Chapter 2

Elementary Row Operations and Linear Systems

2.1 Homogeneous and Nonhomogeneous Systems

Let us consider a matrix $A \in \mathcal{F}^{m \times n}$. We are interested in solving the system

$$Ax = b \tag{2.1.1}$$

for different $b \in \mathcal{F}^m$. In particular, when we take $b = \theta_m$, we get the system

$$Ax = \theta_m \tag{2.1.2}$$

The system (2.1.2) is called the "**Homogeneous System**" corresponding to the matrix A. When $b \neq \theta_m$ the system (2.1.1) is called a "**Nonhomogeneous system**".

Let us now look at the homogeneous system (2.1.2). Clearly $x = \theta_n$ is a solution of this system. Thus the homogeneous system is always consistent, since there exists a solution $x = \theta_n$. This solution is called the "**Trivial Solution**". If $x \neq \theta_n$ is a solution of the homogeneous system then it is called a "**Nontrivial solution**". A homogeneous equation may or may not have nontrivial solutions. Let us now look at some simple examples.

Example 2.1.1 Let

$$A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

Then the corresponding homogeneous system can be written as

$$\left.\begin{array}{rcl} x_1 + x_2 &=& 0 \\ x_1 - x_2 &=& 0 \end{array}\right\} \tag{2.1.3}$$

Clearly, $x_1 = x_2 = 0$, that is, $x = \theta_2$, is a solution of this system, and it is the only solution of this system. Thus this system has only trivial solution

Example 2.1.2 Let

$$A = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}$$

Then the corresponding homogeneous system can be written as

$$\left.\begin{array}{rcl} x_1 - x_3 &=& 0 \\ x_2 - x_3 &=& 0 \end{array}\right\} \tag{2.1.4}$$

Clearly, $x_1 = x_2 = x_3 = 0$, that is, $x = \theta_2$, is a solution of this system. Further if we take $x_1 = x_2 = x_3 = 1$, that is, $x = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, then x is also a solution of this system. In fact any vector of the form, $x = \begin{pmatrix} \alpha \\ \alpha \\ \alpha \end{pmatrix}$, where α is any element of $\mathcal{F}$, is a solution of this system. Thus this system has nontrivial solutions.

The fact whether the homogeneous system does or does not have nontrivial solutions has a bearing on the uniqueness or otherwise of the solution of the nonhomogeneous system, when the nonhomogeneous system is consistent. We shall now investigate this aspect.
Suppose the nonhomogeneous system $Ax = b$ is consistent. Then
The nonhomogeneous system has more than one solution $\Longrightarrow$
There exist $u, v \in \mathcal{F}^n$ such that $u \neq v$ and $Au = b$ and $Av = b$. This $\Longrightarrow$
The vector $w = u - v$ is in $\mathcal{F}^n$ and $A(u - v) = Au - Av = b - b = \theta_m$, and $w \neq \theta_n$. This $\Longrightarrow$
w is a nontrivial solution of the homogeneous system. Thus we have

$$\left.\begin{array}{l} \text{A consistent nonhomogeneous system } Ax = b \\ \text{has more than one solution} \\ \Longrightarrow \\ \text{The Homogeneous system } Ax = \theta_m \\ \text{has a nontrivial solution} \end{array}\right\} \tag{2.1.5}$$

Conversely suppose the homogeneous system has a nontrivial solution. Let $x_H \neq \theta_n$ be a nontrivial solution of the homogeneous system. Hence

$$Ax_H = \theta_m \tag{2.1.6}$$

The nonhomogeneous system $Ax = b$ is consistent and hence there exists a solution $u \in \mathcal{F}^n$ to the nonhomogeneous system, that is,

$$Au = b \tag{2.1.7}$$

If we define

$$v = u + x_H \tag{2.1.8}$$

then we have

$$\begin{aligned} Av &= A(u + x_H) \\ &= Au + Ax_H \\ &= b + \theta_m \text{ by (2.1.7) and (2.1.6)} \\ &= b \end{aligned}$$

Hence v is also a solution of the nonhomogeneous system, and since $x_H \neq \theta_n$, we have $v \neq u$. Thus u and v are two different solutions of the nonhomogeneous system. Thus we get that that the nonhomogeneous equation has more than one solution. Hence we have

$$\left.\begin{array}{l} \text{The Homogeneous system } Ax = \theta_m \\ \text{has a nontrivial solution} \\ \Longrightarrow \\ \text{The consistent nonhomogeneous system } Ax = b \\ \text{has more than one solution} \end{array}\right\} \tag{2.1.9}$$

Combining (2.1.5) and (2.1.9) we get the following proposition:

Proposition 2.1.1 Let the nonhomogeneous system $Ax = b$ be consistent. Then
The Nonhomogeneous system $Ax = b$ has more than one solution
$\Longleftrightarrow$
The homogeneous system has nontrivial solution.

This can also be restated as follows:
When the nonhomogeneous system $Ax = b$ is consistent,
The Nonhomogeneous system $Ax = b$ has a unique solution
$\Longleftrightarrow$
The homogeneous system has only the trivial solution.

Remark 2.1.1 When x_H is a nontrivial solution of the homogeneous system $Ax = \theta_m$ and u is a solution of the nonhomogeneous system $Ax = b$, then for any $\alpha \in \mathcal{F}$ we have $v = u + \alpha x_H$ is also solution of the nonhomogeneous system. Thus, in particular, if $\mathcal{F} = \mathbb{R}$ or $\mathbb{C}$, we see that there are infinite number of solutions for the nonhomogeneous system.

What is the consequence of all this? Suppose we want to solve the nonhomogeneous system $Ax = b$ and suppose it is consistent, that is, suppose b satisfies the consistency conditions. Then there exists a solution for the nonhomogeneous system. Suppose we know one solution, say x_P, of the nonhomogeneous system. Hence

$$Ax_P = b \tag{2.1.10}$$

Let v be any other solution of the nonhomogeneous system. Then we must have

$$Av = b \tag{2.1.11}$$

Hence we get

$$\begin{aligned} A(v - x_P) &= Av - Ax_P \\ &= b - b \\ &= \theta_m \end{aligned}$$

Thus $v - x_P$ is a solution of the homogeneous system. Hence any solution v is of the form

$$v = x_P + x_H \tag{2.1.12}$$

Thus every solution of the nonhomogeneous system is of the above form. Conversely, it is easy to check that any v of the above form is a solution of the nonhomogeneous system. Hence all solutions of the nonhomogeneous system are obtained once we know one solution x_P of the nonhomogeneous system and all solutions of the homogeneous system. Then by adding these solutions of the homogeneous system to x_P we get all the solutions of the nonhomogeneous system. Thus finding the solution of the nonhomogeneous system has two parts, namely,

1. Find all solutions of the homogeneous system, and

2. Find a particular solution of the nonhomogeneous system

Then we can generate all solutions of the nonhomogeneous system as observed above. The main tools that we first introduce for this purpose are the so called Elementary Row Operations.

