Hence the canonical form of N is given by

$$N_{can} = \left(\begin{array}{c|c|c|c|c|c|c|c}
N_5 & & & & & & & \\ \hline
 & N_4 & & & & & & \\ \hline
 & & N_2 & & & & & \\ \hline
 & & & N_2 & & & & \\ \hline
 & & & & N_1 & & & \\ \hline
 & & & & & N_1 & & \\ \hline
 & & & & & & N_1 & \\ \hline
 & & & & & & & N_1
\end{array}\right)$$

6.7 Jordan Canonical Form

We shall now briefly look at the Jordan cannonical form of a general matrix $A \in \mathbb{F}^{n\times n}$ whose characteristic polynomial and minimal polynomial are given as usual by

$$c_{_A}(\lambda) = (\lambda-\lambda_1)^{a_1}(\lambda-\lambda_2)^{a_2}\cdots(\lambda-\lambda_k)^{a_k} \qquad (6.7.1)$$
$$m_{_A}(\lambda) = (\lambda-\lambda_1)^{r_1}(\lambda-\lambda_2)^{r_2}\cdots(\lambda-\lambda_k)^{r_k} \qquad (6.7.2)$$

In Theorem 6.5.3 we have seen that any such A can be written as the sum $D+N$ of a diagonalizable matrix D and a nilpotent matrix N and that this decomposition is unique if D and N commute. The simplest such sum is one where D is a diagonal matrix and N is a canonical nilpotent matrix. The simplest diagonal D is the diagonal matrix all of whose diagonal entries are the same, say a. Thus the simplest such general matrix $D+N$ we can think of is a matrix whose diagonal entries are all a and the entries to the right of the diagonal are 1 and all other entries are zero. Such a matrix is called a "canonical Jordan matrix". We have

Definition 6.7.1 The $\boldsymbol{m\times m}$ matrix given below is called the "CANONICAL $\boldsymbol{m\times m}$ JORDAN MATRIX with diagonal $\boldsymbol{a}$" and is denoted by $\boldsymbol{J_m(a)}$:

$$\boldsymbol{J_m(a)} \;=\; \begin{pmatrix} \boldsymbol{a} & \boldsymbol{1} & & & \\ & \boldsymbol{a} & \boldsymbol{1} & & \\ & & \ddots & \ddots & \\ & & & \boldsymbol{a} & \boldsymbol{1} \\ & & & & \boldsymbol{a} \end{pmatrix}_{\boldsymbol{m\times m}}$$

It turns out that all matrices $A \in \mathbb{F}^{n\times n}$ whose characteristic polynomial and minimal polynomial are given by (6.7.1) and (6.7.2) can be built by such canonical Jordan matrices as follows:
We can find an invertible matrix $P \in \mathbb{F}^{n\times n}$ such that

$$P^{-1}AP \;\; = \;\; A_{can}$$

where A_{can} has the structure which we describe in detail below:
$A_{can} \in \mathbb{F}n \times n$ is divided into k diagonal blocks $A^{(1)}, A^{(2)}, \cdots, A^{(j)}, \cdots, A^{(k)}$, one block corresponding to each of the k distinct eignevalues $\lambda_1, \lambda_2, \cdots \lambda_j, \cdots, \lambda_k$ respectively. Thus we have

$$P^{-1}AP \;\; = \;\; \left(\begin{array}{c|c|c|c|c|c} A^{(1} & & & & & \\ \hline & A^{(2)} & & & & \\ \hline & & \ddots & & & \\ \hline & & & A^{(j)} & & \\ \hline & & & & \ddots & \\ \hline & & & & & A^{(k)} \end{array}\right)$$

How is each of these blocks structured?
We shall describe, in general, how $A^{(j)}$ is structured.
$A^{(j)}$ is further subdivided into diagonal subblocks , each subblock being a canonical Jordan matrix with diagonal λ_j . What remains to be answered are the following questions:

