EXERCISES - SET V

1. State TRUE or FALSE (Give reasons)

   (a) $A \in \mathbb{C}^{n\times n}$ is nilpotent $\Longleftrightarrow c_{_A}(\lambda) = \lambda^n$

   (b) $A \in \mathbb{C}^{n\times n}$ is nilpotent $\Longleftrightarrow$ 0 is the only eigenvalue of A

   (c) $A \in \mathbb{C}^{n\times n}$ is nilpotent $\Longleftrightarrow m_{_A}(\lambda) = \lambda^n$

   (d) Let $A \in \mathbb{C}^{n\times n}$ and $p(\lambda) \in \mathcal{F}[\lambda]$. Then $p(A)$ is nilpotent $\Longleftrightarrow$ every eigenvalue of A is root of the polynomial $p(\lambda)$

2. If the matrix

$$A = \begin{pmatrix} a_1 & 1 & 2 & 3 & 4 \\ 0 & a_2 & 2 & 3 & 4 \\ 0 & 0 & a_3 & 3 & 4 \\ 0 & 0 & 0 & a_4 & 4 \\ 0 & 0 & 0 & 0 & a_5 \end{pmatrix}$$

   has to be nilpotent then what can you say about the values for a_1, a_2, a_3, a_4, a_5?

3. Let $N \in \mathbb{C}^{n\times n}$ be the matrix defined as follows:

$$N \;=\; \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & -1 & 1 \end{pmatrix}$$

   Show that N is nilpotent and order of nilpotency is 2. Find the canonical form of N

4. Let $N \in \mathbb{C}^{n\times n}$ be the matrix defined as follows:

$$N \;=\; \begin{pmatrix} -4 & 1 & 0 & 2 \\ -8 & 2 & 0 & 4 \\ 0 & 0 & 0 & 0 \\ -4 & 1 & 0 & 2 \end{pmatrix}$$

   Show that N is nilpotent and order of nilpotency is 2. Find the canonical form of N

5. The canonical form of a nilpotent matrix $N \in \mathcal{F}^{10\times 10}$ is given as

$$N_{can} = \left(\begin{array}{cccc|cc|cc|c|c}
0 & 1 & 0 & 0 & & & & & & \\
0 & 0 & 1 & 0 & & & & & & \\
0 & 0 & 0 & 1 & & & & & & \\
0 & 0 & 0 & 0 & & & & & & \\
\hline
 & & & & 0 & 1 & & & & \\
 & & & & 0 & 1 & & & & \\
\hline
 & & & & & & 0 & 1 & & \\
 & & & & & & 0 & 1 & & \\
\hline
 & & & & & & & & 0 & \\
\hline
 & & & & & & & & & 0
\end{array}\right)$$

Answer the following:

(a) What is the characteristic polynomial of N?

(b) What is the minimal polynomial of N?

(c) Find the dimension of the following subspaces?

i. Null Space of N

ii. Null Space of N^2

iii. Null Space of N^3

iv. Null Space of N^4

v. Null Space of N^5

6. Consider the matrices given below:

$$A_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$
$$A_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Show that A_1, A_2 are nilpotent, do not commute and their product A_1A_2 and sum $A_1 + A_2$ are nilpotent. Give an example where two matrices do not commute and their sum and product are not nilpotent. (Thus while for commuting nilpotent matrices the product and sum are always nilpotent, for noncommuting nilpotent matrices the product and sum may or may not be nilpotent)

7. The following data is given about $A \in \mathbb{C}^{14\times 14}$:

$$\begin{aligned}
c_{_A}(\lambda) &= (\lambda-2)^5(\lambda+5)^8(\lambda-4)\\
m_{_A}(\lambda) &= (\lambda-2)^2(\lambda+5)^4(\lambda-4)\\
nullity\ of\ (A-2I_{14\times 14}) &= 3\\
nullity\ of\ (A+5I_{14\times 14}) &= 3\\
nullity\ of\ (A+5I_{14\times 14})^3 &= 7
\end{aligned}$$

Answer the following:

(a) What is nullity of $(A+5I_{14\times 14})^2$?

(b) What is the Jordan canonical form of A?

(c) Find the Jordan canonical form of

i. $sin(\pi A)$

ii. $exp.(A)$

(d) The Jordan canonical form of $A \in \mathbb{C}^{21\times 21}$ is given by

$$A_{can} = \left(\begin{array}{c|c|c|c|c|c|c}
J_3(2) & & & & & & \\ \hline
 & J_3(2) & & & & & \\ \hline
 & & J_2(2) & & & & \\ \hline
 & & & J_4(-1) & & & \\ \hline
 & & & & J_2(-1) & & \\ \hline
 & & & & & J_1(-1) & \\ \hline
 & & & & & & J_6(3)
\end{array}\right)$$

(where $J_r(a)$ denotes the canonical $r \times r$ Jordan matrix with diagonal a)

Answer the following:

i. Find the characteristic polynomial and minimal polynomial of A

ii. Find the dimensions of the following subspaces:

A. Null $Space\ of$ (A-2I$_{21\times 21}$)^r for $r = 1, 2, 3, 4$

B. Null $Space\ of$ (A+I$_{21\times 21}$)^r for $r = 1, 2, 3, 4, 5$

C. Null $Space\ of$ (A-3I$_{21\times 21}$)^r for $r = 1, 2$

iii. Find the canonical form of $exp.(A)$