# MATH 8803: Representation Theory

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# Lecture 1

# Aug. 18 — Historical Perspective

## 1.1 Origin of Representation Theory

One motivation for representation theory is symmetries in physics. From a mathematical perspective, we consider *groups* and *algebras* (a vector space with a bilinear operation). In this course, we will study two types of groups:

1. *finite groups*, e.g. the symmetric group;
2. *Lie groups*, e.g. the rotation group.

**Definition 1.1.** A *representation* of a group $G$ is a homomorphism $G \to \mathrm{End}(V)$, where $V$ is some finite-dimensional vector space.

The history of representation theory is as follows:

1. In the late 19th century, people were interested in *crystallography*, in particular crystallographic groups and their classification. There are related objects called *Bieberbach groups* (e.g. $O(n)$ with translations, i.e. $\mathbb{R}^n \rtimes O(n)$).

   Sophus Lie discovered *Lie groups* in his main manuscript "Transformation groups." From Lie groups, one then derives *Lie algebras*.

2. In the early 20th century (1905), *special relativity* was discovered, which involves the *Lorentz group* $SO(1,3)$ (the transformations preserving the form $-t^2 + x^2 + y^2 + z^2$). This is a Lie group.

   Around the same time, E. Cartan developed the modern theory of *semisimple Lie groups* and *Lie algebras*, and H. Weyl studied their representations.

3. In the period 1920–1930, quantum ("matrix") mechanics was discovered. Here one has a Hilbert space $\mathcal{H}$ and a self-adjoint Hamiltonian (energy) operator $H$ on $\mathcal{H}$. The symmetry operator $A$ satisfies the commutator relation $[H, A] = 0$, and if we set $U = e^{iA}$, we have $UHU^\dagger = H$.

4. After the discovery of *spin* by W. Pauli, E. Wigner realized that spin was directly related to the representation theory of the universal cover $\pi : SU(2) \to SO(3)$.

   In the 1960s, there was a "zoo" of elementary particles. M. Gell-Mann and Y. Neeman realized that all of these can be described by representations of $SU(3)$. The led to the discovery of *quarks* and the later notion of grand unified theories and string theory in the 1970s.

   There are also connections to condensed matter theory and quantum information.

This course will cover the following topics:

1. basics about associative algebras and their representations, finite groups and their representations in general, the symmetric group and its representations, Young tableaux;
2. Lie groups and Lie algebras;
3. the structure of semisimple Lie algebras;
4. representations of $SL(n)$.

## 1.2 Introduction to Lie Groups and Lie Algebras

In general, groups are complicated, whereas algebras are less complicated. We begin with finite groups.

**Definition 1.2.** Let $G$ be a finite group and $\mathbb{F}$ a field. The *group algebra* $\mathbb{F}G$ is

$$\mathbb{F}G = \left\{ \sum_g a_g g : a_g \in \mathbb{F} \right\}.$$

This forms an algebra over $\mathbb{F}$ with the obvious multiplication operation.

**Example 1.2.1.** Consider the rotation group, generated by the matrices

$$R_z(\theta) = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad R_x(\phi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{pmatrix}, \quad R_y(\psi) = \begin{pmatrix} \cos\psi & 0 & -\sin\psi \\ 0 & 1 & 0 \\ \sin\psi & 0 & \cos\psi \end{pmatrix}.$$

Letting $\delta$ be an infinitesimal value and using a Taylor expansion, we can write

$$R_z(\delta\theta) = 1 + \delta\theta \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 1 + \delta\theta M_z,$$

$$R_x(\delta\phi) = 1 + \delta\phi \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} = 1 + \delta\phi M_x,$$

$$R_y(\delta\psi) = 1 + \delta\psi \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} = 1 + \delta\psi M_y.$$

We can measure the commutativity of these matrices via

$$\begin{aligned} R_x(\delta\phi)R_y(\delta\psi)R_x^{-1}(\delta\phi)R_y^{-1}(\delta\psi) &= (1 + M_x\delta\phi)(1 + M_y\delta\psi)(1 - M_x\delta\phi)(1 - M_y\delta\psi) \\ &= 1 + \delta\phi\delta\psi(M_xM_y - M_yM_x). \end{aligned}$$

**Exercise 1.1.** Show that $[M_x, M_y] = -M_z$.

**Remark.** Thus we have a vector space spanned by $M_x, M_y, M_z$ with an operation $[\cdot, \cdot]$ satisfying the identity $[M_x, M_y] = -M_z$. Note that this property is satisfied by the cross product on $\mathbb{R}^3$. The cross product also satisfies the following *Jacobi identity*:

$$[A, [B, C]] = [[A, B], C] + [B, [A, C]].$$

The above properties define a *Lie algebra*.

