Chapter 2

Vector Spaces and Subspaces

2.1 Abelian Group

The notion of an Abelian group is obtained by generalizing the algebraic structure we get from the addition operation $+$ on $\mathbb{R}$. We observe the following:

1. $+ : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$, that is $+$ is a **binary operation** on $\mathbb{R}$
2. $(x + y) + z = x + (y + z)$ for all $x, y, z \in \mathbb{R}$, that is $+$ is **Associative**
3. There exists $0 \in \mathbb{R}$ such that $x + 0 = x = 0 + x$ for all $x \in \mathbb{R}$, that is, there exists an **additive identity**
4. For every $x \in \mathbb{R}$ there exists $(-x) \in \mathbb{R}$ such that
$$x + (-x) = 0 = (-x) + x \ ,$$
that is, every element has an **additive inverse**.
5. $x + y = y + x$ for all $x \in \mathbb{R}$, that is $+$ is **commutative**

We now generalize this to the definition of an Abelian Group.

Definition 2.1.1 Let $\mathbb{G}$ be a nonempty set. Let $+$ be a rule of combining an ordered pair of elements (g_1, g_2) where $g_1, g_2 \in \mathbb{G}$, such that

1. $+ : \mathbb{G} \times \mathbb{G} \longrightarrow \mathbb{G}$, that is $+$ is a **binary operation** on $\mathbb{G}$

2. $(x + y) + z = x + (y + z)$ for all $x, y, z \in \mathbb{G}$, that is $+$ is **Associative**

3. There exists an element, which we denote by $\theta_{\mathbb{G}}$, in $\mathbb{G}$ such that $x + \theta_{\mathbb{G}} = x = \theta_{\mathbb{G}} + x$ for all $x \in \mathbb{R}$, that is, there exists an **additive identity**

4. For every $x \in \mathbb{G}$ there exists an element which we denote by $(-x)$ in $\mathbb{G}$ such that $x + (-x) = \theta_{\mathbb{G}} = (-x) + x$, that is every element has an **additive inverse**

5. $x + y = y + x$ for all $x \in \mathbb{G}$, that is $+$ is **commutative**

Then we say that $(\mathbb{G}, +)$ is an Abelian Group

We give below some examples:

Example 2.1.1 We first give below some simple examples from the standard number systems:

1. $(\mathbb{Q}, +)$ is an Abelian Group
2. $(\mathbb{Q} \setminus \{0\}, \times)$ is an Abelian Group
3. $(\mathbb{R}, +)$ is an Abelian Group
4. $(\mathbb{R} \setminus \{0\}, \times)$ is an Abelian Group
5. $(\mathbb{C}, +)$ is an Abelian Group
6. $(\mathbb{C} \setminus \{0\}, \times)$ is an Abelian Group

Example 2.1.2 Let p be any prime number and let $\mathbb{Z}_p = \{0, 1, 2, \cdots, (p-1)\}$ and let $\oplus_p$ denote addition modulo p and $\otimes_p$ denote multiplication modulo p. Then

1. $(\mathbb{Z}_p, \oplus_p)$ is an Abelian group
2. $(\mathbb{Z}_p \setminus \{0\}, \otimes_p)$ is an Abelian group

Example 2.1.3

Let $\mathbb{R}[x]$ denote the set of all polynomials in one variable x with coefficients over $\mathbb{R}$. A polynomial $p(x) \in \mathbb{R}[x]$ is said to be irreducible over $\mathbb{R}$ if $p(x)$ cannot be written as the product of two polynomials in $\mathbb{R}[x]$ each of which has lower degree than $p(x)$. For example consider $p(x) = x^2 + 1$. This is clearly a irreducible polynomial over $\mathbb{R}$ and has degree two. Now consider all polynomials of degree less than two over $\mathbb{R}$, namely,

$$\{ax + b : a, b \in \mathbb{R}\}$$

These are all precisely the possible resmainders that we get when we divide any polynomial in $\mathbb{R}[x]$ by $p(x) = x^2 + 1$. We denote this collection by

$$\mathbb{R}[x]/\sim p(x)$$

We define addition on this set as addition modulo $p(x)$, and denote this by $\oplus_p$. We can easily check that $(\mathbb{R}[x]/\sim p(x), \oplus_p)$ is an Abelian group. The additive identity of this group is the zero polynomial Θ_p and the additive inverse of any $q(x) = ax + b$ in $\mathbb{R}[x]/\sim p(x)$ is $(-q(x)) = (-a)x + (-b)$
Now consider $\mathbb{R}[x]/\sim p(x) \setminus \{\Theta_p\}$. We define $\otimes_p$ as multiplication modulo $p(x)$. Then $(\mathbb{R}[x]/\sim p(x) \setminus \{\Theta_p\}, \otimes_p)$ is an Abelian group. The identity with respect to $\otimes_p$ is the constant polynomial 1. The inverse of any polynomial $q(x) = ax + b$ in $(\mathbb{R}[x]/\sim p(x) \setminus \{\Theta_p\}, \otimes_p)$ is given by

$$q^{-1}(x) \stackrel{def}{=} -\frac{a}{a^2+b^2}x + \frac{b}{a^2+b^2}$$

(Note that for any polynomial $q(x) = ax + b$ in $(\mathbb{R}[x]/\sim p(x) \setminus \{0\}, \otimes_p)$, we have $a^2 + b^2 \neq 0$).
In general, if $p(x)$ is any irreducible polynomial in $\mathbb{R}[x]$ then we define $\mathbb{R}[x]/\sim p(x)$ to be the set of all polynomials of degree less than that of $p(x)$ and define $\oplus_p$ as addition modulo $p(x)$ and $\otimes_p$ as multiplication modulo $p(x)$. Then,

1. $(\mathbb{R}[x]/\sim p(x), \oplus_p)$ is an Abelian group
2. $(\mathbb{R}[x]/\sim p(x) \setminus \{\Theta_p\}, \otimes_p)$ is an Abelian group

We next look at the notion of a Field. By generalizing the algebraic structure we get on $\mathbb{R}$ with the two operations of addition and multiplication together we get the notion of a field. We have,

Definition 2.1.2 Let $\mathbb{F}$ be a nonempty set and $+$ and $\times$ be binary operations on $\mathbb{F}$ such that

1) $(\mathbb{F}, +)$ is an Abelian Group, (whose additive identity we denote by 0),
2) $(\mathbb{F} \setminus \{0\}, \times)$ is an Abelian group, (the identity with respect to $\times$ is denoted by 1),

3) $$\left.\begin{array}{l} \alpha \times (\beta + \gamma) = (\alpha \times \beta) + (\alpha \times \gamma) \\ (\beta + \gamma) \times \alpha = (\beta \times \alpha) + (\gamma \times \alpha) \end{array}\right\} \text{(Distributive Laws)}$$

Then $(\mathbb{F}, +, \times)$ is called a FIELD

($+$ is called the addition operation of the field and $\times$ is called the multiplication operation of the field).

Example 2.1.4 From the examples discussed above it follows that the following are fields:

1. $(\mathbb{Q}, +, \times)$
2. $(\mathbb{R}, +, \times)$
3. $(\mathbb{C}, +, \times)$
4. $(\mathbb{Z}_p, \oplus_p, \otimes_p)$ where p is a prime number
5. $(\mathbb{R}[x]/\sim p(x), \oplus_p, \otimes_p)$ where $p(x)$ is an irreducible polynomial over $\mathbb{R}$

Example 2.1.5 Let $\mathbb{F}$ be any field. Analogous to $\mathbb{R}[x]$ we can consider $\mathbb{F}[x]$ all polynomials in one variable x with coefficients from $\mathbb{F}$. A polynomial $p(x) \in \mathbb{F}[x]$ is said to be irreducible over $\mathbb{F}$ if it cannot be written as the product of two polynomials in $\mathbb{F}[x]$ each of which has degree lower than that of $p(x)$. Then we denote by $\mathbb{F}[x]/\sim p(x)$ the set of all those polynomials in $\mathbb{F}[x]$ whose degree is less than that of $p(x)$. The operations $\oplus_p$ and $\otimes_p$ are defined, as before, as addition modulo p and multiplication modulo p respectively. Then

$$(\mathbb{F}[x]/\sim p(x), \oplus_p, \otimes_p)$$

is a field. For example consider $\mathbb{F} = \mathbb{Z}_2$. Then $p(x) = x^2 + x + 1$ is an irreducible polynomial of degree two in $\mathbb{F}[x]$. Hence we have that $\mathbb{F}[x]/\sim p(x)$ consists of polynomials over $\mathbb{F}$ of degree less than two. Since the coefficients have to be in $\mathbb{F}$ there are only four such polynomials, namely,

$$p_0(x) = 0, p_1(x) = 1, p_2(x) = x, \text{ and } p_3(x) = x + 1$$

The addition and multiplication of this field are given below:

$\oplus_p$	$p_0(x) = 0$	$p_1(x) = 1$	$p_2(x) = x$	$p_3(x) = x + 1$
$p_0(x) = 0$	$p_0(x)$	$p_1(x)$	$p_2(x)$	$p_3(x)$
$p_1(x) = 1$	$p_1(x)$	$p_0(x)$	$p_3(x)$	$p_2(x)$
$p_2(x) = x$	$p_2(x)$	$p_3(x)$	$p_0(x)$	$p_1(x)$
$p_3(x) = x + 1$	$p_3(x)$	$p_2(x)$	$p_1(x)$	$p_0(x)$

Each element is its own additive inverse. The multiplication table is given below:

$\otimes_p$	$p_0(x) = 0$	$p_1(x) = 1$	$p_2(x) = x$	$p_3(x) = x + 1$
$p_0(x) = 0$	$p_0(x)$	$p_0(x)$	$p_0(x)$	$p_0(x)$
$p_1(x) = 1$	$p_0(x)$	$p_1(x)$	$p_2(x)$	$p_3(x)$
$p_2(x) = x$	$p_0(x)$	$p_2(x)$	$p_3(x)$	$p_1(x)$
$p_3(x) = x + 1$	$p_0(x)$	$p_3(x)$	$p_1(x)$	$p_2(x)$

The multiplicative inverses of the nonzero elements are as follows:

$$\begin{aligned} p_1^{-1} &= p_1 \\ p_2^{-1} &= p_3 \text{ and} \\ p_3^{-1} &= p_2 \end{aligned}$$

2.2 Structure of $\mathbb{R}^k$

Let $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$. While analysing the linear system $Ax = b$ we find that we have to deal with two vectors, namely the known vector b given in $\mathbb{R}^m$ and the unknown vector x to be found in $\mathbb{R}^n$. So, in general, we shall consider a positive integer k and look at the structure of $\mathbb{R}^k$, the collection of all $k \times 1$ column vectors.

$$\mathbb{R}^k = \left\{ x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \end{pmatrix} : x_j \in \mathbb{R},\ 1 \le j \le k \right\} \tag{2.2.1}$$

We first consider the simple operation of "Addition" on $\mathbb{R}^k$defined as follows: For $x, y \in \mathbb{R}^k$ we define

$$x + y \quad = \quad \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_k + y_k \end{pmatrix} \tag{2.2.2}$$

We easily see that addition has the following properties:

1. $x, y \in \mathbb{R}^k \Longrightarrow x + y \in \mathbb{R}^k$

2. $x + y = y + x$ for all $x, y \in \mathbb{R}^k$

3. $(x + y) + z = x + (y + z)$ for all $x, y, z \in \mathbb{R}^k$

4. The vector

$$\theta_k = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}_{k \times 1}$$

is in $\mathbb{R}^k$ and is such that

$$x + \theta_k = x = \theta_k + x \text{ for all } x \in \mathbb{R}^k$$

5. For every $x \in \mathbb{R}^k$ the vector

$$(-x) = \begin{pmatrix} -x_1 \\ -x_2 \\ \vdots \\ -x_k \end{pmatrix} \text{ is in } \mathbb{R}^k \text{ and is such that}$$

$$x + (-x) = \theta_k = (-x) + x$$

We next consider the operation of multiplying a vector in $\mathbb{R}^k$ by a scalar $\alpha \in \mathbb{R}$, defined as follows:

$$\alpha \cdot \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \end{pmatrix} = \begin{pmatrix} \alpha x_1 \\ \alpha x_2 \\ \vdots \\ \alpha x_k \end{pmatrix} \tag{2.2.3}$$

It is easy to see that this operation, called scalar multiplication, has the following properties:

1. $\alpha \in \mathbb{R}$ and $x \in \mathbb{R}^k \Longrightarrow \alpha \cdot x \in \mathbb{R}^k$
2. $\alpha \in \mathbb{R}$ and $x, y \in \mathbb{R}^k \Longrightarrow \alpha \cdot (x + y) = (\alpha \cdot x) + (\alpha \cdot y)$
3. $\alpha,\ \beta \in \mathbb{R}$ and $x \in \mathbb{R}^k \Longrightarrow (\alpha + \beta) \cdot x = (\alpha \cdot x) + (\beta \cdot x)$
4. $\alpha,\ \beta \in \mathbb{R}$ and $x \in \mathbb{R}^k \Longrightarrow (\alpha\beta) \cdot x = \alpha \cdot (\beta \cdot x)$
5. $1 \cdot x = x$ for all $x \in \mathbb{R}^k$

2.3 Vector Space

The important ingredients we discussed above in $\mathbb{R}^k$ are the two basic operations of addition and scalar multiplication and their properties. Any system which has such a structure is called a "Vector Space". More precisely we have the following definition of a vector space over $\mathbb{R}$.

Definition 2.3.1 Let $\mathcal{V}$ be any nonempty set and let $+$ be a binary opertaion on V and $\cdot$ be a rule of combining an element of $\mathbb{R}$ and an element of $\mathcal{V}$, (called "scalar multiplication") such that the following properties are satisfied:
(Axioms for the operation + on $\mathcal{V}$)

1. $x, y \in \mathcal{V} \Longrightarrow x + y \in \mathcal{V}$

2. $x + y = y + x$ for all $x, y \in \mathcal{V}$

3. $(x + y) + z = x + (y + z)$ for all $x, y, z \in \mathcal{V}$

4. There exists a vector $\theta_{\mathcal{V}} \in \mathcal{V}$ such that
$$x + \theta_{\mathcal{V}} = x = \theta_{\mathcal{V}} + x \text{ for all } x \in \mathcal{V}$$
5. For every $x \in \mathcal{V}$ there exists a vector in $\mathcal{V}$, which we denote by $(-x)$, such that
$$x + (-x) = \theta_{\mathcal{V}} = (-x) + x$$

<u>**(Axioms for scalar multiplication)**</u>

6. $x \in \mathcal{V}$ and $\alpha \in \mathbb{R} \Longrightarrow \alpha \cdot x \in \mathcal{V}$
7. $\alpha \cdot (x + y) = \alpha \cdot x + \alpha \cdot y$ for all $\alpha \in \mathbb{R}$ and for all $x, y \in \mathcal{V}$
8. $(\alpha + \beta) \cdot x = \alpha \cdot x + \beta \cdot x$ for all $\alpha, \beta \in \mathbb{R}$ and for all $x \in \mathcal{V}$
9. $(\alpha\beta) \cdot x = \alpha \cdot (\beta \cdot x)$ for all $\alpha, \beta \in \mathbb{R}$ and for all $x \in \mathcal{V}$
10. $1x = x$ for all $x \in \mathcal{V}$

Then we say that $\mathcal{V}$ is **vector space over** $\mathbb{R}$ (with these operations of addition and scalar multiplication)

Remark 2.3.1 In the above definition, if we replace $\mathbb{R}$ by any field $\mathbb{F}$, then we get the notion of a vector space over $\mathbb{F}$. We shall be basically concerned about vector spaces over $\mathbb{R}$ and vector spaces over $\mathbb{C}$.

Remark 2.3.2 From the above axioms the following properties in a vector space $\mathcal{V}$ follow:

1. For any $x \in \mathcal{V}$ we have $0x = \theta_{\mathcal{V}}$.
   This can be seen using the axioms of $+$ on $\mathcal{V}$ and those of scalar multiplication, as follows:
$$\begin{aligned} 0x &= (0+0)x \\ &= 0x + 0x \end{aligned}$$

$$
\begin{aligned}
&\Longrightarrow \\
(-0x) + 0x &= (-0x) + (0x + 0x) \\
&\Longrightarrow \\
\theta_{\mathcal{V}} &= (-0x + 0x) + (0x) \\
&\Longrightarrow \\
\theta_{\mathcal{V}} &= \theta_{\mathcal{V}} + 0x \\
&\Longrightarrow \\
\theta_{\mathcal{V}} &= 0x
\end{aligned}
$$

2. Similarly we can show that

$$(-1)x = (-x) \text{ for all } x \in \mathcal{V}$$

Remark 2.3.3 We shall call elements of a vector space as vectors.
The structure of a vector space comes from the two basic operations of addition and scalar multiplication, and whenever we introduce a transformation on a vector space, we have to keep track of the effect of the transformation on these two basic operations.