1. What is the size of $A^{(j)}$?,

2. How many subblocks are there in $A^{(j)}$?,

3. How are these subblocks arranged?, and

4. What are the sizes of these blocks?

We now proceed to answer these questions.
Question 1:
What is the size of $A^{(j)}$?
$A^{(j)}$ will be an $a_j \times a_j$ matrix
Question 2:
How many subblocks are there in $A^{(j)}$?
The number of subblocks in $A^{(j)}$ is equal to the geometric multiplicity g_j of

the eigenvalue λ_j.
<u>**Note**</u>:
Hence, we observe that if $g_j = a_j$ then each subblock is just 1×1 and therefore $A^{(j)}$ will just be an $a_j \times a_j$ diagonal matrix all of whose diagonal entries are λ_j

Question 3:
How are these subblocks arranged?
These subblocks are arranged in such a way that as we go down the diagonal in $A^{(j)}$, the sizes of the subblocks are nonincreasing
It remains to answer the fourth question.

Question 4:
What are the sizes of these blocks?
The structure is analogous to what we did for nilpotent matrices. We define the following subspaces

$$V_r^{(j)} = Null\ Space\ of\ (A - \lambda_j I_{n\times n})^r \text{ for } 1 \le r \le r_j \tag{6.7.3}$$

We then define the dimensions of these subspaces,

$$d_r^{(j)} = dim.(V_r^{(j)}),\ 1 \le j \le r_j \tag{6.7.4}$$

Then we have

$$g_j = d_1^{(j)} < d_2^{(j)} < \cdots < d_{r_j-1}^{(j)} < d_{r_j}^{(j)} = a_j \tag{6.7.5}$$

We then define

$$\alpha_r^{(j)} = d_r^{(j)} - d_{r-1}^{(j)} \text{ for } 2 \le r \le r_j \tag{6.7.6}$$

These $\alpha_r^{(j)}$ give us the extra dimension we get when we move up from $V_{(r-1)}^{(j)}$

to $V_r^{(j)}$. We shall define $\alpha_1^{(j)} = g_j$ and $\alpha_{r_j+1}^{(j)} = 0$. Then we can write

$$\left.\begin{array}{ll}
d_1^{(j)} = & \alpha_1^{(j)} \text{ (the geometric multiplicity of the eigenvalue 0)} \\
d_2^{(j)} = & d_1^{(j)} + \alpha_2^{(j)} \\
d_3^{(j)} = & d_1^{(j)} + \alpha_2^{(j)} + \alpha_3^{(j)} \\
\cdots & \cdots\cdots \\
d_{(r-1)}^{(j)} = & d_1^{(j)} + \alpha_2^{(j)} + \alpha_3^{(j)} + \cdots + \alpha_{(r-1)}^{(j)} \\
d_r^{(j)} = & d_1^{(j)} + \alpha_2^{(j)} + \alpha_3^{(j)} + \cdots + \alpha_{(r-1)}^{(j)} + \alpha_r^{(j)} \\
d_{(r+1)}^{(j)} = & d_1^{(j)} + \alpha_2^{(j)} + \alpha_3^{(j)} + \cdots + \alpha_{(r-1)}^{(j)} + \alpha_r^{(j)} + \alpha_{(r+1)}^{(j)} \\
\cdots & \cdots\cdots\cdots\cdots\cdots \\
d_{(r_j-1)}^{(j)} = & d_1^{(j)} + \alpha_2^{(j)} + \alpha_3^{(j)} + \cdots + \alpha_{(r-1)}^{(j)} + \alpha_r^{(j)} + \alpha_{(r+1)}^{(j)} + \cdots + \alpha_{(r_j-1)}^{(j)} \\
d_{r_j}^{(j)} = & d_1^{(j)} + \alpha_2^{(j)} + \alpha_3^{(j)} + \cdots + \alpha_{(r-1)}^{(j)} + \alpha_r^{(j)} + \alpha_{(r+1)}^{(j)} + \cdots + \alpha_{(r_j-1)}^{(j)} + \alpha_{r_j}^{(j)} = a_j
\end{array}\right\} \tag{6.7.7}$$

