Math 503: Complex Analysis

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[1] I did not attend this class in person, nor do I go to this university.

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Lecture 1

What is "complex analysis"? The complex numbers are:

$$\mathbb{C} = \{x + iy \mid x, y \in \mathbb{R}\}.$$

"Analysis" is a fancy way of saying "calculus".

So why do we want to study calculus over $\mathbb{C}$? Why bother with $\mathbb{C}$ at all?

To solve quadratics? For example, the equation

$$x^2 + 1 = 0$$

yields the solutions $x = \pm i = \pm\sqrt{-1}$. No! This is the same problem as asking where does the function

$$y = x^2 + 1$$

cross the x-axis. But, of course, the graph of this function doesn't intersect the x-axis at all! So there is no reason to expect a solution for x here.

Historically, mathematicians needed $i = \sqrt{-1}$ to solve cubic equations. For example, consider the equation

$$y = x^3 + 12x - 15.$$

$y \to -\infty$ as $x \to -\infty$ and $y \to \infty$ as $x \to \infty$, so there must be a root somewhere by the intermediate value theorem!

1.1.1 Cardano's Story

c. 1495, Paciolli in Italy writes a textbook about all known mathematics at the time. Quadratics had already been solved everywhere, but cubics were still a mystery (Paciolli says "cubics are as unsolvable as squaring the circle").

c. 1510, del Ferro figures out how to solve the depressed cubic (no quadratic term):

$$ax^3 + cx + d = 0$$

At the time, mathematicians were employed by the rich, and to obtain such a position, one must win a *duel* against the current person holding the position. Each contestant would give the other a set of problems, and whoever solves more would win.

Thus, del Ferro doesn't tell anyone about his solution, so he can use it as a secret weapon to win duels, if necessary. Eventually, on his deathbed in the 1520s, he ends up telling his student Fior the secret. Fior then uses this to attack other mathematicians and win duels all over the place. That is, until he attacks Tartaglia, a renowned mathematician at the time. [1]

Tartaglia sends Fior a set of regular problems, whereas Fior sends him 20 depressed cubics. Tartaglia sees this and reasons that there must now exist a solution to the depressed cubic, contrary to common belief at the time (thus explaining Fior's choice of problems). Knowing this, he rediscovers the solution to the depressed cubic and proceeds to win the duel. The public, seeing this, makes the same conclusion that the depressed cubic has likely been solved.

c. 1530, Cardano visits Tartaglia and asks him for the solution, in the name of adding it to his textbook (an update to Paciolli's) and promising to credit Tartaglia. Tartaglia refuses, wanting to write his own book.

However, after inviting Tartaglia to dinner and lots of drinks, Cardano eventually convinces Tartaglia. Tartaglia makes Cardano solemnly swear to not reveal the solution to the public, and he does so.

Later on, Ferrari becomes a student of Cardano and eventually his collaborator. Ferrari eventually learns the secrets, and together Ferrari and

[1] Tartaglia gave one of the first Latin translations of Euclid's *Elements*.

Cardano solves all cubics (and all quartics too)! But they are unable to publish their findings due to the oath.

They later go on a trip to Bologna, where they are shown del Ferro's notes. There they find his original solution to the depressed cubic, sitting in plain sight for the past 30 years (predating Tartaglia)!

Cardano proceeds to publish his book containing the solution, the *Ars Magna*, in 1545. Tartaglia is not happy, and challenges Cardano and Ferrari to a duel. Of course Tartaglia gets decimated as Ferrari already knows how to solve even quartics. They almost get into an actual duel, but Tartaglia manages to escape before that can happen.

Note: Negative numbers

Interestingly, mathematicians cared about i even before they cared about negative numbers! How do we know this? On top of considering cubics geometrically (with actual cubes), Cardano considered the following cases:

$$\begin{aligned} x^3 + c &= dx^2 \\ x^3 &= c + dx^2 \\ x^3 + dx^2 &= c. \end{aligned}$$

It's evident that these 3 cases are all the same if we take into account negative numbers, but he didn't see this!

1.1.2 Solving the Cubic

So, why were they forced to acknowledge i, if they didn't even use negative numbers? Consider the general cubic equation

$$ax^3 + bx^2 + cx + d = 0, \quad a \neq 0.$$

We can divide by a (or equivalently let $a = 1$) to make the equation *monic*. Then we *depress* the cubic by making the change of variables

$$x = y - \frac{b}{3}.$$

So we have

$$\left(y-\frac{b}{3}\right)^3+b\left(y-\frac{b}{3}\right)^2+c\left(y-\frac{b}{3}\right)+d=0.$$

Notice that the y^2 term in this expansion is

$$\binom{3}{1}y^2\left(-\frac{b}{3}\right)+b\binom{2}{0}y^2=-by^2+by^2=0,$$

so we have eliminated the quadratic term.

Thus we need only consider cubics of the form

$$y^3+Ay+B=0.$$

Note: Quadratic equations

How to solve the quadratic $ax^2+bx+c=0$?

First make it monic:

$$x^2+\frac{b}{a}x+\frac{c}{a}=0.$$

Depress it by letting

$$x=y-\frac{b}{2a}.$$

Then we have

$$\begin{aligned}\left(y-\frac{b}{2a}\right)^2+\frac{b}{a}\left(y-\frac{b}{2a}\right)+\frac{c}{a}&=0,\\ y^2-\cancel{2\frac{b}{2a}y}+\frac{b^2}{4a^2}+\cancel{\frac{b}{a}y}-\frac{b^2}{2a^2}+\frac{c}{a}&=0,\\ y^2-\frac{1}{4}\frac{b^2}{a^2}+\frac{4ca}{4a^2}&=0,\\ y^2&=\frac{b^2-4ac}{4a^2}.\end{aligned}$$

From here, taking square roots and shifting by $\frac{b}{2a}$ again yields the usual quadratic formula.

Although depressing the quadratic makes it trivial, the same is not true for cubics! At first sight, the depressed cubic is not any easier than the general cubic. But the key insight is actually to perform some seemingly unnecessary auxiliary calculations.