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CS 6015: Linear Algebra and Random Processes
Assignment: 02
Course Instructor : Dr. Mitesh Khapra

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1. (1 point) Have you read and understood the honor code?

   Solution: Yes.

Count, Count, Count!

2. (1 point) In how many ways can 10 people be seated:

   (a) In a row such that Motu and Patlu sit next to each other (there is only one boy named Motu and only one boy named Patlu in the group)

   Solution: $9! * 2 = 725760$

   (b) In a row such that there are 5 engineers and 5 doctors and no two doctors or no two engineers can sit next to each other.

   Solution: $2 * (5^2 * 4^2 * 3^2 * 2^2) = 28800$

   (c) In a row such that there are 3 engineers, 3 doctors and 4 lawyers and all people of the same proffession should sit in consecutive positions.

   Solution: $3!(3! * 3! * 4!) = 5184$

   (d) In a row such that there are 5 married couples and each couple must sit together.

   Solution: $5! * 2! * 2! * 2! * 2! * 2! = 3840$

3. ($\frac{1}{2}$ points) How many unique 9 letter words can you form using the letters of the word MANMOHANA (the words can be gibberish)?

Solution:

$$\frac{9!}{2! * 2!3!} = 15120$$

4. ($\frac{1}{2}$ points) Suppose you have a class of 7 students (A,B,C,D,E,F,G) who need to be arranged in a line with the following restrictions.

   - A has to be in one of the first 3 slots.
   - B and A are very good friends and insist on being next to each other.
   - B doesn't want to stand immediately behind C.

   In how many different ways can you arrange them?

Solution: **Case 1:** A occupies first slot.

$$p = 5 * 4 * 3 * 2 * 1 = 120$$

Case 2: A occupies the second slot and B is in the first slot.

$$q = 5 * 4 * 3 * 2 * 1 = 120$$

Case 3: A occupies the second slot and B is in the third slot.

$$r = 5 * 4 * 3 * 2 * 1 = 120$$

Case 4: A occupies the third slot and B is in the second slot.

$$s = 5 * 4 * 3 * 2 * 1 = 120$$

Case 5: A occupies the third slot and B is in fourth slot.

$$t = 5 * 4 * 3 * 2 * 1 = 120$$

$$\therefore p + q + r + s + t = 576$$

So, a total of 576 ways of arranging the class of 7 students.

The Birthday Problem

5. (3 points) The days of the year can be numbered 1 to 365. Consider a group of n people, of which you are not a member. Any of the 365 days is equally likely to be the birthday of any member of this group. An element of the sample space Ω will be a sequence of n birthdays (one for each person).

(a) How many elements are there in the sample space?

Solution: $(365)^n$

(b) Let A be the event that at least one member of the group has the same birthday as you. What is the probability of this event A?

Solution:

$$P(A) = 1 - \left(\frac{364}{365}\right)^n$$

(d) What is the minimum value of n such that $P(A) \geqslant 0.5$?

Solution:

$$\begin{aligned} P(A) &\geqslant 0.5 \\ 1 - \left(\frac{364}{365}^n\right) &\geqslant 0.5 \\ \ln(0.5) &\geqslant n * \ln\frac{364}{365} \\ \therefore n &\approx 252.65 \\ \therefore n &\approx 253 \end{aligned}$$

(e) Let B be the event that at least two members of the group share the same birthday. What is the probability of this event B?

