

Roll No: CS23E001

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Collaborators (if any):

References/sources (if any):

- Use \LaTeX to write-up your solutions (in the solution blocks of the source \LaTeX file of this assignment), submit the resulting rollno.asst2.answers.pdf file at Crowdmark by the due date, and properly drag that pdf's answer pages to the corresponding question in Crowdmark (do this properly, otherwise we won't be able to grade!). (Note: **No late submissions** will be allowed, other than one-day late submission with 10% penalty or four-day late submission with 30% penalty.)
- Please upload to moodle a rollno.zip file containing three files: rollno.asst2.answers.pdf file mentioned above, and two code files for the programming question (rollno.ipynb file and rollno.py file). Do not forget to upload to Crowdmark your results/answers (including Jupyter notebook **with output**) for the programming question.
- Collaboration is encouraged, but all write-ups must be done individually and independently, and mention your collaborator(s) if any. Same rules apply for codes written for any programming assignments (i.e., write your own code; we will run plagiarism checks on codes).
- If you have referred a book or any other online material or LLMs (Large Language Models like ChatGPT) for obtaining a solution, please cite the source. Again don't copy the source *as is* - you may use the source to understand the solution, but write-up the solution in your own words (this also means that you cannot copy-paste the solution from LLMs!). Please be advised that *the lesser your reliance on online materials or LLMs for answering the questions, the more your understanding of the concepts will be and the more prepared you will be for the course exams.*
- Points will be awarded based on how clear, concise and rigorous your solutions are, and how correct your answer is. The weightage of this assignment is 12% towards the overall course grade.

1. (8 points) [SPECTRAL CLUSTERING - LAPLACIAN EIGENMAP] Consider a simple undirected graph $G = (V, E)$ with $|V| = n$ nodes and $|E| = m$ edges. Let A be the binary adjacency matrix of the graph (i.e., the symmetric 0-1 matrix where 1 indicates the presence of the corresponding edge; diagonal entries of A are zero). Let $x \in \mathbb{R}^n$ denote the node scores.

Let the graph Laplacian matrix be $L = D - A$ seen in class, with D being the diagonal matrix of node degrees. Let $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$ denote the eigenvalues of the graph Laplacian L (sometimes also referred to as L_G to explicitly mention the graph).

- (a) (2 points) Show that $x^T L x = \sum_{(i,j) \in E} (x_i - x_j)^2$, and hence argue very briefly why $\lambda_i \geq 0$ for $i = 1, 2, \dots, n$?

Solution: Given, $G = (V, E)$ with $|V| = n$, $|E| = m$ and $L = D - A$,

The laplacian matrix L is defined by $L = D - A$ also we,ve to find $x^T L x$

To derive the expression $x^T L x = \sum_{i=1}^n \deg(i) x_i^2 - \sum_{(i,j) \in E} x_i x_j = \sum_{(i,j) \in E} (x_i - x_j)^2$, we'll step through each term.

1. $x^T L x$:

$$x^T L x = x^T (D - A) x$$

2. $x^T D x$:

The term $x^T D x$ is a quadratic form representing the diagonal elements of the matrix product Dx . In this case, D is a diagonal matrix, so the product Dx simply scales each element of x by the corresponding degree of the node. This term becomes:

$$x^T D x = \sum_{i=1}^n \deg(i) x_i^2$$

3. $x^T A x$:

The term $x^T A x$ is also a quadratic form representing the off-diagonal elements of the matrix product Ax . In this case, A is the adjacency matrix of the graph, and A_{ij} is 1 if there is an edge between nodes i and j , and 0 otherwise. This term becomes:

$$x^T A x = \sum_{(i,j) \in E} a_{ij} x_i x_j$$

Combining all three terms:

$$x^T L x = \sum_{i=1}^n \deg(i) x_i^2 - \sum_{(i,j) \in E} a_{ij} x_i x_j$$

Now, we want to express this in terms of the differences between x_i and x_j , also $a_{ij} = 2$

$$\begin{aligned} &= \sum_{i=1}^n \deg(i) x_i^2 - \sum_{(i,j) \in E} a_{ij} x_i x_j \\ &= \sum_{i=1}^n \deg(i) x_i^2 - \sum_{(i,j) \in E} 2x_i x_j \\ &= \sum_{(i,j) \in E} x_i^2 + x_j^2 - \sum_{(i,j) \in E} 2x_i x_j \\ &= \sum_{(i,j) \in E} (x_i - x_j)^2 \end{aligned}$$