2.4 Examples of Vector Spaces

We shall now look at some examples of vector spaces:

1. Clearly $\mathbb{R}^k$ is a vector space over $\mathbb{R}$

2. Similarly $\mathbb{C}^k$ is a vector space over $\mathbb{R}$

3. $\mathbb{C}^k$ is also vector space over $\mathbb{C}$

4. $\mathbb{R}^{m\times n}$, the collection of all $m \times n$ real matrices, is a vector space over $\mathbb{R}$ with the usual laws of addition and scalar multiplication of matrices

5. In particular, $\mathbb{R}^{n\times n}$, the set of all $n \times n$ real square matrices, is a vector space over $\mathbb{R}$

6. Similarly, $\mathbb{C}^{m\times n}$, the set of all $m \times n$ complex matrices, is a vector space over $\mathbb{R}$ as well as over $\mathbb{C}$, and the set $\mathbb{C}^{n\times n}$, the set of all $n \times n$ complex square matrices is a vector space over $\mathbb{R}$ as well as over $\mathbb{C}$

7. Let $\mathcal{I}$ any interval on the real line. Then $\mathcal{V} = F_{\mathbb{R}}[\mathcal{I}]$, the collection of all real valued functions over $\mathcal{I}$, is a vector space over $\mathbb{R}$ with the usual laws of addition and scalar multiplication of functions.

8. Similarly, $\mathcal{V} = \mathcal{C}_{\mathbb{R}}[\mathcal{I}]$, the collection of all real valued continuous functions over $\mathcal{I}$, is a vector space over $\mathbb{R}$

9. $\mathcal{V} = L^2_{\mathbb{R}}[I]$, the collection of all real valued functions over $\mathcal{I}$, for which the integral $\int_{\mathcal{I}} |f(t)|^2 \, dt$ is defined and $< \infty$, is a vector space over $\mathbb{R}$

10. $\mathcal{V} = L^1_{\mathbb{R}}[I]$, the collection of all real valued functions over $\mathcal{I}$, for which the integral $\int_{\mathcal{I}} |f(t)| \, dt$ is defined and $< \infty$, is a vector space over $\mathbb{R}$

11. $\mathcal{V} = F_{\mathbb{C}}[\mathcal{I}]$, the collection of all complex valued functions over $\mathcal{I}$, is a vector space over $\mathbb{R}$ as well as over $\mathbb{C}$, with the usual laws of addition and scalar multiplication of functions.

12. Similarly, $\mathcal{V} = \mathcal{C}_{\mathbb{C}}[\mathcal{I}]$, the collection of all complex valued continuous functions over $\mathcal{I}$, is a vector space over $\mathbb{R}$

13. $\mathcal{V} = L^2_{\mathbb{C}}[I]$, the collection of all complex valued functions over $\mathcal{I}$, for which the integral $\int_{\mathcal{I}} |f(t)|^2 \, dt$ is defined and $< \infty$, is a vector space over $\mathbb{R}$ as well as over $\mathbb{C}$

14. $\mathcal{V} = L^1_{\mathbb{C}}[I]$, the collection of all complex valued functions over $\mathcal{I}$, for which the integral $\int_{\mathcal{I}} |f(t)| \, dt$ is defined and $< \infty$, is a vector space over

$\mathbb{R}$ as well as over $\mathbb{C}$

15. Let $\mathbb{F}$ be the field $\mathcal{Z}_2$. Then

$$\mathcal{V} = \mathcal{Z}_2^k = \left\{ x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \end{pmatrix} : x_j \in \mathcal{Z}_2 \text{ for } 1 \leq j \leq k \right\}$$

is a vector space over $\mathcal{Z}_2$ with the usual laws of addition and scalar multiplication.

16. Similarly if p is any prime number then $\mathcal{Z}_p^k$ is a vector space over $\mathcal{Z}_p$

17. In general, if $\mathbb{F}$ is any field then $\mathbb{F}^k$ is a vector space over $\mathbb{F}$.

2.5 Subspace

Let us consider a plane $\mathcal{P}$ passing through the origin in the vector space $\mathbb{R}^3$. The set of all vectors in $\mathcal{P}$ clearly form a nonempty subset of $\mathcal{P}$. It is easy to see that vectors in this subset have the following simple properties:

1. The "resultant" of two vectors in $\mathcal{P}$ is also in $\mathcal{P}$, that is,

$$x, y \in \mathcal{P} \Longrightarrow x + y \in \mathcal{P}$$

2. If we "scale" any vector in $\mathcal{P}$ by a factor α the resultant is also in $\mathcal{P}$, that is,

$$\alpha \in \mathbb{R} \text{ and } x \in \mathcal{P} \Longrightarrow \alpha x \in \mathcal{P}$$

The above two properties imply that the set $\mathcal{P}$ is self contained (or "closed") with respect to the two basic operations in the vector space, namely, addition and scalar multiplication. Analogously, if we consider a line $\mathcal{L}$, passing through the origin, in the vector space $\mathbb{R}^3$, it is easy to see that $\mathcal{L}$ is also closed with respect to the two basic operations of addition and scalar multiplication. We now generalize this property, of a subset being closed with respect to the two basic operations of the vector space, to get the notion of a subspace.

Definition 2.5.1 Let $\mathcal{V}$ be a vector space over a field $\mathbb{F}$. A nonempty subset $\mathcal{W}$ is said to be a **subspace** of $\mathcal{V}$ if

$$x, y \in \mathcal{W} \implies x + y \in \mathcal{W} \tag{2.5.1}$$

$$x \in \mathcal{W} \text{ and } \alpha \in \mathbb{F} \implies \alpha x \in \mathcal{W} \tag{2.5.2}$$

We now make some simple observations:

Remark 2.5.1 What the above definition says is that when we perform the two basic operations of the vector space with the $\mathcal{W}$ vectors then the resultant vectors are also $\mathcal{W}$ vectors. The main idea in analysing these vector spaces is to break the vector space into smaller subspaces, in a suitable manner, and analyse the problem in each subspace and then put all these together to get to the final analysis on the whole space.

Remark 2.5.2 We have

$$\begin{aligned} \mathcal{W} \text{ is a subspace} &\implies \mathcal{W} \text{ is nonempty} \\ &\implies \exists w \in \mathcal{W} \\ &\implies 0w \in \mathcal{W} \\ &\implies \theta_{\mathcal{V}} \in \mathcal{W} \end{aligned}$$

Thus we see that the zero vector belongs to every subspace

Remark 2.5.3 $\mathcal{V}$ is itself a subspace of $\mathcal{V}$

Remark 2.5.4 $\mathcal{V}$ is the "largest" subspace of $\mathcal{V}$ and $\mathcal{W} = \{\theta_{\mathcal{V}}\}$ is the smallest subspace of $\mathcal{V}$

Remark 2.5.5 Let $\mathcal{W}$ be a subspace of a vector space $\mathcal{V}$ over a field $\mathbb{F}$. Then by the requirements (3.4.1) and (3.4.2) for $\mathcal{W}$ to be a subspace, it follows that, the addition and scalar multiplication in $\mathcal{V}$ induce an addition and scalar multiplication on $\mathcal{W}$ and that $\mathcal{W}$ is itself a vector space over $\mathbb{F}$ with these operations. Thus every subspace $\mathcal{W}$ of $\mathcal{V}$ is a vector space inside the vector space $\mathcal{V}$, and the basic operations on $\mathcal{W}$ being the same as those in $\mathcal{V}$, except that they are now restricted only to the vectors in $\mathcal{W}$.

Example 2.5.1 Consider $\mathcal{V} = \mathbb{R}^3$.

