Honor code: I pledge on my honor that: I have completed all steps in the below homework on my own, I have not used any unauthorized materials while completing this homework, and I have not given anyone else access to my homework.

Name and Signature

1. (1 point) Have you read and understood the honor code?

   Solution: Yes.

Count, Count, Count!

2. (1 point) In how many ways can 10 people be seated:

   (a) [0.25 point] in a row such that Motu and Patlu sit next to each other (there is only one boy named Motu and only one boy named Patlu in the group)

   Solution: $9! \times 2$ ways.

   Consider Motu and Patlu as one entity, then we have to permute 9 such people to ensure Motu and Patlu sit next to each other. Since, Motu and Patlu can exchange positions, we have 2! ways to do this and by multiplicative principle the required answer is $9! \times 2$ or 725760 ways.

   (b) [0.25 point] in a row such that there are 5 engineers and 5 doctors and no two doctors or no two engineers can sit next to each other

   Solution: $5! \times 5! \times 2$

   Doctors and engineers occupy alternative slots. Each of the 5 engineers can be permuted in their 5 slots in 5! ways. Likewise, 5! ways to permute engineers. Since, the first seat could either start with doctors or engineers, there are 2 ways to do this. By multiplicative principle, the required number of ways is $5! \times 5! \times 2$ or 28800 ways.

   (c) [0.25 point] in a row such that there are 3 engineers, 3 doctors and 4 lawyers and all people of the same profession should sit in consecutive positions.

Solution: $3! \times 3! \times 4! \times 3!$

Since, 3 engineers are seated consecutively, there are 3! ways to arrange them. Likewise, 3! and 4! ways respectively to arrange doctors and lawyers. To arrange the 3 groups of professions, there are 3! ways to do this. By multiplicative principle, the required number of ways is $3! \times 3! \times 4! \times 3!$ or 5184 ways.

(d) **[0.25 point]** in a row such that there are 5 married couples and each couple must sit together.

Solution: $5! \times 2^5$

There are 5! ways to arrange the 5 married couples in the row and 2! ways to arrange each couple sitting together, which 2^5 ways for all couples. By multiplicative principle, the required answer is $5! \times 2^5$ or 3840 ways.

3. (½ point) How many unique 9 letter words can you form using the letters of the word MANMOHANA (the words can be gibberish)?

Solution: $\dfrac{9!}{2! \times 3! \times 2!}$

There are 9! ways to arrange the nine letters, but since some letters repeat we use the division principle to ensure we don't recount repeating permutations. To account for the 2 repetitions of M and N, we divide by 2! for each. To account for the 3 repetitions of the letter A, we divide by 3!. The required answer is $\frac{9!}{2! \times 3! \times 2!}$ or 15120 ways.

4. (½ point) Suppose you have a class of 7 students (A,B,C,D,E,F,G) who need to be arranged in a line with the following restrictions:

   1. A has to be in one of the first 3 slots
   2. B and A are very good friends and insist on being next to each other
   3. B doesn't want to stand immediately behind C

   In how many different ways can you arrange them?

Solution: Consider 3 cases for each position A can stand in

- **case (i): A is front of line**

A B _ _ _ _ _

Then, B must be second to satisfy condition (2). The remaining 5 students can be permuted in 5! ways.

- **case (ii): A is second in line**

  _ A _ _ _ _ _

  We consider 2 cases, where B is next to A to satisfy (2).

  - **subcase(ii.1): B is in front**

    B A _ _ _ _ _

    There are 5! ways to permute the remaining 5 students.

  - **subcase(ii.2): B is behind A**

    _ A B _ _ _ _

    There are 5! ways to permute the remaining 5 students.

  By the addition principle, the total number of ways that A is second in line, is by adding the above two cases, which is $2 \times 5!$ ways.

- **case (iii): A is third in line**

  _ _ A _ _ _ _

  We consider 2 cases, where B is next to A to satisfy (2).

  - **subcase(ii.1): B is in front of A**

    _ B A _ _ _ _

    There are 5! ways to permute all 5 students of which there are 4! ways in which C is fixed in front. But C in front violates condition (3). The required number of ways, by subtraction principle, is 5! - 4! ways to arrange the students.

  - **subcase(ii.2): B is behind A**

    _ _ A B _ _ _

    There are 5! ways to permute the remaining 5 students.

  By the addition principle, the total number of ways that A is second in line, is by adding the above two cases, which is $2 \times 5! - 4!$ ways.

Now, by addition principle again, as the cases are all disjoint, the total number ways of arranging the students in the line are $5 \times 5! - 4! = 576$ ways.

