Markov chains

* $S = \{1, \ldots, n\}$ (state space)

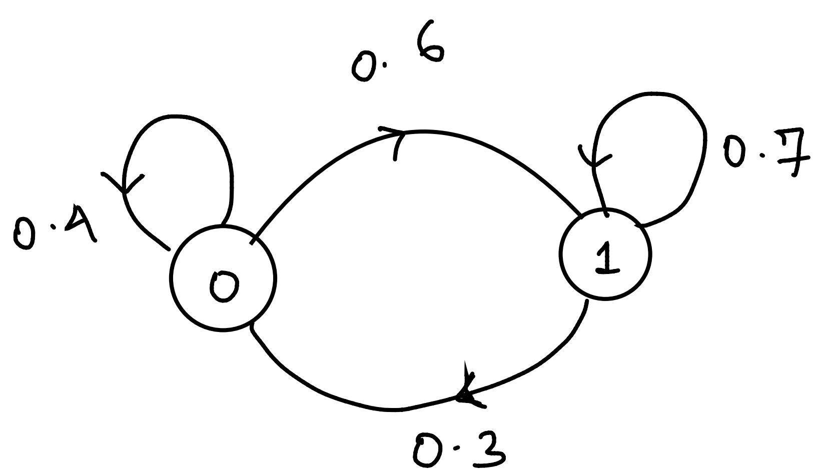
* $\{X_t\}_{t \geq 0}$ is a stochastic process (sequence of random variables)

  it takes values in S ($X_t \in S$)

* Toy Example. $S = \{0, 1\}$

$$\overset{0}{X_1}, \overset{1}{X_2}, \overset{0}{X_3}, \overset{0}{X_4}, \overset{1}{X_5}, \overset{0}{X_6}, \overset{0}{X_7}, \overset{1}{X_8}, \overset{1}{X_9}, \overset{1}{X_{10}}$$

* Probability Model



$$\text{Prob}(X_{t+1} = 0 \mid X_t = 0) = 0.4$$

sum to 1 $\left\{ \begin{array}{l} \text{Prob}(X_{t+1} = 0 \mid X_t = 0) = 0.4 \\ \text{Prob}(X_{t+1} = 1 \mid X_t = 0) = 0.6 \end{array} \right.$

sum to 1 $\left\{ \begin{array}{l} \text{Prob}(X_{t+1} = 0 \mid X_t = 1) = 0.3 \\ \text{Prob}(X_{t+1} = 1 \mid X_t = 1) = 0.7 \end{array} \right.$

✯ Markov Property

Given the current, the future does not depend on the past

$$\text{Prob}(\underbrace{X_{t+1}=j}_{\text{future}} \mid \underbrace{X_t=i}_{\text{current}}, \underbrace{X_{t-1}=k, \ldots, X_0=l}_{\text{past}}) = \text{Prob}(X_{t+1}=j \mid X_t=i)$$

✯ Probability Transition Matrix (one step transition matrix)

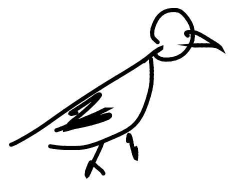
to j

$$\underset{\text{from } i}{P(i, j)} = \text{Prob}(X_{t+1}=j \mid X_t=i)$$

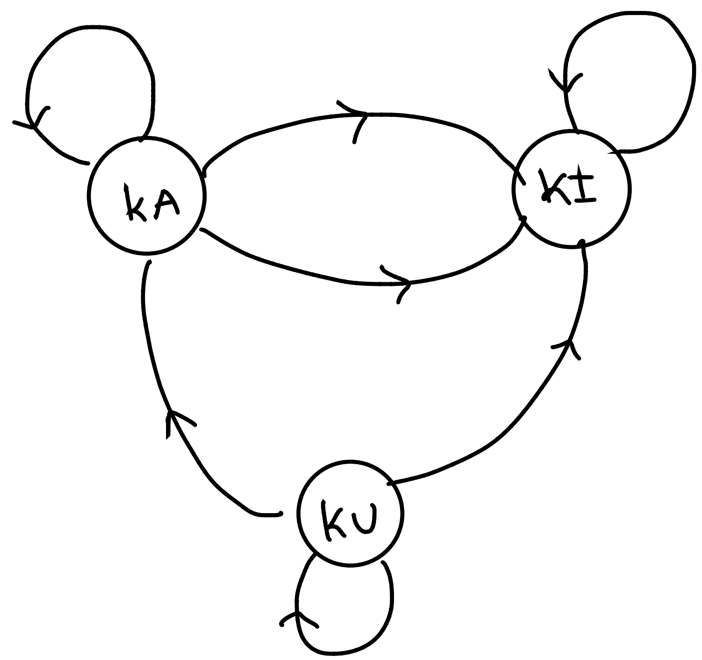
$$P = \begin{array}{c} \\ 0 \\ 1 \end{array}\begin{array}{c} \begin{array}{cc} 0 & 1 \end{array} \\ \begin{bmatrix} 0.4 & 0.6 \\ 0.3 & 0.7 \end{bmatrix} \end{array}$$