1. Let $\mathcal{W}$ be the subset of $\mathbb{R}^3$ defined as

$$\mathcal{W} = \left\{ x = \begin{pmatrix} x_1 \\ x_2 \\ x_1 + x_2 \end{pmatrix} : x_1, x_2 \in \mathbb{R} \right\}$$

Geometrically speaking, this is the plane $z = x + y$ in $\mathbb{R}^3$. It is easy to verify that this is a subspace of $\mathbb{R}^3$. We verify this fact as follows:

(a) Since

$$\theta_3 = \begin{pmatrix} 0 \\ 0 \\ 0 + 0 \end{pmatrix}$$

it follows that $\theta_3 \in \mathcal{W}$ and hence $\mathcal{W}$ is nonempty

(b) We have

$$\begin{aligned} x, y \in \mathcal{W} \quad &\Longrightarrow \quad x = \begin{pmatrix} x_1 \\ x_2 \\ x_1 + x_2 \end{pmatrix}, \; y = \begin{pmatrix} y_1 \\ y_2 \\ y_1 + y_2 \end{pmatrix}, \; x_j, y_j \in \mathbb{R}, \; 1 \le j \le 2 \\ &\Longrightarrow \quad x + y = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ (x_1 + y_1) + (x_2 + y_2) \end{pmatrix} \\ & \quad = \begin{pmatrix} \alpha \\ \beta \\ \alpha + \beta \end{pmatrix} \quad \text{where } \alpha = x_1 + x_2, \; \beta = y_1 + y_2 \in \mathbb{R} \\ &\Longrightarrow \quad x + y \in \mathcal{W} \end{aligned}$$

(c) Further,

$$\begin{aligned} x \in \mathcal{W}, \alpha \in \mathbb{R} \quad &\Longrightarrow \quad x = \begin{pmatrix} x_1 \\ x_2 \\ x_1 + x_2 \end{pmatrix}, \alpha \in \mathbb{R} \\ &\Longrightarrow \quad \alpha x = \begin{pmatrix} \alpha x_1 \\ \alpha x_2 \\ \alpha(x_1 + x_2) \end{pmatrix} \\ & \quad = \begin{pmatrix} a \\ b \\ a + b \end{pmatrix} \quad \text{where } a = \alpha x_1, \; b = \alpha x_2 \in \mathbb{R} \\ &\Longrightarrow \quad \alpha x \in \mathcal{W} \end{aligned}$$

Thus we see that $\mathcal{W}$ is nonempty and is closed with respect to addition and scalar multiplication and hence $\mathcal{W}$ is a subspace of $\mathbb{R}^3$

2. Let $\mathcal{W}$ be defined as

$$\mathcal{W} = \left\{ x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} : 2x_1 - 3x_2 + x_3 = 0 \text{ and } 3x_1 - 4x_2 - x_3 = 0; x_1, x_2, x_3 \in \mathbb{R} \right\}$$

Once again it is easy to verify that this is a subspace of $\mathbb{R}^3$. Geometrically speaking, this subspace is the line of intersection of the two planes $2x - 3y + z = 0$ and $3x - 4y - z = 0$

Remark 2.5.6 In general, a subspace in $\mathbb{R}^3$ will be either $\mathbb{R}^3$ or $\{\theta_3\}$, or a plane through the origin or a line through the origin.

Example 2.5.2 Consider the vector space $\mathbb{R}^k$ (where we assume $k \geq 2$). Then we can easily verify that the following subsets are subspaces of $\mathbb{R}^n$:

1. $\mathcal{W} = \left\{ x \in \mathbb{R}^k : x_1 = 0 \right\}$
2. $\mathcal{W} = \left\{ x \in \mathbb{R}^k : x_k = 3x_1 \right\}$

Example 2.5.3 Consider the vector space $\mathbb{R}^{4\times 4}$. It is easy to verify that the following subsets are subspaces of $\mathbb{R}^{4\times 4}$:

1. $\mathcal{W} = \left\{ A \in \mathbb{R}^{4\times 4} : a_{23} + a_{32} = 0 \right\}$
2. $\mathcal{W} = \left\{ A \in \mathbb{R}^{4\times 4} : a_{ij} = a_{ji} \text{ for all } 1 \leq i, j \leq 4 \right\}$
3. $\mathcal{W} = \left\{ A \in \mathbb{R}^{4\times 4} : a_{ij} = -a_{ji} \text{ for all } 1 \leq i, j \leq 4 \right\}$
4. $\mathcal{W} = \left\{ A \in \mathbb{R}^{4\times 4} : Trace(A) = 0 \right\}$

However the following subset is NOT a subspace of $\mathbb{R}^{4\times 4}$ (Why?):

$$\mathcal{W} = \left\{ A \in \mathbb{R}^{4\times 4} : Trace(A) = 1 \right\}$$

Example 2.5.4 Let us consider the vector space $\mathcal{V} = \mathcal{C}_{\mathbb{R}}[0,1]$, of all real valued, continuous functions, defined on the interval $[0,1]$. We can easily verify that the following subsets are subspaces:

1. $\mathcal{W} = \{f \in \mathcal{V} : f(0.5) = 0\}$
2. $\mathcal{W} = \{f \in \mathcal{V} : f(0.5) = 3f(1) + 4f(0)\}$
3. $\mathcal{W} = \left\{f \in \mathcal{V} : \int_0^1 f(t)dt = 0\right\}$
4. $\mathcal{W} = \left\{f \in \mathcal{V} : \int_0^1 f(t)e^{-t}dt = 0\right\}$

2.6 Putting Subspaces Together

We shall now see how to combine subspaces to form a new subspace.
Let $\mathcal{V}$ be a vector space over a field $\mathbb{F}$ and let $\mathcal{W}_1$ and $\mathcal{W}_2$ be subspaces. The first question is whether their union $\mathcal{W} = \mathcal{W}_1 \cup \mathcal{W}_2$ will also be a subspace. We shall now look at an example to show that this may or may not be the case in general.

Example 2.6.1 Consider the vector space $\mathbb{R}^3$. Let $\mathcal{W}_1$ and $\mathcal{W}_2$ be subspaces defined as follows:

$$\mathcal{W}_1 = \left\{x = \begin{pmatrix} x_1 \\ 0 \\ 0 \end{pmatrix} : x_1 \in \mathbb{R}\right\} \tag{2.6.1}$$

$$\mathcal{W}_2 = \left\{x = \begin{pmatrix} 0 \\ x_2 \\ 0 \end{pmatrix} : x_2 \in \mathbb{R}\right\} \tag{2.6.2}$$

We see that the the vector $u = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ is in $\mathcal{W}_1$ and the vector $v = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ is in $\mathcal{W}_2$. Hence both these vectors are in $\mathcal{W} = \mathcal{W}_1 \cup \mathcal{W}_2$. However we have

$$u + v = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

is neither in $\mathcal{W}_1$ nor in $\mathcal{W}_2$ and hence not in $\mathcal{W} = \mathcal{W}_1 \cup \mathcal{W}_2$. Thus $\mathcal{W}$ is not closed under addition and hence is not a subspace. Thus the sum of these two subspaces is not a subspace. On the other hand, consider the subspaces

$$\mathcal{W}_1 = \left\{ x = \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix} : x_1, x_2 \in \mathbb{R} \right\} \tag{2.6.3}$$

$$\mathcal{W}_2 = \left\{ x = \begin{pmatrix} 0 \\ x_2 \\ 0 \end{pmatrix} : x_2 \in \mathbb{R} \right\} \tag{2.6.4}$$

$$\tag{2.6.5}$$

Then we see that $\mathcal{W} = \mathcal{W}_1 \cup \mathcal{W}_2 = \mathcal{W}_1$ since $\mathcal{W}_2 \subset \mathcal{W}_2$. Hence $\mathcal{W}$ is a subspace since $\mathcal{W} = \mathcal{W}_2$ and $\mathcal{W}_2$ is given to be a subspace. From these two examples we see that the union of two subspaces may or may not be a subspace. It is not difficult to see that, (as suggested by the above example), **the union of two subspaces is also a subspace if and only if one of them is contained in the other**.

