Application of Differential equation in Quantum Mechanics

Order: $$\frac{d^n y}{dx^n} + c_1 \frac{d^{n-1} y}{dx^{n-1}} + \dots + c_{n-1}\frac{dy}{dx} + c_n y = 0$$

The diff eqn has order n.

Linear: $$\frac{d^n y}{dx^n} = \sum_{i=0}^{n-1} f_i(x)\left(\frac{d^i y}{dx^i}\right)^j + g(x)$$

Here if $j=1$, so the diff eqn becomes linear.

for $j=1$, $$\frac{d^n y}{dx^n} = f_0(x)\, y + f_1(x)\frac{dy}{dx} + f_2(x)\frac{d^2y}{dx^2} + \dots + g(x)$$

It is a linear diff eqn because all derivative has a power of unity.

Non linear: if $j \neq 1$ in above case then

$$\frac{d^n y}{dx^n} = f_0(x)\, y + f_1(x)\frac{dy}{dx} + f_2(x)\left(\frac{d^2y}{dx^2}\right)^2 + \dots + g(x)$$

It is a non linear diff. eqn.

Homogeneous & Non homogeneous:

$$\frac{d^n y}{dx^n} + f_1(x)\frac{d^{n-1} y}{dx^{n-1}} + \dots + f_{n-1}(x)\frac{dy}{dx} + f_n(x)\, y = F(x)$$

if $F(x) \neq 0$ then Non homogeneous
if $F(x) = 0$ .. Homogeneous.

Q: State the criteria of the following diff. eqn —

$$\frac{d^2 y}{dx^2} + (1-x^2)\frac{dy}{dx} + xy = 0$$

⇒ It is a 2nd order linear homogeneous diff eqn.

Various solution process of a diff eqn

Type-I: 2nd order linear homogeneous diff eqn with functional coefficient

$$\frac{d^2y}{dx^2} + f_1(x)\frac{dy}{dx} + f_2(x)y = 0$$

To solve this, we have to identify one solution fn with intelligency. Let it is $y_1(x)$. To find another solution $y_2(x)$ we have
$y_2(x) = V(x)\, y_1(x)$ as the two solutions must be linearly independent. Here $V(x)$ will be —

$$V(x) = \int \frac{1}{y_1^2} e^{-\int f_1(x)dx} dx$$

Ex-1a: $\frac{d^2y}{dx^2} + y = 0$. Find the solution $\leftarrow \frac{d^2y}{dx^2} + 0\frac{dy}{dx} + 1\cdot y = 0$

Hence $f_1(x) = 0$
$f_2(x) = 1$

$\Rightarrow$ If we choose $y = \sin x$
then $y' = \frac{dy}{dx} = \cos x$
and $y'' = \frac{d^2y}{dx^2} = -\sin x$

Thus $y = \sin x$ satisfies the diff eqn. Let $y_1(x) = \sin x$. To find another soln we need $V(x) = \int \frac{1}{\sin^2 x} e^{-\int 0\, dx} dx$

$$= \int \operatorname{cosec}^2 x\, dx$$

$$= -\cot x$$

$\therefore y_2(x) = V(x)\, y_1(x) = -\cot x \sin x = -\cos x$

$\therefore$ gen. soln. $y(x) = A\sin x - B\cos x$
$= A\left(\frac{e^{ix} - e^{-ix}}{2i}\right) - B\left(\frac{e^{ix} + e^{-ix}}{2}\right) = Ce^{ix} + De^{-ix}$

Ex-1b: Find the solution of

$$2x\frac{d^2y}{dx^2} - x\frac{dy}{dx} - xy = 0$$

$\Rightarrow$ Here $f_1(x) = -\frac{x}{2x}$

$f_2(x) = -\frac{x}{2x}$

let choose $y = e^x$

$\therefore$ $y' = e^x = y''$

$\therefore$ from the diff eqn.

$2xe^x - xe^x - xe^x = 0$

$\therefore$ $y_1(x) = e^x$ is a solution of it.

$$\therefore V(x) = \int \frac{1}{e^{2x}} e^{-\int -\frac{x}{2x} dx} dx$$

$$= \int \frac{1}{e^{2x}} e^{\int \frac{1}{2} dx} dx$$

$$= \int \frac{1}{e^{2x}} e^{x/2} dx$$

$$= \int e^{-\frac{3x}{2}} dx$$

$$= -\frac{2}{3} e^{-\frac{3x}{2}}$$

$$\therefore y_2(x) = V(x)\, y_1(x) = -\frac{2}{3} e^{-\frac{3x}{2}} e^x = -\frac{2}{3} e^{-x/2}$$

$\therefore$ general solution

$$y(x) = A y_1(x) + B y_2(x)$$

$$= Ae^x - \frac{2}{3} B e^{-x/2}$$

(Pg-4)

Type-II : 2nd order linear homogeneous diff. eqn. with constant coefficient.

$$\boxed{\cancel{\frac{d^2y}{dx^2}+2a\frac{dy}{dx}+b^2y}}$$

$$\frac{d^2y}{dx^2}+c_1\frac{dy}{dx}+c_2y=0$$

It must have an exponential solution e^{mx}. For appropriate value of the constants m, c_1, c_2 the above equation should be satisfied.