**Definition 1.3.** Let $\{e_k\}$ be a basis of a Lie algebra and $[e_i, e_j] = \sum_k c_{ij}^k e_k$. The *universal enveloping algebra* of the Lie algebra is the free associative algebra on $\{e_k\}$, modulo the relations $[e_i, e_j] = \sum_k c_{ij}^k e_k$.

**Remark.** One way to return to the Lie group from the Lie algebra is exponentiation, e.g. $R_z(\theta) = e^{\theta M_z}$.

## 1.3 Algebras and Modules

Let $k$ be a commutative ring (most of the time $k = \mathbb{C}$). All rings will be associative and unital.

**Definition 1.4.** A *(associative and unital) $k$-algebra* is a unital ring $A$ with a homomorphism $i : k \to A$ such that $i(r) \cdot a = a \cdot i(r)$, i.e. the image of $i$ commutes with $A$.

**Example 1.4.1.** Any ring is a $\mathbb{Z}$-algebra.

**Definition 1.5.** A *homomorphism* of $k$-algebras is a $k$-linear homomorphism of unital rings.

**Definition 1.6.** Let $A, B$ be unital rings, and $M$ an abelian group. Then

1. a *left $A$-module structure* on $M$ is a $\mathbb{Z}$-bilinear map $A \times M \to M$, associative in the sense that
$$a_1(a_2 m) = (a_1 a_2)m, \quad \text{for all } a_1, a_2 \in A,\ m \in M,$$
and such that $1_A m = m$ for all $m \in M$;
2. a *right $A$-module structure* on $M$ is a $\mathbb{Z}$-bilinear map $M \times B \to M$, associative in the sense that
$$(mb_1)b_2 = m(b_1 b_2), \quad \text{for all } b_1, b_2 \in B,\ m \in M,$$
and such that $m1_B = m$ for all $m \in M$;
3. an *$A$-$B$-bimodule structure* on $M$ is a left $A$-module and right $B$-module structure on $M$, along with the condition that $(am)b = a(mb)$ for all $a \in A$, $b \in B$, and $m \in M$.

**Remark.** In general, an $A$-module will mean a left $A$-module by default.

**Definition 1.7.** Let $M, N$ be left $A$-modules. An *$A$-module homomorphism* is a map $\varphi : M \to N$ such that $\varphi(am) = a\varphi(m)$ for all $a \in A$ and $m \in M$.

**Example 1.7.1.** A ring $A$ is both a left/right $A$-module and an $A$-$A$-bimodule (the *regular bimodule*).

**Definition 1.8.** The *direct sum* $\bigoplus_{i \in I} M_i$ of left $A$-modules $M_i$ is the collection of $(m_i)_{i \in I}$ with finitely many nonzero entries, with component-wise addition and scalar multiplication.

**Example 1.8.1.** Let $I$ be an index set. Then $A^{\oplus I}$ is the *coordinate $A$-module.*

**Definition 1.9.** A *submodule* of $M$ is a nontrivial subgroup closed under addition and invariant under the action of $A$.

**Example 1.9.1.** Submodules of the regular left/right $A$-module are the left/right ideals of $A$.

**Definition 1.10.** Let $M$ be a left $A$-module and $M_0$ a submodule of $M$. The *quotient module* $M/M_0$ is the set of equivalence classes $m + M_0$, where the action of $A$ is given by $a(m + M_0) = am + M_0$.

**Lemma 1.1.** *Let $M, N$ be $A$-modules and $M_0 \subseteq M$ a submodule. Let $\varphi : M \to N$ be $A$-linear such that $\varphi(M_0) = \{0\}$. Then there exists a unique $A$-linear map $\underline{\varphi} : M/M_0 \to N$ such that $\varphi = \underline{\varphi} \circ \pi$, where $\pi : M \to M/M_0$ is the canonical projection.*