Chapter 3

Vector Spaces and Subspaces

3.1 Structure of $\mathbb{R}^k$

Let $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$. While analysing the linear system $Ax = b$ we find that we have to deal with two vectors, namely the known vector b given in $\mathbb{R}^m$ and the unknown vector x to be found in $\mathbb{R}^n$. So, in general, we shall consider a positive integer k and look at the structure of $\mathbb{R}^k$, the collection of all $k \times 1$ column vectors.

$$\mathbb{R}^k = \left\{ x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \end{pmatrix} : x_j \in \mathbb{R},\ 1 \leq j \leq k \right\} \tag{3.1.1}$$

We first consider the simple operation of "Addition" on $\mathbb{R}^k$defined as follows: For $x, y \in \mathbb{R}^k$ we define

$$x + y = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_k + y_k \end{pmatrix} \tag{3.1.2}$$

We easily see that addition has the following properties:

1. $x, y \in \mathbb{R}^k \Longrightarrow x + y \in \mathbb{R}^k$

2. $x + y = y + x$ for all $x, y \in \mathbb{R}^k$

3. $(x + y) + z = x + (y + z)$ for all $x, y, z \in \mathbb{R}^k$

4. The vector

$$\theta_k = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}_{k \times 1}$$

is in $\mathbb{R}^k$ and is such that

$$x + \theta_k = x = \theta_k + x \text{ for all } x \in \mathbb{R}^k$$

5. For every $x \in \mathbb{R}^k$ the vector

$$(-x) = \begin{pmatrix} -x_1 \\ -x_2 \\ \vdots \\ -x_k \end{pmatrix} \text{ is in } \mathbb{R}^k \text{ and is such that}$$

$$x + (-x) = \theta_k = (-x) + x$$

We next consider the operation of multiplying a vector in $\mathbb{R}^k$ by a scalar $\alpha \in \mathbb{R}$, defined as follows:

$$\alpha \cdot \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \end{pmatrix} = \begin{pmatrix} \alpha x_1 \\ \alpha x_2 \\ \vdots \\ \alpha x_k \end{pmatrix} \tag{3.1.3}$$

It is easy to see that this operation, called scalar multiplication, has the following properties:

1. $\alpha \in \mathbb{R}$ and $x \in \mathbb{R}^k \Longrightarrow \alpha \cdot x \in \mathbb{R}^k$
2. $\alpha \in \mathbb{R}$ and $x, y \in \mathbb{R}^k \Longrightarrow \alpha \cdot (x + y) = (\alpha \cdot x) + (\alpha \cdot y)$
3. $\alpha,\ \beta \in \mathbb{R}$ and $x \in \mathbb{R}^k \Longrightarrow (\alpha + \beta) \cdot x = (\alpha \cdot x) + (\beta \cdot x)$
4. $\alpha,\ \beta \in \mathbb{R}$ and $x \in \mathbb{R}^k \Longrightarrow (\alpha\beta) \cdot x = \alpha \cdot (\beta \cdot x)$
5. $1 \cdot x = x$ for all $x \in \mathbb{R}^k$

3.2 Vector Space

The important ingredients we discussed above in $\mathbb{R}^k$ are the two basic operations of addition and scalar multiplication and their properties. Any system which has such a structure is called a "Vector Space". More precisely we have the following definition of a vector space over $\mathbb{R}$.

Definition 3.2.1 Let $\mathcal{V}$ be any nonempty set and let $+$ be a binary opertaion on V and $\cdot$ be a rule of combining an element of $\mathbb{R}$ and an element of $\mathcal{V}$, (called "scalar multiplication") such that the following properties are satisfied:
(Axioms for the operation + on $\mathcal{V}$)

1. $x, y \in \mathcal{V} \Longrightarrow x + y \in \mathcal{V}$

2. $x + y = y + x$ for all $x, y \in \mathcal{V}$

3. $(x + y) + z = x + (y + z)$ for all $x, y, z \in \mathcal{V}$

4. There exists a vector $\theta_{\mathcal{V}} \in \mathcal{V}$ such that
$$x + \theta_{\mathcal{V}} = x = \theta_{\mathcal{V}} + x \text{ for all } x \in \mathcal{V}$$

5. For every $x \in \mathcal{V}$ there exists a vector in $\mathcal{V}$, which we denote by $(-x)$, such that
$$x + (-x) = \theta_{\mathcal{V}} = (-x) + x$$

(Axioms for scalar multiplication)

6. $x \in \mathcal{V}$ and $\alpha \in \mathbb{R} \Longrightarrow \alpha \cdot x \in \mathcal{V}$

7. $\alpha \cdot (x + y) = \alpha \cdot x + \alpha \cdot y$ for all $\alpha \in \mathbb{R}$ and for all $x, y \in \mathcal{V}$

8. $(\alpha + \beta) \cdot x = \alpha \cdot x + \beta \cdot x$ for all $\alpha, \beta \in \mathbb{R}$ and for all $x \in \mathcal{V}$

9. $(\alpha\beta) \cdot x = \alpha \cdot (\beta \cdot x)$ for all $\alpha, \beta \in \mathbb{R}$ and for all $x \in \mathcal{V}$

10. $1x = x$ for all $x \in \mathcal{V}$

Then we say that $\mathcal{V}$ is **vector space over** $\mathbb{R}$ (with these operations of addition and scalar multiplication)

Remark 3.2.1 In the above definition, if we replace $\mathbb{R}$ by any field $\mathcal{F}$, then we get the notion of a vector space over $\mathcal{F}$. We shall be basically concerned about vector spaces over $\mathbb{R}$ and vector spaces over $\mathbb{C}$.

Remark 3.2.2 From the above axioms the following properties in a vector space $\mathcal{V}$ follow:

1. For any $x \in \mathcal{V}$ we have $0x = \theta_{\mathcal{V}}$.
   This can be seen using the axioms of $+$ on $\mathcal{V}$ and those of scalar multiplication, as follows:

$$\begin{aligned}
0x &= (0+0)x \\
&= 0x + 0x \\
\Longrightarrow & \\
(-0x) + 0x &= (-0x) + (0x + 0x) \\
\Longrightarrow & \\
\theta_{\mathcal{V}} &= (-0x + 0x) + (0x) \\
\Longrightarrow & \\
\theta_{\mathcal{V}} &= \theta_{\mathcal{V}} + 0x \\
\Longrightarrow & \\
\theta_{\mathcal{V}} &= 0x
\end{aligned}$$

2. Similarly we can show that

$$(-1)x = (-x) \text{ for all } x \in \mathcal{V}$$

Remark 3.2.3 We shall call elements of a vector space as vectors.
The structure of a vector space comes from the two basic operations of addition and scalar multiplication, and whenever we introduce a transformation on a vector space, we have to keep track of the effect of the transformation on these two basic operations.

3.3 Examples of Vector Spaces

We shall now look at some examples of vector spaces:

1. Clearly $\mathbb{R}^k$ is a vector space over $\mathbb{R}$

2. Similarly $\mathbb{C}^k$ is a vector space over $\mathbb{R}$

3. $\mathbb{C}^k$ is also vector space over $\mathbb{C}$

4. $\mathbb{R}^{m\times n}$, the collection of all $m \times n$ real matrices, is a vector space over $\mathbb{R}$ with the usual laws of addition and scalar multiplication of matrices

5. In particular, $\mathbb{R}^{n\times n}$, the set of all $n \times n$ real square matrices, is a vector space over $\mathbb{R}$

6. Similarly, $\mathbb{C}^{m\times n}$, the set of all $m \times n$ complex matrices, is a vector space over $\mathbb{R}$ as well as over $\mathbb{C}$, and the set $\mathbb{C}^{n\times n}$, the set of all $n \times n$ complex square matrices is a vector space over $\mathbb{R}$ as well as over $\mathbb{C}$

7. Let $\mathcal{I}$ any interval on the real line. Then $\mathcal{V} = F_{\mathbb{R}}[\mathcal{I}]$, the collection of all real valued functions over $\mathcal{I}$, is a vector space over $\mathbb{R}$ with the usual laws of addition and scalar multiplication of functions.

