MATH 8803: Representation Theory

Frank Qiang
Instructor: Anton Zeitlin

Georgia Institute of Technology
Fall 2025

Contents

1 Aug. 18 — Historical Perspective **2**
- 1.1 Origin of Representation Theory . . . 2
- 1.2 Introduction to Lie Groups and Lie Algebras . . . 3
- 1.3 Algebras and Modules . . . 4

2 Aug. 20 — Algebras and Modules **5**
- 2.1 More on Algebras and Modules . . . 5
- 2.2 Module of Homomorphisms . . . 6
- 2.3 Tensor Product of Modules . . . 7
- 2.4 Tensor-Hom Adjunction . . . 8

3 Aug. 25 — Complete Reducibility **9**
- 3.1 Reducibility of Modules . . . 9
- 3.2 Schur's Lemma . . . 9
- 3.3 Completely Reducible Modules . . . 10
- 3.4 Simple Algebras . . . 12

4 Aug. 27 — Semisimple Algebras **13**
- 4.1 Semisimple Algebras . . . 13
- 4.2 Characterizations of Semisimple Algebras . . . 13
- 4.3 Double Centralizer Theorem . . . 14
- 4.4 Representations of Finite Groups . . . 15

Lecture 1

Aug. 18 — Historical Perspective

1.1 Origin of Representation Theory

One motivation for representation theory is symmetries in physics. From a mathematical perspective, we consider *groups* and *algebras* (a vector space with a bilinear operation). In this course, we will study two types of groups:

1. *finite groups*, e.g. the symmetric group;
2. *Lie groups*, e.g. the rotation group.

Definition 1.1. A *representation* of a group G is a homomorphism $G \to \mathrm{End}(V)$, where V is some finite-dimensional vector space.

The history of representation theory is as follows:

1. In the late 19th century, people were interested in *crystallography*, in particular crystallographic groups and their classification. There are related objects called *Bieberbach groups* (e.g. $O(n)$ with translations, i.e. $\mathbb{R}^n \rtimes O(n)$).

   Sophus Lie discovered *Lie groups* in his main manuscript "Transformation groups." From Lie groups, one then derives *Lie algebras*.

2. In the early 20th century (1905), *special relativity* was discovered, which involves the *Lorentz group* $SO(1,3)$ (the transformations preserving the form $-t^2 + x^2 + y^2 + z^2$). This is a Lie group.

   Around the same time, E. Cartan developed the modern theory of *semisimple Lie groups* and *Lie algebras*, and H. Weyl studied their representations.

3. In the period 1920–1930, quantum ("matrix") mechanics was discovered. Here one has a Hilbert space $\mathcal{H}$ and a self-adjoint Hamiltonian (energy) operator H on $\mathcal{H}$. The symmetry operator A satisfies the commutator relation $[H, A] = 0$, and if we set $U = e^{iA}$, we have $UHU^\dagger = H$.

4. After the discovery of *spin* by W. Pauli, E. Wigner realized that spin was directly related to the representation theory of the universal cover $\pi : SU(2) \to SO(3)$.

   In the 1960s, there was a "zoo" of elementary particles. M. Gell-Mann and Y. Neeman realized that all of these can be described by representations of $SU(3)$. The led to the discovery of *quarks* and the later notion of grand unified theories and string theory in the 1970s.

   There are also connections to condensed matter theory and quantum information.

This course will cover the following topics:

1. basics about associative algebras and their representations, finite groups and their representations in general, the symmetric group and its representations, Young tableaux;
2. Lie groups and Lie algebras;
3. the structure of semisimple Lie algebras;
4. representations of $SL(n)$.

1.2 Introduction to Lie Groups and Lie Algebras

In general, groups are complicated, whereas algebras are less complicated. We begin with finite groups.

Definition 1.2. Let G be a finite group and $\mathbb{F}$ a field. The *group algebra* $\mathbb{F}G$ is

$$\mathbb{F}G = \left\{ \sum_g a_g g : a_g \in \mathbb{F} \right\}.$$

This forms an algebra over $\mathbb{F}$ with the obvious multiplication operation.

Example 1.2.1. Consider the rotation group, generated by the matrices

$$R_z(\theta) = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad R_x(\phi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{pmatrix}, \quad R_y(\psi) = \begin{pmatrix} \cos\psi & 0 & -\sin\psi \\ 0 & 1 & 0 \\ \sin\psi & 0 & \cos\psi \end{pmatrix}.$$

Letting δ be an infinitesimal value and using a Taylor expansion, we can write

$$R_z(\delta\theta) = 1 + \delta\theta \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 1 + \delta\theta M_z,$$

$$R_x(\delta\phi) = 1 + \delta\phi \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} = 1 + \delta\phi M_x,$$

$$R_y(\delta\psi) = 1 + \delta\psi \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} = 1 + \delta\psi M_y.$$

We can measure the commutativity of these matrices via

$$\begin{aligned} R_x(\delta\phi)R_y(\delta\psi)R_x^{-1}(\delta\phi)R_y^{-1}(\delta\psi) &= (1 + M_x\delta\phi)(1 + M_y\delta\psi)(1 - M_x\delta\phi)(1 - M_y\delta\psi) \\ &= 1 + \delta\phi\delta\psi(M_xM_y - M_yM_x). \end{aligned}$$

Exercise 1.1. Show that $[M_x, M_y] = -M_z$.