Solution: Let, B = At least two members of the group share the same birthday,
B' = No members of the group share the same birthday,
and , $P(B') = 1 - P(B)$.
Now, let's find $P(B')$: $\rightarrow$ The first person can have birthday on any of 365 days.
$\rightarrow$ The second person can have birthday on any day except the birthday of the first person i.e, 364 days.
$\rightarrow$ The third person can have birthday on any day except the birthdays of the first and the second person i.e, 363 days.
$\vdots$
$\rightarrow$ The nth person, who can have birthday on any day other than the birthdays of $n - 1$ person, i.e., $365 - (n - 1)$ days.

$$\therefore P(B') = \frac{365}{365} \times \frac{364}{365} \times \frac{363}{365} \times \cdots \times \frac{365 - n + 1}{365}$$

Multiplying numerator and denominator by $(365-n)!$:

$$P(B') = \frac{365!}{365^n \times (365-n)!}$$

Now, we know that:

$$P(B) = 1 - P(B')$$

$$\rightarrow P(B) = 1 - \frac{365!}{365^n \times (365-n)!}$$

(f) What is the minimum value of n such that $P(B) \geqslant 0.5$?

Solution: From above, we know: $\rightarrow P(B) = 1 - \frac{365!}{365^n \times (365-n)!}$

$$\begin{aligned} \because P(B) &\geqslant 0.5 \\ \implies 1 - \frac{365!}{365^n \times (365-n)!} &\geqslant 0.5 \\ \implies \frac{365!}{365^n \times (365-n)!} &\leqslant 0.5 \\ \implies \frac{365}{365} \times \frac{364}{365} \times \cdots \times \frac{365-n+1}{365} &\leqslant 0.5 \\ \implies 1 \times \left(1 - \frac{1}{365}\right) \times \left(1 - \frac{n-1}{365}\right) \ldots \left(\frac{365-n+1}{365}\right) &\leqslant 0.5 \end{aligned}$$

Now, $1 + x \leqslant e^x \implies 1 - \frac{1}{x} \leqslant e^{\frac{1}{x}}$
Using this fact, we get:

$$\begin{aligned} e^{\frac{-n(n-1)}{730}} &\leqslant 0.5 \\ \implies \frac{-n(n-1)}{730} &\leqslant \ln\left(\frac{1}{2}\right) \\ \implies \frac{n(n-1)}{730} &\leqslant -\ln\left(\frac{1}{2}\right) \\ \implies \frac{n(n-1)}{730} &\geqslant \ln\left(\frac{1}{2}\right) \\ \implies n &= 23 \end{aligned}$$

Minimum value of n is 23.

A biased coin

6. (1 point) Your friend Chaman has a coin which is biased (i.e. $P(H) \neq P(T)$). He proposes that he

will toss the coin twice and asks you to bet on one of these events:

A: Both the tosses will result in the same outcome.

B: Both the tosses will result in a different outcome.

Which event will you bet on to maximize your chance of winning the bet. (I am looking for a precise mathematical answer. No marks for answers which do not have an explanation.)

Solution: Let,

> **A:** Be an event of both tosses being same.
>
> **B:** Be an event of both tosses being different.

$\therefore$

$$\begin{aligned} P(H) &= p \\ P(T) &= (1-p) \\ \therefore P(A) &= p * p + (1-p) * (1-p) \\ \therefore P(B) &= p * (1-p) + (1-p) * p \\ \therefore \frac{P(A)}{p(B)} &= \frac{p * p + (1-p) * (1-p)}{p * (1-p) + (1-p) * p} \\ &= \frac{1-2p+2p^2}{2p-2p^2} = \frac{1-(2p-2p^2)}{2p-2p^2} \\ &= \frac{1}{2p-2p^2} - 1 \end{aligned} \tag{1}$$

Since, $0 < p < 1$, the denominator of the first term of eq. (1) will be always less than 1 and greater than 0. Hence the first term is always greater than 1.

$$\begin{aligned} 0 < 2p - 2p^2 < 1 \\ \therefore \frac{1}{2p-2p^2} > 1 \end{aligned} \tag{2}$$

Hence, from eq. (1) and eq. (2) we have,

$$\frac{P(A)}{P(B)} > 1$$

Hence, choosing the event A, i.e. both the tosses are same would be beneficial.

Alice in Wonderland

7. (1 point) A bag contains one ball which could either be green or red. You take another red ball and put it in his pouch. You now close your eyes and pull out a ball from the pouch. It turns out to be red. What is the probability that the original ball in the pouch was red?

