

MATH 8803: Representation Theory

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Contents

1	Aug. 18 — Historical Perspective	2
1.1	Origin of Representation Theory	2
1.2	Introduction to Lie Groups and Lie Algebras	3
1.3	Algebras and Modules	4
2	Aug. 20 — Algebras and Modules	5
2.1	More on Algebras and Modules	5
2.2	Module of Homomorphisms	6
2.3	Tensor Product of Modules	7
2.4	Tensor-Hom Adjunction	8

Lecture 1

Aug. 18 — Historical Perspective

1.1 Origin of Representation Theory

One motivation for representation theory is symmetries in physics. From a mathematical perspective, we consider *groups* and *algebras* (a vector space with a bilinear operation). In this course, we will study two types of groups:

1. *finite groups*, e.g. the symmetric group;
2. *Lie groups*, e.g. the rotation group.

Definition 1.1. A *representation* of a group G is a homomorphism $G \rightarrow \text{End}(V)$, where V is some finite-dimensional vector space.

The history of representation theory is as follows:

1. In the late 19th century, people were interested in *crystallography*, in particular crystallographic groups and their classification. There are related objects called *Bieberbach groups* (e.g. $O(n)$ with translations, i.e. $\mathbb{R}^n \rtimes O(n)$).

Sophus Lie discovered *Lie groups* in his main manuscript “Transformation groups.” From Lie groups, one then derives *Lie algebras*.

2. In the early 20th century (1905), *special relativity* was discovered, which involves the *Lorentz group* $SO(1, 3)$ (the transformations preserving the form $-t^2 + x^2 + y^2 + z^2$). This is a Lie group.

Around the same time, E. Cartan developed the modern theory of *semisimple Lie groups* and *Lie algebras*, and H. Weyl studied their representations.

3. In the period 1920–1930, quantum (“matrix”) mechanics was discovered. Here one has a Hilbert space \mathcal{H} and a self-adjoint Hamiltonian (energy) operator H on \mathcal{H} . The symmetry operator A satisfies the commutator relation $[H, A] = 0$, and if we set $U = e^{iA}$, we have $UHU^\dagger = H$.
4. After the discovery of *spin* by W. Pauli, E. Wigner realized that spin was directly related to the representation theory of the universal cover $\pi : SU(2) \rightarrow SO(3)$.

In the 1960s, there was a “zoo” of elementary particles. M. Gell-Mann and Y. Neeman realized that all of these can be described by representations of $SU(3)$. This led to the discovery of *quarks* and the later notion of grand unified theories and string theory in the 1970s.

There are also connections to condensed matter theory and quantum information.

This course will cover the following topics:

1. basics about associative algebras and their representations, finite groups and their representations in general, the symmetric group and its representations, Young tableaux;
2. Lie groups and Lie algebras;
3. the structure of semisimple Lie algebras;
4. representations of $SL(n)$.

1.2 Introduction to Lie Groups and Lie Algebras

In general, groups are complicated, whereas algebras are less complicated. We begin with finite groups.

Definition 1.2. Let G be a finite group and \mathbb{F} a field. The *group algebra* $\mathbb{F}G$ is

$$\mathbb{F}G = \left\{ \sum_g a_g g : a_g \in \mathbb{F} \right\}.$$

This forms an algebra over \mathbb{F} with the obvious multiplication operation.

Example 1.2.1. Consider the rotation group, generated by the matrices

$$R_z(\theta) = \begin{pmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad R_x(\phi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \phi & -\sin \phi \\ 0 & \sin \phi & \cos \phi \end{pmatrix}, \quad R_y(\psi) = \begin{pmatrix} \cos \psi & 0 & -\sin \psi \\ 0 & 1 & 0 \\ \sin \psi & 0 & \cos \psi \end{pmatrix}.$$

Letting δ be an infinitesimal value and using a Taylor expansion, we can write

$$\begin{aligned} R_z(\delta\theta) &= 1 + \delta\theta \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 1 + \delta\theta M_z, \\ R_x(\delta\phi) &= 1 + \delta\phi \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} = 1 + \delta\phi M_x, \\ R_y(\delta\psi) &= 1 + \delta\psi \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} = 1 + \delta\psi M_y. \end{aligned}$$