The boring questions are done. I hope you find the rest of the assignment to be interesting!

The birthday problem

5. (3 points) The days of the year can be numbered 1 to 365 (ignore leap days). Consider a group of n people, of which you are not a member. Any of the 365 days is equally likely to be the birthday of any member of this group. An element of the sample space Ω will be a sequence of n birthdays (one for each person).

   (a) **[0.5 points]** How many elements are there in the sample space?

   > **Solution:** There are $(365)^n$ elements in the sample space.
   > The first person can have a birthday on any of the 365 days. The second person's birthday is independent of the birthday of the first person and can also be on any of the 365 days. Similarly, for the last person. By the multiplication rule of counting, there are $365 \times 365 \cdots \times 365$ ways in total or $(365)^n$ elements in the sample space.

   (b) **[0.5 point]** Let A be the event that at least one member of the group has the same birthday as you. What is the probability of this event A?

   > **Solution:** Let C be the event that no one has the same birthday as me. The required probability is then,
   >
   > $$P(A) = 1 - P(C)$$
   >
   > We want every other person's birthday to lie on any of the 364 days excluding my birthday. The probability of this is 364/365 and for n people, by multiplication principle, we see that
   >
   > $$P(C) = \left(\frac{364}{365}\right)^n$$
   >
   > Substituting in,
   >
   > $$P(A) = 1 - P(C)$$
   >
   > we get,
   >
   > $$P(A) = 1 - \left(\frac{364}{365}\right)^n$$

   (c) **[Ungraded question]** Write a formula for computing the probability of the event that any two members of the group will have the same birthday?

Solution: -

(d) [0.5 point] What is the minimum value of n such that $P(A) \geq 0.5$?

Solution:

$$
\begin{aligned}
&P(A) \geq 0.5 \\
&\implies 1 - \left(\frac{364}{365}\right)^n \geq 0.5 \\
&\implies 0.5 \geq \left(\frac{364}{365}\right)^n \\
&\implies \log(0.5) \leq n \log\left(\frac{364}{365}\right) \\
&\implies n \geq 252.6519888588
\end{aligned}
$$

So, $n = 253$.

(e) [0.5 point] Let B be the event that at least two members of the group share the same birthday. What is the probability of this event B?

Solution: Let D be the event that no two people share the same birthday. The required probability is then,

$$P(B) = 1 - P(D)$$

Let the first person have a birthday on any given day. The probability of the second person having a different birthday is 364/365 and for the third person is 363/365. By multiplication principle, for n people to not share a birthday, we see that

$$P(D) = \left(\frac{365}{365}\right) \times \left(\frac{364}{365}\right) \times \left(\frac{363}{365}\right) \cdots \left(\frac{365-n+1}{365}\right)$$

Substituting in,

$$P(B) = 1 - P(D)$$

we get,

$$P(B) = 1 - \left(\frac{365}{365}\right) \times \left(\frac{364}{365}\right) \times \left(\frac{363}{365}\right) \cdots \left(\frac{365-n+1}{365}\right)$$

(f) [1 point] What is the minimum value of n such that $P(B) \geq 0.5$?

Solution:

$$
\begin{aligned}
&P(B) \geq 0.5 \\
&\implies 1 - \left(\frac{365}{365}\right) \times \left(\frac{364}{365}\right) \times \left(\frac{363}{365}\right) \cdots \left(\frac{365-n+1}{365}\right) \geq 0.5 \\
&\implies n \geq 23 \qquad \text{(by simulation)}
\end{aligned}
$$

Python code used for simulation:

5e.py > ...

```python
import sys

term = 1
for n in range(1, sys.maxsize):
    term *= (365.0 - n + 1) / 365.0
    if (1 - term) > 0.5:
        print(n)
        exit()
```

PROBLEMS OUTPUT TERMINAL DEBUG CONSOLE

```
(base) ➜ hw6 python3 5e.py
23
```

(g) **[Ungraded question]** Why is there a big gap between the answers to part (c) and part (e)? (although at "first glance" they look very similar problems)

A biased coin

6. (1 point) Your friend Chaman has a coin which is biased (i.e., $P(H) \neq P(T)$). He proposes that he will toss the coin twice and asks you to bet on one of these events: A: both the tosses will result in the same outcome or B: both the tosses will result in a different outcome. Which event will you bet on to maximize your chance of winning the bet. (I am looking for a precise mathematical answer. No marks for answers which do not have an explanation).

Solution: Let $P(H) = p$ where $p \in [0, 1]$ for the biased coin. Then , $P(H) = 1 - P(T) = 1 - p$.