8. Similarly, $\mathcal{V} = \mathcal{C}_{\mathbb{R}}[\mathcal{I}]$, the collection of all real valued continuous functions over $\mathcal{I}$, is a vector space over $\mathbb{R}$

9. $\mathcal{V} = L^2_{\mathbb{R}}[I]$, the collection of all real valued functions over $\mathcal{I}$, for which the integral $\int_{\mathcal{I}} |f(t)|^2 \, dt$ is defined and $< \infty$, is a vector space over $\mathbb{R}$

10. $\mathcal{V} = L^1_{\mathbb{R}}[I]$, the collection of all real valued functions over $\mathcal{I}$, for which the integral $\int_{\mathcal{I}} |f(t)|\, dt$ is defined and $< \infty$, is a vector space over $\mathbb{R}$

11. $\mathcal{V} = F_{\mathbb{C}}[\mathcal{I}]$, the collection of all complex valued functions over $\mathcal{I}$, is a vector space over $\mathbb{R}$ as well as over $\mathbb{C}$, with the usual laws of addition and scalar multiplication of functions.

12. Similarly, $\mathcal{V} = \mathcal{C}_{\mathbb{C}}[\mathcal{I}]$, the collection of all complex valued continuous functions over $\mathcal{I}$, is a vector space over $\mathbb{R}$

13. $\mathcal{V} = L^2_{\mathbb{C}}[I]$, the collection of all complex valued functions over $\mathcal{I}$, for which the integral $\int_{\mathcal{I}} |f(t)|^2\, dt$ is defined and $< \infty$, is a vector space over $\mathbb{R}$ as well as over $\mathbb{C}$

14. $\mathcal{V} = L^1_{\mathbb{C}}[I]$, the collection of all complex valued functions over $\mathcal{I}$, for which the integral $\int_{\mathcal{I}} |f(t)|\, dt$ is defined and $< \infty$, is a vector space over $\mathbb{R}$ as well as over $\mathbb{C}$

15. Let $\mathcal{F}$ be the field $\mathcal{Z}_2$. Then

$$\mathcal{V} = \mathcal{Z}_2^k = \left\{ x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \end{pmatrix} : x_j \in \mathcal{Z}_2 \text{ for } 1 \leq j \leq k \right\}$$

is a vector space over $\mathcal{Z}_2$ with the usual laws of addition and scalar multiplication.

16. Similarly if p is any prime number then $\mathcal{Z}_p^k$ is a vector space over $\mathcal{Z}_p$

17. In general, if $\mathcal{F}$ is any field then $\mathcal{F}^k$ is a vector space over $\mathcal{F}$.

3.4 Subspace

Let us consider a plane $\mathcal{P}$ passing through the origin in the vector space $\mathbb{R}^3$. The set of all vectors in $\mathcal{P}$ clearly form a nonempty subset of $\mathcal{P}$. It is easy to see that vectors in this subset have the following simple properties:

1. The "resultant" of two vectors in $\mathcal{P}$ is also in $\mathcal{P}$, that is,

$$x, y \in \mathcal{P} \Longrightarrow x + y \in \mathcal{P}$$

2. If we "scale" any vector in $\mathcal{P}$ by a factor α the resultant is also in $\mathcal{P}$, that is,

$$\alpha \in \mathbb{R} \text{ and } x \in \mathcal{P} \Longrightarrow \alpha x \in \mathcal{P}$$

The above two properties imply that the set $\mathcal{P}$ is self contained (or "closed") with respect to the two basic operations in the vector space, namely, addition and scalar multiplication. Analogously, if we consider a line $\mathcal{L}$, passing through the origin, in the vector space $\mathbb{R}^3$, it is easy to see that $\mathcal{L}$ is also closed with respect to the two basic operations of addition and scalar multiplication. We now generalize this property, of a subset being closed with respect to the two basic operations of the vector space, to get the notion of a subspace.

Definition 3.4.1 Let $\mathcal{V}$ be a vector space over a field $\mathcal{F}$. A nonempty subset $\mathcal{W}$ is said to be a **subspace** of $\mathcal{V}$ if

$$x, y \in \mathcal{W} \quad \Longrightarrow \quad x + y \in \mathcal{W} \tag{3.4.1}$$

$$x \in \mathcal{W} \text{ and } \alpha \in \mathcal{F} \quad \Longrightarrow \quad \alpha x \in \mathcal{W} \tag{3.4.2}$$

We now make some simple observations:

Remark 3.4.1 What the above definition says is that when we perform the two basic operations of the vector space with the $\mathcal{W}$ vectors then the resultant vectors are also $\mathcal{W}$ vectors. The main idea in analysing these vector spaces is to break the vector space into smaller subspaces, in a suitable manner, and analyse the problem in each subspace and then put all these together to get to the final analysis on the whole space.

Remark 3.4.2 We have

$$\begin{aligned}\mathcal{W} \text{ is a subspace} &\implies \mathcal{W} \text{ is nonempty} \\ &\implies \exists w \in \mathcal{W} \\ &\implies 0w \in \mathcal{W} \\ &\implies \theta_{\mathcal{V}} \in \mathcal{W}\end{aligned}$$

Thus we see that the zero vector belongs to every subspace

Remark 3.4.3 $\mathcal{V}$ is itself a subspace of $\mathcal{V}$

Remark 3.4.4 $\mathcal{V}$ is the "largest" subspace of $\mathcal{V}$ and $\mathcal{W} = \{\theta_{\mathcal{V}}\}$ is the smallest subspace of $\mathcal{V}$

Remark 3.4.5 Let $\mathcal{W}$ be a subspace of a vector space $\mathcal{V}$ over a field $\mathcal{F}$. Then by the requirements (3.4.1) and (3.4.2) for $\mathcal{W}$ to be a subspace, it follows that, the addition and scalar multiplication in $\mathcal{V}$ induce an addition and scalar multiplication on $\mathcal{W}$ and that $\mathcal{W}$ is itself a vector space over $\mathcal{F}$ with these operations. Thus every subspace $\mathcal{W}$ of $\mathcal{V}$ is a vector space inside the vector space $\mathcal{V}$, and the basic operations on $\mathcal{W}$ being the same as those in $\mathcal{V}$, except that they are now restricted only to the vectors in $\mathcal{W}$.

Example 3.4.1 Consider $\mathcal{V} = \mathbb{R}^3$.

