EXERCISES - SET II

1. Let $\mathcal{V}$ be the set of all real numbers > 0. Let addition $x \oplus y$ of any two positive real numbers, and scalar multiplication $\alpha \odot x$ of any positive real number x by a real scalar α, be defined as follows:

$$\begin{aligned} x \oplus y &= xy \text{ for every } x, y \in \mathcal{V} \\ \alpha \odot x &= x^{\alpha} \text{ for every } \alpha \in \mathbb{R} \text{ and for every } x \in \mathcal{V} \end{aligned}$$

   Examine if $\mathcal{V}$ is a vector space over $\mathbb{R}$ with these operations

2. Let $\mathcal{V} = \mathbb{R}^2$ with $+$ as the usual addition. Scalar multiplication $\alpha \odot x$ of $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, by real numbers α is defined as below. In each case determine which of the axioms of scalar multiplication are satisfied and which are not satisfied:

   (a) $\alpha \odot x = \begin{pmatrix} \alpha x_1 \\ 2\alpha x_2 \end{pmatrix}$

   (b) $\alpha \odot x = \begin{pmatrix} \alpha^2 x_1 \\ \alpha^2 x_2 \end{pmatrix}$

   (c) $\alpha \odot x = \begin{pmatrix} \alpha x_1 \\ 0 \end{pmatrix}$

   (d) $\alpha \odot x = \begin{pmatrix} \alpha x_2 \\ \alpha x_1 \end{pmatrix}$

3. Let $\mathbb{C}^2$ denote the collection of all ordered pairs of complex numbers,

$$\mathbb{C}^2 = \left\{ x = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} : z_1,\ z_2 \in \mathbb{C} \right\}$$

   Let addition on $\mathbb{C}^2$ be defined as usual. Consider scalar multiplication $\alpha \odot x$ of an $x = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \in \mathbb{C}^2$ by a scalar $\alpha \in \mathbb{C}$ be defined as follows:

$$\alpha \odot x = \begin{pmatrix} \mathcal{R}e(\alpha) z_1 \\ \mathcal{R}e(\alpha) z_2 \end{pmatrix}$$

   (where $\mathcal{R}e(\alpha)$ stands for Real Part of α). Determine which of the axioms of scalar multiplication are satisfied and which are not satisfied.

4. Which of the following subsets of $\mathbb{R}^n$ $(n > 2)$ are subspaces of $\mathbb{R}^n$?

    (a) $\mathcal{W} = \{x \in \mathbb{R}^n : x_j \geq 0,\ 1 \leq j \leq n\}$

    (b) $\mathcal{W} = \{x \in \mathbb{R}^n : x_1 = 0\}$

    (c) $\mathcal{W} = \{x \in \mathbb{R}^n : x_1 x_2 = 0\}$

    (d) $\mathcal{W} = \left\{x \in \mathbb{R}^n : \sum_{j=1}^{n} x_j = 0\right\}$

5. Which of the following subsets of $\mathbb{R}^{n\times n}$ are subspaces of $\mathbb{R}^{n\times n}$?

    (a) W=$\Big\{A \in \mathbb{R}^{n\times n} : A^T = A\Big\}$, the set of all symmetric matrices

    (b) W=$\{A \in \mathbb{R}^{n\times n}$: A is a diagonal matrix$\}$

    (c) W=$\{A \in \mathbb{R}^{n\times n}$: A is an upper triangular matrix matrix$\}$

    (d) W=$\{A \in \mathbb{R}^{n\times n}$: A is a lower triangular matrix$\}$

    (e) W=$\{A \in \mathbb{R}^{n\times n}$: A is a triangular matrix matrix$\}$

    (f) Let B be a fixed matrix in $\mathbb{R}^{n\times n}$, and

    $$\mathcal{W} = \Big\{A \in \mathbb{R}^{n\times n} : AB = BA\Big\}$$

    (g) $\mathcal{W} = \{A \in \mathbb{R}^{n\times n} : Trace(A) = 0\}$

6. Let $\mathcal{F}_{\mathbb{R}}$ be the vector space, over the field $\mathbb{R}$, of all real valued functions defined on $\mathbb{R}$. Which of the following subsets are subspaces of this vector space?

    (a) $\mathcal{W} = \Big\{f \in \mathcal{F}_{\mathbb{R}} : f(0) = f(-1)\Big\}$

    (b) $\mathcal{W} = \Big\{f \in \mathcal{F}_{\mathbb{R}} : f(x) = f(-x) \text{ for all } x \in \mathbb{R}\Big\}$

    (c) $\mathcal{W} = \Big\{f \in \mathcal{F}_{\mathbb{R}} : f(x) = -f(-x) \text{ for all } x \in \mathbb{R}\Big\}$

    (d) $\mathcal{W} = \Big\{f \in \mathcal{F}_{\mathbb{R}} : f(x^2) = \{f(x)\}^2 \text{ for all } x \in \mathbb{R}\Big\}$

7. Let x_0 be a fixed vector in $\mathbb{R}^n$. Let

    $$\mathcal{W} = \Big\{A \in \mathbb{R}^{m\times n} : A(x_0) = \theta_m\Big\}$$

    Is $\mathcal{W}$ a subspace of $\mathbb{R}^{m\times n}$?

8. Let $\mathcal{S}$ be a a nonempty subset of $\mathbb{R}^n$. Let

$$\mathcal{W} = \Big\{ A \in \mathbb{R}^{m\times n} : A(s) = \theta_m \text{ for all } s \in \mathcal{S} \Big\}$$

   Is $\mathcal{W}$ a subspace of $\mathbb{R}^{m\times n}$?

