Chapter 4

Linear Independence and Basis

4.1 Finitely Generated Spaces

We shall now begin investigating the question of obtaining a spanning set of optimal size. We shall first introduce the notion of finitely generated spaces. We have,

Definition 4.1.1 Let $\mathcal{V}$ be a vector space over a field $\mathcal{F}$. A subspace $\mathcal{W}$ of $\mathcal{V}$ is said to be finitely generated if there exists a finite spanning set for $\mathcal{W}$, that is, if there exists $S \subset \mathcal{W}$ such that S is finite and $\mathcal{L}[S] = \mathcal{W}$

We illustrate this by some examples.

Example 4.1.1 Consider the vector space $\mathbb{R}^3$. Let $\mathcal{W}$ be the subspace defined as

$$\mathcal{W} = \left\{ x = \begin{pmatrix} \alpha \\ \beta \\ \alpha + \beta \end{pmatrix} : \alpha,\ \beta \in \mathbb{R} \right\}$$

Clearly the set of vectors

$$u_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix},\ u_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

form a finite spanning set for $\mathcal{W}$. Hence $\mathcal{W}$ is a finitely generated subspace.

Clearly the set of vectors

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \; e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \; e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

form a finite spanning set for $\mathbb{R}^3$ and hence the vector spsce $\mathbb{R}^3$ is itself finitely generated.

Example 4.1.2 Let $\mathcal{V}$ be the vector space, $\mathcal{F}_{\mathbb{R}}[\mathbb{R}]$ of all functions from $\mathbb{R}$ to $\mathbb{R}$. We have

$$\mathcal{F}_{\mathbb{R}}[\mathbb{R}] \;\; = \;\; \{f : \mathbb{R} \longrightarrow \mathbb{R}\}$$

Consider the subspace $\mathcal{W} = \mathbb{R}[x]$ of all polynomials in x with real coefficients. Then $\mathcal{W}$ is not finitely generated. For, suppose it is finitely generated. This would then mean that there exists a finite spanning set

$$S = p_1, p_2, \cdots, p_k$$

for $\mathcal{W}$. Let

$$d = Max.\{degree \; p_j : 1 \leq j \leq k\}$$

Since S is a spanning set for $\mathcal{W}$ we have

$$\begin{aligned} p \in \mathcal{W} \;\; &\Longrightarrow \;\; p = \alpha_1 p_1 + \alpha_2 p_2 + \cdots + \alpha_k p_k, \; (\alpha_j \in \mathbb{R}, \; 1 \leq j \leq k) \\ &\Longrightarrow \;\; degree \; p \leq d \end{aligned}$$

This means that no polynomial in $\mathcal{W}$ can have degree greater than d. Thus is a contradiction, since for example, $x^{d=1}$ is a polynomial of degree greater than d and is in $\mathcal{W}$. Thus $\mathcal{W}$ is not finitely generated. On the other hand, consider the subspace, $\mathcal{W} = \mathbb{R}_N[x]$, of all polynomials in $\mathcal{V}$ of degree less than or equal to N. Then clearly

$$S = \{p_n = x^n\}_{n=0}^{N}$$

is a finite spanning set for $\mathcal{W}$ and hence this subspace is finitely generated.

4.2 Linear Independence

We shall next introduce the notion of a linearly independent set. Consider a finite set of vectors

$$u_1, u_2, \cdots, u_r$$

in a vector space $\mathcal{V}$. Any linear combination of these vectors is of the form

$$\alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_r u_r$$

where $\alpha_j \in \mathcal{F}$, for $1 \leq j \leq r$. In particular,

$$0u_1 + 0u_2 + \cdots + 0u_r$$

is a linear combination of these vectors and is equal to $\theta_{\mathcal{V}}$. This linear combination is called the trivial linear combination of these vectors. Thus we find that given any finite set of vectors, we can obtain the zero vector $\theta_{\mathcal{V}}$, as a linear combination of these vectors.

Example 4.2.1 Consider the vector space $\mathcal{V} = \mathbb{R}^3$ and the set of vectors,

$$S = u_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \ u_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

Then clearly we can write the zero vector $\theta_{\mathcal{V}}$ as athe trivial linear combination of these vectors as

$$\theta_{\mathcal{V}} = 0u_1 + 0u_2$$

Further this is the only way we can express $\theta_{\mathcal{V}}$ as a linear combination of u_1, u_2. For if a linear combination gives $\theta_{\mathcal{V}}$, then we must have,

$$\begin{aligned} \alpha_1 u_1 + \alpha_2 u_2 &= \theta_{\mathcal{V}} \\ &\Longrightarrow \\ \begin{pmatrix} \alpha_1 \\ \beta_1 \\ \alpha_1 + \beta_1 \end{pmatrix} &= 0 \\ &\Longrightarrow \\ \alpha_1, \text{ and } \alpha_2 &= 0 \end{aligned}$$

On the other hand consider theset of vectors,

$$S = u_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \ u_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \ u_3 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$$

Then we have the trivial linear combination

$$\theta_{\mathcal{V}} \quad = \quad 0u_1 + 0u_2 + 0u_3$$

We also have

$$1u_1 + 1u_2 + (-1)u_3 \quad = \quad \theta_{\mathcal{V}}$$

In fact, for any $\alpha \in \mathbb{R}$ we have

$$\alpha u_1 + \alpha u_2 + (-\alpha)u_3 \quad = \quad \theta_{\mathcal{V}}$$

Thus nontrivial linear combinations of u_1, u_2, u_3 also give rise to the zero vector.

From the above example it follows that given any finite subset S of a vector space $\mathcal{V}$, the following two possibilities arise:

1. EITHER $\theta_{\mathcal{V}}$ can be expressed ONLY as the trivial linear combination of the vectors in S,

2. OR $\theta_{\mathcal{V}}$ can also be expressed as a nontrivial linear combination of the vectors in S

We distinguish these two possibilities with the following definition:

Definition 4.2.1 Let $\mathcal{V}$ be a vector space over a field $\mathcal{F}$. A nonempty finite subset

$$S = u_1, u_2, \cdots, u_r$$

is said to be **linearly independent** if

$$\alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_r u_r = \theta_{\mathcal{V}} \implies \alpha_j = 0, \ 1 \leq \jmath \leq r \qquad (4.2.1)$$

(that is, the only way to express the zero vector as a linear combination of the vectors in S is to express it as the trivial linear combination).
If S is not linearly independent it is said to be **linearly dependent**.

Remark 4.2.1 The set

$$S = u_1, u_2, \cdots, u_r$$

is linearly dependent means that there exist $a_1, \alpha_2, \cdots, \alpha_r \in \mathcal{F}$, at least one of which is not zero, such that

$$\alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_r u_r = \theta_{\mathcal{V}}$$

Example 4.2.2 In Example 4.2.1 above, the set

$$S = u_1, u_2$$

is linearly independent, whereas te set

$$S = u_1, u_2, u_3$$

is linearly dependent.

Example 4.2.3 Consider the vector space $\mathcal{V} = \mathbb{R}[x]$ of all polynomials over $\mathbb{R}$.

1. Consider the set

$$S_1 = p_1, p_2, p_3$$

   where

$$p_1 = 1,\ p_2 = x,\ p_3 = x^2$$

   We have

$$\begin{array}{rcl} \alpha_1 p_1 + \alpha_2 p_2 + \alpha_3 p_3 & = & \theta_{\mathcal{V}} \\ \Longrightarrow & & \\ \alpha_1 + \alpha_2 x + \alpha_3 x^2 & = & \theta_{\mathcal{V}} \\ \Longrightarrow & & \\ \alpha_1,\ \alpha_2 \text{ and } \alpha_3 & = & 0 \end{array}$$

   Hence the set S_1 is linearly independent.

2. Next we consider the set

$$S_2 = f_1, f_2, f_3$$

where

$$f_1 = 1 + x,\ f_2 = 1 + x^2,\ f_3 = 1 + x + x^2$$

We have

$$\begin{aligned}
\alpha_1 f_1 + \alpha_2 f_2 + \alpha_3 f_3 &= \theta_{\mathcal{V}} \\
\Longrightarrow & \\
\alpha_1(1+x) + \alpha_2(1+x^2) + \alpha_3(1+x+x^2) &= \theta_{\mathcal{V}} \\
\Longrightarrow & \\
(\alpha_1+\alpha_2+\alpha_3) + (\alpha_1+\alpha_3)x + (\alpha_2+\alpha_3)x^2 &= \theta_{\mathcal{V}} \\
\Longrightarrow & \\
\left.\begin{array}{lcl} \alpha_1+\alpha_2+\alpha_3 & = & 0 \\ \alpha_1+\alpha_3 & = & 0 \\ \alpha_2+\alpha_3 & = & 0 \end{array}\right\} & \\
\Longrightarrow & \\
\alpha_1,\ \alpha_2 \text{ and } \alpha_3 &= 0
\end{aligned}$$

Hence the set S_2 is linearly independent.

3. Consider the set

$$S_3 = f_1, f_2, f_3$$

where

$$f_1 = 1 + x,\ f_2 = x + x^2,\ f_3 = 1 + x^2$$

We have

$$\begin{aligned}
\alpha_1 f_1 + \alpha_2 f_2 + \alpha_3 f_3 &= \theta_{\mathcal{V}} \\
\Longrightarrow & \\
\alpha_1(1+x) + \alpha_2(x+x^2) + \alpha_3(1+x^2) &= \theta_{\mathcal{V}} \\
\Longrightarrow & \\
(\alpha_1+\alpha_3) + (\alpha_1+\alpha_2)x + (\alpha_2+\alpha_3)x^2 &= \theta_{\mathcal{V}} \\
\Longrightarrow & \\
\left.\begin{array}{lcl} \alpha_1+\alpha_3 & = & 0 \\ \alpha_1+\alpha_2 & = & 0 \\ \alpha_2+\alpha_3 & = & 0 \end{array}\right\} & \\
\Longrightarrow & \\
\alpha_1,\ \alpha_2 \text{ and } \alpha_3 &= 0
\end{aligned}$$

Hence the set S_3 is linearly independent

4. Consider the set
$$S_4 = f_1, f_2, f_3$$
where
$$f_1 = 1 - x,\ f_2 = 1 + x,\ f_3 = 1$$
This set is linearly dependent since we have
$$1f_1 + 1f_2 + (-2)f_3 = \theta_{\mathcal{V}}$$
a nontrivial linear combination giving rise to $\theta_{\mathcal{V}}$.