We can measure the commutativity of these matrices via

$$\begin{aligned} R_x(\delta\phi)R_y(\delta\psi)R_x^{-1}(\delta\phi)R_y^{-1}(\delta\psi) &= (1 + M_x\delta\phi)(1 + M_y\delta\psi)(1 - M_x\delta\phi)(1 - M_y\delta\psi) \\ &= 1 + \delta\phi\delta\psi(M_xM_y - M_yM_x). \end{aligned}$$

Exercise 1.1. Show that $[M_x, M_y] = -M_z$.

Remark. Thus we have a vector space spanned by M_x, M_y, M_z with an operation $[\cdot, \cdot]$ satisfying the identity $[M_x, M_y] = -M_z$. Note that this property is satisfied by the cross product on \mathbb{R}^3 . The cross product also satisfies the following *Jacobi identity*:

$$[A, [B, C]] = [[A, B], C] + [B, [A, C]].$$

The above properties define a *Lie algebra*.

Definition 1.3. Let $\{e_k\}$ be a basis of a Lie algebra and $[e_i, e_j] = \sum_k c_{ij}^k e_k$. The *universal enveloping algebra* of the Lie algebra is the free associative algebra on $\{e_k\}$, modulo the relations $[e_i, e_j] = \sum_k c_{ij}^k e_k$.

Remark. One way to return to the Lie group from the Lie algebra is exponentiation, e.g. $R_z(\theta) = e^{\theta M_z}$.

1.3 Algebras and Modules

Let k be a commutative ring (most of the time $k = \mathbb{C}$). All rings will be associative and unital.

Definition 1.4. A (*associative and unital*) k -algebra is a unital ring A with a homomorphism $i : k \rightarrow A$ such that $i(r) \cdot a = a \cdot i(r)$, i.e. the image of i commutes with A .

Example 1.4.1. Any ring is a \mathbb{Z} -algebra.

Definition 1.5. A *homomorphism* of k -algebras is a k -linear homomorphism of unital rings.

Definition 1.6. Let A, B be unital rings, and M an abelian group. Then

1. a *left A -module structure* on M is a \mathbb{Z} -bilinear map $A \times M \rightarrow M$, associative in the sense that

$$a_1(a_2 m) = (a_1 a_2) m, \quad \text{for all } a_1, a_2 \in A, m \in M,$$

and such that $1_A m = m$ for all $m \in M$;

2. a *right A -module structure* on M is a \mathbb{Z} -bilinear map $M \times A \rightarrow M$, associative in the sense that

$$(m b_1) b_2 = m (b_1 b_2), \quad \text{for all } b_1, b_2 \in A, m \in M,$$

and such that $m 1_A = m$ for all $m \in M$;

3. an *A - B -bimodule structure* on M is a left A -module and right B -module structure on M , along with the condition that $(am)b = a(mb)$ for all $a \in A, b \in B$, and $m \in M$.

Remark. In general, an A -module will mean a left A -module by default.

Definition 1.7. Let M, N be left A -modules. An *A -module homomorphism* is a map $\varphi : M \rightarrow N$ such that $\varphi(am) = a\varphi(m)$ for all $a \in A$ and $m \in M$.

Example 1.7.1. A ring A is both a left/right A -module and an A - A -bimodule (the *regular bimodule*).

Definition 1.8. The *direct sum* $\bigoplus_{i \in I} M_i$ of left A -modules M_i is the collection of $(m_i)_{i \in I}$ with finitely many nonzero entries, with component-wise addition and scalar multiplication.

Example 1.8.1. Let I be an index set. Then $A^{\oplus I}$ is the *coordinate A -module*.

Definition 1.9. A *submodule* of M is a nontrivial subgroup closed under addition and invariant under the action of A .

Example 1.9.1. Submodules of the regular left/right A -module are the left/right ideals of A .

Definition 1.10. Let M be a left A -module and M_0 a submodule of M . The *quotient module* M/M_0 is the set of equivalence classes $m + M_0$, where the action of A is given by $a(m + M_0) = am + M_0$.