2.2 Elementary Row Operations (EROs)

We introduce three types of elementary row operations. These are simple operations performed on the rows of a matrix.
ERO of Type I:
Row interchange:
In this operation we keep all rows except two unchanged, and remaining two rows interchange their positions If the ith row and jth rows are interchanged we denote this operation by R_{ij}.

Example 2.2.1

$$A = \begin{pmatrix} 0 & 1 & 2 \\ 3 & 2 & 1 \end{pmatrix} \xrightarrow{R_{12}} \begin{pmatrix} 3 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$$

Example 2.2.2

$$A = \begin{pmatrix} 1 & -1 & 3 \\ 0 & 1 & 2 \\ -2 & -3 & 1 \\ 0 & 8 & 4 \end{pmatrix} \xrightarrow{R_{23}} B = \begin{pmatrix} 1 & -1 & 3 \\ -2 & -3 & 1 \\ 0 & 1 & 2 \\ 0 & 8 & 4 \end{pmatrix}$$

We observe the following simple, but important, properties of ERO of Type I :

1. If
$$A \xrightarrow{R_{ij}} A_1 \text{ then } A_1 \xrightarrow{R_{ij}} A$$
This means that the ERO R_{ij} is invertible and its inverse is R_{ij} itself. We have $(R_{ij})^{-1} = R_{ij}$

2. If A_1 is obtained from A by an ERO of Type I then the homogeneous system $Ax = \theta_m$ and $A_1 x = \theta_m$ have the same set of solutions. This is because both the systems have the same set of equations, only difference being that they are in different order.

   Example 2.2.3 Consider the matrix A of Example 2.2.1 and the A_1 obtained from it by the ERO R_{12}. The Homogeneous system $Ax = \theta_3$ is given by

$$\left.\begin{array}{lcl} x_1 + 2x_3 & = & 0 \\ 3x_1 + 2x_2 + x_3 & = & 0 \end{array}\right\} \qquad (2.2.1)$$

   The Homogeneous system $A_1 x = \theta_3$is given by

$$\left.\begin{array}{lcl} 3x_1 + 2x_2 + x_3 & = & 0 \\ x_1 + 2x_3 & = & 0 \end{array}\right\} \qquad (2.2.2)$$

   Clearly both the systems have the same set of solutions since both are same systems except that the equations are written in different order.

3. Let A be an $m \times n$ matrix. Consider the $m \times m$ identity matrix I_m. Then if

$$I \xrightarrow{R_{ij}} E$$

   then

$$A \xrightarrow{R_{ij}} B = EA$$

   Thus the ERO R_{ij} on A can be effected by premultiplying A by E where E is the matrix obtained by applying R_{ij} to I_m.

   Example 2.2.4 In Example 2.2.1 we have

$$I_2 \xrightarrow{R_{12}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = E$$

$$\begin{array}{rcl} EA & = & \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 2 \\ 3 & 2 & 1 \end{pmatrix} \\ & = & \begin{pmatrix} 3 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} \\ & = & A_1 \end{array}$$

Example 2.2.5 For the matrix in Example 2.2.2 we have

$$I_4 \xrightarrow{R_{23}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = E$$

We have

$$\begin{aligned} EA &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 3 \\ 0 & 1 & 2 \\ -2 & -3 & 1 \\ 0 & 8 & 4 \end{pmatrix} \\ &= \begin{pmatrix} 1 & -1 & 3 \\ -2 & -3 & 1 \\ 0 & 1 & 2 \\ 0 & 8 & 4 \end{pmatrix} \\ &= A_1 \text{ (as obtained in Example 2.2.2)} \end{aligned}$$

<u>ERO of Type II</u>:
Adding a Multiple of one row to another row:
In this operation we keep all but one row, say the jth row, unchanged, and the jth row is changed by adding a multiple of another row, say the ith row, to it. If α times the ith row is added to the jth row we denote this operation by $R_j + \alpha R_i$.

Example 2.2.6

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 0 & 1 \\ 3 & 1 & 2 \end{pmatrix} \xrightarrow{R_3+2R_1} \begin{pmatrix} 1 & 1 & 0 \\ 2 & 0 & 1 \\ 5 & 3 & 2 \end{pmatrix}$$

Example 2.2.7

$$A = \begin{pmatrix} 1 & -1 & 3 \\ 0 & 1 & 2 \\ -2 & -3 & 1 \\ 0 & 8 & 4 \end{pmatrix} \xrightarrow{R_4+2R_1} A_1 = \begin{pmatrix} 1 & -1 & 3 \\ 0 & 1 & 2 \\ -2 & -3 & 1 \\ 2 & 6 & 10 \end{pmatrix}$$

We observe the following simple, but important, properties of ERO of Type II :

1. If
$$A \stackrel{R_j+\alpha R_i}{\longrightarrow} A_1 \text{ then } A_1 \stackrel{R_j+(-\alpha)R_i}{\longrightarrow} A$$
This means that the ERO $R_i + \alpha R_j$ is invertible and its inverse is $R_i + (-\alpha)R_j$ again an ERO of the same type. We have $(R_j + \alpha R_i)^{-1} = R_j + (-\alpha)R_i$

2. If A_1 is obtained from A by an ERO of Type II then the homogeneous system $Ax = \theta_m$ and $A_1x = \theta_m$ have the same set of solutions. This is because both the systems have the same set of equations, only difference being that the ith equation has been replaced by the adding the jth equation to it.

   Example 2.2.8 For the matrix of Example 2.2.6 we have,
   The Homogeneous system $Ax = \theta_3$ is given by
$$\left.\begin{array}{lcl} x_1 + x_2 & = & 0 \\ 2x_1 + x_3 & = & 0 \\ 3x_1 + x_2 + x_3 & = & 0 \end{array}\right\} \tag{2.2.3}$$
   The Homogeneous system $A_1x = \theta_3$ is given by
$$\left.\begin{array}{lcl} x_1 + x_2 & = & 0 \\ 2x_1 + x_3 & = & 0 \\ 5x_1 + 3x_2 + x_3 & = & 0 \end{array}\right\} \tag{2.2.4}$$
   It is easy to see that these two have the same set of solutions.

3. Let A be an $m \times n$ matrix. Consider the $m \times m$ identity matrix I_m. Then if
$$I \stackrel{R_i+\alpha R_j}{\longrightarrow} E$$
   then
$$A \stackrel{R_i+\alpha R_j}{\longrightarrow} A_1 = EA$$
   Thus the ERO $R_i + \alpha R_j$ on A can be effected by premultiplying A by E where E is the matrix obtained by applying $R_i + \alpha R_j$ to I_m.

