Worksheet on "Multivariate Normal and Bayes Classifier"

PRML – CS5691 (Jul–Nov 2023)

August 11, 2023

1. (a) Consider a continuous random variable X and a discrete random variable Y. Let
   - $P_Y(Y = 1) = 0.5$ and $P_Y(Y = -1) = 0.5$, and
   - $(X|Y = 1) \sim \text{Unif}(1, 2)$ and $(X|Y = -1) \sim \text{Unif}(-2, -1)$.

   Draw the plots for $P(Y = 1|X = x)$ and $P(Y = -1|X = x)$ given the above assumptions.

   (b) Consider the following setting:
   - $P_Y(Y = 1) = 0.7$ and $P_Y(Y = -1) = 0.3$
   - $(X|Y = 1) \sim Unif(-1, 3)$ and $(X|Y = -1) \sim Unif(-2, 0)$

   1. Compute $P(Y = 1|X = x)$ for different possible values of x.
   2. Draw the plot for $P(Y = 1|X = x)$.

2. In this question, you are required to verify if the following probability mass function over its respective support S follows the following properties:

   1. $P(X = x) \geq 0 \quad \forall x \in S$, and
   2. $\sum_{x \in S} P(X = x) = 1$.

   In addition, find the expectation, $\mathbb{E}(X)$ and variance, $Var(X)$ in the following case: A discrete random variable X is said to have a Poisson distribution, with parameter $\lambda > 0$ over the support $S = \{0, 1, 2, \dots\}$ if it has the following probability mass function:

   $$P(X = x) = \frac{\lambda^x e^{-\lambda}}{x!}$$

   Hint: $\sum_{n=1}^{\infty} \frac{a^n}{n!} = e^a$

3. Consider a multivariate normal $X \sim N(\mu, \Sigma)$ where $X = \begin{pmatrix} X_1 & X_2 \end{pmatrix}$, $d = 2$, $\mu \in \mathbb{R}^2$ and $\Sigma \in \mathbb{R}^{2\times 2}$. Then, the density is defined as: $f_X(x) = \frac{1}{(2\pi)\sqrt{|\Sigma|}} exp\left(-\frac{1}{2}(x-\mu)^T\Sigma^{-1}(x-\mu)\right)$.

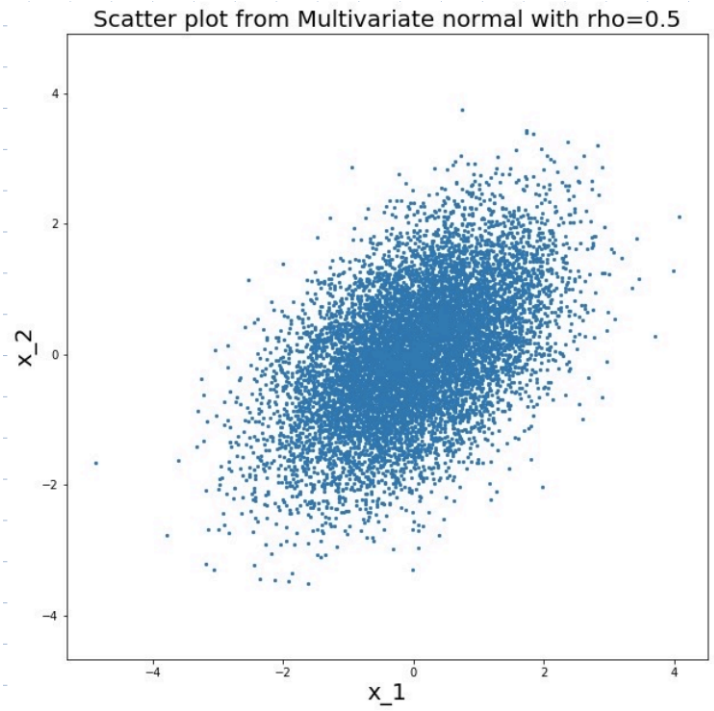
   (a) If $\Sigma = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$, $\Sigma^{-1} = \frac{1}{1-\rho^2}\begin{pmatrix} 1 & -\rho \\ -\rho & 1 \end{pmatrix}$, and $\mu = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. What are the legal values for ρ?

   (b) If $\rho = 0.5$ and $X_1 \sim N(0, 1)$, what is the distribution for $X_2|X_1 = 4$?

   (c) If $\rho = 0.5$ and $X_2 \sim N(0, 1)$, what is the distribution for $X_1|X_2 = 3$?

   (d) Consider the following scatter plot from a multivariate normal with $\rho = 0.5$. Draw the corresponding contour plot and mark the lines depicting the means of X_1 and X_2.

   (e) If $\Sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, and $\mu = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, what is the distribution of $X_2|X_1 = x_1$ and $X_1|X_2 = x_2$?


Scatter plot from Multivariate normal with rho=0.5
x_2
x_1
4
2
0
−2
−4
−4
−2
0
2
4