Remark. Thus we have a vector space spanned by M_x, M_y, M_z with an operation $[\cdot,\cdot]$ satisfying the identity $[M_x, M_y] = -M_z$. Note that this property is satisfied by the cross product on $\mathbb{R}^3$. The cross product also satisfies the following *Jacobi identity*:

$$[A, [B, C]] = [[A, B], C] + [B, [A, C]].$$

The above properties define a *Lie algebra*.

Definition 1.3. Let $\{e_k\}$ be a basis of a Lie algebra and $[e_i, e_j] = \sum_k c_{ij}^k e_k$. The *universal enveloping algebra* of the Lie algebra is the free associative algebra on $\{e_k\}$, modulo the relations $[e_i, e_j] = \sum_k c_{ij}^k e_k$.

Remark. One way to return to the Lie group from the Lie algebra is exponentiation, e.g. $R_z(\theta) = e^{\theta M_z}$.

1.3 Algebras and Modules

Let k be a commutative ring (most of the time $k = \mathbb{C}$). All rings will be associative and unital.

Definition 1.4. A *(associative and unital) k-algebra* is a unital ring A with a homomorphism $i : k \to A$ such that $i(r) \cdot a = a \cdot i(r)$, i.e. the image of i commutes with A.

Example 1.4.1. Any ring is a $\mathbb{Z}$-algebra.

Definition 1.5. A *homomorphism* of k-algebras is a k-linear homomorphism of unital rings.

Definition 1.6. Let A, B be unital rings, and M an abelian group. Then

1. a *left A-module structure* on M is a $\mathbb{Z}$-bilinear map $A \times M \to M$, associative in the sense that
$$a_1(a_2m) = (a_1a_2)m, \quad \text{for all } a_1, a_2 \in A,\ m \in M,$$
and such that $1_Am = m$ for all $m \in M$;
2. a *right A-module structure* on M is a $\mathbb{Z}$-bilinear map $M \times B \to M$, associative in the sense that
$$(mb_1)b_2 = m(b_1b_2), \quad \text{for all } b_1, b_2 \in B,\ m \in M,$$
and such that $m1_B = m$ for all $m \in M$;
3. an *A-B-bimodule structure* on M is a left A-module and right B-module structure on M, along with the condition that $(am)b = a(mb)$ for all $a \in A$, $b \in B$, and $m \in M$.

Remark. In general, an A-module will mean a left A-module by default.

Definition 1.7. Let M, N be left A-modules. An *A-module homomorphism* is a map $\varphi : M \to N$ such that $\varphi(am) = a\varphi(m)$ for all $a \in A$ and $m \in M$.

Example 1.7.1. A ring A is both a left/right A-module and an A-A-bimodule (the *regular bimodule*).

Definition 1.8. The *direct sum* $\bigoplus_{i\in I} M_i$ of left A-modules M_i is the collection of $(m_i)_{i\in I}$ with finitely many nonzero entries, with component-wise addition and scalar multiplication.

Example 1.8.1. Let I be an index set. Then $A^{\oplus I}$ is the *coordinate A-module.*

Definition 1.9. A *submodule* of M is a nontrivial subgroup closed under addition and invariant under the action of A.

Example 1.9.1. Submodules of the regular left/right A-module are the left/right ideals of A.

Definition 1.10. Let M be a left A-module and M_0 a submodule of M. The *quotient module* M/M_0 is the set of equivalence classes $m + M_0$, where the action of A is given by $a(m + M_0) = am + M_0$.

Lemma 1.1. *Let M, N be A-modules and $M_0 \subseteq M$ a submodule. Let $\varphi : M \to N$ be A-linear such that $\varphi(M_0) = \{0\}$. Then there exists a unique A-linear map $\underline{\varphi} : M/M_0 \to N$ such that $\varphi = \underline{\varphi} \circ \pi$, where $\pi : M \to M/M_0$ is the canonical projection.*

Lecture 2

Aug. 20 — Algebras and Modules

2.1 More on Algebras and Modules

Definition 2.1. A *free* module is a module which has a basis.

Example 2.1.1. Consider the coordinate module $A^{\oplus I}$. Then a basis is given by $e_i = \{\delta_{ij}\}_{j \in I}$ for $i \in I$.

Proposition 2.1. *Let M be a left A-module. Let I be an index set and let $m_i \in M$ for $i \in I$. Then*

1. *There exists a unique A-linear map $A^{\oplus I} \to M$ which sends $e_i \mapsto m_i$.*
2. *This map is surjective if and only if the elements m_i span M. In particular, every M is isomorphic to a quotient of a free module.*
3. *This map is an isomorphism if and only if $\{m_i\}$ form a basis of M. In particular, every coordinate module is a free module.*

Proof. Left as an exercise. □

Example 2.1.2. Suppose M is spanned by a single element m. Then $M \cong A/I$, where I is the left ideal

$$I = \{a \in A : am = 0\}.$$

Example 2.1.3. We can now construct the following examples of algebras:

1. Let $\mathrm{Mat}_n(A)$ be the set of $n \times n$ matrices with entries in A. If A is a k-algebra, then $\mathrm{Mat}_n(A)$ is also a k-algebra.
2. If G is a group, then the group algebra kG (for a ring k) given by

$$kG = \left\{ \sum_{g \in G} a_g g : a_g \in k \right\}$$

is a free module with basis identified with the elements of G.

The importance of of this object is as follows: Let G be a group and B an algebra. Consider the set of maps satisfying $1_G \mapsto 1_B$ and respecting the group multiplication. This set is in bijection with maps $kG \to B$ (they extend by linearity). If V is a vector space and $B = \mathrm{End}(V)$, then this statement says that there is a bijection between the representations of the group G and the representations of the group algebra kG.