$\longrightarrow$ row sum = 1

Example



$S = \{ KA, KI, KU \}$



$$P = \begin{array}{c c} & \begin{array}{ccc} KA & KI & KU \end{array} \\ \begin{array}{c} KA \\ KI \\ KU \end{array} & \begin{bmatrix} 0.3 & 0.4 & 0.3 \\ 0.4 & 0.3 & 0.3 \\ 0.3 & 0.3 & 0.4 \end{bmatrix} \end{array}$$

$$\begin{bmatrix} 0.1 & 0.1 & 0.8 \\ 0.1 & 0.1 & 0.8 \\ 0.1 & 0.1 & 0.8 \end{bmatrix}$$

$X_0 = KA$

$X_1 \sim P(\cdot, KA)$

$X_2 \sim P(\cdot, KA)$

$X_3 \sim P(\cdot, KU)$

Day	$X_1^{(i)}$	$X_2^{(i)}$	$X_3^{(i)}$	$X_4^{(i)}$	$X_5^{(i)}$
1	KA	KU	KI		
2	KI				

$\{X_{1:100}^{(i)}\}_{i=1,\dots,10}$

Table is given how will you recover P

Partial ~~A~~ N = For X_1 see how many KA out of total examples

~~A~~

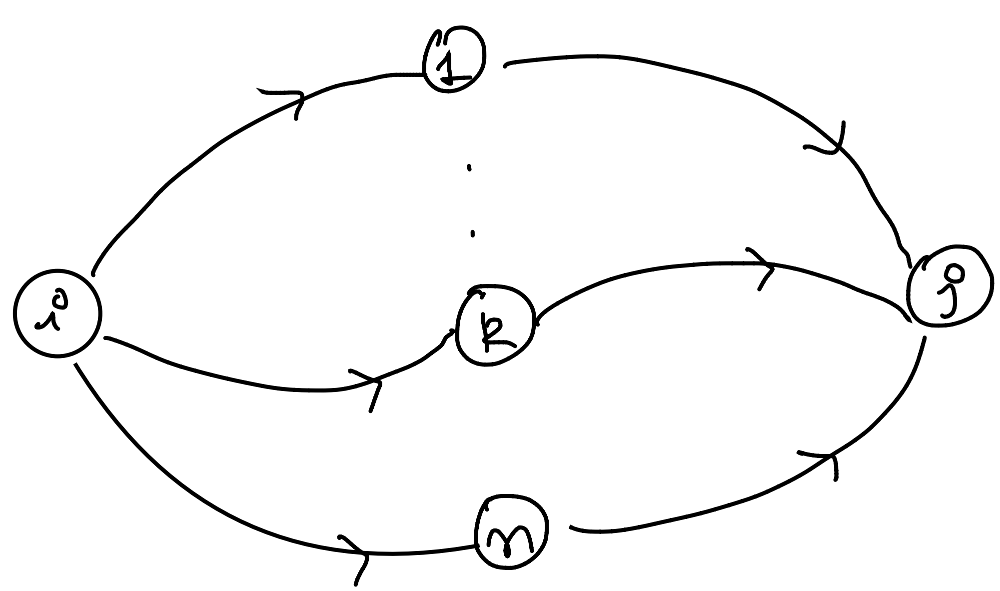
$$P(KA, KU) \approx \frac{N(KA, KU)}{\sum_{s \in S} N(KA, s)}$$