4.3 Properties of Linearly Dependent Sets

We shall now look at an useful property of a linearly dependent set. Consider a linearly dependent set
$$S = u_1, u_2, \cdots, u_r$$
(We arrange these vectors in S in some order as above). By Remark 4.2.1, there exist $\alpha_1, \alpha_2, \cdots, \alpha_r \in \mathcal{F}$, at least one of which is not zero, such that
$$\alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_r u_r = \theta_{\mathcal{V}}$$
Let k be the largest index such that $\alpha_k \neq 0$, that is, $\alpha_k \neq 0$ and $\alpha_j = 0$ if $j > k$. Then we have
$$\alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_k u_k = \theta_{\mathcal{V}}$$
Since $\alpha_k \neq 0$ we get
$$u_k = \beta_1 u_1 + \beta_2 u_2 + \cdots + \beta_{(k-1)} u_{(k-1)}$$
where
$$\beta_j = \alpha^{-1}\alpha_j \text{ for } 1 \leq j \leq (k-1)$$
Thus we see that u_k is a linear combination of the preceeding vectors $u_1, u_2, \cdots, u_{(k-1)}$. Thus we have the folowing property of a linearly dependent set:

Property 1:

If S is a finite linearly dependent set, (in a vector space $\mathcal{V}$), whose vectors are arranged in some order
$$\boldsymbol{S = u_1, u_2, \cdots, u_r}$$
then there exists a vector u_k such that it is a linear combination of the preceeding vectors $u_1, u_2, \cdots, u_{(k-1)}$

Example 4.3.1 Consider the folowing set of vectors in the vector space $\mathbb{R}^4$:

$$S = u_1, u_2, u_3, u_4, u_5$$

where

$$u_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \; u_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \; u_3 = \begin{pmatrix} 2 \\ 3 \\ 5 \\ 3 \end{pmatrix}, \; u_4 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \; u_5 = \begin{pmatrix} 3 \\ 2 \\ 2 \\ 2 \end{pmatrix}$$

This is a linearly dependent set since we can have nontrivial linear combination giving rise to θ_4. For example,

$$2u_1 + 3u_2 + (-1)u_3 + 0u_4 + 0u_5 = \theta_4$$

We see that the vector u_3 can be expressed as linear combination of the preceeding vectors u_1, u_2 as

$$u_3 = 2u_1 + 3u_2$$

We shall now use this property to remove the redundancies from a linearly dependent spanning set for a subspace.
Consider a finite set of vectors

$$S = u_1, u_2, \cdots, u_r$$

Without loss of generality let us assume that these vectors are all nonzero.
Case 1: S is linearly independent
In this case S is a linearly independent spanning set for $\mathcal{L}[S]$.
Case 2; S is linearly dependent
In this case, by the above property of linearly dependent sets, we must have a u_k such that it is a linear combination of the preceeding vectors $u_1, u_2, \cdots, u_{(k-1)}$. Let k_1 be the smallest index such that u_{k_1} is a linear combination of the preceeding vectors. (Since the vectors are all nonzero vectors we have $k_1 > 1$). Ths means that,

1. u_{k_1} is a linear combination of $u_1, u_2, \cdots, u_{(k_1-1)}$, and

2. u_j is NOT a linear combination of $u_1, u_2, \cdots, u_{(j-1)}$ for any $j < k_1 - 1$

Now any vector that can be written as a linear combination of $u_1, u_2, \cdots, u_r$ can also be written as a linear combinationof the set of vectors,

$$S_1 = u_1, u_2, \cdots, u_{(k_1-1)}, u_{(k_1+1)}, \cdots, u_r$$

obtained from S by removing the vector u_k. Thus we have

$$\mathcal{L}[S] = \mathcal{S}_1$$

If S_1 is linearly independent then it is a linearly independent spanning set for $\mathcal{L}[S]$ and $S_1 \subset S$.
If S_1 is linearly dependent, we repeat the above process with S_1 and remove one more vector to get a subset $S_2 \subset S_1 \subset S$ such that

$$\mathcal{L}[S_2] = \mathcal{L}[S_1] = \mathcal{L}[S]$$

If S_2 is linearly independent then it is a linearly independent spanning set for $\mathcal{L}[S]$. If not, we continue this process and in each step we remove one vector, and since S is a finite set, we get, after a finite number of steps, a subset $\tilde{S} \subset S$ such that $\tilde{S}$ is a linearly independent spanning set for $\mathcal{L}[S]$. Thus we have the folowing property of a linearly dependent set:
Property 2:
If S is a finite linearly dependent set in a vector space $\mathcal{V}$, there exists a subset $\tilde{S} \subset S$ such that, $\tilde{S}$ is a linearly independent spanning set for $\mathcal{L}[S]$.

Example 4.3.2 Consider the set of vectors in the Example 4.3.1. We had seen that this is a linearly dependent set. We shall now find a subset $\tilde{S}$ of S which is a linearly independent spanning set for $\mathcal{L}[S]$. We proceed as follows: We observe that all the vectors are nonzero vectors. (If the zero vector is in the set we remove it first).
Then we consider u_1, u_2 and write these as row vectors of a matrix and find the RRE form.

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix}$$

This is already in RRE form and since there are no zero rows it follows that u_1, u_2 are linearly independent. We then append u_3. We have

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 2 & 3 & 5 & 3 \end{pmatrix} \xrightarrow{R_3-2R_1} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 3 & 3 & 3 \end{pmatrix} \xrightarrow{R_3-3R_2} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The zero row gives us that u_3 is a linear combination of u_1 and u_2. In fact the above EROs give us

$$u_3 - 2u_1 - 3u_3 = \theta_4$$

Hence

$$u_3 = 2u_1 + 3u_2$$

as observed in Example . We now this redundant u_3. Next we append u_4 to u_1, u_2.

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_3 - R_1} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \xrightarrow{-R_3} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\xrightarrow{R_1 - R_3,\ R_2 - R_3} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

The absence of zero rows in the RRE form gives us that u_1, u_2, u_3 are linearly independent. Next we append u_5 to these three vectors.

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 3 & 2 & 2 & 2 \end{pmatrix} \xrightarrow{R_4 - 3R_1} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 2 & 2 & 2 \end{pmatrix} \xrightarrow{R_4 - 2R_2} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 \end{pmatrix}$$

$$\xrightarrow{R_4 - 2R_3} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The zero row gives us the fact that u_5 is a linear combination of u_1, u_2, u_4. Hence we remove this redundant u_5. Thus we finally get the linearly independent subset

$$\tilde{S} = u_1, u_2, u_4$$

of S such that $\mathcal{L}[S] = \mathcal{L}[\tilde{S}]$

These ideas lead us to the notion of a basis which we introduce i the next section.

4.4 Basis

Consider a finitely generated subspace $\mathcal{W}$ of a vector space $\mathcal{V}$. Since $\mathcal{W}$ is finitely generated there must be a finite spanning set, say

$$S = u_1, u_2, \cdots, u_r$$

Since S is a spanning set for $\mathcal{W}$, we have $\mathcal{L}[S] = \mathcal{W}$. If S is linearly independent then we have a linearly independent spanning set for $\mathcal{W}$. If S is linearly dependent, then by Property 2 of the previous section we can get a linearly independent subset $\tilde{S} \subset S$ such that $\mathcal{L}[\tilde{S}] = \mathcal{L}[S] = \mathcal{W}$. Hence $\tilde{S}$ is a linearly independent spanning set. Thus, in any case, we see that a finitely generated subspace must possess a linearly independent, finite, spanning set. This leads us to the following definition:

Definition 4.4.1 A finite linearly independent spanning set for a finitely generated subspace is called a **BASIS** for the subspace.

