Exercises - Set III

1. In the vector space $\mathbb{R}^4$ determine whether the following sets of vectors are linearly independent:

   (a) $\mathcal{S} : u_1 = \begin{pmatrix} 1 \\ -3 \\ 0 \\ 2 \end{pmatrix}, \; u_2 = \begin{pmatrix} -2 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \; u_3 = \begin{pmatrix} -1 \\ -2 \\ 1 \\ 3 \end{pmatrix}$

   (b) $\mathcal{S} : u_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}, \; u_2 = \begin{pmatrix} 2 \\ 3 \\ 4 \\ 5 \end{pmatrix}, \; u_3 = \begin{pmatrix} 3 \\ 4 \\ 5 \\ 6 \end{pmatrix}, \; u_4 = \begin{pmatrix} 4 \\ 5 \\ 6 \\ 7 \end{pmatrix}$

2. For the sets $\mathcal{S}$ in the above example find a basis for $\mathcal{L}[\mathcal{S}]$ and the dimension of $\mathcal{L}[\mathcal{S}]$

3. Let $\mathcal{S} = u, \; v, \; w$ be a linearly independent set in $\mathbb{R}^3$. Determine whether the set $\mathcal{S}' = u', \; v', \; w'$ is linearly independent, where

$$u' = u + v, \; v' = v + w, \; w' = w + u$$

4. For what values of $\alpha \in \mathbb{R}$ are the following vectors in $\mathbb{R}^3$ linearly independent?

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \; v_2 = \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix}, \; v_3 = \begin{pmatrix} 5 \\ 3 \\ \alpha \end{pmatrix}$$

5. If u, v, w are linearly independent vectors in a vector space $\mathcal{V}$, for what value(s) of k are the vectors $u', \; v', \; w'$ linearly independent where

$$u' = v - u, \; v' = kw - v, \; w' = u - w$$

6. Let $u, \; v, \; w, \; z$ be vectors in $\mathbb{R}^n$ such that u, v, w is a linearly dependent set and v, w, z is a linearly independent set. Prove the following:

   (a) u is a linear combination of v, w **and**

   (b) z is NOT a linear combination of u, v, w

7. Consider the set S in $\mathbb{R}^4$ defined below:

$$S: v_1 = \begin{pmatrix} 1 \\ 1 \\ 3 \\ 0 \end{pmatrix}, v_2 = \begin{pmatrix} 1 \\ 0 \\ 2 \\ -1 \end{pmatrix}, v_3 = \begin{pmatrix} 3 \\ 2 \\ 8 \\ -1 \end{pmatrix}, v_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, v_5 = \begin{pmatrix} 1 \\ 1 \\ 3 \\ 2 \end{pmatrix}$$

Find a linearly independent subset $\tilde{S}$ of S such that $\tilde{S}$ is a basis for $\mathcal{L}[S]$.

8. Find a subset $\tilde{\mathcal{C}}$, of the set of the column vectors of the following matrix $A \in \mathbb{R}^{4\times 5}$, such that $\tilde{\mathcal{C}}$ is a basis for $\mathcal{R}_{A}$:

$$A = \begin{pmatrix} 1 & 2 & 4 & 8 & 0 \\ 0 & 1 & 2 & 3 & 0 \\ 1 & 0 & 2 & 2 & 1 \\ 1 & 0 & 2 & 2 & 0 \end{pmatrix}$$

9. Show that the following set of vectors form a basis for $\mathbb{R}^3$:

$$\mathcal{B}: v_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \; v_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \; v_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

10. Show that the following set of matrices form a basis for $\mathbb{R}^{2\times 2}$:

$$\mathcal{B}: A_1 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \; A_2 = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \; A_3 = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \; A_4 = \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix}$$

11. Consider the subspace $\mathcal{W}$ of $\mathbb{R}^4$ defined as follows:

$$\mathcal{W} = \left\{ x = \begin{pmatrix} \alpha \\ \beta \\ 2\alpha + \beta \\ 0 \end{pmatrix} : \alpha, \; \beta \in \mathbb{R} \right\}$$

Answer the following:

(a) Find a basis for $\mathcal{W}$ and hence the dimension of $\mathcal{W}$

(b) Extend the basis for $\mathcal{W}$ obtained above to a basis for $\mathbb{R}^4$

12. Consider the vector space $\mathcal{V} = \mathbb{C}^{2\times 2}$ over the field $\mathbb{R}$ of real numbers. Let $\mathcal{W}$ be the subspace of this vector space defined as follows:

$$\mathcal{W} = \left\{ A = \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix} : \alpha,\ \beta,\ \gamma \in \mathbb{R} \right\}$$

Find a basis for $\mathcal{W}$ and hence the dimension of $\mathcal{W}$.

