Honor code: I pledge on my honor that: I have completed all steps in the below homework on my own, I have not used any unauthorized materials while completing this homework, and I have not given anyone else access to my homework.

Name and Signature

1. (1 point) Have you read and understood the honor code?

Solution: Yes

Eigenstory: Special Properties

2. (1 point) Prove that for any square matrix A the eigenvectors corresponding to distinct eigenvalues are always independent.

Solution:
Eigenvector equation,

$$\begin{aligned} Av_i &= \lambda_i v_i \\ Av_i - \lambda_i v_i &= 0 \\ (A - \lambda_i I)v_i &= 0 \end{aligned}$$

Let's prove this by proof by contradiction:
Let's say that, $v_1, v_2, v_3, ..., v_n$ are linearly independent, then

$$c_1v_1 + c_2v_2 + c_3v_3 + ... + c_nv_n = 0 \qquad \text{(eq.1)}$$

Where $c_i = 0$ for all i = 1,2,3,...,n
Now, assume that $v_1, v_2, v_3, ..., v_n$ are linearly dependent so that one of the co-efficient $c_i \neq 0$
Without loss of generality we can assume that $c_1 \neq 0$.
Now,

$$\begin{aligned} (A - \lambda_2 I)v_i &= Av_i - \lambda_2 v_i \\ &= \lambda_i v_i - \lambda_2 v_i \\ (A - \lambda_2 I)v_i &= (\lambda_i - \lambda_2)v_i \\ (A - \lambda_2 I) &= (\lambda_i - \lambda_2) \end{aligned}$$

Multiplying both the side of eq.1 by $(A - \lambda_2 I)$

$$c_1(\lambda_1 - \lambda_2)v_1 + c_2(\lambda_2 - \lambda_2)v_2 + c_3(\lambda_3 - \lambda_2)v_3 + ... + c_n(\lambda_n - \lambda_2)v_n = 0$$
$$c_1(\lambda_1 - \lambda_2)v_1 + c_2(0)v_2 + c_3(\lambda_3 - \lambda_2)v_3 + ... + c_n(\lambda_n - \lambda_2)v_n = 0$$
$$c_1(\lambda_1 - \lambda_2)v_1 + c_3(\lambda_3 - \lambda_2)v_3 + ... + c_n(\lambda_n - \lambda_2)v_n = 0 \quad \text{(eq.2)}$$

Here c_2v_2 term is removed from the above equation,
Now, similarly if we multiply $(A - \lambda_3 I)$ in the eq. 2 c_3v_3 term will be removed and equation will look alike,

$$c_1(\lambda_1 - \lambda_3)(\lambda_1 - \lambda_2)v_1 + ... + c_n(\lambda_n - \lambda_3)(\lambda_n - \lambda_2)v_n = 0$$

if we continue multiplying $(A-\lambda_4 I), (A-\lambda_5 I), ..., (A-\lambda_n I)$ then all terms involving c_iv_i where $i = 2, 3, .., n$ will be removed,
as result,

$$c_1(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)(\lambda_1 - \lambda_4)...(\lambda_1 - \lambda_n)v_1 = 0$$

In the above equation v_1 cant be zero because it is non-zero eigenvector,
Now, if all the eigenvalues are distinct $((\lambda_1 \neq \lambda_2), (\lambda_1 \neq \lambda_3), ...)$ then c_1 is necessarily to be zero $(c_1 = 0)$.
It is contradicting our prior assumption that one of the co-efficient $c_i \neq 0$ so that $v_1, v_2, v_3, ..., v_n$ are linearly dependent.
So our assumption that $v_1, v_2, v_3, ..., v_n$ are linearly dependent is incorrect because we proved that $(c_1 = 0)$.

Hence, proved that if all the eigenvalues are distinct then eigenvectors of A are linearly independent.

3. (2 points) Prove the following.

   (a) **[1 point]** The sum of the eigenvalues of a matrix is equal to its trace.

Solution:
Characteristic root of equation can be written as,

$$(\lambda_1 - \lambda)(\lambda_2 - \lambda)...(\lambda_n - \lambda) = (-\lambda)^n + (\lambda_1 + \lambda_2 + ... + \lambda_n)(-\lambda)^{n-1} + ... + (\lambda_1\lambda_2...\lambda_n) \quad \text{(eq.1)}$$

Now, When we expand $det(A - \lambda I)$ all the terms that contain $(-\lambda)^{n-1}$ comes from the diagonal entries because highest degree that we can get by choosing a nondiagonal entry is $(-\lambda)^{n-2}$.

we can observe this in below equation,

$$(a_{11}-\lambda_1)(a_{22}-\lambda_2)...(a_{(n-2)(n-2)}-\lambda_{(n-2)})\{(a_{(n-1)(n-1)}-\lambda_{n-1})(a_{nn}-\lambda_n) \\ -a_{n(n-1)}a_{(n-1)n}))\}$$

Now if we expand this equation we can see that highest degree that we can get by choosing a nondiagonal entry $(a_{n(n-1)}a_{(n-1)n})$ is $(-\lambda)^{n-2}$.
Thus the co-efficient along with $(-\lambda)^{n-1}$ in the expansion of $det(A-\lambda I)$ is $a_{11}+a_{22}+...+a_{nn}$.
Now if we compare the co-efficient of $(-\lambda)^{n-1}$ from this expansion of $det(A-\lambda I)$ and eq.1,
we get ,

$$\lambda_1+\lambda_2+...+\lambda_n = a_{11}+a_{22}+...+a_{nn}$$

Hence, proved that trace of the Matrix is equal to the sum of eigenvalues.

(b) **[1 point]** The product of the eigenvalues of a matrix is equal to its determinant.

Solution:
Characteristic root of equation can be written as,

$$det(A-\lambda I) = (\lambda_1-\lambda)(\lambda_2-\lambda)...(\lambda_n-\lambda)$$

Choosing $\lambda = 0$ we get,

$$det(A-0) = (\lambda_1-0)(\lambda_2-0)...(\lambda_n-0)$$
$$det(A) = \lambda_1\lambda_2...\lambda_n$$

Hence, proved that product of the eigenvalues of a matrix is equal to its determinant.

4. (2 points) What is the relationship between the rank of a matrix and the number of non-zero eigenvalues? Explain your answer.

Solution: I think the answer to this question is "The rank of a matrix is equal to the number of non-zero eigenvalues if the non-zero eigenvalues are distinct."
Here 3 cases can be possible,
We can have,
(i) One or more eigenvalue as 0 and others as non-zero

If we have atleast one eigenvalue as 0 then,

$$\begin{aligned}(A-\lambda I)x &= 0\\ Ax &= 0\end{aligned}$$

Nullity of (A- λ I) and A will be same.
$\therefore$ Using Rank-nullity theorem we can always be able to find the rank of matrix A.