Solution: Let, A be the event that the original ball is red.
Let, B be the event that you drew red ball.
$\therefore$ According to Bayes' Theorem,

$$P(B|A) = 1$$
$$P(A) = \frac{1}{2}$$
$$P(B) = P(A)P(B|A) + P(A')P(B|A') = \frac{3}{4}$$

So,

$$P(A|B) = \frac{1}{2} \times \frac{4}{3} = \frac{2}{3}$$

Rock, Paper and Scissors

8. (2 points) Your friend Chaman has 3 strange dice: Red, Yellow and Green. Unlike a standard die whose 6 faces are the numbers: 1,2,3,4,5,6, these 3 dice have the following faces: red: 3,3,3,3,3,6; yellow: 5,5,5,2,2,2 and green: 4,4,4,4,4,1. Chaman suggests the following game:

   1. You pick any one die
   2. Chaman then "carefully" picks on of the remaining two dice.
   3. Each of you then roll your own die a 100 times.

   If on a given roll, the score of your die is higher than the score of Chaman's die then you get 1 INR, else Chaman gets 1 INR. You play this game for many days and realise that you lose more often than Chaman.

   (a) Why are you losing more often? or What is Chaman's "carefully" planned strategy? (the key thing to note is that he lets you choose first)

   Solution: Let **X** be a random variable for the outcomes of **Red** die. Let **Y** be a random variable for the outcomes of **Yellow** die. Let **Z** be a random variable for the outcomes of **Green** die.
   Suppose we choose Red die $\rightarrow$ Now, the probability of winning the Red die and other die being one of the two remaining dies:

Case 1: You choose **Red** and other die is **Yellow**.
Let **A** be the event that **Red** die wins.

$$
\begin{aligned}
P(A) &= P_X(3) * P_Y(2) + P_X(6) * P_Y(5) + P_X(6) * P_Y(2) \\
&= \frac{5}{6} \times \frac{1}{2} + \frac{1}{6} \times \frac{1}{2} + \frac{1}{6} \times \frac{1}{2} \\
&= \frac{1}{2} \times \frac{1}{12} = 0.5833 \\
P(A') &= 0.4167 \\
P(A) &> P(A')
\end{aligned}
$$

Here, it's clear that the **Red** die beats **Yellow** die because we have higher probability of **Red** die winning than the **Yellow** die winning. Hence, in the long run **Red** die will beat **Yellow** die.
Case 2: You choose **Yellow** and other die is **Green**.
Let **B** be the event that **Yellow** die wins.

$$
\begin{aligned}
P(B) &= P_Y(5) * P_Z(4) + P_Y(5) * P_Z(1) + P_Y(2) * P_Z(1) \\
&= \frac{1}{2} \times \frac{5}{6} + \frac{1}{2} \times \frac{1}{6} + \frac{1}{2} \times \frac{1}{6} \\
&= 0.5833 \\
P(B') &= 0.4167 \\
P(B) &> P(B')
\end{aligned}
$$

Since the probability of **Yellow** die beating **Green** die is higher than probability of **Green** die beating **Yellow** die, thus in the long run **Yellow** die will beat **Green** die.
Case 3: You choose **Green** and other die is **Red**.
Let **C** be the event that **Green** die wins.

$$
\begin{aligned}
P(C) &= P_Z(4) * P_X(3) \\
&= \frac{5}{6} \times \frac{5}{6} = 0.69444 \\
P(C') &= 0.30556 \\
P(C) &> P(C')
\end{aligned}
$$

Since the probability of **Green** die beating **Red** die is higher than probability of **Red** die beating **Green** die, thus in the long run **Green** die will beat **Red** die.
Conclusion:
From the **Case1, Case 2** and **Case 3** its clear that **Red** die beats **Yellow** die, similarly **Yellow** die beats **Green** die, and similarly **Green** die beats **Red** die.

