# Worksheet on "Probability"

PRML – CS5691 (Jul–Nov 2023)

August 4, 2023

1. For three events A, B, and C, we know that

   - A and C are independent,
   - B and C are independent,
   - A and B are disjoint,

   Given $P(A \cup C) = \frac{2}{3}$, $P(B \cup C) = \frac{3}{4}$, and $P(A \cup B \cup C) = \frac{11}{12}$. Find P(A), P(B), and P(C).

   **Solution:** Let $P(A) = a$,$P(B) = b$, and $P(C) = c$,
   $P(A \cup C) = a+c-(ac) = \frac{2}{3}$; $P(B \cup C) = b+c-(bc) = \frac{3}{4}$; $P(A \cup B \cup C) = a+b+c-(ac)-(bc) = \frac{11}{12}$.
   On solving the three set of equations, $a = \frac{1}{3}$, $b = \frac{1}{2}$ and $c = \frac{1}{2}$.
   [Source: https://www.probabilitycourse.com/chapter1/1_4_5_solved3.php]

2. There are three bags that contain 100 marbles each:

   - Bag 1 contains 75 red and 25 blue marble;
   - Bag 2 contains 60 red and 40 blue marble;
   - Bag 3 contains 45 red and 55 blue marble;

   A bag is chosen uniformly at random and a marble is also chosen uniformly at random. (a) What is the probability that the chosen marble is red? (b) If the chosen marble is red, what is the probability that Bag 1 was chosen?

   **Solution:** Let **R** be an event that the marble chosen is red. Let $B_i$, be the event that the $i^{th}$ bag is chosen.
   For (a),
   $P(R|B_1) = 0.75$, $P(R|B_2) = 0.6$ and $P(R|B_3) = 0.45$
   Using the law of total probability,
   $P(R) = P(R|B_1)P(B_1) + P(R|B_2)P(B_2) + P(R|B_3)P(B_3)$
   $P(R) = (0.75)\frac{1}{3} + (0.6)\frac{1}{3} + (0.45)\frac{1}{3}$
   $P(R) = 0.6$
   For (b), we want $P(B_1|R)$. Using Bayes' rule,
   $P(B_1|R) = P(R|B_1)P(B_1)/P(R) = 0.75/3 * 0.6 = 0.42$.
   [Source: https://www.probabilitycourse.com/chapter1/1_4_2_total_probability.php]

3. You toss a fair coin three times. Given that you have observed at least one tails, what is the probability that you observe at least two tails?

**Solution:** Let A1 be the event that you observe at least one tails, and A2 be the event that you observe at least two tails. Then,
$A_1 = S - \{HHH\}$ , and $P(A_1) = \frac{7}{8}$
$A_2 = \{TTH, THT, HTT, TTT\}$ , and $P(A_2) = \frac{4}{8}$.

$$\begin{aligned} P(A_2|A_1) &= \frac{P(A_2 \cap A_1)}{P(A_1)} \\ &= \frac{P(A_2)}{P(A_1)} \\ &= \frac{4}{8} . \frac{8}{7} = \frac{4}{7} \end{aligned}$$

[Source: `https://www.probabilitycourse.com/chapter1/1_4_5_solved3.php`]

4. It is estimated that 50% of emails are spam emails. Some software has been applied to filter these spam emails before they reach your inbox. A certain brand of software claims that it can detect 99% of spam emails, and the probability for a false positive (a non-spam email detected as spam) is 5%. Now if an email is detected as spam, then what is the probability that it is in fact a non-spam email?

**Solution:**

Define events

$$A = \text{event that an email is detected as spam},$$
$$B = \text{event that an email is spam},$$
$$B^c = \text{event that an email is not spam}.$$

We know $P(B) = P(B^c) = .5$, $\quad P(A \mid B) = 0.99$, $\quad P(A \mid B^c) = 0.05$.
Hence by the Bayes's formula we have

