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CS22E005
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CS 6015: Linear Algebra and Random Processes
Assignment: 01
Course Instructor : Dr. Mitesh Khapra

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1. (1 point) Have you read and understood the honor code?

Solution: Yes.

Eigenstory: Special Properties

2. (1 point) Prove that for any square matrix A the eigenvectors corresponding to distinct eigenvalues are always independent.

Solution: Let, A be $k \times k$ matrix, $\{\lambda_1, \dots, \lambda_m\}$ $(m \leqslant k)$ be the eigenvalues of A, $\{\mathbf{x}_1, \dots, \mathbf{x}_m\}$ be the corresponding eigenvectors.
When $m = k$, there exists k distinct eigenvalues and k linearly independent eigenvectors, as it spans the k-dimensional column vector.
Hence, Proved.

3. (2 points) Prove the following.

   (a) The sum of the eigenvalues of a matrix is equal to its trace.

Solution: The Characteristic Equation of an $n \times n$ matrix A is given by:

$$p(t) = \det(A - tI) = (-1)^n\{t^n - (tr(A))t^{n-1} + \dots + (-1)^n \det(A)\}$$

On the other hand,

$$p(t) = (-1)^n (t - \lambda_1) \dots (t - \lambda_n),$$

when the λ_j are eigenvalues of A. So, by comparing coefficients,

$$tr(A) = \lambda_1 + \lambda_2 + \dots + \lambda_n$$

(b) The product of the eigenvalues of a matrix is equal to its determinant.

Solution: We know that the determinant of a triangular matrix is the product of the diagonal elements. Therefore, given a matrix A , we can find P such that $P^{-1}AP$ is upper triangular with the eigenvalues of A on the diagonal. Thus $\det(P^{-1}AP)$ is the product of the eigenvalues. We also know that $\det(P^{-1}AP) = \det(P^{-1}PA) = \det(A)$. Thus, the determinant of A is the product of the eigenvalues.

4. (2 points) What is the relationship between the rank of a matrix and the number of non-zero eigenvalues? Explain your answer.

Solution: The claim that the rank of a matrix is equal to the number of non-zero eigenvalues will hold true only for diagonalizable matrices. If a matrix is $n \times n$, then diagonalizability is equivalent to having a set of n linearly independent eigenvectors, and those eigenvectors corresponding to non-zero eigenvalues will form a basis for the range of the matrix; hence rank is obtained (including multiplicities).

However, if you look at A^TA, then you can use the eigenvalues of that matrix to obtain the rank, regardless of what A is. This is because A^TA is symmetric, and thus must be diagonalizable, and furthermore one can show that $\mathrm{rank}(A^TA) = \mathrm{rank}(A)$.

5. (1 point) If A is a square symmetric matrix then prove that the number of positive pivots it has is the same as the number of positive eigenvalues it has.

Solution: Let us assume, Matrix A allows the symmetric factorization $A = LDU$. By the law of inertia, A has the same number of positive eigenvalues as D. But, the eigenvalues of D are just its diagonal entries (i.e. the pivots). Thus, the number of positive pivots matches the number of positive eigenvalues of A.

Eigenstory: Special Matrices

6. (2 points) Consider the matrix $R = I - 2\mathbf{u}\mathbf{u}^T$ where $\mathbf{u}$ is a unit vector $\in \mathbb{R}^n$.

(a) Show that R is symmetric and orthogonal. (How many independent vectors will R have?)

Solution:

$$
\begin{aligned}
R &= I - 2\mathbf{uu}^{\mathsf{T}} \\
R^{\mathsf{T}} &= I^{\mathsf{T}} - 2(\mathbf{uu}^{\mathsf{T}})^{\mathsf{T}} \\
&= I - 2\mathbf{uu}^{\mathbf{T}} \\
&= R \\
\therefore R^{\mathsf{T}} &= R \rightarrow R \text{ is Symmetric.}
\end{aligned}
$$

If orthogonal

$$R^{\mathsf{T}}R = I$$

$$
\begin{aligned}
&\therefore (I - \mathbf{uu}^{\mathbf{T}})^{\mathsf{T}}(I - 2\mathbf{uu}^{\mathbf{T}}) \\
&\quad = I \cdot I - 2\mathbf{uu}^{\mathsf{T}} - 2uu^{\mathsf{T}} + 4(\mathbf{uu}^{\mathbf{T}})^{\mathsf{T}}(\mathbf{uu}^{\mathbf{T}}) \\
&\quad = I - 4\mathbf{uu}^{\mathbf{T}} - 4 \cdot \mathbf{u}\underline{\mathbf{u}^{\mathbf{T}}\mathbf{u}}\mathbf{u}^{\mathbf{T}} \qquad [\because \mathbf{u}^{\mathbf{T}}\mathbf{u} = \mathbf{1}] \\
&\quad = I
\end{aligned}
$$

