Math 503: Complex Analysis

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Contents

Lecture 1

Introduction

1.1 Historical Motivation

What is “complex analysis”? The complex numbers are:

$$\mathbb{C} = \{x + iy \mid x, y \in \mathbb{R}\}.$$

“Analysis” is a fancy way of saying “calculus.”

So why do we want to study calculus over $\mathbb{C}$? Why bother with $\mathbb{C}$ at all?

To solve quadratics? For example, the equation

$$x^2 + 1 = 0$$

yields the solutions $x = \pm i = \pm\sqrt{-1}$. No! This is the same problem as asking where does the function

$$y = x^2 + 1$$

cross the x-axis. But, of course, the graph of this function doesn't intersect the x-axis at all! So there is no reason to expect a solution for x here.

Historically, mathematicians needed $i = \sqrt{-1}$ to solve cubic equations. For example, consider the equation

$$y = x^3 + 12x - 15.$$

$y \to -\infty$ as $x \to -\infty$ and $y \to \infty$ as $x \to \infty$, so there must be a root somewhere by the intermediate value theorem!

1.1.1 Cardano's Story

c. 1495, Paciolli in Italy writes a textbook about all known mathematics at the time. Quadratics had already been solved everywhere, but cubics were still a mystery (Paciolli says “cubics are as unsolvable as squaring the circle”).

c. 1510, del Ferro figures out how to solve the depressed cubic (no quadratic term):

$$ax^3 + cx + d = 0$$

At the time, mathematicians were employed by the rich, and to obtain such a position, one must win a *duel* against the current person holding the position. Each contestant would give the other a set of problems, and whoever solves more would win.

Thus, del Ferro doesn't tell anyone about his solution, so he can use it as a secret weapon to win duels, if necessary. Eventually, on his deathbed in the 1520s, he ends up telling his student Fior the secret. Fior then uses this to attack other mathematicians and win duels all over the place. That is, until he attacks Tartaglia, a renowned mathematician at the time.[1]

Tartaglia sends Fior a set of regular problems, whereas Fior sends him 20 depressed cubics. Tartaglia sees this and reasons that there must now exist a solution to the depressed cubic, contrary to common belief at the time (thus explaining Fior's choice of problems). Knowing this, he rediscovers the solution to the depressed cubic and proceeds to win the duel. The public, seeing this, makes the same conclusion that the depressed cubic has likely been solved.

c. 1530, Cardano visits Tartaglia and asks him for the solution, in the name of adding it to his textbook (an update to Paciolli's) and promising to credit Tartaglia. Tartaglia refuses, wanting to write his own book.

However, after inviting Tartaglia to dinner and lots of drinks, Cardano eventually convinces Tartaglia. Tartaglia makes Cardano solemnly swear to not reveal the solution to the public, and he does so.

Later on, Ferrari becomes a student of Cardano and eventually his collaborator. Ferrari eventually learns the secrets, and together Ferrari and Cardano solves all cubics (and all quartics too)! But they are unable to publish their findings due to the oath.

They later go on a trip to Bologna, where they are shown del Ferro's notes. There they find his original solution to the depressed cubic, sitting in plain sight for the past 30 years (predating Tartaglia)!

Cardano proceeds to publish his book containing the solution, the *Ars Magna*, in 1545. Tartaglia is not happy, and challenges Cardano and Ferrari to a duel. Of course Tartaglia gets decimated as Ferrari already knows how to solve even quartics. They almost get into an actual duel, but Tartaglia manages to escape before that can happen.

Aside: Negative numbers

Interestingly, mathematicians cared about i even before they cared about negative numbers! How do we know this? On top of considering cubics geometrically (with actual cubes), Cardano considered the following cases:

$$x^3 + c = dx^2$$
$$x^3 = c + dx^2$$
$$x^3 + dx^2 = c.$$

It's evident that these 3 cases are all the same if we take into account negative numbers, but he didn't see this!

1.1.2 Solving the Cubic

So, why were they forced to acknowledge i, if they didn't even use negative numbers? Consider the general cubic equation

$$ax^3 + bx^2 + cx + d = 0, \quad a \neq 0.$$

[1] Tartaglia gave one of the first Latin translations of Euclid's *Elements*.

We can divide by a (or equivalently let $a = 1$) to make the equation *monic*. Then we *depress* the cubic by making the change of variables

$$x = y - \frac{b}{3}.$$

So we have

$$\left(y - \frac{b}{3}\right)^3 + b\left(y - \frac{b}{3}\right)^2 + c\left(y - \frac{b}{3}\right) + d = 0.$$

Notice that the y^2 term in this expansion is

$$\binom{3}{1} y^2 \left(-\frac{b}{3}\right) + b\binom{2}{0} y^2 = -by^2 + by^2 = 0,$$

so we have eliminated the quadratic term.

Thus we need only consider cubics of the form

$$y^3 + Ay + B = 0.$$

Aside: Quadratic equations

How to solve the quadratic $ax^2 + bx + c = 0$?

