EXERCISES - SET I

1. Consider the nonhomogeneous system $Ax = b$ where

$$\begin{aligned} A &= \begin{pmatrix} 2\alpha & 3\alpha \\ 2\beta & 3\beta \end{pmatrix} \\ b &= \begin{pmatrix} 2\beta \\ 3\alpha \end{pmatrix} \end{aligned}$$

(where $\alpha\beta \neq 0$ and $3\alpha^2 - 2\beta^2 \neq 0$). Answer the following:

   (a) Show that the nonhomogeneous system $Ax = b$ is not consistent

   (b) Find all least square solutions of the system

2. Which of the following matrices are/is in RRE form?

$$A = \begin{pmatrix} 1 & 0 & 1 & 0 & 2 \\ 0 & 1 & 4 & 0 & 3 \\ 0 & 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 4 & 3 \\ 0 & 1 & 0 & 0 & -4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$C = \begin{pmatrix} 1 & 0 & 1 & 1 & 2 \\ 0 & 1 & 4 & 0 & 3 \\ 0 & 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad D = \begin{pmatrix} 1 & 1 & 1 & 0 & 2 \\ 0 & 1 & 4 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -4 \end{pmatrix}$$

3. If the row rank of the matrix $A \in \mathcal{F}^{5\times 4}$ is 4 answer whether the following statements are True or False

   (a) The homogeneous system $Ax = \theta_m$ has only trivial solution

   (b) For any $b \in \mathcal{F}^5$ the row rank of the augmented matrix A_{aug} is also 4

   (c) The nonhomogeneous equation $Ax = b$ is consistent for every $b \in \mathcal{F}^5$ and the solution is unique

4. If the RRE form of $A \in \mathcal{F}^{3\times 4}$ is of the form

$$\begin{pmatrix} 1 & 2 & a & -1 \\ \star & \star & \star & \star \\ \star & \star & \star & \star \end{pmatrix}$$

and if $x = \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \end{pmatrix}$ is a solution of the homogeneous system $Ax = \theta_m$ then what should be the value of a?

5. TRUE or FALSE:
   Let $A \in \mathcal{R}^{m\times n}$ and $b \in \mathbb{R}^m$ then the nonhomogeneous system $Ax = b$ is consistent **if and only if** the normal system $A^T Ax = b$ is consistent

6. If the nonhomogeneous system $Ax = b$ is not consistent and, x_ℓ is a least square solution of $Ax = b$, and x_H is a solution of the homogeneous system $Ax = \theta_m$, then show that $x_1 = x_\ell + x_H$ is also a least square solution of the nonhomogeneous system $Ax = b$.

7. Use the result above to prove the following: (Assume that every inconsistent nonhomogeneous system has a least square solution)
   An inconsistent nonhomogeneous system $Ax = b$ has a unique least square solution if and only if the corresponding homogeneous system $Ax = \theta_m$ has only trivial solution

8. For the nonhomogeneous system, $Ax = b$, where

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 2 & 3 & -1 \end{pmatrix}, \quad \text{and } b = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

   find the corresponding normal system.

9. Consider the matrix A and the vector b as given below:

$$A = \begin{pmatrix} 2 & 2 & 6 \\ 2 & 1 & 7 \\ -2 & -6 & -7 \end{pmatrix}, \; b = \begin{pmatrix} 4 \\ 6 \\ -1 \end{pmatrix}$$

   Answer the folowing:

   (a) Find the RRE form of A_{aug}

   (b) Find the row ranks of A and A_{aug}

   (c) Is the system $Ax = b$ consistent

(d) If $Ax = b$ is consistent find all solutions of $Ax = b$. If it is not consistent find all least square solutions

(e) Will the system be consistent if the b above is changed to some other $b \in \mathcal{F}^3$?