3. If I is a two-sided ideal, then A/I has a natural algebra structure.
4. If A_1, A_2 are k-algebras, then the direct sum $A_1 \oplus A_2$ is again a k-algebra (with component-wise multiplication). One can extend this by induction to a finite direct sum, but note that we lose the multiplicative identity in an infinite direct sum (so we do not get an algebra in the infinite case).

2.2 Module of Homomorphisms

Definition 2.2. Let k be a commutative ring and A a k-algebra. Let M, N be left A-modules. Denote by $\mathrm{Hom}_A(M, N)$ the set of all A-module homomorphisms $M \to N$. Give $\mathrm{Hom}_A(M, N)$ a k-module structure via

$$[\varphi_1 + \varphi_2](m) = \varphi_1(m) + \varphi_2(m), \quad [r\varphi](m) = r\varphi(m)$$

for $\varphi_1, \varphi_2 \in \mathrm{Hom}_A(M, N)$, $r \in k$, and $m \in M$.

Remark. Let L, M, N be left A-modules. Then we can define a k-bilinear map

$$\begin{aligned} \mathrm{Hom}_A(M, N) \times \mathrm{Hom}_A(L, M) &\longrightarrow \mathrm{Hom}_A(L, N) \\ (\varphi, \psi) &\longmapsto \varphi \circ \psi. \end{aligned}$$

Exercise 2.1. Let N_2 be an A-module, $N_1 \subseteq N_2$ an A-submodule, and $N_3 = N_2/N_1$. Let $i : N_1 \hookrightarrow N_2$ be the inclusion and $\pi : N_2 \to N_3$ the projection. Define the maps

$$\begin{aligned} \widetilde{\imath} : \mathrm{Hom}(M, N_1) &\to \mathrm{Hom}(M, N_2) \\ \varphi_1 &\longmapsto i \circ \varphi_1 \\ \widetilde{\pi} : \mathrm{Hom}(M, N_2) &\to \mathrm{Hom}(M, N_3) \\ \varphi_2 &\longmapsto \pi \circ \varphi_2. \end{aligned}$$

Then show that $\widetilde{\imath}$ is injective and $\mathrm{Im}\, \widetilde{\imath} = \ker \widetilde{\pi}$.

Remark. Let B be a k-algebra and M and A-B-bimodule. Then for all A-modules N, we have that $\mathrm{Hom}_A(M, N)$ is a left B-module via

$$[b\varphi](m) = \varphi(mb).$$

Similarly, if N is an A-C-bimodule, then $\mathrm{Hom}_A(M, N)$ is a right C-module via

$$[\varphi c](m) = \varphi(m)c.$$

So if M is an A-B-bimodule and N an A-C-bimodule, then $\mathrm{Hom}_A(M, N)$ is a B-C-bimodule.

Remark. Let M be a left A-module. We write $\mathrm{End}_A(M)$ in place of $\mathrm{Hom}_A(M, M)$, and composition gives $\mathrm{End}_A(M)$ the structure of a k-algebra. If $M = A^{\oplus n}$, then we can identify

$$\mathrm{End}_A(M) = \mathrm{Mat}_n(A^{\mathrm{opp}}),$$

where the opposite algebra exchanges the order of multiplication in the original algebra (this is because $\mathrm{End}_A(M)$ must respect the action by A). Then M becomes an A-$(\mathrm{Mat}_n(A))^{\mathrm{opp}}$-bimodule.

Remark. If M, N are two left A-modules, then $\mathrm{Hom}_A(M, N)$ is an $\mathrm{End}_A(N)$-$\mathrm{End}_A(M)$-bimodule (by taking into account compositions).

2.3 Tensor Product of Modules

Remark. Let A be a k-algebra, M a right A-module, and N a left A-module. We want to produce a k-module $M \otimes_A N$, which will be the *tensor product* of M and N over A.

Definition 2.3. Let L be a k-module. We say that a map $\varphi : M \times N \to L$ is A-*bilinear* if it is k-linear in both arguments and satisfies

$$\varphi(ma, n) = \varphi(m, an)$$

for any $a \in A$, $m \in M$, and $n \in N$.

Definition 2.4 (Universal property of the tensor product)**.** There is an A-bilinear map

$$\begin{aligned} M \times N &\longrightarrow M \otimes_A N \\ (m, n) &\longmapsto m \otimes n \end{aligned}$$

such that for any A-bilinear map $\varphi : M \times N \to L$, there exists a unique k-linear map $\psi : M \otimes_A N \to L$ such that $\varphi(m, n) = \psi(m \otimes n)$. As a diagram, this says that

$$\begin{tikzcd} M \times N \arrow[rr, "(m,n)\mapsto m\otimes n"] \arrow[dr, "\varphi"'] & & M \otimes_A N \arrow[dl, "\psi"] \\ & L & \end{tikzcd}$$

Exercise 2.2. If we choose $M \otimes'_A N$ with bilinear map $(m, n) \mapsto m \otimes' n$, then there exists a unique isomorphism $i : M \otimes_A N \to M \otimes'_A N$ given by $i(m \otimes n) = m \otimes' n$.