Since, each toss of a coin is independent we can multiply probabilities to get,

$$\begin{aligned}
P(HH) &= P(H)P(H) = p^2 \\
P(HT) &= P(H)P(T) = p(1-p) \\
P(TH) &= P(T)P(H) = p(1-p) \\
P(TT) &= P(T)P(T) = (1-p)^2
\end{aligned}$$

Also, getting two heads and getting two tails are two disjoint events, so the probabilty of their union, is the sum of their probabilities. Similarly, for getting one head and then tail versus getting one tail and then head. Using the above reasoning we see that,

$$\begin{aligned}
P(A) &= P(HH \cup TT) \\
&= P(HH) + P(TT) \\
&= p^2 + (1-p)^2 \\
&= 1 - 2p(1-p)
\end{aligned}$$

$$\begin{aligned}
P(B) &= P(HT \cup TH) \\
&= P(HT) + P(TH) \\
&= 2p(1-p)
\end{aligned}$$

We also notice that P(B) attains it maximum value of 0.5 at p=0.5,

$$\begin{aligned}
\frac{\mathrm{d}P(B)}{\mathrm{d}p} &= 2 - 4p = 0 \\
&\implies p = 0.5 \\
\frac{\mathrm{d}^2P(B)}{\mathrm{d}p^2} &= -4 < 0 \\
&\implies \text{p is point of maxima} \\
&\implies 0 \leq P(B) \leq 0.5
\end{aligned}$$

Clearly, P(A) = 1 - P(B), using above result we get,

$$0.5 \leq P(A) \leq 1$$

. Therefore,

$$P(A) \geq P(B)$$

. So, I will choose event A to maximize my chance of winning.

Alice in Wonderland

7. (1 point) A bag contains one ball which could either be green on red. You take another red ball and put it in this pouch. You now close your eyes and pull out a ball from the pouch. It turns out to be red. What is the probability that the original ball in the pouch was red?

> **Solution:** After taking the red ball and placing it in the bag, we have two cases depending on the colour of the original ball -
>
> $$\begin{aligned} &B_1 : \text{Bag has 2 red balls} \\ &B_2 : \text{Bag has 1 red and 1 green ball} \end{aligned}$$
>
> We assume it is equally likely for the original ball to be red or green, so
>
> $$P(B_1) = P(B_2) = 0.5$$
>
> Let R be the event of pulling a red ball from the bag. it is easy to see that,
>
> $$\begin{aligned} &P(R/B_1) = 1 \\ &P(R/B_2) = 0.5 \qquad\qquad [\because \text{bag has one red and one green ball}] \end{aligned}$$
>
> Now, by Bayes' Theorem,
>
> $$\begin{aligned} P(B_1/R) &= \frac{P(R/B_1)P(B_1)}{P(R/B_1)P(B_1) + P(R/B_2)P(B_2)} \\ &= \frac{1 \times 0.5}{1 \times 0.5 + 0.5 \times 0.5} \\ &= \frac{2}{3} \end{aligned}$$
>
> The probability that the original ball was red, given we pulled a red ball out, is $P(B_1/R) = 2/3$.

Rock, paper and scissors

8. (2 points) Your friend Chaman has 3 strange dice: red, yellow and green. Unlike a standard die whose 6 faces are the numbers 1,2,3,4,5,6 these 3 dice have the following faces: red: 3,3,3,3,3,6, yellow: 5,5,5,2,2,2 and green: 4,4,4,4,4,1. Chaman suggests the following game: (i) You pick any one die (ii) Chaman then "carefully" picks one of the remaining two dice. Each of you will then roll your own die a 100 times. If on a given

roll, the score of your die is higher than the score of Chaman's die then you get 1 INR else Chaman gets 1 INR. You play this game for many days and realise that you lose more often than Chaman.

(a) [1 point] Why are you losing more often? or What is Chaman's "carefully" planned strategy? (the key thing to note is that he lets you choose first)

Solution:
Let,

- R be a random variable representing the value (3 or 6) that shows up when the red die is rolled.
- Y be a random variable representing the value (2 or 5) that shows up when the yellow die is rolled.
- G be a random variable representing the value (1 or 4) that shows up when the green die is rolled.