   Example 2.2.9 Corresponding to the Example 2.2.6 we have
$$I_3 \stackrel{R_3+2R_1}{\longrightarrow} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 2 & 0 & 1 \end{pmatrix} = E$$

$$\begin{aligned} EA &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 2 & 0 & 1 \\ 3 & 1 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 1 & 0 \\ 2 & 0 & 1 \\ 5 & 3 & 2 \end{pmatrix} \\ &= A_1 \end{aligned}$$

Example 2.2.10 We have for the matrix of Example 2.2.7,

$$I_4 \xrightarrow{R_4+2R_1} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 \end{pmatrix} = E$$

Further

$$\begin{aligned} EA &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 3 \\ 0 & 1 & 2 \\ -2 & -3 & 1 \\ 0 & 8 & 4 \end{pmatrix} \\ &= \begin{pmatrix} 1 & -1 & 3 \\ 0 & 1 & 2 \\ -2 & -3 & 1 \\ 2 & 6 & 10 \end{pmatrix} \\ &= A_1 \text{ as obtained in Example 2.2.7} \end{aligned}$$

ERO of Type III:
Multiplying a row by a nonzERO scalar:
In this operation we keep all rows except one row unchanged, and one row is multiplied by a nonzERO scalar. If the ith row is multiplied by α we denote this operation by αR_i.

Example 2.2.11

$$A = \begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & 2 \end{pmatrix} \xrightarrow{(-3R_2)} \begin{pmatrix} 1 & 0 & -1 \\ -6 & -3 & -6 \end{pmatrix} = A_1$$

Example 2.2.12

$$A = \begin{pmatrix} 1 & -1 & 3 \\ 0 & 1 & 2 \\ -2 & -3 & 1 \\ 0 & 8 & 4 \end{pmatrix} \xrightarrow{2R_1} A_1 = \begin{pmatrix} 2 & -2 & 6 \\ 0 & 1 & 2 \\ -2 & -3 & 1 \\ 0 & 8 & 4 \end{pmatrix}$$

We observe the following simple, but important, properties of ERO of Type I :

1. If
$$A \xrightarrow{\alpha R_i} A_1 \text{ then } A_1 \xrightarrow{(\alpha^{-1})R_i} A$$
This means that the ERO αR_i is invertible and its inverse is $(\alpha^{-1})R_i$ again an ERO of the same type.

2. If A_1 is obtained from A by an ERO of Type III then the homogeneous system $Ax = \theta_m$ and $A_1 x = \theta_m$ have the same set of solutions. This is because both the systems have the same set of equations, only difference being that the ith equation has been multiplied throughout by the nonzERO scalar α.

   Example 2.2.13 For the Example 2.2.10 we have,
   The Homogeneous system $Ax = \theta_2$ is given by
$$\left.\begin{array}{rcl} x_1 - x_3 & = & 0 \\ 2x_1 + x_2 + 2x_3 & = & 0 \end{array}\right\} \tag{2.2.5}$$
   The Homogeneous system $A_1 x = \theta_3$ is given by
$$\left.\begin{array}{rcl} x_1 - x_3 & = & 0 \\ -6x_1 - 3x_2 - 6x_3 & = & 0 \end{array}\right\} \tag{2.2.6}$$
   Clearly these two have the same set of solutions.

3. Let A be an $m \times n$ matrix. Consider the $m \times m$ identity matrix I_m. Then if
$$I \xrightarrow{\alpha R_i} E$$
then
$$A \xrightarrow{\alpha R_i} A_1 = EA$$
Thus the ERO αR_i on A can be effected by premultiplying A by E where E is the matrix obtained by applying αR_i to I_m.

Example 2.2.14 For the matrix in Example 2.2.10 we have

$$I_2 \xrightarrow{(-3)R_2} \begin{pmatrix} 1 & 0 \\ 0 & -3 \end{pmatrix} = E$$

$$\begin{aligned} EA &= \begin{pmatrix} 1 & 0 \\ 0 & -3 \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & -1 \\ -6 & -3 & -6 \end{pmatrix} \\ &= A_1 \end{aligned}$$

Example 2.2.15 We have for the matrix of Example 2.2.11

$$I_4 \xrightarrow{2R_1} \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = E$$

and

$$\begin{aligned} EA &= \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 3 \\ 0 & 1 & 2 \\ -2 & -3 & 1 \\ 0 & 8 & 4 \end{pmatrix} \\ &= \begin{pmatrix} 2 & -2 & 6 \\ 0 & 1 & 2 \\ -2 & -3 & 1 \\ 0 & 8 & 4 \end{pmatrix} \\ &= A_1 \text{ as obtained in Example 2.2.11} \end{aligned}$$

From the above it follows that all EROs

(a) are invertible and the inverses are also EROs of the same type,

(b) are such that they do not alter the set of solutions of the homogeneous sytem, and

(c) can be effected by premultiplying by a matrix E obtained from the identity matrix I_m by applying the same ERO.

Definition 2.2.1 A matrix $E \in \mathcal{F}^{m\times m}$ obtained by applying an ERO to the identity matrix I_m is called an "**Elementary Matrix** " (of order m)

We denote by $\mathcal{E}_m$ the set of all elementary matrices of order m. From our discussion it follows that,

(a) Every elementary matrix is invertible

(b) EROs on $m\times n$ can be effected by premultiplication by elementary matrices of order m

(c) If E is any elementary matrix of order m and A is any $m \times n$ matrix, then the homogeneous systems $EAx = \theta_m$ and $Ax = \theta_m$ have the same set of solutions

If we apply a finite number os EROs of the above three types in any order and obtain a matrix B then the homogeneous systems $Ax = \theta_m$ and $Bx = \theta_m$ will have the same set of solutions. The basic strategy is to use EROs and obtain the matrix B from A such that $Bx = \theta_m$ is a simple system and hence easy to solve. Then since $Ax = \theta_m$ and $Bx = \theta_m$ have the same set of solutions, by solving the easy system $Bx = \theta_m$, we would have also solved the given system $Ax = \theta_m$. We shall see next what type of simple matrix B we can reduce A to, using EROs.

2.3 Row Equivalence

The notion of EROs and the fact that the EROs do not alter the solution set of the Homogeneous system lead to an equivalence notion among matrices. We have

Definition 2.3.1 A matrix $A \in \mathcal{F}^{m\times n}$ is said to be "**Row Equivalent**" to a matrix $B \in \mathcal{F}^{m\times n}$ if there exists a finite number of EROs $E_1, E_2, \cdots, E_k$ such that B can be obtained from A by applying this finite sequence of EROs

If A is row equivalent to B we write $A \stackrel{R}{\sim} B$. Since EROs do not alter the solution set of a Homogeneous system, it follows that if two matrices are row equivalent they have the same set of solutions for their homogeneous systems. It is easy to see that the following properties hold:

(a) $A \in \mathcal{F}^{m\times n} \Longrightarrow A \stackrel{R}{\sim} A$ (Reflexivity of $\stackrel{R}{\sim}$)

(b) If $A, B \in \mathcal{F}^{m\times n}$ then $A \stackrel{R}{\sim} B \Longleftrightarrow B \stackrel{R}{\sim} A$. (Symmetry of $\stackrel{R}{\sim}$) This follows from the fact that EROs are invertible

(c) If $A, B, C \in \mathcal{F}^{m\times n}$ then $A \stackrel{R}{\sim} B$ and $B \stackrel{R}{\sim} C \Longrightarrow A \stackrel{R}{\sim} C$. (Transitivity of $\stackrel{R}{\sim}$)

A relation $\sim$ on a set is said to be an equivalence relation on that set if it is reflexive, symmetric and transitive. Hence it follows that $\stackrel{R}{\sim}$ is an equivalence relation on $\mathcal{F}^{m\times n}$. As with every equivalence relation, the equivalence relation $\stackrel{R}{\sim}$ partitions the set of matrices $\mathcal{F}^{m\times n}$ into distinct equivalence classes. Any two matrices A and B in the same equivalence class will have the same set of solutions for their Homogeneous systems, since they are row equivalent. What we would therefore like to do is to look at one representative from each class which can be easily studied and try to reduce a given matrix by EROs to one of these representatives. What type of representatives we choose? This leads us to the notion of "Row Reduced Echelon" matrices.