These $\alpha_r^{(j)}$ have a nice structure as follows:

$$\alpha_1^{(j)} \geq \alpha_2^{(j)} \geq \cdots \geq \alpha_r^{(j)} \geq \alpha_{(r+1)}^{(j)} \geq \cdots \geq \alpha_{(r_j-1)}^{(j)} \geq \alpha_{r_j}^{(j)} \tag{6.7.8}$$

This means that the extra dimension that we acquire from moving up from $V_{(r-1)}^{(j)}$ to $V_r^{(j)}$ is at least as much as the extra dimension that we acquire in the next stage of moving up from $V_r^{(j)}$ to $V_{(r+1)}^{(j)}$.
We now look at the differences of the dimensions gained at two successive stages. More precisely we define,

$$\left.\begin{array}{lcl}
n_1^{(j)} & = & \alpha_1^{(j)} - \alpha_2^{(j)} \\
n_2^{(j)} & = & \alpha_2^{(j)} - \alpha_3^{(j)} \\
n_3^{(j)} & = & \alpha_3^{(j)} - \alpha_2^{(j)} \\
\cdots & \cdots & \cdots \\
n_r^{(j)} & = & \alpha_r^{(j)} - \alpha_{(r+1)}^{(j)} \\
\cdots & \cdots & \cdots \\
n_{r_j-1}^{(j)} & = & \alpha_{(r_j-1)}^{(j)} - \alpha_{r_j}^{(j)} \\
n_{r_j}^{(j)} & = & \alpha_{r_j}^{(j)} - \alpha_{(r_j+1)}^{(j)} = \alpha_{r_j}
\end{array}\right\} \tag{6.7.9}$$

(We observe that $\alpha_{r_j}^{(j)}$ is at least 1 and hence $n_{r_j}^{(j)}$ is at least one). We now state the following theorem (without proof):

Theorem 6.7.1 With the above notations, there exists an invertible matrix, $P \in \mathbb{F}^{n\times n}$ such that $P^{-1}AP = A_{can}$, where A_{can} is an $n \times n$ matrix which is built as follows: It is made up of diagonal blocks of matrices $A^{(1)}, A^{(2)}, \cdots, A^{(j)}, \cdots, A^{(k)}$. For each j, $(1 \le j \le k)$, the $A^{(j)}$ is divided into a number of subblocks of different sizes such that,

1. **The subblocks are arranged in non increasing order as we go down the diagonal,**
2. **The leading subblock is $J_{r_j}(\lambda_j)$, the canonical $r_j \times r_j$ Jordan matrix with diagonal λ_j, (and all other blocks can be at most of this size),**
3. **For $1 \le r \le r_j$, there are $n_r^{(j)}$ blocks of size $r \times r$**
4. **The total number of subblocks in $A^{(j)}$ is equal to g_j, the geometric multiplicity of λ_j**

A_{can} is called the "Jordan Canonical Form" of the matrix A.

Analogous to the nilpotent matrices we can easily see that

$$\begin{aligned} n_r^{(j)} &= \textit{number of } r \times r \textit{ subblocks in } A^{(j)} \\ &= 2dim.(V_r^{(j)}) - dim.(V_{(r-1)}^{(j)}) - dim.(V_{(r+1)}^{(j)}) \end{aligned}$$

Reason:
We have

$$\begin{aligned} n_r^{(j)} &= \alpha_r^{(j)} - \alpha_{(r+1)}^{(j)} \\ &= (d_r^{(j)} - d_{(r-1)}^{(j)}) - (d_{(r+1)}^{(j)} - d_r^{(j)}) \\ &= 2d_r^{(j)} - d_{(r-1)}^{(j)} - d_{(r+1)}^{(j)} \\ &= 2dim.(V_r^{(j)}) - dim.(V_{(r-1)}^{(j)}) - dim.(V_{(r+1)}^{(j)}) \end{aligned}$$

