CS6046 Problem Set 1

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1. (Practice) Let F be any non-empty set. Let $\mathcal{F}$ be a collection of subsets of F with the following properties.

   (a) $F \in \mathcal{F}$

   (b) If $X, Y \in \mathcal{F}$ and $Y \subset X$, then $X - Y \in \mathcal{F}$

   (c) If $(X_n)_{n \in \mathbb{N}} \in \mathcal{F}$ and $X_n \subset X_{n+1}$ for each $n \in \mathbb{N}$, then $\bigcup_{n \in \mathbb{N}} X_n \in \mathcal{F}$

   (d) If $X, Y \in \mathcal{F}$, then $X \cap Y \in \mathcal{F}$

   Show the $\mathcal{F}$ is a σ-**algebra**.

2. (Practice) Consider two random variables X and Z. Given that $Y = \mathbb{E}[X \mid Z]$, use the **tower property of expectation** to calculate $\mathbb{E}[(X - Y)^2]$ and $\textsc{cov}(Z,\ \mathbb{E}[X \mid Z])$. Here, $\mathbb{E}[X]$ is the expectation of X and $\textsc{cov}(\cdot, \cdot)$ is the covariance.

3. (8+2 marks) Consider two Gaussian distributions $p \sim \mathcal{N}(\mu_p, \sigma_p^2)$ and $q \sim \mathcal{N}(\mu_q, \sigma_q^2)$. Calculate the **KL divergence** $\mathcal{D}_{KL}(p||q)$ from distribution p to distribution q. Is $\mathcal{D}_{KL}(p||q) = \mathcal{D}_{KL}(q||p)$?

4. (4+3+3 marks) Let X and Y be two independent random variables, where $X \sim \mathcal{N}(0, 1)$ and $Y \sim \text{Exp}(\lambda)$. The **MGF** of a random variable Z is defined as $M_Z(t) = \mathbb{E}[e^{tZ}]$, where t is a real parameter.

   (a) Calculate the MGF $M_X(t)$ and $M_Y(t)$ for random variables X and Y respectively.

   (b) Let $W = X + Y$. Calculate the MGF $M_W(t)$ for the random variable W.

   (c) Using the results from parts (a) and (b), find the MGF $M_Z(t)$ for a random variable $Z = 2X - 3Y$.

5. (Practice) Let U be a continuous random variable following a uniform distribution over the interval $[a, b]$ ($U \sim \text{Uniform}(a, b)$), with $0 < a < b$. Using the **inverse CDF** method, transform U into an exponential random variable X with parameter λ ($X \sim \text{Exp}(\lambda)$).

6. (10+10 marks) Let $(\Omega, \mathcal{F}, P)$ be a probability space, $\mathcal{G}, \mathcal{H} \subseteq \mathcal{F}$ be sub-σ-fields and X, Y be random variables with $\mathbb{E}[|X|]$, $\mathbb{E}[|X|] < \infty$. Then, prove the following properties:

   (a) If $\mathcal{H} \subseteq \mathcal{G}$, then $\mathbb{E}[X|\mathcal{H}] = \mathbb{E}[\mathbb{E}[X|\mathcal{G}]|\mathcal{H}]$.

   (b) If Y is $\mathcal{G}$, then $\mathbb{E}[XY|\mathcal{G}] = Y\mathbb{E}[X|\mathcal{G}]$.