(ii) All n eigenvalues are non-zero and distinct
If all eigenvalues are distinct then we will get n different eigenvector corresponding to every distinct eigenvalue,

$$Ax = \lambda x$$

From this equation we can say λx is in the column space of A.
Now if we get n different eigenvector then those n eigenvector would be in n different direction so we will have n independent eigenvector and those independent eigenvector span whole R^n.
So the Rank of matrix A = n.

(iii) All eigenvalues are non-zero but some eigenvalues are repeating.
If some eigenvalue are repeating then we may get n different eigenvector or we may not.
we can not be sure about number of independent eigenvector we get so we can not tell the exact rank of matrix A.
But we can say,

$$Rank(A) \geq \text{number of distinct eigenvalues}$$

5. (1 point) If A is a square symmetric matrix then prove that the number of positive pivots it has is the same as the number of positive eigenvalues it has.

Solution:
Let's take 2 by 2 example,

$$A = \begin{bmatrix} 1 & 5 \\ 5 & 1 \end{bmatrix}$$

Eigenvalues of matrix A are 6 and -4.

Now let's do LDU factorization of above matrix,

$$A = LDU$$
$$\begin{bmatrix}1 & 5\\5 & 1\end{bmatrix} = \begin{bmatrix}1 & 0\\5 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\0 & -24\end{bmatrix}\begin{bmatrix}1 & 5\\0 & 1\end{bmatrix}$$

Here we can observe A is decomposed into $LDL^\top$ and their pivots are 1 and -24.
Now, eigenvalue of IDI^top will be the diagonal entries of matrix D,

$$IDI^\top = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\0 & -24\end{bmatrix}\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$$

Here 1 and -24 become the eigenvalues of $IDI^\top$.
Here matrix A is non singular matrix so there eigenvalues would be nonzero.

Now, here eigenvalues are changing as entry 5 in matrix L changes to 0.
To change the sign of eigenvalues it is needed that eigenvalues would cross the zero.
But as matrix is non singular their eigenvalues will never touch to zero so sign of the eigenvalues will never cross the zero.
So we can sure that sign will not change as λ's move to d's.
this can be extend to n by n case where we have n non-zero eigenvalues and n non-zero pivots.
Hence proved that number of positive pivots is the same as number of positive eigenvalues.

Eigenstory: Special Matrices

6. (2 points) Consider the matrix $R = I - 2\mathbf{u}\mathbf{u}^\top$ where $\mathbf{u}$ is a unit vector $\in \mathbb{R}^n$.

   (a) **[0.5 point]** Show that R is symmetric and orthogonal. (How many independent vectors will R have?)

   Solution:
   Proof: R is Symmetric,

   $R = I - 2uu^\top$

We know that for symmetric matrix, $R^\top = R$

$$\begin{aligned}
R^\top &= (I - 2uu^\top)^\top \\
&= (I^\top - (2uu^\top)^\top) \\
&= (I - 2(u^\top)^\top u^\top) \\
&= (I - 2uu^\top) \\
R^\top &= R
\end{aligned}$$

Hence R is symmetric.

Proof: R is Orthogonal,
We know that for orthogonal matrices, $R^\top R = RR^\top = I$

$$\begin{aligned}
R^\top R &= (I - 2uu^\top)^\top (I - 2uu^\top) \\
&= (I^\top - (2uu^\top)^\top)(I - 2uu^\top) \\
&= (I - 2(u^\top)^\top u^\top)(I - 2uu^\top) \\
&= (I - 2uu^\top)(I - 2uu^\top) \\
&= I - 2uu^\top - 2uu^\top + 4uu^\top uu^\top \\
&= I - 4uu^\top + 4uu^\top && \text{(u is unit vector } \therefore u^\top u = 1) \\
R^\top R &= I
\end{aligned}$$

$$\begin{aligned}
RR^\top &= (I - 2uu^\top)(I - 2uu^\top)^\top \\
&= (I - 2uu^\top)(I^\top - (2uu^\top)^\top) \\
&= (I - 2uu^\top)(I - 2(u^\top)^\top u^\top) \\
&= (I - 2uu^\top)(I - 2uu^\top) \\
&= I - 2uu^\top - 2uu^\top + 4uu^\top uu^\top \\
&= I - 4uu^\top + 4uu^\top && \text{(u is unit vector } \therefore u^\top u = 1) \\
RR^\top &= I
\end{aligned}$$

Hence R is Orthogonal.

(b) **[0.5 point]** Let $\mathbf{u} = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$. Draw the line passing through this vector in geogebra (or any tool of your choice). Now take any vector in $\mathbf{R}^3$ and multiply it with the matrix R (i.e., the matrix R as defined above with $\mathbf{u} = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$). What do you observe or what do you think the matrix R does or what would you call matrix R? (Hint: the name starts with R)

Solution: Let's compute R,

$$
\begin{aligned}
R &= I - 2uu^\top \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - 2 \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - 2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \\
R &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}
\end{aligned}
$$

Now, Let's take a vector in R^3,

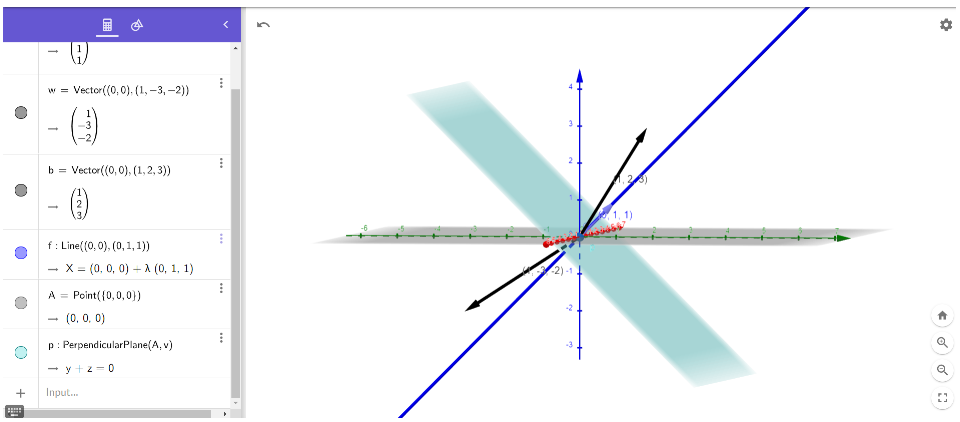
$$
\begin{aligned}
x &= \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \\
Rx &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \\
&= \begin{bmatrix} 1 \\ -3 \\ -2 \end{bmatrix}
\end{aligned}
$$

In general,

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$Rx = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= \begin{bmatrix} x_1 \\ -x_3 \\ x_2 \end{bmatrix}$$



Here we can observe that transformed vector is a reflection of original vector across the plane which is perpendicular to vector u.
Matrix R is called as Reflector.