1. Let $\mathcal{W}$ be the subset of $\mathbb{R}^3$ defined as

$$\mathcal{W} = \left\{ x = \begin{pmatrix} x_1 \\ x_2 \\ x_1 + x_2 \end{pmatrix} : x_1, x_2 \in \mathbb{R} \right\}$$

   Geometrically speaking, this is the plane $z = x + y$ in $\mathbb{R}^3$. It is easy to verify that this is a subspace of $\mathbb{R}^3$. We verify this fact as follows:

   (a) Since

$$\theta_3 = \begin{pmatrix} 0 \\ 0 \\ 0 + 0 \end{pmatrix}$$

   it follows that $\theta_3 \in \mathcal{W}$ and hence $\mathcal{W}$ is nonempty

(b) We have

$$
\begin{aligned}
x, y \in \mathcal{W} \implies& x = \begin{pmatrix} x_1 \\ x_2 \\ x_1 + x_2 \end{pmatrix}, \; y = \begin{pmatrix} y_1 \\ y_2 \\ y_1 + y_2 \end{pmatrix}, \; x_j, y_j \in \mathbb{R}, \; 1 \le j \le 2 \\
\implies& x + y = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ (x_1 + y_1) + (x_2 + y_2) \end{pmatrix} \\
&= \begin{pmatrix} \alpha \\ \beta \\ \alpha + \beta \end{pmatrix} \text{ where } \alpha = x_1 + x_2, \; \beta = y_1 + y_2 \in \mathbb{R} \\
\implies& x + y \in \mathcal{W}
\end{aligned}
$$

(c) Further,

$$
\begin{aligned}
x \in \mathcal{W}, \alpha \in \mathbb{R} \implies& x = \begin{pmatrix} x_1 \\ x_2 \\ x_1 + x_2 \end{pmatrix}, \alpha \in \mathbb{R} \\
\implies& \alpha x = \begin{pmatrix} \alpha x_1 \\ \alpha x_2 \\ \alpha(x_1 + x_2) \end{pmatrix} \\
&= \begin{pmatrix} a \\ b \\ a + b \end{pmatrix} \text{ where } a = \alpha x_1, \; b = \alpha x_2 \in \mathbb{R} \\
\implies& \alpha x \in \mathcal{W}
\end{aligned}
$$

Thus we see that $\mathcal{W}$ is nonempty and is closed with respect to addition and scalar multiplication and hence $\mathcal{W}$ is a subspace of $\mathbb{R}^3$

2. Let $\mathcal{W}$ be defined as

$$
\mathcal{W} = \left\{ x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} : 2x_1 - 3x_2 + x_3 = 0 \text{ and } 3x_1 - 4x_2 - x_3 = 0; x_1, x_2, x_3 \in \mathbb{R} \right\}
$$

Once again it is easy to verify that this is a subspace of $\mathbb{R}^3$. Geometrically speaking, this subspace is the line of intersection of the two planes $2x - 3y + z = 0$ and $3x - 4y - z = 0$

Remark 3.4.6 In general, a subspace in $\mathbb{R}^3$ will be either $\mathbb{R}^3$ or $\{\theta_3\}$, or a plane through the origin or a line through the origin.

Example 3.4.2 Consider the vector space $\mathbb{R}^k$ (where we assume $k \geq 2$). Then we can easily verify that the following subsets are subspaces of $\mathbb{R}^n$:

1. $\mathcal{W} = \left\{x \in \mathbb{R}^k : x_1 = 0\right\}$
2. $\mathcal{W} = \left\{x \in \mathbb{R}^k : x_k = 3x_1\right\}$

Example 3.4.3 Consider the vector space $\mathbb{R}^{4\times4}$. It is easy to verify that the following subsets are subspaces of $\mathbb{R}^{4\times4}$:

1. $\mathcal{W} = \left\{A \in \mathbb{R}^{4\times4} : a_{23} + a_{32} = 0\right\}$
2. $\mathcal{W} = \left\{A \in \mathbb{R}^{4\times4} : a_{ij} = a_{ji} \text{ for all } 1 \leq i, j \leq 4\right\}$
3. $\mathcal{W} = \left\{A \in \mathbb{R}^{4\times4} : a_{ij} = -a_{ji} \text{ for all } 1 \leq i, j \leq 4\right\}$
4. $\mathcal{W} = \left\{A \in \mathbb{R}^{4\times4} : Trace(A) = 0\right\}$

However the following subset is NOT a subspace of $\mathbb{R}^{4\times4}$ (Why?):

$$\mathcal{W} = \left\{A \in \mathbb{R}^{4\times4} : Trace(A) = 1\right\}$$

Example 3.4.4 Let us consider the vector space $\mathcal{V} = \mathcal{C}_{\mathbb{R}}[0,1]$, of all real valued, continuous functions, defined on the interval $[0,1]$. We can easily verify that the following subsets are subspaces:

1. $\mathcal{W} = \{f \in \mathcal{V} : f(0.5) = 0\}$
2. $\mathcal{W} = \{f \in \mathcal{V} : f(0.5) = 3f(1) + 4f(0)\}$
3. $\mathcal{W} = \left\{f \in \mathcal{V} : \int_0^1 f(t)dt = 0\right\}$
4. $\mathcal{W} = \left\{f \in \mathcal{V} : \int_0^1 f(t)e^{-t}dt = 0\right\}$

3.5 Subspace generated by a set of vectors

Let $\mathcal{V}$ be a vector space over a field $\mathcal{F}$, and let $x \neq \theta_{\mathcal{V}}$ be a vector in $\mathcal{V}$. Clearly, the subset,

$$\mathcal{S} = \{x\},$$

of $\mathcal{V}$, consisting of the single vector x is not a subspace. Let us now try to enclose it in a subspace. Obviously $\mathcal{V}$ is one subspace of $\mathcal{V}$ which encloses all subsets of $\mathcal{V}$, and hence $\mathcal{V}$ is, indeed, a subspace that encloses the set $\mathcal{S}$. What we want to do is to enclose $\mathcal{S}$ in as small a subspace as possible. What do we mean by this? We want to see if we can get a subspace $\mathcal{W}_{\mathcal{S}}$ of $\mathcal{V}$ which is such that,

1. $S \subseteq \mathcal{W}_{\mathcal{S}}$, that is $\mathcal{W}_{\mathcal{S}}$ is a subspace that encloses the set $\mathcal{S}$, and
2. If $\mathcal{W}_1$ is any other subspace that encloses $\mathcal{S}$, then $\mathcal{W}_{\mathcal{S}} \subset \mathcal{W}_1$, that is no subspace smaller than $\mathcal{W}$ encloses $\mathcal{S}$.

Let us now analyse to see whether we can get such a subspace. Suppose $\mathcal{W}$ is any subspace that encloses $\mathcal{S}$, that is, $\mathcal{S} \subseteq \mathcal{W}$. Then since $x \in \mathcal{S}$ we must have $x \in \mathcal{W}$. We have,
$\mathcal{W}$ is a subspace $\Longrightarrow$
$\mathcal{W}$ is closed under scalar multiplication $\Longrightarrow$
$\alpha x \in \mathcal{W}$ for all $\alpha \in \mathcal{F}$
Thus we have

$$\left.\begin{array}{l} \mathcal{W} \text{ subspace that encloses } \mathcal{S} \Longrightarrow \\ \mathcal{W} \text{ must contain all scalar multiples of } x \end{array}\right\} \tag{3.5.1}$$

Now consider the set $\mathcal{W}_{\mathcal{S}}$ consisting of only the scalar multiples of x, that is

$$\mathcal{W}_{\mathcal{S}} = \{\alpha x : \alpha \in \mathcal{F}\}$$

We can easily verify that $\mathcal{W}_{\mathcal{S}}$ is a subspace and encloses $\mathcal{S}$, and as observed above in (3.5.1), all vectors in $\mathcal{W}_{\mathcal{S}}$ must be in every subspace that encloses $\mathcal{S}$. Thus $\mathcal{W}_{\mathcal{S}}$ is the smallest subspace enclosing the set $\mathcal{S}$. Thus we have obtained the smallest subspace that encloses $\mathcal{S}$ in the case when $\mathcal{S} = \{x\}$, a set consisting of a single vector x. (Note that this argument works even if $x = \theta_{\mathcal{V}}$ and in this case we get $\mathcal{W}_{\mathcal{S}} = \{\theta_{\mathcal{V}}\}$).
We shall now generalise this idea to sets containing more than one vector. Let us first consider a finite set of vectors

$$\mathcal{S} = \{u_1, u_2, \cdots, u_r\}$$

Let $\mathcal{W}$ be any subspace enclosing $\mathcal{S}$. Then as before, we observe that
$\mathcal{W}$ is a subspace $\Longrightarrow$
$\mathcal{W}$ is closed under scalar multiplication $\Longrightarrow$
$\alpha u_j \in \mathcal{W}$ for all $\alpha \in \mathcal{F}$ and $1 \leq j \leq r$
Thus we have

$$\left.\begin{array}{l} \mathcal{W} \text{ subspace that encloses } \mathcal{S} \Longrightarrow \\ \mathcal{W} \text{ must contain all scalar multiples} \\ \text{of each of the vectors } u_j,\ 1 \leq j \leq r \end{array}\right\} \tag{3.5.2}$$

