Worksheet on "Multivariate Normal and Bayes Classifier"

PRML – CS5691 (Jul–Nov 2023)

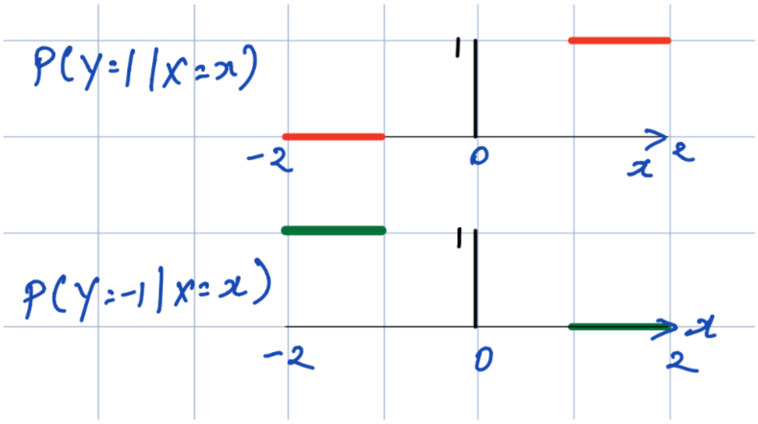
August 11, 2023

1. (a) Consider a continuous random variable X and a discrete random variable Y. Let

- $P_Y(Y=1)=0.5$ and $P_Y(Y=-1)=0.5$, and
- $(X|Y=1) \sim \text{Unif}(1,2)$ and $(X|Y=-1) \sim \text{Unif}(-2,-1)$.

Draw the plots for $P(Y=1|X=x)$ and $P(Y=-1|X=x)$ given the above assumptions.

Solution:



[Source: From [HG]Notes (by Harish Guruprasad; see course moodle for URL)]

(b) Consider the following setting:

- $P_Y(Y=1)=0.7$ and $P_Y(Y=-1)=0.3$
- $(X|Y=1) \sim Unif(-1,3)$ and $(X|Y=-1) \sim Unif(-2,0)$

1. Compute $P(Y=1|X=x)$ for different possible values of x.
2. Draw the plot for $P(Y=1|X=x)$.

Solution:

$$\eta(x) = P(Y=1 \mid X=x) = \frac{f_{X|Y}(x|1) \cdot 0.7}{0.7\, f_{X|Y}(x|1) + 0.3\, f_{X|Y}(x|-1)}$$

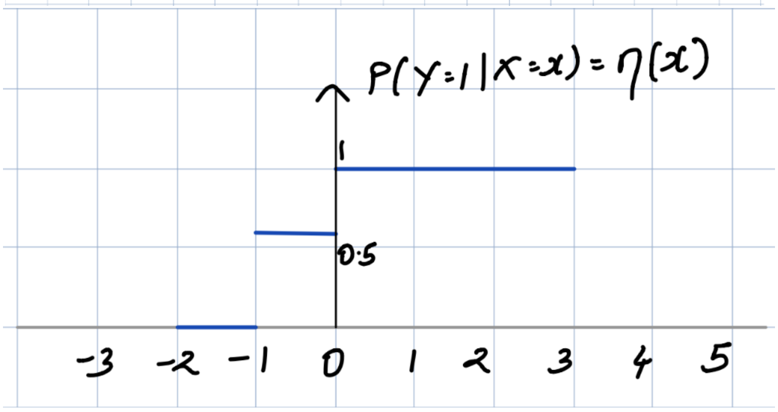
Case 1: $x < -2 \Rightarrow \eta(x) = \frac{0}{0}$ = Doesn't matter

Case 2: $x \in [-2,-1) \Rightarrow \eta(x) = \frac{0}{0+\frac{1}{2}(0.3)} = 0$

Case 3: $x \in [-1,0) \Rightarrow \eta(x) = \frac{1/4 \cdot (0.7)}{1/4\,(0.7) + 1/2\,(0.3)} =$

Case 4: $x \in [0,3) \Rightarrow \eta(x) = \frac{1/4\,(0.7)}{1/4\,(0.7)} = 1$

Case 5: $x \geq 3 \Rightarrow \eta(x) = \frac{0}{0}$ = Doesn't matter



[SOURCE: From [HG]Notes (by Harish Guruprasad; see course moodle for URL)]

2. In this question, you are required to verify if the following probability mass function over its respective support S follows the following properties:

   1. $P(X=x) \geq 0 \ \ \forall x \in S$, and
   2. $\sum_{x \in S} P(X=x) = 1$.