Definition 2.3.2 A matrix $A \in \mathcal{F}^{m\times n}$ is said to be a "**Row Reduced Echelon**", (RRE for short), matrix if it satisfies the following properties:

(a) Zero rows, if any, are below all nonzero rows

(b) The first nonzero entry (read from the left) in each nonzero row is 1. (This is called the pivotal 1 of that row)

(c) The pivotal 1 in any nonzero row is to the right of the pivotal 1s of all the previous rows, that is, if the nonzero rows are

$R_1, R_2, \cdots, R_\rho$, and the pivotal 1 in R_i appears in the column $C_{k_i},\ i = 1, 2, \cdots, \rho$, then,

$$k_1 < k_2 < \cdots < k_{(i-1)} < k_i < k_{(i+1)} < \cdots < k_\rho$$

.

(d) All other entries in the column that has a pivotal 1 are zero, that is, if R_i has pivotal 1 in k_ith column then $a_{jk_i} = 0$ for all $j \neq i$.

Remark 2.3.1 It is not difficult to see that two different RRE matrices cannot be row equivalent.

Example 2.3.1 The matrix

$$A_R = \begin{pmatrix} 1 & 2 & 0 & 2 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

is a RRE matrix. The nonzero rows are R_1 and R_2. The k_1 and k_1 in this case are 1 and 3 since the pivotal 1s appear in columns C_1 and C_3 respectively. We have $k_1 < k_2$ since $1 < 3$. All other entries in columns C_1 and C_3 are zero.

Example 2.3.2 The matrices

$$A = \begin{pmatrix} 1 & 2 & 2 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},\ B = \begin{pmatrix} 0 & 1 & 2 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},\ \text{and } C = \begin{pmatrix} 1 & 2 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

are not RRE matrices. (For each of these examine which requirement for RRE matrix is not satisfied)

2.4 Homogeneous System With RRE Matrices

We shall next look at homogeneous systems corresponding to an RRE matrix. Let $A_R \in \mathcal{F}^{m\times n}$ be a RRE matrix. Let the nonzero rows be $R_1, R_2, \cdots, R_\rho$ and the pivotal 1s be in the columns $C_{k_1}, C_{k_2}, \cdots, C_{k_\rho}$,(where $k_1 < k_2 < \cdots, k_\rho$). Then the variables $x_{k_1}, x_{k_2}, \cdots, x_{k_\rho}$ are called "**Pivotal Variables**". The remaining $(n-\rho)$ variables are called "**Free Variables**".

Example 2.4.1 For the RRE matrix in Example 2.3.1 we have,
Pivotal Variables: x_1 and x_3
Free Variables: x_2,x_4 and x_5

From the homogeneous system $A_R x = \theta_m$ we can eliminate the pivotal variables x_{k_i} $(1 \le i \le \rho)$ from the ith equation in terms of the free variables, and the free variables can be chosen arbitrarily.

Example 2.4.2 In Example 2.3.1 we can eliminate the pivotal variables x_1, x_3 in terms of the free variables x_2, x_4, x_5 as follows:
The system can be written as

$$\left.\begin{array}{lcl} x_1 + 2x_2 + 2x_4 & = & 0 \\ x_3 + x_4 & = & 0 \end{array}\right\}$$

Eliminating x_1 from the first equation and x_3 from the second equation we get

$$\begin{array}{lcl} x_1 & = & -2x_2 - 2x_4 \\ x_3 & = & -x_4 \end{array}$$

Denoting the arbitrary values of $x_2, x_4,$ and x_5 respectively by α, β and γ, we can write the general solution of the system as

$$x = \alpha \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} -2 \\ 0 \\ -1 \\ 1 \\ 0 \end{pmatrix} + \gamma \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Thus we see that it is easy to solve the homogeneous system corresponding to a RRE matrix.

Remark 2.4.1 We note that the number of pivotal variables is the same as the number of pivotal 1s which is the same as the number of nonzero rows in A_R. This is called the "**Row Rank** " of A_R. The number of free variables is called the "**Nullity**" of A_R. We have

$$\left.\begin{array}{l}\text{Row Rank of } A_R + \text{Nullity of } A_R = \\ \text{Number of Columns in } A_R\end{array}\right\} \tag{2.4.1}$$

2.5 Reduction of a General Matrix to RRE Matrix

We shall now see how to reduce a general matrix $A \in \mathcal{F}^{m\times n}$ to RRE form using EROs. We first describe the basic step below:
First Column Operation
Let A be any matrix. We look at the first column of A and ask the question,
Does the first column have a nonzero entry?
The answer may be Yes or No
Case 1: The answer is Yes.
In this case we do the following:
Bring a nonzero entry to the leading position (that is the first row first column position), if necessary by an ERO of Type I.
Then make this leading nonzero entry as 1 by ERO of Type III
Then make all entries in the first column of A below this 1 as zero by ERO of Type II
The resulting matrix is of the form

$$\left(\begin{array}{c|c} 1 & 0_{1\times(n-1)} \\ \hline 0_{(m-1)\times 1} & \mathrm{A}^{(1)} \end{array}\right)$$

Case 2: The Answer is No
In this case, since the first column has all zeros, the matrix A is of the

form

$$\left(\begin{array}{c|c} 0_{1\times m} & \mathrm{A}^{(1)} \end{array} \right)$$

Now given any matrix A apply the above procedure and get the submatrix $A^{(1)}$ and repeat the process on $A^{(1)}$ to get a submatrix $A^{(2)}$ and so on. Continuing this process we reduce the matrix A to a matrix which satisfies all the requirements of RRE except the requirement 4 (see Definition 2.3.2). We now use RRE of Type II to achieve this and reduce to zero all other entries in a column containing a pivotal 1, (starting from the left most pivotal 1). The resulting matrix is an RRE matrix and Row equivalent to A. Since no other RRE matrix can be row equivalent to this we call this the RRE form of A and we denote this by A_R. We shall illustrate this reduction process by means of an example below.