First make it monic:

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0.$$

Depress it by letting

$$x = y - \frac{b}{2a}.$$

Then we have

$$\begin{aligned}
\left(y - \frac{b}{2a}\right)^2 + \frac{b}{a}\left(y - \frac{b}{2a}\right) + \frac{c}{a} &= 0, \\
y^2 - \cancel{2\frac{b}{2a}}y + \frac{b^2}{4a^2} + \cancel{\frac{b}{a}}y - \frac{b^2}{2a^2} + \frac{c}{a} &= 0, \\
y^2 - \frac{1}{4}\frac{b^2}{a^2} + \frac{4ac}{4a^2} &= 0, \\
y^2 &= \frac{b^2 - 4ac}{4a^2}.
\end{aligned}$$

From here, taking square roots and shifting by $\frac{b}{2a}$ again yields the usual quadratic formula.

Although depressing the quadratic makes it trivial, the same is not true for cubics! At first sight, the depressed cubic is not any easier than the general cubic. But the key insight is actually to perform some seemingly unnecessary auxiliary calculations.

Notice that

$$\begin{gathered}
(z + w)^3 = z^3 + 3z^2w + 3zw^2 + w^3 = 3wz(z + w) + z^3 + w^3, \\
(z + w)^3 - 3wz(z + w) - z^3 - w^3 = 0.
\end{gathered}$$

This looks awfully similar to the depressed cubic if we let $y = z + w$ and carefully choose z and w so that

$$\begin{cases} A = -3wz \\ B = -z^3 - w^3. \end{cases}$$

To solve this, notice that

$$w = -\frac{A}{3z}$$

by the first equation. Substituting this into the second equation yields

$$z^3 - \frac{A^3}{27z^3} = -B,$$
$$z^6 + Bz^3 - \frac{A^3}{27} = 0.$$

This equation is now quadratic in z^3! So by the quadratic formula,[2]

$$z^3 = \frac{-B + \sqrt{B^2 + \frac{4A^3}{27}}}{2}.$$

Using $z^3 + w^3 = -B$, we also have

$$w^3 = \frac{-B - \sqrt{B^2 + \frac{4A^3}{27}}}{2}.$$

This means that[3]

$$z = \sqrt[3]{-\frac{B}{2} + \sqrt{\frac{B^2}{4} + \frac{A^3}{27}}}, \quad w = \sqrt[3]{-\frac{B}{2} - \sqrt{\frac{B^2}{4} + \frac{A^3}{27}}},$$

so finally $y = z + w$ as defined above.

1.1.3 A Peculiar Example

Consider

$$x^3 - 15x - 4 = 0.$$

$A = -15$ and $B = -4$, so the discriminant is

$$\frac{B^2}{4} + \left(\frac{A}{3}\right)^3 = 4 - 125 = -121.$$

Now we need to take the square root of a negative number! For now, proceeding anyway gives

$$x = \sqrt[3]{2 + 11i} + \sqrt[3]{2 - 11i}.$$

But soon people realized:

$$(2+i)^3 = 8 + 3(2^2)i - 3(2) - i = 2 + 11i.$$

Then we end up with

$$x = (2+i) + (2-i) = 4,$$

a legitimate *real* solution. The $+i$ and $-i$ canceled!

So people were willing to deal with complex numbers here because they needed them to reach the real answer that they expected.

[2] This is a result of the symmetry group S_3 being *solvable*.

[3] The quantity $\frac{B^2}{4} + \frac{A^3}{27}$ is called the *discriminant* of the cubic.

Lecture 2

Holomorphic Functions

Given a function $f : \mathbb{C} \to \mathbb{C}$, when is f differentiable and what does that imply?

2.1 Revisiting Multivariable Calculus

When is a function $f : \mathbb{R}^2 \to \mathbb{R}^2$ differentiable?

Recall that calculus is *linearization*. That is, given something complicated, zoom in close enough and it looks like a line.

Definition (Little-o)

We say $E = o(|h|)$ if

$$\frac{E}{|h|} \to 0$$

as $|h| \to 0$.

Definition (Differentiability in $\mathbb{R}^2$)

$f : \mathbb{R}^2 \to \mathbb{R}^2$ is *differentiable* at z if

$$f(z+h) = f(z) + Lh + o(|h|),$$

where L is a linear map.

In other words, $f : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable if it is *locally affine*. If $f(x_1, \ldots, x_m) = (f_1(\vec{x}), \ldots, f_m(\vec{x}))$, then

$$f(z+h) = f(z) + Lh + o(|h|).$$

Here L is the *total derivative*, given by

$$L = \left(\frac{\partial f_i}{\partial x_j}\right)_{ij}.$$

2.2 Holomorphic Functions

We can identify a function $f(x, y) = (u, v)$ with the complex-valued function $f(x+iy) = u + iv$, where we write that $u = \mathfrak{Re} f$ and $v = \mathfrak{Im} f$. Here,

$$L = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}.$$

Using the additional structure of the complex numbers (instead of just 2D vectors), we can write

$$\frac{f(z+h)-f(z)}{h} = L + o(1).$$

Notice that dividing by h makes no sense in $\mathbb{R}^n$, but we can over $\mathbb{C}$.

Definition (Holomorphic)

$f : \Omega \to \mathbb{C}$ is *holomorphic at z* if

$$\lim_{h\to 0} \frac{f(z+h)-f(z)}{h}$$

exists.[a] If it does, then we call this limit $f'(z)$.

f is *holomorphic on* Ω if the above is true for all $z \in \Omega$.

