CS6015: Linear Algebra and Random Processes
Mid-Semester

Name : **Roll No** :

Answer all 8 questions. Write only your answers inside the boxes. No steps required. Rough work is to be done on separate sheets.

(1) Let $A \in \mathbb{R}^{4\times4}$. Let B be also be a 4×4 matrix, whose rows are given as follows.

$$\begin{aligned}
\text{Row 1 of } B &= \text{Row 1 of } A + \text{Row 2 of } A\\
\text{Row 2 of } B &= \text{Row 2 of } A + \text{Row 4 of } A\\
\text{Row 3 of } B &= \text{Row 3 of } A + \text{Row 1 of } A\\
\text{Row 4 of } B &= \text{Row 4 of } A + \text{Row 3 of } A
\end{aligned}$$

Give the determinant of B as a function of determinant of A. (**3 points**)

(2) Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$. Let $B = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 2 & 3 \\ 3 & 3 & 3 \end{bmatrix}$.

Give the determinants of A and B. (**2 points**)

(3) Let $A = \begin{bmatrix} 3 & 0 & 3 & 1 \\ 0 & 4 & 0 & 0 \\ 3 & 0 & 3 & -1 \end{bmatrix}$.

i. Give the singular value decomposition of A.
ii. Give the best (in terms of Frobenius norm) rank-2 approximation of A.

(**3+1 points**)

(4) $A \in \mathbb{R}^{5\times5}$ has eigen values $0, 1, 2, 3, 4$. $B \in \mathbb{R}^{5\times5}$ has eigen values $0, 0, 0, 1, 2$.

i. What are the possible values of the rank of A?

ii. Give an example possibility of A for each element in the answer above.

iii. What are the possible values of the rank of B?

iv. Give an example possibility of B for each element in the answer above.

(1+1+1+2 points)

(5) Let $A \in \mathbb{R}^{4\times 4}$ be a matrix composed of 2×2 block matrices B, C, D, E as follows. $A = \begin{bmatrix} B & C \\ D & E \end{bmatrix}$. If C and D are equal to the zero matrix, how are eigen values and eigen vectors of B and E related to the eigen values and eigen vectors of A?

(**4 points**)

(6) Let A, B, M be $n \times n$ matrices such that $A = MBM^{-1}$. How are the eigen values and eigen vectors of A and B related? (**3 points**)

(7) Construct matrix A and vector b such that the set of solutions to $Ax = b$ is given by the affine space $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \operatorname{span}\left(\left\{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right\}\right)$. **(3 points)**

(8) A is an a-dimensional subspace of $\mathbb{R}^5$. B is a b-dimensional subspace of $\mathbb{R}^5$. Let $C = A + B$ be a c-dimensional subspace of $\mathbb{R}^5$. Let $D = A \cap B$ be a d-dimensional subspace of $\mathbb{R}^5$.

Note: The sum of two subspaces A, B of the same vector space W is defined as follows:

$$A + B = \{x \in W : x = u + v \text{ for some } u \in A, v \in B\}.$$

The intersection of two subspaces A, B is defined in the standard way.

Construct subspaces A, B according to each of the following settings. If it is not possible to construct such subspaces mention it, along with a reason. Express the subspaces by writing them down as the span of their basis.

i. Give an example of A, B such that $a = 2, b = 2, c = 4, d = 0$.
ii. Give an example of A, B such that $a = 2, b = 3, c = 5, d = 0$.
iii. Give an example of A, B such that $a = 2, b = 2, c = 2, d = 2$.
iv. Give an example of A, B such that $a = 2, b = 3, c = 3, d = 2$.
v. Give an example of A, B such that $a = 2, b = 2, c = 5, d = 2$.
vi. Give an example of A, B such that $a = 2, b = 2, c = 4, d = 1$.

(6 points)