Example 2.5.1 Consider the matrix

$$A = \begin{pmatrix} 1 & 2 & 2 & 3 \\ 2 & 4 & 1 & 3 \\ 3 & 6 & 1 & 4 \end{pmatrix}$$

Let us now reduce this to its RRE form.

$$A = \begin{pmatrix} 1 & 2 & 2 & 3 \\ 2 & 4 & 1 & 3 \\ 3 & 6 & 1 & 4 \end{pmatrix} \xrightarrow{R_2-2R_1,\ R_3-3R_1} \left(\begin{array}{c|ccc} 1 & 2 & 2 & 3 \\ \hline 0 & 0 & -3 & -3 \\ 0 & 0 & -5 & -5 \end{array} \right)$$

$$\text{(partitioned as)} \left(\begin{array}{c|c|cc} 1 & 2 & 2 & 3 \\ \hline 0 & 0 & -3 & -3 \\ 0 & 0 & -5 & -5 \end{array} \right)$$

$$\xrightarrow{-\frac{1}{3}R_2} \left(\begin{array}{c|c|cc} 1 & 2 & 2 & 3 \\ \hline 0 & 0 & 1 & 1 \\ 0 & 0 & -5 & -5 \end{array} \right) \xrightarrow{R_3+5R_2} \left(\begin{array}{c|c|c|c} 1 & 2 & 2 & 3 \\ \hline 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

$$\xrightarrow{R_1-2R_2} \left(\begin{array}{c|c|c|c} 1 & 2 & 0 & 1 \\ \hline 0 & 0 & 1 & 1 \\ \hline 0 & 0 & 0 & 0 \end{array} \right) = A_R, \text{ the RRE form of } A$$

The RRE form is obtained from A by means of EROs. Since each ERO can be effected by premultiplication by an elementary matrix, by taking the products of all the elementary matrices involved in the reduction, (in the same order as the EROs are performed), we get a matrix E such that

$$EA = A_R, \text{ the RRE form of } A$$

Example 2.5.2 The Elementary Matrices in the above example can be obtained by applying this same sequence of EROs to the identity matrix I_3.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_2 - 2R_1,\ R_3 - 3R_1} \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow{-\frac{1}{3}R_2} \begin{pmatrix} 1 & 0 & 0 \\ \frac{2}{3} & -\frac{1}{3} & 0 \\ -3 & 0 & 1 \end{pmatrix} \xrightarrow{R_3 + 5R_2} \begin{pmatrix} 1 & 0 & 0 \\ \frac{2}{3} & -\frac{1}{3} & 0 \\ \frac{1}{3} & -\frac{5}{3} & 1 \end{pmatrix}$$

$$\xrightarrow{R_1 - 2R_2} \begin{pmatrix} -\frac{1}{3} & \frac{2}{3} & 0 \\ \frac{2}{3} & -\frac{1}{3} & 0 \\ \frac{1}{3} & -\frac{5}{3} & 1 \end{pmatrix} = E$$

It is easy to check that

$$\begin{aligned} EA &= \begin{pmatrix} -\frac{1}{3} & \frac{2}{3} & 0 \\ \frac{2}{3} & -\frac{1}{3} & 0 \\ \frac{1}{3} & -\frac{5}{3} & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 2 & 3 \\ 2 & 4 & 1 & 3 \\ 3 & 6 & 1 & 4 \end{pmatrix} \\ &= \\ &= \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} = A_R, \text{ the RRE form of } A \end{aligned}$$

Remark 2.5.1 Analogous to the EROs we can define Elementary Column Operations (ECOs) and Column Reduced Echelon (CRE) Matrices . We can reduce any matrix to its CRE form by ECOs. Just as an ERO on an $m \times n$ matrix can be effected by premultiplying with an elementary matrix obtained from I_m by applying to it the same ERO, we can effect an ECO on A by postmultiplying A by a matrix obtained from I_n by applying to it the same ECO. (This amounts to doing ERO on A^T and then again transposing). Just as we got EA to be A_R we can get a matrix F obtained by applying ECOs to I_n such that $AF =$ is equal to the CRE of A.

2.6 Solution of Homogeneous System Using RRE Form

Given the homogeneous system $Ax = \theta_m$, we first reduce A to its RRE form A_R. Since A and A_R are row equivalent it follows that the homogeneous system $Ax = \theta_m$ has the same set of solutions as the system $A_R x = \theta_m$ corresponding to the RRE matrix A_R. We can therefore get the solution of the system $Ax = \theta_m$ by solving the RRE system $A_R x = \theta_m$ as described in the Section 2.4 above.

Example 2.6.1 In the Example 2.5.1 above, the Homogeneous System corresponding to A_R is given by

$$\left\{ \begin{array}{lcl} x_1 + 2x_2 + x_4 & = & 0 \\ x_3 + x_4 & = & 0 \end{array} \right\}$$

The Pivotal Variables are x_1 and x_3, and the Free Variables are x_2 and x_4. Eliminating the pivotal vaiable x_1 from the first equation and the pivotal variable x_3 from the second equation and denoting the free variables x_2 and x_4 by α and β we get the general solution of the system $A_R x = \theta_m$ - which is also the general solution of the given system $Ax = \theta_m$, as

$$x = \alpha \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} -1 \\ 0 \\ -1 \\ 1 \end{pmatrix}$$

We now define the Row Rank of a matrix.

Definition 2.6.1 The number of nonzero rows in the RRE form A_R of A is called the "**Row Rank of** A, and the number of free variables for the RRE system $A_R x = \theta_m$ is called the "**Nullity of** A

Example 2.6.2 In the Example above we have the number of Nonzero Rows in RRE form is 2 and hence the Row Rank of A is 2. Similarly, the number of Free Variables is 2 and hence Nullity of A is 2. Thus we see that

$$\begin{aligned}\text{Row rank} + \text{Nullity} &= 2+2=4\\ &= \text{the number of columns in } A\\ &= \text{The number of unknowns in}\\ &\quad\ \text{the homogeneous system}\end{aligned}$$

Remark 2.6.1 Analogously the number of nonzero columns in the CRE form is called the "**Column Rank** " of A. and we have

$$\text{Column Rank of } A = \text{Row Rank of } A^T \tag{2.6.1}$$

Hence we get

$$\left.\begin{array}{l}\text{Column Rank of } A + \text{Nullity of } A^T =\\ \text{Row Rank of } A^T + \text{Nullity of } A^T =\\ \text{Number of columns in } A^T =\\ m\end{array}\right\}$$

2.7 Column Reduction of A_R and Matrix Factorization

We shall now look at the column Operations on A_R. Let the row rank of A be ρ. We can then reduce A_R to its CRE form by ECOs we get a matrix of the form

$$\left(\begin{array}{c|c}\mathcal{I}_{\rho\times\rho} & 0_{\rho\times(n-\rho)}\\ \hline 0_{(m-\rho)\times\rho} & 0_{(m-\rho)\times(n-\rho)}\end{array}\right)$$

Since we can effect these Column reductions by postmultiplying by a matrix $F \in \mathcal{F}^{n\times n}$ we get

$$A_R F = \left(\begin{array}{c|c} \mathcal{I}_{\rho\times\rho} & 0_{\rho\times(n-\rho)} \\ \hline 0_{(m-\rho)\times\rho} & 0_{(m-\rho)\times(n-\rho)} \end{array}\right)$$

Using the fact that $A_R = EA$ we get

$$EAF = \left(\begin{array}{c|c} \mathcal{I}_{\rho\times\rho} & 0_{\rho\times(n-\rho)} \\ \hline 0_{(m-\rho)\times\rho} & 0_{(m-\rho)\times(n-\rho)} \end{array}\right)$$

