

Mathematical Logic (II)

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1 The Syntax of First-order Logic

Example 1.1 (Group Theory).

(G1) For all x, y, z we have $(x \circ y) \circ z = x \circ (y \circ z)$.

(G2) For all x we have $x \circ e = e$.

(G3) For every x there is a y such that $x \circ y = e$.

A group is a triple $\mathfrak{G} = (G, \circ^{\mathfrak{G}}, e^{\mathfrak{G}})$, i.e., a structure \mathfrak{G} , which satisfies (G1)–(G3). \dashv

Example 1.2 (Equivalence Relations).

(E1) For all x we have $(x, x) \in R$.

(E2) For all x and y if $(x, y) \in R$ then $(y, x) \in R$.

(E3) For all x, y, z if $(x, y) \in R$ and $(y, z) \in R$ then $(x, z) \in R$.

An equivalence relation is specified by a structure $\mathfrak{A} = (A, R^{\mathfrak{A}})$ in which $R^{\mathfrak{A}}$ satisfies (E1)–(E3). \dashv

1.1 Alphabets

Definition 1.3. An **alphabet** is a nonempty set of **symbols**. \dashv

Definition 1.4. Let \mathbb{A} be an alphabet. Then a **word** w over \mathbb{A} is a finite sequence of symbols in \mathbb{A} , i.e.,

$$w = w_1 w_2 \cdots w_n$$

where $n \in \mathbb{N}$ and $w_i \in \mathbb{A}$ for every $i \in [n] = \{1, \dots, n\}$. In case $n = 0$, then w is the **empty word**, denoted by ε . The **length** $|w|$ of w is n . In particular, $|\varepsilon| = 0$.

\mathbb{A}^* denotes the set of all words over \mathbb{A} , or equivalently

$$\mathbb{A}^* = \bigcup_{n \in \mathbb{N}} \mathbb{A}^n = \bigcup_{n \in \mathbb{N}} \{w_1 \dots w_n \mid w_1, \dots, w_n \in \mathbb{A}\}. \quad \dashv$$

Countable sets

Later on, we will need to count the number of words over a given alphabet.

Definition 1.5. A set M is **countable** if there exists an **injective** function α from \mathbb{N} **onto** M , i.e., $\alpha : \mathbb{N} \rightarrow M$ is a bijection. Thereby, we can write

$$M = \{\alpha(n) \mid n \in \mathbb{N}\} = \{\alpha(0), \alpha(1), \dots, \alpha(n), \dots\}.$$

A set M is **at most countable** if M is either finite or countable. \dashv

Lemma 1.6. Let M be a non-empty set. Then the following are equivalent.

(a) M is at most countable.

(b) There is a surjective function $f : \mathbb{N} \rightarrow M$.

(c) There is an injective function $f : M \rightarrow \mathbb{N}$. \dashv

Lemma 1.7. Let \mathbb{A} be an alphabet which is at most countable. Then \mathbb{A}^* is countable. \dashv

1.2 The alphabet of a first-order language

Definition 1.8. The **alphabet of a first-order language** consists of the following symbols.

(a) v_0, v_1, \dots (variables).

(b) $\neg, \wedge, \vee, \rightarrow, \leftrightarrow$, (negation, conjunction, disjunction, implication, if and only if).

(c) \forall, \exists , (for all, exists).

(d) \equiv , (equality).

(e) $(,)$, (parentheses).

(f) (1) For every $n \geq 1$ a set of **n -ary relation symbols**.

(2) For every $n \geq 1$ a set of **n -ary function symbols**.

(3) A set of **constants**.

Note any set in (f) can be empty. \dashv

We use \mathbb{A} to denote the set of symbols in (a)–(e), i.e., the set of **logic symbols**, while S is the set of remaining symbols in (f). Then a first-order language has

$$\mathbb{A}_S := \mathbb{A} \cup S$$

as its alphabet and S as its **symbol set**.

Thus every first-order language has the same set \mathbb{A} of logic symbols but might have different symbol set S .

1.3 Terms and formulas

Throughout this section, we fix a symbol set S .

Definition 1.9. The set T^S of **S -terms** contains precisely those words in \mathbb{A}_S^* which can be obtained by applying the following rules finitely many times.

(T1) Every variable is an S -term.

