

Chapter 4

Row Reduced Echelon Matrices and Linear Systems

4.1 Row Reduced Echelon (RRE) Matrices

We shall now look at homogeneous linear systems $Ax = \theta_m$ for a matrix $A \in \mathbb{F}^{m \times n}$ when the matrix A is in a simple form. We first introduce the notion of a Row Reduced Echelon Matrix.

Definition 4.1.1 A matrix $A \in \mathcal{F}^{m \times n}$ is said to be a “**Row Reduced Echelon**”, (RRE for short), matrix if it satisfies the following properties:

1. Zero rows, if any, are below all nonzero rows
2. The first nonzero entry (read from the left) in each nonzero row is 1. (This is called the pivotal 1 of that row)
3. The pivotal 1 in any nonzero row is to the right of the pivotal 1s of all the previous rows, that is, if the nonzero rows are R_1, R_2, \dots, R_ρ , and the pivotal 1 in R_i appears in the column C_{k_i} , $i = 1, 2, \dots, \rho$, then,

$$k_1 < k_2 < \dots < k_{(i-1)} < k_i < k_{(i+1)} < \dots < k_\rho$$

.

4. All other entries in the column that has a pivotal 1 are zero, that is, if R_i has pivotal 1 in k_i th column then $a_{jk_i} = 0$ for all $j \neq i$.

Example 4.1.1 The matrix

$$A = \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

is an RRE matrix

Example 4.1.2 The matrix

$$A_R = \begin{pmatrix} 1 & 2 & 0 & 2 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

is a RRE matrix. The nonzero rows are R_1 and R_2 . The k_1 and k_1 in this case are 1 and 3 since the pivotal 1s appear in columns C_1 and C_3 respectively. We have $k_1 < k_2$ since $1 < 3$. All other entries in columns C_1 and C_3 are zero.

Example 4.1.3 The matrices

$$A = \begin{pmatrix} 1 & 2 & 2 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & 1 & 2 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \text{ and } C = \begin{pmatrix} 1 & 2 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

are not RRE matrices. (For each of these examine which requirement for RRE matrix is not satisfied)

4.2 Row Space and Column Space of RRE Matrices

From the structure of an RRE matrix it follows that the nonzero rows of an RRE matrix form a basis for the Row Space of that matrix. Suppose $A \in \mathbb{F}^{m \times n}$ is an RRE matrix and its first ρ rows are nonzero rows and the remaining $(m - \rho)$ rows are zero rows. Then R_1, R_2, \dots, R_ρ form a basis for $\text{Row}(A)$. (Recall that $\text{Row}(A) = \text{Col}(A^T) = \mathcal{R}_{A^T}$). This gives us that for

an RRE matrix A , the dimension of \mathcal{R}_{A^T} is ρ , the number of nonzero rows in A . Hence we get

$$\rho_{A^T} = \rho \quad (4.2.1)$$

Example 4.2.1 For the RRE matrix of Example 4.1.1 we have the basis for $Row(A)$ given by

$$R_1 = \begin{pmatrix} 1 \\ 2 \\ 0 \\ 1 \end{pmatrix}, R_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}$$

Similarly, for the RRE matrix of Example 4.1.2 basis for $Row(A)$ is given by

$$R_1 = \begin{pmatrix} 1 \\ 2 \\ 0 \\ 2 \\ 0 \end{pmatrix}, R_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

For both these matrices we therefore have

$$\rho_{A^T} = 2 \quad (4.2.2)$$

Similarly we see that for an RRE matrix the columns containing the pivotal 1s form a basis for the Column Space of A and hence for the Range Space of A . Hence we get that the dimension of $Col(A)$ is equal to the number of pivotal 1s. Since each nonzero row supports one pivot, it follows that the dimension of \mathcal{R}_A is also ρ . Thus we get for an RRE matrix A ,

$$\rho_A = \rho, \text{ the number of nonzero rows} \quad (4.2.3)$$

From (4.2.1) and (4.2.3) we get

$$\rho_A = \rho_{A^T} \text{ for any RRE matrix } A \quad (4.2.4)$$

This means that,

for an RRE matrix the matrix and its transpose have the same rank.

Example 4.2.2 For the RRE matrix of Example 4.1.1 we have the basis for \mathcal{R}_A is given by

$$C_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, C_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

Similarly, for the RRE matrix of Example 4.1.2 basis for $Row(A)$ is given by

$$C_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, C_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

For both these matrices we therefore have

$$\rho_A = 2 \quad (4.2.5)$$

Comparing this with (4.2.3) we see that for both these matrices we have

$$\rho_A = \rho_{A^T}$$

4.3 Homogeneous Systems With RRE Matrices

We shall next look at homogeneous systems corresponding to RRE matrices. Let $A \in \mathbb{F}^{m \times n}$ be a RRE matrix. Let the nonzero rows be R_1, R_2, \dots, R_ρ and the pivotal 1s be in the columns $C_{k_1}, C_{k_2}, \dots, C_{k_\rho}$, (where $k_1 < k_2 < \dots, k_\rho$). Then the variables $x_{k_1}, x_{k_2}, \dots, x_{k_\rho}$ are called “**Pivotal Variables**”. The remaining $(n - \rho)$ variables are called “**Free Variables**”.

Example 4.3.1 For the RRE matrix in Example 4.2.1, we have

Pivotal variables: x_1 and x_3

Nonpivotal Variables: x_2 and x_4

Example 4.3.2 For the RRE matrix in Example 4.2.2 we have,

Pivotal Variables: x_1 and x_3

Nonpivotal Variables: x_2, x_4 and x_5

From the homogeneous system $Ax = \theta_m$ we can eliminate the pivotal variables x_{k_i} ($1 \leq i \leq \rho$) from the i th equation in terms of the nonpivotal variables, and the nonpivotal variables can be chosen arbitrarily.

Example 4.3.3 In Example 4.2.1 we can eliminate the pivotal variables x_1, x_3 in terms of the nonpivotal variables x_2 and x_4 as follows:
The system can be written as

$$\left. \begin{array}{rcl} x_1 + 2x_2 + x_4 & = & 0 \\ x_3 + x_4 & = & 0 \end{array} \right\}$$

Eliminating x_1 and x_3 we get

$$\begin{array}{rcl} x_1 & = & -2x_2 - x_4 \\ x_3 & = & -x_4 \end{array}$$

Denoting the arbitrary values of x_2 and x_4 respectively by α and β , we can write the general solution of the system as

$$x = \alpha \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} -1 \\ 0 \\ -1 \\ 1 \end{pmatrix}$$

Example 4.3.4 In Example 4.2.2 we can eliminate the pivotal variables x_1, x_3 in terms of the nonpivotal variables x_2, x_4, x_5 as follows:
The system can be written as

$$\left. \begin{array}{rcl} x_1 + 2x_2 + 2x_4 & = & 0 \\ x_3 + x_4 & = & 0 \end{array} \right\}$$

Eliminating x_1 from the first equation and x_3 from the second equation we get

$$\begin{array}{rcl} x_1 & = & -2x_2 - 2x_4 \\ x_3 & = & -x_4 \end{array}$$

Denoting the arbitrary values of x_2, x_4 , and x_5 respectively by α, β and γ , we can write the general solution of the system as

$$x = \alpha \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} -2 \\ 0 \\ -1 \\ 1 \\ 0 \end{pmatrix} + \gamma \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Thus we see that it is easy to solve the homogeneous system corresponding to a RRE matrix. From the above two examples it follows that corresponding to every nonpivotal variables we get a solution to the Homogeneous system and the set of these solutions corresponding to all the nonpivotal variables are linearly independent. Hence we get that

$$\begin{aligned} \text{Dimension of } \mathcal{N}_A &= \text{number of nonpivotal variables} \\ &= n - \rho \end{aligned}$$

But since we have $\rho = \rho_A = \rho_{A^T}$ we see that

$$\begin{aligned} \rho_A + \nu_A &= n \\ \rho_{A^T} + \nu_A &= n \end{aligned}$$

Example 4.3.5 From Example 4.3.3 we get for RRE the matrix of Example 4.2.1, basis for \mathcal{N}_A is given by

$$u = \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, v = \begin{pmatrix} -1 \\ 0 \\ -1 \\ 1 \end{pmatrix}$$

Similarly from Example 4.3.4 we get for the RRE matrix of Example 4.2.2, basis for \mathcal{N}_A is given by

$$u = \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, v = \begin{pmatrix} -2 \\ 0 \\ -1 \\ 1 \\ 0 \end{pmatrix}, w = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

4.4 Homogeneous and Nonhomogeneous Systems

Let us consider a matrix $A \in \mathbb{F}^{m \times n}$. We are interested in solving the system

$$Ax = b \tag{4.4.1}$$

for different $b \in \mathbb{F}^m$. In particular, when we take $b = \theta_m$, we get the system

$$Ax = \theta_m \tag{4.4.2}$$

The system (2.1.2) is called the “**Homogeneous System**” corresponding to the matrix A . When $b \neq \theta_m$ the system (2.1.1) is called a “**Nonhomogeneous system**”.

