EXERCISES - SET IV

1. Let $A \in \mathcal{F}^{3\times 3}$ be as defined below:

$$A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 1 & 1 & 2 \end{pmatrix}$$

The characteristic polynomial of A is given by

$$c_{_A}(\lambda) = (\lambda+1)^2(\lambda-5)$$

Answer the following:

(a) Find the eigenvalues and their algebraic multiplicities

(b) Find the eigenspaces $\mathcal{W}_1$ corresponding to the eigenvalue 5 and $\mathcal{W}_2$ corresponding to the eigenvalue -1 and hence find the geometeric multiplicities of these eigenvalues

(c) Show that the matrix A diagonalizable over $\mathcal{F}$? Find an invertible matrix $P \in \mathcal{F}^{3\times 3}$ such that

$$P^{-1}AP = \begin{pmatrix} 5 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

(d) Express $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathcal{F}^3$ as $x = X_1 + X_2$ where $X_1 \in \mathcal{W}_1$ and $X_2 \in \mathcal{W}_2$. Is this decomposition unique?

(e) Find the Lagrange interpolation polynomials corresponding to these eigenvalues

(f) Find matrices A_1 and A_2 in $\mathcal{F}^{3\times 3}$ such that

$$A_1 + A_2 = I_{3\times 3},\ 5A_1 - A_2 = A \text{ and } A_1A_2 = A_2A_1 = 0_{3\times 3}$$

(g) For the A_1 and A_2 found above answer the following:

i. Verify that $A_1^2 = A_1$ and $A_2^2 = A_2$

ii. Verify that $\mathit{Range\ of}\ A_1 = \mathcal{W}_1$ and $\mathit{Range\ of}\ A_2 = \mathcal{W}_2$

iii. Verify that ,
$x \in \mathcal{W}_1 \implies A_1 x = x$ and $A_2 x = \theta_3$; and
$x \in \mathcal{W}_2 \implies A_2 x = x$ and $A_1 x = \theta_3$

2. Let $A \in \mathbb{C}^{3\times 3}$ be as given below:

$$A = \begin{pmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{pmatrix} \tag{0.0.1}$$

The characteristic polynomial of A is given by

$$c_{_A}(\lambda) = (\lambda+1)^2(\lambda-3)$$

Answer the following:

(a) What are the distinct eigenvalues of A and their algebraic multiplicities?

(b) Find the eigenspaces and the geometric multiplicities of the eigenvalues

(c) Is A diagonalisable and if so find an invertible $P \in \mathbb{C}^{3\times 3}$ such that

$$P^{-1}AP = \begin{pmatrix} 3 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

(d) Find matrices $A_1, A_2 \in \mathbb{C}^{3\times 3}$ such that

i. $A_1 + A_2 = I_{3\times 3}$,

ii. $3A_1 - A_2 = A$, and

iii. $A_1A_2 = A_2A_1 = 0_{3\times 3}$

(e) For the A_1 and A_2 found above answer following:

i. Prove that $A_1^2 = A_1$ and $A_2^2 = A_2$

ii. Prove that *Range of* A_1 = *Eigenspace corresponding to eigenvalue* 3

iii. Prove that *Range of* A_2 = *Eigenspace corresponding to eigenvalue* -1

iv. What are the characteristic and minimal polynomials of A_1?

v. What are the characteristic and minimal polynomials of A_2?

(f) Find the following:

i. $sin(\pi A)$

ii. $cos(\pi A)$

iii. $exp(A)$

3. Let A be the matrix given below:

$$A = \begin{pmatrix} 6 & -3 & -2 \\ 4 & -1 & -2 \\ 10 & -5 & -3 \end{pmatrix}$$

The characteristic polynomial of this matrix is given by

$$c_{_A}(\lambda) = \lambda^3 - 2\lambda^2 + \lambda - 2$$

If one of the eigenvalues of A is 2 answer the following:

(a) Is A diagonalizable over $\mathbb{R}$?

(b) Is A diagonalizable over $\mathbb{C}$?

(c) In the above, if the answer is YES, find an invertible matrix over that corresponding field such that $P^{-1}AP$ is a diagonal matrix over that field

4. State whether the following are TRUE or FALSE:

(a) If $A, B \in \mathcal{F}^{n\times n}$ have the same characteristic polynomial and A is diagonalizable then B is also diagonalizable

(b) If $A, B \in \mathcal{F}^{n\times n}$ are diagonalizable and have the same characteristic polynomial then there exists an invertible matrix $R \in \mathbb{C}^{n\times n}$ such that $A = R^{-1}BR$

5. Which of the following matrices are similar to the diagonal matrix $D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{pmatrix}$?

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{pmatrix}, \; B = \begin{pmatrix} 4 & 0 & 0 \\ 7 & 1 & 0 \\ 8 & 9 & 6 \end{pmatrix}, \; C = \begin{pmatrix} 1 & 2 & 0 \\ 3 & 4 & 5 \\ 0 & 7 & 6 \end{pmatrix}$$

$$E = \begin{pmatrix} 4 & 7 & 0 \\ 0 & 1 & 0 \\ 8 & 9 & 6 \end{pmatrix}, \; F = \begin{pmatrix} 1 & 2 & 0 \\ 3 & 4 & 0 \\ 0 & 5 & 6 \end{pmatrix}, \; G = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 5 & 1 \\ 3 & 1 & 5 \end{pmatrix}$$

6. TRUE or FALSE (Give Reasons):

   (a) If $A \in \mathcal{F}^{n\times n}$ has n distinct eigenvalues in $\mathcal{F}$ then A is diagonalizable over $\mathcal{F}$

   (b) If $A, B \in \mathcal{F}^{n\times n}$ and if $(I-AB)^{-1}$ exists then verify that $(I-BA)^{-1}$ is given by
   $$(I - BA)^{-1} = I + B(I - AB)^{-1}A$$

   (c) If $A, B \in \mathcal{F}^{n\times n}$ and $\lambda \in \mathcal{F}$ then
   $$\lambda \text{ is an eigenvalue of } AB \Longleftrightarrow \lambda \text{is an eigenvalue of } BA$$

7. TRUE or FALSE? (Give reasons): If E_1, E_2 are in $\mathcal{F}^{n\times n}$ are such that $E_1+E_2 = I_{n\times n}$ and $E_1^2 = E_1$, and $E_2^2 = E_2$ then $E_1E_2 = 0_{n\times n} = E_2E_1$

8. If $A, K_1, K_2, \cdots, K_m \in \mathbb{C}^{n\times n}$ are such that $A = K_1K_2\cdots K_m$ then show that
   $$A \text{ is not invertible} \Longleftrightarrow \exists\ j \text{ such that } K_j \text{ is not invertible}$$

9. Let $A \in \mathbb{C}^{n\times n}$ and $p(\lambda) \in \mathbb{C}[\lambda]$ Prove the following:

   (a) If λ_j is an eigenvalue of A then $p(\lambda_j)$ is an eigenvalue of $p(A)$

   (b) If μ is an eigenvalue of $p(A)$ then there exists an eigenvalue λ of A such that $p(\lambda) = \mu$

10. TRUE or FALSE (Give Reasons): Let $A \in \mathcal{F}^{n\times n}$ be such that $A^2 = A$. Answer the following:

    (a) What are the possibilities for the minimal polynomial of A?

    (b) Should A be diagonalisable?

    (c) If, in addition, A is neither $O_{n\times n}$ nor $I_{n\times n}$ then what is the minimal polynomial of A?