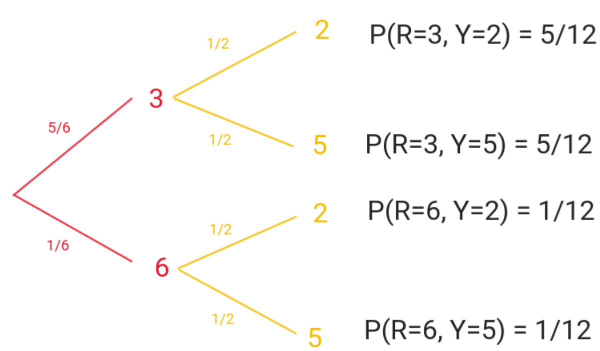
Consider, 3 cases, each corresponding to the strategy Chaman will take, when I choose a particular die -

case (i): I choose the Yellow die

The events of Chaman rolling a red die and me rolling a yellow die are independent. Hence, the probabilities of the intersection of corresponding events can be multiplied. For example, the probability of getting a 3 on red and a 2 on yellow is given by,

$$P(R = 3, Y = 2) = P(R = 3)P(Y = 2) = 5/6 \times 1/2 = 5/12$$

For brevity, the probability tree diagram for rolling a red die and a yellow die is shown below,

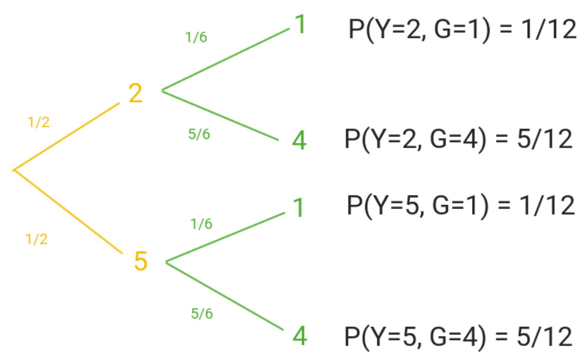


The probability of Red winning is given by,

$$
\begin{aligned}
P(Red\ wins) &= P(R > Y) \\
&= P(R = 3, y = 2) + P(R = 6, Y = 2) + P(R = 6, Y = 5) \\
&\qquad\qquad [\because \text{probabilities of union of disjoint events can be added}] \\
&= 7/12 \\
P(Yellow\ wins) &= 1 - P(Red\ wins) \\
&= 5/12
\end{aligned}
$$

Clearly, $P(Red\ wins) > P(Yellow\ wins)$, so when I pick a yellow die, Chaman can pick the red die to be more likely to win.

case (ii): I choose the Green die
Similarly, the probability tree diagram for rolling a yellow die and a green die is shown below,

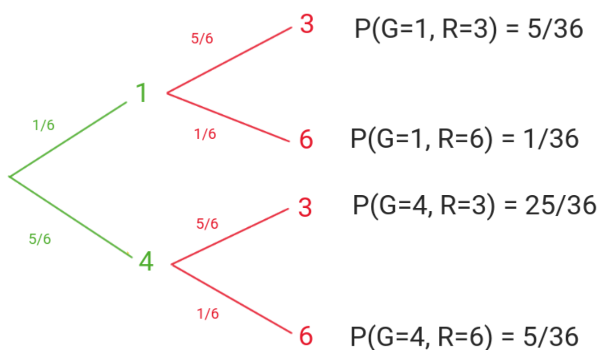


The probability of Yellow winning is given by,

$$
\begin{aligned}
P(Yellow\ wins) &= P(Y > G) \\
&= P(Y = 2, G = 1) + P(Y = 5, G = 1) + P(Y = 5, G = 4) \\
&\qquad\qquad [\because \text{probabilities of union of disjoint events can be added}] \\
&= 7/12 \\
P(Green\ wins) &= 1 - P(Yellow\ wins) \\
&= 5/12
\end{aligned}
$$

Clearly, $P(Yellow\ wins) > P(Greenwins)$, so when I pick a Green die, Chaman can pick the Yellow die to be more likely to win.

case (iii): I choose the Red die
Similarly, the probability tree diagram for rolling a green die and a red die is shown below,



The probability of Green winning is given by,

$$\begin{aligned} P(Green\ wins) &= P(G > R) \\ &= P(G = 4, R = 3) \\ &= 25/36 \\ P(Red\ wins) &= 1 - P(Green\ wins) \\ &= 11/36 \end{aligned}$$

Clearly, $P(Green\ wins) > P(Red\ wins)$, so when I pick a Red die, Chaman can pick the Green die to be more likely to win.

Thus, we can see through Chaman's winning strategy by allowing me to choose first and then choosing an appropriate die and his strategy explains why we were losing more often.

(b) **[1 point]** You realise what is happening and decide to turn the tables on Chaman. You buy 3 dice which are identical to Chaman's red, yellow and green dice. You now propose that instead of rolling a single die each of you will roll two dice of the same color. The rest of the rules remain the same (i) You pick any one color (ii) Chaman then uses his original strategy to carefully pick a different color (he is overconfident and simply uses the same strategy that he used when you were rolling only one die) (iii) If on a given roll, the sum of your two dice is greater than the sum of Chaman's two dice then you get 1 INR else Chaman gets 1 INR. To his horror Chaman realises that now he is loosing more often. Explain why?