Example 6.7.1 Suppose $A \in \mathbb{F}^{14\times 14}$ has characteristic polynomial and minimal polynomial given by,

$$\begin{aligned} c_{_A}(\lambda) &= (\lambda-2)^5(\lambda+4)^3(\lambda-6)^6 \\ m_{_A}(\lambda) &= (\lambda-2)^3(\lambda+4)^2(\lambda-6)^3 \end{aligned}$$

Then the distinct eigenvalues are $\lambda_1 = 2$, $\lambda_2 = -4$ and $\lambda_3 = 6$, with algebraic multiplicities given by

$$a_1 = 5,\ a_2 = 3, a_3 = 6$$

Since there are three distinct eigenvalues there will be three major blocks $A^{(1)}$, $A^{(2)}$ and $A^{(3)}$ corresponding to the three distinct eigenvalues $\lambda_1 = 2, \lambda_2 = -4$ and $\lambda_3 = 6$, We have

$$A_{can} = \left(\begin{array}{c|c|c} A^{(1)} & & \\ \hline & A^{(2)} & \\ \hline & & A^{(3)} \end{array}\right)$$

The algebraic multiplicities of the eigenvalues give us the fact that $A^{(1)}$ is 5×5, $A^{(2)}$ is 3×3 and $A^{(3)}$ is 6×6. So we have

$$A_{can} = \left(\begin{array}{c|c|c} (A^{(1)})_{5\times 5} & & \\ \hline & (A^{(2)})_{3\times 3} & \\ \hline & & (A^{(3)})_{6\times 6} \end{array}\right) \tag{6.7.10}$$

The minimal polynomial gives us

$$r_1 = 3,\ r_2 = 2 \text{ and } r_3 = 3$$

This means that,

1. The leading subblock in $A^{(1)}$ is

$$J_3(\lambda_1) = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

2. The leading subblock in $A^{(2)}$ is

$$J_2(\lambda_2) = \begin{pmatrix} -4 & 1 \\ 0 & -4 \end{pmatrix}$$

3. The leading subblock in $A^{(3)}$ is

$$J_3(\lambda_3) = \begin{pmatrix} 6 & 1 & 0 \\ 0 & 6 & 1 \\ 0 & 0 & 6 \end{pmatrix}$$

To determine the remaining subblocks we need to look at the relevant subspaces $V_r^{(j)} = Null\ Space\ of\ (A - \lambda_j I_{14\times 14})^r$. Suppose we are given the following data:

$$\begin{aligned}
d_1^{(1)} = dim.(V_1^{(1)}) &= Null\ Space\ of\ (A - 2I_{14\times 14}) = 3 \\
d_2^{(1)} = dim.(V_2^{(1)}) &= Null\ Space\ of\ (A - 2I_{14\times 14})^2 = 4 \\
d_3^{(1)} = dim.(V_3^{(1)}) &= Null\ Space\ of\ (A - 2I_{14\times 14})^3 = 5
\end{aligned}$$

$$\begin{aligned}
d_1^{(2)} = dim.(V_1^{(2)}) &= Null\ Space\ of\ (A + 4I_{14\times 14}) = 2 \\
d_2^{(2)} = dim.(V_2^{(2)}) &= Null\ Space\ of\ (A + 4I_{14\times 14})^2 = 3
\end{aligned}$$

$$\begin{aligned}
d_1^{(3)} = dim.(V_1^{(3)}) &= Null\ Space\ of\ (A - 6I_{14\times 14}) = 3 \\
d_2^{(3)} = dim.(V_2^{(3)}) &= Null\ Space\ of\ (A - 6I_{14\times 14})^2 = 5 \\
d_3^{(3)} = dim.(V_3^{(3)}) &= Null\ Space\ of\ (A - 6I_{14\times 14})^3 = 6
\end{aligned}$$

