IITM-CS5691 : Pattern Recognition and Machine Learning — Release Date: October 9, 2023
Assignment II — Due Date : October 23, 2023, 23:59

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Collaborators (if any):
References/sources (if any):

- Use LaTeX to write-up your solutions (in the solution blocks of the source LaTeX file of this assignment), submit the resulting rollno.asst2.answers.pdf file at Crowdmark by the due date, and propery drag that pdf's answer pages to the corresponding question in Crowdmark (do this propery, otherwise we won't be able to grade!). (Note: **No late submissions** will be allowed, other than one-day late submission with 10% penalty or four-day late submission with 30% penalty.)
- Please upload to moodle a rollno.zip file containing three files: rollno.asst2.answers.pdf file mentioned above, and two code files for the programming question (rollno.ipynb file and rollno.py file). Do not forget to upload to Crowdmark your results/answers (including Jupyter notebook **with output**) for the programming question.
- Collaboration is encouraged, but all write-ups must be done individually and independently, and mention your collaborator(s) if any. Same rules apply for codes written for any programming assignments (i.e., write your own code; we will run plagiarism checks on codes).
- If you have referred a book or any other online material or LLMs (Large Language Models like ChatGPT) for obtaining a solution, please cite the source. Again don't copy the source *as is* - you may use the source to understand the solution, but write-up the solution in your own words (this also means that you cannot copy-paste the solution from LLMs!). Please be advised that *the lesser your reliance on online materials or LLMs* for answering the questions, *the more your understanding* of the concepts will be and *the more prepared you will be for the course exams*.
- Points will be awarded based on how clear, concise and rigorous your solutions are, and how correct your answer is. The weightage of this assignment is 12% towards the overall course grade.

1. (8 points) [SPECTRAL CLUSTERING - LAPLACIAN EIGENMAP] Consider a simple undirected graph $G = (V, E)$ with $|V| = n$ nodes and $|E| = m$ edges. Let A be the binary adjacency matrix of the graph (i.e., the symmetric 0-1 matrix where 1 indicates the presence of the corresponding edge; diagonal entries of A are zero). Let $x \in \mathbb{R}^n$ denote the node scores.

   Let the graph Laplacian matrix be $L = D - A$ seen in class, with D being the diagonal matrix of node degrees. Let $\lambda_1 \leqslant \lambda_2 \leqslant \ldots \leqslant \lambda_n$ denote the eigenvalues of the graph Laplacian L (sometimes also referred to as L_G to explicitly mention the graph).

   (a) (3 points) If G is a complete graph on n nodes, we know that the multiplicity of eigen value 0 of L_G is 1; prove in this case that the multiplicity of eigen value n of L_G is $n - 1$.
   (Hint: Let v be an eigen vector of L_G orthogonal to the (all-ones) eigen vector of L corresp.

to eigen value 0. Assume, without loss of generality, that $v(1) \neq 0$. Now compute the first coordinate of $L_G v$, and then divide by $v(1)$ to compute eigen value λ.)

Solution: Given a complete graph on n nodes, the Laplacian matrix L_G can be written as:

$$L_G = nI - J$$

Where I is the identity matrix of size $n \times n$, and J is a matrix of all ones of size $n \times n$.
We know that the multiplicity of eigenvalue 0 of L_G is 1. Let v be an eigenvector of L_G orthogonal to the all-ones vector **1** corresponding to eigenvalue 0. Without loss of generality, we can assume that $v(1) \neq 0$.
Now, the first coordinate of $L_G v$:

$$(L_G v)_1 = (nI - J)\, v_1 - \sum_{i=2}^{n} v_i = nv_1 - (v_2 + v_3 + \ldots + v_n)$$

Since v is orthogonal to **1**, we know that $\sum_{i=1}^{n} v_i = 0$. Therefore, the first coordinate simplifies to:

$$(L_G v)_1 = nv_1$$

Now, divide by v_1 to compute the eigenvalue λ:

$$\lambda = \frac{(L_G v)_1}{v_1} = n$$

This shows that the eigenvalue n has multiplicity $n-1$, as it arises from considering eigenvectors orthogonal to the all-ones vector, excluding the eigenvector with eigenvalue 0.