We shall now look at the intersection of two subspaces. If $\mathcal{W}_1$ and $\mathcal{W}_2$ are subspaces let $\mathcal{W} = \mathcal{W}_1 \cap \mathcal{W}_2$. then we have the folowing:

1. $\theta_{\mathcal{V}} \in \mathcal{W}_1$ since $\mathcal{W}_1$ is a subspace and similarly $\theta_{\mathcal{V}} \in \mathcal{W}_2$ since $\mathcal{W}_2$ is a subspace.
   $\Longrightarrow \theta_{\mathcal{V}} \in \mathcal{W}_1 \cap \mathcal{W}_2 = \mathcal{W}$. Thus $\mathcal{W}$ is nonempty

2. $u, v \in \mathcal{W} \Longrightarrow u, v \in \mathcal{W}_1$ and $u, v \in \mathcal{W}_2$
   $\Longrightarrow u + v \in \mathcal{W}_1$ and $u + v \in \mathcal{W}_2$ since $\mathcal{W}_1$ and $\mathcal{W}_2$ are subspaces
   $\Longrightarrow u + v \in \mathcal{W}_1 \cap \mathcal{W}_2 = \mathcal{W}$
   $\Longrightarrow \mathcal{W}$ is closed under addition

3. Similarly we see that,
   $\alpha \in \mathbb{F}$ and $x \in \mathcal{W} \Longrightarrow \alpha \in \mathbb{F}$ and $x \in \mathcal{W}_1$ and $\alpha \in \mathbb{F}$ and $x \in \mathcal{W}_2$
   $\Longrightarrow \alpha x \in \mathcal{W}_1$ and $\alpha x \in \mathcal{W}_2$ since $\mathcal{W}_1$ and $\mathcal{W}_2$ are subspaces
   $\Longrightarrow \alpha x \in \mathcal{W}_1 \cap \mathcal{W}_2 = \mathcal{W}$
   $\mathcal{W}$ is closed under scalar multiplication

The above three properties show that the intersection $\mathcal{W} = \mathcal{W}_1 \cap \mathcal{W}_2$ of two subspaces $\mathcal{W}_1$ and $\mathcal{W}_2$ is also a subspace.
We now look at another process of building up a new subspace out of two

subspaces. In Example 2.6.1 above we saw that the union of two subspaces need not be a subspace. The reason that the union failed to be a subspace in that example was that we could take a vector u in one of the subspaces and a vector v in the other subspace such that the resultant or the sum of two vectors was not in either. We shall now correct this defect by taking all possible sums of vectors, one from each of the subspaces. More precisely, if $\mathcal{W}_1$ and $\mathcal{W}_2$ are two subspaces we define

$$\mathcal{W}_1 + \mathcal{W}_2 \stackrel{def}{=} \{w = w_1 + w_2 : w_1 \in \mathcal{W}_1 \text{ and } w_2 \in \mathcal{W}_2\} \tag{2.6.6}$$

We shall now see that $\mathcal{W}_1 + \mathcal{W}_2$ is also a subspace. We have

1. $\theta_{\mathcal{V}} \in \mathcal{W}_1$ and $\theta_{\mathcal{V}} \in \mathcal{W}_2 \Longrightarrow \theta_{\mathcal{V}} = \theta_{\mathcal{V}} + \theta_{\mathcal{V}} \in \mathcal{W}_1 + \mathcal{W}_2$
$\Longrightarrow \mathcal{W}_1 + \mathcal{W}_2$ is nonempty

2. $x, y \in \mathcal{W}_1 + \mathcal{W}_2$
$\Longrightarrow \exists u_1, u_2 \in \mathcal{W}_1$ and $v_1, v_2 \in \mathcal{W}_2$ such that $x = u_1 + v_1$ and $y = v_1 + v_2$
$\Longrightarrow x + y = (u_1 + v_1) + (u_2 + v_2)$ where $u_1 + v_1 \in \mathcal{W}_1$ and $u_2 + v_2 \in \mathcal{W}_2$ since $\mathcal{W}_1$ and $\mathcal{W}_2$ are subspaces
$\Longrightarrow x + y \in \mathcal{W}_1 + \mathcal{W}_2$
$\Longrightarrow \mathcal{W}_1 + \mathcal{W}_2$ is closed under addition

3. Further we have
$\alpha \in \mathbb{F}$ and $x \in \mathcal{W}_1 + \mathcal{W}_2$
$\Longrightarrow \alpha \in \mathbb{F}$ and $\exists w_1 \in \mathcal{W}_1$ and $w_2 \in \mathcal{W}_2$ such that $x = w_1 + w_2$
$\Longrightarrow \alpha x = (\alpha w_1) + (\alpha w_2)$
$\Longrightarrow \alpha x \in \mathcal{W}_1 + \mathcal{W}_2$ since $\alpha w_1 \in \mathcal{W}_1$ and $\alpha w_2 \in \mathcal{W}_2$ because $\mathcal{W}_1$ and $\mathcal{W}_2$ are subspaces
$\Longrightarrow \mathcal{W}_1 + \mathcal{W}_2$ is closed under scalar multiplication

The above three properties show that **if $\mathcal{W}_1$ and $\mathcal{W}_2$ are subspaces then $\mathcal{W}_1 + \mathcal{W}_2$ is also a subspace** . It is now easy to generalize this to finite number of subspaces as follows:
If $\mathcal{W}_1, \mathcal{W}_2, \cdots, \mathcal{W}_k$ are finite number of subspaces of $\mathcal{V}$ then

$$\mathcal{W}_1 + \mathcal{W}_2 + \cdots + \mathcal{W}_k \stackrel{def}{=} \{x = w_1 + w_2 + \cdots + w_k : w_j \in \mathcal{W}_j \text{ for } 1 \leq j \leq k\} \tag{2.6.7}$$

is also a subspace

Example 2.6.2 Consider the vector space $\mathcal{V} = \mathbb{R}^4$.
a) We first consider the following two subspaces:

$$\begin{aligned} \mathcal{W}_1 &= \left\{ x = \begin{pmatrix} \alpha \\ 0 \\ \alpha \\ 0 \end{pmatrix} : \alpha \in \mathbb{R} \right\} \\ \mathcal{W}_2 &= \left\{ x = \begin{pmatrix} 0 \\ \beta \\ \beta \\ 0 \end{pmatrix} : \beta \in \mathbb{R} \right\} \end{aligned}$$

The subsspace $\mathcal{W} = \mathcal{W}_1 + \mathcal{W}_2$ consists of all those vectors $w \in \mathbb{R}^4$ which can be written in the form

$$w = \begin{pmatrix} \alpha \\ 0 \\ \alpha \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \beta \\ \beta \\ 0 \end{pmatrix}$$

for suitable values of $\alpha, \beta \in \mathbb{R}$. Thus we have

$$\mathcal{W} = \mathcal{W}_1 + \mathcal{W}_2 = \left\{ w = \begin{pmatrix} \alpha \\ \beta \\ \alpha + \beta \\ o \end{pmatrix} : \alpha, \beta \in \mathbb{R} \right\}$$

Any $x \in \mathcal{W}$ is such that its fourth component is zero and the third component is the sum of the first two components, and conversely, any such vector is in $\mathcal{W}$. Thus $\mathcal{W}$ consists precisely of all those vectors in $\mathbb{R}^4$ which are such that their fourth component is zero and the third component is the sum of the first two components. Not that, in this case, any vector in $\mathcal{W} = \mathcal{W}_1 + \mathcal{W}_2$ can be written UNIQUELY as the sum of a vctor in $\mathcal{W}_1$ and a vector in $\mathcal{W}_2$.
b) We shall next consider the following two subspaces of $\mathbb{R}^n$:

$$\mathcal{W}_1 = \left\{ x = \begin{pmatrix} \alpha \\ \beta \\ \alpha + \beta \\ 0 \end{pmatrix} : \alpha, \beta \in \mathbb{R} \right\}$$

$$\mathcal{W}_2 = \left\{ x = \begin{pmatrix} 0 \\ \gamma \\ \delta \\ 0 \end{pmatrix} : \gamma, \delta \in \mathbb{R} \right\}$$

We have $w \in \mathcal{W} = \mathcal{W}_1 + \mathcal{W}_2 \Longleftrightarrow$
$w = w_1 + w_2$ where $w_1 \in \mathcal{W}_1$ and $w_2 \in \mathcal{W}_2 \Longleftrightarrow$
There exist $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ such that $w = \begin{pmatrix} \alpha \\ \beta \\ \alpha+\beta \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \gamma \\ \delta \\ 0 \end{pmatrix} \Longrightarrow$
There exist $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ such that $w = \begin{pmatrix} \alpha \\ \beta+\gamma \\ \alpha+\beta+\delta \\ 0 \end{pmatrix}$
From the above we observe th following facts:

1. Any vector in $\mathcal{W} = \mathcal{W}_1 + \mathcal{W}_2$ has its fourth component as zer0
2. We shall see now that any vector in $\mathbb{R}^4$ that has its fourth component as zero must be in $\mathcal{W}$. To see this w must show that given any $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ 0 \end{pmatrix} \in \mathbb{R}^4$ there exists $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ such that $x = \begin{pmatrix} \alpha \\ \beta+\gamma \\ \alpha+\beta+\delta \\ 0 \end{pmatrix}$.
   We shall find such $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ given any x in $\mathbb{R}^4$ as follows:
   $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ 0 \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta+\gamma \\ \alpha+\beta+\delta \\ 0 \end{pmatrix} \Longrightarrow$

$$\begin{aligned} \alpha &= x_1 \\ \beta+\gamma &= x_2 \\ \alpha+\beta+\delta &= x_3 \end{aligned}$$