Remark 4.4.1 If the vector space $\mathcal{V}$ is itself finitely generated then it will have finite, linearly independent, spanning set and such a spanning set is called a basis for $\mathcal{V}$

We shall now study some properties of linearly independent sets and basis. Suppose now $\mathcal{V}$ is a finitely generated space and so has a basis, say,

$$\mathcal{B} = u_1, u_2, \cdots, u_d$$

Let

$$S = v_1, v_2, \cdots, v_r$$

be any linearly independent set in $\mathcal{V}$. Consider the set

$$S_1 = v_1, u_1, u_2, \cdots, u_d$$

Since $v_1 \in \mathcal{V}$ and $\mathcal{B}$ is a basis we must have v_1 as a linear combination of the vectors in $\mathcal{B}$. Hence S_1 must be linearly dependent. Hence by Property 2 of linearly dependent spanning sets obtained in the previous section, we must have a subset $\tilde{S}_1 \subset S_1$ such that S_1 is a linearly independent spanning set for $\mathcal{V}$, that is, $\tilde{S}_1$ is a basis for $\mathcal{V}$. This is got by the process of removing the redundancy in the linearly dependent spanning set, S_1, using the procedure described in the previous section. Clearly the process does not remove v_1

from the set S_1. Hence there must be a proper subset $\mathcal{B}'$ of $\mathcal{B}$, (obtained by removing at least one vector from $\mathcal{B}$), such that

$$\mathcal{B}_1 = v_1, \mathcal{B}'$$

is a basis for $\mathcal{V}$. Now we let

$$S_2 = v_2, v_1, \mathcal{B}'$$

Since this is a linearly dependent set we can repeat the above argument to S_2 to obtain a proper subset $\mathcal{B}_1'$ of $\mathcal{B}_1$, (and hence a proper subset of $\mathcal{B}$), such that

$$\mathcal{B}_2 = v_2, v_1, \mathcal{B}_1'$$

is a basis for $\mathcal{V}$. We continue this process. There arise two possibilities:
Possibility 1: The process continues for r steps
In this case we get a basis

$$\mathcal{B}_r = v_r, v_{(r-1)}, \cdots, v_1, \mathcal{B}'_{(r-1)}$$

where $\mathcal{B}'_{(r-1)}$ is a proper subset of $\mathcal{B}$. Since in each step we remove at least one of the vectors in $\mathcal{B}$ we must have at least r vectors in $\mathcal{B}$, that is,

$$r \leq d \tag{4.4.1}$$

Possibility 2: The process terminates at the kth step where $k < r$
In this case we have a basis

$$\mathcal{B}_k = v_k, v_{(k-1)}, \cdots, v_1$$

for $\mathcal{V}$, and $k < r$. hence we have

$$\mathcal{V} = \mathcal{L}[\mathcal{B}_k] \text{ and } v_{(k+1)} \in \mathcal{V}$$

Hence $v_{(k+1)}$ must be a linear combination of $v_1, v_2, \cdots, v_k$, which is a contradiction, since S is linearly independent. Thus this possibility cannot arise. Hence we have (4.4.1). Thus we have
Property 1:
If a vector space $\mathcal{V}$ has a basis consisting of d vectors then any linearly independent set in $\mathcal{V}$ can have at most d vectors

We shall now apply this to get another important property of a basis.
Suppose $\mathcal{V}$ is finitely generated vector space. Then it has a finite basis. Let

$$\mathcal{B} = u_1, u_2, \cdots, u_d$$

be a basis for $\mathcal{V}$. If $\mathcal{B}'$ is any oter basis for $\mathcal{V}$ then since $\mathcal{B}'$ must be linearly independent it can have at most d vectors in it and hence it must be finite. Thus every basis for $\mathcal{V}$ will be finite. Further, let

$$\begin{aligned}\mathcal{B} &= u_1, u_2, \cdots, u_m \\ \mathcal{B}' &= v_1, v_2, \cdots, v_n\end{aligned}$$

be any two bases for $\mathcal{V}$. Since $\mathcal{B}$ is a basis for $\mathcal{V}$ and $\mathcal{B}'$ is linearly independent we get by Property 1 above,

$$n \leq m \tag{4.4.2}$$

Similarly, since $\mathcal{B}'$ is a basis for $\mathcal{V}$ and $\mathcal{B}$ is linearly independent we get by Property 1 above,

$$m \leq n \tag{4.4.3}$$

From (4.4.2) and (4.4.3) we get

$$m = n \tag{4.4.4}$$

Summarising we get
Property 2:
In a finitely generated vector space all bases are finite and all bases have the same number of vectors
This leads us to the following definition:

Definition 4.4.2 For a finitely generated vector space, the number of vectors in any basis is called the **DIMENSION** of the space

Remark 4.4.2 From now on we shall, therefore, refer to a finitely generated vector space as a **Finite Dimensional Vector Space**.

Definition 4.4.3 For a matrix $A \in \mathcal{F}^{m\times n}$, the dimension of $\mathcal{R}_A$, the Range of A, is defined as the "**Rank of** A" and is denoted by ρ_A, and the dimension of $\mathcal{N}_A$, the Null Space of A, is defined as the "Nullity of A" and is denoted by ν_A.

Remark 4.4.3 Similarly the rank of A^T is the dimension of $\mathcal{R}_{A^T}$, the Range of A^T, and is denoted by ρ_{A^T}, and the dimension of $\mathcal{N}_{A^T}$, the Null space of A^T, is denoted by ν_{A^T}

Remark 4.4.4 It is easy to see that for any matrix, the Nullity of a matrix as defined above is the same as the nullity as defined in

Now consider a n dimensional vector space. Hence bases have exactly n vectors. By Property 1 above, any linearly independent set can have at most n vectors. Hence any set having more than n vectors must be linearly dependent. Thus we have

Property 3:

In an n dimensional vector space any subset having more than n vectors must be linearly dependent

An immediate consequence is the following:

Let $\mathcal{B} = u_1, u_2, \cdots, u_n$ be a basis for an n dimensional vector space. Then any set S in $\mathcal{V}$ such that $\mathcal{B}$ is a proper subset of S must contain at least one vector more that that in $\mathcal{B}$. Hence S must have more than n vectors and by the above property it follows that S must be linearly dependent. Thus any set S of which $\mathcal{B}$ is a proper subset must be linearly dependent. This means that $\mathcal{B}$ cannot be a proper subset of any linearly independent set. In other words, we say that $\mathcal{B}$ is a maximal linearly independent set in $\mathcal{V}$. Thus we have,

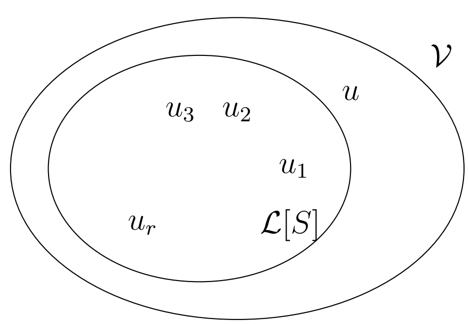
Property 4:

A basis for $\mathcal{V}$ is a maximal linearly independent set in $\mathcal{V}$

Next, consider a linearly independent set,

$$S = u_1, u_2, \cdots, u_r$$

Suppose $u \in \mathcal{V}$ is such that $u \notin \mathcal{L}[S]$



Now consider the set

$$S_1 = u_1, u_2, \cdots, u_r, u$$

We shall se that this is linearly independent. For,

$$\begin{aligned}
\alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_r u_r + \alpha u &= \theta_{\mathcal{V}} \\
\Longrightarrow & \\
\alpha &= 0
\end{aligned}$$

For, if not, then $\alpha \neq 0$ and hence α^{-1} exists and we get

$$u = \alpha^{-1}\alpha_1 u_1 + \alpha^{-1}\alpha_2 u_2 + \cdots + \alpha^{-1}\alpha_r u_r$$

and hence $u \in \mathcal{L}[S]$ - a contradiction. Hence $\alpha = 0$. But then

$$\begin{aligned}
\alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_r u_r + \alpha u &= \theta_{\mathcal{V}} \\
\Longrightarrow & \\
\alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_r u_r &= \theta_{\mathcal{V}} \\
\Longrightarrow & \\
\alpha_j &= 0 \text{ for } 1 \leq j \leq r \\
& \quad \text{(since } S \text{ is linearly independent)}
\end{aligned}$$

Thus we have

$$\begin{aligned}
\alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_r u_r + \alpha u = \theta_{\mathcal{V}} &\Longrightarrow \alpha = \alpha_1 = \alpha_2 = \cdots = \alpha_r = 0 \\
\Longrightarrow &
\end{aligned}$$

S_1 is linearly independent. Thus we have

Property 5:

If $S = u_1, u_2, \cdots, u_r$ is linearly independent and $u \in \mathcal{V}$ is such that $u \notin \mathcal{L}[S]$ then $S_1 = u_1, u_2, \cdots, u_r, u$ is also linearly independent

An immediate consequence of this property is the following: Suppose $\mathcal{W}$ is a subspace and $S = u_1, u_2, \cdots, u_r$ linearly independent in $\mathcal{W}$ and $u \in \mathcal{V}$ such that $u \notin \mathcal{W}$. Then we have

$$\begin{aligned}
\mathcal{L}[S] \subset \mathcal{W} &\Longrightarrow u \notin \mathcal{L}[S] \\
\Longrightarrow & \\
S_1 = u_1, u_2, \cdots, u_r, u \quad \text{is} &\quad \text{linearly independent}
\end{aligned}$$

Thus we have

Property 6:

If $\boldsymbol{S = u_1, u_2, \cdots, u_r}$ is a linearly independent subset of a subspace $\boldsymbol{\mathcal{W}}$ and $\boldsymbol{u \in \mathcal{V}}$ is such that $\boldsymbol{u \notin \mathcal{W}}$ then $\boldsymbol{S_1 = u_1, u_2, \cdots, u_r, u}$ is a linearly independent subset of $\boldsymbol{\mathcal{V}}$

We shall now look at a consequence of this property.

Consider an n dimensional vector space V and let

$$S = u_1, u_2, \cdots, u_n$$

be a linearly independent set in $\mathcal{V}$. Since this is a linearly independent set, the only way this can fail to be a basis is that it is not a spanning set for $\mathcal{V}$. Now,

S is not a spanning set for $\mathcal{V} \Longrightarrow \mathcal{W} = \mathcal{L}[S] \neq \mathcal{V}$

$\Longrightarrow$

There exists a vector $u \in \mathcal{V}$ such that $u \notin \mathcal{W}$

$\Longrightarrow$

(By Property 5 above) $u_1, u_2, \cdots, u_n, u$ is a linearly independent set

- a contradiction, since by Property 1, no linearly independent set in an n dimensional space can have more than n vectors. Thus we must have $\mathcal{L}[S] = \mathcal{V}$ and hence S will be a linearly independent spanning set for $\mathcal{V}$ and therefore a basis for $\mathcal{V}$. Thus we have

Property 7:

In an n dimensional vector space $\mathcal{V}$ any set of n linearly independent vectors must form a basis for $\mathcal{V}$.

