IITM-CS5691 : Pattern Recognition and Machine Learning | Release Date: August 30, 2023
Assignment 1 | Due Date : September 14, 2023, 23:59

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Collaborators (if any):
References/sources (if any):

- Use LaTeX to write-up your solutions (in the solution blocks of the source LaTeX file of this assignment), and submit the resulting pdf files (one per question) at Crowdmark by the due date. (Note: **No late submissions** will be allowed, other than one-day late submission with 10% penalty or four-day late submission with 30% penalty! Instructions to join Crowdmark and submit your solution to each question within Crowdmark **TBA** later).
- For the programming question, please submit your code (rollno.ipynb file and rollno.py file in rollno.zip) directly in moodle, but provide your results/answers (including Jupyter notebook **with output**) in the pdf file you upload to Crowdmark.
- Collaboration is encouraged, but all write-ups must be done individually and independently, and mention your collaborator(s) if any. Same rules apply for codes written for any programming assignments (i.e., write your own code; we will run plagiarism checks on codes).
- If you have referred a book or any other online material or LLMs (Large Language Models like ChatGPT) for obtaining a solution, please cite the source. Again don't copy the source *as is* - you may use the source to understand the solution, but write-up the solution in your own words (this also means that you cannot copy-paste the solution from LLMs!). Please be advised that *the lesser your reliance on online materials or LLMs* for answering the questions, *the more your understanding* of the concepts will be and *the more prepared you will be for the course exams*.
- Points will be awarded based on how clear, concise and rigorous your solutions are, and how correct your answer is. The weightage of this assignment is 12% towards the overall course grade.

1. (8 points) [BAYESIAN DECISION THEORY]

(a) (4 points) [Optimal Classifier by Pen/Paper] Let L be the loss matrix defined by $L = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}$, where L_{ij} indicates the loss for an input x with i being the true class and j the predicted class. Given the data:

x	-2.8	1.5	0.4	-0.3	-0.7	0.9	1.8	0.8	-2.4	-1.3	1.1	2.5	2.6	-3.3
y	1	3	2	2	1	3	3	2	1	1	2	3	3	1

find the optimal Bayes classifier $h(x)$, and provide its decision boundaries/regions. Assume that the class conditionals are Gaussian distributions with a known variance of 1 and unknown means (to be estimated from the data).

(b) (4 points) Consider a classification problem in which the loss incurred on mis-classifying an input vector from class C_k as C_j is given by loss matrix entry L_{kj}, and for which the loss incurred in selecting the reject option is ψ. Find the decision criterion that will give minimum expected loss, and then simplify it for the case of 0-1 loss (i.e., when $L_{kj} = \mathbb{1}_{k \neq j}$).

Solution: The solution of question (a):
To find the optimal Bayes classifier $h(x)$ and its decision boundaries/regions, we need to estimate the class means for the Gaussian distributions for each class using the given data and then apply Bayes' rule to classify new data points. Bayes' rule states:

$$h(x) = \underset{c \in \mathcal{C}}{\operatorname{argmax}}\, P(C = c | X = x)$$

The class means for each class based on the data:

For Class 1 (y = 1):

$$\mu_1 = \frac{\sum_{i=1}^{N} x_i}{N} = \frac{-2.8 - 0.7 - 2.4 - 1.3 - 3.3}{5} = -2.1$$

For Class 2 (y = 2):

$$\mu_2 = \frac{\sum_{i=1}^{N} x_i}{N} = \frac{0.4 - 0.3 + 0.8 + 1.1}{4} = 0.5$$

For Class 3 (y = 3):

$$\mu_3 = \frac{\sum_{i=1}^{N} x_i}{N} = \frac{1.5 + 0.9 + 1.8 + 2.5 + 2.6}{5} = 1.86$$

The decision boundaries are where the posterior probabilities are equal for two neighboring classes. Let's calculate the posterior probabilities for each class:

For Class 1:

$$P(C = 1|X = x) \propto \exp\left(-\frac{(x-\mu_1)^2}{2\sigma^2}\right)$$

For Class 2:

$$P(C = 2|X = x) \propto \exp\left(-\frac{(x-\mu_2)^2}{2\sigma^2}\right)$$

For Class 3:

$$P(C = 3|X = x) \propto \exp\left(-\frac{(x-\mu_3)^2}{2\sigma^2}\right)$$

Assuming $\sigma^2 = 1$ (known variance), we can compare these probabilities for each class at each data point x. The class with the highest probability will be the classification.

The Bayes classifier minimizes the expected loss, which can be expressed as:

$$\mathcal{E}(L(y, h(x))) = \sum_{i,j} P(C = i, h(x) = j) \cdot L(i, j)$$

1. Decision boundary between Class 1 (y = 1) and Class 2 (y = 2):

At each data point x, calculate the expected loss for both classes and compare them. The boundary occurs where the expected losses are equal:

$$\mathcal{E}(L(y = 1, h(x))) = \mathcal{E}(L(y = 2, h(x)))$$

The values of x where these equations hold are the decision boundary between Class 1 and Class 2.

2. Decision boundary between Class 2 (y = 2) and Class 3 (y = 3): Similar to the previous step, compare the expected losses between Class 2 and Class 3 to find the boundary:

$$\mathcal{E}(L(y = 2, h(x))) = \mathcal{E}(L(y = 3, h(x)))$$

The values of x where these equations hold are the decision boundary between Class 2 and Class 3.

3. Decision boundary between Class 1 (y = 1) and Class 3 (y = 3): Compare the expected losses between Class 1 and Class 3 to find the boundary:

$$\mathcal{E}(L(y = 1, h(x))) = \mathcal{E}(L(y = 3, h(x)))$$

The values of x where these equations hold are the decision boundary between Class 1 and Class 3.

Now, calculate these boundaries using the estimated means and the loss matrix L. The boundaries occur where the expected losses for the respective classes are equal at each data point x.

The solution of question (b)

The decision criterion that minimizes the expected loss is to choose the class C_j that minimizes the expected risk:

$$\text{Decision: } j = \arg\min_j \sum_k P(C_k|\text{input})L_{kj} + P(\text{reject}|\text{input})\psi$$

Here, $P(C_k|\text{input})$ is the posterior probability of the true class C_k given the input data, and $P(\text{reject}|\text{input}$ is the probability of selecting the reject option given the input data.

Now, let's simplify this decision criterion for the case of 0-1 loss, which means L_{kj} is 0 if $k = j$ (correct classification) and 1 if $k \neq j$ (incorrect classification), and ψ is a constant representing the loss for rejecting.

In this case, the decision criterion becomes:

$$\text{Decision: } j = \arg\min_j \sum_k P(C_k|\text{input}) \cdot \mathbb{I}(k \neq j) + P(\text{reject}|\text{input})\psi \tag{1}$$

Here, $\mathbb{I}(k \neq j)$ is an indicator function that equals 1 when $k \neq j$ (incorrect classification) and 0 when $k = j$ (correct classification).

So, for the case of 0-1 loss, the decision criterion becomes select the class with the highest posterior probability, and if the probability of all classes is below a certain threshold (determined by ψ), then choose the reject option.