[a]Here, Ω is an *open subset* of $\mathbb{C}$.

Definition (Continuity)

f is *continuous* if for all $\epsilon > 0$, there exists a $\delta > 0$ such that whenever $|h| < \delta$, we have

$$|f(z+h) - f(z)| < \epsilon.$$

How can we visualize these functions? To graph a function $f : \mathbb{R}^2 \to \mathbb{R}^2$, we would need $\mathbb{R}^4$ with 4 dimensions. This is hard. Instead, we show a "before" and "after" image.

2.2.1 Classic Example: Smooth but Not Analytic

Let

$$f(z) = \begin{cases} e^{-1/z^2}, & z \neq 0 \\ 0, & z = 0. \end{cases}$$

Over $\mathbb{R}$, as $z \to 0$, $-\frac{1}{z^2} \to -\infty$ and $f \to 0$. And as $|z| \to \infty$, $f \to e^0 = 1$. So f is continuous. f is also differentiable, with

$$f'(z) = 2z^{-3}e^{-\frac{1}{z^2}}.$$

Notice that as $z \to 0$, $f' \to 0$ as well. What about further derivatives?

$$f^{(n)}(z) = P_n(z^{-1})e^{-\frac{1}{z^2}},$$

so $f^{(n)} \to 0$ also as $z \to 0$. So $f \in C^\infty(\mathbb{R})$.[1]

Now take

$$g(z) = \begin{cases} e^{-1/z^2}, & z > 0 \\ 0, & z \leq 0. \end{cases}$$

This is an example of a *smooth bump function*.[2] All derivatives of g are 0 for $z > 0$ as before, and they are also 0 for $z \leq 0$ by definition. So g is also smooth, with $f \equiv g$ on $z \geq 0$ and $f \neq g$ on $z < 0$.

[1]Meaning f is *smooth*, or infinitely differentiable.

[2]These are used in real analysis, particularly for *partitions of unity*.

Notice that f and g both extend $e^{-1/z^2}\Big|_{\mathbb{R}^+}$ to all of $\mathbb{R}$, yet $f \neq g$. We will see later that this cannot happen for complex-analytic functions.[3]

Definition (Analytic)

A function f is *analytic* at z if there exists an $\epsilon > 0$ and $a_0, a_1, a_2, \ldots$ such that for all $|h| < \epsilon$, the power series

$$a_0 + a_1 h + a_2 h^2 + \cdots = \sum_{i=0}^{\infty} a_i h^i$$

converges absolutely and is equal to $f(z+h)$.

In simpler words, f is analytic at z if it has a locally convergent Taylor series expansion around z.

Now, is

$$f(z) = \begin{cases} e^{-1/z^2}, & z \neq 0 \\ 0, & z = 0 \end{cases}$$

analytic at 0? All the Taylor coefficients are zero:

$$a_0 = a_1 = a_2 = \cdots = 0.$$

Thus the series expansion of f converges to 0, but there is no neighborhood of 0 where $f \equiv 0$. So f is *not* analytic. This also cannot happen for complex-valued functions.[4]

What about $f(z) = e^{-1/z^2}$ for $z \in \mathbb{C}$? Now, f isn't even continuous! If $z \to 0$ from $\mathbb{R}$, $f \to 0$. But if $z = iy$ with $y \to 0$,

$$f(z) = e^{-\frac{1}{(iy)^2}} = e^{\frac{1}{y^2}},$$

which approaches ∞. The two paths give conflicting limits![5]

2.3 The Cauchy-Riemann Equations

Recall that f being holomorphic implies

$$\frac{f(z+h) - f(z)}{h} \to f'(z)$$

as $h \to 0$ in $\mathbb{C}$. In $\mathbb{R}^2$ the equivalent is $f(z+h) = f(z) + Lh + o(|h|)$ where $f = u + iv$ and

$$L = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}.$$

The power of $\mathbb{C}$ is that the previous limit is the same no matter how h approaches 0. If we let $h = h_1 + ih_2$, what happens when $h_2 = 0$? So $h = h_1 \in \mathbb{R}$. Then as $h_1 \to 0$,

$$\frac{f(z+h_1) - f(z)}{h_1} \to f'(z) = \frac{\partial}{\partial x} f = u_x + iv_x.$$

[3]This is the fact that *analytic continuations* are *unique*.
[4]This is why the terms *holomorphic* and *analytic* are often used interchangeably in complex analysis.
[5]In fact, you can make f converge to any value by picking some path. This is because f has an *essential singularity* at 0.

On the other hand, if $h_1 = 0$ so that $h = ih_2$ where $h_2 \in \mathbb{R}$. Then as $h_2 \to 0$,

$$\frac{f(z+ih_2)-f(z)}{ih_2} \to f'(z) = \frac{1}{i}\left(\frac{\partial}{\partial y} f\right) = \frac{1}{i}(u_y + iv_y).$$

So these two quantities must be equal!

Theorem (Cauchy-Riemann equations)

If $f = u + iv$ is holomorphic at $z = x + iy$, then $u_x = v_y$ and $v_x = -u_y$.