$\therefore\ m^2e^{mx}+mc_1e^{mx}+c_2e^{mx}=0$

or, $(m^2+c_1m+c_2)e^{mx}=0$

or, $m^2+c_1m+c_2=0$ [if $e^{mx}=0$ the soln. doesn't make any sence]

Now it is a simple quadratic equation which is solved by Sridhar Acharya's law. The two roots m_1 and m_2 will be —

$$m_{\pm}=\frac{-c_1\pm\sqrt{c_1^2-4c_2}}{2}$$

where + and − sign denotes 1 and 2 respectively or vice versa. Now there are three possibility

① $c_1^2>4c_2 \Rightarrow$ roots are real but not equal

② $c_1^2=4c_2 \Rightarrow$ roots are real and equal

③ $c_1^2<4c_2 \Rightarrow$ roots are purely imaginary. and one is complex conjugate of the other.

Let $c_1=2a$ and $c_2=b^2$

$\therefore\ m_1=\dfrac{-2a+\sqrt{4a^2-4b^2}}{2}=-a+\sqrt{a^2-b^2}$

$m_2=\dfrac{-2a-\sqrt{4a^2-4b^2}}{2}=-a-\sqrt{a^2-b^2}$

~~Case~~ ① corresponds to m_1 & m_2 are real and $m_1\neq m_2$

② " " m_1 & m_2 " " " $m_1=m_2$

③ " " m_1 & m_2 " imaginary " $m_1^*=m_2$

Ex:-2a: Discuss about the diff equation

$$\frac{d^2y}{dx^2} + 2a\frac{dy}{dx} + b^2 y = 0$$

$\Rightarrow$ It is linear 2nd order homogeneous diff eqn with constant coeff. If you ~~notice~~ observe carefully, you will find that the above diff eqn is the eqn of motion of a damped harmonic oscillator where a is damping constant and $b = \omega_0$ is the frequency. The three possibilities ~~are~~ ~~it~~ denotes three types of damping.

$a > b$ corresponds to overdamped motion.
$a = b$ " " critical damping
$a < b$ " " under damped.

<u>Solutions</u>

If the roots ~~are~~ of auxiliary eqn are m_1 & m_2 then for $a > b$, the soln is

$$y(x) = A_1 e^{m_1 x} + A_2 e^{m_2 x}$$

for $a = b$, ~~one sol~~ $m_1 = m_2 = -a$ (say).

$\therefore$ one soln. is $y_1(x) = e^{-ax}$.

To find other soln, obtain $v(x) = \int \frac{1}{[y_1(x)]^2} e^{-\int 2a\,dx} dx$

$$= \int \frac{1}{e^{-2ax}} e^{-2ax} dx$$

$$= x$$

the other soln is $y_2(x) = v(x)\, y_1(x)$.

$$= x e^{-ax}$$

general soln $y(x) = A e^{-ax} + B x e^{-ax}$

$$= (A + Bx) e^{-ax}$$

For $a < b$, m_1 and m_2 are complex and ~~or~~ $m_1^* = m_2$. Let $m_1 = \alpha + i\beta$
& $m_2 = \alpha - i\beta$.

The general solution:

$$y(x) = A_1 e^{m_1 x} + A_2 e^{m_2 x}$$
$$= A_1 e^{(\alpha+i\beta)x} + A_2 e^{(\alpha-i\beta)x}$$
$$= A_1 e^{\alpha x} e^{i\beta x} + A_2 e^{\alpha x} e^{-i\beta x}$$
$$= e^{\alpha x}\left(A_1 e^{i\beta x} + A_2 e^{-i\beta x}\right);$$
$$= e^{\alpha x}\left(\tilde{A}_1 \cos\beta x + i\tilde{A}_2 \sin\beta x\right)$$

where
$A_1 + A_2 = \tilde{A}_1$
$A_1 - A_2 = \tilde{A}_2$

Type-III: 2nd order linear inhomogeneous differential equation

A 2nd order linear inhomogeneous diff eqn ~~is~~ looks like —

$$\frac{d^2y}{dx^2} + f_1(x)\frac{dy}{dx} + f_0(x)\,y = g(x)$$

observe that if $g(x) = 0$, the diff eqn looks like a homogeneous 2nd order linear diff eqn and the processes of solving that type are allready disscussed in Type-I and Type-II.