Hence, Chaman will always ask the other player to choose any die, and once one die has been picked up, Chaman will pick the other die "carefully", choosing which one beats the other player's chosen die.

(b) You realise what is happening and decide to turn the tables on Chaman. You buy 3 dice which are identical to Chaman's red, yellow and green dice. You now propose that instead of rolling a single die, each of you will roll two dice of the same color. The rest of the rules remain the same. Chaman realises that now he is loosing more often, Explain Why?

Solution:
Case 1: You choose **Red** die then by previous logic Chaman will choose **Green** die.
Since, there will be 16 cases overall, Let **A** be event that **Red** die beats **Green** die. Computing the $P(\mathbf{A})$,

$$\begin{aligned} P(A) &= 1 - P(A') \\ &= 1 - P_X(3) \times P_X(3) \times P_Z(4) \times P_Z(4) \\ &= 1 - \frac{5}{6} * \frac{5}{6} * \frac{5}{6} * \frac{5}{6} \\ \therefore P(A) &= 1 - 0.4822 = 0.51774 \end{aligned}$$

Here its clear that **Red** die beats **Green** die because we have higher probability of **Red** die winning than the **Green** die winning. Hence in the long run **Red** die will beat **Green** die. Look how the Tables have turned.
Case 2: You choose **Yellow** die then by previous logic Chaman will choose **Red** die.
Since, there will be 16 cases overall, Let **A** be event that **Yellow** die beats **Red** die. Computing the $P(\mathbf{A})$,

$$\begin{aligned} P(A) &= 2 \times (P_Y(2)P_Y(5)P_X(3)P_X(3)) \\ &+ 2 \times (P_Y(5)P_Y(5)P_X(6)P_X(3)) \\ &+ (P_Y(5)P_Y(5)P_X(3)P_X(3)) \\ \therefore P(A) &= 0.5902 \\ \therefore P(A') &= 0.4097 \end{aligned}$$

Here, its clear that **Yellow** die beats **Red** die because we have higher probability of **Yellow** die winning than the **Red** die winning. Hence in the long run **Yellow** die will beat **Red** die.
Case 3: You choose **Green** die then by previous logic Chaman will choose **Yellow** die.
Since, there will be 16 cases overall, Let **A** be event that **Green** die beats **Yellow** die. Computing the $P(\mathbf{A})$,

$$
\begin{aligned}
P(A) &= 2 \times (P_Z(4)P_Z(1)P_X(2)P_X(2)) \\
&\quad + 2 \times (P_Z(4)P_Z(4)P_X(2)P_X(5)) \\
&\quad + (P_Z(4)P_Z(4)P_X(2)P_X(2)) \\
\therefore P(A) &= 0.5902 \\
\therefore P(A') &= 0.4097
\end{aligned}
$$

Here, its clear that **Green** die beats **Yellow** die because we have higher probability of **Green** die winning than the **Yellow** die winning. Hence in the long run **Green** die will beat **Yellow** die.

Conclusion: From **Case 1, Case 2** and **Case 3** it is clear that the order has been reversed from what it was initially, and now a **Red** die beats **Green** die, **Green** die beats **Yellow** die, and, **Yellow** die beats **Red** die.
Hence, Chaman's strategy will fail and for that reason he started loosing more frequently.

9. (1 point) Which of the following has a greater chance of success?

A. Six fair dice are tossed independently and at least one 6 appears.

B. Twelve fair dice are tossed independently and at least two 6s appear.

C. Eighteen fair dice are tossed independently and at least three 6s appear.

Explain your answer.

Solution: Let **X**, be a random variable, representing number of 6s.