2.4 Conformal Maps

The Cauchy-Riemann equations imply that for holomorphic f,

$$L = \begin{pmatrix} u_x & u_y \\ -u_y & u_x \end{pmatrix} = \begin{pmatrix} A & B \\ -B & A \end{pmatrix}.$$

What does this mean geometrically?

Theorem (*QR* decomposition)

Any matrix L can be decomposed into $L = QR$ where Q is orthogonal[a] and R is upper-triangular.

[a] One way to define *orthogonal* is that Q is orthogonal if $QQ^T = I$.

In other words, given any matrix, we can rotate it so that it is upper-triangular. In particular, in the case of a 2×2 matrix L, we can write[6]

$$L = \underbrace{\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}}_{K} \underbrace{\begin{pmatrix} x_1 & 0 \\ 0 & x_2 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} 1 & C \\ 0 & 1 \end{pmatrix}}_{N},$$

where K is "compact," A is "abelian,"[7] and N is "unipotent."[8] This is the *Iwasawa decomposition*.

Notably, K is a rotation, A is a dilation, and N is a shear.

Exercise

If

$$L = \begin{pmatrix} A & B \\ -B & A \end{pmatrix},$$

then $C = 0$ and $x_1 = x_2$. In other words, there is no shearing and the dilation is uniform.

The above exercise implies that locally, angles are *preserved* by L. This is called *conformal*.

[6] Note that we factored R by pulling out the diagonal entries.
[7] Or "diagonal."
[8] Same as saying all eigenvalues are 1.

2.4.1 Orientation

Consider the Jacobian of f:

$$J = \det L = u_x^2 + u_y^2 > 0$$

since $u_x, u_y \in \mathbb{R}$. So in addition to being conformal, L also preserves *orientations*.

Note that in general, conformal maps need not preserve orientation. The map $f(z) = \overline{z}$ is perfectly conformal, but it reverses orientation.

2.4.2 Dilation

By the Cauchy-Riemann equations,

$$\det L = u_x^2 + u_y^2 = u_x^2 + (-v_x)^2 = u_x^2 + v_x^2 = |\underbrace{u_x + iv_x}_{\frac{\partial}{\partial x} f = f'(z)}|^2.$$

So $\det L = |f'(z)|^2$.

The important takeaway is that the extra geometry from division by complex numbers (in addition to just being a usual vector space) adds a lot of restrictions (and in turn nice properties).

2.5 Change of Variables Perspective

Recall the differential operator $\frac{\partial}{\partial x}$:

$$\frac{\partial}{\partial x} f = u_x + iv_y.$$

This is if we think of f as $f(x,y) = (u(x,y), v(x,y))$.

Now, think of f as a function of two other auxiliary variables, z and $\overline{z}$.[9] Let

$$f_1(z, \overline{z}) = f\left(\underbrace{\frac{z+\overline{z}}{2}}_{x}, \underbrace{\frac{z-\overline{z}}{2i}}_{y}\right).$$

Then we can try to define the operation $\frac{\partial}{\partial z}$:

$$\frac{\partial}{\partial z} f_1(z, \overline{z}) = \frac{\partial}{\partial z} f\left(\frac{z+\overline{z}}{2}, \frac{z-\overline{z}}{2i}\right) = f_x \frac{1}{2} + f_y \frac{1}{2i}$$

by the chain rule. So as a differential operator,

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} + \frac{1}{i}\frac{\partial}{\partial y}\right).$$

Similarly, we can define $\frac{\partial}{\partial \overline{z}}$ by the chain rule as

$$\frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} - \frac{1}{i}\frac{\partial}{\partial y}\right).$$

If f is holomorphic, then $u_x = v_y$ and $u_y = -v_x$ by the Cauchy-Riemann equations. So

$$\frac{\partial}{\partial \overline{z}} f = \frac{1}{2}\left[u_x + iv_x - \frac{1}{i}(u_y + iv_y)\right] = \frac{1}{2}\left[(u_x - v_y) + i(v_x + u_y)\right] = 0.$$

In other words, if f is holomorphic, then it is "only a function of z," in the sense that $\frac{\partial}{\partial \overline{z}} f = 0$.

[9] Here, think of z and $\overline{z}$ as independent variables. They are simply the result of the change of variables $(z, \overline{z}) = (x + iy, x - iy)$.

Lecture 3

Power Series

3.1 Last Time

Recall that a function $f : \Omega \to \mathbb{C}$, where $\Omega \subseteq \mathbb{C}$ is open, is holomorphic at $z \in \Omega$ if

$$f'(z) := \lim_{h \to 0} \frac{f(z+h) - f(z)}{h}.$$

exists. f is holomorphic on Ω if it is holomorphic at all $z \in \Omega$.

Theorem

If a function $f = u + iv$ is holomorphic, then:

(i). The complex derivative $f'(z)$ satisfies $f'(z) = \frac{\partial}{\partial z} f = \frac{\partial}{\partial x} f = u_x + iv_x$.

(ii). The Jacobian $|J| = |f'(z)|^2 = u_x^2 + v_x^2$.

(iii). (Cauchy-Riemann) $u_x = v_y$ and $u_y = -v_x$.

(iv). The partial derivative with respect to $\overline{z}$ satisfies $\frac{\partial}{\partial \overline{z}} f = 0$.

