Bernoulli Distribution

Posterior

Weather (W) → good / bad

$W \in \{Bad, Good\}$ (0, 1)

problem: have data

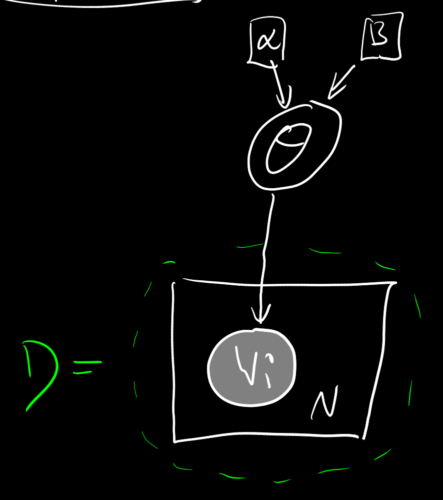
$D = \{Bad, Bad, Good, Bad, Good, \dots\}$

but no θ

$W \sim Bern(\theta)$

we want: $p(\theta \mid D)$

Graphical Model



$\theta \sim Beta(\alpha, \beta)$

$W_i \sim Bern(\theta)$

$$\text{joint: } p(\theta, D) = \underbrace{p(\theta)}_{\text{prior}} \underbrace{p(D \mid \theta)}_{\text{likelihood}}$$

$$= \frac{1}{B(\alpha,\beta)} \cdot \theta^{\alpha-1} \cdot (1-\theta)^{\beta-1} \cdot \prod_{i=0}^{N-1} \left(\theta^{w[i]} \cdot (1-\theta)^{1-w[i]}\right)$$

$$p(\theta \mid D) \overset{\text{Bayes}}{\underset{\text{Rule}}{=}} \frac{p(D \mid \theta)\, p(\theta)}{p(D)} = \frac{p(\theta, D)}{p(D)}$$

$p(D)$ — difficult

$$\sim p(\theta, D)$$

[conditional is proportional to the joint]

$$p(\theta \mid D) = \frac{1}{\cancel{B(\alpha,\beta)}} \cdot \theta^{\alpha-1} \cdot (1-\theta)^{\beta-1} \cdot \prod_{i=0}^{N-1} \left(\theta^{w[i]} \cdot (1-\theta)^{1-w[i]}\right)$$

$$\sim \theta^{\alpha-1} \cdot \prod_{i=0}^{N-1} \left(\theta^{w[i]}\right) \cdot (1-\theta)^{\beta-1} \cdot \prod_{i=0}^{N-1} (1-\theta)^{1-w[i]}$$

$$\theta^0 \cdot \theta^1 \cdot \theta^0 \cdot \theta^1 \cdot \theta^1 = \theta^3$$

$$= \theta^{\alpha-1} \cdot \theta^{\sum_{i=0}^{N-1} w[i]} \cdot (1-\theta)^{\beta-1} \cdot (1-\theta)^{\sum_{i=0}^{N-1} (1-w[i])}$$

$$= \theta^{\alpha + \sum_{i=0}^{N-1} (w[i]) - 1} \cdot (1-\theta)^{\beta + N - \sum_{i=0}^{N-1} w[i] - 1}$$

$$\alpha' = \alpha + \sum_{i=0}^{N-1} w[i] \qquad \beta' = \beta + N - \sum_{i=0}^{N-1} w[i]$$

$$= \theta^{\alpha'-1} \cdot (1-\theta)^{\beta'-1}$$

$$\sim Beta(\alpha', \beta') = p(\theta \mid D)$$

Conjugate prior

prior: Beta

Model: Bernoulli

posterior: Beta

$$p(\theta \mid D) = Beta(\alpha', \beta') = \frac{1}{B(\alpha', \beta')} \cdot \theta^{\alpha'-1} \cdot (1-\theta)^{\beta'-1}$$