(c) **[0.5 point]** Compute the eigenvalues and eigenvectors of the matrix R as defined above with $\mathbf{u} = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

Solution:

$$R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}$$

Let's find the eigenvalues and eigenvectors for matrix R,

$$det(R - \lambda I) = 0$$
$$\begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & -\lambda & -1 \\ 0 & -1 & -\lambda \end{vmatrix} = 0$$
$$(1-\lambda)(\lambda^2 - 1) = 0$$
$$\lambda = 1, \lambda = 1, \lambda = -1$$

Putting $\lambda = 1$ in eigenvector equation,

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Solving above linear system we get,

$$x = c \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + d \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

Putting $\lambda = -1$ in eigenvector equation,

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Solving above linear system we get,

$$x = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

(d) [0.5 point] I believe that irrespective of what **u** is any such matrix R will have the same eigenvalues as you obtained above (with one of the eigenvalues repeating). Can you reason why this is the case? (Hint: think about how we reasoned about the eigenvectors of the projection matrix P even without computing them.)

Solution:
In the above figure we can see the there is line passing through the vector u and there is a plane which is perpendicular to the line.
Now Reflector matrix R transform a vector x such that the transformed vector (Rx) become the reflection of the given vector (x) across the orthogonal plane.

Now if we take any vector x which is in the plane then reflection (Rx) will be in the same plane which is the vector itself (Rx = x). Here the λ would be 1 so that we can write as,

$$Rx = x$$
$$Rx = (1)x$$
$$Rx = \lambda x$$

So every vector which is in the plane will have eigenvalue as 1.

Now if we take any vector (x) from which the line is passing through then reflected vector (Rx) would be in the exact opposite direction of x.
Here λ would be -1 so that we can write as,

$$Rx = -x$$
$$Rx = (-1)x$$
$$Rx = \lambda x$$

So every vector from which the line is passing through will have $\lambda = -1$.

7. (2 points) Let Q be a $n \times n$ real orthogonal matrix (i.e., all its elements are real and its columns are orthonormal). State with reason whether the following statements are True or False (provide a proof if the statement is True and a counter-example if it is False).

   (a) **[0.5 point]** If λ is an eigenvalue of Q then $|\lambda| = 1$

   Solution: True.
   Let x be some eigenvector of Q and λ be its corresponding eigenvalue. $\therefore$ we have $Qx = \lambda x$. Pre-multiplying this equation by $(Qx)^\top$ we get, $(Qx)^\top(Qx) = x^\top Q^\top Qx = \lambda x^\top \cdot \lambda x = |\lambda|^2 x^\top x$. Since Q is orthogonal, $Q^\top Q = I$ and the equation boils down to $x^\top x = |\lambda|^2 x^\top x$. This implies $|\lambda|^2 = 1$ or $|\lambda| = 1$

   (b) **[0.5 point]** The eigenvectors of Q are orthogonal

Solution: False.
We can take simple Counter-Example as identity matrix,

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

We know that eigenvalue for identity matrix is 1 and every vector in R^2 will be the eigenvector of A.
Let's say u and v are two eigenvector of A,
$u = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $v = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ are not orthogonal.

(c) **[1 point]** Q is always diagonalizable.

Solution: False.
Q is not always diagonalisable under Real set R.
as we have seen in part (a) of this question that we may get imaginary eigenvalues which may lead to imaginary eigenvectors which are not diagonalisable under Real set R.

But Q is always diagonalisable under Complex set C.

Proof:
We know that,

$$Q^\top Q = I$$

Let's say, v and u are some nonzero eigenvector of Q, Now Let's take dot product between transformed vectors (Qv) and (Qw).

$$\begin{aligned}
(Qv)^\top (Qw) &= v^\top Q^\top Q w \\
(\lambda_1 v)^\top (\lambda_2 w) &= v^\top I w && (Q^\top Q = I) \\
\lambda_1 \lambda_2 v^\top w &= v^\top w \\
\lambda_1 \lambda_2 v^\top w - v^\top w &= 0 \\
(\lambda_1 \lambda_2 - 1) v^\top w &= 0
\end{aligned}$$

Here from the above equation we can say,
If $(\lambda_1 \lambda_2 \neq 1)$ then it force $v^\top w = 0$.
Hence v and w are orthogonal to each other.
$\therefore$ v and w are independent eigenvectors.

Hence we can say that whenever ($\lambda_1\lambda_2 \neq 1$) then we get all eigenvector of Q are orthogonal hence independent and hence we can say that Q is diagonalisable.
Now, if ($\lambda_1\lambda_2 = 1$) then two case is possible either ($\lambda_1 = 1$ and $\lambda_2 = 1$) where (Q=I) or ($\lambda_1 = -1$ and $\lambda_2 = -1$) where (Q = -I).
In both the case as eigenvectors of Q will span whole R^n hence we Q will be diagonalisable.
Hence proved.

8. (1½ points) Any rank one matrix can be written as $\mathbf{u}\mathbf{v}^\top$.

(a) **[0.5 point]** Prove that the eigenvalues of any rank one matrix are $\mathbf{v}^\top\mathbf{u}$ and 0.

Solution:
Here, $uv^\top$ is Rank 1 matrix.
$uv^\top$ matrix has all its column to be some multiple of u and those multiples are corresponding element of v.
Let's compute,

$$
\begin{aligned}
Au &= (uv^\top)u \\
Au &= u(v^\top u) \\
Au &= u(\lambda) && \text{(Here c is some constant where } \lambda = v^\top u) \\
Au &= \lambda u
\end{aligned}
$$

Above equation represents the eigenvector formula where u is eigenvector and $\lambda = v^\top u$ is eigenvalue.
Hence we are assure that one eigenvalue of $uv^\top$ is $v^\top u$.

Now, if dimension of $uv^\top$ is more than 1 let's say (n x n) where $n > 1$ then the second eigenvalue of matrix will be 0 because (n x n) matrix which have rank=1 is singular and singular matrix have eigenvalue 0.

(b) **[0.5 point]** How many times does the value 0 repeat?

Solution: Eigenvalue 0 will repeat (n-1) times where n is dimension of the vector u.

(c) **[0.5 point]** What are the eigenvectors corresponding to these eigenvalues?

Solution:
Here $uv^\top$ matrix have all the vectors as some multiple of u and there corresponding scalar are the elements of v.
Now, column space of this matrix will span the line passing through the vector u in the n dimensional space.
if we multiply any vector with matrix $uv^\top$ then transformed vector will be some multiple of u which is in the span of matrix $uv^\top$ (line).
So here we can have atmost one independent eigenvector which have corresponding eigenvalue as non-zero.
We cant have another eigenvector which stays in the same direction as before after transformation applied because we have only a 1D span of vector u.

Now, we can have (n-1) eigenvectors which are Orthogonal vector u and all these eigenvector have corresponding eigenvalue as 0.

9. (2 points) Consider a $n \times n$ Markov matrix.

