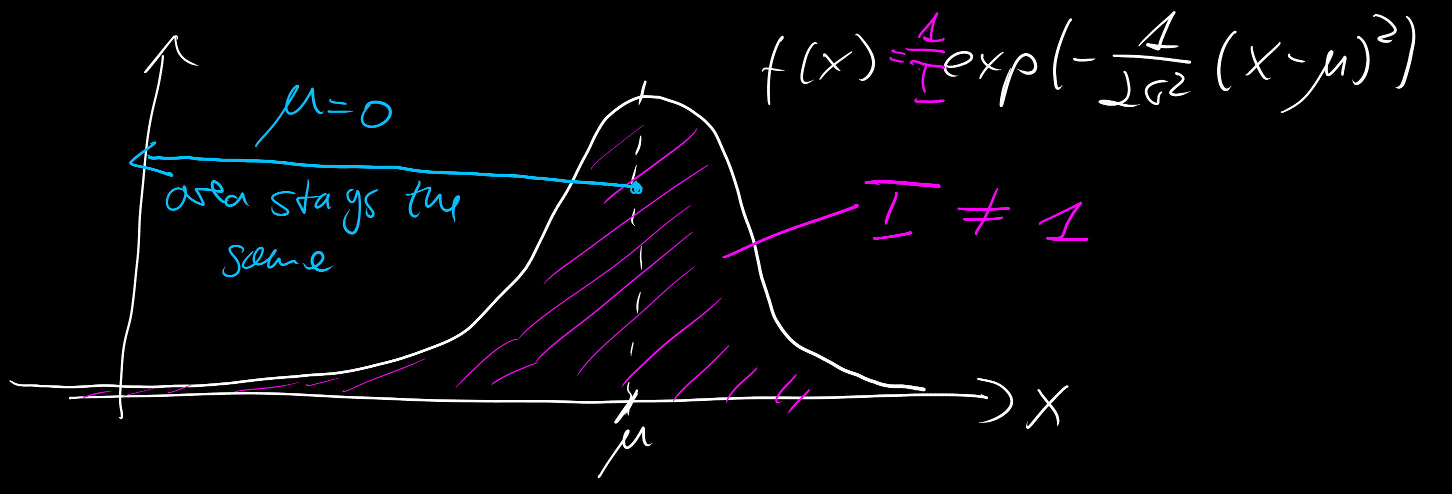
Normalization Constant of the Normal / Gaussian

$$X \sim N(\mu,\sigma) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2\sigma^2}(X-\mu)^2\right)$$

?

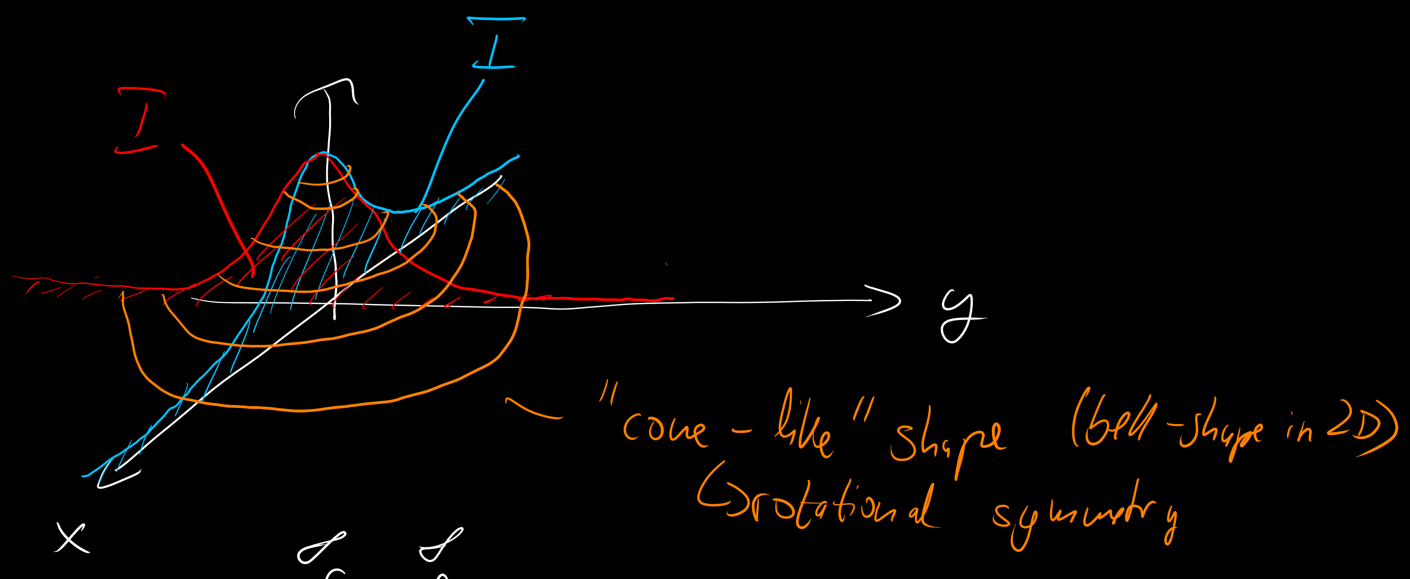


$$I = \int_{-\infty}^{\infty} \exp\left(-\frac{1}{2\sigma^2}(x-\mu)^2\right)dx$$

$$= \int_{-\infty}^{\infty} \exp\left(-\frac{1}{2\sigma^2}x^2\right)dx$$

Antiderivative
↳ by a trick

$$I^2 = I \cdot I = \int_{-\infty}^{\infty} \exp\left(-\frac{1}{2\sigma^2}x^2\right)dx \cdot \int_{-\infty}^{\infty} \exp\left(-\frac{1}{2\sigma^2}y^2\right)dy$$



$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\left(-\frac{1}{2\sigma^2}x^2\right)\exp\left(-\frac{1}{2\sigma^2}y^2\right)dx\,dy$$

$$= \underbrace{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}}_{\mathbb{R}^2} \exp\left(-\frac{1}{2\sigma^2}\overbrace{(x^2+y^2)}^{r^2}\right)dx\,dy$$

↳ polar coordinates

$$r^2 = x^2 + y^2$$
$$\varphi = \arctan\left(\frac{y}{x}\right)$$

$$= \int_0^{2\pi}\int_0^{\infty} \exp\left(-\frac{1}{2\sigma^2}r^2\right)\cdot r \cdot dr\,d\varphi$$

↳ antiderivative

$$= \int_0^{2\pi}\left[-\sigma^2 \exp\left(-\frac{1}{2\sigma^2}r^2\right)\right]_0^{\infty} d\varphi$$

$$-\frac{1}{\cancel{2}\sigma^2}\cancel{2}r\exp(\ldots))$$

$$= \int_0^{2\pi}\left(-\sigma^2\cdot 0 - (-\sigma^2\cdot 1)\right)d\varphi$$

$$= \int_0^{2\pi}\sigma^2 d\varphi = \sigma^2 2\pi$$

$$I^2 = \sigma^2 2\pi$$

$$I = \sqrt{\sigma^2 2\pi} = \sigma\sqrt{2\pi}$$