Further we have
$\mathcal{W}$ is a subspace $\Longrightarrow$
$\mathcal{W}$ is closed under addition $\Longrightarrow$
$\alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_r u_r \in \mathcal{W}$ for all $\alpha_j \in \mathcal{F}$
Thus we have

$$\left.\begin{array}{l} \mathcal{W} \text{ subspace that encloses } \mathcal{S} \Longrightarrow \\ \mathcal{W} \text{ must contain all vectors of the form} \\ \alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_r u_r \ \text{ where } \alpha_j \in \mathcal{F},\ 1 \leq j \leq r \end{array}\right\} \tag{3.5.3}$$

Let us now consider the set $\mathcal{W}_{\mathcal{S}}$, consisting only all the vectors of this form, that is

$$\mathcal{W}_{\mathcal{S}} \quad = \quad \{x = \alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_r u_r : \alpha_j \in \mathcal{F},\ 1 \leq j \leq r\} \tag{3.5.4}$$

It is now easy to verify that this is a subspace of $\mathcal{V}$, that obviously encloses the given $\mathcal{S}$, and by (3.5.3), no subspace smaller than this can enclose $\mathcal{S}$. Thus the $\mathcal{W}_{\mathcal{S}}$ defined in (3.5.4) is the smallest subspace that encloses $\mathcal{S}$. This leads us to the following definition:

Definition 3.5.1 Let $u_1, u_2, \cdots, u_r$ be a finite set of vectors in a vector space $\mathcal{V}$ over the field $\mathcal{F}$. Any vector of the form

$$\alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_r u_r \text{ where } \alpha_j \in \mathcal{F},\ 1 \leq j \leq r$$

is called a "**Linear Combination**" of the vectors $u_1, u_2, \cdots, u_r$.

Thus it follows that the smallest subspace $\mathcal{W}_{\mathcal{S}}$ that encloses a finite set of vectors $\mathcal{S}$ obtained in (3.5.4) is the collection of all linear combinations of the vectors in $\mathcal{S}$.
If $\mathcal{S}$ is an infinite set, then we can verify, (using similar ideas as above), that

the set, $\mathcal{W}_{\mathcal{S}}$, of all finite linear combinations of vectors in $\mathcal{S}$ is the smallest subspace enclosing $\mathcal{S}$.

Thus given any subset $\mathcal{S}$ of $\mathcal{V}$ there is a subspace which is the smallest subspace enclosing $\mathcal{S}$ and this subspace is called the **Subspace generated by** $\mathcal{S}$ or the **Subspace spanned by** $\mathcal{S}$ or the **Linear Span of** $\mathcal{S}$ and we shall denote this by $\mathcal{L}[\mathcal{S}]$. We have

$$\mathcal{L}[\mathcal{S}] = \begin{cases} \text{The set of all linear combinations} \\ \text{of vectors in } \mathcal{S} \text{ if } \mathcal{S} \text{ is a finite set.} \\ \text{The set of all finite linear combinations of} \\ \text{the vectors in } \mathcal{S} \text{ if } \mathcal{S} \text{ is an infinite set.} \end{cases} \tag{3.5.5}$$

Example 3.5.1 Consider $\mathcal{V} = \mathbb{R}^3$ and let

$$\mathcal{S} = u_1, u_2 \text{ where } u_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \ u_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

Then

$$\begin{aligned} \mathcal{L}[\mathcal{S}] &= \left\{ x \in \mathbb{R}^3 : x = \alpha_1 u_1 + \alpha_2 u_2, \text{ where } \alpha_1, \ \alpha_2 \in \mathbb{R} \right\} \\ &= \left\{ x \in \mathbb{R}^3 : x = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_1 + \alpha_2 \end{pmatrix}, \ \alpha_1, \alpha_2 \in \mathbb{R} \right\} \end{aligned}$$

3.6 Column and Row Subspaces Associated with a Matrix

We shall use the idea of a subspace generated by a set of vectors, introduced above, to define two important subspaces associated with a matrix. Let $A \in \mathcal{F}^{m \times n}$. Consider the column vectors,

$$C_j = \begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{pmatrix}, \ j = 1, 2, \cdots, n$$

These are n vectors in $\mathcal{F}^m$. Consider the set $\mathcal{C}$ of these n vectors in $\mathcal{F}^m$

$$\mathcal{C} = C_1, C_2, \cdots, C_n$$

The set, $\mathcal{L}[\mathcal{C}]$, of all linear combinations of these column vectors, is a subspace of $\mathcal{F}^m$. This subspace is called the **<u>COLUMN SPACE OF *A*</u>**. We denote this space by $Col(A)$.
Similarly we can define the column space of A^T and we denote this by $Col(A^T)$. This is, obviously, a subspace of $\mathcal{F}^n$.
Let us denote by R_j the transpose of the j-th row of A, that is,

$$R_j = \begin{pmatrix} a_{j1} \\ a_{j2} \\ \vdots \\ a_{jn} \end{pmatrix}$$

These are m vectors in $\mathcal{F}^n$. Let us denote the collection of these m vectors as $\mathcal{R}$. Thus

$$\mathcal{R} = \{R_j\}_{j=1}^m$$

The collection, $\mathcal{L}[\mathcal{R}]$, of all linear combinations of these vectors, is a subspace of $\mathcal{F}^n$. This subspace is called the **<u>ROW SPACE OF *A*</u>**. We denote this subspace by $Row(A)$. Similarly we can define the row space, $Row(A^T)$, of A^T. Clearly we have

1. $Col(A) = Row(A^T)$

2. $Col(A^T) = Row(A)$

3.7 Null Space and Range of a Matrix

We shall now introduce four more subspaces associated with a matrix. Consider $A \in \mathcal{F}^{m\times n}$. Consider the homogeneous system $Ax = \theta_m$. Let us denote by $\mathcal{N}_{_A}$ the set of all solutions of this homogeneous system. We have

$$\mathcal{N}_{_A} \;=\; \{x \in \mathcal{F}^n : Ax = \theta_m\} \tag{3.7.1}$$

This is a subset of $\mathcal{F}^n$. Will this be a subspace of $\mathcal{F}^n$? We have,

1. $\mathcal{N}_{_A}$ is nonempty since $\theta_n \in \mathcal{N}_{_A}$

2. $x, y \in \mathcal{N}_{_A} \implies Ax = \theta_m$ and $Ay = \theta_m$
   $\implies Ax + Ay = \theta_m$

$\Longrightarrow A(x+y) = \theta_m$
$\Longrightarrow x+y \in \mathcal{N}_{_A}$
$\Longrightarrow \mathcal{N}_{_A}$ is closed under addition

3. Similarly we have
$x \in \mathcal{N}_{_A}$ and $\alpha \in \mathcal{F} \Longrightarrow Ax = \theta_m$ and $\alpha \in \mathcal{F}$
$\Longrightarrow \alpha Ax = \theta_m$
$\Longrightarrow A(\alpha x) = \theta_m$
$\Longrightarrow \alpha x \in \mathcal{N}_{_A}$
$\Longrightarrow \mathcal{N}_{_A}$ is closed under scalar multiplication

