(CS5020, Jul-Nov 2023) Nonlinear Optimisation: Theory and Algorithms
Worksheet - 6

Convexity

$$f(\theta x + (1-\theta)y) \le \theta f(x) + (1-\theta) f(y)$$

First Order Condition:
1-dimension

$$f(y) \ge f(x) + f'(x)(y-x)$$

d-dimensions

$$f(y) \ge f(x) + \langle \nabla f(x), (y-x) \rangle$$

Second Order Condition:
1-dimension

$$f''(x) \ge 0$$

d-dimensions

$$u^\top \nabla^2 f(x) u \ge 0, \forall x, u \in \mathbb{R}^d$$

Composition: If f(x) = h(g(x)), then

$$\begin{aligned} f'(x) &= h'(g(x))g'(x) \\ f''(x) &= h''(g(x))(g'(x))^2 + h'(g(x))g''(x) \end{aligned}$$

So f(x) is convex (i.e. $f''(x) \ge 0$) if either (a) or (b) holds
(a) $h(x)$ is convex (i.e., $h''(g(x))(g'(x))^2 \ge 0$) and $h(x)$ non-decreasing and $g(x)$ is convex (i.e., $h'(g(x))g''(x) \ge 0$).
(b) $h(x)$ is convex (i.e., $h''(g(x))(g'(x))^2 \ge 0$) and $h(x)$ non-increasing and $g(x)$ is concave (i.e., $h'(g(x))g''(x) \ge 0$).

(1) Find out whether the following functions are convex, concave or neither
(a) $f(x) = e^{ax}$ (here $a \in \mathbb{R}$, state the result for various values of a)
(b) $f(x) = e^{ax^2}$ (here $a \in \mathbb{R}$, state the result for various values of a)
(c) $f(x) = e^{ax^2}$ (here $a \in \mathbb{R}$, state the result for various values of a)
(d) $f(x) = x^a, x > 0$ (here $a \in \mathbb{R}$, state the result for various values of a)
(d) $f(x) = \ln(x), x > 0$
(e) $f(x) = x\ln(x), x > 0$
(f) $f(x) = \ln(1 + \exp(x)), x > 0$

(2) Find out whether the following functions are convex, concave or neither
(a) $f(x(1), x(2)) = \max\{x(1), x(2)\}$
(b) $f(x(1), x(2)) = \sqrt{x(1)x(2)}$
(c) $f(x(1), x(2)) = \frac{x(1)^2}{x(2)}, x(2) > 0$

Note: The above problems are from Chapter 3 of S. Boyd and L. Vandenberghe, Convex Optimization.

(3) Let $f : \mathbb{R} \to \mathbb{R}$ be a convex function. We only know the following things about f:

$$f(0) = 3, f(4) = 1 \text{ and } f'(0) = -1$$

(a) What are the smallest and largest values that $f(2)$ can take?
(b) What are the smallest and largest values that $f'(2)$ can take?
(c) Give a candidate convex function f satisfying $f(0) = 3, f(4) = 1$ and $f'(0) = -1$.

(4) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be convex. We only know the following things about f:

$$f(0,0) = 0, \quad f(1,0) = 3, \quad f(0,1) = 5, \quad f(1,1) = 2.$$

(a) What are the smallest and largest values that $f(\frac{1}{2}, \frac{1}{2})$ can take?
(b) Give a candidate convex function f satisfying $f(0,0) = 0, f(1,0) = 3, f(0,1) = 5, f(1,1) = 2$.

(5) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be convex. We only know the following things about f:

$$\nabla f(0,0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

and f is differentiable at $(0,0)$, $(1,0)$ and $(0,1)$.
(a) Give the set of values the gradient of f can take at $(1,0)$.
(b) Give the set of values the gradient of f can take at $(0,1)$.
(c) Give a candidate convex function f satisfying $\nabla f(0,0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \nabla f(1,0) = \begin{bmatrix} 2 \\ 4 \end{bmatrix}, \nabla f(0,1) = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$.

(6) Find the first order approximation to $f(x) = x^2$ at $x = 0$, $x = 1$, $x = -1$, which are denoted by f_1, f_2 and f_3 respectively. Sketch the function $f(x) = x^2$ and its approximation $\hat{f}(x) = \max f_1(x), f_2(x), f_3(x)$ obtained by taking point-wise maximum of f_1, f_2, f_3.