Now consider an n dmensional vector space $\mathcal{V}$, and let

$$S = u_1, u_2, \cdots, u_r$$

be any linearly independent set in $\mathcal{V}$ and $r < n$. Then S is not a basis for $\mathcal{V}$ since any basis must have n vectors. Hence we have $\mathcal{W} = \mathcal{L}[S]$ is a proper subspace of $\mathcal{V}$. Therfore, there exists a vector $u_{(r+1)} \in \mathcal{V}$ such that $u_{(r+1)} \notin \mathcal{W}$. By Property 5 we have, the set

$$S_1 = u_1, u_2, \cdots, u_r, u_{(r+1)}$$

is linearly independent in $\mathcal{V}$. If $r + 1 = n$ this will be a basis for $\mathcal{V}$. If $(r + 1) < n$ then we have $\mathcal{L}[S_1]$ is a proper subspace of $\mathcal{V}$. We can get a vector $u_{(r+2)} \in \mathcal{V}$ such that $u_{(r+2)} \notin \mathcal{L}[S_1]$. Continuiung this process $n - r$

times we get vectors $u_{(r+1)}, u_{(r+2)}, \cdots, u_n$ such that

$$\mathcal{B} = u_1, u_2, \cdots, u_r, u_{(r+1)}, u_{(r+2)}, \cdots, u_n$$

is a basis for $\mathcal{V}$. Thus we are able to get a basis of which the given linearly independent set is a part. We state this property as follows:

Property 8:

In an n dimensional vector space any linearly independent set having less than n vectors can be "extended" to be a basis for $\mathcal{V}$.

We shall now look at an application of the above result.

Let $A \in \mathcal{F}^{m\times n}$ be a nonzero matrix. Then $\mathcal{N}_A$ is a proper subspace of $\mathcal{F}^n$ and its dimension is denoted by ν_A. Let

$$\mathcal{B}_{\mathcal{N}_A} = \varphi_1, \varphi_2, \cdots, \varphi_{\nu_A}$$

be a basis for $\mathcal{N}_A$, (where $\nu_A < n$). By the Property 8 above, we can extend this to a basis

$$\mathcal{B} = \varphi_1, \varphi_2, \cdots, \varphi_{\nu_A}, v_1, v_2, \cdots, v_{(n-\nu_A)}$$

for $\mathcal{F}^n$, by appending suitable vectors $v_1, v_2, \cdots, v_{(n-\nu_A)}$. Now any vector $b \in \mathcal{R}_A$ is of the form Ax for some $x \in \mathcal{F}^n$, and any $x \in \mathcal{F}^n$ is a linear combination of the vectors in the basis $\mathcal{B}$. Therefore we have,

$$\begin{aligned}
b \in \mathcal{R}_A \implies & \exists x \in \mathcal{F}^n \ni b = Ax \\
\implies & b = A\left(\sum_{j=1}^{\nu}{}_A \alpha_j\varphi_j + \sum_{k=1}^{(n-\nu_A)} \beta_k v_k\right) \\
& (\text{where } \alpha_j, \beta_k \in \mathcal{F},\ 1 \le j \le \nu_A,\ 1 \le k \le n-\nu_A) \\
& b = \sum_{j=1}^{\nu}{}_A \alpha_j(A\varphi_j) + \sum_{k=1}^{(n-\nu_A)} \beta_k(Av_k) \ (\text{since } A\varphi_j = \theta_n \text{ as } \varphi_j \in \mathcal{N}_A \\
& b = \sum_{k=1}^{(n-\nu_A)} \beta_k u_k \text{ where } u_k = Av_k \in \mathcal{R}_A
\end{aligned}$$

Thus we see that the set of vectors,

$$S = u_1, u_2, \cdots, u_k$$

is in $\mathcal{R}_A$ and every vector in $\mathcal{R}_A$ is a linear combination of these vectors. Hence S is a spanning set for $\mathcal{R}_A$. If we show that S is also linearly independent then it will become a linearly independent spanning set and hence a basis for $\mathcal{R}_A$. We now proceed to prove that S is linearly independent. We have,

$$\begin{aligned}
\sum_{k=1}^{(n-\nu_A)} \beta_k u_= \theta_n &\Longrightarrow \sum_{k=1}^{(n-\nu_A)} \beta_k (Av_k) = \theta_n \text{ (since } u_k = Av_k \\
&\Longrightarrow A\left(\sum_{k=1}^{(n-\nu_A)} \beta_k v_k\right) = \theta_n \\
&\Longrightarrow \sum_{k=1}^{(n-\nu_A)} \beta_k v_k \in \mathcal{N}_A \\
&\Longrightarrow \sum_{k=1}^{(n-\nu_A)} \beta_k v_k = \sum_{j=1}^{\nu_A} \gamma_j \varphi_j, \text{ since } \mathcal{B}_{\mathcal{N}_A} \text{ is a basis for } \mathcal{N}_A \\
&\Longrightarrow \sum_{j=1}^{\nu_A} \gamma_j \varphi_j + \sum_{k=1}^{(n-\nu_A)} (-\beta_k) v_k = \theta_n \\
&\Longrightarrow \gamma_j = 0,\ \beta_k = 0,\ 1 \leq j \leq \nu_A,\ 1 \leq k \leq \nu_A \\
&\Longrightarrow S \text{ is linearly independent}
\end{aligned}$$

Thus S is a linearly independent spanning set for $\mathcal{R}_A$ and hence basis for $\mathcal{R}_A$. Since there are $n - \nu_A$ vectos in S we get

$$\textit{Dimension}\ \mathcal{R}_A = n - \nu_A$$

But the dimension of $\mathcal{R}_A$ is ρ_A, the rank of A. Thus we get

$$\rho_A + \nu_A = \text{number of columns of } A \tag{4.4.5}$$

Similarly we get

$$\rho_{A^T} + \nu_{A^T} = \text{number of columns of } A^T \tag{4.4.6}$$

Thus we have,

Theorem 4.4.1 **Rank Nullity Theorem**:
For any matrix $A \in \mathcal{F}^{m\times n}$, we have

$$\boldsymbol{Rank\ of\ A + Nullity\ of\ A = Number\ of\ Columns\ in\ A}$$

Remark 4.4.5 We had observed in that

$$Row\ Rank\ of\ A + Nullity\ of\ A = Number\ of\ Columns\ in\ A$$

Comparing this with the above theorem we get

$$\boldsymbol{Row\ Rank\ of\ A = Rank\ of\ A} \tag{4.4.7}$$

Since $col(A) = \mathcal{R}_{A}$ we have the column rank of A to be the same as the rank of A. Hence we get that all the three ranks of A are the same. that is,
For any matrix $\boldsymbol{A}$,

$$\boldsymbol{Row\ Rank\ of\ A = Column\ Rank\ of\ A = Rank\ of\ A} \tag{4.4.8}$$

From the above equations we also get,

$$\begin{aligned} Rank\ A &= Row\ Rank\ A \\ &= Column\ Rank\ A^T \\ &= Rank\ A^T \end{aligned}$$

Thus we get
For any matrix $\boldsymbol{A \in \mathcal{F}^{m\times n}}$

$$\boldsymbol{\rho_{A}}, \textbf{ the Rank of } \boldsymbol{A} \ = \ \boldsymbol{\rho_{A^T}}, \textbf{ the Rank of } \boldsymbol{A^T} \tag{4.4.9}$$

Example 4.4.1 Clearly, the set of vectors

$$\mathcal{B} \ = \ \{e_j\}_{j=1}^{n} \text{ where } v_j \text{ has } j\text{th component as 1 and all other components as zero}$$

is a linearly independent spanning set for $\mathcal{F}^n$ and hence form a basis for $\mathcal{F}^n$. Since $\mathcal{F}^2$ has a basis with n vectors, the dimension of $\mathcal{F}^n$ is n. Any linearly independent set in $\mathcal{F}^n$ which has n vectors must be a basis for $\mathcal{F}^n$. For example, the dimension of $\mathcal{F}^2$ is 2 and the set,

$$S \ : \ v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \ v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

is a linearly independent set in $\mathcal{F}^2$ and hence is a basis for $\mathcal{F}^2$

Example 4.4.2 Consider the subspace,

$$\mathcal{W} = \left\{ \left\{ x = \begin{pmatrix} \alpha \\ \beta \\ 2\alpha + \beta \end{pmatrix} : \alpha,\ \beta \in \mathbb{R} \right\} \right.$$

We have,

$$\begin{aligned} x \in \mathcal{W} &\iff x = \begin{pmatrix} \alpha \\ \beta \\ 2\alpha + \beta \end{pmatrix} \text{ for some } \alpha,\ \beta \in \mathcal{F} \\ &\iff x = \alpha \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix},\ \alpha,\ \beta \in \mathbb{R} \\ &\iff x \in \mathcal{L}[v_1, v_2] \text{ where} \\ v_1 &= \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix},\ \text{and} \\ v_2 &= \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \end{aligned}$$

Now it is easy to see that $\mathcal{B} : v_1, v_2$ is a linearly independent in $\mathcal{W}$. Hence $\mathcal{B}$ is a linearly independent spanning set for $\mathcal{W}$ and hence a basis for $\mathcal{W}$. Since $\mathcal{W}$ has a basis with two vectors we have,

$$\textit{Dimension}\, \mathcal{W} = 2$$

Further any two linearly independent vectors in $\mathcal{W}$ will form a basis for $\mathcal{W}$. For example the set,

$$\mathcal{B}' : v_1' = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix},\ v_2' = \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}$$

is a linearly independent set in $\mathcal{W}$ and this is a set with two linearly independent vectors , and the dimension of $\mathcal{W}$ is two, this set will also form a basis for $\mathcal{W}$.