(a) **[0.5 point]** Prove that the dominant eigenvalue of a Markov matrix is 1

Solution:
Proof (part 1): 1 is an eigenvalue of a Markov matrix
Let's say A is n by n matrix,

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ . & . & . & \dots & . \\ . & . & . & \dots & . \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{bmatrix}$$

Now, by definition of markov matrix all the elements in rows add up to 1.
if we subtract 1 from any element of the row of A and then add up all the elements, it will be zero.
So, we can say matrix $(A - I)$ have all the rows which sums up to 0,

$$a_{i1} + a_{i2} + a_{i3} + ... + +a_{in} = 0$$

Now if we compare above equation with,

$$a_{i1}v_1 + a_{i2}v_2 + a_{i3}v_3 + ... + +a_{in}v_n = 0$$

then we can write,

$$(A - I)v = 0$$

where v is the vector of all one's.
Now if we compare the above equation with eigenvalue equation,

$$(A - \lambda I)v = 0$$

We can prove that $\lambda = 1$ such that,

$$(A - 1 \cdot I)v = 0$$

Hence, proved that $\lambda = 1$ is an eigenvalue of markov matrix.

Proof (part 2): all other eigenvalues are less than 1
Let's say λ is eigenvalue of (n x n) markov matrix A.

$$Av = \lambda v$$

Comparing i^{th} row of both the side we obtain,

$$a_{i1}v_1 + a_{i2}v_2 + a_{i3}v_3 + ... + a_{in}v_n = \lambda v_i \qquad \text{(eq. 1)}$$

Let v_k is the maximum entry of vector v,
$v_k = max\{v_1, v_2, v_3, ..., v_n\}$,

Here $v_k > 0$ otherwise we have $v = 0$ and this contradict that v is nonzero vector,

Now, from eq.1

$$\lambda v_k = a_{k1}v_1 + a_{k2}v_2 + a_{k3}v_3 + ... + a_{kn}v_n$$

since we v_k is maximum we can also write this way,

$$\begin{aligned}
\lambda v_k &\leq a_{k1}v_k + a_{k2}v_k + a_{k3}v_k + ... + a_{kn}v_k \\
\lambda v_k &\leq (a_{k1} + a_{k2} + a_{k3} + ... + a_{kn})v_k \\
\lambda v_k &\leq (1)v_k \qquad \text{(for markov matrix rows sum to 1)} \\
\lambda &\leq 1
\end{aligned}$$

Hence Proved.

(b) **[0.5 point]** Consider any 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ such that $a + b = c + d$. Show that one of the eigenvalues of such a matrix is 1. (I hope you notice that a Markov matrix is a special case of such a matrix where $a + b = c + d = 1$.)

Note: For this sub-question and the rest, there have been two variations of answers by the students. The original question (as given above) asks to use $a + b = c + d$ and prove 1 is an eigenvalue. A modification of this question was posted in the discussion forum which is $a + b = c + d = k$ is given and it is required to prove that k is an eigenvalue. We will provide the solutions for both variations in case the you have answered with either assumption. Please grade your answer accordingly.

Solution:

- **Original as given above:** Disproved. Consider the matrix $A = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}$. The eigenvalues of A are -1 and 4 and none of them are equal to 1.

- **Modification as given in Discussion Forum:** Consider $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the eigenvalues of A are roots of $det(A - \lambda I) = 0$. $det(A - \lambda I) = \begin{bmatrix} a - \lambda & b \\ c & d - \lambda \end{bmatrix} = (a-\lambda)(d-\lambda) - bc = \lambda^2 - (a+d)\lambda + (ad-bc) = 0$. Discriminant of this quadratic is $(a+d)^2 - 4(ad-bc) = a^2 + d^2 + 2ad - 4ad + 4bc = a^2 + d^2 - 2ad + 4bc = (a-d)^2 + 4bc$. Since, $a + b = c + d$, $a - d = c - b$), $(a-d)^2 + 4bc = (c-b)^2 + 4bc = c^2 + b^2 + 2bc = (c+b)^2$. Hence, one of roots of this quadratic (which an eigenvalue) is $\frac{a+d+\sqrt{(c+b)^2}}{2} = \frac{a+b+c+d}{2} = \frac{2k}{2} = k$.

(c) **[0.5 point]** Does the result extend to $n \times n$ matrices where the sum of the elements of a row is the same for all the n rows? (Explain with reason)

Solution:

- **Original as given above:** No, it cannot be extended. Any simple counter-example will suffice.

- **Modification as given in Discussion Forum:** Let A be an $n \times n$ matrix such that for each row, all the row elements add up to a constant value, say k, i.e $\forall i$, $\sum_{j=0}^{n} a_{ij} = k$. The eigenvalues for this matrix can be obtained as solutions to $det(A - \lambda I) = 0$. This determinant will have $a_{ii} - \lambda$ along the principle diagonal. As determinant does not alter with column-ops, we perform $C_1 \leftarrow \sum_{i=1}^{n} C_i$. Now, the first element of each row contains the sum of the elements of that row, which will be $\forall i$, $a_{i1} + a_{i2} + \cdots + a_{ii} - \lambda + \cdots + a_{in} = (k - \lambda)$.

Since after this column-op, all the elements of the first column will be $k - \lambda$, we can pull it out of the determinant as a common factor and the first column will be all $1s$. Let this newly formed determinant have a value $g(\lambda)$ (since it still has λ terms in it). $\therefore$ as per our steps, $det(A - \lambda I) = (k - \lambda) \cdot g(\lambda)$. If we have $\lambda = k$, $det(A - kI) = (k - k) \cdot g(k) = 0 \cdot g(k) = 0$. Since the determinant of $A - \lambda I = 0$ for $\lambda = k$, i.e. the sum of the values of any row, it is an eigenvalue.

(d) **[0.5 point]** What is the corresponding eigenvector?

Solution:

- **Original as given above:** An eigenvector may not always exist.
- **Modification as given in Discussion Forum:** eigenvector $v = \begin{bmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$.

Eigenstory: Special Relations

10. (4 points) For each of the statements below state True or False with reason.

(a) **[0.5 point]** The eigenvalues of A^T are **always** the same as that of A.

Solution: True
Eigenvalues of a matrix are roots of characteristic polynomial.
characteristic polynomial of $A^\top$ is given by,

$$\begin{aligned} det(A^\top - \lambda I) &= det(A^\top - \lambda I^\top) && \text{(because } I = I^\top\text{)} \\ &= det(A - \lambda I)^\top && ((A - B)^\top = A^\top - B^\top) \\ det(A^\top - \lambda I) &= det(A - \lambda I) && \text{(because } det(A^\top) = det(A)\text{)} \end{aligned}$$

Here from above equation we can see that A and $A^\top$ both have same characteristic polynomial.
$\therefore$ A and $A^\top$ both have same roots of characteristic polynomial.
$\therefore$ A and $A^\top$ both have same eigenvalues.