Thus, R is also orthogonal.
Number of independent vectors of R = n

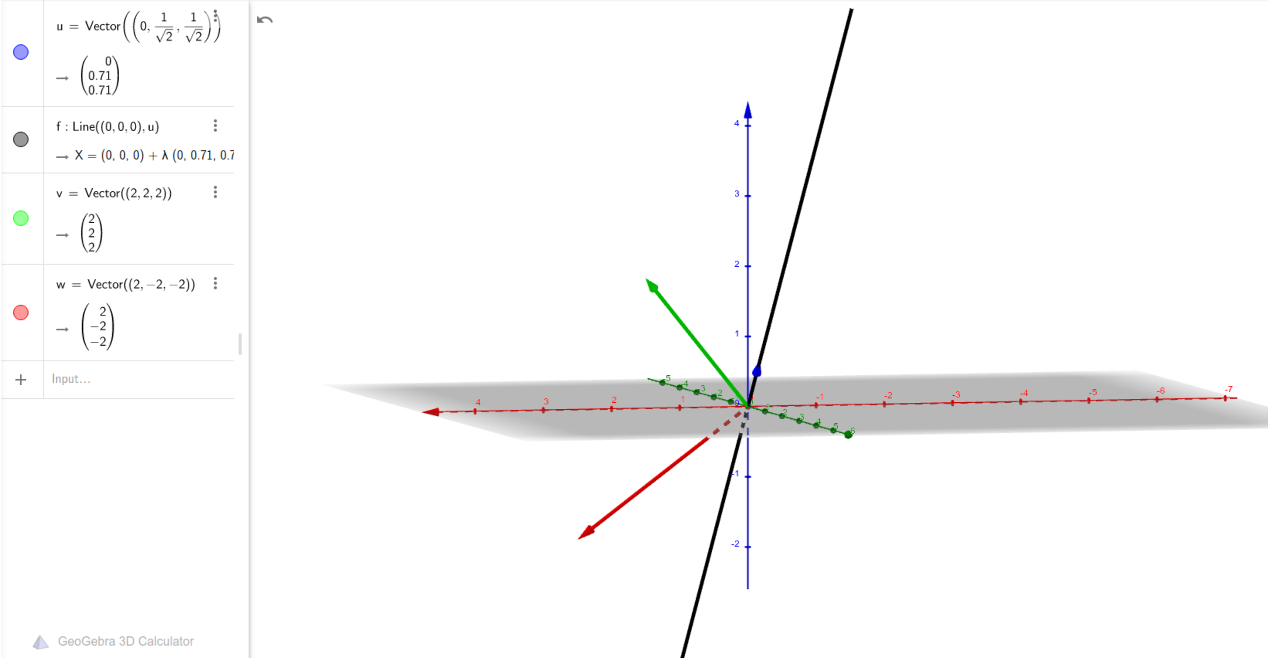
(b) Let $\mathbf{u} = \frac{1}{\sqrt{2}}\begin{bmatrix}0\\1\\1\end{bmatrix}$. Draw the line passing through this vector in geogebra (or any tool of your choice). Now take any vector in $\mathbf{R}^3$ and multiply it with the matrix R (i.e., the matrix R as defined above with $\mathbf{u} = \frac{1}{\sqrt{2}}\begin{bmatrix}0\\1\\1\end{bmatrix}$). What do you observe or what do you think the matrix R does or what would you call matrix R? (Hint: the name starts with R)

Solution:

$$\mathbf{u} = \frac{1}{\sqrt{2}}\begin{bmatrix}0\\1\\1\end{bmatrix}$$

$$
\begin{aligned}
R &= I - 2\mathbf{uu}^{\mathbf{T}} \\
&= \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} - 2 \cdot \frac{1}{\sqrt{2}}\begin{bmatrix}0\\1\\1\end{bmatrix} \cdot \frac{1}{\sqrt{2}}\begin{bmatrix}0&1&1\end{bmatrix} \\
&= \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} - \begin{bmatrix}0&0&0\\0&1&1\\0&1&1\end{bmatrix} = \begin{bmatrix}1&0&0\\0&0&-1\\0&-1&0\end{bmatrix}
\end{aligned}
$$

Therefore, it can be observed that multiplying any vector in $\mathbb{R}^3$ with R rotates it counterclockwise by an angle of 90° about the x-Axis. Thus, R is a Rotation Matrix in 3D.



(c) Compute the eigenvalues and eigenvectors of the matrix R as defined above with $\mathbf{u} = \frac{1}{\sqrt{2}}\begin{bmatrix}0\\1\\1\end{bmatrix}$.

Solution:

$$R = \begin{bmatrix}1 & 0 & 0\\0 & 0 & -1\\0 & -1 & 0\end{bmatrix} \qquad R - \lambda I = \begin{bmatrix}1-\lambda & 0 & 0\\0 & -\lambda & -1\\0 & -1 & -\lambda\end{bmatrix}$$

$$\begin{aligned}\therefore \det(R - \lambda I) &= (1-\lambda)(\lambda^2 - 1) = 0\\ \implies (1-\lambda)(\lambda-1)(\lambda+1) &= 0\\ \implies \lambda &= 1, -1, 1\end{aligned}$$

When, $\lambda = 1$,

$$\begin{aligned}(R - I)\mathbf{x} &= 0\\ \implies \begin{bmatrix}0 & 0 & 0\\0 & -1 & -1\\0 & -1 & -1\end{bmatrix}\mathbf{x} &= 0\end{aligned}$$

$$\therefore \begin{bmatrix}1 & 0 & 0\\0 & 0 & -1\\0 & -1 & 0\end{bmatrix} \to \begin{bmatrix}1 & 0 & 0\\0 & 0 & -1\\0 & -1 & 0\end{bmatrix} \to \begin{bmatrix}1 & 0 & 0\\0 & 0 & -1\\0 & -1 & 0\end{bmatrix}$$

Here,

$$\mathbf{x}_1 \rightarrow \text{free} ; \mathbf{x}_2 \rightarrow \text{fixed} ; \mathbf{x}_3 \rightarrow \text{free}$$

Now,

$$\begin{aligned} \therefore -x_2 - x_3 &= 0; \text{ Let, } x_1 = 0, x_3 = 1 \\ \implies -x_2 &= 1 \\ \implies x_2 &= -1 \end{aligned} \qquad \therefore \mathbf{v}_1 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

$$\begin{aligned} \therefore -x_2 - x_3 &= 0; \text{ Let, } x_1 = 1, x_3 = 0 \\ \implies -x_2 &= 0 \end{aligned} \qquad \therefore \mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

When $\lambda = -1$,

$$(R + I)\mathbf{x} = 0$$

$$\implies \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \end{bmatrix} \mathbf{x} = 0$$

$$\therefore \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Here,

$$\mathbf{x}_1 \rightarrow \text{pivot} ; \mathbf{x}_2 \rightarrow \text{pivot} ; \mathbf{x}_3 \rightarrow \text{free}$$

Now,

$$\begin{aligned} \therefore 2x_1 &= 0 \\ x_2 - x_3 &= 0 \end{aligned} \qquad \therefore \mathbf{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

(d) I believe that irrespective of what $\mathbf{u}$ is any such matrix R will have the same eigenvalues as you obtained above (with one of the eigenvalues repeating). Can you reason why this is the case? (Hint: think about how we reasoned about the eigenvectors of the projection matrix P even without computing them.)

