

Roll No: CS23E001

Name: Shuvrajeet Das

Collaborators (if any):

References/sources (if any):

- Use \LaTeX to write-up your solutions (in the solution blocks of the source \LaTeX file of this assignment), and submit the resulting pdf files (one per question) at Crowdmark by the due date. (Note: **No late submissions** will be allowed, other than one-day late submission with 10% penalty or four-day late submission with 30% penalty! Instructions to join Crowdmark and submit your solution to each question within Crowdmark **TBA** later).
- For the programming question, please submit your code (rollno.ipynb file and rollno.py file in rollno.zip) directly in moodle, but provide your results/answers (including Jupyter notebook **with output**) in the pdf file you upload to Crowdmark.
- Collaboration is encouraged, but all write-ups must be done individually and independently, and mention your collaborator(s) if any. Same rules apply for codes written for any programming assignments (i.e., write your own code; we will run plagiarism checks on codes).
- If you have referred a book or any other online material or LLMs (Large Language Models like ChatGPT) for obtaining a solution, please cite the source. Again don't copy the source *as is* - you may use the source to understand the solution, but write-up the solution in your own words (this also means that you cannot copy-paste the solution from LLMs!). Please be advised that *the lesser your reliance on online materials or LLMs for answering the questions, the more your understanding of the concepts will be and the more prepared you will be for the course exams.*
- Points will be awarded based on how clear, concise and rigorous your solutions are, and how correct your answer is. The weightage of this assignment is 12% towards the overall course grade.

Solution: The solution of question(a)

Given the conditions, $\mu = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $\Sigma = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

from this we can get, $\Sigma^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ and $|\Sigma| = (ad - bc)$

The multivariate Gaussian distribution can be defined as:-

$$f_{\mathbf{x}}(\mathbf{x}) = \frac{1}{(2\pi)^{d/2}|\Sigma|^{1/2}} \cdot \exp\left(-\frac{1}{2}(\mathbf{x} - \mu)^T \Sigma^{-1}(\mathbf{x} - \mu)\right)$$

Putting all the necessary values we get:

$$\begin{aligned} &= \frac{1}{(2\pi)(ad-bc)^{1/2}} \exp\left(-\frac{1}{2(ad-bc)} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^T \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) \\ &= \frac{1}{(2\pi)(ad-bc)^{1/2}} \exp\left(-\frac{1}{2(ad-bc)} [x_1 d - x_2 c \quad -x_1 b + x_2 a] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) \\ &= \frac{1}{(2\pi)(ad-bc)^{1/2}} \exp\left(-\frac{1}{2(ad-bc)} (x_1^2 d - x_1 x_2 c - x_1 x_2 b + x_2^2 a)\right) \\ &= \frac{1}{(2\pi)(ad-bc)^{1/2}} \exp\left(-\frac{1}{2(ad-bc)} (x_1^2 d - x_1 x_2 (c+b) + x_2^2 a)\right) \end{aligned}$$

For question (1) $P(X_2|X_1 = x_1)$ is

$$= \frac{1}{(2\pi)(ad-bc)^{1/2}} \exp\left(-\frac{a}{2(ad-bc)} \left(\frac{x_1^2 d}{a} - 2x_1 x_2 \left(\frac{c+b}{2a}\right) + x_2^2\right)\right)$$