From these we get

$$\begin{aligned} \alpha &= x_1 \\ \gamma &= x_2 - \beta \end{aligned}$$

$$\begin{aligned}\delta &= x_3 - \alpha - \beta \\ &= x_3 - x_1 - \beta\end{aligned}$$

We can choose β arbitrarily. Thus we hav any $x \in \mathbb{R}^4$ whose fourth componnt is zero as

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ 0 \end{pmatrix} = \begin{pmatrix} x_1 \\ \beta \\ x_1 + \beta \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ x_2 - \beta \\ x_3 - x_1 - \beta \\ 0 \end{pmatrix} \tag{2.6.8}$$

where β can be given any value. Thus w find that

$$\mathcal{W} = \mathcal{W}_1 + \mathcal{W}_2 = \left\{ x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \in \mathbb{R}^4 : x_4 = 0 \right\}$$

and every vector in $\mathcal{W}$ has an infinite number of representations, (given by (2.6.8)) as the sum of a vector in $\mathcal{W}_1$ and a vector in $\mathcal{W}_2$. Thus in this case we could represent every vector in $\mathcal{W} = \mathcal{W}_1 + \mathcal{W}_2$ as the sum of a vector in $\mathcal{W}_1$ and a vector in $\mathcal{W}_2$ in a NONUNIQUE way

The above example leads us to the following definition:

Definition 2.6.1 If $\mathcal{W}_1$ and $\mathcal{W}_2$ are subspaces of a vector space $\mathcal{V}$ then the sum $\mathcal{W} = \mathcal{W}_1 + \mathcal{W}_2$ is said to be a **Direct Sum** if every vector in $\mathcal{W}$ has a unique representation as the sum of a vector in $\mathcal{W}_1$ and a vector in $\mathcal{W}_2$. We then write

$$\mathcal{W} = \mathcal{W}_1 \oplus \mathcal{W}_2$$

Similarly, if $\mathcal{W}_1, \mathcal{W}_2, \cdots, \mathcal{W}_k$ are finite number of subspaces of the vector space $\mathcal{V}$ we then say that th sum $\mathcal{W} = \mathcal{W}_1 + \mathcal{W}_2 + \cdots + \mathcal{W}_k$ is a Direct Sum if every vector in $\mathcal{W}$ can be represented uniquely as the sum of a vector in $\mathcal{W}_1$, a vector in $\mathcal{W}_2$, $\cdots$, and a vector in $\mathcal{W}_k$. We then write

$$\mathcal{W} = \mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \cdots \oplus \mathcal{W}_k$$

Example 2.6.3 Let I be the interval $-1 \leq t \leq 1$ in $\mathbb{R}$. Let $\mathcal{V}$ be the vector space $\mathcal{F}_{\mathbb{R}}[I]$ of all real valued functions defined on the interval I. We define

$\mathcal{F}_e$ and $\mathcal{F}_o$ respectively as the set of all even functions and odd in $\mathcal{F}_{\mathbb{R}}[I]$. We have

$$\begin{aligned} \mathcal{F}_e &= \left\{ f(t) \in \mathcal{F}_{\mathbb{R}}[I] : f(t) = f(-t) \right\} && (2.6.9) \\ \mathcal{F}_o &= \left\{ f(t) \in \mathcal{F}_{\mathbb{R}}[I] : f(t) = -f(-t) \right\} && (2.6.10) \end{aligned}$$

It is easy to verify that $\mathcal{F}_e$ and $\mathcal{F}_o$ are subspaces of $\mathcal{F}_{\mathbb{R}}[I]$. We first observe that every function f in $\mathcal{F}_{\mathbb{R}}[I]$ is in $\mathcal{F}_e + \mathcal{F}_o$. For, if $f(t) \in \mathcal{F}_{\mathbb{R}}[I]$ we define,

$$\begin{aligned} f_(t) &= \frac{f(t) + f(-t)}{2} && (2.6.11) \\ f_o(t) &= \frac{f(t) - f(-t)}{2} && (2.6.12) \end{aligned}$$

It is easy to see that $f_e(t) \in \mathcal{F}_e$ and $f_o(t) \in \mathcal{F}_o$. Further we have

$$f_e(t) + f_o(t) = f(t) \qquad (2.6.13)$$

Thus every $f(t) \in \mathcal{F}_{\mathbb{R}}[I]$ is the sum of a function in $\mathcal{F}_e$ and a function in $\mathcal{F}_o$ and hence

$$\mathcal{F}_{\mathbb{R}}[I] = \mathcal{F}_e + \mathcal{F}_o$$

We shall now see that this is a Direct Sum. For this we must show that the representation (2.6.13) is unique. We have
$f(t) = g(t) + h(t)$ where $g(t) \in \mathcal{F}_e$ and $h(t) \in \mathcal{F}_o \Longrightarrow$
$f(-t) = g(-t) + h(-t) = g(t) - h(-t)$ (since $g(t) \in \mathcal{F}_e$ and $h(t) \in \mathcal{F}_o$) $\Longrightarrow$
$g(t) = \dfrac{f(t) + f(-t)}{2} = f_e(t)$ and $h(t) = \dfrac{f(t) - f(-t)}{2} = f_o(t)$
Thus the representation is unique and we have

$$\mathcal{F}_{\mathbb{R}}[I] = \mathcal{F}_e \oplus \mathcal{F}_o$$

2.7 Four Fundamental Subspaces Associated With A Matrix

We shall now introduce four fundamental subspaces associated with a matrix. These subspaces play a pivotal role in analysing the structure of a matrix. Consider $A \in \mathbb{F}^{m \times n}$. We can think of A as a transformation from $\mathbb{F}^n$ to $\mathbb{F}^m$ which transforms a vector $x \in \mathbb{F}^n$ to the vector $Ax \in \mathbb{F}^m$. Let us denote by

$\mathcal{N}_A$ the set of all vectors $x \in \mathbb{F}^n$ which get transformed to the zero vector θ_m in $\mathbb{F}^m$. We have

$$\mathcal{N}_A = \{x \in \mathcal{F}^n : Ax = \theta_m\} \tag{2.7.1}$$

This is a subset of $\mathbb{F}^n$. Will this be a subspace of $\mathbb{F}^n$? We have,

1. $\mathcal{N}_A$ is nonempty since $\theta_n \in \mathcal{N}_A$
2. $x, y \in \mathcal{N}_A \Longrightarrow Ax = \theta_m$ and $Ay = \theta_m$
   $\Longrightarrow Ax + Ay = \theta_m$
   $\Longrightarrow A(x + y) = \theta_m$
   $\Longrightarrow x + y \in \mathcal{N}_A$
   $\Longrightarrow \mathcal{N}_A$ is closed under addition
3. Similarly we have
   $x \in \mathcal{N}_A$ and $\alpha \in \mathbb{F} \Longrightarrow Ax = \theta_m$ and $\alpha \in \mathbb{F}$
   $\Longrightarrow \alpha Ax = \theta_m$
   $\Longrightarrow A(\alpha x) = \theta_m$
   $\Longrightarrow \alpha x \in \mathcal{N}_A$
   $\Longrightarrow \mathcal{N}_A$ is closed under scalar multiplication

From the above three properties we see that $\mathcal{N}_A$ is a subspace of $\mathbb{F}^n$. This subspace is called the "**NULL SPACE**" of the matrix A. Analogously we can define the Null Space of A^T, denoted by $\mathcal{N}_{A^T}$, as

$$\mathcal{N}_{A^T} = \left\{y \in \mathbb{F}^m : A^T y = \theta_n\right\} \tag{2.7.2}$$

The null space $\mathcal{N}_{A^T}$ is a subspace of $\mathbb{F}^m$.
Let us next consider all those vectors $b \in \mathbb{F}^m$ which are the images of some $x \in \mathbb{F}^n$ under the transformation A. This means we collect all those $b \in \mathbb{F}^m$ for which there exists an $x \in \mathbb{F}^n$ such that $b = Ax$. In other words, we are collecting all those $b \in \mathbb{F}^m$ which can be written in the form Ax for some $x \in \mathbb{F}^n$. We denote this collection by $\mathcal{R}_A$. We have

$$\mathcal{R}_A = \{b \in \mathbb{F}^m : \exists x \in \mathbb{F}^n \ni Ax = b\} \tag{2.7.3}$$

