Worksheet on "Background on calculus/optimization, Density Estimation"

PRML – CS5691 (Jul–Nov 2023)

September 11, 2023

1. a Find the linear approximation of $f(x) = \sqrt{x}$ at $x = 16$
   b Use it to approximate $\sqrt{15.9}$
2. Find the tangent plane to $f(x, y) = 2 - x^2 - y^2$ at $(\frac{1}{2}, -\frac{1}{2})$
3. Prove if the statement if true [or] Provide counter-example if the statement is false:
   a Sum of two convex functions is a convex function
   b Product of two convex functions is a convex function
   c Difference of two convex functions is a convex function
4. Suppose that a particular gene occurs as one of two alleles (A and a), where allele A has frequency θ in the population. That is, a random copy of the gene is A with probability θ and a with probability $1 - \theta$. Since a diploid genotype consists of two genes, the probability of each genotype is given by:

Genotype	Probability
AA	θ^2
Aa	$2\theta(1-\theta)$
aa	$(1-\theta)^2$

   Suppose we test a random sample of people and find that k1 are AA, k2 are Aa, and k3 are aa. Find the MLE of θ.
5. a Complete the derivation of MLE of Bernoulli Distribution seen in class
   b Similarly complete the derviation of MLE of Multinoulli Distribution.

   Hint: You can use log likelihood LL seen in class and follow the below steps:
   i Compute the gradient of log likelihood LL
   ii Equate it to zero to find the stationary points
   iii Argue the stationary point is global maxima - e.g., by verifying if the LL is concave
6. Prove that $\frac{\partial}{\partial x}(x^T A\ x) = A^T x + Ax$ (or $2Ax$ if A is Symmetric)

   (Hint: $x^T A\ x = \sum_{j=1}^{n} \sum_{i=1}^{n} a_{ij} x_i x_j$)
7. a Prove that $f(x) = x_1 \cdot x_2$ is not convex.
   b Prove that $f(x) = {x_1}^2 + {x_2}^2$ is not convex.

   Hint: Use the property, f is convext iff $H(x)$ is positive semidefinite $\forall x \in \mathbb{R}^d$