(T2) Every constant in S is an S -term.

(T3) If t_1, \dots, t_n are S -terms and f is a n -ary function symbol in S , then $ft_1 \dots t_n$ is an S -term. \dashv

Definition 1.10. The set L^S of **S -formulas** contains precisely those words in \mathbb{A}_S^* which can be obtained by applying the following rules finitely many times.

(A1) Let t_1 and t_2 be two S -terms. Then $t_1 \equiv t_2$ is an S -formula.

(A2) Let t_1, \dots, t_n be S -terms and R an n -ary relation symbol in S . Then $Rt_1 \dots t_n$ is also an S -formula.

(A3) If φ is an S -formula, then so is $\neg\varphi$.

(A4) If φ and ψ are S -formulas, then so is $(\varphi * \psi)$ where $*$ $\in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$.

(A5) Let φ be an S-formula and x a variable. Then $\forall x\varphi$ and $\exists x\varphi$ are S-formulas, too.

The formulas in (A1) and (A2) are **atomic**, as they don't contain any other S-formulas as subformulas.

- $\neg\varphi$ is the **negation** of φ .
- $(\varphi \wedge \psi)$ is the **conjunction** of φ and ψ .
- $(\varphi \vee \psi)$ is the **disjunction** of φ and ψ .
- $(\varphi \rightarrow \psi)$ is the **implication** from φ to ψ .
- $(\varphi \leftrightarrow \psi)$ is the **equivalence** between φ and ψ . ⊢

Lemma 1.11. Let S be at most countable. Then both T^S and L^S are countable.

Definition 1.12. Let t be an S-term. Then $\text{var}(t)$ is the set of variables in t . Or inductively,

$$\begin{aligned}\text{var}(x) &:= \{x\}, \\ \text{var}(c) &:= \emptyset, \\ \text{var}(ft_1 \dots t_n) &:= \bigcup_{i \in [n]} \text{var}(t_i).\end{aligned}$$
⊢

Definition 1.13. Let φ be an S-formula and x a variable. We say that **an occurrence of x in φ is free** if it is not in the scope of any $\forall x$ or $\exists x$. Otherwise, the occurrence is **bound**.

Definition 1.14. Let φ be an S-formula. Then $\text{free}(\varphi)$ is the set of variables which have free occurrences in φ . Or inductively,

$$\begin{aligned}\text{free}(t_1 \equiv t_2) &:= \text{var}(t_1) \cup \text{var}(t_2), \\ \text{free}(Rt_1 \dots t_n) &:= \bigcup_{i \in [n]} \text{var}(t_i), \\ \text{free}(\neg\varphi) &:= \text{free}(\varphi), \\ \text{free}(\varphi * \psi) &:= \text{free}(\varphi) \cup \text{free}(\psi) \quad \text{with } * \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}, \\ \text{free}(\forall x\varphi) &:= \text{free}(\varphi) \setminus \{x\}, \\ \text{free}(\exists x\varphi) &:= \text{free}(\varphi) \setminus \{x\}.\end{aligned}$$
⊢

Example 1.15. The formula below shows that a variable might have both free and bound occurrences in the same formula.

$$\begin{aligned}\text{free}((Rxy \rightarrow \forall y \neg y \equiv z)) &= \text{free}(Rxy) \cup \text{free}(\forall y \neg y \equiv z) \\ &= \{x, y\} \cup (\text{free}(y \equiv z) \setminus \{y\}) = \{x, y, z\}.\end{aligned}$$
⊢

Definition 1.16. An S-formula is an **S-sentence** if $\text{free}(\varphi) = \emptyset$. ⊢

Recall that **actual** variables we can use are v_0, v_1, \dots

Definition 1.17. Let $n \in \mathbb{N}$. Then

$$L_n^S := \{\varphi \mid \varphi \text{ an S-formula with } \text{free}(\varphi) \subseteq \{v_0, \dots, v_{n-1}\}\}.$$

In particular, L_0^S is the set of S-sentences. ⊢

2 The Semantics of First-order Logic

2.1 Structures and interpretations

We fix a symbol set S .

S 结构

Definition 2.1. An S -**structure** is a pair $\mathfrak{A} = (A, \alpha)$ which satisfies the following conditions.