Let us now look at the homogeneous system (2.1.2). Clearly $x = \theta_n$ is a solution of this system. Thus the homogeneous system is always consistent, since there exists a solution $x = \theta_n$. This solution is called the “**Trivial Solution**”. If $x \neq \theta_n$ is a solution of the homogeneous system then it is called a “**Nontrivial solution**”. A homogeneous equation may or may not have nontrivial solutions. Let us now look at some simple examples.

Example 4.4.1 Let

$$A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

Then the corresponding homogeneous system can be written as

$$\left. \begin{aligned} x_1 + x_2 &= 0 \\ x_1 - x_2 &= 0 \end{aligned} \right\} \quad (4.4.3)$$

Clearly, $x_1 = x_2 = 0$, that is, $x = \theta_2$ is a solution of this system, and it is the only solution of this system. Thus this system has only trivial solution

Example 4.4.2 Let

$$A = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}$$

Then the corresponding homogeneous system can be written as

$$\left. \begin{aligned} x_1 - x_3 &= 0 \\ x_2 - x_3 &= 0 \end{aligned} \right\} \quad (4.4.4)$$

Clearly, $x_1 = x_2 = x_3 = 0$, that is, $x = \theta_2$ is a solution of this system. Further if we take $x_1 = x_2 = x_3 = 1$, that is, $x = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, then x is also a solution

of this system. In fact any vector of the form, $x = \begin{pmatrix} \alpha \\ \alpha \\ \alpha \end{pmatrix}$, where α is any element of \mathbb{F} , is a solution of this system. Thus this system has nontrivial solutions.

The fact whether the homogeneous system does or does not have nontrivial solutions has a bearing on the uniqueness or otherwise of the solution of the nonhomogeneous system, when the nonhomogeneous system is consistent. We shall now investigate this aspect.

Suppose the nonhomogeneous system $Ax = b$ is consistent. Then

The nonhomogeneous system has more than one solution \implies

There exist $u, v \in \mathbb{F}^n$ such that $u \neq v$ and $Au = b$ and $Av = b$. This \implies

The vector $w = u - v$ is in \mathbb{F}^n and $A(u - v) = Au - Av = b - b = \theta_m$, and $w \neq \theta_n$. This \implies

w is a nontrivial solution of the homogeneous system. Thus we have

$$\left. \begin{array}{l} \text{A consistent nonhomogeneous system } Ax = b \\ \text{has more than one solution} \\ \implies \\ \text{The Homogeneous system } Ax = \theta_m \\ \text{has a nontrivial solution} \end{array} \right\} \quad (4.4.5)$$

Conversely suppose the homogeneous system has a nontrivial solution. Let $x_H \neq \theta_n$ be a nontrivial solution of the homogeneous system. Hence

$$Ax_H = \theta_m \quad (4.4.6)$$

The nonhomogeneous system $Ax = b$ is consistent and hence there exists a solution $u \in \mathbb{F}^n$ to the nonhomogeneous system, that is,

$$Au = b \quad (4.4.7)$$

If we define

$$v = u + x_H \quad (4.4.8)$$

then we have

$$\begin{aligned} Av &= A(u + x_H) \\ &= Au + Ax_H \\ &= b + \theta_m \text{ by (2.1.7) and (2.1.6)} \\ &= b \end{aligned}$$

Hence v is also a solution of the nonhomogeneous system, and since $x_H \neq \theta_n$, we have $v \neq u$. Thus u and v are two different solutions of the nonhomogeneous system. Thus we get that the nonhomogeneous equation has more than one solution. Hence we have

$$\left. \begin{array}{l} \text{The Homogeneous system } Ax = \theta_m \\ \text{has a nontrivial solution} \\ \implies \\ \text{The consistent nonhomogeneous system } Ax = b \\ \text{has more than one solution} \end{array} \right\} \quad (4.4.9)$$

Combining (2.1.5) and (2.1.9) we get the following proposition:

Theorem 4.4.1 Let the nonhomogeneous system $Ax = b$ be consistent. Then

The Nonhomogeneous system $Ax = b$ has more than one solution

\iff

The homogeneous system has nontrivial solution.

This can also be restated as follows:

When the nonhomogeneous system $Ax = b$ is consistent,

The Nonhomogeneous system $Ax = b$ has a unique solution

\iff

The homogeneous system has only the trivial solution.

Remark 4.4.1 When x_H is a nontrivial solution of the homogeneous system $Ax = \theta_m$ and u is a solution of the nonhomogeneous system $Ax = b$, then for any $\alpha \in \mathbb{F}$ we have $v = u + \alpha x_H$ is also solution of the nonhomogeneous system. Thus, in particular, if $\mathbb{F} = \mathbb{R}$ or \mathbb{C} , we see that there are infinite number of solutions for the nonhomogeneous system.

What is the consequence of all this? Suppose we want to solve the nonhomogeneous system $Ax = b$ and suppose it is consistent, that is, suppose b satisfies the consistency conditions. Then there exists a solution for the nonhomogeneous system. Suppose we know one solution, say x_P , of the nonhomogeneous system. Hence

$$Ax_P = b \quad (4.4.10)$$

Let v be any other solution of the nonhomogeneous system. Then we must have

$$Av = b \quad (4.4.11)$$

Hence we get

$$\begin{aligned} A(v - x_p) &= Av - Ax_p \\ &= b - b \\ &= \theta_m \end{aligned}$$

Thus $v - x_p$ is a solution of the homogeneous system. Hence any solution v is of the form

$$v = x_p + x_H \tag{4.4.12}$$

Thus every solution of the nonhomogeneous system is of the above form. Conversely, it is easy to check that any v of the above form is a solution of the nonhomogeneous system. Hence all solutions of the nonhomogeneous system are obtained once we know one solution x_p of the nonhomogeneous system and all solutions of the homogeneous system. Then by adding these solutions of the homogeneous system to x_p we get all the solutions of the nonhomogeneous system. Thus finding the solution of the nonhomogeneous system has two parts, namely,

1. Find all solutions of the homogeneous system, and
2. Find a particular solution of the nonhomogeneous system

Then we can generate all solutions of the nonhomogeneous system as observed above. We have seen in the previous section that the solutions to the Homogeneous system can be easily found if the coefficient matrix is an RRE matrix. We shall, therefore, try to reduce a given matrix A to an RRE matrix A_R such that both Homogeneous systems the systems $Ax = \theta_m$ and $A_Rx = \theta_m$ have the same set of solutions. The main tools in this reduction process are the Elementary Row Operations which we introduce in the next section.

4.5 Elementary Row Operations (EROs)

We introduce three types of elementary row operations. These are simple operations performed on the rows of a matrix. The three types we shall look at are Row Interchange, Adding a multiple of a row to another row and multiplying a row by a nonzero scalar.

ERO of Type I:**Row interchange:**

In this operation we keep all rows except two unchanged, and remaining two rows interchange their positions. If the i th row and j th rows are interchanged we denote this operation by R_{ij} .