Thus we have,

$$\begin{aligned}
\alpha_1^{(1)} &= d_1^{(1)} = 3 \\
\alpha_2^{(1)} &= d_2^{(1)} - d_1^{(1)} = 1 \\
\alpha_3^{(1)} &= d_3^{(1)} - d_2^{(1)} = 1
\end{aligned}$$

Hence we get

$$\begin{aligned}
n_3^{(1)} &= \alpha_3^{(1)} = 1 \\
n_2^{(1)} &= \alpha_2^{(1)} - \alpha_3^{(1)} = 0 \\
n_1^{(1)} &= \alpha_1^{(1)} - \alpha_2^{(1)} = 2
\end{aligned}$$

Thus in $A^{(1)}$, there is one subblock of 3×3, (because $n_3^{(1)} = 1$), zero sublocks of size 2×2, (because $n_2^{(1)} = 0$) and two subblocks of size 1×1, (because

$n_1^{(1)} = 2)$. Thus we get,

$$A^{(1)} = \left(\begin{array}{ccc|c|c} 2 & 1 & 0 & & \\ 0 & 2 & 1 & & \\ 0 & 0 & 2 & & \\ \hline & & & 2 & \\ \hline & & & & 2 \end{array}\right) \tag{6.7.11}$$

$$= \left(\begin{array}{c|c|c} J_3(2) & & \\ \hline & J_1(2) & \\ \hline & & J_1(2) \end{array}\right) \tag{6.7.12}$$

Similarly we get

$$\begin{aligned} \alpha_1^{(2)} &= d_1^{(2)} = 2 \\ \alpha_2^{(2)} &= d_2^{(2)} - d_1^{(2)} = 1 \end{aligned}$$

Hence we get

$$\begin{aligned} n_2^{(2)} &= \alpha_2^{(2)} = 1 \\ n_1^{(1)} &= \alpha_1^{(2)} - \alpha_2^{(2)} = 1 \end{aligned}$$

Thus in $A^{(2)}$, there is one subblock of 2×2, (because $n_2^{(2)} = 1$), and one subblock of size 1×1, (because $n_1^{(2)} = 1$). Thus we get,

$$A^{(2)} = \left(\begin{array}{rr|r} -4 & 1 & \\ 0 & -4 & \\ \hline & & -4 \end{array}\right) \tag{6.7.13}$$

$$= \left(\begin{array}{c|c} J_2(-4) & \\ \hline & J_1(-4) \end{array}\right) \tag{6.7.14}$$

For the third eigenvalue we get,

$$\begin{aligned} \alpha_1^{(3)} &= d_1^{(3)} = 3 \\ \alpha_2^{(2)} &= d_2^{(3)} - d_1^{(3)} = 2 \\ \alpha_3^{(3)} &= d_3^{(3)} - d_2^{(3)} = 1 \end{aligned}$$

Hence we get

$$\begin{aligned} n_3^{(3)} &= \alpha_3^{(3)} = 1 \\ n_2^{(3)} &= \alpha_2^{(3)} - \alpha_3^{(3)} = 1 \\ n_1^{(3)} &= \alpha_1^{(3)} - \alpha_2^{(3)} = 1 \end{aligned}$$

Thus in $A^{(3)}$, there is one subblock of 3×3, (because $n_3^{(3)} = 1$), one sublock of size 2×2, (because $n_2^{(3)} = 1$) and one subblock of size 1×1, (because $n_1^{(3)} = 1$). Thus we get,

$$A^{(3)} = \left(\begin{array}{ccc|cc|c} 6 & 1 & 0 & & & \\ 0 & 6 & 1 & & & \\ 0 & 0 & 6 & & & \\ \hline & & & 6 & 1 & \\ & & & 0 & 1 & \\ \hline & & & & & 0 \end{array}\right) \tag{6.7.15}$$

$$= \left(\begin{array}{c|c|c} J_3(6) & & \\ \hline & J_2(6) & \\ \hline & & J_1(6) \end{array}\right) \tag{6.7.16}$$