Corollary 2.0.1. *Assume $M \otimes_A N$ satisfies the universal property. Then $\{m \otimes n\}$ span $M \otimes_A N$.*

Theorem 2.1. *The tensor product $M \otimes_A N$ exists for all right A-modules M and left A-modules N.*

Proof. We sketch the proof. First take M to be free. Then we can define $M \otimes_A N$ as $N^{\oplus I}$, where we have $(e_i a_i) \otimes n = (a_i n)_{i \in I}$. The universal property is easy to check for this case, and the general case can be done by writing M as a quotient of a free module. □

Example 2.4.1. If M, N are both free and $\{e_i\}_{i \in I}$, $\{f_j\}_{j \in J}$ are bases of M, N, respectively, then $M \otimes_A N$ is a free k-module with basis vectors $\{e_i \otimes f_j\}_{i \in I, j \in J}$.

Exercise 2.3. Let $M = A/I$, where I is a right ideal. Show that $M \otimes_A N = N/IN$. Find out what happens when $N = A/J$, where J is a left ideal, what can you say about $M \otimes_A N$ in terms of A, I, J?

Proposition 2.2. *Assume B is a k-algebra and M a B-A-module. Then $M \otimes_A N$ is a left B-module.*

Proof. Define $\varphi_b : M \times N \to M \otimes_A N$ by $(m, n) \mapsto bm \otimes n$. This is bilinear, so by the universal property, there exists $\psi_b : M \otimes_A N \to M \otimes_A N$ such that $\psi_b(m \otimes n) = bm \otimes n$, which gives the B-action. □

Definition 2.5. Let L be a B-module. A map $\varphi : M \times N \to L$ is called B-A-*linear* if it is k-linear in both arguments and

$$\varphi(ma, n) = \varphi(m, an), \quad \varphi(bm, n) = b\varphi(m, n)$$

for all $m \in M$, $n \in N$, $b \in B$, and $a \in A$.

Proposition 2.3. *The left B-module $M \otimes_A N$ has the following universal property:*

Let L be any left B-module and $\varphi : M \times N \to L$ a B-A-linear map. Then there exists a unique B-linear map $\psi : M \otimes_A N \to L$ such that $\psi(m \otimes n) = \varphi(m, n)$.

Example 2.5.1. Let A_1, A_2 be k-algebras. Then

1. $A_1 \otimes_k A_2$ has the structure of a k-algebra via

$$(a_1 \otimes a_2)(b_1 \otimes b_2) = (a_1 b_1) \otimes (a_2 b_2),$$

where $1 \otimes 1$ is a unit element.

2. Let M_i be a left A_i-module for $i = 1, 2$. Then $M_1 \otimes_k M_2$ is a module for $A_1 \otimes_k A_2$.

2.4 Tensor-Hom Adjunction

Proposition 2.4 (Tensor-Hom adjunction)**.** *Let A, B be associative algebras, N a B-module, M an A-module, and L an A-B-bimodule. Then*

1. *$L \otimes_B N$ is an A-module;*
2. *$\mathrm{Hom}_A(L, M)$ is a B-module.*

Moreover, there is a natural k-linear isomorphism

$$\mathrm{Hom}_A(L \otimes_B N, M) \xrightarrow{\cong} \mathrm{Hom}_B(N, \mathrm{Hom}_A(L, M)).$$

Proof. By the universal property, there is a natural map

$$\mathrm{Hom}_A(L \otimes_B N, M) \xrightarrow{\cong} \mathrm{Bilin}_{A,B}(L \times N, M).$$

So it suffices to find

$$\begin{aligned} \mathrm{Hom}_B(N, \mathrm{Hom}_A(L, M)) &\xrightarrow{\cong} \mathrm{Bilin}_{A,B}(L \times N, M) \\ f &\longmapsto \varphi_f. \end{aligned}$$

Construct this map by $\psi_f(e, n) = [f(n)](e)$, with inverse $h \mapsto \psi(\cdot, h)$ for $\psi \in \mathrm{Bilin}_{A,B}(L \times N, M)$. □

Example 2.5.2. If we have an algebra homomorphism $B \to A$, where A is a an A-B-bimodule. One can show as an exercise that $\mathrm{Hom}_A(A, M)$ is naturally identified with M as an A-module and B-module. Thus by the Tensor-Hom adjunction, we have a natural isomorphism

$$\mathrm{Hom}_A(A \otimes_B N, M) \xrightarrow{\cong} \mathrm{Hom}_B(N, M).$$

Definition 2.6. The A-module $A \otimes_B N$ is said to be *induced* from N.

Remark. Assume there is Hom from $A \to B$. Then B is an A-B-bimodule. Take it as L in the Tensor-Hom adjunction. Note that $B \otimes_B N \cong N$ as A-modules, and we have a natural isomorphism

$$\mathrm{Hom}_A(N, M) \xrightarrow{\cong} \mathrm{Hom}_B(N, \mathrm{Hom}_A(B, M)).$$

Definition 2.7. The B-module $\mathrm{Hom}_A(B, M)$ is said to be *coinduced* from M.

Lecture 3

Aug. 25 — Complete Reducibility

3.1 Reducibility of Modules

Remark. Consider an associative algebra A over a field $\mathbb{F}$. We proceed to study completely reducible representations of A. Let U be an A-module.

Definition 3.1. An A-module U is *irreducible* if it only has two distinct submodules ($\{0\}$ and U).

Remark. With this definition, $\{0\}$ is not irreducible.

Definition 3.2. An A-module U is *completely reducible* if for any submodule $U' \subseteq U$, there exists an A-submodule U'' such that $U = U' \oplus U''$.

Exercise 3.1. Show that any submodule and any quotient module of a completely reducible A-module is also completely reducible.