To solve the inhomogeneous linear ~~2nd order~~ diff eqn we have to put $g(x) = 0$ at first and solve that homogeneous eqn. This is complementary function (CF). After that depending on the form of $g(x)$ the solution have to be determined and it is called particular integral (PI). The general soln is

CF + PI.

Now we shall discuss the solution depending on the form of $g(x)$.

$$\frac{d^2y}{dx^2} + f_1(x)\frac{dy}{dx} + f_0(x)y = g(x)$$

or, $\left[D^2 + f_1(x)D + f_0(x)\right]y = g(x)$ [where $D = \frac{d}{dx}$]

or, $F(D)\, y = g(x)$

The particular integral is

$$y_p = \frac{1}{F(D)} g(x)$$

Where D^{-1} represents the integration and $\frac{1}{F(D)} = F(D)^{-1}$. It is known as integration operator function.

① $g(x) = x^m$

$$y_p = \frac{1}{F(D)} x^m$$

We have to expand $F(D)^{-1}$ bionomially and then that operates on x^m.

Ex: Find y_p for $\frac{d^2y}{dx^2} + a^2y = c(1-x)$

$\Rightarrow \frac{d^2y}{dx^2} + a^2y = c(1-x)$

or, $(D^2+a^2)y = c(1-x)$

$\therefore y_p = c\frac{1}{(D^2+a^2)}(1-x)$

$$= \frac{c}{a^2}\frac{1}{(1+D^2/a^2)}(1-x)$$

$$= \frac{c}{a^2}(1 - D^2/a^2)(1-x)$$

$$\therefore y_p = \frac{c}{a^2}\left[1(1-x) - \frac{D^2}{a^2}(1-x)\right]$$
$$= \frac{c}{a^2}\left[1-x - \frac{1}{a^2}\frac{d^2}{dx^2}(1-x)\right]$$
$$= \frac{c}{a^2}[1-x]$$

② $g(x) = e^{mx}$, where m is any integer.

$$\Rightarrow y_p = \frac{1}{F(D)} e^{mx}$$
$$= \frac{1}{F(m)} e^{mx}$$

where $F(m) \neq 0$. But

if $F(m) = 0$, $y_p = \frac{1}{F(D)} e^{mx} = x \frac{1}{F'(m)} e^{mx}$ $[F'(m) \neq 0]$

" $F'(m) = 0$, $y_p = \frac{1}{F(D)} e^{mx} = x^2 \frac{1}{F''(m)} e^{mx}$ $[F''(m) \neq 0]$

<u>Ex:</u> Find PI of $\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y = 5e^{3x}$

$$\Rightarrow F(D) = D^2 + 6D + 9$$
$$\therefore y_p = \frac{1}{F(D)} 5e^{3x}$$
$$= 5 \frac{1}{D^2+6D+9} e^{3x}$$
$$= 5 \frac{e^{3x}}{3^2+6\cdot3+9}$$
$$= \frac{5e^{3x}}{36}$$

③ $g(x) = e^{mx}\, v(x)$

$$y_p = \frac{1}{F(D)}\, e^{mx}\, v(x)$$

$$= e^{mx}\, \frac{1}{F(D+m)}\, v(x)$$

$$= e^{mx}\, F(D+m)^{-1}\, v(x)$$

$F(D+m)^{-1}$ has to be expanded binomially.

__Ex:__ Find PI of $(D^2+4)y = e^x \cos x$

$$\Rightarrow \quad y_p = \frac{1}{(D^2+4)}\, e^x \cos x$$

$$= e^x\, \frac{1}{(D^2+4+1)} \cos x$$

$$= e^x\, \frac{1}{D^2+5+2D} \cos x$$

$$= \frac{e^x}{5}\, \frac{1}{1+\frac{D^2+2D}{5}} \cos x$$

$$= \frac{e^x}{5}\left(1 - \frac{D^2+2D}{5}\right) \cos x$$

$$= \frac{e^x}{5}\left(\cos x - \frac{1}{5}\frac{d^2}{dx^2}\cos x\} + \frac{2}{5}\frac{d}{dx}\cos x\right)$$