3.2 Practice with Holomorphic Functions

Example 1

Let $f = (u, v) = u + iv$ where $u = x^2 + y^2$ and $v = 2xy$. Is f holomorphic?

Solution. We can calculate

$$u_y = 2y, \quad v_x = 2y.$$

The Cauchy-Riemann equations are not satisfied since $u_y \neq v_x$, so f is *not* holomorphic. □

Example 2

Let $f = (u, v) = u + iv$ where $u = x^2 - y^2$ and $v = 2xy$. Is f holomorphic?

Solution. Notice that f is actually

$$f(z) = z^2 = (x + iy)^2 = x^2 - y^2 + 2ixy.$$

So f *is* holomorphic and it indeed satisfies the Cauchy-Riemann equations. $\square$

Example 3

Let $f = (u, v) = u + iv$ where $u = x^2 - y^2$ and $v = -2xy$. Is f holomorphic?

Solution. Compute
$$\frac{\partial}{\partial \bar{z}} f = \frac{1}{2}\left(\frac{\partial}{\partial x}(u+iv) - \frac{1}{i}\frac{\partial}{\partial y}(u+iv)\right) = 2x - 2yi = 2\bar{z}.$$
So $f = \bar{z}^2$, which is not holomorphic. This is actually called *anti-holomorphic*, when $\frac{\partial}{\partial z} f = 0$. $\square$

Example 4

Let $f = (u, v) = u + iv$ where $u = x^2 + y^2$ and $v = 0$. Is f holomorphic?

Solution. Compute

$$\frac{\partial}{\partial z} f = \frac{1}{2}\left(\frac{\partial}{\partial x}(u+iv) + \frac{1}{i}\frac{\partial}{\partial y}(u+iv)\right) = \frac{1}{2}\left(2x + \frac{1}{i}2y\right) = x - iy = \bar{z}.$$

Similarly we can find that $\frac{\partial}{\partial \bar{z}} = z$. Note that $f(z) = |z|^2 = z\bar{z}$, which is not holomorphic. $\square$

3.3 Converse of Cauchy-Riemann

Theorem (Converse of Cauchy-Riemann equations)

If u and v are $\mathbb{R}^2$-differentiable near z and satisfy the Cauchy-Riemann equations at z, then the complex function $f = u + iv$ is holomorphic at z.

Proof. For $h = h_1 + ih_2$, look at

$$f(z+h) - f(z) = u(z+h) - u(z) + i(v(z+h) - v(z)) = u_x h_1 + u_y h_2 + i(v_x h_1 + v_y h_2) + o(|h|)$$

Then by the Cauchy-Riemann equations (exchange $u_y \to -v_x$ and $v_y \to u_x$),

$$\begin{aligned} f(z+h) - f(z) &= u_x h_1 - v_x h_2 + i(v_x h_1 + u_x h_2) + o(|h|) = (u_x + iv_x)h_1 + i(u_x + iv_x)h_2 + o(|h|) \\ &= (u_x + iv_x)(h_1 + ih_2) + o(|h|) = (u_x + iv_x)h + o(|h|) \end{aligned}$$

Dividing by h gives
$$\frac{f(z+h) - f(z)}{h} = \frac{\partial}{\partial x} f + o(1) = f'(z) + o(1).$$
So $f'(z)$ exists and f is holomorphic at z. $\square$

Remark. It is not enough for u_x, u_y, v_x, v_y to exist and satisfy the Cauchy-Riemann equations for holomorphicity. We really need u and v to be $\mathbb{R}^2$-differentiable. The classic counterexample here is

$$f = \frac{z^5}{|z|^4} = \frac{z^5}{z^2\bar{z}^2} = \frac{z^3}{\bar{z}^2}.$$

3.4 Analytic Functions

Recall that $f : \Omega \to \mathbb{C}$ is analytic near $z_0 \in \Omega$ if there exists $R > 0$ and a power series representation

$$f(z) = \sum_{n \geq 0} a_n (z - z_0)^n$$

which converges to f absolutely for each z in an open ball $B_R(z_0)$.

Example

$$\sum_{n \geq 0} \frac{z^n}{n!} = \exp(z), \quad \text{radius of convergence } R = \infty.$$

$$\sum \frac{z^{2n}}{(2n)!}(-1)^n = \cos(z) = \frac{e^{iz} + e^{-iz}}{2}, \quad R = \infty.$$

$$\sum \frac{z^{2n+1}}{(2n+1)!}(-1)^n = \sin(z) = \frac{e^{iz} - e^{-iz}}{2i}, \quad R = \infty.$$

$$\sum z^n = \frac{1}{1-z}, \quad \text{converges for } |z| < R = 1.$$

$$\sum 2^n z^n = \frac{1}{1-2z}, \quad |z| < R = \frac{1}{2}.$$

So even exponential growth in a_n can result in a positive radius of convergence!

$$\sum n! z^n, \quad R = 0, \text{ i.e. it diverges everywhere.}$$

This grows too fast. For $n > |z|$, $n! z^n \to \infty$, and $a_n \to 0$ is a necessary condition for convergence.

So what happens in general?

