Deriving Normalization Constant for the Beta

$$\Theta \sim Beta(\Theta; \alpha, \beta) = \frac{1}{B(\alpha,\beta)} \Theta^{\alpha-1}(1-\Theta)^{\beta-1}$$

$$\int_0^1 Beta(\Theta; \alpha, \beta)\, d\Theta = 1$$

$\Theta \in [0,1]$

$$B(\alpha,\beta) = \int_0^1 \Theta^{\alpha-1} \cdot (1-\Theta)^{\beta-1} dt$$

to show $\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$

gamma function: generalization of factorial to real-valued numbers

$5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 20 \cdot 6 = 120$

$5! = 5 \cdot 4!$

$\Gamma(5.5) = 5.5 \cdot \Gamma(4.5)$ ($\to$ Gamma Distribution)

$$\Gamma(\alpha) = \int_0^\infty e^{-x} x^{\alpha-1} dx$$

$$\Gamma(\beta) = \int_0^\infty e^{-x} x^{\beta-1} dx$$

$$\Gamma(\alpha)\Gamma(\beta) = \int_0^\infty e^{-x} x^{\alpha-1} dx \int_0^\infty e^{-y} y^{\beta-1} dy$$

$$= \int_0^\infty \int_0^\infty e^{-x} x^{\alpha-1} e^{-y} y^{\beta-1} dx\,dy$$

$$= \int_0^\infty \int_0^\infty e^{-(x+y)} x^{\alpha-1} y^{\beta-1} dx\,dy$$

$t = x+y$ ($\Leftrightarrow y = t-x$)

$\frac{dy}{dt} = 1$

$$= \int_0^\infty \int_0^t e^{-t} x^{\alpha-1} (t-x)^{\beta-1} dx\,dt$$

$x = t \cdot \mu$ ($\Leftrightarrow t = \frac{x}{\mu}$)

$\mu = \frac{x}{t}$ $\quad \frac{dx}{d\mu} = t$

$dx = t \cdot d\mu$

$$= \int_0^\infty \int_0^1 e^{-t} (t\mu)^{\alpha-1} (t - t\mu)^{\beta-1} t\,d\mu\,dt$$

$$= \int_0^\infty \int_0^1 e^{-t} t^{\alpha-1} \mu^{\alpha-1} t^{\beta-1} \cdot (1-\mu)^{\beta-1} t\,d\mu\,dt$$

$\to$ limits of integration do no longer depend on each other

$\to$ split into 2 integrals

$$= \int_0^\infty e^{-t} t^{\alpha-1} t^{\beta-1} t\,dt \cdot \int_0^1 \mu^{\alpha-1} (1-\mu)^{\beta-1} d\mu$$

$$= \int_0^\infty e^{-t} t^{\alpha+\beta-1} dt \cdot \int_0^1 \mu^{\alpha-1} (1-\mu)^{\beta-1} d\mu$$

$$= \Gamma(\alpha+\beta) \quad B(\alpha,\beta) = \Gamma(\alpha)\Gamma(\beta)$$

$$\Leftrightarrow B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

$\to$ we expressed the Beta Function in terms of the Gamma Function