Understanding Analysis (Stephen Abbott) Exercises

Frank Qiang

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1 The Real Numbers

1.1 Discussion: The Irrationality of $\sqrt{2}$

1.2 Some Preliminaries

Exercise 1.2.1.

(a) Prove that $\sqrt{3}$ is irrational. Does a similar argument work to show $\sqrt{6}$ is irrational?
(b) Where does the proof of Theorem 1.1.1 break down if we try to use it to prove $\sqrt{4}$ is irrational?

Proof of (a). Suppose by way of contradiction that $\sqrt{3}$ is rational. Then there exist $p, q \in \mathbb{Z}$ coprime such that $\sqrt{3} = p/q$. Square both sides to get

$$\left(\frac{p}{q}\right)^2 = 3.$$

Rewriting the above equality yields

$$p^2 = 3q^2.$$

Then $3|p^2$, and since 3 is prime, we also have $3|p$. By the definition of divisibility there exists $k \in \mathbb{Z}$ such that $p = 3k$. Substituting into the above equation yields

$$(3k)^2 = 9k^2 = 3q^2 \implies 3k^2 = q^2.$$

By the same logic as above, $3|q^2$ and $3|q$. Thus we have reached a contradiction as p and q are coprime yet they share a common factor of 3. Therefore the original assumption must have been false and $\sqrt{3}$ must be irrational.

A similar argument works for $\sqrt{6}$. Although 6 is not prime, it is square-free, so we can still make the argument that $6|p^2$ implies $6|p$. $\square$

Proof of (b). The proof fails when we argue that $4|p^2$ implies $4|p$. $p = 2$ is a counterexample as $p^2 = 4$ is a multiple of 4 yet p is not. $\square$

Exercise 1.2.2.

Show that there is no rational number r satisfying $2^r = 3$.

Proof. Suppose by way of contradiction that there exists such $r \in \mathbb{Q}$. Express this r as p/q where $p, q \in \mathbb{Z}$ are coprime and $q \neq 0$. Then

$$2^{p/q} = 3 \implies 2^p = 3^q.$$

after raising both sides to the q-th power. And here we have reached a contradiction as an integer power of 2 can never equal a non-zero integer power of 3. So there cannot exist such an $r \in \mathbb{Q}$. $\square$