From the above three properties we see that $\mathcal{N}_{_A}$ is a subspace of $\mathcal{F}^n$. This subspace is called the "**NULL SPACE**" of the matrix A. Analogously we can define the Null Space of A^T, denoted by $\mathcal{N}_{_{A^T}}$, as

$$\mathcal{N}_{_{A^T}} \quad = \quad \left\{ y \in \mathcal{F}^m : A^T y = \theta_n \right\} \tag{3.7.2}$$

Let us next consider the nonhomogeneous system $Ax = b$. We have seen that this system may be consistent for some $b \in \mathcal{F}^m$ and may not be consistent for some other $b \in \mathcal{F}^m$. Let us now collect all those $b \in \mathcal{F}^m$ for which this nonhomogeneous system is consistent. This means we collect all those $b \in \mathcal{F}^m$ for which there exists an $x \in \mathcal{F}^n$ such that $b = Ax$. In other words, we are collecting all those $b \in \mathcal{F}^m$ which can be written in the form Ax for some $x \in \mathcal{F}^n$. We denote this collection by $\mathcal{R}_{_A}$. We have

$$\mathcal{R}_{_A} \quad = \quad \{ b \in \mathcal{F}^m : \exists x \in \mathcal{F}^n \ni Ax = b \} \tag{3.7.3}$$

This is a subspace of $\mathcal{F}^m$. Will this a subspace of $\mathcal{F}^m$? We have

1. Clearly $\mathcal{R}_{_A}$ is nonempty, since $\theta_m \in \mathcal{R}_{_A}$, for, θ_m can be written as $A\theta_n$.

2. $b_1, b_2 \in \mathcal{R}_{_A} \Longrightarrow \exists\ x_1,\ x_2 \in \mathcal{F}^n$ such that $Ax_1 = b_1$ and $Ax_2 = b_2$
$\Longrightarrow \exists\ x = x_1 + x_2 \in \mathcal{F}^n$ such that $Ax = A(x_1+x_2) = Ax_1 + Ax_2 = b_1 + b_2$
$\Longrightarrow b_1 + b_2 \in \mathcal{R}_{_A}$
$\Longrightarrow \mathcal{R}_{_A}$ is closed under addition

3. $b \in \mathcal{R}_{_A}$ and $\alpha \in \mathcal{F} \Longrightarrow \exists\ x \in \mathcal{F}^n$ such that $Ax = b$ and $\alpha \in \mathcal{F}$
$\Longrightarrow \exists\ u = \alpha x \in \mathcal{F}^n$ such that $Au = A(\alpha x) = \alpha Ax = \alpha b$
$\Longrightarrow \alpha b \in \mathcal{R}_{_A}$
$\Longrightarrow \mathcal{R}_{_A}$ is closed under scalar multiplication.

From the above three properties we see that $\mathcal{R}_A$ is a subspace of $\mathcal{F}^m$. This subspace is called the "**RANGE SPACE**" of the matrix A. Analogously we can define the Range Space of A^T, denoted by $\mathcal{R}_{A^T}$, as

$$\mathcal{R}_{A^T} = \left\{x \in \mathcal{F}^n : \exists y \in \mathcal{F}^n \ni A^T y = x\right\} \tag{3.7.4}$$

Thus, in addition to the two subspaces introduced in the previous section, namely,
$Col(A)$, which is a subspace of $\mathcal{F}^m$ and
$Row(A)$, which is a subspace of $\mathcal{F}^n$,
we have now four more subspaces, namely,
$\mathcal{N}_A$, and $\mathcal{R}_{A^T}$, which are subspaces of $\mathcal{F}^n$, and
$\mathcal{N}_{A^T}$ and $\mathcal{R}_A$, which are subspaces of $\mathcal{F}^m$.

Thus we have so far seen the following subspaces associated with a matrix:

Subspaces of $\mathbb{R}^n$	Subspaces of $\mathbb{R}^m$
$Col(A^T) = Row(A)$	$Col(A) = Row(A^T)$
$\mathcal{N}_A$	$\mathcal{N}_{A^T}$
$\mathcal{R}_{A^T}$	$\mathcal{R}_A$

3.8 Connection Between the Subspaces Associated With a Matrix

We shall now look at the connection between some of the subspaces obtained above. We shall be seeing more and more connections as we proceed.

Connection Between $Col(A)$ and $\mathcal{R}_A$

We shall now see that the subspaces $Col(A)$ and $\mathcal{R}_A$ are one and the same. Recall that

$Col(A) = \mathcal{L}[\mathcal{C}]$, the subspace generated by the columns $C_1, C_2, \cdots, C_n$ of the mateix A

We have

$$
\begin{aligned}
b \in Col(A) &\iff \exists \alpha_1, \alpha_2, \cdots, \alpha_n \in \mathcal{F} \text{ such that } b = \alpha_1 C_1 + \alpha_2 C_2 \cdots + \alpha_n C_n \\
&\iff b = \alpha_1 \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} + \alpha_2 \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix} + \cdots + \alpha_n \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix} \\
&\iff b = \begin{pmatrix} \alpha_1 a_{11} + \alpha_2 a_{12} + \cdots + \alpha_n a_{1n} \\ \alpha_1 a_{21} + \alpha_2 a_{22} + \cdots + \alpha_n a_{2n} \\ \vdots \\ \alpha_1 a_{m1} + \alpha_2 a_{m2} + \cdots + \alpha_n a_{mn} \end{pmatrix} \\
&\iff b = Ax \text{ where } x = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix} \in \mathcal{F}^n \\
&\iff b \in \mathcal{R}_A
\end{aligned}
$$

Thus we have

$$Col(A) \;=\; \mathcal{R}_A \tag{3.8.1}$$

Analogously we have

$$Col(A^T) \;=\; \mathcal{R}_{A^T} \tag{3.8.2}$$

Thus the subspaces we have obtained are the following

Subspaces of $\mathbb{R}^n$	Subspaces of $\mathbb{R}^m$
$\mathcal{R}_{A^T} = Col(A^T) = Row(A)$	$\mathcal{R}_A = Col(A) = Row(A^T)$
$\mathcal{N}_A$	$\mathcal{N}_{A^T}$

Thus there are four basic subspaces associated with a matrix. Of these, two subspaces, $\mathcal{R}_{A^T}$ and $\mathcal{N}_A$, are subspaces of $\mathcal{F}^n$, and the other two, namely $\mathcal{R}_A$

and $\mathcal{N}_{A^T}$, are subspaces of $\mathcal{F}^m$. These are called the "**Four Fundamental Subspaces**" associated with a matrix $A \in \mathcal{F}^{m \times n}$

Connection Between $\mathcal{N}_A$ and $\mathcal{R}_{A^T}$

We have

$$\begin{aligned} x \in \mathcal{N}_A &\iff Ax = \theta_m \\ &\iff (Ax)^T b = 0 \text{ for all } b \in \mathcal{F}^m \\ &\iff (x^T A^T) b = 0 \text{ for all } b \in \mathcal{F}^m \\ &\iff x^T (A^T b) = 0 \text{ for all } b \in \mathcal{F}^m \\ &\iff x^T u = 0 \text{ for all } u \in \mathcal{R}_{A^T} \end{aligned}$$

Thus we have

$$x \in \mathcal{N}_A \iff x^T u = 0 \text{ for all } u \in \mathcal{R}_{A^T} \tag{3.8.3}$$

We can also write this as

$$u \in \mathcal{R}_{A^T} \iff u^T x = 0 \text{ for all } x \in \mathcal{N}_A \tag{3.8.4}$$

Analogously we have

$$x \in \mathcal{N}_{A^T} \iff x^T u = 0 \text{ for all } u \in \mathcal{R}_A \tag{3.8.5}$$

We can also write this as

$$b \in \mathcal{R}_A \iff b^T x = 0 \text{ for all } x \in \mathcal{N}_{A^T} \tag{3.8.6}$$

3.9 A First Look Into $\mathcal{R}_A$

We shall now take a first look into the structure of $\mathcal{R}_A$ where $A \in \mathcal{F}^{m \times n}$. The first question that arises is the following:

Question

Given a vector $b \in \mathcal{F}^m$ how do we determine whether b is in $\mathcal{R}_A$ or not?