13. Let $\mathcal{V}$ and $\mathcal{U}$ be any two vector spaces over a field $\mathcal{F}$. The addition and scalar multiplication in these two spaces are denoted respectively by, $x \oplus_{\mathcal{V}} y$, $\alpha \odot_{\mathcal{V}} x$, and $x \oplus_{\mathcal{U}} y$ and $\alpha \odot_{\mathcal{U}} x$. Let $\mathcal{W}$ be defined as follows:

$$\mathcal{W} = \{(v, u) : v \in \mathcal{V},\ u \in \mathcal{U}\}$$

Define addition on $\mathcal{W}$ as follows:

$$\begin{aligned} x \oplus_{\mathcal{W}} y &= (v_1 \oplus_{\mathcal{V}} v_2, u_1 \oplus_{\mathcal{U}} u_2) \text{ for any } x = (v_1, u_1),\ y = (v_2, u_2) \in \mathcal{W} \\ \alpha \odot_{\mathcal{W}} x &= (\alpha \odot_{\mathcal{V}} v, \alpha \odot_{\mathcal{U}} u) \text{ for any } x = (v, u) \in \mathcal{W} \text{ and any } \alpha \in \mathcal{F} \end{aligned}$$

Answer the following:

(a) Show that, with these operations, $\mathcal{W}$ is vector space over $\mathcal{F}$

(b) If $\mathcal{B}_{\mathcal{V}} = v_1, v_2, \cdots, v_m$ is a basis for $\mathcal{V}$ and $\mathcal{B}_{\mathcal{U}} = u_1, u_2, \cdots, u_n$ is a basis for $\mathcal{U}$, find a basis for $\mathcal{W}$

(c) True or False :?

$$\textit{dimension of } \mathcal{W} \ = \ \textit{dimension of } \mathcal{V} \ + \textit{dimension of } \mathcal{U}$$

14. For the subspace $\mathbb{R}^3$ consider the ordered basis given below:

$$\mathcal{B} : v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},\ v_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix},\ v_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \text{ and}$$

$$\mathcal{B}' : v_1' = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix},\ v_2' = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix},\ v_3' = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

Answer the following:

(a) For any $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3$ find $[x]_{\mathcal{B}}$ and $[x]_{\mathcal{B}'}$

(b) Find invertible matrices $\mathcal{M}_{\mathcal{B}\mathcal{B}'}$ and $\mathcal{M}_{\mathcal{B}'\mathcal{B}}$ such that

i. $[x]_{\mathcal{B}'} = \mathcal{M}_{\mathcal{B}\mathcal{B}'}[x]_{\mathcal{B}}$ and $[x]_{\mathcal{B}} = \mathcal{M}_{\mathcal{B}'\mathcal{B}}[x]_{\mathcal{B}'}$, and

ii. $\mathcal{M}_{\mathcal{B}\mathcal{B}'}\mathcal{B}_{\mathcal{B}'\mathcal{B}} = I_{3\times 3}$

15. Let $\mathcal{V}$ be the vector space $\mathbb{R}^{2\times 2}$ over the field $\mathbb{R}$. Consider the subspace,

$$\mathcal{W} = \left\{ A = \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix} : \alpha,\ \beta,\ \gamma \in \mathbb{R} \right\}$$

Answer the following:

(a) Show that the following are basis for $\mathcal{W}$:

$$\begin{aligned} \mathcal{B} &= v_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},\ v_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ v_3 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \\ \mathcal{B}' &= v_1' = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ v_2' = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\ v_3' = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \end{aligned}$$

(b) For the matrix

$$A = \begin{pmatrix} 2 & -3 \\ -3 & 4 \end{pmatrix}$$

find $[A]_{\mathcal{B}}$ and $[A]_{\mathcal{B}'}$

(c) If $M \in \mathcal{W}$ is the such that,

$$[M]_{\mathcal{B}'} = \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix}$$

find the matrix M

16. Let $\mathcal{V} = \mathbb{R}_2[x]$ be the vector space of all polynomials in x of degree less than or equal to two. Answer the following:

(a) Show that the following set is a basis for $\mathcal{V}$:

$$\mathcal{B} : p_1, p_2, p_3 \text{ where } p_1(x) = 1, p_2(x) = x, p_3(x) = x^2$$

(b) What is the dimension of $\mathcal{V}$?

(c) Show that the following set is also a basis for $\mathcal{V}$:

$$\mathcal{B}' \;:\; p'_1, p'_2, p'_3 \text{ where } p'_1(x) = 1,\ p'_2(x) = (x-1),\ p'_3(x) = (x-1)^2$$

(d) For any polynomial $p(x) = a_0 + a_1 x + a_2 x^2$ find $[p]_{\mathcal{B}}$ and $[p]_{\mathcal{B}'}$

(e) Find $\mathcal{M}_{\mathcal{B}\mathcal{B}'}$ and $\mathcal{B}_{\mathcal{B}'\mathcal{B}}$ in $\mathbb{R}^{3\times 3}$ such that

$$[p]_{\mathcal{B}'} = \mathcal{M}_{\mathcal{B}\mathcal{B}'}[p]_{\mathcal{B}} \text{ and } [p]_{\mathcal{B}} = \mathcal{M}_{\mathcal{B}'\mathcal{B}}[p]_{\mathcal{B}'}$$