* Markov Property

Given the current, the future does not depend on the past

$$\text{Prob}(\underbrace{X_{t+2}}_{\text{future}} = j \mid \underbrace{X_t = i}_{\text{current}}, \underbrace{X_{t-1} = k, \ldots, X_0 = \ell}_{\text{past}}) = \text{Prob}(X_{t+2} = j \mid X_t = i)$$

Need to jump from i to j in 2 steps



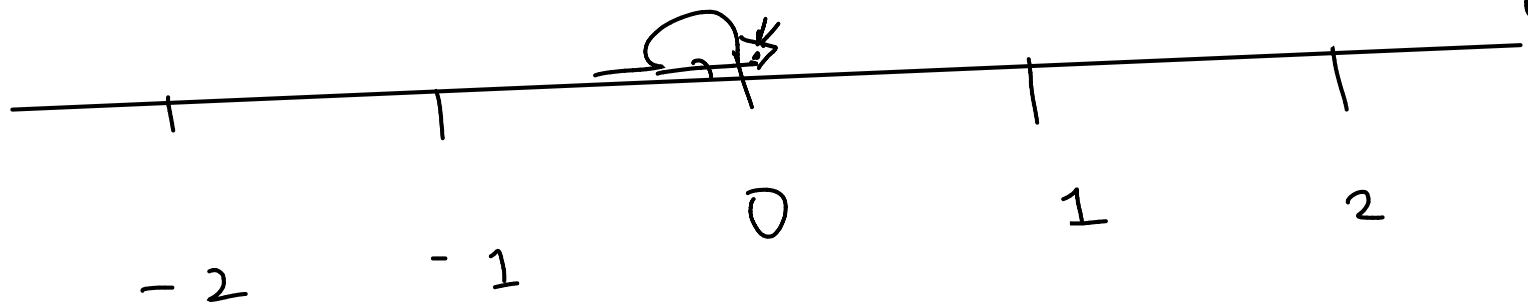
$$\text{Prob}(X_{t+2} = j \mid X_t = i) = \sum_{k=1}^{n} P(X_{t+1} = k \mid X_t = i)\ \text{Prob}(X_{t+2} = j \mid X_{t+1} = k, X_t = i)$$

$$= \sum_{k=1}^{n} P(X_{t+1} = k \mid X_t = i)\ \text{Prob}(X_{t+2} = j \mid X_{t+1} = k)$$

$$= \sum_{k=1}^{n} P(i,k)\ P(k,j) = P^2(i,j)$$

$$P \cdot P = \begin{array}{cc} & \\ i^{th} \text{ row} \rightarrow & \begin{bmatrix} P(i,1) \cdots P(i,k) \cdots P(i,n) \\ \end{bmatrix} \end{array} \begin{array}{c} j^{th} \text{ column} \\ \begin{bmatrix} P(1,j) \\ \vdots \\ P(k,j) \\ \vdots \\ P(n,j) \end{bmatrix} \end{array}$$

Meaning of P^t $\quad P^t(i,j)$ = I saw the mice in position i at $t=0$, I closed my eyes, I only reopen at t, where is the mice?



Mice move left w.p $= \frac{1}{2}$, right w.p $= \frac{1}{2}$

P

	-2	-1	0	1	2
-2	0.5	0.5	0	0	0
-1	0.5	0	0.5	0	0
0	0	0.5	0	0.5	0
1	0	0	0.5	0	0.5
2	0	0	0	0.5	0.5

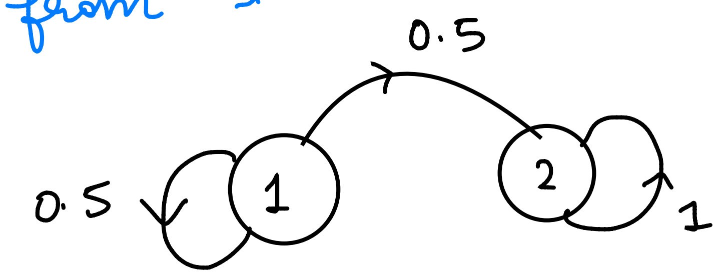
P^2

	-2	-1	0	1	2
-2					
-1					
0	0.25	0	0.25 + 0.25	0	0.25
1					
2					

★ A state j is accessible from state i if $i \longmapsto j$

$$P^t(i,j) > 0 \quad \text{for some } t \geq 0$$

can hop from i to j in some t steps.