Theorem (Hadamard)

Let $\sum a_n z^n$ be a power series. Then there exists some $0 \leq R < \infty$ such that:

(i) For all $|z| < R$, the series converges absolutely.

(ii) For all $|z| > R$, the series diverges.

Furthermore, $R = \frac{1}{L}$ where $L = \limsup |a_n|^{1/n}$.

Proof of (i). Assume $0 < R < \infty$ (the cases $R = 0$ and $R = \infty$ are left as exercises). Let $L = \limsup |a_n|^{1/n}$ and let $|z| < R$. Note that $L = \limsup |a_n|^{1/n}$ means that for all ϵ, there exists N such that for all $n > N$,

$$|a_n|^{1/n} < L + \epsilon.$$

There exists ϵ such that $(L + \epsilon)|z| \leq r < R = \frac{1}{L}$. So $L|z| < 1$. Let ϵ be small enough such that $(L + \epsilon)|z| \leq r < 1$. Now look at the tail of the series:

$$\sum_{n > N} |a_n||z|^n \leq \sum_{n > N} (L + \epsilon)^n |z|^n \leq \sum_{n > N} r^n < \infty$$

since $r < 1$. So the series converges absolutely for $|z| < R$. The proof for (ii) is left as an exercise. $\square$

For example, if $a_n = n!$, recall that $n! \approx n^n e^{-n}\sqrt{n}$ by Stirling's formula. So

$$|n!|^{\frac{1}{n}} \approx \frac{n}{e} \to \infty = L$$

and $R = 0$. So we should expect this to diverge everywhere.

Theorem (Analytic implies holomorphic)

If $f = \sum a_n z^n$ has a radius of convergence $R > 0$, then f is holomorphic on $|z| < R$. Moreover,

$$f'(z) = \sum n a_n z^{n-1},$$

which has the same radius R of absolute convergence.

Sketch of proof. We know f is analytic, so let

$$S_N(z) = \sum_{n \le N} a_n z^n, \quad E_N(z) = \sum_{n > N} a_n z^n$$

be the partial sums and error terms, respectively. Let

$$g(z) = \sum n a_n z^{n-1}.$$

Note that $n^{1/n} \to 1$, so clearly g has absolute convergence for $|z| < R$.[1] Then want to show that

$$\left| \frac{f(z+h) - f(z)}{h} - g(z) \right| \to 0.$$

The key is to use a 3ϵ argument. Since $f(z) = S_N + E_N$, we have

$$\left| \frac{f(z+h) - f(z)}{h} - g(z) \right| \le \left| \frac{S_N(z+h) - S_N(z)}{h} - S_N' \right| + |S_N' - g| + \left| \frac{E_N(z+h) - E_N(z)}{h} \right|.$$

The first term is less than ϵ since S_N is a polynomial, which are holomorphic. The second term is small since g is a convergent power series and $g \to S_N'$. The final term is small since E_N is an error term. □

Remark. Really the miracle about complex analysis is that holomorphicity implies analyticity, and analytic functions are the ones that have all these nice properties (e.g. infinite differentiability, etc.).

[1] To show this limit, let $f = n^{1/n}$ and look at its log. We have $\ln f = \ln(n^{1/n}) = \frac{1}{n} \ln n \to 0$, so $f \to 1$.

Lecture 4

Integration on Curves

4.1 Last Time

Theorem

$u, v : \mathbb{R}^2 \to \mathbb{R}$ are $\mathbb{R}^2$-differentiable and satisfy the Cauchy-Riemann equations if and only if $f = u + iv$ is holomorphic.

Theorem (Hadamard)

Let $f(z) = \sum_{n\geq 0} a_n z^n$ be a power series. Then there exists an $R \in [0, \infty]$ (in fact we know $R = 1/L$, where $L = \limsup_{n\to\infty} |a_n|^{1/n}$) such that

1. for all $|z| < R$, the series converges absolutely and
2. for all $|z| > R$, the series diverges.

Moreover, the convergence is uniform for all $|z| < r < R$.

For the last part, recall that for $|z| < r < R = 1/L$, we can make ϵ so small that

$$L|z| < r_1 < 1.$$

From here there's enough room to make $(L+\epsilon)|z| < r_1 < 1$. Then

$$\sum_{n\geq N} |a_n||z|^n \leq \sum ((L+\epsilon)|z|)^n = \sum_{n\geq N} r_1^n < \infty,$$

so the convergence is uniform.

Theorem (Analytic $\implies$ holomorphic)

If $f(z) = \sum a_n z^n$ with radius of convegence $R > 0$, then $f'(z)$ exists and

$$f'(z) = g(z) = \sum n a_n z^{n-1},$$

and g has the same radius of convergence R.