(b) **[0.5 point]** The eigenvectors of A^T are **always** the same as that of A

Solution: False
$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$
Here A and $A^{\top}$ have same eigenvalue as 1.
But A has only one eigenvector as $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $A^{\top}$ has only one eigenvector as $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

(c) [0.5 point] The eigenvalues of A^{-1} are **always** the reciprocal of the eigenvalues of A.

Solution: True (Only if A is invertible)

$$\begin{aligned} Av &= \lambda v \\ A^{-1}Av &= A^{-1}\lambda v \\ v &= A^{-1}\lambda v \\ \frac{1}{\lambda}v &= A^{-1}v \\ A^{-1}v &= \frac{1}{\lambda}v \end{aligned}$$

Here in above equation we can see that $\frac{1}{\lambda}$ will always be a eigenvalue of A^{-1}.

(d) [0.5 point] The eigenvectors of A^{-1} are **always** the same as the eigenvectors of A.

Solution: True
Let's say v is eigenvector and λ is eigenvalue of A,

$$\begin{aligned} Av &= \lambda v \\ A^{-1}Av &= A^{-1}\lambda v && (\because \text{ multiplying } A^{-1} \text{ both the side.}) \\ Iv &= \lambda A^{-1}v \\ \frac{1}{\lambda}v &= A^{-1}v \\ A^{-1}v &= \frac{1}{\lambda}v \end{aligned}$$

From the above equation we can say that same v is the eigenvector for both A and A^{-1} and $\frac{1}{\lambda}$ is eigenvalue for A^{-1}.
Hence, proved.

(e) **[0.5 point]** If $\mathbf{x}$ is an eigenvector of A and B then it is also an eigenvector of both AB and BA, even if the eigenvalues of A and B corresponding to $\mathbf{x}$ are different.

Solution: True
Let's say,
$Ax = \lambda_1 x$
$Bx = \lambda_2 x$
Now,

$$\begin{aligned}(AB)x &= A(Bx) \\ &= A(\lambda_2 x) \\ &= \lambda_2 (Ax) \\ &= \lambda_2 (\lambda_1 x) \\ (AB)x &= (\lambda_1 \lambda_2) x\end{aligned}$$

Here we can see that given x is eigenvector of A and B it is also the eigenvector of (AB).
Now,

$$\begin{aligned}(BA)x &= B(Ax) \\ &= B(\lambda_1 x) \\ &= \lambda_1 (Bx) \\ &= \lambda_1 (\lambda_2 x) \\ (BA)x &= (\lambda_1 \lambda_2) x\end{aligned}$$

Here we can see that given x is eigenvector of A and B it is also the eigenvector of (BA).

(f) **[0.5 point]** If $\mathbf{x}$ is and eigenvector of A and B then it is also an eigenvector of $A + B$

Solution: True
Let's say,
$Ax = \lambda_1 x$
$Bx = \lambda_2 x$
Now,

$$\begin{aligned}(A+B)x &= Ax + Bx \\ &= \lambda_1 x + \lambda_2 x \\ (A+B)x &= (\lambda_1 + \lambda_2) x\end{aligned}$$

Here we can see that given x is eigenvector of A and B it is also the eigenvector of (A+B).

(g) [0.5 point] If λ is an eigenvalue of A then $\lambda + k$ is an eigenvalue of $A + kI$.

Solution: True
Let's say,
$Av = \lambda v$ $\quad eq.1$
Now Let's take the same v for (A + kI),

$$\begin{aligned}(A + kI)v &= Av + kIv \\ &= \lambda v + kv && \text{(from eq.1)} \\ (A + kI)v &= (\lambda + k)v && \text{(eq.2)}\end{aligned}$$

From the eq.2 we can say that $(\lambda + k)$ is eigenvalue of $(A + kI)$.

(h) [0.5 point] The non-zero eigenvalues of $AA^\top$ and $A^\top A$ are equal.

Solution: True

Let's say x is an eigenvector of matrix $A^\top A$ and λ is eigenvalue then,

$$\begin{aligned}A^\top Ax &= \lambda x \\ AA^\top Ax &= A\lambda x && (\therefore \text{ multiplying A both the side}) \\ AA^\top(Ax) &= \lambda(Ax)\end{aligned}$$

From the above equation we can say that (Ax) is eigenvector and same λ is the eigenvalue for $AA^\top$.
Here above λ is the eigenvalue for $AA^\top$ and $A^\top A$.

Hence proved that non-zero eigenvalue of $AA^\top$ and $A^\top A$ are equal.

Eigenstory: Change of basis

11. (2 points) Consider the following two basis. Basis 1: $\mathbf{u_1} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}, \mathbf{u_2} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}$, and Basis 2: $\mathbf{u_1} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}, \mathbf{u_2} = \frac{1}{\sqrt{2}}\begin{bmatrix}-1\\-1\end{bmatrix}$,. Consider a vector $\mathbf{x} = \begin{bmatrix}a\\b\end{bmatrix}$ in Basis 1 (i.e., $\mathbf{x} = a\mathbf{u_1} + b\mathbf{u_2}$). How would you represent it in Basis 2?

Solution:

Here vector $x = \begin{bmatrix} a \\ b \end{bmatrix}$ in Basis 1 is represented in standard coordinate system as,

$x = au_1 + au_2$

$x = a \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} + b \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$

Now, this vector x is represented in Basis 2 as,

$$
\begin{aligned}
\left(\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \right)^{-1} x &= (-1) \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} x \\
&= \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} x \\
&= \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \left(a \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} + b \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} \right) \\
&= \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \left(\begin{bmatrix} a\frac{1}{\sqrt{2}} \\ a\frac{1}{\sqrt{2}} \end{bmatrix} + \begin{bmatrix} b\frac{1}{\sqrt{2}} \\ -b\frac{1}{\sqrt{2}} \end{bmatrix} \right) \\
&= \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \left(\frac{1}{\sqrt{2}} \begin{bmatrix} (a+b) \\ (a-b) \end{bmatrix} \right) \\
&= \frac{1}{\sqrt{2}}(a+b) \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} + \frac{1}{\sqrt{2}}(a-b) \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} \\
&= \frac{1}{2}(a+b) \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \frac{1}{2}(a-b) \begin{bmatrix} -1 \\ -1 \end{bmatrix} \\
&= \frac{1}{2} \begin{bmatrix} (a+b) \\ -(a+b) \end{bmatrix} + \frac{1}{2} \begin{bmatrix} -(a-b) \\ -(a-b) \end{bmatrix} \\
&= \frac{1}{2} \begin{bmatrix} a+b+b-a \\ -a-b+b-a \end{bmatrix} \\
&= \frac{1}{2} \begin{bmatrix} 2b \\ -2a \end{bmatrix} \\
&= \begin{bmatrix} b \\ -a \end{bmatrix}
\end{aligned}
$$

So, vector $\begin{bmatrix} b \\ -a \end{bmatrix}$ is representation of x in Basis 2.