Example 4.4.3 Consider the vector space $\mathbb{C}^2$ over the field $\mathbb{C}$. Clearly, the set

$$\mathcal{B} \ : \ e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \ e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

is a linearly independent spanning set for $\mathbb{C}^2$ and hence a basis for $\mathbb{C}^2$. Thus dimension of this vector space is 2. The set

$$\mathcal{B}' \ : \ v_1 = \begin{pmatrix} i \\ 0 \end{pmatrix}, \ v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

is a linearly independent set in $\mathbb{C}^2$ and hence is also a basis for this vector space.

Example 4.4.4 Now consider the vector space $\mathcal{V} = \mathbb{C}^2$ over the field $\mathcal{F} = \mathbb{R}$. Clearly, the set

$$S \ : \ e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \ e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

is a linearly independent set in this vector space. However this does not span this vector space. For, we cannot express the vectors $x = \begin{pmatrix} i \\ 0 \end{pmatrix}$ in this vector space, as a linear combination

$$\alpha e_1 + \beta e_2 \text{ where } \alpha, \ \beta \in \mathbb{R}$$

Hence the set S is only a linearly independent set in this vector space but not a basis for this vector space. We can extend S to a basis for this vector space by appending the vectors

$$e_3 = \begin{pmatrix} i \\ 0 \end{pmatrix} \text{ and } e_4 = \begin{pmatrix} 0 \\ i \end{pmatrix}$$

The set

$$\mathcal{B} \ : \ e_1, e_2, e_3, e_4 \text{ (as defined above)}$$

is clearly a linearly independent set in this vector space. Moreover,

$$\begin{aligned}
x \in \mathcal{V} \quad &\Longrightarrow \quad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \text{ where } x_1, x_2 \in \mathbb{C} \\
&\Longrightarrow \quad x = \begin{pmatrix} a+ib \\ c+id \end{pmatrix} \text{ where } a, b, c, d \in \mathbb{R} \\
&\Longrightarrow \quad x = ae_1 + ce_2 + be_3 + de_4 \text{ where } a, b, c, d \in \mathbb{R} \\
&\Longrightarrow \quad x \text{ is a linear combination of the vectors in } \mathcal{B}
\end{aligned}$$

Thus $\mathcal{B}$ is a linearly independent spanning set for this vector space, and hence a basis for this vector space. Hence the dimension of this vector space is 4. In general, $\mathbb{C}^n$ as a vector space over $\mathbb{R}$ has dimension $2n$.

Example 4.4.5 In the vector space $\mathcal{F}^{m\times n}$, of all $m \times n$ matrices over $\mathcal{F}$, let $\mathcal{B}$ be the set of matrices,

$$\mathcal{B} = \{A_{ij}\}_{1\leq i\leq m,\ 1\leq j\leq n}$$

where A_{ij} is the $m \times n$ matrix whose (p, q)th entry is 0 if $(p, q) \neq (i, j)$ and (i, j)th entry is 1 . Then the set S is clearly linearly independent. It is also a spanning set for this vector space since any $A = (a_{ij}) \in \mathcal{F}^{m\times n}$ can be expressed as a linear combination of the vectors in $\mathcal{B}$ as follows:

$$A = \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}A_{ij}$$

Hence S is a basis for $\mathcal{F}^{m\times n}$, and therefore the dimension of this space is $m \times n$. For instance, the set of matrices

$$\mathcal{B} : A_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},\ A_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ A_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},\ A_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

is a basis for $\mathbb{R}^{2\times 2}$, and the dimension of this vector space is

$$2 \times 2 = 4$$

.

Example 4.4.6 For the vector space of all complex $m \times n$ matrices over the field of complex numbers, the set of matrices, $\{A_{ij}\}$ as defined above is a basis, and hence the dimension of this vector space is $m \times n$. However, analogous to our observations in Example 4.4.4, this set is only a linearly independent set in the vector space of all complex $m \times n$ matrices over the field of real numbers. For this subspace, this set appended with the set,

$$S_1 = \{B_{ij}\}_{1 \leq i \leq m,\ 1 \leq j \leq n}$$

where, B_{ij} is the $m \times n$ matrix whose (p, q)th entry is 0 if $(p, q) \neq (i, j)$ and (i, j)th entry is i , is a basis, that is $S \cup S_1$ is a basis for this vector space. Hence the dimension of this vector space is $2(m \times n)$. For instance,

$$\begin{aligned} dimension\ of\ \mathbb{C}^{3 \times 4}\ ober\ \mathbb{C} &= 3 \times 4 = 12 \\ dimension\ of\ \mathbb{C}^{3 \times 4}\ over\ \mathbb{R} &= 2(3 \times 4) = 24 \end{aligned}$$

Example 4.4.7 Let $\mathcal{V}$ be the vector space over $\mathbb{R}$ of all positive real numbers endowed with the operations,

$$\begin{aligned} x \oplus y &= xy \text{ for all } x, y \in \mathbb{R}^+ \\ \alpha \odot x &= x^\alpha \text{ for all } \alpha \in \mathbb{R} \text{ and for all } x \in \mathbb{R}^+ \end{aligned}$$

The zero vector of this vector space is $\theta_{\mathcal{V}} = 1$. Consider any $b \in \mathbb{R}^+$ such that $b \neq \theta_{\mathcal{V}}$, that is $b \neq 1$. Then we have that the set consisting of the single vector b is linearly independent, since b is a nonzero vector. Now let $x \in \mathbb{R}^+$. Can we express x as a linear combination of b? We have,

$$\begin{gathered} x = \alpha \odot b \text{ where } \alpha \in \mathbb{R} \qquad x = b^\alpha \\ \alpha = log_b(x) \end{gathered}$$

For any $x \in \mathbb{R}^+$ we have $log_b(x)$ is well defined and hence we can write

$$\begin{aligned} x &= b^{log_b(x)} \\ &= log_b(x) \odot b \end{aligned}$$

Thus every x in $\mathcal{V}$ is a linear combination of b and hence b is a spanning set for $\mathcal{V}$. Since it is also linearly independent, as observed above, we see that the set consisting of the single vector b is a basis for $\mathcal{V}$. Hence dimension of this space is 1, and any positive real number b, other than 1, is a basis for this space.

Example 4.4.8 Consider the vector space $\mathbb{R}_2[x]$, of all polynomials over $\mathbb{R}$ whose degree is ≤ 2. Consider the following set of vectors in $\mathbb{R}_2[x]$:

$$\mathcal{B} : p_1, p_2, p_3 \text{ where } p_1(x) = 1,\ p_2(x) = x,\ p_3(x) = x^2$$

We have

$$\begin{aligned}
\alpha_1 p_1 + \alpha_2 p_2 + \alpha_3 p_3 = \theta_{\mathcal{V}} \quad &\Longleftrightarrow \quad \alpha_1 + \alpha_2 x + \alpha_3 x^2 = \theta_{\mathcal{V}} \\
&\Longleftrightarrow \quad \alpha_1 = \alpha_2 = \alpha_3 = 0 \\
&\Longrightarrow \quad \mathcal{B} \text{ is linearly independent}
\end{aligned}$$

Moreover, any $p(x) = a_0 + a_1 x + a_2 x^2$ in $\mathbb{R}_2[x]$, can be expressed as a linear combination of the vectors in $\mathcal{B}$ as,

$$p \ = \ a_0 p_1 + a_1 p_1 + a_2 p_2$$

Hence $\mathcal{B}$ is a spanning set for $\mathbb{R}_2[x]$. Since $\mathcal{B}$ is both linearly independent set and spanning set, it follows that $\mathcal{B}$ is a basis for $\mathbb{R}_2[x]$. Consequently the dimension of this space is 3. Hence any linearly independent set in $\mathbb{R}_2[x]$ having three vectors will be a basis for $\mathbb{R}_2[x]$. For example, consider the set,

$$\mathcal{B}' \ : \ p_1', p_2', p_3' \text{ where } p_1(x) = 1 + x,\ p_2(x) = x + x^2,\ p_3(x) = 1 + x^2$$

We have

$$\begin{aligned}
\alpha_1 p_1' + \alpha_2 p_2' + \alpha_3 p_3' = \theta_{\mathcal{V}} \quad &\Longleftrightarrow \quad \alpha_1(1 + x) + \alpha_2(x + x^2) + \alpha_3(1 + x^2) = \theta_{\mathcal{V}} \\
&\Longleftrightarrow \quad (\alpha_1 + \alpha_3) + (\alpha_1 + \alpha_2)x + (\alpha_2 + \alpha_3)x^2 = \theta_{\mathcal{V}} \\
&\Longleftrightarrow \quad \alpha_1 + \alpha_3 = 0,\ \alpha_1 + \alpha_2 = 0,\ \alpha_2 + \alpha_3 = 0 \\
&\Longleftrightarrow \quad \alpha_1 = \alpha_2 = \alpha_3 = 0 \\
&\Longrightarrow \quad \mathcal{B}' \text{ is linearly independent}
\end{aligned}$$

Since $\mathcal{B}'$ is linearly independent, has three vectors, and the dimension of $\mathbb{R}_2[x]$ is 3, it follows that $\mathcal{B}'$ is also a basis for $\mathbb{R}_2[x]$.