Solution: Let,

- R be a random variable representing the value (6 or 9 or 12) that shows up when the 2 red dice are rolled.
- Y be a random variable representing the value (4 or 7 or 9) that shows up when 2 yellow die are rolled.

- G be a random variable representing the value (2 or 5 or 8) that shows up when 2 green die are rolled.

The PMFs are given by:

$$P(R = 6) = 25/36 \qquad P(R = 9) = 10/36 \qquad P(R = 12) = 1/36$$
$$P(Y = 4) = 9/36 \qquad P(Y = 7) = 18/36 \qquad P(Y = 10) = 9/36$$
$$P(G = 2) = 1/36 \qquad P(G = 5) = 10/36 \qquad P(G = 8) = 25/36$$

Consider, 3 cases, each corresponding to the strategy Chaman will take, when I choose a particular die -

case (i): I choose 2 Yellow die

The events of Chaman rolling 2 red dice and me rolling a yellow die are independent. Hence, the probabilities of the intersection of corresponding events can be multiplied. For example, the probability of getting a 6 on two red and a 4 on two yellow dice is given by,

$$P(R = 6, Y = 4) = P(R = 6)P(Y = 4) = 25/36 \times 9/36 = 225/1296$$

The probability of Yellow winning is given by,

$$\begin{aligned}
P(Yellow\ wins) &= P(Y > R) \\
&= P(Y = 7, R = 6) + P(Y = 10, R = 6) + P(Y = 10, R = 9) \\
&\qquad [\because \text{probabilities of union of disjoint events can be added}] \\
&= 765/1296 \\
P(Red\ wins) &= 1 - P(Yellow\ wins) \\
&= 531/1296
\end{aligned}$$

Clearly, $P(Red\ wins) < P(Yellow\ wins)$, so when I pick 2 yellow dice, if Chaman picks 2 red dice, I am still more likely to win.

case (ii): I choose 2 Green die

The probability of Green winning is given by,

$$\begin{aligned}
P(Green\ wins) &= P(G > Y) \\
&= P(G = 5, Y = 4) + P(G = 8, Y = 4) + P(G = 8, Y = 7) \\
&\qquad [\because \text{probabilities of union of disjoint events can be added}] \\
&= 765/1296 \\
P(Green\ wins) &= 1 - P(Yellow\ wins) \\
&= 531/1296
\end{aligned}$$

Clearly, $P(Yellow\ wins) < P(Green\ wins)$, so when I pick 2 Green dice, if Chaman picks 2 Yellow dice, I am still more likely to win.

case (iii): I choose 2 Red die
The probability of Green winning is given by,

$$\begin{aligned} P(Green\ wins) &= P(G > R) \\ &= P(G = 8, R = 6) \\ &= 625/1296 \\ P(Red\ wins) &= 1 - P(Green\ wins) \\ &= 671/1296 \end{aligned}$$

Clearly, $P(Red\ wins) > P(Green\ wins)$, so when I pick 2 Red dice, if Chaman picks 2 Green dice, I am still more likely to win.
This explains why I am winning more often now.

Sitting under an apple tree

9. (1 point) Which of the following has a greater chance of success?

   A. **[0.3 point]** Six fair dice are tossed independently and at least one "6" appears.

   B. **[0.3 point]** Twelve fair dice are tossed independently and at least two "6"s appear.

   C. **[0.4 point]** Eighteen fair dice are tossed independently and at least three "6"s appear.

   Explain your answer.

Solution: Let A' be the event that no 6 appears when 6 dice are tossed. For each die, the probability of not getting a 6 is 5/6. Since, tossing 6 dices are independent, the probability of their intersection is given by multiplication. Then,

$$\begin{aligned} P(A) &= 1 - P(A') \\ &= 1 - (5/6)^6 \\ &= 0.665 \end{aligned}$$

Let the probability of getting a 6 be given by, $p = 1/6$. Let X be a random variable indicating the number of 6's appearing when 12 fair dice are tossed. Clearly, this can be modelled using a binomial distribution, with the above p and n=12. Let B' be the event of getting zero or one 6's.

$$\begin{aligned} P(B) &= 1 - P(B') \\ &= 1 - P(X < 2) \\ &= 1 - \sum_{x=0}^{1} \binom{12}{x} (1/6)^x (5/6)^{(12-x)} \\ &= 0.619 \end{aligned}$$

Let the probability of getting a 6 be given by, $p = 1/6$. Let X be a random variable indicating the number of 6's appearing when 18 fair dice are tossed. Clearly, this can be modelled using a binomial distribution, with the above p and n=12. Let C' be the event of getting zero or one or two 6's.

$$\begin{aligned} P(C) &= 1 - P(C') \\ &= 1 - P(X < 3) \\ &= 1 - \sum_{x=0}^{2} \binom{18}{x} (1/6)^x (5/6)^{(18-x)} \\ &= 0.597 \end{aligned}$$

Therefore, $P(A) > P(B) > P(C)$. P(A) has the greatest chance of success.