Proof. Note that g clearly has the same radius of convergence R since $n^{1/n} \to 1$. Now for $|z|, |z+h| < r < R$,

$$\frac{f(z+h)-f(z)}{h} - g(z) = \frac{f(z+h)-f(z)}{h} - \frac{S_N(z+h)-S_N(z)}{h} + \frac{S_N(z+h)-S_N(z)}{h} - S_N'(z) + S_N'(z) - g(z).$$

Here $S_N(z) = \sum_{n\le N} a_n z^n$ are the partial sums. Note that S_N is a finite polynomial in z. Then by the triangle inequality,

$$\left|\frac{f(z+h)-f(z)}{h} - g(z)\right| \le \left|\frac{f(z+h)-f(z)}{h} - \frac{S_N(z+h)-S_N(z)}{h}\right| + \left|\frac{S_N(z+h)-S_N(z)}{h} - S_N'(z)\right| + \left|S_N'(z) - g(z)\right|.$$

For the last term, note that $S_N' \to g$ uniformly by Hadamard's theorem. Then for any $\epsilon > 0$, there exists N_1 such that for all $|z| < r$ and $N \ge N_1$,

$$\left|S_N'(z) - g(z)\right| < \epsilon.$$

For the second term, note that S_N is a polynomial. So for all N and $\epsilon > 0$, there exists $\delta > 0$ such that for all $|h| < \delta$ (pick δ small enough so that $|z+h| < r < R$),

$$\left|\frac{S_N(z+h)-S_N(z)}{h} - S_N'(z)\right| < \epsilon.$$

For the first term, note that f is the infinite sum while S_N is a finite sum, so their difference is the tail:

$$\begin{aligned}\left|\frac{f(z+h)-f(z)}{h} - \frac{S_N(z+h)-S_N(z)}{h}\right| &= \left|\frac{1}{h}\sum_{n>N} a_n\left[(z+h)^n - z^n\right]\right| \\ &= \left|\frac{1}{h}\sum_{n>N} a_n h((z+h)^{n-1} + (z+h)^{n-2}z + \cdots + z^{n-1})\right|,\end{aligned}$$

using the fact that $a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + \cdots + b^{n-1})$. Then we can cancel the h to get

$$\left|\frac{f(z+h)-f(z)}{h} - \frac{S_N(z+h)-S_N(z)}{h}\right| = \left|\sum_{n>N} a_n((z+h)^{n-1} + (z+h)^{n-2}z + \cdots + z^{n-1})\right| \le \sum_{n>N} |a_n| n r^{n-1}$$

since $|z+h|, |z| < r < R$. Now by the absolute convergence of g on $|z| < R$, for all $\epsilon > 0$ we can find N_2 such that for all $N \ge N_2$,

$$\left|\frac{f(z+h)-f(z)}{h} - \frac{S_N(z+h)-S_N(z)}{h}\right| \le \sum_{n>N} |a_n| n r^{n-1} < \epsilon.$$

So for a fixed $\epsilon > 0$, let $N > \max\{N_1, N_2\}$. Then we can find $\delta > 0$ such that

$$\left|\frac{f(z+h)-f(z)}{h} - g(z)\right| \le \epsilon + \epsilon + \epsilon = 3\epsilon,$$

so the limit exists and is equal to $g(z)$. □

Corollary

f analytic implies $f \in C^\infty_{\mathbb{C}}$, i.e. it is infinitely differentiable in the complex sense. Also each $f^{(n)}$ has the same radius of convergence R and is given by termwise differentiation.

4.2 Integration Over Curves

4.2.1 Parametric Curves

Definition (Parametric curve)

A *parametric curve* is a function $z : [a,b] \to \mathbb{C}$ which is continuously differentiable. Furthermore, one-sided derivatives exist at a^+ and b^- and are continuous.

In this area, this is usually what mathematicians mean when they say *smooth*, as opposed to being infinitely differentiable. We say that z is *piecewise smooth* if $z : [a,b] \to \mathbb{C}$ is continuous and continuously differentiable on finitely many pieces $a = a_0 < a_1 < \cdots < a_n = b$.

Example

We can parametrize the unit circle by $z : [0,1] \to \mathbb{C}$ with $z(t) = e^{2\pi i t}$ or $z : [0, 2\pi] \to \mathbb{C}$ with $z(t) = e^{it}$.

This is kind of annoying since they are really the exact same curve, so we would like to define some kind of equivalence relation on these parametrizations.

Definition (Equivalence of parametric curves)

We call two parametric curves $z : [a,b] \to \mathbb{C}$ and $\tilde{z} : [c,d] \to \mathbb{C}$ *equivalent* if there exists a continuously differentiable bijection $t : [a,b] \to [c,d]$ such that $\tilde{z}(s) = z(t(s))$ and preserves orientation, i.e. $t'(s) > 0$.

Note that we need $t'(s) \geq 0$ since we do care about the orientation of our curves. Here the opposite curve $z^- : [a,b] \to \mathbb{C}$ is given by $z^-(t) = z(a+b-t)$. Furthermore we need $t'(s) \neq 0$ to guarantee that t^{-1} is differentiable. This ensures that we have actually defined an equivalence relation.

Definition (Smooth curve)

A *smooth curve* γ is an equivalence class $\gamma = [z] = \text{parametric curve} / \sim$ of parametric curves.

4.2.2 Integrals and Immediate Results

Definition (Integral of a function along a curve)

If $f : \Omega \to \mathbb{C}$ is continuous and $\gamma \subseteq \Omega$ is a smooth curve, then the *integral* of f along γ is defined as

$$\int_\gamma f(z)dz = \int_a^b f(z(t)) \cdot z'(t)\,dt$$

for some parametrization $z : [a,b] \to \mathbb{C}$ of γ.

Lemma

The above definition of the integral of f along γ is independent of the choice of z.