4.5 Ordered Basis and Coordinates

Consider a finite dimensional vector space $\mathcal{V}$. A basis for $\mathcal{V}$ is a linearly independent set. When we list the elements of a set, the order in which we

list these elements, is not relevant. However when we prescibe a particular order in which these elements are to be listed then we get the notion of an ordered set. Thus a basis for $\mathcal{V}$ in which the vectors are arranged in a prescribed order is called an "**ordered basis**" for $\mathcal{V}$. Thus for example the sets $\mathcal{B} = \left\{ e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix},\ e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$ and $\mathcal{B}' = \left\{ e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix},\ e_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$ represent the same basis, but as ordered basis they are different.
Now conisder an n dimensional vector space $\mathcal{V}$ and an ordered basis for $\mathcal{V}$,

$$\mathcal{B} \quad = \quad v_1, v_2, \cdots, v_n$$

Since $\mathcal{B}$ is a basis, it is a spanning set for $\mathcal{V}$, and hence every vector $x \in \mathcal{V}$ is a linear combination of the vectors in $\mathcal{B}$. Let

$$x \quad = \quad \sum_{j=1}^{n} x_j v_j \text{ where } x_j \in \mathcal{F},\ 1 \leq j \leq n \tag{4.5.1}$$

Using the fact that $\mathcal{B}$ is linearly independent, we can easily see that the above representation of any $x \in \mathcal{V}$ is unique. Thus, corresponding to every $x \in \mathcal{V}$ we get a unique sequence of n scalars $x_1, x_2, \cdots, x_n \in \mathcal{F}$ such that (4.5.1) holds. Now consider the column vector $[x]_{\mathcal{B}} \in \mathcal{F}^n$ defined as,

$$[x]_{\mathcal{B}} \quad = \quad \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \tag{4.5.2}$$

This is constructed using the vector x and its representation as a linear combination of the vectors in the ordered basis $\mathcal{B}$. Thus, corresponding to every $x \in \mathcal{V}$ we have a unique $[x]_{\mathcal{B}} \in \mathcal{F}^n$. This gives us a transformation,

$$T : \mathcal{V} \longrightarrow \mathcal{F}^n$$

defined as

$$T_{\mathcal{B}}(x) \quad = \quad [x]_{\mathcal{B}} \tag{4.5.3}$$

WE now observe some simple properties of this transformation which converts every $x \in \mathcal{V}$ to the language of column vectors in $\mathcal{F}^n$. We have

$$x, y \in \mathcal{V} \implies x = \sum_{j=1}^{n} x_j v_j \text{ and } y = \sum_{j=1}^{n} y_j v_j$$

$$
\begin{aligned}
&\Longrightarrow \quad x + y = \sum_{j=1}^{n} (x_j + y_j) v_j \\
&\Longrightarrow \\
T_{\mathcal{B}}(x) = [x]_{\mathcal{B}} \quad &= \quad \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \\
T_{\mathcal{B}}(y) = [y]_{\mathcal{B}} \quad &= \quad \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} \\
T_{\mathcal{B}}(x+y) = [x+y]_{\mathcal{B}} \quad &= \quad \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_n + y_n \end{pmatrix} \\
&= \quad \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} \\
&= \quad T_{\mathcal{B}}(x) + T_{\mathcal{B}}(y)
\end{aligned}
$$

Thus we see that

$$
T_{\mathcal{B}}(x+y) \quad = \quad T_{\mathcal{B}}(x) + T_{\mathcal{B}}(y) \text{ for all } x, y \in \mathcal{V} \tag{4.5.4}
$$

Thus, the transformation T "preserves" the addition operation.
Similarly we have

$$
\begin{aligned}
x \in \mathcal{V},\ \alpha \in \mathcal{F} \quad &\Longrightarrow \quad \alpha x = \sum_{j=1}^{n} (\alpha x_j) v_n \\
&\Longrightarrow \\
T_{\mathcal{B}}(\alpha x) = [\alpha x]_{\mathcal{B}} \quad &= \quad \begin{pmatrix} \alpha x_1 \\ \alpha x_2 \\ \vdots \\ \alpha x_n \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
&= \alpha \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \\
&= \alpha T_{\mathcal{B}}(x)
\end{aligned}
$$

Thus we have

$$T_{\mathcal{B}}(\alpha x) \;=\; \alpha T_{\mathcal{B}}(x) \text{ for all } a \in \mathcal{F} \text{ and for all } x \in \mathcal{V} \tag{4.5.5}$$

Thus, the transformation $T_{\mathcal{B}}$ preserves scalar multiplication.
Thus, the transformation T preserves the two basic operations of the vector space, and hence preserves superpositions. We generalize this to get the notion of a linear transformation.

Definition 4.5.1 Let $\mathcal{V}$ and $\mathcal{W}$ be vector spaces over a field $\mathcal{F}$. A transformation

$$T : V \longrightarrow W$$

is said to be a "**linear transformation**" from $\mathcal{V}$ to $\mathcal{W}$ if,

$$T(x+y) \;=\; T(x) + T(y) \text{ for all } x, y \in \mathcal{V} \text{ and,} \tag{4.5.6}$$

$$T(\alpha x) \;=\; \alpha T(x) \text{ for all } a \in \mathcal{F} \text{ and for all } x \in \mathcal{V} \tag{4.5.7}$$

Hence from the above discussion, it follows that, every ordered basis $\mathcal{B}$ to $\mathcal{V}$ gives rise to a linear transformation $T_{\mathcal{B}}$ from $\mathcal{V}$ to $\mathcal{F}^n$ which maps $x \in \mathcal{V}$ to $T_{\mathcal{B}}(x) = [x]_{\mathcal{B}}$ in $\mathcal{F}^n$. This transformation $T_{\mathcal{B}}$ can be interpreted as a coding of a vectors x in the abstarct vector space $\mathcal{V}$, to a column vectors $T_{\mathcal{B}}(x) = [x]_{\mathcal{B}}$ in $\mathcal{F}^n$. We shall now look at some more properties of this code.
We have,

$$
\begin{aligned}
x, y \in \mathcal{V}, x \neq y \;&\iff\; [x]_{\mathcal{B}} \neq [y]_{\mathcal{B}} (Why?) \\
&\iff\; T_{\mathcal{B}}(x) \neq T_{\mathcal{B}}(y)
\end{aligned}
$$

We acn also state the above property as

$$T_{\mathcal{B}}(x) = T_{\mathcal{B}}(y) \iff x = y$$

Thus this coding assigns different column vectors codes to different vectors. We say that the transformation $T_{\mathcal{B}}$ is "one-one". We generalize this to abstract linear transformations as follows:

Definition 4.5.2 Let $\mathcal{V}$ and $\mathcal{W}$ be vector spaces over a field $\mathcal{F}$. A linear transformation

$$T : \mathcal{V} \longrightarrow \mathcal{W}$$

is said to be "**one-one**" if

$$T(x) = T(y) \quad \Longleftrightarrow \quad x = y \tag{4.5.8}$$

Further we have,

$$\begin{aligned}
u \in \mathcal{F} \quad &\Longrightarrow \quad u = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} \text{ where } u_j \in \mathcal{F},\ 1 \le j \le n \\
&\Longrightarrow \quad x = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n \in \mathcal{V} \\
&\Longrightarrow \quad \exists x \in \mathcal{V} \text{as defined above and } [x]_{\mathcal{B}} = u \\
&\Longrightarrow \quad \exists x \in \mathcal{V} \text{ such that } T_{\mathcal{B}}(x) = u
\end{aligned}$$

Thus, we have the following property:

$$\text{For every } u \in \mathcal{F}^n \quad \exists \text{ a } x \in \mathcal{V} \text{ such that} \quad T_{\mathcal{B}}(x) = u$$

We say that $T_{\mathcal{B}}$ is "onto". What this says is that the coding of $x \in \mathcal{V}$ to $T_{\mathcal{B}}(x) \in \mathcal{F}^n$ uses every code available in $\mathcal{F}^n$.
We generalize this as follows:

Definition 4.5.3 Let $\mathcal{V}$ and $\mathcal{W}$ be vector spaces over a field $\mathcal{F}$. A linear transformation

$$T : \mathcal{V} \longrightarrow \mathcal{W}$$

is said to be "**onto**" if

$$\text{For every } u \in \mathcal{F}^n \quad \exists \text{ a } x \in \mathcal{V} \text{ such that} \quad T(x) = u \tag{4.5.9}$$

The transformation $T_{\mathcal{B}}$ is both one-one and onto, that is, every vectors $x \in \mathcal{V}$ has a unique column vector code $T_{\mathcal{B}}(x) = [x]_{\mathcal{B}}$, and every column vector $u \in \mathcal{F}^n$ is the code of a vector $x \in \mathcal{V}$. We call such transformations as isomorphisms. We have

Definition 4.5.4 Let $\mathcal{V}$ and $\mathcal{W}$ be vector spaces over a field $\mathcal{F}$. A linear transformation

$$T : \mathcal{V} \longrightarrow \mathcal{W}$$

is said to be an "**isomorphism**" of $\mathcal{V}$ onto $\mathcal{W}$ if it is both one-one and onto.

Thus $T_{\mathcal{B}}$ is an isomrphism of $\mathcal{V}$ ono $\mathcal{F}^n$.