We shall now find the answer this question. We have

$$\begin{aligned} b \in \mathcal{R}_A &\iff \exists\, x \in \mathcal{F}^n \ni\ Ax = b \\ &\iff \text{The Nonhomogeneous System } Ax = b \text{ is consistent} \\ &\iff \text{Row rank of the augmented matrix } \left(\begin{array}{c|c} A & b \end{array} \right) \text{ is equal to the Row rank of } A \end{aligned}$$

Thus in order to determine whether a given $b \in \mathcal{F}^m$ is in $\mathcal{R}_{_A}$, we first write the augmented matrix $A_{aug} = \left(\begin{array}{c|c} A & b \end{array} \right)$ and then use EROs to RRE form to get

$$\left(\begin{array}{c|c} A & b \end{array} \right) \xrightarrow{EROs} \left(\begin{array}{c|c} A_R & \tilde{b} \end{array} \right)$$

If Row rank of A is ρ, that is the nonzero rows of A_R are $R_1, R_2, \cdots, R_\rho$, then

$$b \in \mathcal{R}_{_A} \Longleftrightarrow \tilde{b}_j = 0 \text{ for } \rho + 1 \le j \le m$$

Example 3.9.1 . Let $A \in \mathbb{R}^{3\times 4}$ be the matrix given below:

$$A \;=\; \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 7 \\ 1 & 2 & 3 & 6 \end{pmatrix}$$

Let us now determine whether the vector

$$b = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

is in $\mathcal{R}_{_A}$. We now write the augmented matrix and row reduce this to RRE form. We have

$$\left(\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 1 \\ 2 & 4 & 6 & 7 & 2 \\ 1 & 2 & 3 & 6 & 1 \end{array} \right) \xrightarrow{R_2 - 2R_1,\, R_3 - R_1} \left(\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 1 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 2 & 0 \end{array} \right) \xrightarrow{-R_2}$$

$$\left(\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 & 0 \end{array} \right) \xrightarrow{R_3 - 2R_2} \left(\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right) \xrightarrow{R_1 - 4R_2}$$

$$\left(\begin{array}{cccc|c} 1 & 2 & 3 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right)$$

Hence we see that the Row Rank of the Augmented Matrix, A_{aug}, is the same as that of A. Hence the given b is in $\mathcal{R}_{_A}$. On the other hand if we take

$$b = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

then we see that the above EROs reduce A_{aug} as

$$A_{aug} = (A|b) = \left(\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 1 \\ 2 & 4 & 6 & 7 & 1 \\ 1 & 2 & 3 & 6 & 1 \end{array}\right) \overset{above\ EROs}{\longrightarrow}$$

$$\left(\begin{array}{cccc|c} 1 & 2 & 3 & 0 & -3 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & -2 \end{array}\right)$$

Hence this b is not in $\mathcal{R}_A$ since the Augmented matrix A_{aug} and the matrix A have different ranks.

We now look at the natural question that arises about $\mathcal{R}_A$, namely,
Question
How can we determine the subspace $\mathcal{R}_A$, that is how do we characterise all vectors in $\mathcal{R}_A$?
The above example suggests what we should do. We now look at the Augmented matrix $\left(A \mid b \right)$ with a general

$$b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ \mathrm{b}_m \end{pmatrix}$$

We then do the row reduction to get

$$(A|b) \overset{EROs}{\longrightarrow} \left(A_R \mid \tilde{b} \right)$$

If the row rank of A is ρ then we have the first ρ rows of A_R as nonzero rows and the remaining $m - \rho$ rows as zero rows. Thus we have

$$\left(A_R \mid \tilde{b} \right) = \left(\begin{array}{c|c} R_1 & \tilde{b}_1 \\ R_2 & \tilde{b}_2 \\ \vdots & \vdots \\ R_\rho & \tilde{b}_\rho \\ 0_{1\times n} & \tilde{b}_{(\rho+1)} \\ \vdots & \vdots \\ 0_{1\times n} & \tilde{b}_m \end{array}\right)$$

Hence row rank of the Augmented Matrix, A_{aug}, will be equal to the the row rank of A if and only if $\tilde{b}_j = 0$ for $\rho + 1 \le j \le m$. Hence $b \in \mathcal{R}_A$ if and only if,

$$\left.\begin{array}{lcl} \tilde{b}_{(\rho+1)} & = & 0 \\ \tilde{b}_{(\rho+2)} & = & 0 \\ \cdots & \cdots & \cdots \\ \tilde{b}_m & = & 0 \end{array}\right\}$$

The above system of equations is an $(m - \rho) \times m$ homogeneous system of equations for determining the vector b. Solving this homogeneous system we get the vectors in $\mathcal{R}_A$.

Remark 3.9.1 Since the $Col(A)$ is the same as $\mathcal{R}_A$, the same method can be used to determine the Column space of a matrix.

It is easy to see that EROs on a matrix do not alter the Row Space of the matrix. This means that if

$$A \xrightarrow{EROs} A_1$$

then

$$Row(A) = Row(A_1)$$

Hence, in particular, we see that a matrix and its RRE form must have the same Row Space. Thus we have

$$Row(A^T) = Row((A^T)_R)$$

But using the fact that the $Row(A^T) = Col(A) = \mathcal{R}_A$ we see that

$$\mathcal{R}_A = Row((A^T)_R)$$

However, the transpose of the nonzero rows of $(A^T)_R)$ form a basis for $Row((A^T)_R)$ and hence also form a basis for $\mathcal{R}_A$. Similarly the transpose of the nonzero rows of A_R form a basis for $\mathcal{R}_{A^T}$.

Example 3.9.2 For the matrix of Example 3.9.1 we have

$$A_{aug} = \left(\begin{array}{cccc|c} 1 & 2 & 3 & 4 & b_1 \\ 2 & 4 & 6 & 7 & b_2 \\ 1 & 2 & 3 & 6 & b_3 \end{array}\right) \xrightarrow{R_2 - 2R_1,\ R_3 - R_1} \left(\begin{array}{cccc|c} 1 & 2 & 3 & 4 & b_1 \\ 0 & 0 & 0 & -1 & -2b_1 + b_2 \\ 0 & 0 & 0 & 2 & -b_1 + b_3 \end{array}\right)$$

$$\xrightarrow{-R_2} \left(\begin{array}{cccc|c} 1 & 2 & 3 & 4 & b_1 \\ 0 & 0 & 0 & 1 & 2b_1 - b_2 \\ 0 & 0 & 0 & 2 & -b_1 + b_3 \end{array}\right) \xrightarrow{R_3 - 2R_2} \left(\begin{array}{cccc|c} 1 & 2 & 3 & 4 & b_1 \\ 0 & 0 & 0 & 1 & 2b_1 - b_2 \\ 0 & 0 & 0 & 0 & -5b_1 + 2b_2 + b_3 \end{array}\right)$$

$$\xrightarrow{R_1 - 4R_2} \left(\begin{array}{cccc|c} 1 & 2 & 3 & 0 & -7b_1 + 4b_2 \\ 0 & 0 & 0 & 1 & 2b_1 - b_2 \\ 0 & 0 & 0 & 0 & -5b_1 + 2b_2 + b_3 \end{array}\right)$$