Example 4.5.1

$$A = \begin{pmatrix} 0 & 1 & 2 \\ 3 & 2 & 1 \end{pmatrix} \xrightarrow{R_{12}} \begin{pmatrix} 3 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$$

Example 4.5.2

$$A = \begin{pmatrix} 1 & -1 & 3 \\ 0 & 1 & 2 \\ -2 & -3 & 1 \\ 0 & 8 & 4 \end{pmatrix} \xrightarrow{R_{23}} B = \begin{pmatrix} 1 & -1 & 3 \\ -2 & -3 & 1 \\ 0 & 1 & 2 \\ 0 & 8 & 4 \end{pmatrix}$$

We observe the following simple, but important, properties of Type I ERO:

1) If

$$A \xrightarrow{R_{ij}} A_1 \text{ then } A_1 \xrightarrow{R_{ij}} A$$

This means that the ERO R_{ij} is invertible and its inverse is R_{ij} itself. We have $(R_{ij})^{-1} = R_{ij}$

2) If A_1 is obtained from A by an ERO of Type I then the homogeneous system $Ax = \theta_m$ and $A_1x = \theta_m$ have the same set of solutions. This is because both the systems have the same set of equations, only difference being that they are in different order.

Example 4.5.3 Consider the matrix A of Example 4.5.1 and the A_1 obtained from it by the ERO R_{12} . The Homogeneous system $Ax = \theta_3$ is given by

$$\left. \begin{aligned} x_1 + 2x_3 &= 0 \\ 3x_1 + 2x_2 + x_3 &= 0 \end{aligned} \right\} \quad (4.5.1)$$

The Homogeneous system $A_1x = \theta_3$ is given by

$$\left. \begin{aligned} 3x_1 + 2x_2 + x_3 &= 0 \\ x_1 + 2x_3 &= 0 \end{aligned} \right\} \quad (4.5.2)$$

Clearly both the systems have the same set of solutions since both are same systems except that the equations are written in different order.

3) If A_1 is obtained from A by an ERO of Type I then they both have the same Row spaces, that is, $Row(A) = Row(A_1)$.

4) Let A be an $m \times n$ matrix. Consider the $m \times m$ identity matrix I_m . Then if

$$I \xrightarrow{R_{ij}} E$$

then

$$A \xrightarrow{R_{ij}} B = EA$$

Thus the ERO R_{ij} on A can be effected by premultiplying A by E where E is the matrix obtained by applying R_{ij} to I_m .

Example 4.5.4 . In Example 4.5.1 we have

$$I_2 \xrightarrow{R_{12}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = E$$

$$\begin{aligned} EA &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 2 \\ 3 & 2 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 3 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} \\ &= A_1 \end{aligned}$$

Example 4.5.5 For the matrix in Example 4.5.2 we have

$$I_4 \xrightarrow{R_{23}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = E$$

We have

$$\begin{aligned} EA &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 3 \\ 0 & 1 & 2 \\ -2 & -3 & 1 \\ 0 & 8 & 4 \end{pmatrix} \\ &= \begin{pmatrix} 1 & -1 & 3 \\ -2 & -3 & 1 \\ 0 & 1 & 2 \\ 0 & 8 & 4 \end{pmatrix} \\ &= A_1 \text{ (as obtained in Example 4.5.2)} \end{aligned}$$

ERO of Type II:

Adding a Multiple of one row to another row:

In this operation we keep all but one row, say the j th row, unchanged, and the j th row is changed by adding a multiple of another row, say the i th row, to it. If α times the i th row is added to the j th row we denote this operation by $R_j + \alpha R_i$.

Example 4.5.6

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 0 & 1 \\ 3 & 1 & 2 \end{pmatrix} \xrightarrow{R_3+2R_1} \begin{pmatrix} 1 & 1 & 0 \\ 2 & 0 & 1 \\ 5 & 3 & 2 \end{pmatrix}$$

Example 4.5.7

$$A = \begin{pmatrix} 1 & -1 & 3 \\ 0 & 1 & 2 \\ -2 & -3 & 1 \\ 0 & 8 & 4 \end{pmatrix} \xrightarrow{R_4+2R_1} A_1 = \begin{pmatrix} 1 & -1 & 3 \\ 0 & 1 & 2 \\ -2 & -3 & 1 \\ 2 & 6 & 10 \end{pmatrix}$$

We observe the following simple, but important, properties of Type II ERO

1) If

$$A \xrightarrow{R_j+\alpha R_i} A_1 \text{ then } A_1 \xrightarrow{R_j+(-\alpha)R_i} A$$

This means that the ERO $R_i + \alpha R_j$ is invertible and its inverse is $R_i + (-\alpha)R_j$ again an ERO of the same type. We have $(R_j + \alpha R_i)^{-1} = R_j + (-\alpha)R_i$

2) If A_1 is obtained from A by an ERO of Type II then the homogeneous system $Ax = \theta_m$ and $A_1x = \theta_m$ have the same set of solutions. This is because both the systems have the same set of equations, only difference being that the i th equation has been replaced by the adding the j th equation to it.

Example 4.5.8 For the matrix of Example 4.5.6 we have,

The Homogeneous system $Ax = \theta_3$ is given by

$$\left. \begin{array}{lcl} x_1 + x_2 & = & 0 \\ 2x_1 + x_3 & = & 0 \\ 3x_1 + x_2 + x_3 & = & 0 \end{array} \right\} \quad (4.5.3)$$

The Homogeneous system $A_1x = \theta_3$ is given by

$$\left. \begin{array}{lcl} x_1 + x_2 & = & 0 \\ 2x_1 + x_3 & = & 0 \\ 5x_1 + 3x_2 + x_3 & = & 0 \end{array} \right\} \quad (4.5.4)$$

It is easy to see that these two have the same set of solutions.

3) If A_1 is obtained from A by an ERO of Type II then both the matrices have the same Row space, that is, $Row(A) = Row(A_1)$

4) Let A be an $m \times n$ matrix. Consider the $m \times m$ identity matrix I_m . Then if

$$I \xrightarrow{R_i + \alpha R_j} E$$

then

$$A \xrightarrow{R_i + \alpha R_j} A_1 = EA$$

Thus the ERO $R_i + \alpha R_j$ on A can be effected by premultiplying A by E where E is the matrix obtained by applying $R_i + \alpha R_j$ to I_m .

Example 4.5.9 Corresponding to the Example 4.5.6 we have

$$I_3 \xrightarrow{R_3 + 2R_1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 2 & 0 & 1 \end{pmatrix} = E$$

$$\begin{aligned} EA &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 2 & 0 & 1 \\ 3 & 1 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 1 & 0 \\ 2 & 0 & 1 \\ 5 & 3 & 2 \end{pmatrix} \\ &= A_1 \end{aligned}$$

Example 4.5.10 We have for the matrix of Example 4.5.7,

$$I_4 \xrightarrow{R_4 + 2R_1} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 \end{pmatrix} = E$$

Further

$$EA = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 3 \\ 0 & 1 & 2 \\ -2 & -3 & 1 \\ 0 & 8 & 4 \end{pmatrix}$$

$$\begin{aligned}
&= \begin{pmatrix} 1 & -1 & 3 \\ 0 & 1 & 2 \\ -2 & -3 & 1 \\ 2 & 6 & 10 \end{pmatrix} \\
&= A_1 \text{ as obtained in Example 4.5.7}
\end{aligned}$$

ERO of Type III:

Multiplying a row by a nonzero scalar:

In this operation we keep all rows except one row unchanged, and one row is multiplied by a nonzero scalar. If the i th row is multiplied by α we denote this operation by αR_i .

Example 4.5.11

$$A = \begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & 2 \end{pmatrix} \xrightarrow{(-3R_2)} \begin{pmatrix} 1 & 0 & -1 \\ -6 & -3 & -6 \end{pmatrix} = A_1$$

Example 4.5.12

$$A = \begin{pmatrix} 1 & -1 & 3 \\ 0 & 1 & 2 \\ -2 & -3 & 1 \\ 0 & 8 & 4 \end{pmatrix} \xrightarrow{2R_1} A_1 = \begin{pmatrix} 2 & -2 & 6 \\ 0 & 1 & 2 \\ -2 & -3 & 1 \\ 0 & 8 & 4 \end{pmatrix}$$

We observe the following simple, but important, properties of Type III ERO

1) If

$$A \xrightarrow{\alpha R_i} A_1 \text{ then } A_1 \xrightarrow{(\alpha^{-1})R_i} A$$

This means that the ERO αR_i is invertible and its inverse is $(\alpha^{-1})R_i$ again an ERO of the same type.

