial weight* is an element of $\mathbb{C}^n$ such that every element $(w_1, \dots, w_n) \in \mathbb{C}^n$ combinatorially equivalent to it satisfies:

1. $w_1 = 0$;
2. for all $i = 1, \dots, n-1$, $w_i \neq w_{i+1}$;
3. for all $i = 1, \dots, n-2$, we have $w_{i+1} = w_i \pm 1$ implies $w_{i+2} \neq w_i$.

Denote the set of combinatorial weights by $c\mathrm{Wt}_n$.

Corollary 9.0.1. *We have the following:*

1. $\mathrm{Wt}_n \subseteq c\mathrm{Wt}_n$*, so* Wt_n *is a collection of c-equivalence classes.*
2. *c-equivalence implies r-equivalence. Moreover,* $|\mathrm{Wt}_n/{\sim_r}| \leq |\mathrm{Wt}_n/{\sim_c}| \leq |c\mathrm{Wt}_n/{\sim_c}|$.
3. *There is a one-to-one correspondence* $\mathrm{Wt}_n/{\sim_r} \longleftrightarrow \mathrm{Irr}(\mathbb{C}S_n)$.

Lemma 9.1. *Every c-equivalence class contains elements of the form*

$$(0, 1, \dots, n_1 - 1, -1, 0, 1, \dots, n_2 - 2, -2, \dots, (1-k), \dots, n_k - k),$$

where $n_1 \geq n_2 \geq \cdots \geq n_k$ *and* $n_1 + \cdots + n_k = n$.

Proof. First we show that all components of combinatorial weights are integers. Suppose not, and let i be the minimal number such that $w_i \notin \mathbb{Z}$. Then we can make admissible transformations from right to left until it reaches the first slot, which is a contradiction since $w_1 = 0 \in \mathbb{Z}$.

Consider the lexicographic order on $c\mathrm{Wt}_n$, i.e. $(w_1, \dots, w_n) > (w'_1, \dots, w'_n)$ if there exists i such that $w_j = w'_j$ for each $1 \leq j < i$ and $w_i > w'_i$. Let $(w_1, \dots, w_n)$ be a maximal element in this equivalence class. We need to show that this maximal element is of the desired form.

To do this, first take n_1 such that $n_1 - 1 = \max\{w_i\}$. Let k be the smallest index such that $w_k = n_1 - 1$. We claim that $k = n_1$ and $w_i = i - 1$ for all $i < n_1$. Assume not. Then pick the largest index $j < k$ with $w_j \neq n_1 - 1 - (k - j)$. By the choice of k, we have $w_j < n_1$. We also have $w_j \geq j - 1$ (otherwise one can permute j and $j+1$, which increases the order). Note that if $w_j \geq j$, then we can make admissible transformations to the left until we arrive to (w_j, w_j), $(w_j, w_{j\pm 1}, w_j)$, or w_j in the first position, which are all impossible. Thus $w_j = n_1 - 1 - (k - j)$ for all $j < k$. But $w_1 = 0$, so $k = n_1$.

Thus we have shown that we can take an element starting with $0, 1, \dots, n_1 - 1$. Now if $n_1 = n$, then we are done. Otherwise, we need to prove that $w_{n_1+1} = -1$. Note that $w_{n_1+1} \leq n_1 - 1$ by our choice of n_1, and $w_{n_1} \neq n_1 - 1$ since $w_{n+1} \neq w_n$. If we move w_{n+1} to the left, then we encounter

$$(w_{n_1+1}, w_{n_1+1} + 1, w_{n_1+1})$$

for any $w_{n+1} \geq 0$. If $w_{n+1} < -1$, then we can move it to the first position, which is impossible since we always have $w_1 = 0$. So the only possibility is $w_{n+1} = -1$.

Now we can repeat the above argument to get the rest of the form. $\square$

Remark. Lemma 9.1 implies the following:

1. $c\mathrm{Wt}_n = \mathrm{Wt}_n$;

2. $\sim_c = \sim_r$;
3. $n_1, \ldots, n_k$ uniquely characterize the equivalence class.

Example 9.2.1. Consider the following:

1. triv_4 for S_4, i.e. (x, x, x, x). Here $(w_1, w_2, w_3, w_4) = (0, 1, 2, 3)$, so $k = 1$, $n_1 = 4$.
2. refl_4 with path $P = \mathrm{triv}_1 \to \cdots \to \mathrm{triv}_i \to \mathrm{refl}_{i+1} \to \cdots \to \mathrm{refl}_n$. In this case, we have seen that
$$(w_1, \ldots, w_n) = (0, 1, \ldots, i-1, i, \ldots, n-2).$$
For refl_4, we can get $(0, -1, 1, 2)$, $(0, 1, -1, 2)$, $(0, 1, 2, -1)$. The last one has $k = 2$, $n_1 = 3$, $n_2 = 1$.

Exercise 9.1. Compute the combinatorial weights for $\mathbb{C}^2$ (for S_4).

9.3 Standard Young Tableaux

Remark. Recall there is a one-to-one correspondence between partitions $(n_1, \ldots, n_k)$ of n (satisfying $n_1 \leq \cdots \leq n_k$ and $n_1 + \cdots + n_k$) and $\mathrm{Irr}(\mathbb{C}S_n)$. Also recall *Young tableaux* for partitions.

Definition 9.3. A *standard Young tableau* is a Young tableau filled with numbers $\{1, \ldots, n\}$ so that they strictly increase from bottom to top and from left to right. Denote by $\mathrm{STY}(n)$ the set of standard Young tableaux (corresponding to a partition of n).

Definition 9.4. To a Young tableau T, assign its *content* as follows. Let (x_i, y_i) be the coordinates of the box numbered i. Then the content of the box is $x_i - y_i$. The content of the tableau is
$$c(T) = (x_1 - y_1, x_2 - y_2, \ldots, x_n - y_n).$$

Exercise 9.2. Show that the map $T \mapsto cT$ is injective.