This is a subset of $\mathbb{F}^m$. Will this a subspace of $\mathbb{F}^m$? We have

1. Clearly $\mathcal{R}_A$ is nonempty, since $\theta_m \in \mathcal{R}_A$, for, θ_m can be written as $A\theta_n$.

2. $b_1, b_2 \in \mathcal{R}_A \Longrightarrow \exists\ x_1,\ x_2 \in \mathbb{F}^n$ such that $Ax_1 = b_1$ and $Ax_2 = b_2$
$\Longrightarrow \exists\ x = x_1+x_2 \in \mathbb{F}^n$ such that $Ax = A(x_1+x_2) = Ax_1+Ax_2 = b_1+b_2$
$\Longrightarrow b_1 + b_2 \in \mathcal{R}_A$
$\Longrightarrow \mathcal{R}_A$ is closed under addition

3. $b \in \mathcal{R}_A$ and $\alpha \in \mathbb{F} \Longrightarrow \exists\ x \in \mathbb{F}^n$ such that $Ax = b$ and $\alpha \in \mathbb{F}$
$\Longrightarrow \exists\ u = \alpha x \in \mathbb{F}^n$ such that $Au = A(\alpha x) = \alpha Ax = \alpha b$
$\Longrightarrow \alpha b \in \mathcal{R}_A$
$\Longrightarrow \mathcal{R}_A$ is closed under scalar multiplication.
From the above three properties we see that $\mathcal{R}_A$ is a subspace of $\mathbb{F}^m$. This subspace is called the "**RANGE SPACE**" of the matrix A. Analogously we can define the Range Space of A^T, denoted by $\mathcal{R}_{A^T}$, as

$$\mathcal{R}_{A^T} \quad = \quad \left\{x \in \mathbb{F}^n : \exists y \in \mathbb{F}^m \ni A^T y = x\right\} \tag{2.7.4}$$

Thus we have the four subspaces $\mathcal{N}_A$, $\mathcal{R}_A$, $\mathcal{N}_{A^T}$ and $\mathcal{R}_{A^T}$. Of these four subspaces,

(a) $\mathcal{N}_A$ and $\mathcal{R}_{A^T}$ are subspaces of $\mathbb{F}^n$ and,
(b) $\mathcal{N}_{A^T}$ and $\mathcal{R}_A$ are subspaces of $\mathbb{F}^m$.

2.8 Subspace generated by a set of vectors

Let $\mathcal{V}$ be a vector space over a field $\mathcal{F}$, and let $x \neq \theta_{\mathcal{V}}$ be a vector in $\mathcal{V}$. Clearly, the subset,

$$\mathcal{S} = \{x\},$$

of $\mathcal{V}$, consisting of the single vector x is not a subspace. Let us now try to enclose it in a subspace. Obviously $\mathcal{V}$ is one subspace of $\mathcal{V}$ which encloses all subsets of $\mathcal{V}$, and hence $\mathcal{V}$ is, indeed, a subspace that encloses the set $\mathcal{S}$. What we want to do is to enclose $\mathcal{S}$ in as small a subspace as possible. What do we mean by this? We want to see if we can get a subspace $\mathcal{W}_{\mathcal{S}}$ of $\mathcal{V}$ which is such that,

(a) $S \subseteq \mathcal{W}_{\mathcal{S}}$, that is $\mathcal{W}_{\mathcal{S}}$ is a subspace that encloses the set $\mathcal{S}$, and
(b) If $\mathcal{W}_1$ is any other subspace that encloses $\mathcal{S}$, then $\mathcal{W}_{\mathcal{S}} \subset \mathcal{W}_1$, that is no subspace smaller than $\mathcal{W}$ encloses $\mathcal{S}$.

Let us now analyse to see whether we can get such a subspace. Suppose $\mathcal{W}$ is any subspace that encloses $\mathcal{S}$, that is, $\mathcal{S} \subseteq \mathcal{W}$. Then since $x \in \mathcal{S}$ we must have $x \in \mathcal{W}$. We have,
$\mathcal{W}$ is a subspace $\Longrightarrow$
$\mathcal{W}$ is closed under scalar multiplication $\Longrightarrow$
$\alpha x \in \mathcal{W}$ for all $\alpha \in \mathcal{F}$
Thus we have

$$\left.\begin{array}{l}\mathcal{W} \text{ subspace that encloses } \mathcal{S} \Longrightarrow \\ \mathcal{W} \text{ must contain all scalar multiples of } x\end{array}\right\} \tag{2.8.1}$$

Now consider the set $\mathcal{W}_{\mathcal{S}}$ consisting of only the scalar multiples of x, that is

$$\mathcal{W}_{\mathcal{S}} = \{\alpha x : \alpha \in \mathcal{F}\}$$

We can easily verify that $\mathcal{W}_{\mathcal{S}}$ is a subspace and encloses $\mathcal{S}$, and as observed above in (3.5.1), all vectors in $\mathcal{W}_{\mathcal{S}}$ must be in every subspace that encloses $\mathcal{S}$. Thus $\mathcal{W}_{\mathcal{S}}$ is the smallest subspace enclosing the set $\mathcal{S}$. Thus we have obtained the smallest subspace that encloses $\mathcal{S}$ in the case when $\mathcal{S} = \{x\}$, a set consisting of a single vector x. (Note that this argument works even if $x = \theta_{\mathcal{V}}$ and in this case we get $\mathcal{W}_{\mathcal{S}} = \{\theta_{\mathcal{V}}\}$).
We shall now generalise this idea to sets containing more than one vector. Let us first consider a finite set of vectors

$$\mathcal{S} = \{u_1, u_2, \cdots, u_r\}$$

Let $\mathcal{W}$ be any subspace enclosing $\mathcal{S}$. Then as before, we observe that
$\mathcal{W}$ is a subspace $\Longrightarrow$
$\mathcal{W}$ is closed under scalar multiplication $\Longrightarrow$
$\alpha u_j \in \mathcal{W}$ for all $\alpha \in \mathcal{F}$ and $1 \leq j \leq r$
Thus we have

$$\left.\begin{array}{l}\mathcal{W} \text{ subspace that encloses } \mathcal{S} \Longrightarrow \\ \mathcal{W} \text{ must contain all scalar multiples} \\ \text{of each of the vectors } u_j,\ 1 \leq j \leq r\end{array}\right\} \tag{2.8.2}$$

Further we have
$\mathcal{W}$ is a subspace $\Longrightarrow$
$\mathcal{W}$ is closed under addition $\Longrightarrow$

$\alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_r u_r \in \mathcal{W}$ for all $\alpha_j \in \mathcal{F}$
Thus we have

$$\left.\begin{array}{l} \mathcal{W} \text{ subspace that encloses } \mathcal{S} \Longrightarrow \\ \mathcal{W} \text{ must contain all vectors of the form} \\ \alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_r u_r \ \text{ where } \alpha_j \in \mathcal{F},\ 1 \leq j \leq r \end{array}\right\} \tag{2.8.3}$$

Let us now consider the set $\mathcal{W}_{\mathcal{S}}$, consisting only all the vectors of this form, that is

$$\mathcal{W}_{\mathcal{S}} = \{x = \alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_r u_r : \alpha_j \in \mathcal{F},\ 1 \leq j \leq r\} \tag{2.8.4}$$

It is now easy to verify that this is a subspace of $\mathcal{V}$, that obviously encloses the given $\mathcal{S}$, and by (3.5.3), no subspace smaller than this can enclose $\mathcal{S}$. Thus the $\mathcal{W}_{\mathcal{S}}$ defined in (3.5.4) is the smallest subspace that encloses $\mathcal{S}$. This leads us to the following definition:

Definition 2.8.1 Let $u_1, u_2, \cdots, u_r$ be a finite set of vectors in a vector space $\mathcal{V}$ over the field $\mathcal{F}$. Any vector of the form

$$\alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_r u_r \text{ where } \alpha_j \in \mathcal{F},\ 1 \leq j \leq r$$

is called a "**Linear Combination**" of the vectors $u_1, u_2, \cdots, u_r$.