A.

$$P(X \geqslant 1) = 1 - \binom{6}{0} \times \left(\frac{5}{6}\right)^6 = 0.6651$$

B.

$$
\begin{aligned}
P(X \geqslant 2) &= 1 - (P(X=0) + P(X=1)) \\
&= 1 - \left(\frac{5}{6}\right)^{12} - \binom{12}{1}\left(\frac{1}{6}\right)^1\left(\frac{5}{6}\right)^{11} \\
&= 0.61866
\end{aligned}
$$

C.

$$\begin{aligned}
P() &= 1 - \Big(P(X=0) + P(X=1) + P(X=2)\Big) \\
&= 1 - \left(\left(\frac{5}{6}\right)^{18} + \binom{18}{1}\left(\frac{1}{6}\right)^{1}\left(\frac{5}{6}\right)^{17} + \binom{18}{2}\left(\frac{1}{6}\right)^{2}\left(\frac{5}{6}\right)^{16}\right) \\
&= 0.59734
\end{aligned}$$

The probability of event A is highest, so it has the highest chance of occuring.

With love from Poland

10. (1 point) A Chainsmoker carries two matchboxes — one in his left pocket and another in his right pocket. Every time he wants to light a cigarette he randomly selects a matchbox from one of the two pockets and then uses a matchbox from that box to light his cigarette. Suppose he takes out a matchbox and sees for the first time that it is empty, what is the probability that the matchbox on the other pocket has only one matchstick left?

Solution: Let both left and right pocket matchboxes have n matchsticks initially. Consider a case where the smoker has his left pocket empty and then has 1 matchstick left in his right pocket.
Let p be the probability that its a left pocket and q be the probability that its a right pocket since both are equally likely, $\therefore p = q = \frac{1}{2}$.
Since the left pocket is empty, it means that all n matchsticks have been used from that pocket — There must be n trials in which the smoker chose the left pocket and 1 trial will correspond to finding out its empty, also the right pocket has only 1 matchstick left, meaning it must have been used $n-1$ matchsticks before means it must have choosen the right pocket $n-1$ times.
Therefore total number of trials would be $2n-1$ followed by one final check on left pocket.

$$\begin{aligned}
\therefore P(X=1) &= \binom{2n-1}{n} p^n q^{n-1} p \\
&= \binom{2n-1}{n}\left(\frac{1}{2}\right)^{n}\left(\frac{1}{2}\right)^{n-1}\frac{1}{2} \\
&= \binom{2n-1}{n}\left(\frac{1}{2}\right)^{2n}
\end{aligned}$$

Similarly, for the right pocket is empty, and left pocketr has 1 matchstick left.

$$\begin{aligned}
P(Y=1) &= \binom{2n-1}{n} q^n p^{n-1} q \\
&= \binom{2n-1}{n}\left(\frac{1}{2}\right)^{2n}
\end{aligned}$$

Hence,

$$\begin{aligned} P(Total) &= P(X=1) + P(Y=1) \\ &= 2 \times \binom{2n-1}{n} \left(\frac{1}{2}\right)^{2n} \\ &= \binom{2n-1}{n} \left(\frac{1}{2}\right)^{2n-1} \end{aligned}$$

A Paradox

11. (1 point) Suppose there are 3 boxes:

    1. A box containing two gold coins,
    2. A box containing two silver coins,
    3. A box containing one gold coin and one silver coin.

    You select one box at random and draw a coin from it. The coin turns out to be a gold coin. You remove this coin and draw another coin from the same box. What is the probability that the second coin is also a gold coin?

Solution: Let, G_1 be event that you get gold in the first draw from randomly chosen box. G_2 be the event that you get gold in the second draw from the same box as in Draw 1.

$$\begin{aligned} P(G_2|G_1) &= \frac{P(\text{both draws are gold})}{P(G_1)} \\ &= \frac{\frac{1}{3} \times 1}{\frac{1}{3} \times 1 + \frac{1}{3} \times 0 + \frac{1}{3} \times \frac{1}{2}} = \frac{2}{3} \end{aligned}$$

Once upon a time in Goa

12. (1 point) You are in one of the famous casinos in Goa. You are observing the game of roulette. A roulette has 36 slots of which 18 are red and the remaining 18 are black. Each slot is equally likely. The manager places a ball on the roulette and then spins the roulette. When the roulette stops spinning, the ball lands in one of the 36 slots. If it lands in a slot which has the same color as what you bet on then you win. You do not believe in gambling but you are a student of probability. You observe that the ball has landed in a vlack slot for the 26th consecutive round. Based on what you havge learned in CS6015 you predict that there is a much higher chance of ball landing in a red slot

in the next round (since the probability of 27 consecutive black slots is very very low). You bet all your life's savings on red. What is the probability that you will win?