12. (1 point) Let $\mathbf{u}$ and $\mathbf{v}$ be two vectors in the standard basis. Let $T(\mathbf{u})$ and $T(\mathbf{v})$ be the representation of these vectors in a different basis. Prove that $\mathbf{u} \cdot \mathbf{v} = T(\mathbf{u}) \cdot T(\mathbf{v})$ if and only if the basis represented by T is an orthonormal basis (i.e., dot products are preserved only when the new basis is orthonormal).

Solution:
Let's say there is a matrix A which describes the transformation T(x) where x is any vector.
So the Matrix A will be orthogonal matrix.

$$A^\top A = AA^\top = I \qquad (eq.1)$$

Let's take two vectors u and v in standard basis,

$$T(u) = Au$$
$$T(v) = Av$$

We can also wrtie as,

$$u \cdot v = u^\top v$$

Now,

$$\begin{aligned} T(u) \cdot T(v) &= (Au) \cdot (Av) \\ &= (Au)^\top (Av) \\ &= u^\top A^\top Av \\ &= u^\top Iv && (\because \text{from eq.1}) \\ &= u^\top v \\ T(u) \cdot T(v) &= u \cdot v \end{aligned}$$

Hence, proved that Orthonormal transformation preserves the dot product.

Eigenstory: PCA and SVD

13. (1 point) How are PCA and SVD related? (no vague answers please, think and answer very precisely with mathematical reasoning)

Solution:
Let's say we have (n x d) data matrix X where n is number of data points and d is

the dimension of data point.
then the covariance matrix,

$$S = \frac{1}{n}\Sigma_{i=1}^{n}(x_i - \mu_i)^\top (x_i - \mu_i)$$
$$S = \frac{1}{n}X^\top X$$

Covariance matrix capture the variation of data Now, the eigenvector decomposition of the matrix S,

$$S = V\Lambda V^\top \qquad \text{(eq.1)}$$

Now, SVD decomposition of matrix X can be written as,

$$\begin{aligned} X &= U\Sigma V^\top \\ &= \Sigma_{i=1}^{r}\sigma_i u_i v_i^\top \end{aligned}$$

Now,

$$\begin{aligned} X^\top X &= (U\Sigma V^\top)^\top U\Sigma V^\top \\ &= V\Sigma^\top U^\top U\Sigma V^\top \\ &= V\Sigma^\top I\Sigma V^\top \\ X^\top X &= V(\Sigma^\top\Sigma)V^\top \end{aligned}$$

Now if we put the value in equation of covariance matrix S,

$$S = \frac{1}{n}X^\top X$$
$$S = \frac{1}{n}V(\Sigma^\top\Sigma)V^\top \qquad \text{(eq. 2)}$$

from eq.1 and eq.2 we get,

$$\begin{aligned} \frac{1}{n}V(\Sigma^\top\Sigma)V^\top &= V\Lambda V^\top \\ (\Sigma^\top\Sigma) &= n\Lambda \\ \sigma_i^2 &= n\lambda_i \end{aligned}$$
$$\sigma_i = \sqrt{n\lambda_i} \qquad \text{(eq.3)}$$

Now, right singular vectors,

$$V^\top = \begin{bmatrix} -- & v_1^\top & -- \\ -- & v_2^\top & -- \\ -- & .. & -- \\ -- & v_r^\top & -- \end{bmatrix}$$

Now from eq.1 we can say this $v_i's$ are principle components,
So right singular vector of svd can be written as,

$$Xv_i = \sigma_i u_i$$

putting the values from eq.3,

$$\begin{aligned} Xv_i &= \sqrt{n\lambda_i}u_i \\ u_i &= \frac{1}{\sqrt{n\lambda_i}}Xv_i \end{aligned}$$

Here we have established a relation between singular vector of SVD to principle components of PCA.

14. (1½ points) Consider the matrix $\begin{bmatrix} 4 & 4 \\ -3 & 3 \end{bmatrix}$

(a) **[0.5 point]** Find Σ and V, *i.e.*, the eigenvalues and eigenvectors of $A^\top A$

Solution:

$$\begin{aligned} A^\top A &= \begin{bmatrix} 4 & -3 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 4 & 4 \\ -3 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 25 & 7 \\ 7 & 25 \end{bmatrix} \end{aligned}$$

Eigenvalues of $A^\top A$,

$$
\begin{aligned}
det(A^\top A - \lambda I) &= 0 \\
det\left(\begin{bmatrix} 25 & 7 \\ 7 & 25 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) &= 0 \\
det\left(\begin{bmatrix} 25-\lambda & 7 \\ 7 & 25-\lambda \end{bmatrix}\right) &= 0 \\
(25-\lambda)^2 - 49 &= 0 \\
\lambda^2 - 50\lambda + 576 &= 0 \\
\lambda &= 32, \lambda = 18
\end{aligned}
$$

Eigenvectors of $A^\top A$,
Eigenvector equation,

$$(A^\top A - \lambda I)x = 0$$

putting $\lambda = 32$ in the equation,
we get,

$$\begin{bmatrix} -7 & 7 \\ 7 & -7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

By solving the above equation we get eigenvector corresponding to $\lambda = 32$ is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.
putting $\lambda = 18$ in the equation,
we get,

$$\begin{bmatrix} 7 & 7 \\ 7 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

By solving the above equation we get eigenvector corresponding to $\lambda = 18$ is $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.
So, $\Sigma = \begin{bmatrix} 32 & 0 \\ 0 & 18 \end{bmatrix}$ and after normalising the eigenvector $V = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$.

(b) **[0.5 point]** Find Σ and U, *i.e.*, the eigenvalues and eigenvectors of $AA^\top$

Solution:

$$
\begin{aligned}
AA^\top &= \begin{bmatrix} 4 & 4 \\ -3 & 3 \end{bmatrix} \begin{bmatrix} 4 & -3 \\ 4 & 3 \end{bmatrix} \\
&= \begin{bmatrix} 32 & 0 \\ 0 & 18 \end{bmatrix}
\end{aligned}
$$

Eigenvalues of $AA^\top$,

$$
\begin{aligned}
det(AA^\top - \lambda I) &= 0 \\
det\left(\begin{bmatrix} 32 & 0 \\ 0 & 18 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right) &= 0 \\
det\left(\begin{bmatrix} 32-\lambda & 0 \\ 0 & 18-\lambda \end{bmatrix} \right) &= 0 \\
(32-\lambda)(18-\lambda) &= 0 \\
\lambda = 32, \lambda = 18
\end{aligned}
$$

Eigenvectors of $AA^\top$,
Eigenvector equation,

$$
(AA^\top - \lambda I)x = 0
$$

putting $\lambda = 32$ in the equation,
we get,

$$
\begin{bmatrix} 0 & 0 \\ 0 & -14 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
$$

By solving the above equation we get eigenvector corresponding to $\lambda = 32$ is $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.
putting $\lambda = 18$ in the equation,
we get,

$$
\begin{bmatrix} 14 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
$$

By solving the above equation we get eigenvector corresponding to $\lambda = 18$ is $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.
So, $\Sigma = \begin{bmatrix} 32 & 0 \\ 0 & 18 \end{bmatrix}$ and after normalising the eigenvector $U = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

(c) **[0.5 point]** Now compute $U\Sigma V^\top$. Did you get back A? If yes, good! If not, what went wrong?