2) If A_1 is obtained from A by an ERO of Type III then the homogeneous system $Ax = \theta_m$ and $A_1x = \theta_m$ have the same set of solutions. This is because both the systems have the same set of equations, only difference being that the i th equation has been multiplied throughout by the nonzero scalar α .

Example 4.5.13 For the Example 4.5.10 we have,

The Homogeneous system $Ax = \theta_2$ is given by

$$\left. \begin{aligned} x_1 - x_3 &= 0 \\ 2x_1 + x_2 + 2x_3 &= 0 \end{aligned} \right\} \quad (4.5.5)$$

The Homogeneous system $A_1x = \theta_3$ is given by

$$\left. \begin{array}{rcl} x_1 - x_3 & = & 0 \\ -6x_1 - 3x_2 - 6x_3 & = & 0 \end{array} \right\} \quad (4.5.6)$$

Clearly these two have the same set of solutions.

3) If A_1 is obtained from A by an ERO of Type III then both the matrices have the same Row Space, that is, $Row(A) = Row(A_1)$.

4) Let A be an $m \times n$ matrix. Consider the $m \times m$ identity matrix I_m . Then if

$$I \xrightarrow{\alpha R_i} E$$

then

$$A \xrightarrow{\alpha R_i} A_1 = EA$$

Thus the ERO αR_i on A can be effected by premultiplying A by E where E is the matrix obtained by applying αR_i to I_m .

Example 4.5.14 For the matrix in Example 2.2.10 we have

$$I_2 \xrightarrow{(-3)R_2} \begin{pmatrix} 1 & 0 \\ 0 & -3 \end{pmatrix} = E$$

$$\begin{aligned} EA &= \begin{pmatrix} 1 & 0 \\ 0 & -3 \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & -1 \\ -6 & -3 & -6 \end{pmatrix} \\ &= A_1 \end{aligned}$$

Example 4.5.15 We have for the matrix of Example 4.5.11

$$I_4 \xrightarrow{2R_1} \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = E$$

and

$$\begin{aligned}
EA &= \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 3 \\ 0 & 1 & 2 \\ -2 & -3 & 1 \\ 0 & 8 & 4 \end{pmatrix} \\
&= \begin{pmatrix} 2 & -2 & 6 \\ 0 & 1 & 2 \\ -2 & -3 & 1 \\ 0 & 8 & 4 \end{pmatrix} \\
&= A_1 \text{ as obtained in Example 4.5.11}
\end{aligned}$$

From the above it follows that all EROs

1. are invertible and the inverses are also EROs of the same type,
2. are such that they do not alter the set of solutions of the homogeneous system,
3. are such that they do not alter the Row Space of the matrix, and
4. can be effected by premultiplying by a matrix E obtained from the identity matrix I_m by applying the same ERO.

Definition 4.5.1 A matrix $E \in \mathbb{F}^{m \times m}$ obtained by applying an ERO to the identity matrix I_m is called an “**Elementary Matrix**” (of order m)

We denote by \mathcal{E}_m the set of all elementary matrices of order m . From our discussion it follows that,

1. Every elementary matrix is invertible
2. EROs on $m \times n$ can be effected by premultiplication by elementary matrices of order m
3. If E is any elementary matrix of order m and A is any $m \times n$ matrix, then the homogeneous systems $EAx = \theta_m$ and $Ax = \theta_m$ have the same set of solutions

If we apply a finite number of EROs of the above three types in any order and obtain a matrix B then the homogeneous systems $Ax = \theta_m$ and $Bx = \theta_m$ will have the same set of solutions. The basic strategy is to use EROs and obtain the matrix B from A such that $Bx = \theta_m$ is a simple system and hence easy to solve. Then since $Ax = \theta_m$ and $Bx = \theta_m$ have the same set of solutions, by solving the easy system $Bx = \theta_m$, we would have also solved the given system $Ax = \theta_m$. We shall see we can reduce A to an RRE matrix A_R , using EROs.

Row Equivalence

The notion of EROs and the fact that the EROs do not alter the solution set of the Homogeneous system lead to an equivalence notion among matrices. We have

Definition 4.5.2 A matrix $A \in \mathbb{F}^{m \times n}$ is said to be “**Row Equivalent**” to a matrix $B \in \mathbb{F}^{m \times n}$ if there exists a finite number of EROs E_1, E_2, \dots, E_k such that B can be obtained from A by applying this finite sequence of EROs

If A is row equivalent to B we write $A \stackrel{R}{\sim} B$. Since EROs do not alter the solution set of a Homogeneous system, it follows that if two matrices are row equivalent they have the same set of solutions for their homogeneous systems. It is easy to see that the following properties hold:

1. $A \in \mathbb{F}^{m \times n} \implies A \stackrel{R}{\sim} A$ (Reflexivity of $\stackrel{R}{\sim}$)
2. If $A, B \in \mathbb{F}^{m \times n}$ then $A \stackrel{R}{\sim} B \iff B \stackrel{R}{\sim} A$. (Symmetry of $\stackrel{R}{\sim}$) This follows from the fact that EROs are invertible
3. If $A, B, C \in \mathbb{F}^{m \times n}$ then $A \stackrel{R}{\sim} B$ and $B \stackrel{R}{\sim} C \implies A \stackrel{R}{\sim} C$. (Transitivity of $\stackrel{R}{\sim}$)

A relation \sim on a set is said to be an equivalence relation on that set if it is reflexive, symmetric and transitive. Hence it follows that $\stackrel{R}{\sim}$ is an equivalence relation on $\mathbb{F}^{m \times n}$. As with every equivalence relation, the equivalence relation $\stackrel{R}{\sim}$ partitions the set of matrices $\mathbb{F}^{m \times n}$ into distinct equivalence classes. Any two matrices A and B in the same equivalence class will have the same set of solutions for their Homogeneous systems, since they are row equivalent. What we would therefore like to do is to look at one representative from each class which can be easily studied and try to reduce a given matrix by EROs to one of these representatives. What type of representatives we choose? The

right representative we want is the RRE form which makes solution of the homogeneous system easy.

4.6 Reduction of a General Matrix to RRE Matrix

We shall now see how to reduce a general matrix $A \in \mathcal{F}^{m \times n}$ to RRE form using EROs. We first describe the basic step below:

First Column Operation

Let A be any matrix. We look at the first column of A and ask the question, Does the first column have a nonzero entry?

The answer may be Yes or No

Case 1: The answer is Yes.

In this case we do the following:

Bring a nonzero entry to the leading position (that is the first row first column position), if necessary by an ERO of Type I.

Then make this leading nonzero entry as 1 by ERO of Type III

Then make all entries in the first column of A below this 1 as zero by ERO of Type II

The resulting matrix is of the form

$$\left(\begin{array}{c|c} 1 & K_{1 \times (n-1)} \\ \hline 0_{(m-1) \times 1} & A^{(1)} \end{array} \right)$$

Case 2: The Answer is No

In this case, since the first column has all zeros, the matrix A is of the form

$$\left(0_{1 \times m} \mid A^{(1)} \right)$$

Now given any matrix A apply the above procedure and get the submatrix $A^{(1)}$ and repeat the process on $A^{(1)}$ to get a submatrix $A^{(2)}$ and so on. Continuing this process we reduce the matrix A to a matrix which satisfies all the requirements of RRE except the requirement 4 (see Definition 2.3.2). We now use RRE of Type II to achieve this and reduce to zero all other entries in a column containing a pivotal 1, (starting from the left most pivotal 1).

The resulting matrix is an RRE matrix and Row equivalent to A . Since no other RRE matrix can be row equivalent to this we call this the RRE form of A and we denote this by A_R . We shall illustrate this reduction process by means of an example below.