Thus the canonical form of A is given by (6.7.10), where $A^{(1)}$,$A^{(2)}$ and $A^{(3)}$ are given by (6.4.12),(6.4.14) and (6.4.16). Thus we have

$$A_{can} = \left(\begin{array}{c|c|c||c|c||c|c|c} J_3(2) & & & & & & & \\ \hline & J_1(2) & & & & & & \\ \hline & & J_1(2) & & & & & \\ \hline\hline & & & J_2(-4) & & & & \\ \hline & & & & J_1(-4) & & & \\ \hline\hline & & & & & J_3(6) & & \\ \hline & & & & & & J_2(6) & \\ \hline & & & & & & & J_1(6) \end{array}\right)$$

6.8 Functions of a General Matrix

Just as we used the diagonal form to define $f(A)$ for a diagonalizable matrix $A \in \mathbb{C}^{n \times n}$, for any function $f(\lambda)$ analytic in a disc containing the eigenvalues of A, we can use the Jordan Canonical Form to define $f(A)$ for a general matrix $A \in \mathbb{C}^{n\times n}$. We first observe that, if $J_m(a)$ is a canonical Jordan matrix, then

$$(J_m(a))^2 = \begin{pmatrix} a^2 & 2a & 1 & & & & \\ & a^2 & 2a & 1 & & & \\ & & a^2 & 2a & 1 & & \\ & & & \ddots & \ddots & & \\ & & & & a^2 & 2a & 1 \\ & & & & & a^2 & 2a \\ & & & & & & a^2 \end{pmatrix}_{m\times m}$$

By induction we can then show that, $(J_m(a))^r$ is the $m \times m$ matrix whose first row is such that the jth entry in the first row is given by

$$[(J_m(a))^r]_{1j} = \begin{pmatrix} r \\ (j-1) \end{pmatrix} a^{r-(j-1)} \tag{6.8.1}$$

Thus the first row of $(J_m(a))^r$ is of the form,

$$\left(a^r \quad ra^{(r-1)} \quad \begin{pmatrix} r \\ 2 \end{pmatrix} a^2 \cdots\cdots \begin{pmatrix} r \\ (j-1) \end{pmatrix} a^{r-(j-1)} \cdots \begin{pmatrix} r \\ (n-1) \end{pmatrix} a^{r-(n-1)} \right)$$

The next row is obtained by shifting the above to the right and so on. Thus the ith row of $(J_m(a))^r$ will have all entries upto the diagonal entry as zero and then start off with a^r and proceed as above. We have the ith row of $(J_m(a))^r$ is give by

$$\left(0\ 0\ \cdots \underbrace{a^r}_{i-th\ entry} \quad ra^{(r-1)} \quad \begin{pmatrix} r \\ 2 \end{pmatrix} a^{r-2} \cdots\cdots \left(\begin{matrix} r \\ (n-i) \end{matrix} \right. a^{r-(n-i)} \right)$$

Using this we can show that if $p(\lambda)$ is any poynomial in $\mathbb{F}[\lambda]$ then $p(J_m(a))$ is the matrix whose ith row is of the form

$$0\ 0\ \cdots \underbrace{p(a)}_{i-th\ entry} \begin{pmatrix} r \\ 1 \end{pmatrix} p'(a) \quad \begin{pmatrix} r \\ 1 \end{pmatrix} p''(a) \cdots \begin{pmatrix} r \\ (n-i) \end{pmatrix} p^{(n-i)}(a)$$

where $p^{(m)}$ denotes the mth derivative of p.
Finally, using the fact that any function $f(\lambda)$ analytic in a disc containing a

can be approximated by a sequence of polynomials, we see that we can define $f(J_m(a))$ as the $m \times m$ matrix whose ith row is of the form

$$0\ 0\ \cdots\ \underbrace{f(a)}_{i-th\ entry}\ \binom{r}{1} f'(a)\quad \binom{r}{2} f''(a)\ \cdots\cdots \binom{r}{(n-i)} f^{(n-i)}(a)$$