With love from Poland

10. (1 point) A chain smoker carries two matchboxes - one in his left pocket and another in his right pocket. Every time he wants to light a cigarette he randomly selects a matchbox from one of the two pockets and then uses a matchbox from that box to light his cigarette. Suppose he takes out a matchbox and sees for the first time that it is empty, what is the probability that the matchbox in the other pocket has exactly one matchstick left?

Solution: Let us assume each match box contains n matchsticks. Consider two cases, **case(i) Right pocket is empty and Left pocket contains one match:** Let X be a random variable representing the number of matchsticks drawn from right pocket. The random variable is modelled by a binomial distribution with $p = 0.5$ as each pocket is equally likely to be picked and total number of trials as $2n - 1$ as one matchstick is left. Since, right pocket is empty we set number of trials $= n$.

Therefore,

$$P(Right\ empty) = P(X = n) = \binom{2n-1}{n} 0.5^n * 0.5^{n-1}$$

Although, his right pocket is empty now, he still needs to make one more toss to find out that it is empty. (We assume that the chain smoker does not keep track of his matchsticks in memory). Therefore,

$$P(Find\ right\ empty) = \binom{2n-1}{n} 0.5^n * 0.5^{n-1} * 0.5$$

case(i) Right pocket is empty and Left pocket contains one match:
This case is exactly same as before, except that right and left pockets are swapped. Therefore,

$$P(Find\ left\ empty) = \binom{2n-1}{n} 0.5^n * 0.5^{n-1} * 0.5$$

The probability that the match box in the other pocket has exactly one left is given by

$$\begin{aligned}
&P(Find\ right\ empty) + P(Find\ left\ empty)\\
&= \binom{2n-1}{n} 0.5^n * 0.5^{n-1} * 0.5 + \binom{2n-1}{n} 0.5^n * 0.5^{n-1} * 0.5\\
&= \binom{2n-1}{n} 0.5^{2n-1}
\end{aligned}$$

A paradox

11. (1 point) Suppose there are 3 boxes:

    1. a box containing two gold coins,
    2. a box containing two silver coins,
    3. a box containing one gold coin and one silver coin.

    You select one box at random and draw a coin from it. The coin turns out to be a gold coin. You remove this coin and draw another coin from the same box. What is the probability that the second coin is also a gold coin?

Solution: Let,

- B_1 be the event that chosen box has 2 gold coins.
- B_2 be the event that chosen box has 2 silver coins.
- B_3 be the event that chosen box has 1 gold coins and 1 silver coin.
- G be event first coin drawn is gold coin.

Since, it is equally likely for any box to be chosen, $P(B_1) = P(B_2) = P(B_3) = 1/3$.
By Bayes' Theorem, we want to find,

$$\begin{aligned} P(B_1/G) &= \frac{P(G/B_1)P(B_1)}{P(G/B_1)P(B_1) + P(G/B_2)P(B_2) + P(G/B_3)P(B_3)} \\ &= \frac{1 \times 1/3}{1 \times 1/3 + 0 \times 1/3 + 1/2 \times 1/3} \\ &= \frac{2}{3} \end{aligned}$$

Therefore, probability second coin is also a gold coin is same as $P(B_1/G) = 2/3$.

Once upon a time in Goa

12. (1 point) You are in one of the famous casinos in Goa[1]. You are observing the game of roulette. A roulette has 36 slots of which 18 are red and the remaining 18 are black. Each slot is equally likely. The manager places a ball on the roulette and then spins the roulette. When the roulette stops spinning, the ball lands in one of the 36 slots. If it lands in a slot which has the same color as what you bet on then you win. You do not believe in gambling but you are a student of probability[2]. You observe that the ball has landed in a black slot for the 26 consecutive rounds. Based on what you have learned in CS6015 you predict that there is a much higher chance of the ball landing in a red slot in the next round (since the probability of 27 consecutive black slots is very very low). You bet all your life's savings on red. What is the probability that you will win?