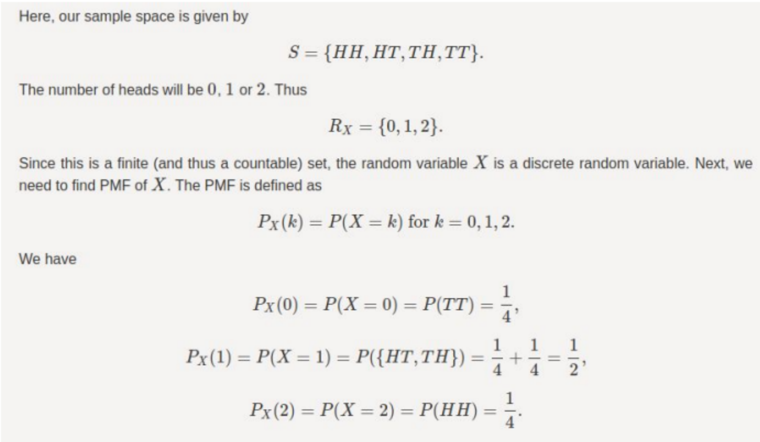
$$\begin{aligned} P(B^c \mid A) &= \frac{P(A \mid B^c)P(B^c)}{P(A \mid B)P(B) + P(A|B^c)P(B^c)} \\ &= \frac{0.05 \times 0.5}{0.05 \times 0.5 + 0.99 \times 0.5} \\ &= \frac{5}{104}. \end{aligned}$$

Figure 1:

[Source: `https://www.studocu.com/row/document/united-international-university/data-structure-an`
`ai-final-spring-2021/31515566`]

5. A fair coin is tossed twice, and **X** is defined as the number of heads that are observed. Find the range of **X** ($R_X$) and the probability mass function ($P_X$).

**Solution:**



Here, our sample space is given by

$$S = \{HH, HT, TH, TT\}.$$

The number of heads will be 0, 1 or 2. Thus

$$R_X = \{0, 1, 2\}.$$

Since this is a finite (and thus a countable) set, the random variable $X$ is a discrete random variable. Next, we need to find PMF of $X$. The PMF is defined as

$$P_X(k) = P(X = k) \text{ for } k = 0, 1, 2.$$

We have

$$P_X(0) = P(X = 0) = P(TT) = \frac{1}{4},$$

$$P_X(1) = P(X = 1) = P(\{HT, TH\}) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2},$$

$$P_X(2) = P(X = 2) = P(HH) = \frac{1}{4}.$$

Figure 2:

[SOURCE: `https://www.probabilitycourse.com/chapter3/3_1_3_pmf.php`]

6. (RECOMMENDED FOR NEXT WEEK CLASSES) Consider a continuous random variable X and a discrete random variable Y. Let

- $P_Y(Y = 1) = 0.5$ and $P_Y(Y = -1) = 0.5$, and
- $(X|Y = 1) \sim \text{Unif}(1, 2)$ and $(X|Y = -1) \sim \text{Unif}(-2, -1)$.

a. What is the marginal distribution of $X$? Specifically, plot the pdf of $X$ denoted $f_X(x)$.

b. Write down the pdf $f_X(x)$ of $X$.

c. Let $Z = X^2$. What is the pdf of $Z$? Use it to compute $E[Z]$. (Hint: To obtain pdf of $Z$, you could simply derive the cdf of $Z$ and differentiate it (or you could also use the change-of-variables formula).)

d. Now, use the pdf of $X$ directly to compute $E[X^2]$ (using the law of the unconscious statistician). Does this give the same answer as the previous question?

**Solution:**

a. Solution

Class conditionals $f_{X|Y}(x)$:

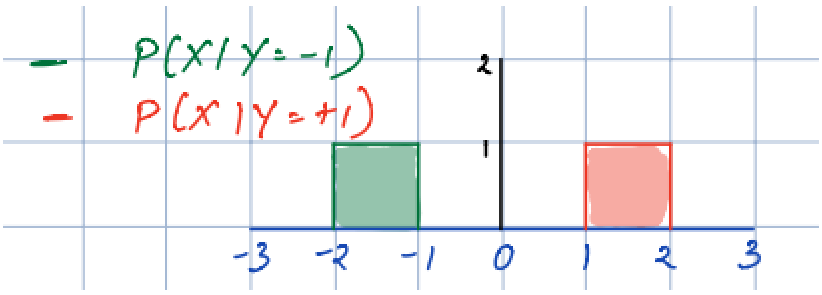



Figure 3:

Marginal $f_X(x)$:

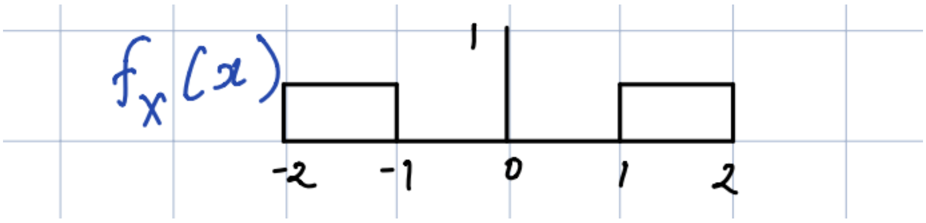



Figure 4:

b. See Figure 5.