Solution: Observing the way R has been constructed, it is true that irrespective of $\mathbf{u}$, Matrix R will have the same eigenvalues with one of them repeating, because ultimately the vectors lie in the columnspace of $\mathbf{u}$.

7. (2 points) Let Q be a $n \times n$ real orthogonal matrix (i.e., all its elements are real and its columns

are orthonormal). State with reason whether the following statements are True or False (provide a proof if the statement is True and a counter-example if it is False).

(a) If λ is an eigenvalue of Q then $|\lambda| = 1$

Solution: True. Let, Q be an orthogonal matrix, with an eigenvalue λ and $\mathbf{x}$ be an eigenvector belonging to λ. Since, $\mathbf{x}$ is non-zero and Q is orthogonal, it follows:

$$\langle \mathbf{x}, \mathbf{x} \rangle = \langle \mathbf{Qx}, \mathbf{Qx} \rangle = \langle \lambda\mathbf{x}, \lambda\mathbf{x} \rangle = \lambda^2 \langle \mathbf{x}, \mathbf{x} \rangle$$

$$\because \langle \mathbf{x}, \mathbf{x} \rangle > 0$$
$$\implies \lambda^2 = 1$$
$$\therefore \lambda = |1|$$

(b) The eigenvectors of Q are orthogonal

Solution: False. Two Eigenvectors of the Identity matrix need not be orthogonal.

(c) Q is always diagonalizable.

Solution: False. Let, S be the eigenvectors of Q.

$$\begin{aligned} QS &= \begin{bmatrix} Q_1S_1 & Q_2S_2 & \dots & Q_nS_n \end{bmatrix} \\ &= \begin{bmatrix} \lambda_1S_1 & \lambda_2S_2 & \dots & \lambda_nS_n \end{bmatrix} \\ &= \begin{bmatrix} S_1 & S_2 & \dots & S_n \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & 0 & \dots & 0 \\ 0 & \lambda_2 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \lambda_n \end{bmatrix} \end{aligned}$$

$$\implies QS = S\lambda$$
$$\implies S^{-1}QS = \lambda$$

But, S might not be invertible as the eigenvectors may be dependent.
Hence, **not** diagonalisable.

8. (1.5 points) Any rank one matrix can be written as $\mathbf{uv}^T$.

(a) Prove that the eigenvalues of any rank one matrix are $\mathbf{v}^T u$ and 0.

Solution: Given, $A = \mathbf{u}\mathbf{v}^{\mathbf{T}}$.
$\because A$ is rank 1, and not a full rank matrix,
$\therefore 0$ is an eigenvalue of that matrix A.
Again,

$$A\mathbf{u} = (\mathbf{u}\mathbf{v}^{\mathbf{T}})\mathbf{u} = \mathbf{u}(\mathbf{v}^{\mathbf{T}}\mathbf{u})$$
$$A\mathbf{u} = (\mathbf{v}^{\mathbf{T}}\mathbf{u})\mathbf{u}$$

1 $\therefore \mathbf{v}^{\mathbf{T}}\mathbf{u}$ is another eigenvalue.
Hence, Proved.

(b) How many times does the value 0 repeat?

Solution: '0' will repeat $(n-1)$ times.

(c) What are the eigenvectors corresponding to these eigenvalues?

Solution: Let, x be a non-zero vector, such that:

$$A\mathbf{x} = \lambda\mathbf{x}$$
$$(\mathbf{u}\mathbf{v}^{\mathbf{T}})\mathbf{x} = \lambda\mathbf{x}$$
$$\mathbf{u}(\mathbf{v}^{\mathbf{T}}\mathbf{x}) = \lambda\mathbf{x}$$

9. (2 points) Consider a $n \times n$ Markov matrix.

(a) Prove that the dominant eigenvalue of a Markov matrix is 1.

Solution: Let, λ be an eigenvalue of A and $\mathbf{x}$ be the corresponding eigenvector.

$$A\mathbf{x} = \lambda\mathbf{x} \tag{1}$$

Let, k be such that $|x_j| \leqslant |x_k|, \forall\, 1 \leqslant j, k \leqslant n$.
Then equating the k_{th} component of each side of the above $eq^n(1)$ gives,

$$\Sigma_{j=1}^{n} a_{kj}x_j = \lambda x_k$$

Hence,

$$|\lambda x_k| = |\lambda| \cdot |x_k| = |\Sigma_{j=1}^{n} a_{kj}x_j| \leqslant \Sigma_{j=1}^{n} a_{kj}|x_j|$$
$$\leqslant \Sigma_{j=1}^{n} a_{kj}|x_k| = |x_k|$$

Hence, $\lambda \leqslant 1$
This proves that 1 is an eigenvalue and all other eigenvalues are less than 1. Hence, Proved!

(b) Consider any 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ such that $a + b = c + d$. Show that one of the eigenvalues of such a matrix is 1. (I hope you notice that a Markov matrix is a special case of such a matrix where $a + b = c + d = 1$.)