state 2 is accessible from 1

state 1 is not accessible from 2

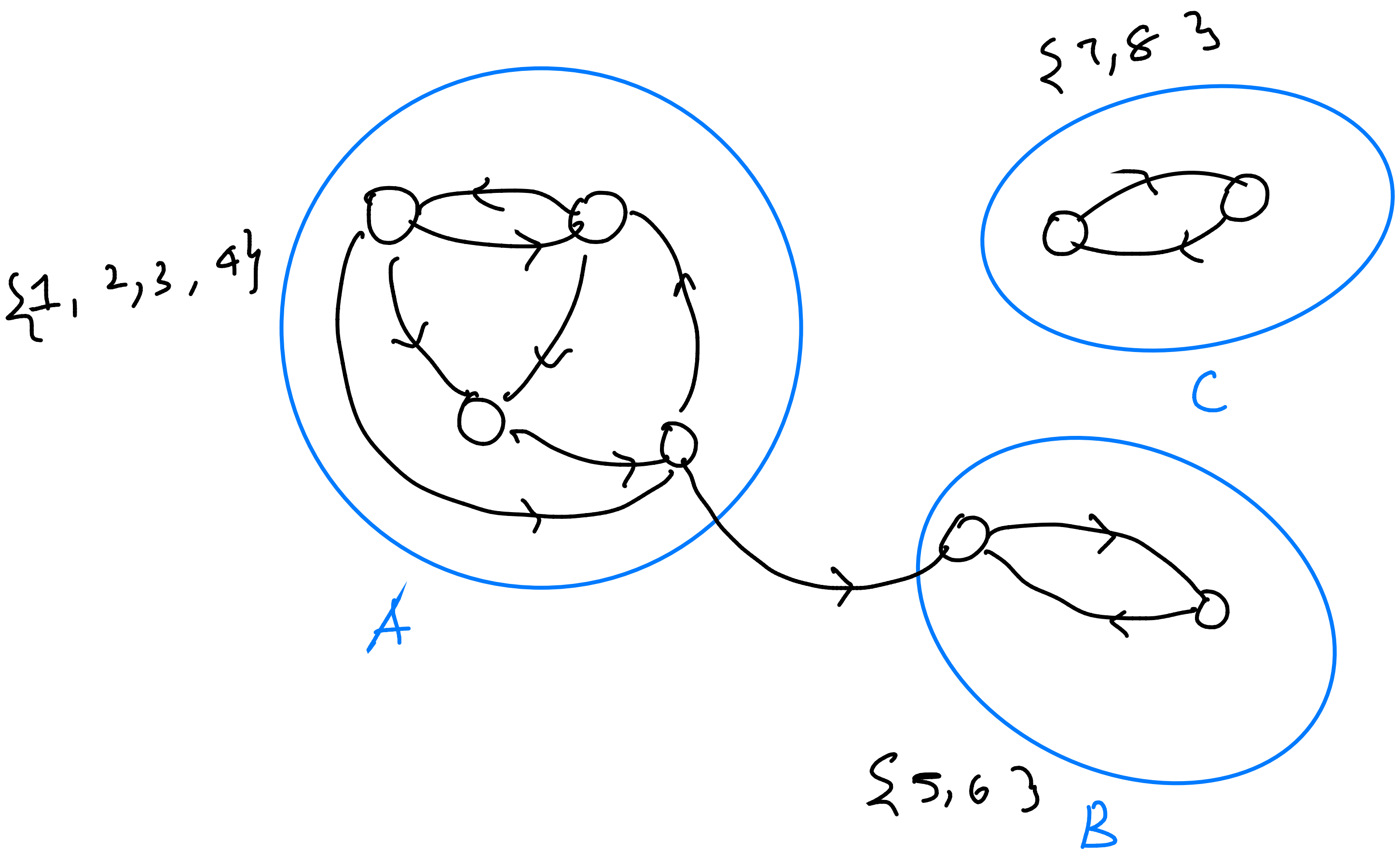
★ i and j are mutually communicating states if $i \longmapsto j$ and $j \longmapsto i$