Since E and F are both invertible we finally get the factorization

$$A = Q\left(\begin{array}{c|c} \mathcal{I}_{\rho\times\rho} & 0_{\rho\times(n-\rho)} \\ \hline 0_{(m-\rho)\times\rho} & 0_{(m-\rho)\times(n-\rho)} \end{array}\right)P \qquad (2.7.1)$$

where

$$Q = E^{-1} \in \mathcal{F}^{m\times m} \text{ and } P = F^{-1} \in \mathcal{F}^{n\times n}$$

Example 2.7.1 We have, from Example ,

$$
\begin{array}{l|l}
\begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
\downarrow C_{32} & \downarrow C_{32} \\
\begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
\downarrow C_3 - 2C_1,\ C_4 - C_1 & \downarrow C_3 - 2C_1,\ C_4 - C_1 \\
\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 1 & 0 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
\downarrow C_4 - C_2 & \downarrow C_4 - C_2 \\
\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 1 & 0 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix} = F
\end{array}
$$

Thus

$$A_R F = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Using $EA = A_R$ we get

$$EAF = A_R F = \left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right)$$

From Example we can get

$$Q = E^{-1}$$

and from above we have

$$P = F^{-1}$$

Hence we get the factorization for the matrix A of Example as

$$A = Q\left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right)P$$

where Q and P are as above.

Conclusion:
Given any $A \in \mathcal{F}^{m\times n}$ of row rank ρ, there exists an invertible matrix $Q \in \mathcal{F}^{m\times m}$ and an invertible matrix $P \in \mathcal{F}^{n\times n}$ such that,

$$A = Q\left(\begin{array}{c|c} \mathcal{I}_{\rho\times\rho} & 0_{\rho\times(n-\rho)} \\ \hline 0_{(m-\rho)\times\rho} & 0_{(m-\rho)\times(n-\rho)} \end{array}\right)P$$

where

$$\begin{aligned} Q &\in \mathcal{F}^{m\times m} \text{ and} \\ P &\in \mathcal{F}^{n\times n} \end{aligned}$$

In particular, given any square matrix $A \in \mathcal{F}^{n\times n}$ of row rank ρ, there exist invertible matrices Q and $P \in \mathcal{F}^{n\times n}$ such that,

$$A = QDP$$

where D is a diagonal matrix whose first ρ diagonal entries are 1 and the remaining entries are all zero.

2.8 EROs and Inverse of a Matrix

Suppose $A \in \mathcal{F}^{n\times n}$, and the Row Rank of A is n. Then we have $A_R = I_n$, the $n \times n$ identity matrix. This means that

$$A \xrightarrow{EROs} A_R$$

Let

$$I_n \overset{\textit{same EROs}}{\longrightarrow} E$$

Then we have

$$\begin{aligned} EA &= A_R \\ \Longrightarrow & \\ EA &= I_n \text{ (since } A_R = I_n) \\ \Longrightarrow & \\ A^{-1} &= E \end{aligned}$$

<u>Conclusion</u>:
When $A \in \mathcal{F}^{n\times n}$ then A is invertible if and only if, Row Rank of A is equal to n, and the inverse can be obtained by applying to I_n the same sequence of EROs that were applied to A to obtain A_R

Example 2.8.1 Consider the matrix

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 6 & 4 \\ -3 & -8 & -4 \end{pmatrix}$$

We have the reduction as follows:

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 2 & 6 & 4 & 0 & 1 & 0 \\ -3 & -8 & -4 & 0 & 0 & 1 \end{array}\right) \overset{R_2-2R_1,\ R_3+3R_1}{\longrightarrow}$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 2 & 2 & -2 & 1 & 0 \\ 0 & -2 & -1 & 3 & 0 & 1 \end{array}\right) \overset{\frac{1}{2}R_2}{\longrightarrow}$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & \frac{1}{2} & 0 \\ 0 & -2 & -1 & 3 & 0 & 1 \end{array}\right) \overset{R_3+2R_2}{\longrightarrow}$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{array}\right) \overset{R_1-2R_2}{\longrightarrow}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 3 & -1 & 0 \\ 0 & 1 & 1 & -1 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{array}\right) \overset{R_1+R_3,\ R_2-R_3}{\longrightarrow} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 4 & 0 & 0 \\ 0 & 1 & 0 & -2 & -\frac{1}{2} & -1 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{array}\right)$$

Hence we get

$$A^{-1} = E \;\; = \begin{pmatrix} 4 & 0 & 1 \\ -2 & -\frac{1}{2} & -1 \\ 1 & 1 & 1 \end{pmatrix}$$

We shall next look at Nonhomogeneous systems.

2.9 Nonhomogeneous Systems and EROs

We shall now study the use of EROs in solving a consistent nonhomogeneous system $Ax = b$. We first introduce the notion of equivalence of two consistent homogeneous systems.

Definition 2.9.1 Let A and A_1 be in $\mathcal{F}^{m\times n}$, and b and $b_1 \in \mathcal{F}^m$. Then the two systems,

$$Ax = b \text{ and } A_1x = b_1$$

when consistent, are said to be equivalent if both have the same set of solutions

While dealing with homogeneous systems we found that EROs do not affect the solution set of the homogeneous system. What is the reason? The b for a homogeneous system is θ_m and EROs on this give back θ_m only. However, for a nonhomogeneous system $b \neq \theta_m$ and EROs on such a nonzero b alter the b.

Example 2.9.1 Consider

$$A = \begin{pmatrix} 2 & 3 & -1 \\ 1 & -1 & 1 \end{pmatrix} \text{ and } b = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Then

$$A \xrightarrow{R_{21}} A_1 = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 3 & -1 \end{pmatrix}$$

Clearly the systems $Ax = b$ and $A_1x = b$ are consistent and not equivalent. However,

$$b \xrightarrow{R_{21}} b_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

It is easy to see that the systems $Ax = b$ and $A_1x = b_1$ are consistent and equivalent. In general, if $Ax = b$ is consistent, and

$$A \xrightarrow{ERO} A_1 \text{ and } b \xrightarrow{same\ ERO} b_1$$

then $Ax = b$ and $A_1x = b_1$ are consistent and equivalent
<u>**Moral**</u>:
Let $Ax = b$ be consistent. When any ERO is performed on the coefficient matrix to get A_1, if the same ERO is performed on b to get a b_1, then the system $Ax = b$ will be "equivalent" to the system $A_1x = b_1$. In order to carry out this type of EROs simultaneously on A and b in a systematic manner, we introduce the notion of "**AUGMENTED MATRIX**". If $A \in \mathcal{F}^{m\times n}$ and $b \in \mathcal{F}^m$, we define the augmented matrix as

$$A_{aug} \quad = \quad \left(\begin{array}{c|c} \mathrm{A} & \mathrm{b} \end{array} \right)$$

A_{aug} is obtained from A by appending b as an additional $(n+1)$th column. Clearly A_{aug} is an $m \times (n+1)$ matrix.