1. $A \neq \emptyset$ is the **universe** of \mathfrak{A} . 比如我们考虑 \mathbb{N}^2 , $S = \{0, 1, +, *, <\}$
2. α is a function defined on S such that: 则这里的 α 干了什么? 把语法上的符号解释成语义上的一个符号。
 - (a) Let $R \in S$ be an n -ary relation symbol. Then $\alpha(R) \subseteq A^n$. 例子: $\{(a, b) \mid a < b, a, b \in \mathbb{N}\} = \alpha(<) \subseteq \mathbb{N}^2$
 - (b) Let $f \in S$ be an n -ary function symbol. Then $\alpha(f) : A^n \rightarrow A$. 例子: $x + (\text{语义}) y = \alpha(+(\text{语法}))(x, y) \in \mathbb{N}$
 - (c) $\alpha(c) \in A$ for every constant $c \in S$. 我甚至可以把小于号定义成大于, 这也是合理的

For better readability, we write $R^\mathfrak{A}$, $f^\mathfrak{A}$, and $c^\mathfrak{A}$, or even R^A , f^A , and c^A , instead of $\alpha(R)$, $\alpha(f)$, and $\alpha(c)$. Thus for $S = \{R, f, c\}$ we might write an S -structure as 这样写就相当于是规定了 S 中每个符号的数学对象

$$\mathfrak{A} = (A, R^\mathfrak{A}, f^\mathfrak{A}, c^\mathfrak{A}) = (A, R^A, f^A, c^A). \quad \dashv$$

Examples 2.2. 1. For $S_{Ar} := \{+, \cdot, 0, 1\}$ the S_{Ar} -structure

$$\mathfrak{N} = (\mathbb{N}, +^\mathbb{N}, \cdot^\mathbb{N}, 0^\mathbb{N}, 1^\mathbb{N})$$

is the standard model of natural numbers with addition, multiplication, and constants 0 and 1.

2. For $S_{Ar}^< := \{+, \cdot, 0, 1, <\}$ we have an $S_{Ar}^<$ -structure

$$\mathfrak{N}^< = (\mathbb{N}, +^\mathbb{N}, \cdot^\mathbb{N}, 0^\mathbb{N}, 1^\mathbb{N}, <^\mathbb{N}),$$

i.e., the standard model of \mathbb{N} with the natural ordering $<$. \dashv

Definition 2.3. An **assignment** in an S -structure \mathfrak{A} is a mapping 变元通过 Assignment 定义

$$\beta : \{v_i \mid i \in \mathbb{N}\} \rightarrow A. \quad \dashv$$

Definition 2.4. An S -**interpretation** \mathcal{I} is a pair (\mathfrak{A}, β) where \mathfrak{A} is an S -structure and β is an assignment in \mathfrak{A} . 解释——顾名思义, 能够 from what defined in S -structure and assignment to yield 很多东西

Definition 2.5. Let β be an assignment in \mathfrak{A} , $a \in A$, and x a variable. Then $\beta \frac{a}{x}$ is the **assignment** defined by

$$\beta \frac{a}{x}(y) := \begin{cases} a, & \text{if } y = x, \\ \beta(y), & \text{otherwise.} \end{cases} \quad \text{相当于是对 Assignment 做一个更新, 让 } x = a$$

Then, for the S -interpretation $\mathcal{I} = (\mathfrak{A}, \beta)$ we use $\mathcal{I} \frac{a}{x}$ to denote the S -interpretation $(\mathfrak{A}, \beta \frac{a}{x})$. \dashv

2.2 The satisfaction relation $\mathcal{I} \models \varphi$

We fix an S -interpretation $\mathcal{I} = (\mathfrak{A}, \beta)$.

Definition 2.6. For every S -term t we define its **interpretation** $\mathcal{I}(t)$ by induction on the construction of t .

- (a) $\mathcal{I}(x) = \beta(x)$ for a variable x .

(b) $\mathcal{I}(c) = c^{\mathfrak{A}}$ for a constant $c \in S$.