Example 4.5.1 Consider the vector space $\mathcal{V} = \mathbb{R}^2$. Consider the ordered basis

$$\mathcal{B} = v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix},\ v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Then $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2$ can bewritten as

$$x = x_1 e_1 + x_2 e_2$$

Hence we have

$$[x]_{\mathcal{B}} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Now consider the ordered basis

$$\mathcal{B}' = v_1' = \begin{pmatrix} 1 \\ 1 \end{pmatrix},\ v_2' = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Then we have

$$x = \frac{x_1 + x_2}{2} v_1' + \frac{x_1 - x_2}{2} v_2'$$

Hence we have

$$[x]_{\mathcal{B}'} = \begin{pmatrix} \frac{x_1+x_2}{2} \\ \frac{x_1-x_2}{2} \end{pmatrix}$$

Thus we have

$$\begin{aligned} T_{\mathcal{B}}(x) &= \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ T_{\mathcal{B}'}(x) &= \begin{pmatrix} \frac{x_1+x_2}{2} \\ \frac{x_1-x_2}{2} \end{pmatrix} \end{aligned}$$

We can easily verify that both $T_{\mathcal{B}}$ and $T_{\mathcal{B}'}$ are isomorphisms of $\mathbb{R}^2$ onto $\mathbb{R}^2$

Example 4.5.2 Let $\mathcal{W}$ be he subspace of $\mathbb{R}^3$ defined as follows:

$$\mathcal{W} = \left\{ x = \begin{pmatrix} \alpha \\ \beta \\ 2\alpha + \beta \end{pmatrix} : \alpha,\ \beta \in \mathbb{R} \right\}$$

Consider the following ordered basis for $\mathcal{W}$:

$$\mathcal{B} : v_1 = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \; v_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

Then any $x = \begin{pmatrix} \alpha \\ \beta \\ 2\alpha + \beta \end{pmatrix} \in \mathcal{W}$ can be written as

$$x = \alpha v_1 + \beta v_2$$

Hence we have

$$[x]_{\mathcal{B}} \;=\; \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

Note that, since the subspace $\mathcal{W}$ is two dimensional, its coding is done as a (2×1) column vectors. We therefore have

$$T_{\mathcal{B}}(x) \;=\; [x]_{\mathcal{B}} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

is an isomorphism of $\mathcal{W}$ onto $\mathbb{R}^2$. Consider now the following ordered basis foe $\mathcal{W}$:

$$\mathcal{B}' : \; v_1' = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}, v_2' = \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix},$$

Then any $x = \begin{pmatrix} \alpha \\ \beta \\ 2\alpha + \beta \end{pmatrix} \in \mathcal{W}$ can be written as

$$x = \frac{2\alpha + \beta}{4} v_1' + \frac{2\alpha - \beta}{4} v_2'$$

Hence we have

$$[x]_{\mathcal{B}'} \;=\; \begin{pmatrix} \frac{2\alpha+\beta}{4} \\ \frac{2\alpha-\beta}{4} \end{pmatrix}$$

Thus we have

$$T_{\mathcal{B}}(x) \;=\; [x]_{\mathcal{B}'} = \begin{pmatrix} \frac{2\alpha+\beta}{4} \\ \frac{2\alpha-\beta}{4} \end{pmatrix}$$

For example, if $x \in \mathcal{W}$ is the vector

$$x = \begin{pmatrix} 2 \\ 3 \\ 7 \end{pmatrix}$$

then we have

$$\alpha = 2 \text{ and } \beta = 3$$

Hence we have

$$\begin{aligned} T_{\mathcal{B}}(x) &= [x]_{\mathcal{B}} = \begin{pmatrix} 2 \\ 3 \\ 7 \end{pmatrix} \\ T_{\mathcal{B}'} &= [x]_{\mathcal{B}'} = \begin{pmatrix} \frac{7}{4} \\ \frac{1}{4} \end{pmatrix} \end{aligned}$$

The transformations $T_{\mathcal{B}}$ and $T_{\mathcal{B}'}$ are both isomorphisms of $\mathcal{W}$ onto $\mathbb{R}^2$.

Example 4.5.3 Consider the vector space $\mathcal{V} = \mathbb{R}_2[x]$, of all polynomials in x, of degree less than or equal to two, with real coefficients. Consider the following ordered basis for $\mathbb{R}_2[x]$:

$$\mathcal{B} : p_1, p_2, p_3 \text{ where } p_1(x) = 1,\ p_2(x) = x,\ p_3(x) = x^2$$

Then any polynomial $p(x) = a_0 + a_1 x + a_2 x^2$ can be written as

$$p = a_0 p_1 + a_1 p_2 + a_2 p_3$$

Hence we get

$$[p]_{\mathcal{B}} = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix}$$

Now consider the ob,

$$\mathcal{B}' : p_1', p_2', p_3' \text{ where } p_1(x) = 1 + x,\ p_2(x) = x + x^2,\ p_3(x) = 1 + x^2$$

Then p can be written as

$$p = \frac{a_0 + a_1 - a_2}{2} p_1' + \frac{-a_0 + a_1 + a_2}{2} p_2' + \frac{a_0 - a_1 + a_2}{2} p_3'$$

Hence we have

$$[p]_{\mathcal{B}'} = \begin{pmatrix} \frac{a_0+a_1-a_2}{2} \\ \\ \frac{-a_0+a_1+a_2}{2} \\ \\ \frac{a_0-a_1+a_2}{2} \end{pmatrix}$$

For example, if $p = 4x^2 - 3x + 2$ then $a_0 = 2, a_1 = -3, a_2 = 4$ and hence we get

$$[p]_{\mathcal{B}} = \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix}$$

$$[p]_{\mathcal{B}'} = \begin{pmatrix} -\frac{5}{2} \\ \\ -\frac{1}{2} \\ \\ \frac{9}{2} \end{pmatrix}$$

The linear transformations $T_{\mathcal{B}}$ and $T_{\mathcal{B}'}$ defined respectively as $T_{\mathcal{B}}(p) = [p]_{\mathcal{B}}$ and $T_{\mathcal{B}'}(p) = [p]_{\mathcal{B}'}$ are isomorphisms of $\mathbb{R}_2[x]$ onto $\mathbb{R}^3$.

4.6 Relation Between Different Bases Representations

Let $\mathcal{V}$ be an n dimensional vector space over a field $\mathcal{F}$ and let

$$\begin{aligned} \mathcal{B} &= u_1, u_2, \cdots, u_n \\ \mathcal{B}' &= u'_1, u'_2, \cdots, u'_n \end{aligned}$$

be any two ordered bases for $\mathcal{V}$. Any vector $x \in \mathcal{V}$ can be represented as column vectors

$$[x]_{\mathcal{B}} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \text{ and } [x]_{\mathcal{B}'} = \begin{pmatrix} x'_1 \\ x'_2 \\ \vdots \\ x'_n \end{pmatrix}$$

respectively w.r.t the ordered bases $\mathcal{B}$ and $\mathcal{B}'$. Since both these column vectors represent the same vector there must be some relationship between these two column vectors. What is this connection? We shall first look at a simple example.

Example 4.6.1 Consider the vector space $\mathbb{R}^2$ and consider the two ordered bases

$$\begin{aligned}\mathcal{B} &= e_1, e_2 \text{ where } e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix},\ e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ \mathcal{B}' &= u_1, u_2 \text{ where } u_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix},\ u_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}\end{aligned}$$

Any vector $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2$ has the representations

$$[x]_{\mathcal{B}} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$[x]_{\mathcal{B}'} = \begin{pmatrix} \frac{x_1+x_2}{2} \\ \frac{x_1-x_2}{2} \end{pmatrix}$$

Now consider the matrix

$$\mathcal{M}_{\mathcal{B}\mathcal{B}'} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$$

Then it is easy to verify that

$$\mathcal{M}_{\mathcal{B}\mathcal{B}'}[x]_{\mathcal{B}} = [x]_{\mathcal{B}'}$$

Thus we find that by premultiplying $[x]_{\mathcal{B}}$, the representation of x w.r.t. the ordered basis $\mathcal{B}$, by the matrix $\mathcal{M}_{\mathcal{B}\mathcal{B}'}$, we can get $[x]_{\mathcal{B}'}$, the representation of x w.r.t. the ordered basis $\mathcal{B}'$. Similarly if we define $\mathcal{M}_{\mathcal{B}'\mathcal{B}}$ to be the matrix,

$$\mathcal{M}_{\mathcal{B}'\mathcal{B}} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

it is easy to verify that

$$\mathcal{M}_{\mathcal{B}'\mathcal{B}}[x]_{\mathcal{B}'} = [x]_{\mathcal{B}}$$

Hence premultiplying $[x]_{\mathcal{B}'}$, the representation of x w.r.t. the ordered basis $\mathcal{B}'$, by the matrix $\mathcal{M}_{\mathcal{B}'\mathcal{B}}$, we can get $[x]_{\mathcal{B}}$, the representation of x w.r.t. the ordered basis $\mathcal{B}$. Further, it is easy to check that

$$\mathcal{M}_{\mathcal{B}\mathcal{B}'}\mathcal{M}_{\mathcal{B}'\mathcal{B}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2$$

Thus there are two matrices one converting $\mathcal{B}$ representations to $\mathcal{B}'$ representations, and the other converting $\mathcal{B}'$ representations to $\mathcal{B}$ representations, and these two matrices are inverses of each other.