9. Let x_0 be a fixed vector in $\mathbb{R}^n$. Let

$$\mathcal{W} = \Big\{ u \in \mathbb{R}^n : u^T x_0 = 0 \Big\}$$

   Is $\mathcal{W}$ a subspace of $\mathbb{R}^n$?

10. Let $\mathcal{S}$ be a nonempty subset of $\mathbb{R}^n$. Let

$$\mathcal{W} = \Big\{ u \in \mathbb{R}^n : u^T s = 0 \text{ for all } s \in \mathcal{S} \Big\}$$

   Is $\mathcal{W}$ a subspace of $\mathbb{R}^n$?

11. Let $A \in \mathbb{R}^{n\times n}$ and

$$\mathcal{W} = \{ x \in \mathbb{R}^n : Ax = 3x \}$$

   Is $\mathcal{W}$ a subspace of $\mathbb{R}^n$?

12. Let $A \in \mathbb{R}^{n\times n}$ and

$$\mathcal{W} = \Big\{ x \in \mathbb{R}^n : A^2 x = x \Big\}$$

   Is $\mathcal{W}$ a subspace of $\mathbb{R}^n$?

13. Let $\mathcal{W}$ a subspace of $\mathbb{R}^n$. Let $A \in \mathbb{R}^{n\times n}$ and

$$\mathcal{W}_1 = \{ x \in \mathbb{R}^n : Ax \in \mathcal{W} \}$$

   Is $\mathcal{W}_1$ subspace of $\mathbb{R}^n$?

14. Let $\mathcal{V}$ be a vector space over a field $\mathcal{F}$. Are the following statements TRUE or FALSE. If TRUE give a proof and if FALSE give an example where it is false:

    (a) If S and T are subsets of $\mathcal{V}$ such that $S \subseteq T$ then $\mathcal{L}[S] \subseteq \mathcal{L}[T]$

(b) If S is a subset of $\mathcal{V}$ and $x \notin \mathcal{L}[S]$ then

$$\mathcal{L}[S] \subset \mathcal{L}[T] \text{ where } T = S \cup \{x\}$$

(c) If S and T are subsets of $\mathcal{V}$ and $T \subset \mathcal{L}[S]$ then $\mathcal{L}[S] = \mathcal{L}[S \cup T]$

(d) If S and T are subsets of $\mathcal{V}$ such that T is not a subset of $\mathcal{L}[S]$ then $\mathcal{L}[S] \subset \mathcal{L}[S \cup T]$

(e) If S and T are subsets of $\mathcal{V}$ such that $S \subset T$ then $\mathcal{L}[S] \subset \mathcal{L}[T]$

15. Let $A \in \mathbb{R}^{4\times 5}$ be as given below:

$$A = \begin{pmatrix} 1 & 2 & 1 & 0 & 1 \\ 1 & 2 & 2 & -1 & 3 \\ 2 & 4 & 3 & -1 & 4 \\ 3 & 6 & 1 & 2 & -1 \end{pmatrix}$$

Answer the following:

(a) Find $\mathcal{R}_{A}$

(b) Find $\mathcal{N}_{A^T}$

16. Let $A \in \mathbb{R}^{5\times 4}$ be such that the RRE form of A^T is given by

$$(A^T)_R = \begin{pmatrix} 1 & 2 & 0 & 2 & -1 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Answer the following :

(a) Determine which of the following vectors are in $\mathcal{R}_{A}$:

$$b = \begin{pmatrix} 1 \\ 2 \\ 2 \\ -2 \\ 1 \end{pmatrix}, \qquad y = \begin{pmatrix} 1 \\ 2 \\ 0 \\ 2 \\ 0 \end{pmatrix}$$

(b) Find a spanning set for $\mathcal{R}_{A}$ directly from the RRE form of A^T

(c) Find a spanning set for $\mathcal{R}_{A}$ using the property that $b \in \mathcal{R}_{A}$ if and only if $b^T x = 0$ for every $x \in \mathcal{N}_{A^T}$

17. The matrix $A \in \mathbb{R}^{4\times 5}$ is such that the following sequince of EROs give the RRE form A_R of A as follows:

$$A \xrightarrow{R_2+R_1} \xrightarrow{R_3+R_1} \xrightarrow{R_4-R_1} \xrightarrow{R_3-R_2} \xrightarrow{R_4-R_2} \xrightarrow{R_4+3R_3} A_R = \begin{pmatrix} 1 & 1 & 2 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Answer the following:

(a) Find $\mathcal{R}_{A}$

(b) Hence find $\mathcal{N}_{A^T}$

18. Prove the following:

(a) If $A \in \mathcal{F}^{n\times n}$ then $\mathcal{R}_{A^2} \subseteq \mathcal{R}_{A}$

(b) If $A \in \mathcal{F}^{m\times n}$ and $B \in \mathcal{F}^{n\times p}$ then $\mathcal{R}_{AB} \subseteq \mathcal{R}_{A}$

(c) If $A \in \mathcal{F}^{m\times n}$ and $B \in \mathcal{F}^{n\times p}$ then $\mathcal{N}_{B} \subseteq \mathcal{N}_{AB}$

(d) If $A \in \mathbb{R}^{m\times n}$ then $\mathcal{N}_{A^T A} = \mathcal{N}_{A}$