   In addition, find the expectation, $\mathbb{E}(X)$ and variance, $Var(X)$ in the following case: A discrete random variable X is said to have a Poisson distribution, with parameter $\lambda > 0$ over the support $S = \{0, 1, 2, \dots\}$ if it has the following probability mass function:

$$P(X=x) = \frac{\lambda^x e^{-\lambda}}{x!}$$

   Hint: $\sum_{n=1}^{\infty} \frac{a^n}{n!} = e^a$

Solution:

Verifying that for probability mass function, $P(X=x) = \frac{\lambda^x e^{-\lambda}}{x!}$, follow the following properties:

1)$P(X=x) \geq 0 \ \ \forall x \in S$

This statement says that for every element x in the support S, all the probabilities must be positive.

Proof:
Given parameter $\lambda > 0$
$\implies \lambda^x > 0$ As any power of positive number is positive
As $x \in S$ and $S = \{0, 1, 2, \dots\}$ So, x ≥ 0
$\implies x! \geq 0$
As we know that any e is a constant with a positive value 2.71828.
$\implies e^{-\lambda} > 0$ As any power of positive number is positive

$\implies P(X = x) = \frac{\lambda^x e^{-\lambda}}{x!} \geq 0$ As multiplication and division of 2 positive numbers is positive
Hence, $P(X = x) \geq 0 \quad \forall x \in S$

2)$\sum_{x \in S} P(X = x) = 1$

This statement says that if we add up all the probabilities for all the possible values of x, in the support S, then that sum equals 1.

Proof:
Given $P(X = x) = \frac{\lambda^x e^{-\lambda}}{x!}$

$\implies \sum_{n=0}^{\infty} P(X = x) = \sum_{x=0}^{\infty} \frac{\lambda^x e^{-\lambda}}{x!}$

$\implies e^{-\lambda} \sum_{x=0}^{\infty} \frac{\lambda^x}{x!}$

Since We know that, $\sum_{x=0}^{\infty} \frac{\lambda^x}{x!} = e^{\lambda}$

$\implies \sum_{x=0}^{\infty} P(X = x) = e^{-\lambda} e^{\lambda}$

$\implies \sum_{x=0}^{\infty} P(X = x) = 1$
Hence proved.

Calculating Expectation,

E(X) $= \sum_{x=1}^{\infty} x \frac{\lambda^x e^{-\lambda}}{x!}$

$\implies \lambda e^{-\lambda} \sum_{x=1}^{\infty} x \frac{\lambda^{x-1}}{x!}$

$\implies \lambda e^{-\lambda} \sum_{x=1}^{\infty} \frac{\lambda^{x-1}}{(x-1)!}$

As, $\sum_{n=1}^{\infty} \frac{a^n}{n!} = e^a$

So, $\sum_{x=1}^{\infty} \frac{\lambda^{x-1}}{(x-1)!} = e^{\lambda}$

Thus, E(X) $= \lambda e^{-\lambda} e^{\lambda} = \lambda$

Calculating variance
Var(X) $= \sigma^2 = E(X^2) - (E(X))^2$

$E(X^2) = \sum_{x=1}^{\infty} x^2 \frac{\lambda^x e^{-\lambda}}{x!}$

$\implies \lambda e^{-\lambda} \sum_{x=1}^{\infty} x^2 \frac{\lambda^{x-1}}{x!}$

$\implies \lambda e^{-\lambda} \sum_{x=1}^{\infty} x \frac{\lambda^{x-1}}{(x-1)!}$