Hence we see that row rank of the augmented matrix A_{aug} is equal to the row rank of A if and only if $-5b_1 + 2b_2 + b_3 = 0$ and hence

$$b \in \mathcal{R}_A \iff -5b_1 + 2b_2 + b_3 = 0$$

Solving the above 1×3 homogeneous system we get the general solution as

$$b = \alpha \begin{pmatrix} 2 \\ 5 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 1 \\ 0 \\ 5 \end{pmatrix} \text{ (where } \alpha,\ \beta \in \mathbb{R}).$$

Hence $\mathcal{R}_A$ can also be written as

$$\mathcal{R}_A = \mathcal{L}[u, v]$$

where

$$u = \begin{pmatrix} 2 \\ 5 \\ 0 \end{pmatrix} \text{ and } v = \begin{pmatrix} 1 \\ 0 \\ 5 \end{pmatrix}$$

Since $Col(A)$ is equal to $\mathcal{R}_A$ we can also write

$$Col(A) = \mathcal{L}[u, v]$$

On the other hand $Col(A) = \mathcal{L}[\mathcal{C}]$ where

$$\mathcal{C} = C_1, C_2, C_3, C_4$$

where

$$C_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix},\ C_2 = \begin{pmatrix} 2 \\ 4 \\ 2 \end{pmatrix}, C_3 = \begin{pmatrix} 3 \\ 6 \\ 3 \end{pmatrix}, C_4 = \begin{pmatrix} 4 \\ 7 \\ 6 \end{pmatrix}, \text{ the four columns of } A$$

Hence we can write

$$\mathcal{R}_{_A} = Col(A) = \mathcal{L}[u, v] = \mathcal{L}[C_1, C_2, C_3, C_4]$$

Further the transpose of the nonzero rows of A_R are

$$\varphi = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 0 \end{pmatrix}, \; \psi = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

These give a spanning set for $Row(A) = Col(A^T) = \mathcal{R}_{_{A^T}}$. Hence we can write

$$Row(A) = Col(A^T) = \mathcal{R}_{_{A^T}} = \mathcal{L}[\varphi, \psi]$$

Example 3.9.3 Let $A \in \mathbb{R}^{4\times 5}$ be the matrix given below:

$$A = \begin{pmatrix} 1 & 2 & 0 & 1 & 0 \\ 1 & -1 & 1 & 0 & 1 \\ 3 & 3 & 1 & 2 & 1 \\ 3 & 0 & 2 & 1 & 2 \end{pmatrix}$$

Let us find the Range, $\mathcal{R}_{_A}$, for this matrix. We have

$$A_{aug} = A = \left(\begin{array}{ccccc|c} 1 & 2 & 0 & 1 & 0 & \mathrm{b}_1 \\ 1 & -1 & 1 & 0 & 1 & \mathrm{b}_2 \\ 3 & 3 & 1 & 2 & 1 & \mathrm{b}_3 \\ 3 & 0 & 2 & 1 & 2 & \mathrm{b}_4 \end{array}\right) \xrightarrow{R_2 - R_1,\ R_3 - 3R_1,\ R_4 - 3R_1}$$

$$\left(\begin{array}{ccccc|c} 1 & 2 & 0 & 1 & 0 & \mathrm{b}_1 \\ 0 & -3 & 1 & -1 & 1 & -b_1 + b_2 \\ 0 & -3 & 1 & -1 & 1 & -3b_1 + b_3 \\ 0 & -6 & 2 & -2 & 2 & -3b_1 + b_4 \end{array}\right) \xrightarrow{-\frac{1}{3}R_2}$$

$$\left(\begin{array}{ccccc|c} 1 & 2 & 0 & 1 & 0 & \mathrm{b}_1 \\ 0 & 1 & -\frac{1}{3} & \frac{1}{3} & -\frac{1}{3} & \frac{1}{3}b_1 - \frac{1}{3}b_2 \\ 0 & -3 & 1 & -1 & 1 & -3b_1 + b_3 \\ 0 & -6 & 2 & -2 & 2 & -3b_1 + b_4 \end{array}\right) \xrightarrow{R_3 + 3R_2,\ R_4 + 6R_2}$$

$$\left(\begin{array}{ccccc|c} 1 & 2 & 0 & 1 & 0 & \mathrm{b}_1 \\ 0 & 1 & -\frac{1}{3} & \frac{1}{3} & -\frac{1}{3} & \frac{1}{3}b_1 - \frac{1}{3}b_2 \\ 0 & 0 & 0 & 0 & 0 & -2b_1 - b_2 + b_3 \\ 0 & 0 & 0 & 0 & 0 & -b_1 - 2b_2 + b_4 \end{array}\right) \xrightarrow{R_1 - 2R_2}$$

$$\left(\begin{array}{ccccc|c} 1 & 0 & \frac{2}{3} & \frac{1}{3} & \frac{2}{3} & \frac{1}{3}b_1 + \frac{2}{3}b_2 \\ 0 & 1 & -\frac{1}{3} & \frac{1}{3} & -\frac{1}{3} & \frac{1}{3}b_1 - \frac{1}{3}b_2 \\ 0 & 0 & 0 & 0 & 0 & -2b_1 - b_2 + b_3 \\ 0 & 0 & 0 & 0 & 0 & -b_1 - 2b_2 + b_4 \end{array}\right)$$

Hence we see that $b \in \mathcal{R}_{_A}$ if and only if

$$\left.\begin{array}{rcl} -2b_1 - b_2 + b_3 & = & 0 \\ -b_1 - 2b_2 + b_4 & = & 0 \end{array}\right\}$$

This is a 2×3 homogeneous system. Solving this homogeneous system we get

$$b = \alpha \begin{pmatrix} 2 \\ -1 \\ 3 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} -1 \\ 2 \\ 0 \\ 3 \end{pmatrix} \text{ where } \alpha,\ \beta \in \mathbb{R}$$

Hence we see that

$$\mathcal{R}_{_A} = Col(A) = \mathcal{L}[u, v]$$

where

$$u = \begin{pmatrix} 2 \\ -1 \\ 3 \\ 0 \end{pmatrix} \text{ and } v = \begin{pmatrix} -1 \\ 2 \\ 0 \\ 3 \end{pmatrix}$$

Further the transpose of the nonzero rows of A_R are given by

$$\varphi = \begin{pmatrix} 1 \\ 0 \\ \frac{2}{3} \\ \frac{1}{3} \\ \frac{2}{3} \end{pmatrix}, \ \psi = \begin{pmatrix} 0 \\ 1 \\ -\frac{1}{3} \\ \frac{1}{3} \\ -\frac{1}{3} \end{pmatrix}$$

and hence we can write

$$Row(A) = Col(A^T) = \mathcal{R}_{A^T} = \mathcal{L}[\varphi, \psi]$$

Thus we see that we can get the $\mathcal{R}_A$ through the augmented matrix.
We shall now look at $\mathcal{R}_A$ from another point of view. We shall now exploit the connection between $\mathcal{N}_{A^T}$ and $\mathcal{R}_A$ as obtained in 3.8.5. From 3.8.5 we have

$$b \in \mathcal{R}_A \Longleftrightarrow b^T x = 0 \text{ for all } x \in \mathcal{N}_{A^T}$$

Given A we first find $\mathcal{N}_{A^T}$ by solving the homogeneous system $A^T x = \theta_n$. Suppose there are r free variables and $m-r$ pivotal variables, then the general solution the homogeneous system $A^T x = \theta_n$ will be of the form

$$x =_1 u_1 + \alpha_{2u2} + \cdots + \alpha_r u_r$$

We then get

$$\begin{aligned} b \in \mathcal{R}_A &\Longleftrightarrow b^T x = 0 \text{ for all } x \in \mathcal{N}_{A^T} \\ &\Longleftrightarrow b^T u_j = 0 \text{ for } 1 \leq j \leq r \end{aligned}$$

Thus the vectors b in $\mathcal{R}_A$ are determined by the $r \times m$ homogeneous system

$$\left.\begin{array}{ccc} u_1^T b & = & 0 \\ u_2^T b & = & 0 \\ \cdots & \cdots & \cdots \\ u_r^T b & = & 0 \end{array}\right\}$$

Solving this homogeneous system we get all the vectors in $\mathcal{R}_A$.