(f) Find an invertible matrix E such that EA is equal to A_R, the RRE form of A

(g) Is the matrix invertible? If so find the inverse of A

10. Let $A \in \mathbb{R}^{3\times 4}$ be the matrix given below:

$$A = \begin{pmatrix} 1 & 2 & 1 & 1 \\ 2 & 4 & 2 & 2 \\ 3 & 6 & 3 & 4 \end{pmatrix}$$

Answer the following:

(a) Find the RRE form A_R of A

(b) What is the row rank of A?

(c) Find the general soltion of the homogeneous system $Ax = \theta_3$

(d) Find an invertible matrix E such that $EA = A_R$

(e) Find an invertible matrix $Q \in \mathbb{R}^{3\times 3}$ and an invertible matrix $P \in \mathbb{R}^{4\times 4}$ such that

$$A = Q \left(\begin{array}{c|c} \mathrm{I}_{2\times 2} & 0_{2\times 2} \\ \hline 0_{1\times 2} & 0_{1\times 2} \end{array} \right) P$$

11. Using EROs determine whether the following matrix is invertible and if so find its inverse:

$$A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 5 & 7 \\ 3 & 6 & 8 \end{pmatrix}$$

12. TRUE or FALSE?

(a) Two matrices $A, B \in \mathcal{F}^{m\times n}$ are row equivalent if and only if they have the same RRE form

(b) If two matrices $A, B \in \mathcal{F}^{m\times n}$ are row equivalent then they have the same row rank

(c) If two matrices $A, B \in \mathcal{F}^{m\times n}$ are row equivalent then the homogeneous systems $Ax = \theta_m$ and $Bx = \theta_m$ have the same set of solutions

(d) If two matrices $A, B \in \mathcal{F}^{m\times n}$ are row equivalent then there exists an invertible matrix E such that both EA and EB are in RRE form

(e) If two matrices $A, B \in \mathcal{F}^{m\times n}$ are row equivalent then for $b \in \mathcal{F}^m$, the nonhomogeneous systems $Ax = b$ is consistent if and only if $Bx = b$ is consistent

(f) If $A \in \mathcal{F}^{m\times n}$ and $b \in \mathcal{F}^m$ is such that $Ax = b$ is consistent then A and A_{aug} have the same row rank

(g) If $A \in \mathcal{F}^{m\times n}$ and row rank of A is ρ then there exist invertible matrices $E, F \in \mathcal{F}^{m\times m}$ such that

$$EAF = \left(\begin{array}{c|c} \mathrm{I}_{\rho\times\rho} & 0_{\rho\times(n-\rho)} \\ \hline 0_{(m-\rho)\times\rho} & 0_{(m-\rho)\times(n-\rho)} \end{array}\right)$$

(h) If $A \in \mathcal{F}^{m\times n}$ and row rank of A is ρ then there exist invertible matrices $E, F \in \mathcal{F}^{n\times n}$ such that

$$EAF = \left(\begin{array}{c|c} \mathrm{I}_{\rho\times\rho} & 0_{\rho\times(n-\rho)} \\ \hline 0_{(m-\rho)\times\rho} & 0_{(m-\rho)\times(n-\rho)} \end{array}\right)$$

(i) If $A \in \mathcal{F}^{m\times n}$ and row rank of A is ρ then there exist invertible matrices $E \in \mathcal{F}^{m\times m}$ and $F \in \mathcal{F}^{n\times n}$ such that

$$EAF = \left(\begin{array}{c|c} \mathrm{I}_{\rho\times\rho} & 0_{\rho\times(n-\rho)} \\ \hline 0_{(m-\rho)\times\rho} & 0_{(m-\rho)\times(n-\rho)} \end{array}\right)$$

(j) If $A \in \mathcal{F}^{m\times n}$ and row rank of A is ρ then there exist invertible matrices $E \in \mathcal{F}^{n\times n}$ and $F \in \mathcal{F}^{m\times m}$ such that

$$EAF = \left(\begin{array}{c|c} \mathrm{I}_{\rho\times\rho} & 0_{\rho\times(n-\rho)} \\ \hline 0_{(m-\rho)\times\rho} & 0_{(m-\rho)\times(n-\rho)} \end{array}\right)$$