Example 3.2.1. Consider $A = \mathrm{End}_{\mathbb{F}}(U)$. Then U is an A-module and is irreducible (there is a linear operator $\alpha : U \to U$ taking $u \mapsto v$ for any $u, v \in U$, so there are no nontrivial invariant subspaces).

Proposition 3.1. *Let U_1, U_2 be completely reducible A-modules. Then $U_1 \oplus U_2$ is completely reducible.*

Proof. Left as an exercise. □

Corollary 3.0.1. *Let U be a finite-dimensional A-module. Then the following are equivalent:*

1. *U is completely reducible;*
2. *U is isomorphic to a direct sum of irreducible submodules.*

Exercise 3.2. Show that every irreducible A-module is isomorphic to a quotient module for a regular module (i.e. one isomorphic to A). In particular, every irreducible module over a finite-dimensional associative $\mathbb{F}$-algebra is finite-dimensional.

3.2 Schur's Lemma

Theorem 3.1 (Schur's lemma)**.** *Let A be an associative $\mathbb{F}$-algebra and U, V irreducible A-modules. Then*

1. *if U, V are not isomorphic, then $\mathrm{Hom}_A(U, V) = 0$;*
2. *$\mathrm{End}_A(U)$ is a skew field (i.e. a division ring). Furthermore, if U is finite-dimensional and $\mathbb{F}$ is algebraically closed, then $\dim \mathrm{End}_A(U) = 1$.*

Proof. (1) Assume we have a nonzero homomorphism $\varphi : U \to V$. Then $\ker\varphi \subsetneq U$, and $\operatorname{Im}\varphi \subseteq V$ is nontrivial, so by irreducibility φ must be an isomorphism.

(2) Let $\varphi \in \operatorname{End}_A(U)$. From (1), we know that φ is an isomorphism, so φ has an inverse, i.e. $\operatorname{End}_A(U)$ is a skew field. For the second part, since $\mathbb{F}$ is algebraically closed, we can find an eigenvalue z for φ. Then $\varphi - z\,\mathrm{Id}_U$ is not invertible, so we have $\varphi - z\,\mathrm{Id}_U = 0$ by (1). $\square$

Exercise 3.3. Consider $1, i, j, k$, where $i^2 = j^2 = k^2 = -1$ and $ij = -ji = k$. The *quaternion algebra* over $\mathbb{R}$ is given by

$$\mathbb{H}_{\mathbb{R}} = \{q = w + xi + yj + zk : w, x, y, z \in \mathbb{R}\}$$

Note that $\overline{q} = w - xi - yj - zk$ satisfies $q\overline{q} = w^2 + x^2 + y^2 + z^2$, so $q^{-1} = \overline{q}/(w^2 + x^2 + y^2 + z^2)$, i.e. $\mathbb{H}_{\mathbb{R}}$ is a skew field. Show that $\operatorname{End}_{\mathbb{H}_{\mathbb{R}}}(\mathbb{H}_{\mathbb{R}}) \cong \mathbb{H}_{\mathbb{R}}^{\mathrm{opp}}$.

Remark. We have an embedding $\mathbb{H}_{\mathbb{R}} \hookrightarrow \mathrm{Mat}_2(\mathbb{C})$ given by

$$q \longmapsto \begin{pmatrix} w + xi & y + zi \\ -y + zi & w - xi \end{pmatrix}.$$

If we replace $\mathbb{R}$ with $\mathbb{C}$, then $\mathbb{H}_{\mathbb{C}} \cong \mathrm{Mat}_2(\mathbb{C})$, which is reducible (consider the sum of column spaces).

Definition 3.3. Let U be an A-module. We say that U is *endotrivial* if $\operatorname{End}_A(U)$ consists only of scalar maps, i.e. maps of the form $z\,\mathrm{Id}$.

Remark. Suppose $\mathbb{F}$ is algebraically closed and uncountable (e.g. $\mathbb{C}$), A has countable dimension over $\mathbb{F}$, and U an irreducible A-module. Then U is endotrivial.

Definition 3.4. Define the *center* of A to be

$$\mathcal{Z}(A) = \{z \in A : za = az \text{ for all } a \in A\}.$$

Note that this is a commutative algebra.

Exercise 3.4. Schur's lemma gives a description of the center of A. Let U be an endotrivial A-module (e.g. a finite-dimensional module over if $\mathbb{F}$ is algebraically closed). Show that $z \in \mathcal{Z}(A)$ acts as a scalar on U. We call the algebra homomorphism $\mathcal{Z}(A) \to \mathbb{F}$ the *central character* of U.

3.3 Completely Reducible Modules

Remark. Consider finite direct sums of endotrivial irreducible modules:

$$\bigoplus_{i=1}^{k} U_i \otimes M_i,$$

where the U_i are endotrivial modules and the M_i are vector spaces known as *multiplicity spaces*. Note that $U_1^{\oplus i} = U_1 \otimes \mathbb{F}^i$. The A-action on the direct sum for $a \in A$ is given by

$$a(u_1 \otimes m_1, \ldots, u_k \otimes m_k) = (au_1 \otimes m_1, \ldots, au_k \otimes m_k).$$

We will use Schur's lemma to understand homomorphisms between such modules.