Example 4.6.1 Consider the matrix

$$A = \begin{pmatrix} 1 & 2 & 2 & 3 \\ 2 & 4 & 1 & 3 \\ 3 & 6 & 1 & 4 \end{pmatrix}$$

Let us now reduce this to its RRE form.

$$\begin{aligned} A &= \begin{pmatrix} 1 & 2 & 2 & 3 \\ 2 & 4 & 1 & 3 \\ 3 & 6 & 1 & 4 \end{pmatrix} \xrightarrow{R_2-2R_1, R_3-3R_1} \left(\begin{array}{c|cc|cc} 1 & 2 & 2 & 3 \\ 0 & 0 & -3 & -3 \\ 0 & 0 & -5 & -5 \end{array} \right) \\ &\quad \text{(partitioned as)} \quad \left(\begin{array}{c|c|cc} 1 & 2 & 2 & 3 \\ 0 & 0 & -3 & -3 \\ 0 & 0 & -5 & -5 \end{array} \right) \\ &\xrightarrow{-\frac{1}{3}R_2} \left(\begin{array}{c|c|cc} 1 & 2 & 2 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -5 & -5 \end{array} \right) \xrightarrow{R_3+5R_2} \left(\begin{array}{c|c|cc} 1 & 2 & 2 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array} \right) \\ &\xrightarrow{R_1-2R_2} \left(\begin{array}{c|c|cc} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array} \right) = A_R, \text{ the RRE form of } A \end{aligned}$$

The RRE form is obtained from A by means of EROs. Since each ERO can be effected by premultiplication by an elementary matrix, by taking the products of all the elementary matrices involved in the reduction, (in the same order as the EROs are performed), we get a matrix E such that

$$EA = A_R, \text{ the RRE form of } A$$

Example 4.6.2 The Elementary Matrices in the above example can be obtained by applying this same sequence of EROs to the identity matrix I_3 .

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_2-2R_1, R_3-3R_1} \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & 0 & 1 \end{pmatrix}$$

$$\begin{aligned}
& \xrightarrow{-\frac{1}{3}R_2} \begin{pmatrix} 1 & 0 & 0 \\ \frac{2}{3} & -\frac{1}{3} & 0 \\ -3 & 0 & 1 \end{pmatrix} \xrightarrow{R_3+5R_2} \begin{pmatrix} 1 & 0 & 0 \\ \frac{2}{3} & -\frac{1}{3} & 0 \\ \frac{1}{3} & -\frac{5}{3} & 1 \end{pmatrix} \\
& \xrightarrow{R_1-2R_2} \begin{pmatrix} -\frac{1}{3} & \frac{2}{3} & 0 \\ \frac{2}{3} & -\frac{1}{3} & 0 \\ \frac{1}{3} & -\frac{5}{3} & 1 \end{pmatrix} = E
\end{aligned}$$

It is easy to check that

$$\begin{aligned}
EA &= \begin{pmatrix} -\frac{1}{3} & \frac{2}{3} & 0 \\ \frac{2}{3} & -\frac{1}{3} & 0 \\ \frac{1}{3} & -\frac{5}{3} & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 2 & 3 \\ 2 & 4 & 1 & 3 \\ 3 & 6 & 1 & 4 \end{pmatrix} \\
&= \\
&= \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} = A_R, \text{ the RRE form of } A
\end{aligned}$$

Remark 4.6.1 Analogous to the EROs we can define Elementary Column Operations (ECOs) and Column Reduced Echelon (CRE) Matrices. We can reduce any matrix to its CRE form by ECOs. Just as an ERO on an $m \times n$ matrix can be effected by premultiplying with an elementary matrix obtained from I_m by applying to it the same ERO, we can effect an ECO on A by postmultiplying A by a matrix obtained from I_n by applying to it the same ECO. (This amounts to doing ERO on A^T and then again transposing). Just as we got EA to be A_R we can get a matrix F obtained by applying ECOs to I_n such that AF is equal to the CRE of A .

4.7 Solution of Homogeneous System Using RRE Form

Given the homogeneous system $Ax = \theta_m$, we first reduce A to its RRE form A_R . Since A and A_R are row equivalent it follows that the homogeneous system $Ax = \theta_m$ has the same set of solutions as the system $A_Rx = \theta_m$ corresponding to the RRE matrix A_R . We can therefore get the solution of

the system $Ax = \theta_m$ by solving the RRE system $A_Rx = \theta_m$ as described in the Section 2.4 above.

Example 4.7.1 In the Example 4.6.1 above, the Homogeneous System corresponding to A_R is given by

$$\begin{cases} x_1 + 2x_2 + x_4 = 0 \\ x_3 + x_4 = 0 \end{cases}$$

The Pivotal Variables are x_1 and x_3 , and the Free Variables are x_2 and x_4 . Eliminating the pivotal variable x_1 from the first equation and the pivotal variable x_3 from the second equation and denoting the free variables x_2 and x_4 by α and β we get the general solution of the system $A_Rx = \theta_m$ - which is also the general solution of the given system $Ax = \theta_m$, as

$$x = \alpha \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} -1 \\ 0 \\ -1 \\ 1 \end{pmatrix}$$

We now define the Row Rank of a matrix.

Definition 4.7.1 The number of nonzero rows in the RRE form A_R of A is called the “**Row Rank** of A ”, and the number of free variables for the RRE system $A_Rx = \theta_m$ is called the “**Nullity** of A ”

Example 4.7.2 In the Example above we have the number of Nonzero Rows in RRE form is 2 and hence the Row Rank of A is 2. Similarly, the number of Free Variables is 2 and hence Nullity of A is 2. Thus we see that

$$\begin{aligned} \text{Row rank} + \text{Nullity} &= 2 + 2 = 4 \\ &= \text{the number of columns in } A \\ &= \text{The number of unknowns in} \\ &\quad \text{the homogeneous system} \end{aligned}$$

Remark 4.7.1 Analogously the number of nonzero columns in the CRE form is called the “**Column Rank**” of A . and we have

$$\text{Column Rank of } A = \text{Row Rank of } A^T \quad (4.7.1)$$

Hence we get

$$\left. \begin{array}{l} \text{Column Rank of } A + \text{Nullity of } A^T = \\ \text{Row Rank of } A^T + \text{Nullity of } A^T = \\ \text{Number of columns in } A^T = \\ m \end{array} \right\}$$

Example 4.7.3 Consider the matrix $A \in \mathbb{R}^{4 \times 5}$ given below:

$$A = \begin{pmatrix} 1 & 2 & 2 & 9 & -1 \\ 1 & 2 & 3 & 13 & -2 \\ -1 & -2 & -1 & -5 & 1 \\ 1 & 2 & 0 & 1 & 2 \end{pmatrix}$$

We can write the row reduction of the matrix as follows:

$$A_{aug} = \begin{pmatrix} 1 & 2 & 2 & 9 & -1 \\ 1 & 2 & 3 & 13 & -2 \\ -1 & -2 & -1 & -5 & 1 \\ 1 & 2 & 0 & 1 & 2 \end{pmatrix} \xrightarrow{R_2 - R_1, R_3 + R_1, R_4 - R_1}$$

$$\begin{pmatrix} 1 & 2 & 2 & 9 & -1 \\ 0 & 0 & 1 & 4 & -1 \\ 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & -2 & -8 & 3 \end{pmatrix} \xrightarrow{R_3 - R_2, R_4 + 2R_2}$$

$$\begin{pmatrix} 1 & 2 & 2 & 9 & -1 \\ 0 & 0 & 1 & 4 & -1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_4 - R_3}$$

$$\begin{pmatrix} 1 & 2 & 2 & 9 & -1 \\ 0 & 0 & 1 & 4 & -1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_1 - 2R_2}$$

$$\begin{pmatrix} 1 & 2 & 0 & 1 & 1 \\ 0 & 0 & 1 & 4 & -1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_1 - R_3, R_2 + R_3}$$

$$\begin{pmatrix} 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} = A_R$$

The above matrix A_R is the RRE form of the matrix A . The homogeneous system corresponding to A_R is given by

$$\begin{aligned} x_1 + 2x_2 + x_4 &= 0 \\ x_3 + x_4 &= 0 \\ x_5 &= 0 \end{aligned}$$

The pivotal variables are x_1, x_3 and x_5 and these are eliminated in terms of the nonpivotal variables x_2 and x_4 as follows:

$$\begin{aligned} x_1 &= -2x_2 - x_4 \\ x_3 &= -x_4 \\ x_5 &= 0 \end{aligned}$$

Hence the general solution of the Homogeneous system $Ax = \theta_4$, which is the same as the general solution of the Homogeneous system $A_Rx = \theta_4$, is given by

$$x = \alpha \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} -1 \\ 0 \\ -1 \\ 1 \\ 0 \end{pmatrix}$$

where $\alpha, \beta \in \mathbb{R}$. A basis for \mathcal{N}_A , the Null Space of A , is given by

$$u = \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, v = \begin{pmatrix} -1 \\ 0 \\ -1 \\ 1 \\ 0 \end{pmatrix}$$