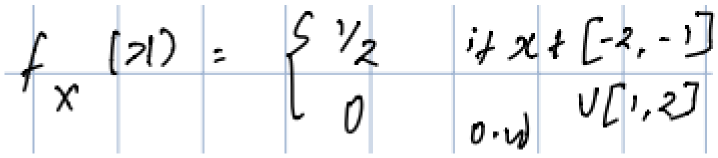



Figure 5:

[SOURCE: Parts a,b from [HG]Notes (by Harish Guruprasad; see course moodle for URL)]

c. Follow hint to obtain $f_Z(z)$; then use standard expectation formula, $E[Z] = \int_{z=-\infty}^{\infty} z f_Z(z) dz$.

The cdf of $Z = X^2$ is given by:

$$\begin{aligned}
F_Z(z) &= P(Z \leq z) \\
&= P(X^2 \leq z) \\
&= P(X \in [-\sqrt{z}, +\sqrt{z}]) \\
&= 2\ P(X \in [0, \sqrt{z}]) \\
&= \begin{cases} 2\ \frac{\sqrt{z}-1}{2} & \text{if } \sqrt{z} \in [1,2] \\ 2\ \frac{1}{2} & \text{if } \sqrt{z} > 2 \\ 0 & \text{o.w.} \end{cases} \\
&= \begin{cases} \sqrt{z} - 1 & \text{if } z \in [1,4] \\ 1 & \text{if } z > 4 \\ 0 & \text{o.w.} \end{cases}
\end{aligned}$$

Differentiating the above cdf gives:

$$f_Z(z) = F'_Z(z) = \begin{cases} \frac{1}{2\sqrt{z}} & \text{if } z \in [1,4] \\ 0 & \text{o.w.} \end{cases}$$

Therefore, $E[Z] = \int_{z=1}^{4} z \frac{1}{2\sqrt{z}} dz = [\frac{z^{3/2}}{3}]_{z=1}^{z=4} = (8-1)/3 \approx 2.33$.

Another way of deriving $f_Z(z)$ using change-of-variables formula: Let $Z = g(X) = X^2$. Assume $z \geq 0$ (otherwise, pdf is zero). Then, $dz = 2x\ dx \implies |dx/dz| = |1/(2x)| = 1/(2\sqrt{z})$.

Since $g(.)$ is a two-to-one function, we use the following change-of-variables formula:

$$\begin{aligned}
f_Z(z) &= f_X(x)\left|\frac{dx}{dz}\right| \text{ (at } x=-\sqrt{z}) + f_X(x)\left|\frac{dx}{dz}\right| \text{ (at } x=+\sqrt{z}) \\
&= f_X(\sqrt{z})\frac{1}{2\sqrt{z}} + f_X(-\sqrt{z})\frac{1}{2\sqrt{z}} \\
&= \begin{cases} 2\left(\frac{1}{2}\ \frac{1}{2\sqrt{z}}\right) & \text{if } \sqrt{z}\in[1,2] \\ 0 & \text{o.w.} \end{cases} \\
&= \begin{cases} \frac{1}{2\sqrt{z}} & \text{if } z\in[1,4] \\ 0 & \text{o.w.} \end{cases}
\end{aligned}$$

d. By law of the unconscious statistician,

$$\begin{aligned}
E[X^2] &= \int_{x=-\infty}^{\infty} f_X(x)x^2dx \\
&= \int_{-2}^{-1}(x^2/2)dx + \int_{1}^{2}(x^2/2)dx \\
&= [x^3/6]_{-2}^{-1} + [x^3/6]_{1}^{2} \\
&= (-1+8+8-1)/6 \\
&= (8-1)/3 \approx 2.33
\end{aligned}$$

The two answers match as expected.