$\implies \lambda e^{-\lambda} (\sum_{x=1}^{\infty} (x-1) \frac{\lambda^{x-1}}{(x-1)!} + \sum_{x=1}^{\infty} \frac{\lambda^{x-1}}{(x-1)!})$

$\implies \lambda e^{-\lambda} (\sum_{x=1}^{\infty} \frac{\lambda^{x-1}}{(x-2)!} + \sum_{x=1}^{\infty} \frac{\lambda^{x-1}}{(x-1)!})$

$\implies \lambda e^{-\lambda} (\lambda \sum_{x=2}^{\infty} \frac{\lambda^{x-2}}{(x-2)!} + \sum_{x=1}^{\infty} \frac{\lambda^{x-1}}{(x-1)!})$

Let y = x - 1, and z = x - 2

then, $\implies \lambda e^{-\lambda} (\lambda \sum_{z=0}^{\infty} \frac{\lambda^{z}}{z!} + \sum_{y=0}^{\infty} \frac{\lambda^{y}}{y!})$

As, $\sum_{n=1}^{\infty} \frac{a^n}{n!} = e^a$

So, $\implies \lambda e^{-\lambda} (\lambda \sum_{z=0}^{\infty} \frac{\lambda^{z}}{z!} + \sum_{y=0}^{\infty} \frac{\lambda^{y}}{y!}) = \lambda e^{-\lambda} (\lambda e^{\lambda} + e^{\lambda})$

$\implies \lambda e^{-\lambda} e^{\lambda} (\lambda + 1)$

$\implies \lambda(\lambda + 1)$

$\implies \lambda^2 + \lambda$

Var(X)= $E(X^2) - (E(X))^2$ And, $(E(X))^2 = \lambda^2$

$\implies Var(X) = \lambda^2 + \lambda - \lambda^2$

$\implies Var(X) = \lambda$

3. Consider a multivariate normal $X \sim N(\mu, \Sigma)$ where $X = \begin{pmatrix} X_1 & X_2 \end{pmatrix}$, $d = 2$, $\mu \in \mathbb{R}^2$ and $\Sigma \in \mathbb{R}^{2\times 2}$. Then, the density is defined as: $f_X(x) = \frac{1}{(2\pi)\sqrt{|\Sigma|}} exp\left(-\frac{1}{2}(x-\mu)^T \Sigma^{-1}(x-\mu)\right)$.

   (a) If $\Sigma = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$, $\Sigma^{-1} = \frac{1}{1-\rho^2}\begin{pmatrix} 1 & -\rho \\ -\rho & 1 \end{pmatrix}$, and $\mu = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. What are the legal values for ρ?

   Solution: $-1 <= \rho <= +1$

   (b) If $\rho = 0.5$ and $X_1 \sim N(0, 1)$, what is the distribution for $X_2 | X_1 = 4$?

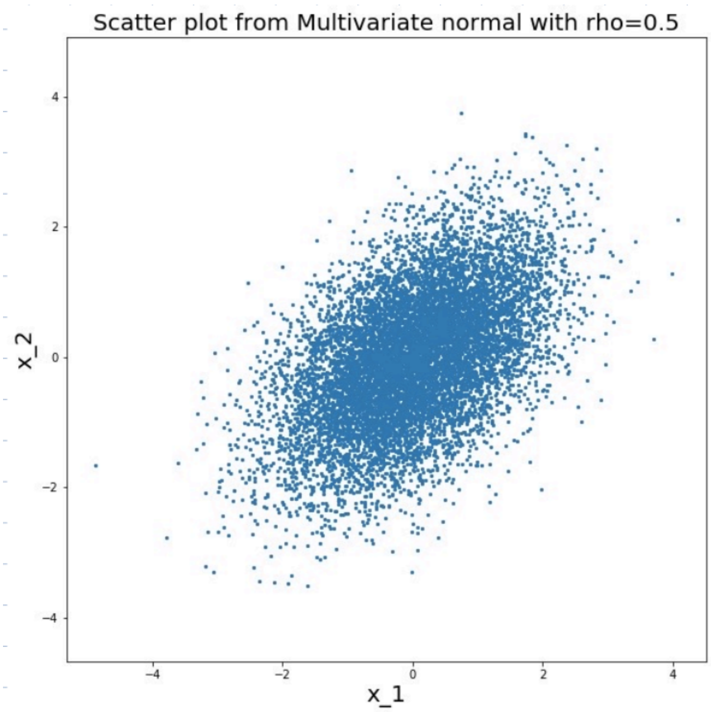
   Solution: $X_2 \sim N(\rho x_1, 1 - \rho^2) \implies X_2 \sim N(2, 0.75)$