Thus it follows that the smallest subspace $\mathcal{W}_{\mathcal{S}}$ that encloses a finite set of vectors $\mathcal{S}$ obtained in (3.5.4) is the collection of all linear combinations of the vectors in $\mathcal{S}$.
If $\mathcal{S}$ is an infinite set, then we can verify, (using similar ideas as above), that the set, $\mathcal{W}_{\mathcal{S}}$, of all finite linear combinations of vectors in $\mathcal{S}$ is the smallest subspace enclosing $\mathcal{S}$.
Thus given any subset $\mathcal{S}$ of $\mathcal{V}$ there is a subspace which is the smallest subspace enclosing $\mathcal{S}$ and this subspace is called the **Subspace generated by $\mathcal{S}$** or the **Subspace spanned by $\mathcal{S}$** or the **Linear Span of $\mathcal{S}$** and we shall denote this by $\mathcal{L}[\mathcal{S}]$. We have

$$\mathcal{L}[\mathcal{S}] = \left\{\begin{array}{l} \text{The set of all linear combinations} \\ \text{of vectors in } \mathcal{S} \text{ if } \mathcal{S} \text{ is a finite set.} \\ \text{The set of all finite linear combinations of} \\ \text{the vectors in } \mathcal{S} \text{ if } \mathcal{S} \text{ is an infinite set.} \end{array}\right. \tag{2.8.5}$$

Given any subspace $\mathcal{W}$ of $\mathcal{V}$, a subset $\mathcal{S}$ of $\mathcal{V}$ is said to be a "**SPANNING SET**" for $\mathcal{W}$ if $\mathcal{L}[\mathcal{S}] = \mathcal{W}$. We have $\mathcal{L}[\mathcal{W}] = \mathcal{W}$. Hence $\mathcal{W}$ itself can be thought of as a spanning set for $\mathcal{W}$. Our endeavour will be to find as small a spanning set as possible given a subspace $\mathcal{W}$. Let $\mathcal{V}$ be a vector space over $\mathbb{F}$. A finite set $\mathcal{S} = u_1, u_2, \cdots, u_r$ of vectors in $\mathcal{V}$ is a spanning set for a subspace $\mathcal{W}$ if

(a) $u_j \in \mathcal{W}$, $1 \leq j \leq r$, that is all the vectors in $\mathcal{S}$ are in $\mathcal{W}$, and

(b) every vector in $\mathcal{W}$ is a linear combination of the vectors in $\mathcal{S}$, that is every vector $w \in \mathcal{W}$ can be writtn as

$$w = \alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_r u_r$$

for suitable values of $\alpha_1, \alpha_2, \cdots, \alpha_r$ (which will depend on w) $\in \mathbb{F}$

Similarly, Let $\mathcal{V}$ be a vector space over $\mathbb{F}$. An infinite set $\mathcal{S}$ of vectors in $\mathcal{V}$ is a spanning set for a subspace $\mathcal{W}$ if

(a) $\mathcal{S} \subseteq \mathcal{W}$, that is all the vectors in $\mathcal{S}$ are in $\mathcal{W}$, and

(b) every vector in $\mathcal{W}$ is a finite linear combination of the vectors in $\mathcal{S}$, that is every vector $w \in \mathcal{W}$ can be writtn as

$$w = \alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_r u_r$$

for suitable finite number of vectors $u_1, u_2, \cdots, u_r$ (which will depend on w) and suitable values of $\alpha_1, \alpha_2, \cdots, \alpha_r \in \mathbb{F}$ (which will depend on w and the set of vectors $u_1, u_2, \cdots, u_r$).

We shall call a subspace $\mathcal{W}$ as "**FINITELY GNERATED**" if there exists a finite set $\mathcal{S}$ in $\mathcal{V}$ such that $\mathcal{L}[\mathcal{S}] = \mathcal{W}$, that is, if $\mathcal{W}$ has a finite spanning set.

Example 2.8.1 Consider $\mathcal{V} = \mathbb{R}^3$ and let

$$\mathcal{S} = u_1, u_2 \text{ where } u_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \ u_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

Then

$$\begin{aligned}\mathcal{L}[\mathcal{S}] &= \Big\{x \in \mathbb{R}^3 : x = \alpha_1 u_1 + \alpha_2 u_2,\ \text{where } \alpha_1,\ \alpha_2 \in \mathbb{R}\Big\} \\ &= \left\{x \in \mathbb{R}^3 : x = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_1 + \alpha_2 \end{pmatrix},\ \alpha_1, \alpha_2 \in \mathbb{R}\right\}\end{aligned}$$

2.9 Column and Row Subspaces Associated with a Matrix

We shall use the idea of a subspace generated by a set of vectors, introduced above, to define two important subspaces associated with a matrix. Let $A \in \mathcal{F}^{m\times n}$. Consider the column vectors,

$$C_j = \begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{pmatrix},\ j = 1, 2, \cdots, n$$

These are n vectors in $\mathcal{F}^m$. Consider the set $\mathcal{C}$ of these n vectors in $\mathcal{F}^m$

$$\mathcal{C} = C_1, C_2, \cdots, C_n$$

The set, $\mathcal{L}[\mathcal{C}]$, of all linear combinations of these column vectors, is a subspace of $\mathcal{F}^m$. This subspace is called the **COLUMN SPACE OF A**.
We denote this space by $Col(A)$.
Similarly we can define the column space of A^T and we denote this by $Col(A^T)$. This is, obviously, a subspace of $\mathcal{F}^n$.
Let us denote by R_j the transpose of the j-th row of A, that is,

$$R_j = \begin{pmatrix} a_{j1} \\ a_{j2} \\ \vdots \\ a_{jn} \end{pmatrix}$$

These are m vectors in $\mathcal{F}^n$. Let us denote the collection of these m vectors as $\mathcal{R}$. Thus

$$\mathcal{R} = \{R_j\}_{j=1}^m$$

The collection, $\mathcal{L}[\mathcal{R}]$, of all linear combinations of these vectors, is a subspace of $\mathcal{F}^n$. This subspace is called the **ROW SPACE OF *A***. We denote this subspace by $Row(A)$. Similarly we can define the row space, $Row(A^T)$, of A^T. Clearly we have

(a) $Col(A) = Row(A^T)$

(b) $Col(A^T) = Row(A)$

2.10 Connection Between the Subspaces Associated With a Matrix

We shall now look at the connection between some of the subspaces obtained above. We shall be seeing more and more connections as we proceed.

Connection Between $Col(A)$ and $\mathcal{R}_A$

We shall now see that the subspaces $Col(A)$ and $\mathcal{R}_A$ are one and the same. Recall that

$Col(A) = \mathcal{L}[\mathcal{C}]$, the subspace generated by the columns $C_1, C_2, \cdots, C_n$ of the mateix A

We have

$$
\begin{aligned}
b \in Col(A) &\iff \exists \alpha_1, \alpha_2, \cdots, \alpha_n \in \mathcal{F} \text{ such that } b = \alpha_1 C_1 + \alpha_2 C_2 \cdots + \alpha_n C_n \\
&\iff b = \alpha_1 \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} + \alpha_2 \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix} + \cdots + \alpha_n \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix} \\
&\iff b = \begin{pmatrix} \alpha_1 a_{11} + \alpha_2 a_{12} + \cdots + \alpha_n a_{1n} \\ \alpha_1 a_{21} + \alpha_2 a_{22} + \cdots + \alpha_n a_{2n} \\ \vdots \\ \alpha_1 a_{m1} + \alpha_2 a_{m2} + \cdots + \alpha_n a_{mn} \end{pmatrix} \\
&\iff b = Ax \text{ where } x = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix} \in \mathcal{F}^n
\end{aligned}
$$

$$\iff \quad b \in \mathcal{R}_{A}$$

Thus we have

$$Col(A) \quad = \quad \mathcal{R}_{A} \tag{2.10.1}$$

Analogously we have

$$Col(A^T) \quad = \quad \mathcal{R}_{A^T} \tag{2.10.2}$$

Thus the subspaces we have obtained are the following

Subspaces of $\mathbb{R}^n$	Subspaces of $\mathbb{R}^m$
$\mathcal{R}_{A^T} = Col(A^T) = Row(A)$	$\mathcal{R}_{A} = Col(A) = Row(A^T)$
$\mathcal{N}_{A}$	$\mathcal{N}_{A^T}$

Thus there are four basic subspaces associated with a matrix. Of these, two subspaces, $\mathcal{R}_{A^T}$ and $\mathcal{N}_{A}$, are subspaces of $\mathcal{F}^n$, and the other two, namely $\mathcal{R}_{A}$ and $\mathcal{N}_{A^T}$, are subspaces of $\mathcal{F}^m$. These are called the "**Four Fundamental Subspaces**" associated with a matrix $A \in \mathcal{F}^{m \times n}$