Let's solve the inner part in the exponent,

$$\begin{aligned} &= \frac{-a}{2(ad-bc)} \left(\frac{x_1^2 d}{a} - 2x_1 x_2 \left(\frac{c+b}{2a}\right) + x_2^2\right) \\ &= \frac{-a}{2(ad-bc)} \left(\frac{x_1^2 d}{a} - \left(\frac{c+b}{2a}\right)^2 x_1^2 + \left(\frac{c+b}{2a}\right)^2 x_1^2 - 2x_1 x_2 \left(\frac{c+b}{2a}\right) + x_2^2\right) \\ &= \frac{-a}{2(ad-bc)} \left(\frac{x_1^2 d}{a} - \left(\frac{c+b}{2a}\right)^2 x_1^2 + \left(\left(\frac{c+b}{2a}\right) x_1 - x_2\right)^2\right) \\ &= \frac{-a}{2(ad-bc)} \left(\left(\frac{4ad - (c+b)^2}{4a^2}\right) x_1^2 + \left(\left(\frac{c+b}{2a}\right) x_1 - x_2\right)^2\right) \\ &= \frac{-a}{2(ad-bc)} \left(\frac{4ad - (c+b)^2}{4a^2}\right) x_1^2 + \frac{-a}{2(ad-bc)} \left(\left(\frac{c+b}{2a}\right) x_1 - x_2\right)^2 \end{aligned}$$

Simply filling the equation again we get, a product of two quantities,

$$\frac{1}{(2\pi)^{1/2}(ad-bc)^{1/2}} \exp\left(\frac{-a}{2(ad-bc)} \left(\left(\frac{c+b}{2a}\right) x_1 - x_2\right)^2\right)$$

and

$$\frac{1}{(2\pi)^{1/2}} \exp\left(\frac{-a}{2(ad-bc)} \left(\frac{4ad - (c+b)^2}{4a^2}\right) x_1^2\right)$$

For question (2) $P(X_1|X_2 = x_2)$ is

$$= \frac{1}{(2\pi)(ad-bc)^{1/2}} \exp\left(-\frac{d}{2(ad-bc)} \left(x_1^2 - 2x_1 x_2 \left(\frac{c+b}{2d}\right) + \frac{x_2^2 a}{d}\right)\right)$$

Let's solve the inner part in the exponent,

$$\begin{aligned} &= -\frac{d}{2(ad-bc)} \left(x_1^2 - 2x_1 x_2 \left(\frac{c+b}{2d}\right) + \frac{x_2^2 a}{d}\right) \\ &= \frac{-d}{2(ad-bc)} \left(x_1^2 - 2x_1 x_2 \left(\frac{c+b}{2d}\right) + \left(\frac{c+b}{2d}\right)^2 x_2^2 - \left(\frac{c+b}{2d}\right)^2 x_2^2 + \frac{x_2^2 a}{d}\right) \end{aligned}$$

Simply filling the equation again we get, a product of two quantities,

$$\frac{1}{(2\pi)^{1/2}(ad-bc)^{1/2}} \exp\left(\frac{-d}{2(ad-bc)} \left(x_1 - \left(\frac{c+b}{2d}\right) x_2\right)^2\right)$$

and

$$\frac{1}{(2\pi)^{1/2}} \exp\left(\frac{-1}{2(ad-bc)} \left(\frac{4ad - (c+b)^2}{4d^2}\right) x_2^2\right)$$

we can say that $d > 0$, $ad - bc > 0$, $a > 0$ also $a = d = 1$ filling up the values we get,

$$\begin{aligned} P(X_1|X_2) &= \frac{1}{(2\pi)^{1/2}(1-bc)^{1/2}} \exp\left(\frac{-1}{2(1-bc)} \left(x_1 - \left(\frac{c+b}{2}\right) x_2\right)^2\right) \\ P(X_2) &= \frac{1}{(2\pi)^{1/2}} \exp\left(\frac{-1}{2(1-bc)} \left(\frac{4 - (c+b)^2}{4}\right) x_2^2\right) \\ P(X_2|X_1) &= \frac{1}{(2\pi)^{1/2}(1-bc)^{1/2}} \exp\left(\frac{-1}{2(1-bc)} \left(\left(\frac{c+b}{2}\right) x_1 - x_2\right)^2\right) \\ P(X_1) &= \frac{1}{(2\pi)^{1/2}} \exp\left(\frac{-1}{2(1-bc)} \left(\frac{4 - (c+b)^2}{4}\right) x_1^2\right) \end{aligned}$$

we also get $1 - bc > 0$