Write $U^j = \bigoplus_{i=1}^k U_i \otimes M_i^j$ for $j = 1, 2$. We can produce a linear map

$$\bigoplus_{i=1}^k \mathrm{Hom}_{\mathbb{F}}(M_i^1, M_i^2) \longrightarrow \mathrm{Hom}_A(U^1, U^2)$$

in the following manner: For $\underline{\varphi} = (\varphi_1, \ldots, \varphi_k) \in \bigoplus_{i=1}^k \mathrm{Hom}_{\mathbb{F}}(M_i^1, M_i^2)$, we can define

$$\psi_{\underline{\varphi}}\left(\sum_{i=1}^k u_i \otimes m_i^1\right) = \sum_{i=1}^k u_i \otimes \varphi_i(m_i^1).$$

Theorem 3.2. *We have the following:*

1. *The map $\underline{\varphi} \mapsto \psi_{\underline{\varphi}}$ defines a vector space isomorphism*

$$\bigoplus_{i=1}^k \mathrm{Hom}_{\mathbb{F}}(M_i^1, M_i^2) \xrightarrow{\cong} \mathrm{Hom}_A(U^1, U^2).$$

2. *Every A-module homomorphism $U_1 \to U_2$ sends $U_i \otimes M_i^1$ to $U_i \otimes M_i^2$ for any i.*

Proof. Left as an exercise (use Schur's lemma). □

Corollary 3.2.1. *We have the following:*

1. *there is an isomorphism $\mathrm{Hom}_A(U_i, U) \xrightarrow{\cong} M_i$;*
2. *there is an isomorphism $\bigoplus_{i=1}^k U_i \otimes \mathrm{Hom}_A(U_i, U) \cong U$ given by*

$$\sum_{i=1}^k u_i \otimes \varphi_i \longmapsto \sum_{i=1}^k \varphi_i(u_i).$$

Proposition 3.2. *For any A-submodule $U' \subseteq U$, there exists a unique collection of determined subspaces $M_i' \subseteq M_i$ such that $U' = \bigoplus_{i=1}^k U_i \otimes M_i'$ as submodules of U.*

Proof. Note that $\mathrm{Hom}_A(U_i, U') \subseteq \mathrm{Hom}_A(U_i, U)$, set $M_i' = \mathrm{Hom}_A(U_i, U')$, and use Corollary 3.2.1. □

Theorem 3.3. *Let U_i be irreducible modules for A and consider maps $\beta_i : A \to \mathrm{End}_{\mathbb{F}}(U_i)$. Set*

$$\beta = \beta_1 \oplus \cdots \oplus \beta_k : A \longrightarrow \bigoplus_{i=1}^k \mathrm{End}_{\mathbb{F}}(U_i),$$

where the U_i are pairwise non-isomorphic. Then the homomorphism β is surjective.

Proof. Replace A by $A/\ker\beta$, so that β is injective. Then β equips $\bigoplus_{i=1}^k \mathrm{End}(U_i)$ with an A-bimodule structure, and there is a natural isomorphism $\mathrm{End}_{\mathbb{F}}(U_i) \cong U_i \otimes U_i^*$. View U_i as the multiplicity space for the right A-module and U_i^* as the multiplicity space for the left A-module. By Proposition 3.2,

$$A = \bigoplus_{i=1}^k U_i \otimes V_i$$

as a left A-module for some $V_i \subseteq U_i^*$. Similarly for the right A-module, we have

$$A = \bigoplus_{i=1}^{k} W_i \otimes U_i^*$$

for some $W_i \subseteq U_i$. Then we must have $U_i \oplus V_i = W_i \oplus U_i^*$, so $U_i \cong W_i$ and $V_i \cong U_i^*$ (the identity $1 \in A$ guarantees that no component is zero). Thus β is surjective. $\square$

Corollary 3.3.1. *Let $\mathbb{F}$ be algebraically closed and A a finite-dimensional $\mathbb{F}$-algebra. Then the set of isomorphism classes of irreducible A-modules is finite and non-empty.*

Proof. First this set is nonempty since A is nonzero, so it has an irreducible subrepresentation. To see that it is finite, note that for all collections $U_1, \dots, U_k$, the map $A \to \bigoplus_{i=1}^k \mathrm{End}_{\mathbb{F}}(U_i)$ is surjective, so

$$\dim A \geq \sum_{i=1}^{k} (\dim U_i)^2.$$

This proves the desired result, since A is finite-dimensional. $\square$

3.4 Simple Algebras

Definition 3.5. An algebra A is *simple* if the only two-sided ideals are $\{0\}$ and A (i.e. A is irreducible as a bimodule over itself).

Theorem 3.4. *Let $\mathbb{F}$ be an algebraically closed field and A a finite-dimensional $\mathbb{F}$-algebra. Then the following are equivalent:*

1. *A is simple;*
2. *$A \cong \mathrm{End}_{\mathbb{F}}(U)$ for some finite-dimensional vector space U.*

Proof. ($1 \Rightarrow 2$): The algebra A has an irreducible representation U, i.e. we have a map $A \to \mathrm{End}_{\mathbb{F}}(U)$. Since A is simple, this map must have trivial kernel, i.e. it is injective. We also already know that it is surjective, so this map is an isomorphism.