Solution: Let's compute,

$$\begin{aligned} A &= U\Sigma V^\top \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{32} & 0 \\ 0 & \sqrt{18} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \\ &= \begin{bmatrix} 4 & 4 \\ -3 & 3 \end{bmatrix} \\ &= A \end{aligned}$$

15. (2 points) Prove that the matrices U and V that you get from the SVD of a matrix A contain the basis vectors for the four fundamental subspaces of A. (this is where the whole course comes together: fundamental subspaces, basis vectors, orthonormal vectors, eigenvectors, and our special symmetric matrices $AA^\top$, $A^\top A$!)

Solution:
Given,

$$A = U\Sigma V^\top$$

Where A is (m x n) matrix,
Then U will be (m x m) matrix,
And V will be (n x n) matrix,
And Σ will be (m x n) matrix,

Now,

$$\begin{aligned} AA^\top &= (U\Sigma V^\top)(U\Sigma V^\top)^\top \\ &= U\Sigma V^\top V\Sigma^\top U^\top \\ &= U\Sigma I\Sigma^\top U^\top \\ AA^\top &= U\Sigma\Sigma^\top U^\top \end{aligned}$$

Here $AA^\top$ is symmetric matrix
$\therefore$ u's are orthonormal eigenvectors of $AA^\top$. conclusion 1

Now, SVD Equation for $v_1, ..., v_r$ and $u_1, ..., u_r$,

$$Av_i = \sigma_i u_i$$

we can say that u_i's will be some linear combination of vector of A.
$\therefore$ u_i's are in the column space of A.
$\therefore u_i \in C(A)$ where $i = 1, 2, , , , r$.
$\therefore (\mathrm{u}_1, u_2, ..., u_r) \in C(A)$ eq. 1

Now from the above conclusion 1 we know all the u's $(u_1, u_2, ..., u_r, u_{r+1}, ..., u_m)$ are orthogonal to each other.
$\therefore$ we can say $(u_1, u_2, ..., u_r)$ are orthogonal to $(u_{r+1}, ..., u_m)$.
$\therefore (u_1, u_2, ..., u_r) \perp (u_{r+1}, ..., u_m)$

We know that $C(A) \perp N(A^\top)$ and eq.1
$\therefore$ we can conclude that $(u_{r+1}, ..., u_m) \in N(A^\top)$.

Now,

$$\begin{aligned}
A^\top A &= (U\Sigma V^\top)^\top (U\Sigma V^\top) \\
&= V\Sigma^\top U^\top U\Sigma V^\top \\
&= V\Sigma^\top I\Sigma V^\top \\
&= V\Sigma^\top \Sigma V^\top
\end{aligned}$$

Here $A^\top A$ is symmetric matrix
$\therefore$ v's are orthonormal eigenvectors of $A^\top A$ conclusion 2. Now,

$$\begin{aligned}
A &= U\Sigma V^\top \\
AV &= U\Sigma V^\top V \\
AV &= U\Sigma I && \text{(V is Orthogonal matrix)} \\
AV &= U\Sigma
\end{aligned}$$

we know that $AA^\top$ and $A^\top A$ have same eigenvalues.
So, in Σ we have r non-zero eigenvalues which are equal for both $AA^\top$ and $A^\top A$.
So (m-r) rows and (n-r) columns of Σ have 0's.
So we will get,

$$A(v_{r+1}, v_{r+2}, ..., v_n) = (0, .., 0)$$

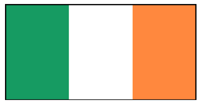
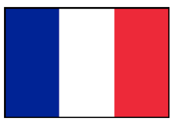
So, we can say $(v_{r+1}, v_{r+2}, ..., v_n) \in N(A)$ eq.2
from conclusion 2 we know that all v's are orthogonal.
$\therefore (v_1, v_2, .., v_r, v_{r+1}, .., v_n)$ are orthogonal to each other.
$\therefore$ we can write as $(v_1, v_2, .., v_r) \perp (v_{r+1}, .., v_n)$.
We know that $N(A) \perp C(A^\top)$ and eq.2
we can conclude that $(v_{r+1}, .., v_n) \in C(A^\top)$,

Hence we proved that vector of U and V are the basis vector's of four fundamental subspace of A.

16. (2 points) Fun with flags.

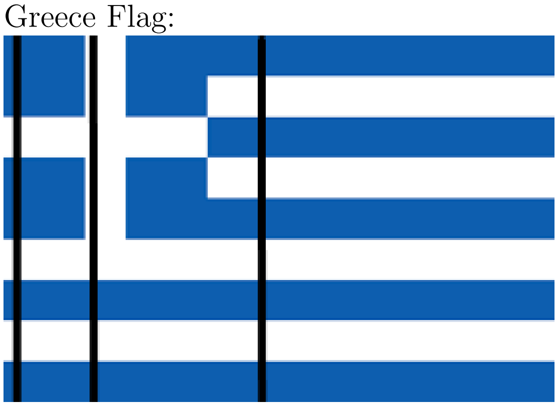
(a) [1 point] Browse through the flags of all countries and paste 5 rank one flags below.

Solution:

1. Ireland Flag: 
2. Poland Flag: 
3. Italy Flag: 
4. Russia Flag: 
5. France Flag: 

(b) [1 point] What is the rank of the flag of Greece?

Solution:
Greece Flag:



Rank of flag of Greece is 3.
Here i have depicted this 3 independent columns with black lines in the above image. All other columns will be dependent on this 3 columns.

17. (2 points) Consider the LFW dataset (Labeled Faces in the Wild).