Solution: Let **A** be the event that you get Red in the next draw.
Let **B** be the event that you get 26 consecutive blacks.

$$\begin{aligned} P(A|B) &= \frac{P(A \cap B)}{P(B)} \\ &= P(A) = \frac{1}{2} \\ \therefore P(Winning) &= \frac{1}{2} \end{aligned}$$

The event of getting **Red** doesn't depend on the past outcomes, hence the chance of winning remains same, i.e. 0.5.

Oh Gambler! Thy shall be ruined!

13. (2 points) You play a game in a casino where your chance of winning the game is p. Every time you win, you get Re. 1 and every time you lose the casino gets Re. 1. You have i rupees at the start of the game and the casino has $N - i$ rupees (obviously, $N >> i$). The game ends when you go bankrupt or the casino goes bankrupt. In either case, the winner will walk away with a total of N rupees.

(a) Find the probability p_i of winning when you start the game with i rupees.

Solution: Let, Probability of the event, I win the first game = $P(A) = p$
$\therefore P(A') = 1 - p$
Let, $P(B_i) =$ probability of me winning when I start the game with i rupees.
Let, $P(B_i) = p_i$ By total probability, we can write:

$$\begin{aligned} &\implies P(B_i) = P(A) \times P(B_i|A) + P(A') \times P(B_i|A) \\ &\implies p_i = p \times p_{i+1} + (1-p) \times p_{i-1} \\ &\implies p \times p_{i+1} - p_i + (1-p) \times p_{i-1} = 0 \end{aligned} \tag{3}$$

We also know that, $p_0 = 0$ and $p_N = 1$. (If I have no money then I will definitely loose, and if I have all the money then I already won.) Equation 3 is a recursive equation , so let's solve it.
Putting $i - 1$, in place of i in equation 3:

$$\begin{aligned} &\implies p_{i-1} = (1-p) \times p_{i-2} + p \times p_i \\ &\implies p \times p_i - p_{i-1} + (1-p) \times p_{i-2} = 0 \end{aligned} \tag{4}$$

Putting $p_i = x^2$, $p_{i-1} = x$ and $p_{i-2} = x^0$ in equation 4:

$$\begin{aligned} &\implies px^2 - x + (1-p) = 0 \\ &\implies x = 1, \frac{(1-p)}{p} \end{aligned} \tag{5}$$

$$\therefore p_i = c_1(1)^i + c_2\left(\frac{1-p}{p}\right)^i \tag{6}$$

$$\begin{aligned} &\because p_0 = 1 \\ &\therefore p_0 = c_1(1)^0 + c_2\left(\frac{1-p}{p}\right)^0 = c_1 + c_2 \\ &\implies c_1 = -c_2 \end{aligned} \tag{7}$$

Also, $p_N = 1$,

$$\begin{aligned} &\therefore c_1(1)^N + c_2\left(\frac{1-p}{p}\right)^N = 1 \\ &\implies c_1 + c_2\left(\frac{1-p}{p}\right)^N = 1 \end{aligned} \tag{8}$$