($2 \Rightarrow 1$): Assume I is a two-sided ideal in $\mathrm{End}_{\mathbb{F}}(U) \cong U \otimes U^*$ and view $I \subseteq U \otimes U^*$. Show as an exercise that we must have $I = \{0\}$. $\square$

Theorem 3.5. *Every finite-dimensional module V for $A = \mathrm{End}_{\mathbb{F}}(U)$ is isomorphic to a direct sum of several copies of U.*

Proof. Recall that every finitely generated module V is a quotient of $A^{\oplus \ell}$ for some $\ell \in \mathbb{N}$. We can write $A = U \otimes U^*$. Let $A^{\oplus \ell} = U \otimes M$ and consider the quotient map $\pi : U \otimes M \to V$. Then $\ker \pi \subseteq U \otimes M$ must be of the form $U \oplus M_0$, so we have $V \cong (U \otimes M)/(U \otimes M_0) = U \otimes (M/M_0)$. $\square$

Lecture 4

Aug. 27 — Semisimple Algebras

4.1 Semisimple Algebras

Definition 4.1. A finite-dimensional $\mathbb{F}$-algebra A is called *semisimple* if it is isomorphic to a direct sum of simple algebras.

Remark. If $\mathbb{F}$ is algebraically closed, then A is a direct sum of matrix algebras, i.e. $\bigoplus_{i=1}^k \operatorname{End}_{\mathbb{F}}(U_i)$.

Theorem 4.1. *Let $U_1, \dots, U_k$ be finite-dimensional vector spaces over $\mathbb{F}$. Let $A = \bigoplus_{i=1}^k \operatorname{End}_{\mathbb{F}}(U_i)$, so that U_i is an irreducible A-module. Then every finite-dimensional A-module V is isomorphic to a direct sum of several copies of $U_1, \dots, U_k$.*

Proof. Left as an exercise. □

Corollary 4.1.1. *Let $\mathbb{F}$ be algebraically closed, and A be semisimple and finite-dimensional. Then*

1. *The number of isomorphism classes of irreducible A-modules is equal to* $\dim \mathcal{Z}(A)$.
2. *Different irreducible modules have different central characters.*

Proof. (1) Let $A = \bigoplus_{i=1}^k \operatorname{End}_{\mathbb{F}}(U_i)$. By Theorem 4.1, the number of irreducible representations is k. We can also write

$$\mathcal{Z}\left(\bigoplus_{i=1}^k A_k\right) = \bigoplus_{i=1}^k \mathcal{Z}(A_i),$$

where $A_i = \operatorname{End}_{\mathbb{F}}(U_i)$. Since $\dim \mathcal{Z}(A_i) = 1$, we have $\dim \mathcal{Z}(\bigoplus_{i=1}^k A_k) = k$ as well.

(2) Use the projections $\mathcal{Z} \to \mathcal{Z}(A_i) \to \mathbb{F}$, which correspond to the central characteres. □

4.2 Characterizations of Semisimple Algebras

Definition 4.2. Let A be a finite-dimensional algebra. We say that a two-sided ideal $I \subseteq A$ is *nilpotent* if $I^n = \{0\}$ for some n.

Exercise 4.1. If I, J are nilpotent, then show that $I + J$ is also nilpotent.

Definition 4.3. The maximal nilpotent ideal of A, denoted $\operatorname{rad}(A)$, is called the *radical* of A.

Theorem 4.2. *Let $\mathbb{F}$ be algebraically closed and A a finite-dimensional algebra. Then the following are equivalent:*

1. *A is semisimple;*
2. *all finite-dimensional representations of A are completely reducible;*
3. $\mathrm{rad}(A) = \{0\}$.

Proof. ($1 \Rightarrow 2$) We have already shown this.

($2 \Rightarrow 3$) Let $I = \mathrm{rad}(A)$, so $I^n = \{0\}$ for some $n \in \mathbb{N}$. Let N be a finite-dimensional A-module. Then $I^\ell N$ is an A-submodule for $\ell = 0, \dots, n$. Since N is completely reducible and $I^{\ell+1}N \subseteq I^\ell N$, we have

$$I^\ell N = N_\ell \oplus I^{\ell+1}.$$

Acting on both sides by I, we get $IN_\ell \subseteq I^{\ell+1}$, so $IN_\ell = \{0\}$. Continuing, we get $IN = 0$, so $A = N$.

($3 \Rightarrow 1$) Take $N_1, \dots, N_k$ to be pairwise non-isomorphic irreducible A-modules. We have an epimorphism $A \to \bigoplus_{i=1}^k \mathrm{End}_\mathbb{F}(N_i)$. Let I be the kernel, so I acts trivially on every irreducible A-module. We claim that I is nilpotent. Take A to be the regular module. Take a filtration

$$A = A_0 \supseteq A_1 \supseteq \cdots \supseteq A_n = \{0\},$$

where A_i/A_{i+1} is irreducible. Now I acts trivially on A_i/A_{i+1}, so $IA_i \subseteq A_{i+1}$ for all i, thus $I^n = \{0\}$. □

Remark. Assume $\mathrm{char}(\mathbb{F}) = 0$. Consider the following bilinear form on A:

$$(a, b)_U = \mathrm{tr}_U(ab),$$

where U is any A-module. Note that U could be A.

Theorem 4.3. *Let $\mathrm{char}(\mathbb{F}) = 0$, and let A be a finite-dimensional $\mathbb{F}$-algebra. Then A is semisimple if and only if $(a, b)_A$ is nondegenerate.*

Proof. ($\Rightarrow$) Assume A is semisimple, so $A = \bigoplus_{i=1}^k \mathrm{End}(U_i)$. Note that the restriction of $(\cdot,\cdot)_A$ to the direct summand $\mathrm{End}_\mathbb{F}(U_i)$ coincides with $(\cdot,\cdot)_{\mathrm{End}_\mathbb{F}(U_i)}$. Let $E_{j\ell}$ denote the matrix with all 0s except a single 1 in the (j, ℓ) entry. Then we can compute that

$$(E_{j\ell}, E_{j'\ell'})_{\mathrm{End}_\mathbb{F}(U_i)} = \delta_{ej'} \,\mathrm{tr}_{\mathrm{End}_\mathbb{F}(U_i)}(E_{j\ell'}) = \delta_{e\ell}\delta_{j\ell'} \dim U_i.$$

So if $\{E_{j\ell}\}$ is a basis, then $\{(\dim U_i)^{-1}E_{j\ell}\}$ is the dual basis. This is nondenegerate if $\mathrm{char}(\mathbb{F}) = 0$.