Since the Jordan canonical form of any matrix A consists of diagonal subblocks of canonical Jordan matrices we define the Jordan canonical form of $f(A)$ by taking $f(subblock))$ for each of the subblocks, provided f is analytic in a disc containing all the eigenvalues of A. We note that if $\mu_1, \mu_2, \cdots, \mu_n$ are the eigenvalues of A, written including their multiplicities, the eigenvalues of $f(A)$ are $f(\mu_1), f(\mu_2), \cdots, f(\mu_n)$.

Example 6.8.1 For the matrix of Example 6.7.1 we had the eigenvalues $2, -4$ and 6. Hence if f is analytic in a disc D_R of radius R centred at the origin, where $R > 6$, then we can define $f(A)$ as follows:
We have

$$f(A^{(1)}) = \left(\begin{array}{ccc|c|c} f(2) & f'(2) & f''^{(2)} & & \\ 0 & f(2) & f'(2) & & \\ 0 & 0 & f(2) & & \\ \hline & & & f(2) & \\ \hline & & & & f(2) \end{array}\right) \tag{6.8.2}$$

$$f(A^{(2)}) = \left(\begin{array}{cc|c} f(-4) & f'(-4) & \\ 0 & f(-4) & \\ \hline & & f(-4) \end{array}\right) \tag{6.8.3}$$

$$f(A^{(3)}) = \left(\begin{array}{ccc|cc|c} f(6) & f'(6) & \frac{f''(6)}{2} & & & \\ 0 & f(6) & f'(6) & & & \\ 0 & 0 & f(6) & & & \\ \hline & & & f(6) & f'(6) & \\ & & & 0 & f(6) & \\ \hline & & & & & f(6) \end{array}\right) \tag{6.8.4}$$

Then the canonical form of $f(A)$ is given by

$$f(A)_{can} = \left(\begin{array}{c|c|c} f(A^{(1)}) & & \\ \hline & f(A^{(2)}) & \\ \hline & & f(A^{(3)}) \end{array}\right) \tag{6.8.5}$$

where $f(A^{(1)})$,$f(A^{(2)})$ and $f(A^{(3)})$ are as in (6.8.1),(6.8.2) and (6.8.3). Thus, for example, if $f(\lambda) = sin(\pi\lambda)$ we get from above,

$$sin(\pi A^{(1)}) \quad = \quad \left(\begin{array}{ccc|c|c} 0 & \pi & 0 & & \\ 0 & 0 & \pi & & \\ 0 & 0 & 0 & & \\ \hline & & & 0 & \\ \hline & & & & 0 \end{array}\right) \tag{6.8.6}$$

$$sin(\pi A^{(2)}) \quad = \quad \left(\begin{array}{cc|c} 0 & \pi & \\ 0 & 0 & \\ \hline & & 0 \end{array}\right) \tag{6.8.7}$$

$$sin(\pi A^{(3)}) \quad = \quad \left(\begin{array}{ccc|cc|c} 0 & \pi & 0 & & & \\ 0 & 0 & \pi & & & \\ 0 & 0 & 0 & & & \\ \hline & & & 0 & \pi & \\ & & & 0 & 0 & \\ \hline & & & & & 0 \end{array}\right) \tag{6.8.8}$$

Finally we have

$$sin(\pi A) \quad = \quad \left(\begin{array}{c|c|c} sin(\pi A^{(1)} & & \\ \hline & sin(\pi A^{(2)}) & \\ \hline & & sin(\pi A^{(3)}) \end{array}\right) \tag{6.8.9}$$