(a) [1 point] Perform PCA using this dataset and plot the first 25 eigenfaces (in a

5 × 5 grid)

Solution: Here is something to get you started.

```
import matplotlib.pyplot as plt
from mpl_toolkits.axes_grid1 import ImageGrid
from sklearn.datasets import fetch_lfw_people
from sklearn.decomposition import PCA
from PIL import Image
import numpy as np

# Load data
lfw_dataset = fetch_lfw_people(min_faces_per_person=100)

_, h, w = lfw_dataset.images.shape
X = lfw_dataset.data

# Compute a PCA
n_components = 100
pca = PCA(n_components=n_components, whiten=True).fit(X)



 # reverse of flatten
principle_components = 25
eigenfaces = np.zeros(shape=[principle_components,h,w])

for l in range(principle_components):
    k=0
    for i in range(h):
        for j in range(w):
            eigenfaces[l][i][j] = pca.components_[l][k]
            k+=1

fig = plt.figure(figsize=(7., 7.))
grid = ImageGrid(fig, 111,  # similar to subplot(111)
                 nrows_ncols=(5, 5),  # creates 2x2 grid of axes
                 axes_pad=0.1,  # pad between axes in inch.
                 )

for ax, im in zip(grid, eigenfaces):
    # Iterating over the grid returns the Axes.
    ax.imshow(im,cmap = 'gray')
```

```
plt.show()


# img_dir = "image dir path"
test_img = Image.open(img_dir)
test_img = test_img.convert('L')
test_img = test_img.resize((h,w), Image.ANTIALIAS)
test_img_array = np.array(test_img)
test_img_vector = test_img_array.flatten(order = 'C')


eigenbasis = pca.components_[:principle_components]

projection_scalar = np.matmul(eigenbasis,test_img_vector.reshape(h*w,1))
projected_image_vectors = np.zeros(shape=[principle_components,h*w])

projected_image_vectors[0] = projection_scalar[0] * eigenbasis[0]

for i in range(1,principle_components):
    projected_image_vector = projection_scalar[i] * eigenbasis[i]
    projected_image_vectors[i] = projected_image_vector + projected_image_vecto

# reverse of flatten
projected_faces = np.zeros(shape=[principle_components,h,w])

for l in range(principle_components):
    k=0
    for i in range(h):
        for j in range(w):
            projected_faces[l][i][j] = projected_image_vectors[l][k]
            k+=1


plt.figure(figsize=(3., 3.))
plt.imshow(test_img,cmap = 'gray')
plt.show()  # display it

fig = plt.figure(figsize=(7., 7.))
grid = ImageGrid(fig, 111,  # similar to subplot(111)
                 nrows_ncols=(5, 5),  # creates 2x2 grid of axes
                 axes_pad=0.1,  # pad between axes in inch.
                 )
```

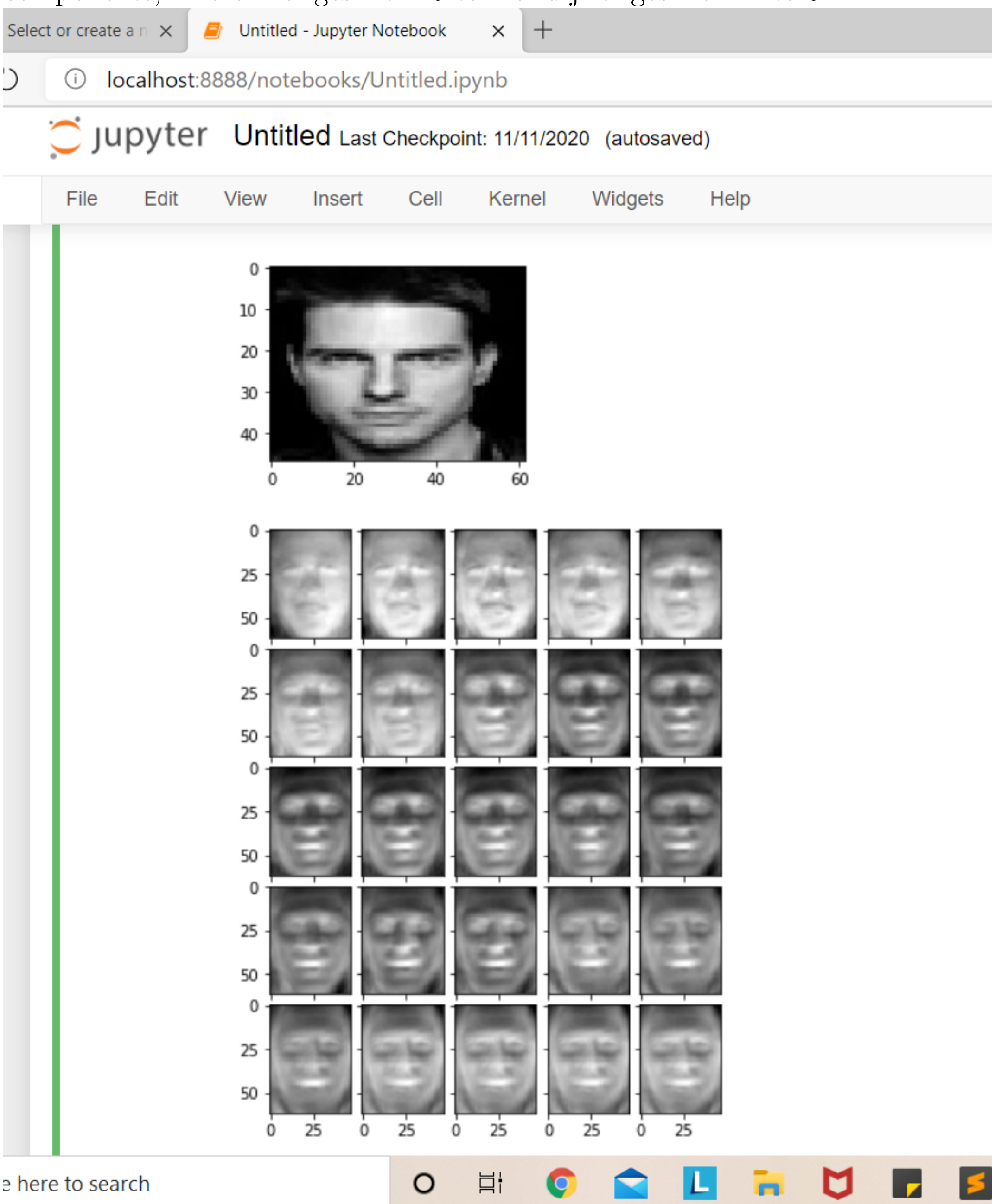
```
for ax, im in zip(grid, projected_faces):
    # Iterating over the grid returns the Axes.
    ax.imshow(im,cmap = 'gray')

plt.show()
```

Here is the plot of first 25 eigenfaces.

(b) [1 point] Take your close-up photograph (face only) and reconstruct it using the first 25 eigenfaces :-). If due to privacy concerns, you do not want to to use your own photo then feel free to use a publicly available close-up photo (face only) of your favorite celebrity.

Solution: I computed the 5 by 5 grid as every image represents the projection of test image on 25 components in cumulative way.
where image $g(i, j)$ in grid represents projection of test image on $(i * 5 + j)$ components, where i ranges from 0 to 4 and j ranges from 1 to 5.



...And that concludes the story of *How I Met Your Eigenvectors :-)* (I hope you enjoyed it!)