($\Leftarrow$) Suppose $(\cdot,\cdot)_A$ is nondegenerate. If I is a nilpotent ideal, then for any $a \in I$ such that $a^n = 0$. Then $\mathrm{tr}_A(a) = 0$ for any $a \in I$, so $I \in \ker(\cdot,\cdot) = 0$. Since $(\cdot,\cdot)$ is nondegenerate, we have $I = \{0\}$. □

4.3 Double Centralizer Theorem

Theorem 4.4 (Double centralizer theorem)**.** *Let V be a finite-dimensional vector space over $\mathbb{F}$. Let $A \subseteq \mathrm{End}_\mathbb{F}(V)$ be a semisimple algebra, and set $B = \mathrm{End}_A(V)$. Then $A = \mathrm{End}_B(V)$.*

Proof. Let $A = \bigoplus_{i=1}^k \mathrm{End}(U_i)$ and V be a faithful representation of A, so V is completely reducible:

$$V \cong \bigoplus_{i=1}^k U_i \oplus M_i,$$

where the M_i are multiplicity spaces. Let $a = (\varphi_1, \dots, \varphi_k) \in A$ (for $\varphi_i \in \mathrm{End}(U_i)$) act on $\mathrm{End}_{\mathbb{F}}(V)$ by

$$(\varphi_1, \dots, \varphi_k) \longmapsto \sum_{i=1}^{k} \varphi_i \otimes \mathrm{Id}_{M_i}.$$

Note that the M_i are nonzero since V is faithful. Then $B = \bigoplus_{i=1}^{n} \mathrm{End}(M_i)$ embeds into $\mathrm{End}_{\mathbb{F}}(V)$ via

$$(\psi_1, \dots, \psi_k) \longmapsto \sum_{i=1}^{k} \mathrm{Id}_{U_i} \otimes \psi_i,$$

which completes the proof. □

4.4 Representations of Finite Groups

Remark. Recall that to any group G we can associate the group algebra $\mathbb{F}G$. For any representation of G, there is a representation of $\mathbb{F}G$ and vice versa.

Remark. Consider the following operations with representations. Let U, V be representations of G.

1. the *tensor product* $U \otimes_{\mathbb{F}} V$, where $g(U \otimes V) = (gU \otimes gV)$;
2. the *dual* U^* defined by $\langle g\alpha, u\rangle = \langle \alpha, g^{-1}u\rangle$ for $u \in U$, $\alpha \in U^*$, $g \in G$;
3. $\mathrm{Hom}_{\mathbb{F}}(U, V)$, with action given by $[g\varphi](h) = g[\varphi(g^{-1}u)]$ for $\varphi \in \mathrm{Hom}_{\mathbb{F}}(U, V)$.

Exercise 4.2. Show the following:

1. The tensor product of representations satisfies associativity, distributivity, and commutativity.
2. There is an isomorphism of representations $U^* \otimes V \to \mathrm{Hom}(U, V)$.
3. $\mathrm{Hom}_G(U, V) \subseteq \mathrm{Hom}(U, V)$ coincides with the space of G-invariant elements.

Remark. For the rest of this section, assume $\mathbb{F}$ is algebraically closed and $\mathrm{char}\,\mathbb{F} = 0$.

Theorem 4.5. *The group algebra $\mathbb{F}G$ is semisimple.*

Proof. It suffices to show that $(\cdot, \cdot)_{\mathbb{F}G}$ is nondegenerate. Take $g, g' \in G$, and note that $gg' : h \mapsto gg'h$, so

$$(g, g')_{\mathbb{F}G} = \mathrm{tr}_{\mathbb{F}G}(gg') = \delta_{1,gg'}|G|,$$

which is nondegenerate. Moreover, the basis $\{g\}$ in $\mathbb{F}G$ corresponds to the dual basis $\{|G|^{-1}g^{-1}\}$. □

Corollary 4.5.1. *(Let $\mathbb{F}$ be algebraically closed and* $\mathrm{char}\,\mathbb{F} = 0$.*)*

1. *Every finite-dimensional representation of G is completely reducible.*
2. *The number of isomorphism classes of irreducible representations is equal to the number of conjugacy classes of G.*
3. *If $U_1, \dots, U_k$ are all of the pairwise non-isomorphic irreducible representations of G, then*

$$|G| = \sum_{i=1}^{k} (\dim U_i)^2.$$

Proof. (1) This follows from the semisimplicity of $\mathbb{F}G$.

(2) It suffices to show that $\dim \mathcal{Z}(\mathbb{F}G)$ equals the number of conjugacy classes of G. We have

$$\mathcal{Z}(\mathbb{F}G) = \left\{ \sum_{g \in G} a_g g : a_g \text{ is constant on conjugacy classes} \right\},$$

i.e. we must have $a_{hgh^{-1}} = a_g$ for any $h \in G$. So the dimension is the number of conjugacy classes.

(3) This automatically follows from looking at the dimension of $\mathbb{F}G$. $\square$