Solution: Let, $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $a + b = c + d$.

$$\therefore \lambda^2 - (a+d)\lambda + (ad - bc) = 0$$
$$\lambda = \frac{(a+d) \mp \sqrt{(a+d)^2 - 4ad + 4bc}}{2}$$
$$= \frac{(a+d) \mp \sqrt{(a-d)^2 + 4bc}}{2} = \frac{(a+d) \mp (b+c)}{2}$$
$$\therefore \lambda = \frac{a+b+c+d}{2} \text{ or } \frac{a+d-b-c}{2}$$

Now, Since its a markov matrix, from $\lambda = \frac{a+b+c+d}{2}$,

$$\therefore a + b = c + d = 1$$
$$\text{Thus, } \lambda = \frac{2(a+b)}{2} = 1$$

And, from $\lambda = \frac{a+d-b-c}{2}$, it can be inferred that other eigenvalues are less than 1.
Hence, Proved!

(c) Does the result extend to $n \times n$ matrices where the sum of the elements of a row is the same for all the n rows? (Explain with reason)

Solution: Yes, it holds true.
Because, one of the eigenvalues will have $\frac{\Sigma_{i=1}^{n} \mathbf{1}}{n} = 1$
Hence, the following holds true for any $n \times n$ Markov Matrix, following the above mentioned property.

(d) What is the corresponding Eigenvector?

Solution: For the given matrix, $a + b = c + d = 1$ and $\lambda = 1$

$$\begin{aligned}
\begin{bmatrix} a-1 & b \\ c & d-1 \end{bmatrix} \mathbf{x} &= 0 \\
\implies \begin{bmatrix} (a-1) & b \\ 0 & (d-1) - \frac{bc}{a-1} \end{bmatrix} \mathbf{x} &= 0 \\
\implies \begin{bmatrix} (a-1) & b \\ 0 & -c + \frac{bc}{b} \end{bmatrix} \mathbf{x} &= 0 \\
\implies \begin{bmatrix} (a-1) & b \\ 0 & 0 \end{bmatrix} \mathbf{x} &= 0 \\
\implies \begin{bmatrix} (a-1) & b \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} &= 0
\end{aligned}$$

Solving the $X_{nullspace}$,

$$\mathbf{x}_2 = c$$

$$\mathbf{x}_1(a-1) = -bc$$

$$\mathbf{x}_1 = -\frac{bc}{(a-1)} = \frac{bc}{b} = c$$

$$\therefore X_{nullspace} = c \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

and

$$\text{Eignevector} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Thus, for $n \times n$ matrix, we will have eigenvector $\mathbf{x}$ such that,

$$\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}_{nx1}$$

EigenStory: Special Relations

10. (4 points) For each of the statements below state True or False with reason.

(a) The eigenvalues of A^T are always the same as that of A.

Solution: True. We find the eigenvalues of a matrix by computing the characteristic polynomial; that is, we find $\det(A - \lambda I)$.

$$\begin{aligned}\det(A^T - \lambda I) &= \det(A^T - \lambda I^T) \\ &= \det((A - \lambda I)^T) \\ &= \det(A - \lambda I)\end{aligned}$$

Therefore, the characteristic polynomial equation of A^T is the same as that for A. So, they must have the same eigenvalues.

(b) The eigenvectors of A^T are always the same as that of A.

Solution: False. The matrix $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, and its transpose $A^T = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$, have only one Eigenvalue, namely 1. However, the Eigenvectors of A are of the form $\begin{bmatrix} c \\ 0 \end{bmatrix}$, whereas the eigenvectors of A^T are of the form $\begin{bmatrix} 0 \\ c \end{bmatrix}$.

(c) The eigenvalues of A^{-1} are always the reciprocal of the eigenvalues of A.

Solution: True.

$$\begin{aligned} & Av && = \lambda v \\ \implies & A^{-1}Av && = \lambda A^{-1} v \\ \implies & A^{-1} v && = \frac{1}{\lambda} v \end{aligned}$$

(d) The eigenvectors of A^{-1} are always the same as the eigenvectors of A.

Solution: True. Consider an invertible matrix A with eigenvalue λ and eigenvector $\mathbf{x}$. Then, by definition, we know that

$$A\mathbf{x} = \lambda \mathbf{x}$$

Now multiplying both sides by A^{-1}:

$$\begin{aligned} A^{-1}A\mathbf{x} &= A^{-1}\lambda\mathbf{x} \\ \mathbf{x} &= A^{-1}\lambda\mathbf{x} \\ \frac{1}{\lambda}\mathbf{x} &= A^{-1}\mathbf{x} \end{aligned}$$

Thus, $A^{-1}\mathbf{x} = \frac{1}{\lambda}\mathbf{x}$. This, shows that $\mathbf{x}$ is an eigenvector of A^{-1} with eigenvalue $\frac{1}{\lambda}$.

(e) If x is an eigenvector of A and B then it is also an eigenvector of both AB and BA, even if the eigenvalues of A and B corresponding to x are different.

Solution: True. If $\mathbf{v}$ is an eigenvector of AB for some nonzero λ, then $B\mathbf{v} \neq 0$ and

$$\lambda B\mathbf{v} = B(AB\mathbf{v}) = (BA)B\mathbf{v}$$

, so $B\mathbf{v}$ is an eigenvector for BA with the same eigenvalue.
If 0 is an eigenvalue of AB then

$$0 = \det(AB) = \det(A)\det(B) = \det(BA)$$

so 0 is also an eigenvalue of BA.

(f) If x is and eigenvector of A and B then it is also an eigenvector of $A + B$.

Solution: True.

$$\begin{aligned} &Ax = \lambda x, Bx = \mu x, x \neq 0 \\ \implies\ &(A + B)x = (\lambda + \mu)x \end{aligned}$$

That is, $\lambda + \mu$ is an eigenvalue of $A + B$ and the corresponding eigenvector is x.