$$= \frac{e^x}{5}\left(\cos x + \frac{1}{5}\cos x\} - \frac{2}{5}\sin x\right)$$

$$= \frac{\cancel{6}e^x}{25}(6\cos x - 2\sin x)$$

(4) (i) $g(x) = \sin(ax+b)$

$$y_p = \frac{1}{F(D)}\sin(ax+b)$$

$$= \frac{1}{\phi(D^2)}\sin(ax+b)$$

$$= \frac{1}{\phi(-a^2)}\sin(ax+b)$$

(ii) $g(x) = \cos(ax+b)$

$$y_p(x) = \frac{1}{F(D)}\cos(ax+b)$$

$$= \frac{1}{\phi(D^2)}\cos(ax+b)$$

$$= \frac{1}{\phi(-a^2)}\cos(ax+b)$$

<u>Ex</u>! Find PI of $(D^2-4D+5)y = 2\sin x$

$$y_p = \frac{1}{(D^2-4D+5)}2\sin x$$

$$= 2\frac{1}{-1^2-4D+5}\sin x$$

$$= 2\frac{1}{4(1-D)}\sin x$$

$$= \frac{1}{2}\frac{(1+D)}{1-D^2}\sin x$$

$$= \frac{1}{2}\frac{(1+D)}{2}\sin x$$

$$= \frac{1}{4}(\sin x + \cos x)$$

(5) (i) $g(x) = x^m \sin(ax+b)$

$$y_p = \frac{1}{F(D)} x^m \sin(ax+b)$$

$$= \text{im.}\left[e^{iax} \frac{1}{F(D+ia)} x^m\right]$$

(ii) $g(x) = x^m \cos(ax+b)$

$$y_p = \frac{1}{F(D)} x^m \cos(ax+b)$$

$$= \text{Re}\left[e^{iax} \frac{1}{F(D+ia)} x^m\right]$$

Ex: Find PI of $(D^2-4D+5)y = x^2\cos^2 x$

$\Rightarrow g(x) = x^2 \cos^2 x$

$$= \frac{1}{2} x^2 (1+\cos 2x)$$

$$= \frac{1}{2}(x^2 + x^2\cos 2x)$$

$$y_p = \frac{1}{D^2-4D+5} \frac{1}{2}(x^2 + x^2\cos 2x)$$

$$= \frac{1}{2}\left[\frac{1}{D^2-4D+5} x^2 + \frac{1}{D^2-4D+5} x^2\cos 2x\right]$$

Now $\frac{1}{D^2-4D+5} = \frac{1}{5} \frac{1}{1+\frac{D^2-4D}{5}}$

$$= \frac{1}{5}\left(1+\frac{D^2-4D}{5}\right)^{-1}$$

$$= \frac{1}{5}\left(1-\frac{D^2-4D}{5}\right)$$

$$= \frac{1}{5} - \frac{D^2-4D}{25}$$

The 1st term in y_p will be

$$\frac{1}{2}\left(\frac{1}{5} - \frac{D^2-4D}{25}\right)x^2 = \frac{1}{2}\left(\frac{1}{5}x^2 - \frac{1}{25}(2-8x)\right)$$

The 2nd term in y_p will be

$$\frac{1}{(D^2-4D+5)} x^2 \cos 2x$$

$$D^2+i4D-4$$
$$-4D-8i+5$$
$$D^2+i4D-4D+1-8i$$

$$= \text{Re.}\left[e^{i2x} \frac{1}{(D+2i)^2-4(D+2i)+5} x^2\right]$$

$$= \text{Re.}\left[e^{2ix} \frac{1}{D^2+i4D-4D+(1-8i)} x^2\right]$$

$$= \text{Re.}\left[e^{i2x} \frac{1}{(1-8i)\left(1+\frac{D^2+(4i-4)D}{1-8i}\right)} x^2\right]$$

$$= \text{Re.}\left[e^{2ix} \frac{1}{1-8i}\left(1-\frac{D^2+(4i-4)D}{1-8i}\right) x^2\right]$$

$$= \text{Re.}\left[e^{2ix} \frac{1}{1-8i}\left(x^2 - \frac{1}{1-8i}\left\{2+(4i-4)\cdot 2x\right\}\right)\right]$$

$$= \text{Re.}\left[(\cos 2x + i\sin 2x)\frac{(1+8i)}{65}\left(x^2 - \frac{(1+8i)}{65}\left\{2+8ix-8x\right\}\right)\right]$$

$$= \cos 2x\left[\frac{x^2}{65} + \frac{536x-126}{65^2}\right] - \sin 2x\left[\frac{8x^2}{65} + \frac{32+652x}{65^2}\right]$$