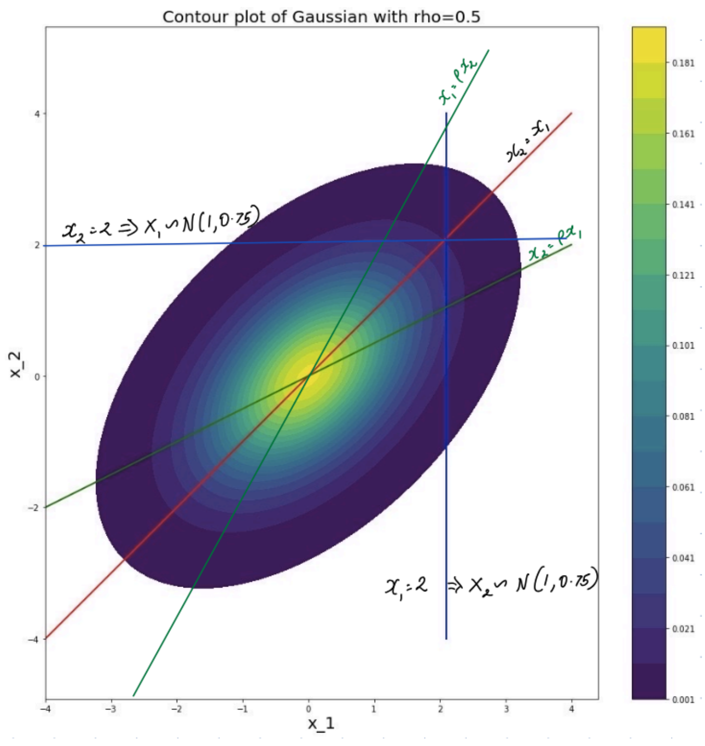
   (c) If $\rho = 0.5$ and $X_2 \sim N(0, 1)$, what is the distribution for $X_1 | X_2 = 3$?

   Solution: $X_1 \sim N(\rho x_2, 1 - \rho^2) \implies X_1 \sim N(1.5, 0.75)$

   (d) Consider the following scatter plot from a multivariate normal with $\rho = 0.5$. Draw the corresponding contour plot and mark the lines depicting the means of X_1 and X_2.

   Solution: Green lines are the means for X_1 and X_2. Ignore other lines.





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(e) If $\Sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, and $\mu = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, what is the distribution of $X_2|X_1 = x_1$ and $X_1|X_2 = x_2$?

Solution: $\Sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$\Sigma^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

$f_X(x) = \frac{1}{2\pi\sqrt{ad-bc}} exp\left[-\frac{1}{2}\frac{1}{ad-bc}\begin{pmatrix} x_1 & x_2 \end{pmatrix}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right]$

$f_X(x) = \frac{1}{2\pi\sqrt{ad-bc}} exp\left[-\frac{1}{2}\frac{1}{ad-bc}(dx_1^2 + ax_2^2 - (b+c)x_1x_2)\right]$

Assume $\Sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \sigma_X^2 & \rho\sigma_X\sigma_Y \\ \rho\sigma_X\sigma_Y & \sigma_X^2 \end{pmatrix}$.

Substitute and simplify.