(CS5020, Jul-Nov 2023) Nonlinear Optimisation: Theory and Algorithms
Worksheet - 5

(1) Let $x \in \mathbb{R}^3$ be a some given vector. It is known that for this vector, $||x||_\infty \leq 5$. Find the smallest constant C such that $||x||_1 \leq C$

(2) Draw the set $B_\infty(x_0, 3) \cap B_2(x_1, 3)$, where $x_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

(3) Consider a point $x = \begin{bmatrix} 11 \end{bmatrix}$ in the standard basis. Let xV be its representation in the basis $V = \{v_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}\}$. Find x_V

(4) Let T be a linear transformation from $\mathbb{R}^3 \to \mathbb{R}^3$. We know that $T(e_1) = e2, T(e_2) = e3$ and $T(e_3) = e_1$. Let M_T be the matrix representation of T in the standard basis. Find M_T and M_T^2.

(5) A real symmetric $H \in \mathbb{R}^{d \times d}$ can be decomposed as $H = VDV^\top$ where D is a diagonal matrix whose i^{th} diagonal entry is the i^{th} eigenvalue of H and $V = [v_1 ... v_d]$ is the matrix whose i^{th} is the corresponding eigenvector. For the matrix $H = \begin{bmatrix} 0.5 & 1.5 \\ 1.5 & 0.5 \end{bmatrix}$ find V and D. Is the matrix H positive definite?

(6) Given 5 examples of 3×3 positive definite real symmetric matrices.

(7) Let $f_1(x) = \frac{1}{2}(5x(1)^2 + x(2)^2)$, and $f_2(x) = \frac{1}{2}(5x(1)^2 - x(2)^2)$.

(a) Give first and second order approximation for f_1 and f_2 at $\hat{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

(b) Give the set of directions at $\hat{x} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ that will decrease f_1.

(c) Give the set of directions at $\hat{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ that will increase f_2.

(d) Draw the contours of f_1 and f_2.

(8) For $f(x) = x(1) \exp\left(-(x(1)^2 + x(2)^2)\right)$

(a) Plot the function in a Jupyter notebook to get an idea of how it looks like (plotting question will not be examination).

(b) Find the gradient and stationary points (i.e, x_* such that $\nabla f(x_*) = 0$).

(c) Find the first and second order approximation of f at $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ -1 \end{bmatrix}$.

(d) Find the local minima and local maxima.

(e) Draw the contours of f.

(f) Give the set of directions at $\hat{x} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ that will decrease f.

(g) Give the set of directions at $\hat{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ that will increase f.

(9) $f(x) = (x(2) - x(1)^2)^2 + x(1)^5$

(a) Plot the function in a Jupyter notebook to get an idea of how it looks like (plotting question will not be examination).

(b) Find the gradient and stationary points (i.e, x_* such that $\nabla f(x_*) = 0$).

(c) Find the first and second order approximation of f at $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ -1 \end{bmatrix}$.

(d) Find the local minima and local maxima.

(e) Give the set of directions at $\hat{x} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ that will decrease f.

(f) Give the set of directions at $\hat{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ that will increase f.

(10) Consider the gradient descent update for the function
$f(x) = \frac{1}{2}(x(1)^2 + 2x(2)^2 + 4x(3)^2)$.

(a) Find the allowable range of step size choices.

(b) The best stepsize.

(c) Split the ranges of stepsizes based on the different ways (oscillatory or non-oscillatory) in which the 3 variables converge to $x_* = 0$.