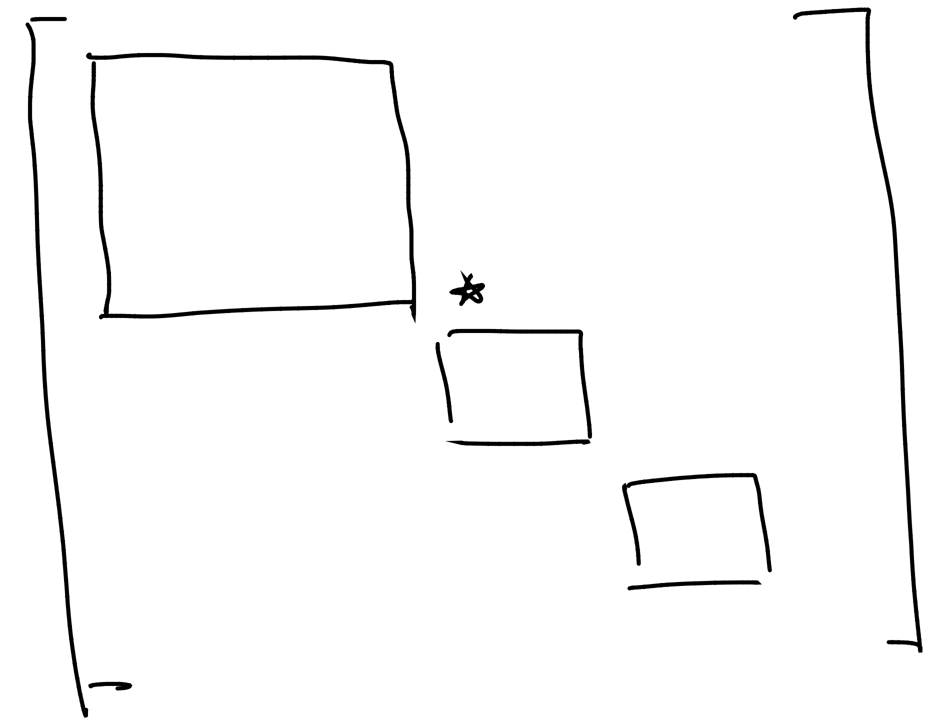
★ Reflexive : $i \longleftrightarrow i$

★ Symmetric : $i \longleftrightarrow j$

★ Transitive : $i \longleftrightarrow j$, $j \longleftrightarrow k$, $i \longleftrightarrow k$

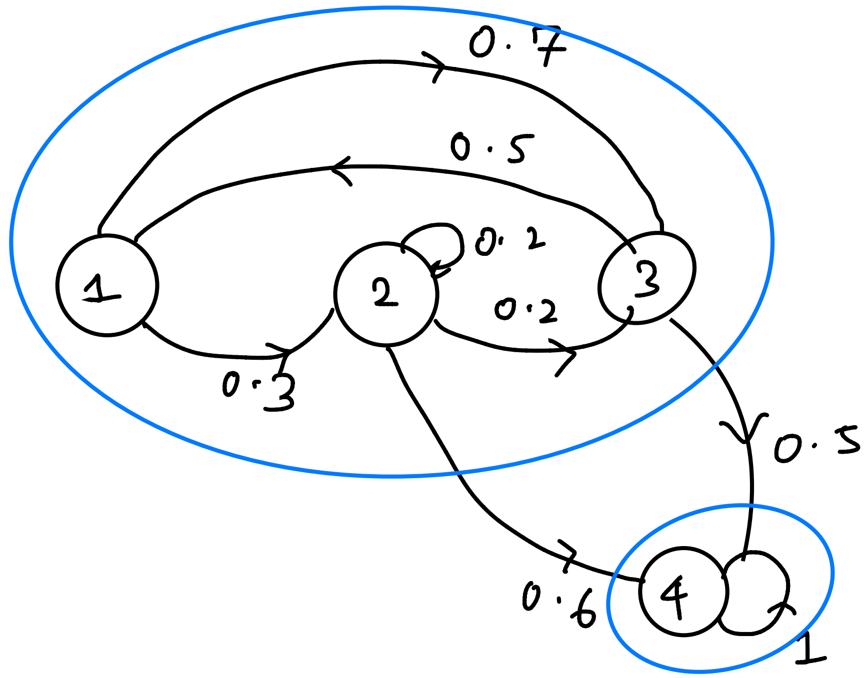
* Communicating classes / Groups





* Irreducible Markov chain :