(g) If λ is an eigenvalue of A then $\lambda + k$ is an eigenvalue of $A + kI$.

Solution: True. Let, λ be the Eigenvalue of A.

$$\begin{aligned} (A + kI)\mathbf{x} &= A\mathbf{x} + kI\mathbf{x} \\ &= \lambda\mathbf{x} + k\mathbf{x} \\ &= (\lambda + k)\mathbf{x} \end{aligned}$$

Thus, $(\lambda + k)$ is an eigenvalue of $(A + kI)$.

(h) The non-zero eigenvalues of AA^T and A^TA are equal.

Solution: True. Let, $\mu \neq 0$ be an eigenvalue of $A^T A$. Therefore,

$$\begin{aligned}
&\det(A^T A - \mu I) &&= 0\\
\implies &\det(I + (-1/\mu)A^T A) &&= 0\\
\implies &\det(I + A(-1/\mu)A^T) &&= 0\\
\implies &\det(AA^T - \mu I) &&= 0
\end{aligned}$$

Thus, $\mu \neq 0$ is an eigenvalue of AA^T.

Eigenstory: Change of basis

11. (2 points) Consider the following two basis. Basis 1: $\mathbf{u}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}$, $\mathbf{u}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}$, and Basis 2: $\mathbf{u}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}$, $\mathbf{u}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix}-1\\-1\end{bmatrix}$. Consider a vector $\mathbf{x} = \begin{bmatrix}a\\b\end{bmatrix}$ in Basis 1 (i.e., $\mathbf{x} = a\mathbf{u}_1 + b\mathbf{u}_2$). How would you represent it in Basis 2?

Solution: Given,
In Basis 1:

$$\mathbf{u}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}, \mathbf{u}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}$$

In Basis 2:

$$\mathbf{u}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}, \mathbf{u}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix}-1\\-1\end{bmatrix}$$

Let, $\mathbf{X} = \begin{bmatrix}a\\b\end{bmatrix}$ in Base 1.

$$\begin{aligned}
\therefore \mathbf{X_{Std}} &= \begin{bmatrix}\mathbf{u}_1 & \mathbf{u}_2\end{bmatrix}\begin{bmatrix}\mathbf{a}\\\mathbf{b}\end{bmatrix}\\
&= \begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\end{bmatrix}\mathbf{X}
\end{aligned}$$

$$\therefore \mathbf{X_{Base\ 2}} = \begin{bmatrix}\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\end{bmatrix}^{-1}\begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\end{bmatrix}\begin{bmatrix}a\\b\end{bmatrix}$$

12. (1 point) Let $\mathbf{u}$ and $\mathbf{v}$ be two vectors in the standard basis. Let $T(\mathbf{u})$ and $T(\mathbf{v})$ be the representation of these vectors in a different basis. Prove that $\mathbf{u} \cdot \mathbf{v} = T(\mathbf{u}) \cdot T(\mathbf{v})$ if and only if the basis represented by

T is an orthonormal basis (i.e., dot products are preserved only when the new basis is orthonormal).

Solution: In New Basis, let the points be $Q^{-1} \cdot \mathbf{u}$ and $Q^{-1} \cdot \mathbf{v}$.

$$\begin{aligned}
\langle Q^{-1} \cdot \mathbf{u}, Q^{-1} \cdot \rangle &= (Q^{-1} \cdot \mathbf{u})^{\mathsf{T}} (Q^{-1} \cdot \mathbf{v}) \\
&= (\mathbf{u}^{\mathsf{T}} (Q^{-1})^{\mathsf{T}}) \cdot (Q^{-1} \cdot \mathbf{v}) \\
&= \mathbf{u}^{\mathsf{T}} Q \cdot Q^{-1} \cdot \mathbf{v} \\
&= \mathbf{u}^{\mathbf{T}} \mathbf{v} \\
&= \langle \mathbf{u}, \mathbf{v} \rangle
\end{aligned}$$

Eigenstory: PCA and SVD

13. (1 point) How are PCA and SVD related? (no vague answers please, think and answer very precisely with mathematical reasoning)

Solution: Simply put, the PCA viewpoint requires that one compute the eigenvalues and eigenvectors of the covariance matrix, which is the product $\frac{\mathbf{1}}{\mathbf{n-1}}\mathbf{X}\mathbf{X}^{\mathbf{T}}$, where $\mathbf{X}$ is the data matrix. Since the covariance matrix is symmetric, the matrix is diagonalizable, and the eigenvectors can be normalized such that they are orthonormal:

$$\frac{1}{n-1}\mathbf{X}\mathbf{X}^{\top} = \frac{1}{n-1}\mathbf{W}\mathbf{D}\mathbf{W}^{\top}$$

On the other hand, applying SVD to the data matrix $\mathbf{X}$ as follows:

$$\mathbf{X} = \mathbf{U}\Sigma\mathbf{V}^{\top}$$

and attempting to construct the covariance matrix from this decomposition gives

$$\frac{1}{n-1}\mathbf{X}\mathbf{X}^{\top} = \frac{1}{n-1}(\mathbf{U}\Sigma\mathbf{V}^{\top})(\mathbf{U}\Sigma\mathbf{V}^{\top})^{\top} = \frac{1}{n-1}(\mathbf{U}\Sigma\mathbf{V}^{\top})(\mathbf{V}\Sigma\mathbf{U}^{\top})$$

and since $\mathbf{V}$ is an orthogonal matrix ($\mathbf{V}^{\mathbf{T}}\mathbf{V} = \mathbf{I}$),

$$\frac{1}{n-1}\mathbf{X}\mathbf{X}^{\top} = \frac{1}{n-1}\mathbf{U}\Sigma^2\mathbf{U}^{\top}$$

and the correspondence is easily seen (the square roots of the eigenvalues of $\mathbf{X}\mathbf{X}^{\mathbf{T}}$ are the singular values of $\mathbf{X}$, etc.)