Lemma 1.1. Let M, N be A -modules and $M_0 \subseteq M$ a submodule. Let $\varphi : M \rightarrow N$ be A -linear such that $\varphi(M_0) = \{0\}$. Then there exists a unique A -linear map $\underline{\varphi} : M/M_0 \rightarrow N$ such that $\varphi = \underline{\varphi} \circ \pi$, where $\pi : M \rightarrow M/M_0$ is the canonical projection.

Lecture 2

Aug. 20 — Algebras and Modules

2.1 More on Algebras and Modules

Definition 2.1. A *free* module is a module which has a basis.

Example 2.1.1. Consider the coordinate module $A^{\oplus I}$. Then a basis is given by $e_i = \{\delta_{ij}\}_{j \in I}$ for $i \in I$.

Proposition 2.1. Let M be a left A -module. Let I be an index set and let $m_i \in M$ for $i \in I$. Then

1. There exists a unique A -linear map $A^{\oplus I} \rightarrow M$ which sends $e_i \mapsto m_i$.
2. This map is surjective if and only if the elements m_i span M . In particular, every M is isomorphic to a quotient of a free module.
3. This map is an isomorphism if and only if $\{m_i\}$ form a basis of M . In particular, every coordinate module is a free module.

Proof. Left as an exercise. □

Example 2.1.2. Suppose M is spanned by a single element m . Then $M \cong A/I$, where I is the left ideal

$$I = \{a \in A : am = 0\}.$$

Example 2.1.3. We can now construct the following examples of algebras:

1. Let $\text{Mat}_n(A)$ be the set of $n \times n$ matrices with entries in A . If A is a k -algebra, then $\text{Mat}_n(A)$ is also a k -algebra.
2. If G is a group, then the group algebra kG (for a ring k) given by

$$kG = \left\{ \sum_{g \in G} a_g g : a_g \in k \right\}$$

is a free module with basis identified with the elements of G .

The importance of this object is as follows: Let G be a group and B an algebra. Consider the set of maps satisfying $1_G \mapsto 1_B$ and respecting the group multiplication. This set is in bijection with maps $kG \rightarrow B$ (they extend by linearity). If V is a vector space and $B = \text{End}(V)$, then this statement says that there is a bijection between the representations of the group G and the representations of the group algebra kG .

3. If I is a two-sided ideal, then A/I has a natural algebra structure.
4. If A_1, A_2 are k -algebras, then the direct sum $A_1 \oplus A_2$ is again a k -algebra (with component-wise multiplication). One can extend this by induction to a finite direct sum, but note that we lose the multiplicative identity in an infinite direct sum (so we do not get an algebra in the infinite case).

2.2 Module of Homomorphisms

Definition 2.2. Let k be a commutative ring and A a k -algebra. Let M, N be left A -modules. Denote by $\text{Hom}_A(M, N)$ the set of all A -module homomorphisms $M \rightarrow N$. Give $\text{Hom}_A(M, N)$ a k -module structure via

$$[\varphi_1 + \varphi_2](m) = \varphi_1(m) + \varphi_2(m), \quad [r\varphi](m) = r\varphi(m)$$

for $\varphi_1, \varphi_2 \in \text{Hom}_A(M, N)$, $r \in k$, and $m \in M$.

Remark. Let L, M, N be left A -modules. Then we can define a k -bilinear map

$$\begin{aligned} \text{Hom}_A(M, N) \times \text{Hom}_A(L, M) &\longrightarrow \text{Hom}_A(L, N) \\ (\varphi, \psi) &\longmapsto \varphi \circ \psi. \end{aligned}$$

Exercise 2.1. Let N_2 be an A -module, $N_1 \subseteq N_2$ an A -submodule, and $N_3 = N_2/N_1$. Let $i : N_1 \hookrightarrow N_2$ be the inclusion and $\pi : N_2 \rightarrow N_3$ the projection. Define the maps

$$\begin{aligned} \tilde{\iota} : \text{Hom}(M, N_1) &\rightarrow \text{Hom}(M, N_2) \\ \varphi_1 &\longmapsto i \circ \varphi_1 \\ \tilde{\pi} : \text{Hom}(M, N_2) &\rightarrow \text{Hom}(M, N_3) \\ \varphi_2 &\longmapsto \pi \circ \varphi_2. \end{aligned}$$

Then show that $\tilde{\iota}$ is injective and $\text{Im } \tilde{\iota} = \ker \tilde{\pi}$.