Example 2.9.2 Let

$$A = \begin{pmatrix} 2 & 3 & -1 \\ 1 & -1 & 1 \end{pmatrix} \text{ and } b = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Then

$$A_{aug} = \left(\begin{array}{ccc|c} 2 & 3 & -1 & 2 \\ 1 & -1 & 1 & 1 \end{array} \right)$$

If $A \in \mathcal{F}^{m\times n}$, $b \in \mathcal{F}^m$ and we perform EROs on A to get its RRE form A_R, that is,

$$A \xrightarrow{EROs} A_R, \text{ Row reduced Echelon Form of } A$$

then

$$A_{aug} \xrightarrow{same\ EROs} \left(\begin{array}{c|c} \mathrm{A}_R & \tilde{b} \end{array} \right)$$

The nonhomogeneous system

$$Ax = b$$

is consistent if and only if the system

$$A_R x = \tilde{b}$$

is consistent. Let us now analyse the structure of the system $A_R x = \tilde{b}$. Suppose the Row Rank of A is ρ. Then the first ρ rows $R_1, R_2, \cdots, R_\rho$ of A_R are nonzero rows, and the remaining $m-\rho$ rows, $R_{\rho+1}, R_{\rho+2}, \cdots, R_m$ of A_R are all zero rows. How does $\left(\begin{array}{c|c} A_R & \tilde{b} \end{array} \right)$ look like? We have

$$\left(\begin{array}{c|c} A_R & \tilde{b} \end{array} \right) = \left(\begin{array}{c|c} R_1 & \tilde{b}_1 \\ R_2 & \tilde{b}_2 \\ \cdots & \cdots \\ \cdots & \cdots \\ R_\rho & \tilde{b}_\rho \\ \hline & \\ 0_{1\times n} & \tilde{b}_{\rho+1} \\ 0_{1\times n} & \tilde{b}_{\rho+2} \\ \cdots & \cdots \\ \cdots & \cdots \\ 0_{1\times n} & \tilde{b}_m \end{array} \right) \tag{2.9.1}$$

Whenever all terms on the lhs of an equation are zero, for the equation to be consistent, we must have the rhs also as zero. Thus from the above structure it follows that, for the system $A_R x = \tilde{b}$ to be consistent we need

$$\tilde{b}_j = 0 \text{ for } \rho + 1 \le j \le m \tag{2.9.2}$$

These are the consistency conditions that b has to satisfy in order that $Ax = b$ is consistent. These conditions can equivalently be stated as follows:

$$\left. \begin{array}{l} Ax = b \text{ is consistent if and only if} \\ \text{Row Rank of } A \text{ is equal to the Row rank of } A_{aug} \end{array} \right\} \tag{2.9.3}$$

We shall now see how to solve the nonhomogeneous system $A_R x = \tilde{b}$ when the consistency condition is satisfied. (The set of solutions we obtain will also be precisely the set of solutions of the original nonhomogeneous system $Ax = b$, since $Ax = b$ and $A_R x = \tilde{b}$ will then be equivalent). When the consistency conditions (2.14.2) are satisfied, $\left(\begin{array}{c|c} \mathrm{A}_R & \tilde{b} \end{array} \right)$ is of the form

$$\left(\begin{array}{c|c} A_R & \tilde{b} \end{array} \right) = \left(\begin{array}{c|c} R_1 & \tilde{b}_1 \\ R_2 & \tilde{b}_2 \\ \cdots & \cdots \\ \cdots & \cdots \\ R_\rho & \tilde{b}_\rho \\ \hline \\ 0_{1\times n} & 0 \\ 0_{1\times n} & 0 \\ \cdots & \cdots \\ \cdots & \cdots \\ 0_{1\times n} & 0 \end{array} \right) \tag{2.9.4}$$

Let the pivotal 1 in the R_ith row be in the k_i th column. Then from the rows $R_1, R_2, \cdots, R_\rho$ we can respectively eliminate the pivotal variables $x_{k_1}, x_{k_2}, \cdots, x_{k_\rho}$ in terms of $\tilde{b}_1, \tilde{b}_2, \cdots, \tilde{b}_\rho$ and the remaining $n - \rho$ variables, and these remaining $n - \rho$ variables can be chosen arbitrarily. We can write the solution as the sum of two parts, firstly the part of the solution which involves the $\tilde{b}$ components and this will be a Particular Solution of the nonhomogeneous system, and secondly the part of the solution which involves the arbitrary free variables, which is the general solution of the homogeneous system. We shall now illustrate these aspects with an example.

Example 2.9.3 Consider the matrix $A \in \mathcal{F}^{4\times 5}$ and $b \in \mathcal{F}^4$ given below:

$$A = \begin{pmatrix} 1 & 2 & 2 & 9 & -1 \\ 1 & 2 & 3 & 13 & -2 \\ -1 & -2 & -1 & -5 & 1 \\ 1 & 2 & 0 & 1 & 2 \end{pmatrix}$$

$$b = \begin{pmatrix} \mathrm{b}_1 \\ \mathrm{b}_2 \\ \mathrm{b}_3 \\ \mathrm{b}_4 \end{pmatrix}$$

We can write the reduction of the Augmented matrix as follows:

$$A_{aug} = \left(\begin{array}{rrrrr|l} 1 & 2 & 2 & 9 & -1 & \mathrm{b}_1 \\ 1 & 2 & 3 & 13 & -2 & \mathrm{b}_2 \\ -1 & -2 & -1 & -5 & 1 & \mathrm{b}_3 \\ 1 & 2 & 0 & 1 & 2 & \mathrm{b}_4 \end{array}\right) \xrightarrow{R_2-R_1,} \xrightarrow{R_3+R_1,} R_4-R_1$$

$$\left(\begin{array}{rrrrr|c} 1 & 2 & 2 & 9 & -1 & \mathrm{b}_1 \\ 0 & 0 & 1 & 4 & -1 & \mathrm{b}_2 - b_1 \\ 0 & 0 & 1 & 4 & 0 & \mathrm{b}_3 + b_1 \\ 0 & 0 & -2 & -8 & 3 & \mathrm{b}_4 - b_1 \end{array}\right) \xrightarrow{R_3-R_2,} \xrightarrow{R_4+2R_2}$$

$$\left(\begin{array}{rrrrr|c} 1 & 2 & 2 & 9 & -1 & \mathrm{b}_1 \\ 0 & 0 & 1 & 4 & -1 & \mathrm{b}_2 - b_1 \\ 0 & 0 & 0 & 0 & 1 & \mathrm{b}_3 - b_2 + 2b_1 \\ 0 & 0 & 0 & 0 & 1 & -3\mathrm{b}_1 + 2b_2 + b_4 \end{array}\right) \xrightarrow{R_4-R_3}$$

$$\left(\begin{array}{rrrrr|c} 1 & 2 & 2 & 9 & -1 & \mathrm{b}_1 \\ 0 & 0 & 1 & 4 & -1 & \mathrm{b}_2 - b_1 \\ 0 & 0 & 0 & 0 & 1 & \mathrm{b}_3 - b_2 + 2b_1 \\ 0 & 0 & 0 & 0 & 0 & -5\mathrm{b}_1 + 3b_2 - b_3 + b_4 \end{array}\right) \xrightarrow{R_1-2R_2}$$