14. (1.5 points) Consider the matrix $\begin{bmatrix} 4 & 4 \\ -3 & 3 \end{bmatrix}$

(a) Find Σ and V , i.e., the eigenvalues and eigenvectors of $A^T A$

Solution: Let, $A = \begin{bmatrix} 4 & 4 \\ -3 & 3 \end{bmatrix}$

$$A^T A = \begin{bmatrix} 4 & -3 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 4 & 4 \\ -3 & 3 \end{bmatrix} = \begin{bmatrix} 25 & 7 \\ 7 & 25 \end{bmatrix}$$
$$A^T A - \lambda I = \begin{bmatrix} 25 & 7 \\ 7 & 25 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 25-\lambda & 7 \\ 7 & 25-\lambda \end{bmatrix}$$

Calculating EigenValues:

$$\begin{aligned} & \det(A^A - \lambda I) && = 0 \\ \implies & (25-\lambda)^2 - 49 && = 0 \\ \implies & \lambda^2 + 50\lambda + 625 - 49 && = 0 \\ \implies & \lambda^2 - 50\lambda + 576 && = 0 \\ \implies & \lambda && = 32, 18 \end{aligned}$$

Eigenvector for Eigenvalue $\lambda = 32$:

$$\begin{aligned} & (A^T A - 32I)\mathbf{v} && = 0 \\ \implies & \left(\begin{bmatrix} 25 & 7 \\ 7 & 25 \end{bmatrix} - \begin{bmatrix} 32 & 0 \\ 0 & 32 \end{bmatrix} \right) \mathbf{v} && = 0 \\ \implies & \begin{bmatrix} -7 & 7 \\ 7 & -7 \end{bmatrix} \mathbf{v} && = 0 \\ \implies & \mathbf{v} && = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \end{aligned}$$

Eigenvector for Eigenvalue $\lambda = 18$:

$$\begin{aligned} & (A^T A - 18I)\mathbf{v} && = 0 \\ \implies & \left(\begin{bmatrix} 25 & 7 \\ 7 & 25 \end{bmatrix} - \begin{bmatrix} 18 & 0 \\ 0 & 18 \end{bmatrix} \right) \mathbf{v} && = 0 \\ \implies & \begin{bmatrix} 7 & 7 \\ 7 & 7 \end{bmatrix} \mathbf{v} && = 0 \\ \implies & \mathbf{v} && = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \end{aligned}$$

Say,
$$V = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \text{ and } D = \begin{bmatrix} 32 & 0 \\ 0 & 18 \end{bmatrix}$$
So, Eigenvalue decomposition of $A^TA = V\Sigma^T\Sigma V^T$, then $\Sigma^T\Sigma = D$.
Hence, $\Sigma = \begin{bmatrix} \sqrt{32} & 0 \\ 0 & \sqrt{18} \end{bmatrix}$ and $V = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$.

(b) Find Σ and U, i.e., the eigenvalues and eigenvectors of AA^T

Solution:
$$\begin{aligned} AA^T &= \begin{bmatrix} 4 & 4 \\ -3 & 3 \end{bmatrix} \begin{bmatrix} 4 & -3 \\ 4 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 32 & 0 \\ 0 & 18 \end{bmatrix} \\ (AA^T - \lambda I) &= \begin{bmatrix} 32 & 0 \\ 0 & 18 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \quad = \begin{bmatrix} 32-\lambda & 0 \\ 0 & 18-\lambda \end{bmatrix} \end{aligned}$$
Calculating Eigenvalues and Eigenvectors:
$$\begin{aligned} \det(AA^T - \lambda I) &= 0 \\ \implies (32-\lambda)(18-\lambda) &= 0 \\ \implies \lambda &= 18, 32 \end{aligned}$$
Eigenvector for $\lambda = 32$:
$$(AA^T - 32I)\mathbf{v} = 0 \implies \left(\begin{bmatrix} 32 & 0 \\ 0 & 18 \end{bmatrix} - \begin{bmatrix} 32 & 0 \\ 0 & 32 \end{bmatrix} \right) \mathbf{v} = 0$$
$$\implies \begin{bmatrix} 0 & 0 \\ 0 & -14 \end{bmatrix} \mathbf{v} = 0 \implies \mathbf{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$
Eigenvector for $\lambda = 18$:
$$(AA^T - 18I)\mathbf{v} = 0 \implies \left(\begin{bmatrix} 32 & 0 \\ 0 & 18 \end{bmatrix} - \begin{bmatrix} 18 & 0 \\ 0 & 18 \end{bmatrix} \right) \mathbf{v} = 0$$
$$\implies \begin{bmatrix} 14 & 0 \\ 0 & 0 \end{bmatrix} \mathbf{v} = 0 \implies \mathbf{v} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$
Say, $U = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 32 & 0 \\ 0 & 18 \end{bmatrix}$. So, the eigenvalue decomposition of $AA^T = UDU^T$.
We know,
$$AA^T = U\Sigma\Sigma^T U^T, \text{ then } \Sigma\Sigma^T = D$$

Hence, $\Sigma = \begin{bmatrix} \sqrt{32} & 0 \\ 0 & \sqrt{18} \end{bmatrix}$ and $U = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(c) Now compute $U\Sigma V^T$. Did you get back A? If yes, good! If not, what went wrong?