Remark. Let B be a k -algebra and M and A - B -bimodule. Then for all A -modules N , we have that $\text{Hom}_A(M, N)$ is a left B -module via

$$[b\varphi](m) = \varphi(mb).$$

Similarly, if N is an A - C -bimodule, then $\text{Hom}_A(M, N)$ is a right C -module via

$$[\varphi c](m) = \varphi(m)c.$$

So if M is an A - B -bimodule and N an A - C -bimodule, then $\text{Hom}_A(M, N)$ is a B - C -bimodule.

Remark. Let M be a left A -module. We write $\text{End}_A(M)$ in place of $\text{Hom}_A(M, M)$, and composition gives $\text{End}_A(M)$ the structure of a k -algebra. If $M = A^{\oplus n}$, then we can identify

$$\text{End}_A(M) = \text{Mat}_n(A^{\text{opp}}),$$

where the opposite algebra exchanges the order of multiplication in the original algebra (this is because $\text{End}_A(M)$ must respect the action by A). Then M becomes an A -($\text{Mat}_n(A))^{\text{opp}}$ -bimodule.

Remark. If M, N are two left A -modules, then $\text{Hom}_A(M, N)$ is an $\text{End}_A(N)$ - $\text{End}_A(M)$ -bimodule (by taking into account compositions).

2.3 Tensor Product of Modules

Remark. Let A be a k -algebra, M a right A -module, and N a left A -module. We want to produce a k -module $M \otimes_A N$, which will be the *tensor product* of M and N over A .

Definition 2.3. Let L be a k -module. We say that a map $\varphi : M \times N \rightarrow L$ is *A -bilinear* if it is k -linear in both arguments and satisfies

$$\varphi(ma, n) = \varphi(m, an)$$

for any $a \in A$, $m \in M$, and $n \in N$.

Definition 2.4 (Universal property of the tensor product). There is an A -bilinear map

$$\begin{aligned} M \times N &\longrightarrow M \otimes_A N \\ (m, n) &\longmapsto m \otimes n \end{aligned}$$

such that for any A -bilinear map $\varphi : M \times N \rightarrow L$, there exists a unique k -linear map $\psi : M \otimes_A N \rightarrow L$ such that $\varphi(m, n) = \psi(m \otimes n)$. As a diagram, this says that

$$\begin{array}{ccc} M \times N & \xrightarrow{(m,n) \mapsto m \otimes n} & M \otimes_A N \\ & \searrow \varphi & \swarrow \psi \\ & L & \end{array}$$

Exercise 2.2. If we choose $M \otimes'_A N$ with bilinear map $(m, n) \mapsto m \otimes' n$, then there exists a unique isomorphism $i : M \otimes_A N \rightarrow M \otimes'_A N$ given by $i(m \otimes n) = m \otimes' n$.

Corollary 2.0.1. Assume $M \otimes_A N$ satisfies the universal property. Then $\{m \otimes n\}$ span $M \otimes_A N$.

Theorem 2.1. The tensor product $M \otimes_A N$ exists for all right A -modules M and left A -modules N .

Proof. We sketch the proof. First take M to be free. Then we can define $M \otimes_A N$ as $N^{\oplus I}$, where we have $(e_i a_i) \otimes n = (a_i n)_{i \in I}$. The universal property is easy to check for this case, and the general case can be done by writing M as a quotient of a free module. \square

Example 2.4.1. If M, N are both free and $\{e_i\}_{i \in I}, \{f_j\}_{j \in J}$ are bases of M, N , respectively, then $M \otimes_A N$ is a free k -module with basis vectors $\{e_i \otimes f_j\}_{i \in I, j \in J}$.