(c) Let $f \in S$ be an n -ary function symbol and t_1, \dots, t_n S -terms. Then

$$\mathcal{I}(ft_1 \cdots t_n) = f^{\mathfrak{A}}(\mathcal{I}(t_1), \dots, \mathcal{I}(t_n)). \quad \dashv$$

Example 2.7. Let $S := S_{Gr} = \{\circ, e\}$ and $\mathcal{I} := (\mathfrak{A}, \beta)$ with $\mathfrak{A} = (\mathbb{R}, +, 0)$, $\beta(v_0) = 2$, and $\beta(v_2) = 6$. Then

$$\begin{aligned} \mathcal{I}(v_0 \circ (e \circ v_2)) &= \mathcal{I}(v_0) + \mathcal{I}(e \circ v_2) \\ &= 2 + (\mathcal{I}(e) + \mathcal{I}(v_2)) = 2 + (0 + 6) = 2 + 6 = 8. \end{aligned} \quad \dashv$$

Definition 2.8. Let φ be an S -formula. We define $\mathcal{I} \models \varphi$ by induction on the construction of φ .

- (a) $\mathcal{I} \models t_1 \equiv t_2$ if $\mathcal{I}(t_1) = \mathcal{I}(t_2)$. \models 指的是在特定解释下, 如果一组前提都为真, 那么结论也一定为真。
- (b) $\mathcal{I} \models Rt_1 \cdots t_n$ if $(\mathcal{I}(t_1), \dots, \mathcal{I}(t_n)) \in R^{\mathfrak{A}}$.
- (c) $\mathcal{I} \models \neg \varphi$ if $\mathcal{I} \not\models \varphi$ (i.e., it is **not** the case that $\mathcal{I} \models \varphi$).
- (d) $\mathcal{I} \models (\varphi \wedge \psi)$ if $\mathcal{I} \models \varphi$ and $\mathcal{I} \models \psi$.
- (e) $\mathcal{I} \models (\varphi \vee \psi)$ if $\mathcal{I} \models \varphi$ or $\mathcal{I} \models \psi$.
- (f) $\mathcal{I} \models (\varphi \rightarrow \psi)$ if $\mathcal{I} \models \varphi$ implies $\mathcal{I} \models \psi$.
- (g) $\mathcal{I} \models (\varphi \leftrightarrow \psi)$ if $(\mathcal{I} \models \varphi$ if and only if $\mathcal{I} \models \psi)$.
- (h) $\mathcal{I} \models \forall x \varphi$ if for all $a \in A$ we have $\mathcal{I}_x^a \models \varphi$. 对于论域中的每一个元素 a , 将 x 解释为 a 后, 公式 φ 都为真。
- (i) $\mathcal{I} \models \exists x \varphi$ if for some $a \in A$ we have $\mathcal{I}_x^a \models \varphi$.

If $\mathcal{I} \models \varphi$, then \mathcal{I} is a **model** of φ , of \mathcal{I} **satisfies** φ .

Let Φ be a set of S -formulas. Then $\mathcal{I} \models \Phi$ if $\mathcal{I} \models \varphi$ for all $\varphi \in \Phi$. Similarly as above, we say that \mathcal{I} is a model of Φ , or \mathcal{I} satisfies Φ . \dashv

Example 2.9. Let $S := S_{Gr}$ and $\mathcal{I} := (\mathfrak{A}, \beta)$ with $\mathfrak{A} = (\mathbb{R}, +, 0)$ and $\beta(x) = 9$ for all variables x . Then

$$\begin{aligned} \mathcal{I} \models \forall v_0 v_0 \circ e \equiv v_0 &\iff \text{for all } r \in \mathbb{R} \text{ we have } \mathcal{I}_{v_0}^r \models v_0 \circ e \equiv v_0, \\ &\iff \text{for all } r \in \mathbb{R} \text{ we have } r + 0 = r. \end{aligned} \quad \dashv$$

Definition 2.10. Let Φ be a set of S -formulas and φ an S -formula. Then φ is a **consequence** of Φ , written $\Phi \models \varphi$, if for any interpretation \mathcal{I} it holds that $\mathcal{I} \models \Phi$ implies $\mathcal{I} \models \varphi$.