Solution: We Know,

$$U = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \Sigma = \begin{bmatrix} \sqrt{32} & 0 \\ 0 & \sqrt{18} \end{bmatrix} \text{ and } V = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$\begin{aligned}
\therefore U\Sigma V^T &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{32} & 0 \\ 0 & \sqrt{18} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \\
&= \begin{bmatrix} \sqrt{32} & 0 \\ 0 & \sqrt{18} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \\
&= \begin{bmatrix} \sqrt{32} \times \frac{1}{\sqrt{2}} & \sqrt{32} \times \frac{1}{\sqrt{2}} \\ -\sqrt{18} \times \frac{1}{\sqrt{2}} & \sqrt{18} \times \frac{1}{\sqrt{2}} \end{bmatrix} \\
&= \begin{bmatrix} 4 & 4 \\ -3 & 3 \end{bmatrix} = A \\
U\Sigma V^T &= A
\end{aligned}$$

Hence, $A = U\Sigma V^T$. Proved!

15. (2 points) Prove that the matrices U and V that you get from the SVD of a matrix A contain the basis vectors for the four fundamental subspaces of A. (this is where the whole course comes together: fundamental subspaces, basis vectors, orthonormal vectors, eigenvectors, and our special symmetric matrices AA^T, A^TA!)

Solution: Let, A be an $m \times n$ real matrix, and $A = U\Sigma V^T$ be any SVD for A where U and V are orthogonal of size $m \times m$ and $n \times n$. Along with that let,

$$\Sigma = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}_{m \times n} \quad \text{where } D = diag(\lambda_1, \lambda_2, \ldots, \lambda_r), \text{ with each } \lambda_i > 0$$

$$U = \begin{bmatrix} \mathbf{u_1} & \ldots & \mathbf{u_r} & \ldots & \mathbf{u_m} \end{bmatrix} \text{ and } V = \begin{bmatrix} \mathbf{v_1} & \ldots & \mathbf{v_r} & \ldots & \mathbf{v_n} \end{bmatrix}$$

, with $\{\mathbf{u_1}, \ldots, \mathbf{u_r}, \ldots, \mathbf{u_m}\}$ and $\{\mathbf{v_1}, \ldots, \mathbf{v_r}, \ldots, \mathbf{v_n}\}$ are orthonormal bases of $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively.

The Four fundamental spaces are:

(a) $\{\mathbf{u}_1, \dots, \mathbf{u}_r\}$ is an orthonormal basis of $C(A)$.

Solution: As $C(A) = C(AV)$ and $AV = U\Sigma$, then the above statement can be inferred from:

$$U\Sigma = \begin{bmatrix} u_1 & \dots & u_r & \dots & u_m \end{bmatrix} \begin{bmatrix} diag(\lambda_1, \dots, \lambda_r) & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \lambda_1 \mathbf{u}_1 & \dots & \lambda_r \mathbf{u}_r & 0 & \dots & 0 \end{bmatrix}$$

(b) $\{\mathbf{u}_{r+1}, \dots, \mathbf{u}_m\}$ is an orthonormal basis of $\mathcal{N}(A^T)$.

Solution: We know,

$$(C(A)^{\perp} = (span\{\mathbf{u}_1, \dots, \mathbf{u}_m\})^{\perp} = span\{\mathbf{u}_{r+1}, \dots, \mathbf{u}_m\}$$

. This proves the above statement because

$$(C(A))^{\perp} = \mathcal{N}(A^T)$$

.

(c) $\{\mathbf{v}_{r+1}, \dots, \mathbf{v}_n\}$ is an orthonormal basis of $\mathcal{N}(A)$.

Solution: We know, $\dim(\mathcal{N}(A)) + \dim(im(A)) = n$ by Rank Nullity Theorem. Also, $imA = C(A)$:

$$\dim(\mathcal{N}(A)) = n - \dim(C(A)) = n - r = \dim(span\{\mathbf{v}_{r+1}, \dots, \mathbf{v}_n\})$$

So, to prove the above statement, it is enough to show that $\mathbf{v}_j \in \mathcal{N}(A)$, whenever $j > r$.

$$\therefore \lambda_{r+1} = \dots = \lambda_n = 0$$

So, $E^T E = diag(\lambda_1^2, \dots, \lambda_r^2, \lambda_{r+1}^2, \dots, \lambda_n^2)$
Each λ_i is an eigenvalue of $\Sigma^T\Sigma$ with eigenvector $\mathbf{x_j} =$ Column j of I_n. Thus, $\mathbf{v_j} = V\mathbf{x_j}$ for each j. As $A^TA = V\Sigma^T\Sigma V^T$:

$$(A^TA)\mathbf{v_j} = (V\Sigma^T\Sigma V^T)(V\mathbf{x_j}) = V(\Sigma^T\Sigma\mathbf{x_j}) = V(\lambda_j^2\mathbf{x_j}) = \lambda_j^2 V\mathbf{x_j} = \lambda_j^2\mathbf{v_j}$$

for $1 \leqslant j \leqslant n$. Thus, each $\mathbf{v_j}$ is an eigenvector of A^TA corresponding to λ_j^2. But,

$$\|A\mathbf{v_j}\|^2 = (A\mathbf{v_j})^T A\mathbf{v_j} = \mathbf{v}_j^T(A^TA\mathbf{v}_j) = \mathbf{v}_j^T(\lambda_j^2\mathbf{v}_j) = \lambda_j^2\|\mathbf{v}_j\| = \lambda_j^2$$

for $i = 1, \dots, n$. Particularly $A\mathbf{v}_j = 0$, whenever $j > r$, so $\mathbf{v}_j \in \mathcal{N}(A)$, if $j > r$. This is what we desired initially.
Hence, Proved!

(d) $\{\mathbf{v}_1, \dots, \mathbf{v}_r\}$ is an orthonormal basis of $C(A^T)$.