All states belong to same communicating class



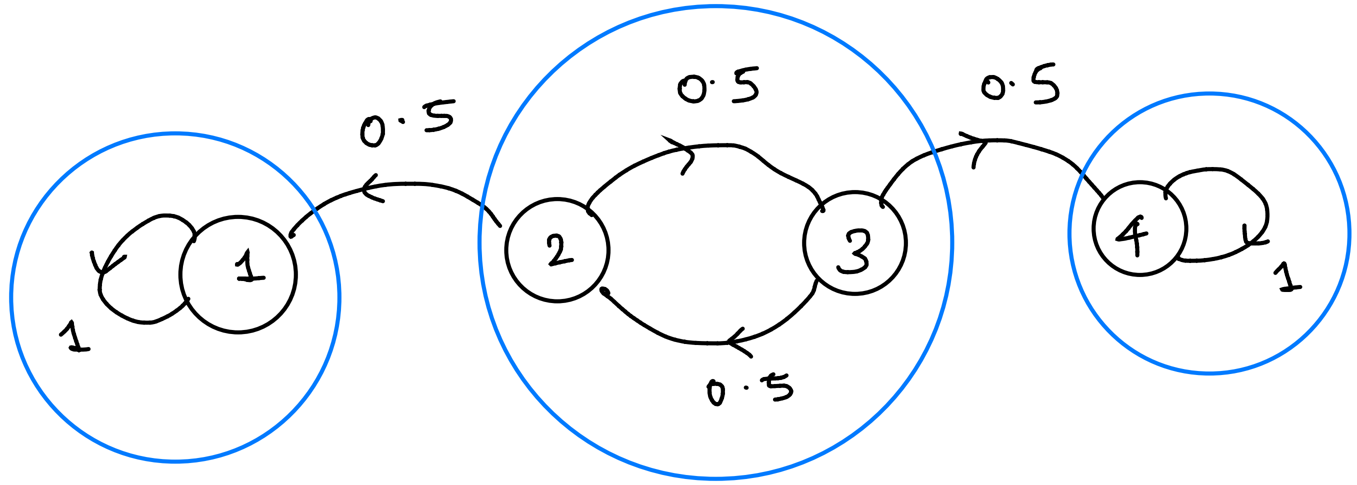
$$\begin{bmatrix} 0 & 0.3 & 0.7 & 0 \\ 0 & 0.2 & 0.2 & 0.6 \\ 0.5 & 0 & 0 & 0.5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

★ Recurrent and Transient State

If a state i is accessible from all other states then it is recurrent

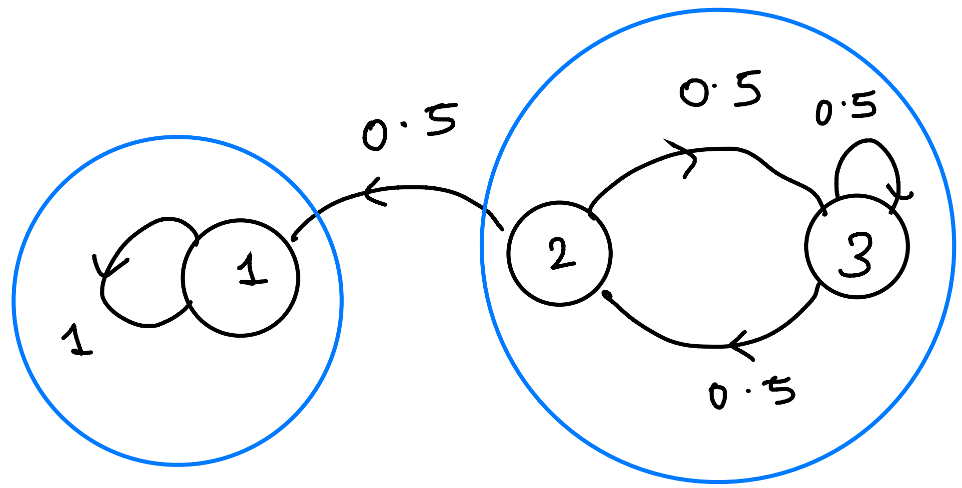
A state is transient if it is not recurrent

Example 1:



No state is recurrent

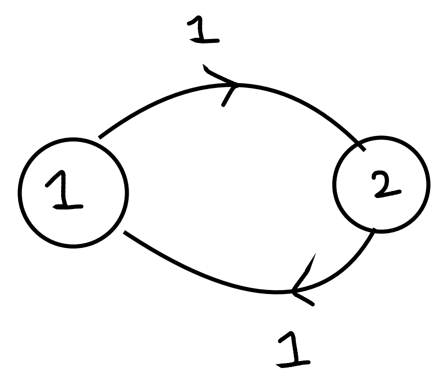
Example 2



1 is recurrent , {2,3} are transient

* Period of state i the greatest common divisor of

$$\{ t \geq 1 : P_{ii}^{t} > 0 \}$$

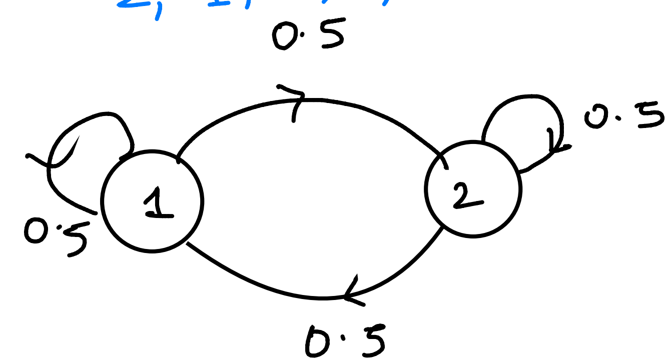


Period is 2.

2, 1, 2

2, 1, 1, 2

2, 1, 1, 1, 2



Period of both states is 1

= Aperiodic

* States of same class

  * same period

  * same recurrence / transience

* Ergodic class = Recurrent + Aperiodic

  Ergodic chain = Irreducible + Recurrent + Aperiodic

★ Stationary Distribution :

- $\pi(i) \geq 0 \ , \ \forall i \in S$

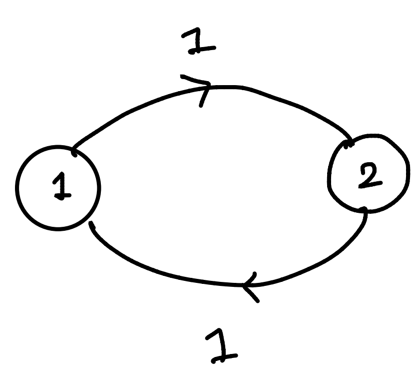
- $\sum_{i \in S} \pi(i) = 1$

- $\pi P = \pi$

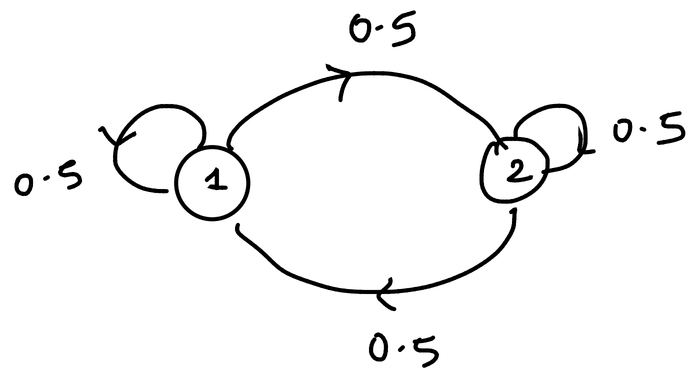
$\pi_t P = \pi_{t+1}$

$$[\pi(1) \cdots \pi(i) \cdots \pi(n)] \begin{array}{cl} \begin{bmatrix} p(1,1) \cdots p(1,j) \cdots p(1,n) \\ p(i,1) \cdots p(i,j) \cdots p(i,n) \\ p(n,1) \cdots \cdots \cdots p(n,n) \end{bmatrix} & \begin{array}{l} \leftarrow 1^{st} \\ \leftarrow i^{th} \text{ state} \\ \\ \end{array} \end{array}$$

★



$$[0.5 \quad 0.5] \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = [0.5 \quad 0.5]$$



* If we an ergodic chain

$$\mu_0 P^t \longrightarrow \pi \quad , \text{ where } \pi P = \pi$$