Solution: The roullete spins can be given by a binomial distribution where the random variable X represents the number of times the ball landed in a black slot.

[1] I know about casinos in Goa purely out of academic interest.
[2] Ah! That's why you are in a casino! That makes perfect sense!

We see that, the probability of the ball landing on the black slot on the 27th time,

$$P(B) = P(X = 27) = \binom{27}{27} 0.5^{27} * 0.5^{(27-27)} = 0.5^{27}$$

The probability of the ball landing on the red slot on the 27th time,

$$P(R) = P(X = 26) * P(Red\ on\ 27th) = \binom{26}{26} 0.5^{26} * 0.5^{(26-26)} * 0.5 = 0.5^{27}$$

We see that, P(B) = P(R), so I have a 50-50 chance of winning, ie, the probability of me winning is 0.5

Oh Gambler! Thy shall be ruined!

13. (2 points) You play a game in a casino[3] where your chance of winning the game is p. Every time you win, you get 1 rupee and every time you lose the casino gets 1 rupee. You have i rupees at the start of the game and the casino has $N - i$ rupees (obviously, $N >> i$). The game ends when you go bankrupt or the casino goes bankrupt. In either case, the winner will walk away with a total of N rupees.

    (a) **[1 point]** Find the probability p_i of winning when you start the game with i rupees.

Solution: Let P_i denote the probability of winning when we start the game with i rupees. Since, our chance of winning a game is p,

$$P_i = p.P_{i+1} + (1-p).P_{i-1}$$

Using the above equation we can write,

$$\begin{aligned} p.P_{i+1} + (1-p).P_{i-1} &= P_i \\ p.P_{i+1} &= P_i - (1-p).P_{i-1} \\ p.P_{i+1} - p.P_i &= P_i - P_{i-1} + p.P_{i-1} - p.P_i \\ P_{i+1} - P_i &= \frac{1-p}{p}(P_i - P_{i-1}) \end{aligned}$$

[3] Again, my interest in casinos in purely academic

When we start the game with 0 rupees, then we lose and so $P_0 = 0$.

$$\begin{aligned} P_2 - P_1 &= \frac{1-p}{p}(P_1 - 0) \\ &= \frac{1-p}{p}(P_1) \end{aligned}$$

$$\begin{aligned} P_3 - P_2 &= \frac{1-p}{p}(P_2 - P_1) \\ &= \left(\frac{1-p}{p}\right)^2 (P_1) \end{aligned}$$

We now see that,

$$\begin{aligned} P_i - P_1 &= \sum_{j=1}^{i-1}(P_{i+1} - P_i) \\ &= \sum_{j=1}^{i-1}\left(\frac{1-p}{p}\right)^j P_1 \end{aligned}$$

Rearranging, we get,

$$P_i = P_1 + P_1\sum_{j=1}^{i-1}\left(\frac{1-p}{p}\right)^j \tag{1}$$

$$P_i = P_1\sum_{j=0}^{i-1}\left(\frac{1-p}{p}\right)^j \tag{2}$$

When $p \neq 0.5$,

$$P_i = P_1\frac{1-((1-p)/p)^i}{1-(1-p)/p} \quad [\because \text{ using formula for summation of geometric series}]$$

When $i = N$, we know, $P_N = 1$, as casino goes bankrupt then.

$$\begin{aligned} 1 = P_N &= P_1\frac{1-((1-p)/p)^N}{1-(1-p)/p} \\ \implies P_1 &= \frac{1-(1-p)/p}{1-((1-p)/p)^N} \end{aligned}$$

Substituting in our above formula for P_i, we get,

$$P_i = \frac{1-((1-p)/p)^i}{1-((1-p)/p)^N}$$

which is the probability we wished to find.

(b) **[1 point]** What happens if $p = \frac{1}{2}$?

Solution: Continuing from (2), when p=0.5,

$$P_i = P_1 \times i \tag{3}$$

When i = N, then $P_N = 1$ and,

$$\begin{aligned} 1 = P_N &= P_1 N \\ \implies P_1 &= 1/N \end{aligned}$$

Substituting in equation (3) that relates P_i and P_1, we get,

$$P_i = \frac{i}{N}$$

(c) **[Ungraded question]** Can you reason why it does not make sense to take on a casino ($N >> i$)? Will you always go bankrupt in the long run?

Solution: -

The disappointed professor

14. (1 point) A particular class has had a history of low attendance. The dejected professor decides that he will not lecture unless at least k of the n students enrolled in the class are present. Each student will independently show up with probability p if the weather is good, and with probability q if the weather is bad. Given that the probability of bad weather on a given day is r, obtain an expression for the probability that the professor will teach his class on that day. [Bertsekas and Tsitsikilis, Introduction to Probability, 2nd edition.]