Solution: We know,

$$span\{\mathbf{v}_{r+1}, \dots, \mathbf{v}_n\} = \mathcal{N}(A) = (row(A))^T$$

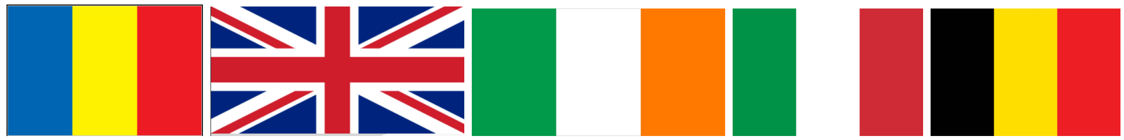
But,

$$row(A) = ((row(A)^\perp)^\perp = (span\{\mathbf{v}_{r+1}, \dots, \mathbf{v}_n\})^\perp = span\{\mathbf{v}_1, \dots, \mathbf{v}_r\}$$

This proves our above statement and hence the complete proof for all four fundamental subspaces.

16. (2 points) Fun with Flags.

(a) Browse through the flags of all countries and paste 5 rank one flags below.

Solution: 

(b) What is the rank of the flag of Greece?

Solution:  The flag of Greece is having a Rank of 3.

17. (2 points) Consider the LFW dataset (Labeled Faces in the Wild).

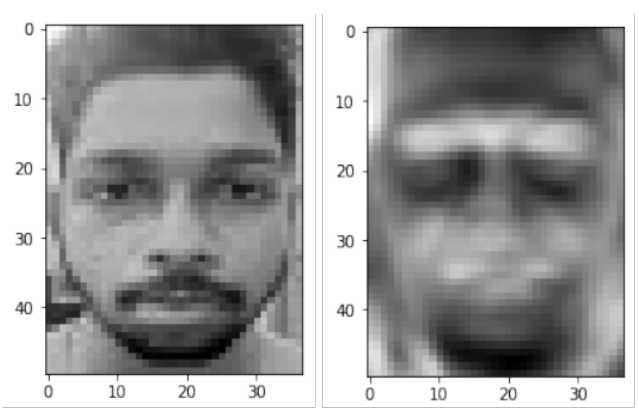
(a) Perform PCA using this dataset and plot the first 25 eigenfaces (in a 5×5 grid)

Figure 1: Original Dataset

Figure 2: Top 25 Eigenvectors

(b) Take your close-up photograph (face only) and reconstruct it using the first 25 eigenfaces :-). If due to privacy concerns, you do not want to to use your own photo then feel free to use a publicly available close-up photo (face only) of your favorite celebrity.

Solution:



. . . And that concludes the story of *How I Met Your Eigenvectors* :-) (And Yes, I have enjoyed it so far!)

CODE:

```
from sklearn.datasets import fetch_lfw_people
from sklearn.decomposition import PCA
import matplotlib.pyplot as plt
import numpy as np
```

```
def plot_gallery(images, h, w, n_row = 5, n_col = 5):
    plt.figure(figsize =(1.8 * n_col, 2.4 * n_row))
    plt.subplots_adjust(bottom = 0, left =.01, right =.99, top =.90, hspace =.35)
    for i in range(n_row * n_col):
        plt.subplot(n_row, n_col, i + 1)
        plt.imshow(images[i].reshape((h, w)), cmap = plt.cm.gray)
        plt.xticks(())
        plt.yticks(())
```

```
lfw_dataset = fetch_lfw_people(min_faces_per_person=100, resize=0.4)
_, h, w = lfw_dataset.images.shape
X = lfw_dataset.data
print(X.shape)
plot_gallery(X,h,w)
```

```
pca = PCA(n_components=n_components, whiten=True).fit(X)
plot_gallery(pca.components_,h,w)
```

EigenFaces on My Face:

```
import os
import cv2
import numpy as np
from matplotlib import pyplot as plt

im = cv2.imread('test2.jpg',0)

plt.imshow(im,cmap='gray')
print(im.shape)
print(im.flatten().shape)
```

```
im_transform = pca.transform(im.flatten().reshape((1,1850)))
print(im_transform.shape)
im_recons = np.matmul(pca.components_.T,test.T)
im_recons = im_recons.reshape(50,37)
print(im_recons.shape)
plt.imshow(im_recons, cmap='gray')
```

```
"""
# EigenFaces (PCA) on LFW Dataset
"""

from sklearn.datasets import fetch_lfw_people
from sklearn.decomposition import PCA
import matplotlib.pyplot as plt
import numpy as np

def plot_gallery(images, h, w, n_row = 5, n_col = 5):
    plt.figure(figsize =(1.8 * n_col, 2.4 * n_row))
    plt.subplots_adjust(bottom = 0, left =.01, right =.99, top =.90, hspace =.35)
    for i in range(n_row * n_col):
        plt.subplot(n_row, n_col, i + 1)
        plt.imshow(images[i].reshape((h, w)), cmap = plt.cm.gray)
        plt.xticks(())
        plt.yticks(())

lfw_dataset = fetch_lfw_people(min_faces_per_person=100, resize=0.4)
_, h, w = lfw_dataset.images.shape
X = lfw_dataset.data
print(X.shape)
plot_gallery(X,h,w)

n_components = 25
pca = PCA(n_components=n_components, whiten=True).fit(X)
plot_gallery(pca.components_,h,w)

"""
# EigenFaces on My Face
"""

import cv2

im = cv2.imread('test2.jpg',0)

plt.imshow(im,cmap='gray')
print(im.shape)
print(im.flatten().shape)

im_transform = pca.transform(im.flatten().reshape((1,1850)))
print(im_transform.shape)
im_recons = np.matmul(pca.components_.T,test.T)
im_recons = im_recons.reshape(50,37)
print(im_recons.shape)
plt.imshow(im_recons, cmap='gray')
plt.show()
```