Example 3.9.4 For the matrix A of Example 3.8.1 we have

$$A^T = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 3 & 6 & 3 \\ 4 & 7 & 6 \end{pmatrix} \xrightarrow{R_2-2R_1,\ R_3-3R_1,\ R_4-4R_1} \begin{pmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 2 \end{pmatrix} \xrightarrow{R_{24}} \begin{pmatrix} 1 & 2 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{-R_2} \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_1-2R_2} \begin{pmatrix} 1 & 0 & 5 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = (A^T)_R$$

From the above RRE form of A^T we get the general solution of the homogeneous system $A^T x = \theta_n$ as

$$x = \alpha \begin{pmatrix} -5 \\ 2 \\ 1 \end{pmatrix}$$

Hence $b \in \mathcal{R}_A$ if and only if,

$$b^T \begin{pmatrix} -5 \\ 2 \\ 1 \end{pmatrix} = 0$$

that is, if and only if

$$-5b_1 + 2b_2 + b_3 = 0$$

as obtaned in Example 3.9.2. Solving this homogeneous system we get the same $\mathcal{R}_A$ as we got in Example 3.9.2. Further the transpose of the nonzero rows of $((A^T)_R)$ are given by

$$u = \begin{pmatrix} 1 \\ 0 \\ 5 \end{pmatrix},\ v = \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}$$

and hence we also get

$$Row(A^T) = Col(A) = \mathcal{R}_A = \mathcal{L}[u, v]$$

Example 3.9.5 We shall find $\mathcal{R}_{_A}$ for the matrix A in Example 3.9.3 using the $\mathcal{N}_{_{A^T}}$. We have

$A^T =$

$$\begin{pmatrix} 1 & 1 & 3 & 3 \\ 2 & -1 & 3 & 0 \\ 0 & 1 & 1 & 2 \\ 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 2 \end{pmatrix} \overset{R_2-2R_1,\ R_4-R_1}{\longrightarrow\longrightarrow} \begin{pmatrix} 1 & 1 & 3 & 3 \\ 0 & -3 & -3 & -6 \\ 0 & 1 & 1 & 2 \\ 0 & -1 & -1 & 2 \\ 0 & 1 & 1 & 2 \end{pmatrix} \overset{-\frac{1}{3}R_2}{\longrightarrow}$$

$$\begin{pmatrix} 1 & 1 & 3 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 1 & 1 & 2 \\ 0 & -1 & -1 & 2 \\ 0 & 1 & 1 & 2 \end{pmatrix} \overset{R_3-R_2,\ R_4+R_2,\ R_5-R_2}{\longrightarrow\longrightarrow} \begin{pmatrix} 1 & 1 & 3 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \overset{R_1-R_2}{\longrightarrow}$$

$$\begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = (A^T)_R$$

Using the above RRE form of A^T we get the general solution of the homogeneous system $A^T x = \theta_5$ as

$$\begin{aligned} x &= \alpha_1 \begin{pmatrix} -2 \\ -1 \\ 1 \\ 0 \end{pmatrix} + \alpha_2 \begin{pmatrix} 1 \\ -2 \\ 0 \\ 1 \end{pmatrix} \text{ where } \alpha_1,\ \alpha_2 \in \mathbb{R} \\ &= \alpha_1 u_1 + \alpha_2 u_2 \end{aligned}$$

where

$$u_1 = \begin{pmatrix} -2 \\ -1 \\ 1 \\ 0 \end{pmatrix} \text{ and } u_2 = \begin{pmatrix} 1 \\ -2 \\ 0 \\ 1 \end{pmatrix}$$

We have $b \in \mathcal{R}_{_A}$ if and only if

$$b^T u_1 = 0$$

$$b^T u_2 = 0$$

Hence we get the homogeneous system

$$\left.\begin{array}{rcl} -2b_1 - b_2 + b_3 & = & 0 \\ b_1 - 2b_2 + b_4 & = & 0 \end{array}\right\}$$

which is the same as that obtained in Example 3.9.3. Hence we get the $\mathcal{R}_A$ again the same as in Example 3.9.3. Moreover, the transpose of the nonzero rows of $(A^T)_R$ are given by,

$$\varphi = \begin{pmatrix} 1 \\ 0 \\ 2 \\ 1 \end{pmatrix}, \ \psi = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 2 \end{pmatrix}$$

Hence we can write

$$Row(A^T) = Col(A) = \mathcal{R}_A = \mathcal{L}[\varphi, \psi]$$

3.10 Spanning Set

Let $\mathcal{V}$ be a vector space over a field $\mathcal{F}$. We have seen that if S is any subset of V then we can define a subspace $\mathcal{L}[S]$, the subspace generated by S, which is the set of all (finite) linear combinations of the vectors in S. Now we ask the converse question, namely, the following:

Question

Given a subspace $\mathcal{W}$ of $\mathcal{V}$, can we find a subset S of $\mathcal{V}$ such that $\mathcal{L}[S] = \mathcal{W}$? We first observe that, obviously, if such a set S exists then since

$$S \subseteq \mathcal{L}[S] = \mathcal{W}$$

we must have $S \subseteq W$. Further, we can then express every vector in $\mathcal{W}$ as a finite linear combination of vectors in S, and hence, essentially, we can think of S as a sampling set for $\mathcal{W}$.

Definition 3.10.1 If $\mathcal{W}$ is a subspace of $\mathcal{V}$ then a subset S of $\mathcal{W}$ such that $\mathcal{L}[S] = \mathcal{W}$ is called a **Spanning Set** for the subspace $\mathcal{W}$

Next we observe that obviously we can take $S = \mathcal{W}$ and we then get clearly $\mathcal{L}[s] = \mathcal{L}[\mathcal{W}] = \mathcal{W}$. Thus certainly we can take $\mathcal{W}$ itself as a spanning set for $\mathcal{W}$. However, interpreted as a sampling set, this does not make much sense since we will be sampling the whole subspace. What we would like naturally is to get a spanning set as small as possible. We shall first look at an example.

Example 3.10.1 In the vector space $\mathbb{R}^3$ consider the subspace

$$\mathcal{W} = \left\{ x = \begin{pmatrix} \alpha \\ \beta \\ \alpha + \beta \end{pmatrix} \right\}$$

Consider the following vectors in $\mathcal{W}$:

$$u_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \qquad u_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \qquad u_3 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$$

We can easily see that the following subsets of $\mathcal{W}$ are all spanning sets for $\mathcal{W}$:

$$\begin{aligned} S_1 &= \{u_1, u_2\} \\ S_2 &= \{u_1, u_3\} \\ S_3 &= \{u_2, u_3\} \\ S_4 &= \{u_1, u_2, u_3\} \end{aligned}$$

From the above example, it follows that a subspace may have many spanning sets, and that different spanning sets may have different "sizes". What we should be looking for, is an optimally sized spanning set, that is, we want to ask the following question:

Question

Given a subspace $\mathcal{W}$ of $\mathcal{V}$, can we find a spanning set S for $\mathcal{W}$, such that any other spanning set for $\mathcal{W}$, must have at least "as many" vectors as in S?

The above question has been posed in an intuitive manner and we have to make it more precise. Seeking the answer to this question leads to the notions of "linear independence" and "basis", which we shall treat in the next chapter.