Proof. Suppose that we use some other parametrization $\tilde{z} : [c,d] \to \mathbb{C}$ of γ. Then by definition $\tilde{z}(s) = z(t(s))$

for some $t : [a,b] \to [c,d]$. Noting that

$$\tilde{z}'(s)\,ds = z'(t(s))t'(s)\,ds = z'(t(s))\,dt,$$

we can make the change of variables $t \mapsto t(s)$ to get

$$\int_a^b f(z(t)) \cdot z'(t)\,dt = \int_c^d f(z(t(s))) \cdot z'(t(s))t'(s)\,ds = \int_c^d f(\tilde{z}(s)) \cdot \tilde{z}'(s)\,ds,$$

as desired. Thus the integral is indeed well-defined. □

Definition (Length of a curve)

The length of a curve γ is

$$\text{length}(\gamma) = \ell(\gamma) = \int_a^b |z'(t)|\,dt$$

for some parametrization $z : [a,b] \to \mathbb{C}$ of γ.

Lemma

If $f : \Omega \to \mathbb{C}$ is continuous and $\gamma \subseteq \Omega$ is a smooth curve, then

$$\left|\int_\gamma f(z)\,dz\right| \le \sup_\gamma |f| \cdot \ell(\gamma).$$

Proof. Pick some parametrization $z : [a,b] \to \mathbb{C}$ of γ. Then

$$\begin{aligned}\left|\int_\gamma f(z)\,dz\right| = \left|\int_a^b f(z(t))z'(t)\,dt\right| &\le \int_a^b |f(z(t))||z'(t)|\,dt \le \int_a^b \sup_\gamma |f| \cdot |z'(t)|\,dt \\ &= \sup_\gamma |f| \int_a^b |z'(t)|\,dt = \sup_\gamma |f| \cdot \ell(\gamma),\end{aligned}$$

By the triangle inequality. Note that $\sup_\gamma |f|$ is independent of t, so we can pull it out of the integral. □

Definition (Primitive)

Let $\Omega \subseteq \mathbb{C}$ be open. If $f : \Omega \to \mathbb{C}$ and there exists some holomorphic $F : \Omega \to \mathbb{C}$ with $F' = f$, then we say that F is a *primitive* of f.

Theorem

If $f : \Omega \to \mathbb{C}$ is continuous with primitive F and $\gamma \subseteq \Omega$ is a smooth curve, then

$$\int_\gamma f(z)\,dz = F(w_1) - F(w_0),$$

where w_0, w_1 are the start and end points of γ, respectively.

Proof. Pick some parametrization $z : [a, b] \to \mathbb{C}$ of γ. Then

$$\int_\gamma f(z)\,dz = \int_a^b f(z(t))z'(t)\,dt = \int_a^b F'(z(t))z'(t)\,dt = \int_a^b (F \circ z)'(t)\,dt = F(z(b)) - F(z(a)) = F(w_1) - F(w_0)$$

by the fundamental theorem of calculus. $\square$

Corollary

If a continuous function $f : \Omega \to \mathbb{C}$ has a primitive and $\gamma \subseteq \Omega$ is a closed curve, then

$$\oint_\gamma f(z)\,dz = 0.$$

Proof. We have $w_0 = w_1$ since γ is closed, after which the result follows from the previous theorem. $\square$

Example

Let $f(z) = 1/z$ and γ be the unit circle. Parametrize γ by $z : [0, 1] \to \mathbb{C}$ with $t \mapsto e^{2\pi i t}$. Then

$$\int_\gamma f(z)\,dz = \int_\gamma \frac{1}{z}\,dz = \int_0^1 \frac{1}{e^{2\pi i t}} \cdot 2\pi i e^{2\pi i t}\,dt = \int_0^1 2\pi i\,dt = 2\pi i \neq 0.$$

This implies that $f(z)$ does not have a primitive for any $\Omega \supseteq S^1$, where S^1 is the unit circle.

Corollary

Let $\Omega \subseteq \mathbb{C}$ be open and connected. If $f : \Omega \to \mathbb{C}$ is holomorphic and $f' \equiv 0$ on Ω, then $f \equiv C$.

Proof. Note that Ω being open and connected means that it is path-connected. Then fix any $w_0 \in \Omega$. Since Ω is path-connected, for any other $w_1 \in \Omega$ there exists a curve $\gamma : w_0 \to w_1$. Then

$$f(w_1) - f(w_0) = \int_\gamma f'(z)\,dz = \int_\gamma 0\,dz = 0,$$

so we have $f(w_1) = f(w_0)$ for all $w_1 \in \Omega$. Thus f is constant. $\square$

Example

Let $f(z) = 1$ and $\gamma : 0 \to 1 + i$. Parametrize $\gamma : [0, 1] \to \mathbb{C}$ by $t \mapsto t + it$. Then

$$\int_\gamma f(z)\,dz = \int_\gamma 1\,dz = \int_0^1 1 \cdot (1 + i)\,dt = 1 + i.$$

Another way to proceed is to see that f has a primitive $F(z) = z$. Then by the previous theorem,

$$\int_\gamma f(z)\,dz = F(1 + i) - F(0) = 1 + i.$$

Notice that the two computations match.