Solution: Let,

- T be the event professor conducts the lecture.
- W be the event weather is good and $\overline{W}$ the event weather is bad
- X be random variable indicating number of students showing up when weather is good.
- Y be random variable indicating number of students showing up when weather is bad.

X and Y are each modelled by a binomial distributions with success probabilities p and q respectively, of n trials as there are n students. By Law of total probability,

$$\begin{aligned} P(T) &= P(T/W)P(W) + P(T/\overline{W})P(\overline{W}) \\ &= P(X \geq k)(1-r) + P(Y \geq k)r \\ &= (1-r)\sum_{j=k}^{n}\binom{n}{j}(p)^j(1-p)^{n-j} + (r)\sum_{j=k}^{n}\binom{n}{j}(q)^j(1-q)^{n-j} \end{aligned}$$

The John von architecture

15. (1 point) Suppose you have a biased coin ($P(H) \neq P(T)$). How will you use it to make unbiased decision. (hint: you can toss the coin multiple times)

Solution: Consider, throwing the coin twice. Let H denote heads and T denote tails. Let the coin have bias p, ie, $P(H) = p$.

$$P(HT) = P(H)P(T) = pq$$

$$P(TH) = P(T)P(H) = pq$$

Clearly, P(HT) = P(TH) and we can use it to make an unbiased decision. If on tossing the coin twice, we get Heads first followed by tails we take decision 1 and if we get Tails first followed by heads we take decision 2. When we get 2 heads or 2 tails, we do nothing and restart the experiment.

Pascal to the rescue

16. (1 point) A six-side die is rolled three times independently. What is more likely: a sum of 11 or 12?

Solution: We assume the die is fair and it is equally likely to get any of the 6 faces. To get a sum of 11, we refer to the below table where the left triplet shows the combination of values showing up on the 3 die rolls and the right triplet shows the number of ways that triplet can occur.

$$\begin{aligned}
&(1,4,6) \to 3! \\
&(1,5,5) \to 3!/2! \\
&(2,3,6) \to 3! \\
&(2,4,5) \to 3! \\
&(3,3,5) \to 3!/2! \\
&(3,4,4) \to 3!/2!
\end{aligned}$$

Therefore, the total number of ways for 11 is $K_{11} = 6+3+6+6+3+3 = 27$ To get a sum of 12, we refer to the below table where the left triplet shows the combination of values showing up on the 3 die rolls and the right triplet shows the number of ways that triplet can occur.

$$\begin{aligned}
&(1,5,6) \to 3! \\
&(2,4,6) \to 3! \\
&(2,5,5) \to 3!/2! \\
&(3,3,6) \to 3!/2! \\
&(3,4,5) \to 3! \\
&(4,4,4) \to 1
\end{aligned}$$

Therefore, the total number of ways for sum 12 is $K_{12} = 6+6+3+3+6+1 = 25$. Since, the probabilities of the getting any of the 6 sides is equal, their values taken t the power, will not matter when calculating the exact probability using a multinomial distribution. However, the numbers in K_{11} and K_{12} influence the co-efficients that appear in the multinomial PMF.
Since, $K_{11} > K_{12}$, it is more likely to get a sum of 11.

Enemy at the gates

17. (1 point) There are 41 soldiers surrounded by the enemy. They would rather die than get captured. They sit around in a circle and devise the following plan. Each soldier will kill the person to his immediate left. They will continue this till only one soldier remains who would then commit suicide. For example, if there are 7 soldiers numbered 1, 2, 3, 4, 5, 6, 7 sitting in a circle then they proceed as follows: 1 kills 2, 3 kills 4, 5 kills 6, 7 kills 1, 3 kills 5, 7 kills 3, 7 commits suicide.

    (a) **[0.5 point]** In how many ways can 41 soldiers be arranged around a circle?

Solution: 40!
In general, n people can be permuted in $n!$ ways in a line, but in a circle, the arrangements of $1, 2, \ldots, n$ is the same as $2, 3, \ldots, n, 1$. There are n such ways which repeat. So, the number ways of arranging n people in circle is $n!/n = (n-1)!$.

(b) [**0.5 point**] If you were one of the 41 soldiers and the soldiers were randomly arranged in the circle, what is the probability that you would survive?

Solution: We know only one person will survive in the end out of 41. Since, the soldiers are randomly arranged it is equally likely for everyone to survive, and the probability that the particular soldier would survive is 1/41.

(c) [**Ungraded question**] Is there a specific position in which you can sit so that you are the last surviving soldier?

Solution: -