(f) Show that,

$$\mathcal{W} \;=\; \{p \in \mathcal{V} : p(1) = 0\}$$

is a subspace of $\mathcal{V}$. Find a basis for $\mathcal{W}$ and hence dimension of $\mathcal{W}$

17. Let $\lambda_1, \lambda_2, \lambda_3$ be three distinct complex numbers and define the polynomials $m(\lambda), m_1(\lambda), m_2(\lambda), m_3(\lambda), l_1(\lambda), l_2(\lambda), l_3(\lambda)$ as follows:

$$\begin{aligned}
m(\lambda) &= (\lambda-\lambda_1)(\lambda-\lambda_2)(\lambda-\lambda_3)\\
m_1(\lambda) &= (\lambda-\lambda_2)(\lambda-\lambda_3) = \frac{m(\lambda)}{(\lambda-\lambda_1)}\\
m_2(\lambda) &= (\lambda-\lambda_1)(\lambda-\lambda_3) = \frac{m(\lambda)}{(\lambda-\lambda_2)}\\
m_3(\lambda) &= (\lambda-\lambda_1)(\lambda-\lambda_2) = \frac{m(\lambda)}{(\lambda-\lambda_3)}\\
l_1(\lambda) &= \frac{m_1(\lambda)}{m_1(\lambda_1)}\\
l_2(\lambda) &= \frac{m_2(\lambda)}{m_2(\lambda_2)}\\
l_3(\lambda) &= \frac{m_3(\lambda)}{m_3(\lambda_3)}
\end{aligned}$$

Answer the following:

(a) Show that

$$l_i(\lambda_j) = \begin{cases} 0 & \text{if } \; i \neq j \\ 1 & \text{if } \; i = j \end{cases}$$

(b) Using the above result show that l_1, l_2, l_3 are linearly independent in the vector space $\mathbb{C}_2[\lambda]$, of all polynomials in λ of degree less than or equal to 2

(c) Do l_1, l_2, l_3 form a basis for $\mathbb{C}_3[\lambda]$?

18. Let $\mathcal{V}$ be an n dimensional vector space over a field $\mathcal{F}$. TRUE or FALSE?

(a) Every nonempty subset of a linearly dependent subset in $\mathcal{V}$ must be linearly dependent

(b) Every superset of a linearly dependent subset in $\mathcal{V}$ must be linearly dependent

(c) Every nonempty subset of a linearly independent subset in $\mathcal{V}$ must be linearly independent

(d) Every superset of a linearly independent subset in $\mathcal{V}$ must be linearly independent

(e) There exists a basis $\mathcal{B}$ for $\mathcal{V}$ such that we can find a linearly independent superset of $\mathcal{B}$

(f) Every proper sub space of $\mathcal{V}$ must have dimension $< n$

(g) For every k, $1 \leq k \leq n$, there exists a k dimensional subspace

(h) If $u_1, u_2, \cdots, u_r$ is a linearly independent set in $\mathcal{V}$ and $u_{(r+1)} \in \mathcal{V}$ is not a linear combination of $u_1, u_2, \cdots, u_r$ then $u_1, u_2, \cdots, u_r, u_{(r+1)}$ is linearly independent

(i) If $S = u_1, u_2, \cdots, u_r$ is a linearly independent set then none of the u_j is a linear combination of the remaining vectors in S

(j) If $S = u_1, u_2, \cdots, u_r$ is a set of vectors in $\mathcal{V}$ such that none of the u_j is a linear combination of the remaining vectors in S, then S must be linearly independent

(k) If $x \in \mathcal{V}$ is a linear combination vectors

$$u_1, u_2, \cdots, u_r$$

and each of the u_j is a linear combination of the vectors

$$v_1, v_2, \cdots, v_k$$

then x is a linear combination of the vectors $v_1, v_2, \cdots, v_k$

(l) If $u_1, u_2, \cdots, u_r$ is a linearly dependent set then every one of the u_j is a linear combination of the remaining

(m) If $S = u_1, u_2, \cdots, u_r$ is any set of vectors in $\mathcal{V}$ and $x \notin \mathcal{L}[S]$ then $S_1 = u_1, u_2, \cdots, u_r, x$ is a linearly independent set

(n) If $S = u_1, u_2, \cdots, u_r$ is a set of vectors such that, every subset of S having $(r-1)$ vectors is linearly independent, then S must be linearly independent

19. For any $A \in \mathcal{F}^{m \times n}$ prove that

$$Nullity\ of\ A^T A = Nullity\ of\ A$$

and

$$Rank\ of\ A^T A = Rank\ of\ A$$

(Hint: Use Exercise 18(d) of Exercises - Set II)