$$\left(\begin{array}{rrrrr|c} 1 & 2 & 0 & 1 & 1 & 3\mathrm{b}_1 - b_2 \\ 0 & 0 & 1 & 4 & -1 & \mathrm{b}_2 - b_1 \\ 0 & 0 & 0 & 0 & 1 & \mathrm{b}_3 - b_2 + 2b_1 \\ 0 & 0 & 0 & 0 & 0 & -5\mathrm{b}_1 + 3b_2 - b_3 + b_4 \end{array}\right) \xrightarrow{R_1-R_3,} \xrightarrow{R_2+R_3}$$

$$\left(\begin{array}{rrrrr|c} 1 & 2 & 0 & 1 & 0 & -\mathrm{b}_1 + b_2 - b_3 \\ 0 & 0 & 1 & 4 & 0 & \mathrm{b}_1 + b_3 \\ 0 & 0 & 0 & 0 & 1 & 2\mathrm{b}_1 - b_2 + b_3 \\ 0 & 0 & 0 & 0 & 0 & -5\mathrm{b}_1 + 3b_2 - b_3 + b_4 \end{array}\right)$$

Thus as in the notation of (2.9.1) we can write

$$A_{aug} \quad \xrightarrow{above\ EROs} \quad \left(\begin{array}{ccccc|c} 1 & 2 & 0 & 1 & 0 & \tilde{b}_1 \\ 0 & 0 & 1 & 4 & 0 & \tilde{b}_2 \\ 0 & 0 & 0 & 0 & 1 & \tilde{b}_3 \\ 0 & 0 & 0 & 0 & 0 & \tilde{b}_4 \end{array}\right)$$

Let

$$\tilde{b} = \begin{pmatrix} \tilde{b}_1 \\ \tilde{b}_2 \\ \tilde{b}_3 \\ \tilde{b}_4 \end{pmatrix}$$

where

$$\begin{aligned} \tilde{b}_1 &= -b_1 + b_2 - b_3 \\ \tilde{b}_2 &= b_1 + b_3 \\ \tilde{b}_3 &= 2b_1 - b_2 + b_3 \\ \tilde{b}_4 &= -5b_1 + 3b_2 - b_3 + b_4 \end{aligned}$$

From (2.14.2) we get the consistency condition as

$$-5b_1 + 3b_2 - b_3 + b_4 = 0$$

When consistency condition satisfied, the system $A_R x = \tilde{b}$ can be written as

$$\left.\begin{aligned} x_1 + 2x_2 + x_4 &= \tilde{b}_1 \\ x_3 + 4x_4 &= \tilde{b}_2 \\ x_5 &= \tilde{b}_3 \end{aligned}\right\}$$

The Pivotal variables are x_1, x_3 and x_5, and these can be eliminated in terms of $\tilde{b}_1, \tilde{b}_2, \tilde{b}_3$ and the free variables x_2 and x_4 to get the solution as

$$x = \begin{pmatrix} \tilde{b}_1 \\ 0 \\ \tilde{b}_2 \\ 0 \\ \tilde{b}_3 \end{pmatrix} + \alpha \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} -1 \\ 0 \\ -4 \\ 1 \\ 0 \end{pmatrix}$$

where α and β can be chosen arbitrarily. A particular solution x_P of the nonhomogeneous system is obtained from the part above involving the $\tilde{b}_j$, that is

$$x_P = \begin{pmatrix} \tilde{b}_1 \\ 0 \\ \tilde{b}_2 \\ 0 \\ \tilde{b}_3 \end{pmatrix}$$

The general solution of the homogeneous system can be obtained from the above by taking the part involving the free variables, that is,

$$x_H = \alpha \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} -1 \\ 0 \\ -4 \\ 1 \\ 0 \end{pmatrix}$$

If for example, we take

$$b = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}$$

the consistency cndition

$$-5b_1 + 3b_2 - b_3 + b_4 \quad = \quad 0$$

is satisfied, and

$$\begin{aligned} \tilde{b_1} &= -b_1 + b_2 - b_3 = -1 \\ \tilde{b_2} &= b_1 + b_3 = 1 \\ \tilde{b_3} &= 2b_1 - b_2 + b_3 = 1 \end{aligned}$$

Hence we get Particular Solution as

$$x_P \quad = \quad \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

It is easy to check that this is a solution of the given Nonhomogeneous system $Ax = b$.

2.10 Inconsistent Systems and EROs

We shall now see that Elementary Row Operations DO NOT preserve the least square solutions of an inconsistent system, that is, if $Ax = b$

is inconsistent, and

$$A_{aug} \xrightarrow{EROs} \left(\begin{array}{c|c} A_R & \tilde{b} \end{array} \right)$$

it does not imply that $Ax = b$ and $A_R x = \tilde{b}$ have same set of least square solutions. We shall illustrate this by means of an example below:

Example 2.10.1 The system

$$\begin{aligned} x_1 + x_2 &= 1 \\ 2x_1 + 2x_2 &= 1 \end{aligned}$$

is obviously inconsistent. We have

$$\begin{aligned} A &= \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} \\ b &= \begin{pmatrix} 1 \\ 1 \end{pmatrix} \end{aligned}$$

We shall now find the least square solutions of this inconsistent system. We want to find x_1, x_2 such that

$$\|Ax - b\|^2 = [(Ax)_1 - b_1]^2 + [(Ax)_2 - b_2]^2$$

is minimum. We have

$$\begin{aligned} \|Ax - b\|^2 &= [(Ax)_1 - b_1]^2 + [(Ax)_2 - b_2]^2 \\ &= (x_1 + x_2 - 1)^2 + (2x_1 + 2x_2 - 1)^2 \end{aligned}$$

The minimum occurs when the derivatives with respect to x_1 and x_2 are zero, that is,

$$\begin{aligned} 5x_1 + 5x_2 &= 3 \\ &\Longrightarrow \\ x_1 + x_2 &= \frac{3}{5} \end{aligned} \tag{2.10.1}$$

Thus the least square solutions must satisfy (2.10.1).
Let us now look at effect of EROs. We have

$$A_{aug} \xrightarrow{R_2 - 2R_1} = \left(\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 0 & -1 \end{array} \right) = \left(\begin{array}{c|c} A_1 & b_1 \end{array} \right)$$

The corresponding inconsistent nonhomogeneous system $A_1x = b_1$:

$$\begin{aligned} x_1 + x_2 &= 1 \\ 0x_1 + 0x_2 &= -1 \end{aligned}$$

Let us now find the least square solutions of this inconsistent system. We have

$$\begin{aligned} \|A_1x - b_1\|^2 &= [(A_1x)_1 - (b_1)_1]^2 + [(A_1x)_2 - (b_1)_2]^2 \\ &= (x_1 + x_2 - 1)^2 + (-1)^2 \end{aligned}$$

Using derivatives we find that the minimum occurs when

$$x_1 + x_2 = 1 \tag{2.10.2}$$

Hence the least square solutions of the inconsistent system obtained after EROs must satisfy (2.10.2). Comparing (2.10.1) and (2.10.2) we see that that Least square solutions have been altered by ERO.