From the RRE form A_R we also get that the Row rank of A is three since there are three nonzero rows in the RRE form A_R of A

4.8 EROs and Inverse of a Matrix

Suppose $A \in \mathbb{F}^{n \times n}$, and the Row Rank of A is n . Then we have $A_R = I_n$, the $n \times n$ identity matrix. This means that

$$A \xrightarrow{EROs} A_R$$

Let

$$I_n \xrightarrow{\text{same EROs}} E$$

Then we have

$$\begin{aligned} EA &= A_R \\ \implies \\ EA &= I_n \text{ (since } A_R = I_n) \\ \implies \\ A^{-1} &= E \end{aligned}$$

Conclusion:

When $A \in \mathbb{F}^{n \times n}$ then A is invertible if and only if, Row Rank of A is equal to n , and the inverse can be obtained by applying to I_n the same sequence of EROs that were applied to A to obtain A_R

Example 4.8.1 Consider the matrix

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 6 & 4 \\ -3 & -8 & -4 \end{pmatrix}$$

We have the reduction as follows:

$$\begin{aligned} & \left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 2 & 6 & 4 & 0 & 1 & 0 \\ -3 & -8 & -4 & 0 & 0 & 1 \end{array} \right) \xrightarrow{R_2-2R_1, R_3+3R_1} \\ & \left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 2 & 2 & -2 & 1 & 0 \\ 0 & -2 & -1 & 3 & 0 & 1 \end{array} \right) \xrightarrow{\frac{1}{2}R_2} \\ & \left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & \frac{1}{2} & 0 \\ 0 & -2 & -1 & 3 & 0 & 1 \end{array} \right) \xrightarrow{R_3+2R_2} \end{aligned}$$

$$\begin{aligned}
& \left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{array} \right) \xrightarrow{R_1 - 2R_2} \\
& \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 3 & -1 & 0 \\ 0 & 1 & 1 & -1 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{array} \right) \xrightarrow{R_1 + R_3, R_2 - R_3} \\
& \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 4 & 0 & 0 \\ 0 & 1 & 0 & -2 & -\frac{1}{2} & -1 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{array} \right)
\end{aligned}$$

Hence we get

$$A^{-1} = E = \begin{pmatrix} 4 & 0 & 1 \\ -2 & -\frac{1}{2} & -1 \\ 1 & 1 & 1 \end{pmatrix}$$

Remark 4.8.1 It is easy to see that $A \in \mathbb{F}^{n \times n}$ is invertible if and only if its RRE form is the identity matrix $I_n \in \mathbb{F}^{n \times n}$

We shall next look at Nonhomogeneous systems.

4.9 Nonhomogeneous Systems and EROs

We shall now study the use of EROs in solving a consistent nonhomogeneous system $Ax = b$. We first introduce the notion of equivalence of two consistent homogeneous systems.

Definition 4.9.1 Let A and A_1 be in $\mathbb{F}^{m \times n}$, and b and $b_1 \in \mathbb{F}^m$. Then the two systems,

$$Ax = b \text{ and } A_1x = b_1$$

when consistent, are said to be equivalent if both have the same set of solutions

While dealing with homogeneous systems we found that EROs do not affect the solution set of the homogeneous system. What is the reason? The b for a homogeneous system is θ_m and EROs on this give back θ_m only. However, for a nonhomogeneous system $b \neq \theta_m$ and EROs on such a nonzero b alter the b .

Example 4.9.1 Consider

$$A = \begin{pmatrix} 2 & 3 & -1 \\ 1 & -1 & 1 \end{pmatrix} \text{ and } b = \begin{pmatrix} 2 & 1 \end{pmatrix}$$

Then

$$A \xrightarrow{R_{21}} A_1 = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 3 & -1 \end{pmatrix}$$

Clearly the systems $Ax = b$ and $A_1x = b$ are consistent and not equivalent. However,

$$b \xrightarrow{R_{21}} b_1 = \begin{pmatrix} 1 & 2 \end{pmatrix}$$

It is easy to see that the systems $Ax = b$ and $A_1x = b_1$ are consistent and equivalent. In general, if $Ax = b$ is consistent, and

$$A \xrightarrow{ERO} A_1 \text{ and } b \xrightarrow{\text{same ERO}} b_1$$

then $Ax = b$ and $A_1x = b_1$ are consistent and equivalent

Moral:

Let $Ax = b$ be consistent. When any ERO is performed on the coefficient matrix to get A_1 , if the same ERO is performed on b to get a b_1 , then the system $Ax = b$ will be “equivalent” to the system $A_1x = b_1$. In order to carry out this type of EROs simultaneously on A and b in a systematic manner, we introduce the notion of “**AUGMENTED MATRIX**”. If $A \in \mathbb{F}^{m \times n}$ and $b \in \mathbb{F}^m$, we define the augmented matrix as

$$A_{aug} = \left(A \mid b \right)$$

A_{aug} is obtained from A by appending b as an additional $(n + 1)$ th column. Clearly A_{aug} is an $m \times (n + 1)$ matrix.

Example 4.9.2 Let

$$A = \begin{pmatrix} 2 & 3 & -1 \\ 1 & -1 & 1 \end{pmatrix} \text{ and } b = \begin{pmatrix} 2 & 1 \end{pmatrix}$$

Then

$$A_{aug} = \left(\begin{array}{ccc|c} 2 & 3 & -1 & 2 \\ 1 & -1 & 1 & 1 \end{array} \right)$$

If $A \in \mathbb{F}^{m \times n}$, $b \in \mathbb{F}^m$ and we perform EROs on A to get its RRE form A_R , that is,

$$A \xrightarrow{EROs} A_R, \text{ Row reduced Echelon Form of } A$$

then

$$A_{aug} \xrightarrow{\text{same EROs}} \left(A_R \mid \tilde{b} \right)$$

The nonhomogeneous system

$$Ax = b$$

is consistent if and only if the system

$$A_R x = \tilde{b}$$

is consistent. Let us now analyse the structure of the system $A_R x = \tilde{b}$. Suppose the Row Rank of A is ρ . Then the first ρ rows R_1, R_2, \dots, R_ρ of A_R are nonzero rows, and the remaining $m - \rho$ rows, $R_{\rho+1}, R_{\rho+2}, \dots, R_m$ of A_R are all zero rows. How does $\left(A_R \mid \tilde{b} \right)$ look like? We have

$$\left(A_R \mid \tilde{b} \right) = \left(\begin{array}{c|c} R_1 & \tilde{b}_1 \\ R_2 & \tilde{b}_2 \\ \dots & \dots \\ \dots & \dots \\ R_\rho & \tilde{b}_\rho \\ \hline 0_{1 \times n} & \tilde{b}_{\rho+1} \\ 0_{1 \times n} & \tilde{b}_{\rho+2} \\ \dots & \dots \\ \dots & \dots \\ 0_{1 \times n} & \tilde{b}_m \end{array} \right) \quad (4.9.1)$$

Whenever all terms on the lhs of an equation are zero, for the equation to be consistent, we must have the rhs also as zero. Thus from the above structure it follows that, for the system $A_R x = \tilde{b}$ to be consistent we need

$$\tilde{b}_j = 0 \text{ for } \rho + 1 \leq j \leq m \quad (4.9.2)$$

These are the consistency conditions that b has to satisfy in order that $Ax = b$ is consistent. These conditions can equivalently be stated as follows:

$$\left. \begin{array}{l} Ax = b \text{ is consistent if and only if} \\ \text{Row Rank of } A \text{ is equal to the Row rank of } A_{aug} \end{array} \right\} \quad (4.9.3)$$