Exercise 2.3. Let $M = A/I$, where I is a right ideal. Show that $M \otimes_A N = N/IN$. Find out what happens when $N = A/J$, where J is a left ideal, what can you say about $M \otimes_A N$ in terms of A, I, J ?

Proposition 2.2. Assume B is a k -algebra and M a B - A -module. Then $M \otimes_A N$ is a left B -module.

Proof. Define $\varphi_b : M \times N \rightarrow M \otimes_A N$ by $(m, n) \mapsto bm \otimes n$. This is bilinear, so by the universal property, there exists $\psi_b : M \otimes_A N \rightarrow M \otimes_A N$ such that $\psi_b(m \otimes n) = bm \otimes n$, which gives the B -action. \square

Definition 2.5. Let L be a B -module. A map $\varphi : M \times N \rightarrow L$ is called *B - A -linear* if it is k -linear in both arguments and

$$\varphi(ma, n) = \varphi(m, an), \quad \varphi(bm, n) = b\varphi(m, n)$$

for all $m \in M$, $n \in N$, $b \in B$, and $a \in A$.

Proposition 2.3. The left B -module $M \otimes_A N$ has the following universal property:

Let L be any left B -module and $\varphi : M \times N \rightarrow L$ a B - A -linear map. Then there exists a unique B -linear map $\psi : M \otimes_A N \rightarrow L$ such that $\psi(m \otimes n) = \varphi(m, n)$.

Example 2.5.1. Let A_1, A_2 be k -algebras. Then

1. $A_1 \otimes_k A_2$ has the structure of a k -algebra via

$$(a_1 \otimes a_2)(b_1 \otimes b_2) = (a_1 b_1) \otimes (a_2 b_2),$$

where $1 \otimes 1$ is a unit element.

2. Let M_i be a left A_i -module for $i = 1, 2$. Then $M_1 \otimes_k M_2$ is a module for $A_1 \otimes_k A_2$.

2.4 Tensor-Hom Adjunction

Proposition 2.4 (Tensor-Hom adjunction). *Let A, B be associative algebras, N a B -module, M an A -module, and L an A - B -bimodule. Then*

1. $L \otimes_B N$ is an A -module;
2. $\text{Hom}_A(L, M)$ is a B -module.

Moreover, there is a natural k -linear isomorphism

$$\text{Hom}_A(L \otimes_B N, M) \xrightarrow{\cong} \text{Hom}_B(N, \text{Hom}_A(L, M)).$$

Proof. By the universal property, there is a natural map

$$\text{Hom}_A(L \otimes_B N, M) \xrightarrow{\cong} \text{Bilin}_{A,B}(L \times N, M).$$

So it suffices to find

$$\begin{aligned} \text{Hom}_B(N, \text{Hom}_A(L, M)) &\xrightarrow{\cong} \text{Bilin}_{A,B}(L \times N, M) \\ f &\longmapsto \varphi_f. \end{aligned}$$

Construct this map by $\psi_f(e, n) = [f(n)](e)$, with inverse $h \mapsto \psi(\cdot, h)$ for $\psi \in \text{Bilin}_{A,B}(L \times N, M)$. \square

Example 2.5.2. If we have an algebra homomorphism $B \rightarrow A$, where A is an A - B -bimodule. One can show as an exercise that $\text{Hom}_A(A, M)$ is naturally identified with M as an A -module and B -module. Thus by the Tensor-Hom adjunction, we have a natural isomorphism

$$\text{Hom}_A(A \otimes_B N, M) \xrightarrow{\cong} \text{Hom}_B(N, M).$$

Definition 2.6. The A -module $A \otimes_B N$ is said to be *induced* from N .

Remark. Assume there is Hom from $A \rightarrow B$. Then B is an A - B -bimodule. Take it as L in the Tensor-Hom adjunction. Note that $B \otimes_B N \cong N$ as A -modules, and we have a natural isomorphism

$$\text{Hom}_A(N, M) \xrightarrow{\cong} \text{Hom}_B(N, \text{Hom}_A(B, M)).$$

Definition 2.7. The B -module $\text{Hom}_A(B, M)$ is said to be *coinduced* from M .