By solving equation 7 and equation 8, we get:

$$c_1 = -c_2 = -\frac{p^N}{(1-p)^N - p^N} \tag{9}$$

Using equation 13 in equation 13, we get:

$$\begin{aligned} p_i &= \frac{p^N}{(1-p)^N - p^N} - \frac{p^N}{(1-p)^N - p^N}\left(\frac{1-p}{p}\right)^i \\ &= \left(\frac{p^N}{(1-p)^N - p^N}\right)\left(1 - \left(\frac{1-p}{p}\right)^i\right) \\ &= \left(\frac{p^N}{(1-p)^N - p^N}\right)\left(\frac{p^i - (1-p)^i}{p^i}\right) \\ &= \frac{p^N}{p_i} \times \frac{p_i - (1-p)^i}{(1-p)^N - p^N} \\ \implies p_i &= p^{N-i}\left(\frac{p_i - (1-p)^i}{(1-p)^N - p^N}\right) \end{aligned}$$

Required Answer

(b) What happens if $p = \frac{1}{2}$?

Solution: When $p = \frac{1}{2}$, then both roots of the equation $(px^2 - x + (1-p) = 0)$ is 1, so, p_i will be as follows:

$$\begin{aligned} p_i &= C \times i \times 1^N + D \times 1^N \\ &= C \times i + D \\ p_0 &= 0 \implies D = 0 \end{aligned}$$

and,

$$p_N = 1 \implies C = \frac{1}{N}$$
$$\implies p_i = \frac{i}{N}$$

Therefore, when $p = \frac{1}{2}$, the probability of me winning when starting with i rupees is $\frac{i}{N}$.

The Disappointed Professor

14. (1 point) A particular class has had a history of low attendance. The dejected professor decides that he will not lecture unless at least k of the n students enrolled in the class are present. Each student will independently show up with probability p if the weather is good, and with probability q if the weather is bad. Given that the probability of bad weather on a given day is r, obtain an expression for the probability that the professor will teach his class on that day.

Solution: Let, $P(A) =$ Probability that weather is good $= (1-r)$
$\therefore P(A') =$ Probability that the weather is bad $= r$

Let, $P(B)$ be the event at least k students goes to the class on a given day.

Given, each student will independently show up with probability p if the weather is good. So, we can calculate $P(B|A)$ as follows:

$$P(B|A) = \sum_{i=k}^{n} \binom{n}{i} p_i (1-p)^{n-i} = 1 - F(k-1; p, n)$$

Here, $F(k-1; p, n)$ is a cumulative function for binomial distribution.

The probability that professor will teach his class on that day will be the probability of the event that at least k students attends the class on a given day. So, expression for the probability can be written as the following by using total probability:

$$\begin{aligned} P(A) \times P(B|A) + P(A') \times P(B|A') &= (1-r) \times \sum_{i=k}^{n} \binom{n}{i} p^i (1-p)^{n-i} + r \times \sum_{i=k}^{n} \binom{n}{i} q^i (1-q)^{n-i} \\ &= (1-r) \times (1 - F(k-1; p, n)) + r \times (1 - F(k-1; q, n)) \end{aligned}$$

So the expression of the probability that the professor will teach on a given day is:

$$(1-r) \times (1 - F(k-1; p, n)) + r \times (1 - F(k-1; q, n))$$

$F(k-1; p, n)$ and $F(k-1; q, n)$ are the cumulative functions for Binomial distribution.

The John von architecture

15. (1 point) Suppose you have a biased coin ($P(H) \neq P(T)$). How will you use it to make unbiased decision. (*Hint:* you can toss the coin multiple times)

Solution: Let probability of heads = $P(H) = p$
$\therefore$ Probability of tails $P(T) = (1-p)$

Now , if we toss a coin two times, there can be the following possibilities: **HH, HT, TH, TT**.

Now, probabilities the events are:

$$\implies P(HH) = p \times p = p^2$$
$$\implies P(HT) = p \times (1-p) = p(1-p)$$
$$\implies P(TH) = (1-p) \times p = p(1-p)$$
$$\implies P(TT) = (1-p) \times (1-p) = (1-p)^2$$

We can see that $P(HT)$ and $P(TH)$ have same probability i.e, $p(1-p)$. So, we can define two events A and B. Event A denotes getting heads in first toss and tails in second toss. Event B denotes getting tails in first toss and tails in second toss.