For simplicity, in case $\Phi = \{\psi\}$ we write $\psi \models \varphi$ instead of $\{\psi\} \models \varphi$. \dashv

Example 2.11. Let Φ 是一组公理, 在这些数学性质下, 我们推导一些定理

$$\begin{aligned} \Phi_{Gr} := \{ &\forall v_0 \forall v_1 \forall v_2 (v_0 \circ v_1) \circ v_2 \equiv v_0 \circ (v_1 \circ v_2), \\ &\forall v_0 v_0 \circ e \equiv v_0, \forall v_0 \exists v_1 v_0 \circ v_1 \equiv e \}. \end{aligned}$$

这是在说结合律, 有零元, 有逆元

Then it can be shown that

$$\Phi_{Gr} \models \forall v_0 e \circ v_0 \equiv v_0.$$

and

$$\Phi_{Gr} \models \forall v_0 \exists v_1 v_1 \circ v_0 \equiv e. \quad \dashv$$

Definition 2.12. An S -formula φ is **valid**, written $\models \varphi$, if $\emptyset \models \varphi$. Or equivalently, $\mathcal{I} \models \varphi$ for any \mathcal{I} . 没有解释符号, “对任意解释都成立”, e.g. $v_0 = v_0$. \dashv

E.g. Proof: $\varphi := \exists x(Rx \rightarrow \forall xRx)$ is valid
 \Leftrightarrow for some $a \in A$, $\mathcal{I}_a^x = Rx \rightarrow \forall xRx$
 $\Leftrightarrow \mathcal{I}_x^a = Rx$ implies $\mathcal{I}_x^a = \forall xRx$ for some $a \in A$
 $\Leftrightarrow \mathcal{I}_x^a \neq Rx$ or $\mathcal{I}_x^a = \forall xRx$ for some $a \in A$
 $\Leftrightarrow \mathcal{I}_x^a \neq Rx$ or $(\mathcal{I}_x^a)^b = Rx$ for some $a \in A$, for all $b \in A$
 $\Leftrightarrow \mathcal{I}_x^a \neq Rx$ or $\mathcal{I}_x^b = Rx$ for some $a \in A$, for all $b \in A$
 $\Leftrightarrow R = A$ or $R \neq A$

Definition 2.13. An S-formula φ is **satisfiable**, if there exists an S-interpretation \mathcal{I} with $\mathcal{I} \models \varphi$. A set Φ of S-formulas is satisfiable if there exists an S-interpretation \mathcal{I} such that $\mathcal{I} \models \varphi$ for every $\varphi \in \Phi$. 可满足的 \dashv

The next lemma is essentially the method of **proof by contradiction**.

Lemma 2.14. Let Φ be a set of S-formulas and φ an S-formula. Then $\Phi \models \varphi$ if and only if $\Phi \cup \{\neg\varphi\}$ is not satisfiable. 有点像反证法。如何证 $\Phi \rightarrow \varphi$? 假设 φ 不成立, 反证. \dashv

Proof:

$\Phi \models \varphi \iff$ Every model of Φ is a model of φ ,
 \iff there is no model \mathcal{I} with $\mathcal{I} \models \Phi$ and $\mathcal{I} \not\models \varphi$,
 \iff there is no model \mathcal{I} with $\mathcal{I} \models \Phi \cup \{\neg\varphi\}$,
 $\iff \Phi \cup \{\neg\varphi\}$ is not satisfiable. \square

Definition 2.15. Two S-formulas φ and ψ are **logic equivalent** if $\varphi \models \psi$ and $\psi \models \varphi$. \dashv

Example 2.16. Let φ be an S-formula. We define a logic equivalent φ^* which does not contain the logic symbols $\wedge, \rightarrow, \leftrightarrow, \forall$.

$\varphi^* := \varphi$ if φ is atomic,
 $(\neg\varphi)^* := \neg\varphi^*$,
 $(\varphi \wedge \psi)^* := \neg(\neg\varphi^* \vee \neg\psi^*)$, de Morgan
 $(\varphi \vee \psi)^* := (\varphi^* \vee \psi^*)$,
 $(\varphi \rightarrow \psi)^* := (\neg\varphi^* \vee \psi^*)$,
 $(\varphi \leftrightarrow \psi)^* := \neg(\varphi^* \vee \psi^*) \vee \neg(\neg\varphi^* \vee \neg\psi^*)$,
 $(\forall x\varphi)^* := \neg\exists x\neg\varphi^*$,
 $(\exists x\varphi)^* := \exists x\varphi^*$.

Thus, it suffices to consider \neg, \vee, \exists as the only logic symbols in any given φ . \dashv