(CS5020, Jul-Nov 2023) Nonlinear Optimisation: Theory and Algorithms
Worksheet - 4

Definition: A real valued sequence $x_0, x_1, \ldots, x_t$ is said to converge to a real number x_* with order q and rate μ if

$$\lim_{t\to\infty} \frac{|x_{t+1} - x_*|}{|x_t - x_*|^q} = \mu \tag{1}$$

(1) Find with justification the order and rate of convergence of
   (a) $x_t = \frac{1}{t}$,
   (b) $x_t = \frac{1}{t^2}$
   (c) $x_t = \frac{1}{2^t}$
   (d) $x_t = \frac{1}{2^{t^2}}$
   (e) $x_t = \frac{1}{2^{2^t}}$

(2) Consider an algorithm whose error $e_t = x_t - x_*$ obeys, $|e_{t+1}| = |e_t|^q$ ($q \geq 2$). Let $|e_0| = 0.1$, and $\epsilon > 0$ be a very small real number. What is the minimum t that is required such that $e_t \leq \epsilon$?

(3) Write down the gradient descent update for the function $f(x) = ax^2+bx+c$, and derive a closed form expression for $x_t - x_*$, and the allowable range of α for convergence.

(4) For the following functions (i) Write gradient descent update and comment whether or not convergence happens with constant step size. (ii) Write Newton update. Comment whether or not convergence happens, and if so linear or quadratic.
   (a)

$$f(x) = \begin{cases} x^2, & x \in [-0.1, 0.1] \\ x^4 + (0.1)^2 - (0.1)^4 & \text{otherwise.} \end{cases} \tag{2}$$

   (b) $f(x) = |x|^3$
   (c) $f(x) = \frac{1}{3}|x|^3 - |x|$
   (d) $f(x) = x^{4/5}$
   (e) $f(x) = x^{6/4}$
   (f) $f(x) = x^4 - x^2$