We now look at how to find such matrices in the general situation.
Let $\mathcal{V}$ be an n dimensional vector space over a field $\mathcal{F}$ and

$$\begin{aligned} \mathcal{B} &= v_1, v_2, \cdots, v_n \\ \mathcal{B}' &= v'_1, v'_2, \cdots, v'_n \end{aligned}$$

be any two ordered bases for $\mathcal{V}$. Any vector $x \in \mathcal{V}$ can be represented as column vectors

$$[x]_{\mathcal{B}} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \text{ and } [x]_{\mathcal{B}'} = \begin{pmatrix} x'_1 \\ x'_2 \\ \vdots \\ x'_n \end{pmatrix}$$

In particular, let us consider the vector v_j in $\mathcal{V}$ w.r.t. the ordered basis $\mathcal{B}'$ to get the representation

$$[v_j]_{\mathcal{B}'} = \begin{pmatrix} v_{1j} \\ v_{2j} \\ \vdots \\ v_{nj} \end{pmatrix}$$

This means

$$v_j = \sum_{i=1}^{n} v_{ij} v'_i \tag{4.6.1}$$

We can do this for $j = 1, 2, \cdots, n$ and get n column vectors

$$\left\{[v_j]_{\mathcal{B}'}\right\}_{j=1}^{n}$$

Now consider $x \in \mathcal{V}$ We have

$$x = \sum_{j=1}^{n} x_j v_j$$

Substituting for v_j from (4.6.1) we get

$$\begin{aligned} x &= \sum_{j=1}^{n} x_j \left\{ \sum_{i=1}^{n} v_{ij} v'_i \right\} \\ &= \sum_{i=1}^{n} \left\{ \sum_{j=1}^{n} v_{ij} x_j \right\} v'_i \end{aligned}$$

From the above we get the ith coordinate of x w.r.t. the ordered basis $\mathcal{B}'$ is given by

$$\sum_{j=1}^{n} v_{ij} x_j$$

Hence we get

$$\begin{aligned} [x]_{\mathcal{B}'} &= \begin{pmatrix} \sum_{j=1}^{n} v_{1j} x_j \\ \sum_{j=1}^{n} v_{2j} x_j \\ \vdots \\ \sum_{j=1}^{n} v_{ij} x_j \\ \vdots \\ \sum_{j=1}^{n} v_{nj} x_j \end{pmatrix} \\ &= \mathcal{M}_{\mathcal{B}\mathcal{B}'} [x]_{\mathcal{B}} \end{aligned}$$

where $\mathcal{M}_{\mathcal{B}\mathcal{B}'}$ is the matrix whose (i,j)-th entry is v_{ij}. Thus the columns of $\mathcal{M}_{\mathcal{B}\mathcal{B}'}$ are respectively $[v_1]_{\mathcal{B}'}, [v_2]_{\mathcal{B}'}, \cdots, [v_j]_{\mathcal{B}'}, \cdots, [v_n]_{\mathcal{B}'}$. Thus we have a matrix which converts $\mathcal{B}$ representations to $\mathcal{B}'$ representations. This matrix arises from the representation of the vectors in the ordered basis $\mathcal{B}$ w.r.t. the ordered basis $\mathcal{B}'$. Analogously, by representing the vectors in the ordered basis $\mathcal{B}'$ w.r.t. the ordered basis $\mathcal{B}$ we get a matrix $\mathcal{M}_{\mathcal{B}'\mathcal{B}}$ whose jth column is the column vector $[v'_j]_{\mathcal{B}}$. This matrix converts $\mathcal{B}'$ representations to $\mathcal{B}$ representations as

$$[x]_{\mathcal{B}} = \mathcal{M}_{\mathcal{B}'\mathcal{B}}[x]_{\mathcal{B}'}$$

We shall now illustrate this with some examples.

Example 4.6.2 Consider $\mathcal{V} = \mathbb{R}^2$ and the two ordered bases

$$\begin{aligned} \mathcal{B} &= v_1, v_2 \text{ where } v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix},\ v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ \mathcal{B}' &= v'_1, v'_2 \text{ where } v'_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix},\ v'_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \end{aligned}$$

We have

$$\begin{aligned} v_1 &= \frac{1}{2}v'_1 + \frac{1}{2}v'_2 \\ v_2 &= \frac{1}{2}v'_1 - \frac{1}{2}v'_2 \end{aligned}$$

Hence we have

$$[v_1]_{\mathcal{B}'} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$$

$$[v_2]_{\mathcal{B}'} = \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}$$

Taking these as the two columns of $\mathcal{M}_{\mathcal{B}\mathcal{B}'}$ we get

$$\mathcal{M}_{\mathcal{B}\mathcal{B}'} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$$

For any $x \in \mathbb{R}^2$ we have

$$[x]_{\mathcal{B}} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$[x]_{\mathcal{B}'} = \begin{pmatrix} \frac{x_1+x_2}{2} \\ \frac{x_1-x_2}{2} \end{pmatrix}$$

It is easy to see that

$$\begin{aligned}
\mathcal{M}_{\mathcal{B}\mathcal{B}'}[x]_{\mathcal{B}} &= \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\
&= \begin{pmatrix} \frac{x_1+x_2}{2} \\ \frac{x_1-x_2}{2} \end{pmatrix} \\
&= [x]_{\mathcal{B}'}
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
v_1' &= 1v_1 + 1v_2 \\
v_2' &= 1v_1 - 1v_2
\end{aligned}$$

Hence we get

$$[v_1']_{\mathcal{B}} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$[v_1']_{\mathcal{B}} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Hence we have

$$\mathcal{M}_{\mathcal{B}'\mathcal{B}} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

It is easy to see that

$$\begin{aligned}
\mathcal{M}_{\mathcal{B}'\mathcal{B}}[x]_{\mathcal{B}'} &= \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \frac{x_1+x_2}{2} \\ \frac{x_1-x_2}{2} \end{pmatrix} \\
&= \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\
&= [x]_{\mathcal{B}}
\end{aligned}$$

Further, we have,

$$\begin{aligned}
\mathcal{M}_{\mathcal{B}\mathcal{B}'}\mathcal{M}_{\mathcal{B}'\mathcal{B}} &= \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\end{aligned}$$

Example 4.6.3 Consider the subspace,

$$\mathcal{W} = \left\{ \begin{pmatrix} \alpha \\ \beta \\ 2\alpha+\beta \end{pmatrix} : \alpha,\ \beta \in \mathbb{R} \right\}$$

of $\mathbb{R}^3$. We have seen in Example that w.r.t. the two ordered basis

$$\begin{aligned} \mathcal{B} &: \ v_1 = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, v_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \text{ and} \\ \mathcal{B}' &: \ v_1' = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}, v_2' = \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}, \text{ and} \end{aligned}$$

we have for any $x = \begin{pmatrix} \alpha \\ \beta \\ 2\alpha+\beta \end{pmatrix} \in \mathcal{W}$,

$$\begin{aligned} [x]_{\mathcal{B}} &= \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \\ [x]_{\mathcal{B}'} &= \begin{pmatrix} \frac{2\alpha+\beta}{4} \\ \frac{2\alpha-\beta}{4} \end{pmatrix} \end{aligned}$$

We have, therefore,

$$[v_1]_{\mathcal{B}'} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$$

$$[v_2]_{\mathcal{B}'} = \begin{pmatrix} \frac{1}{4} \\ -\frac{1}{4} \end{pmatrix}$$

Hence we get,

$$\begin{aligned} \mathcal{M}_{\mathcal{B}\mathcal{B}'} &= \begin{pmatrix} [v_1]_{\mathcal{B}'} & [v_2]_{\mathcal{B}'} \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & -\frac{1}{4} \end{pmatrix} \end{aligned}$$

It is easy to check that,

$$\mathcal{M}_{\mathcal{B}\mathcal{B}'}[x]_{\mathcal{B}} \;=\; [x]_{\mathcal{B}'} \text{ for any } x \in \mathcal{W}$$

Similarly we have,

$$[v_1']_{\mathcal{B}} \;=\; \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$[v_2']_{\mathcal{B}} \;=\; \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

Hence we get,

$$\begin{aligned} \mathcal{M}_{\mathcal{B}'\mathcal{B}} &= \begin{pmatrix} [v_1']_{\mathcal{B}} & [v_2']_{\mathcal{B}} \end{pmatrix} \\ &= \begin{pmatrix} 1 & 1 \\ 2 & -2 \end{pmatrix} \end{aligned}$$

It is easy to check that,

$$\mathcal{M}_{\mathcal{B}'\mathcal{B}}[x]_{\mathcal{B}'} \;=\; [x]_{\mathcal{B}} \text{ for all } x \in \mathcal{W}$$

Example 4.6.4 Consider the vector space $\mathbb{R}_2[x]$, and the two ordered basis,

$$\begin{aligned} \mathcal{B} \;&:\; p_1, p_2, p_3 \text{ where } p_1(x) = 1, p_2(x) = x, p_3(x) = x^2 \\ \mathcal{B}' \;&:\; p_1', p_2', p_3' \text{ where } p_1'(x) = 1 + x,\; p_2'(x) = x + x^2,\; p_3'(x) = 1 + x^2 \end{aligned}$$

We have seen in Example 4.5.3 that any $p(x) = a_0 + a_1x + a_2x^2 \in \mathbb{R}_2[x]$ can be represented w.r.t. these two ordered basis as,

$$[p]_{\mathcal{B}} \;=\; \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix}$$

$$[p]_{\mathcal{B}'} \;=\; \begin{pmatrix} \frac{a_0+a_1-a_2}{2} \\ \\ \frac{-a_0+a_1+a_2}{2} \\ \\ \frac{a_0-a_1+a_2}{2} \end{pmatrix}$$

Hence we get

$$[p_1]_{\mathcal{B}'} = \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$$

$$[p_2]_{\mathcal{B}'} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}$$

$$[p_3]_{\mathcal{B}'} = \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$$

Hence we get,

$$\begin{aligned} \mathcal{M}_{\mathcal{B}\mathcal{B}'} &= \begin{pmatrix} [p_1]_{\mathcal{B}'} & [p_2]_{\mathcal{B}'} & [p_3]_{\mathcal{B}'} \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \end{aligned}$$

It is easy to check that

$$\mathcal{M}_{\mathcal{B}\mathcal{B}'}[x]_{\mathcal{B}} = [x]_{\mathcal{B}'} \text{ for all } x \in \mathbb{R}_2[x]$$

Similarly, we have,

$$[p'_1]_{\mathcal{B}} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$$[p'_2]_{\mathcal{B}} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$$[p'_3]_{\mathcal{B}} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

Hence we get,

$$\begin{aligned}
\mathcal{M}_{\mathcal{B}'\mathcal{B}} &= \begin{pmatrix} [p_1']_{\mathcal{B}} & [p_2']_{\mathcal{B}} & [p_3']_{\mathcal{B}} \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}
\end{aligned}$$

It is easy to check that

$$\mathcal{M}_{\mathcal{B}'\mathcal{B}}[x]_{\mathcal{B}'} = [x]_{\mathcal{B}} \text{ for all } x \in \mathbb{R}_2[x]$$