We shall now see how to solve the nonhomogeneous system $A_R x = \tilde{b}$ when the consistency condition is satisfied. (The set of solutions we obtain will also be precisely the set of solutions of the original nonhomogeneous system $Ax = b$, since $Ax = b$ and $A_R x = \tilde{b}$ will then be equivalent). When the consistency conditions (4.9.2) are satisfied, $\left(A_R \mid \tilde{b} \right)$ is of the form

$$\left(A_R \mid \tilde{b} \right) = \left(\begin{array}{c|c} R_1 & \tilde{b}_1 \\ R_2 & \tilde{b}_2 \\ \cdots & \cdots \\ \cdots & \cdots \\ R_\rho & \tilde{b}_\rho \\ \hline 0_{1 \times n} & 0 \\ 0_{1 \times n} & 0 \\ \cdots & \cdots \\ \cdots & \cdots \\ 0_{1 \times n} & 0 \end{array} \right) \quad (4.9.4)$$

Let the pivotal 1 in the R_i th row be in the k_i th column. Then from the rows R_1, R_2, \dots, R_ρ we can respectively eliminate the pivotal variables $x_{k_1}, x_{k_2}, \dots, x_{k_\rho}$ in terms of $\tilde{b}_1, \tilde{b}_2, \dots, \tilde{b}_\rho$ and the remaining $n - \rho$ variables, and these remaining $n - \rho$ variables can be chosen arbitrarily. We can write the solution as the sum of two parts, firstly the part of the solution which involves the \tilde{b} components and this will be a Particular Solution of the non-homogeneous system, and secondly the part of the solution which involves the arbitrary free variables, which is the general solution of the homogeneous system. We shall now illustrate these aspects with an example.

Example 4.9.3 Consider the matrix $A \in \mathbb{F}^{4 \times 5}$ and $b \in \mathbb{F}^4$ given below:

$$A = \begin{pmatrix} 1 & 2 & 2 & 9 & -1 \\ 1 & 2 & 3 & 13 & -2 \\ -1 & -2 & -1 & -5 & 1 \\ 1 & 2 & 0 & 1 & 2 \end{pmatrix}$$

$$b = \begin{pmatrix} b_1 & b_2 & b_3 & b_4 \end{pmatrix}$$

We can write the reduction of the Augmented matrix as follows:

$$\begin{aligned}
A_{aug} &= \left(\begin{array}{ccccc|c} 1 & 2 & 2 & 9 & -1 & b_1 \\ 1 & 2 & 3 & 13 & -2 & b_2 \\ -1 & -2 & -1 & -5 & 1 & b_3 \\ 1 & 2 & 0 & 1 & 2 & b_4 \end{array} \right) \xrightarrow{R_2-R_1, R_3+R_1, R_4-R_1} \\
&\left(\begin{array}{ccccc|c} 1 & 2 & 2 & 9 & -1 & b_1 \\ 0 & 0 & 1 & 4 & -1 & b_2 - b_1 \\ 0 & 0 & 1 & 4 & 0 & b_3 + b_1 \\ 0 & 0 & -2 & -8 & 3 & b_4 - b_1 \end{array} \right) \xrightarrow{R_3-R_2, R_4+2R_2} \\
&\left(\begin{array}{ccccc|c} 1 & 2 & 2 & 9 & -1 & b_1 \\ 0 & 0 & 1 & 4 & -1 & b_2 - b_1 \\ 0 & 0 & 0 & 0 & 1 & b_3 - b_2 + 2b_1 \\ 0 & 0 & 0 & 0 & 1 & -3b_1 + 2b_2 + b_4 \end{array} \right) \xrightarrow{R_4-R_3} \\
&\left(\begin{array}{ccccc|c} 1 & 2 & 2 & 9 & -1 & b_1 \\ 0 & 0 & 1 & 4 & -1 & b_2 - b_1 \\ 0 & 0 & 0 & 0 & 1 & b_3 - b_2 + 2b_1 \\ 0 & 0 & 0 & 0 & 0 & -5b_1 + 3b_2 - b_3 + b_4 \end{array} \right) \xrightarrow{R_1-2R_2} \\
&\left(\begin{array}{ccccc|c} 1 & 2 & 0 & 1 & 1 & 3b_1 - b_2 \\ 0 & 0 & 1 & 4 & -1 & b_2 - b_1 \\ 0 & 0 & 0 & 0 & 1 & b_3 - b_2 + 2b_1 \\ 0 & 0 & 0 & 0 & 0 & -5b_1 + 3b_2 - b_3 + b_4 \end{array} \right) \xrightarrow{R_1-R_3, R_2+R_3} \\
&\left(\begin{array}{ccccc|c} 1 & 2 & 0 & 1 & 0 & -b_1 + b_2 - b_3 \\ 0 & 0 & 1 & 4 & 0 & b_1 + b_3 \\ 0 & 0 & 0 & 0 & 1 & 2b_1 - b_2 + b_3 \\ 0 & 0 & 0 & 0 & 0 & -5b_1 + 3b_2 - b_3 + b_4 \end{array} \right)
\end{aligned}$$

Thus as in the notation of (4.9.1) we can write

$$A_{aug} \xrightarrow{\text{above EROs}} \left(\begin{array}{ccccc|c} 1 & 2 & 0 & 1 & 0 & \tilde{b}_1 \\ 0 & 0 & 1 & 4 & 0 & \tilde{b}_2 \\ 0 & 0 & 0 & 0 & 1 & \tilde{b}_3 \\ 0 & 0 & 0 & 0 & 0 & \tilde{b}_4 \end{array} \right)$$

Let

$$\tilde{b} = \begin{pmatrix} \tilde{b}_1 \\ \tilde{b}_2 \\ \tilde{b}_3 \\ \tilde{b}_4 \end{pmatrix}$$

where

$$\begin{aligned} \tilde{b}_1 &= -b_1 + b_2 - b_3 \\ \tilde{b}_2 &= b_1 + b_3 \\ \tilde{b}_3 &= 2b_1 - b_2 + b_3 \\ \tilde{b}_4 &= -5b_1 + 3b_2 - b_3 + b_4 \end{aligned}$$

From (4.9.2) we get the consistency condition as

$$-5b_1 + 3b_2 - b_3 + b_4 = 0$$

When consistency condition satisfied, the system $A_R x = \tilde{b}$ can be written as

$$\left. \begin{aligned} x_1 + 2x_2 + x_4 &= \tilde{b}_1 \\ x_3 + 4x_4 &= \tilde{b}_2 \\ x_5 &= \tilde{b}_3 \end{aligned} \right\}$$

The Pivotal variables are x_1 , x_3 and x_5 , and these can be eliminated in terms of $\tilde{b}_1, \tilde{b}_2, \tilde{b}_3$ and the free variables x_2 and x_4 to get the solution as

$$x = \begin{pmatrix} \tilde{b}_1 \\ 0 \\ \tilde{b}_2 \\ 0 \\ \tilde{b}_3 \end{pmatrix} + \alpha \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} -1 \\ 0 \\ -4 \\ 1 \\ 0 \end{pmatrix}$$

where α and β can be chosen arbitrarily. A particular solution x_P of the nonhomogeneous system is obtained from the part above involving the \tilde{b}_j , that is

$$x_P = \begin{pmatrix} \tilde{b}_1 \\ 0 \\ \tilde{b}_2 \\ 0 \\ \tilde{b}_3 \end{pmatrix}$$

The general solution of the homogeneous system can be obtained from the above by taking the part involving the free variables, that is,

$$x_H = \alpha \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} -1 \\ 0 \\ -4 \\ 1 \\ 0 \end{pmatrix}$$

If for example, we take

$$b = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}$$

the consistency condition

$$-5b_1 + 3b_2 - b_3 + b_4 = 0$$

is satisfied, and

$$\begin{aligned} \tilde{b}_1 &= -b_1 + b_2 - b_3 = -1 \\ \tilde{b}_2 &= b_1 + b_3 = 1 \\ \tilde{b}_3 &= 2b_1 - b_2 + b_3 = 1 \end{aligned}$$

Hence we get Particular Solution as

$$x_P = \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

It is easy to check that this is a solution of the given Nonhomogeneous system $Ax = b$.