So, one has to choose an event A or B. Then toss a coin repeatedly until two consecutive outcomes are different. Then compare the outcomes with event A and B, and then decide who wins.

Pascal to the rescue

16. (1 point) A six-side die is rolled three times independently. What is more likely: a sum of 11 or 12?

Solution: Let's first calculate the probability to get a sum of 11. Below is the tabular format of all possible combinations. The first column denotes the outcome of the 1st roll. The 2nd column denotes the possible pairs of outcomes from 2nd and 3rd roll to get the sum as 11.

1st roll	(2nd roll, 3rd roll)	Probability
1	(5, 5), (4, 6), (6, 4)	$\frac{3}{6^3}$
2	(3, 6), (6, 3), (4, 5), (5, 4)	$\frac{4}{6^3}$
3	(2, 6), (6, 2), (3, 5), (5, 3), (4, 4)	$\frac{5}{6^3}$
4	(1, 6), (6, 1), (2, 5), (5, 2), (3, 4), (4, 3)	$\frac{6}{6^3}$
5	(1, 5), (5, 1), (2, 4), (4, 2), (3, 3)	$\frac{5}{6^3}$
6	(1, 4), (4, 1), (2, 3), (3, 2)	$\frac{4}{6^3}$

Any one of the above combination gives a sum of 11.
So the $\mathrm{P}(\text{sum of } 11) = \frac{3}{6^3} + \frac{4}{6^3} + \frac{5}{6^3} + \frac{6}{6^3} + \frac{5}{6^3} + \frac{4}{6^3} = \frac{27}{216} = 0.125$

Now let's calcualte the probability of getting a (sum of 12) in the similar way.

1st roll	(2nd roll, 3rd roll)	Probability
1	(5, 6), (6, 5)	$\frac{2}{6^3}$
2	(4, 6), (6, 4), (5, 5)	$\frac{3}{6^3}$
3	(3, 6), (6, 3), (4, 5), (5, 4)	$\frac{4}{6^3}$
4	(2, 6), (6, 2), (3, 5), (5, 3), (4, 4)	$\frac{5}{6^3}$
5	(1, 6), (6, 1), (2, 5), (5, 2), (3, 4), (4, 3)	$\frac{6}{6^3}$
6	(1, 5), (2, 4), (3, 3), (4, 2), (5, 1)	$\frac{5}{6^3}$

Any one of the above combination gives a sum of 12.
So the probability of (sum of 12) $= \frac{2}{6^3} + \frac{3}{6^3} + \frac{4}{6^3} + \frac{5}{6^3} + \frac{6}{6^3} + \frac{5}{6^3} = 0.115$

We can see that, $\mathrm{P}(\text{Getting sum } 11) < \mathrm{P}(\text{Getting sum } 12)$.
$\therefore$ A sum of 11 is more likely.

Enemy at the gates

17. (2 points) There are 41 soldiers surrounded by the enemy. They would rather die than get captured. They sit around in a circle and devise the following plan. Each soldier will kill the person to his immediate left. They will continue this till only one soldier remains who would then commit suicide. For example, if there are 7 soldiers numbered 1, 2, 3, 4, 5, 6, 7 sitting in a circle then they proceed as follows: 1 kills 2, 3 kills 4, 5 kills 6, 7 kills 1, 3 kills 5, 7 kills 3, 7 commits suicide.

    (a) In how many ways can 41 soldiers be arranged around a circle?

    Solution: We have studied in circular permutation that n objects can be arranged around a circle in $(n-1)!$ ways. So, 41 soldiers can be arranged around a circle in $(41-1)! = 40!$ ways.

    (b) If you were one of the 41 soldiers and the soldiers were randomly arranged in the circle, what is the probability that you would survive?

Solution: We know that out of 41 soldiers only 1 will survive. So the probability of me surviving = $\frac{1}{41}$ = 0.024390