Remark 4.9.1 If the matrix $A \in \mathbb{F}^{m \times n}$ has rank ρ_A then the RRE form of the augmented matrix will have the structure as in (4.9.1). From this it follows that for $Ax = b$ to be consistent, b has to satisfy $(m - \rho_A)$ consistency conditions given by (4.9.3). If the rank of the matrix A is m , that is $\rho_A = m$,

then b does not have to satisfy any consistency conditions and hence the system $Ax = b$ will be consistent for every $b \in \mathbb{F}^m$ if and only if $\rho_A = m$. Further the nonuniqueness in the solution occurs because of the arbitrary choice for the $(n - \rho_A)$ nonpivotal variables. Hence the solution will be unique if and only if $\rho_A = n$. Thus,

**If $A \in \mathbb{F}^{m \times n}$ then $Ax = b$ will be consistent for all $b \in \mathbb{F}^m$ if and only if $\rho_A = m$, and
if $Ax = b$ is consistent for a $b \in \mathbb{F}^m$, solution will be unique if and only if $\rho_A = n$.**

4.10 Inconsistent Systems and EROs

We shall now see that Elementary Row Operations DO NOT preserve the least square solutions of an inconsistent system, that is, if $Ax = b$ is inconsistent, and

$$A_{aug} \xrightarrow{EROs} \left(A_R \mid \tilde{b} \right)$$

it does not imply that $Ax = b$ and $A_R x = \tilde{b}$ have same set of least square solutions. We shall illustrate this by means of an example below:

Example 4.10.1 The system

$$\begin{aligned} x_1 + x_2 &= 1 \\ 2x_1 + 2x_2 &= 1 \end{aligned}$$

is obviously inconsistent. We have

$$\begin{aligned} A &= \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} \\ b &= \begin{pmatrix} 1 \\ 1 \end{pmatrix} \end{aligned}$$

We shall now find the least square solutions of this inconsistent system. We want to find x_1, x_2 such that

$$\|Ax - b\|^2 = [(Ax)_1 - b_1]^2 + [(Ax)_2 - b_2]^2$$

is minimum. We have

$$\begin{aligned} \|Ax - b\|^2 &= [(Ax)_1 - b_1]^2 + [(Ax)_2 - b_2]^2 \\ &= (x_1 + x_2 - 1)^2 + (2x_1 + 2x_2 - 1)^2 \end{aligned}$$

The minimum occurs when the derivatives with respect to x_1 and x_2 are zero, that is,

$$\begin{aligned} 5x_1 + 5x_2 &= 3 \\ \implies \\ x_1 + x_2 &= \frac{3}{5} \end{aligned} \tag{4.10.1}$$

Thus the least square solutions must satisfy (4.10.1).

Let us now look at effect of EROs. We have

$$A_{aug} \xrightarrow{R_2 - 2R_1} \left(\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 0 & -1 \end{array} \right) = \left(\begin{array}{c|c} A_1 & b_1 \end{array} \right)$$

The corresponding inconsistent nonhomogeneous system $A_1x = b_1$:

$$\begin{aligned} x_1 + x_2 &= 1 \\ 0x_1 + 0x_2 &= -1 \end{aligned}$$

Let us now find the least square solutions of this inconsistent system. We have

$$\begin{aligned} \|A_1x - b_1\|^2 &= [(A_1x)_1 - (b_1)_1]^2 + [(A_1x)_2 - (b_1)_2]^2 \\ &= (x_1 + x_2 - 1)^2 + (-1)^2 \end{aligned}$$

Using derivatives we find that the minimum occurs when

$$x_1 + x_2 = 1 \tag{4.10.2}$$

Hence the least square solutions of the inconsistent system obtained after EROs must satisfy (4.10.2). Comparing (4.10.1) and (4.10.2) we see that that Least square solutions have been altered by ERO.

4.11 Row Operations and Linear Independence of Columns

Consider a matrix $A \in \mathbb{F}^{m \times n}$. Let us consider the effects of the Elementary Row Operations on the independence of the columns. Suppose the first j

columns of A are linearly independent and the $(j + 1)$ th column is a linear combination of the first j columns. This means that if we apply the scanning process to the columns of A the first column that will be removed will be the $(j + 1)$ th column. Consider the matrix A_1 obtained from A by retaining only the first $(j + 1)$ columns. We have

$$\begin{aligned} A_1 &= [C_1 \ C_2 \ C_3 \ \cdots \ C_j \ C_{(j+1)}] \\ A &= [A_1 | C_{(j+2)} \ C_{(j+3)} \ \cdots \ C_n] \end{aligned}$$

Since the rank of A_1 is j we have the RRE form of A as

$$(A_1)_R = \left(\begin{array}{c|c|c} I_{j \times j} & 0_{j \times 1} & 0_{j \times (n-(j+1))} \\ \hline 0_{(n-j) \times j} & 0_{(n-j) \times 1} & L_{(n-j) \times (n-j)} \end{array} \right)$$

We then get that A is row equivalent to a matrix of the form

$$\hat{A} = \left((A_1)_R \mid K_{j \times (n-j)} \right)$$

We can start with this matrix \hat{A} and apply Elementary Row Operations to get the RRE form of A . Hence the first pivotal entries in the RRE for A_R of A will be in the first j columns of A_R and the $(j + 1)$ th column of A_R will be a zero column and hence the next pivotal entry cannot be in the $(j + 1)$ th column. Thus it follows that in the RRE form of A also the first j columns will be linearly independent and the $(j + 1)$ th column will be a linear combination of the first j columns. Thus while transforming A to A_R the column space might have changed the column index of the linearly independent columns obtained by the scanning process will remain the process. Thus we have the following for any $A \in \mathbb{F}^{m \times n}$:

The columns in A , corresponding to the columns supporting the pivotal 1s of A_R , form a basis for $Col(A)$, that is for \mathcal{R}_A .

Example 4.11.1 In Exampe 4,7.3, for the matrix

$$A = \begin{pmatrix} 1 & 2 & 2 & 9 & -1 \\ 1 & 2 & 3 & 13 & -2 \\ -1 & -2 & -1 & -5 & 1 \\ 1 & 2 & 0 & 1 & 2 \end{pmatrix}$$

we had the RRE form as

$$\begin{pmatrix} 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Hence the columns supporting the pivots are Columns one, three and five. Hence the columns one, three and five of the matrix A give a basis for \mathcal{R}_A . Hence a basis for \mathcal{R}_A is given by

$$u = \begin{pmatrix} 1 \\ 1 \\ -1 \\ 1 \end{pmatrix}, v = \begin{pmatrix} 2 \\ 3 \\ -1 \\ 0 \end{pmatrix}, w = \begin{pmatrix} -1 \\ -2 \\ 1 \\ 2 \end{pmatrix}$$

4.12 Summary

Summarizing we see that some of the important information that we get from Elementary Row Operations and the RRE form A_R of a matrix $A \in \mathbb{F}^{m \times n}$ are the following:

1. The nonzero rows of A_R give a basis for $Row(A) = Col(A^T) = \mathcal{R}_{A^T}$
2. The number of nonzero rows of A_R gives the Rank of the matrix A , (which is also equal to the Column Rank of A , Row rank of A , Row Rank of A^T , Rank of A^T , Column Rank Of A^T)
3. $\rho_A = \rho_{A^T}$
4. $\rho_A + \nu_A = n$
5. $\rho_{A^T} + \nu_{A^T} = m$
6. By eliminating the pivotal variables in terms of the nonpivotal variables, we get the geeral solution structure for the homogeneous system $Ax = \theta_m$
7. The columns in A , corresponding to the columns supporting the pivotal 1s of A_R , form a basis for $Col(A) = \mathcal{R}_A = Row(A^T)$

8. If A is a square matrix then it is invertible if and only if $A_R = I_n$. In such a case

$$A \xrightarrow{E_k E_{(k-1)} \cdots E_2 E_1} A_R = I_n$$

$$\implies$$

$$I_n \xrightarrow{E_k E_{(k-1)} \cdots E_2 E_1} A^{-1}$$

where E_j is the j th ERO applied in transforming A to A_R .

9. If $Ax = b$ is consistent and $A_{aug} = \left(A \mid b \right) \xrightarrow{ERO} \left(A_R \mid b_R \right)$, then the systems $Ax = b$ and $A_R x = b_R$ have the same set of solutions and we can get the solutions easily from the RRE system $A_R x = b_R$.