Deriving the unbiased estimator for the variance

$$\mu_{MLE} = \frac{1}{N}\sum_{i=0}^{N-1} x^{[i]}$$

$$\sigma^2_{MLE} = \frac{1}{N}\sum_{i=0}^{N-1}\left(x^{[i]} - \mu_{MLE}\right)^2$$

?

$$\underbrace{\sigma^2_{ub} = \frac{1}{N-1}\sum_{i=0}^{N-1}\left(x^{[i]} - \mu_{MLE}\right)^2}_{\text{unbiased estimator}}$$

Bessel's correction

$$\sigma^2_{ub} = \frac{N}{N-1}\,\sigma^2_{MLE}$$

What is unbiasedness?

estimator is unbiased if the Expectation of the Estimator equals what was supposed to be estimated

$\mathbb{E}[\mu_{MLE}]$ has to equal $\mu = \mathbb{E}[X]$

Disclaimer: $\mathbb{E}[X] = \mathbb{E}_{X\sim p(x)}[X]$

$V[X] = V_{X\sim p(x)}[X]$

underlying distribution from which we have N samples

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① test the mean estimator

$$\mathbb{E}[\mu_{MLE}] \stackrel{?}{=} \mu = \mathbb{E}[X]$$

plug in

$$\mathbb{E}\left[\frac{1}{N}\sum_{i=0}^{N-1} x^{[i]}\right] = \frac{1}{N}\sum_{i=0}^{N-1}\mathbb{E}[x^{[i]}]$$

$$\stackrel{\text{iid}}{=} \frac{1}{N}\sum_{i=0}^{N-1}\mathbb{E}[X]$$

$$= \frac{1}{N}\mathbb{E}[X]\cdot\underbrace{\sum_{i=0}^{N-1}1}_{N}$$

$$= \frac{1}{\cancel{N}}\mathbb{E}[X]\cdot\cancel{N}$$

$$\stackrel{\checkmark}{=} \mathbb{E}[X] = \mu$$

→ mean estimator is unbiased

② Test the variance estimator

$$\mathbb{E}[\sigma^2_{MLE}] \stackrel{?}{=} \sigma^2 = V[X]$$

plug in

$$\mathbb{E}\left[\frac{1}{N}\sum_{i=0}^{N-1}\left(x^{[i]} - \mu_{MLE}\right)^2\right] =$$

$\mu_{MLE} = \bar{x}$

Squared differences from the mean

$$\sum_{i=0}^{N-1}\left(x^{[i]}-\mu\right)^2 = \sum_{i=0}^{N-1}\left(x^{[i]}-\mu_{MLE}\right)^2 + N\left(\mu-\mu_{MLE}\right)^2$$

$$\Leftrightarrow \quad \frac{1}{N}\sum_{i=0}^{N-1}\left(x^{[i]}-\mu_{MLE}\right)^2 = \frac{1}{N}\sum_{i=0}^{N-1}\left(x^{[i]}-\mu\right)^2 - \left(\mu-\mu_{MLE}\right)^2$$

$$= \mathbb{E}\left[\frac{1}{N}\sum_{i=0}^{N-1}\left(x^{[i]}-\mu\right)^2\right] + \mathbb{E}\left[-\left(\mu-\mu_{MLE}\right)^2\right]$$

$$= \frac{1}{N}\sum_{i=0}^{N-1}\mathbb{E}\left[\left(x^{[i]}-\mu\right)^2\right] - \mathbb{E}\left[\left(\mu-\mu_{MLE}\right)^2\right]$$

$$= \frac{1}{N}\sum_{i=0}^{N-1}\left(\mathbb{E}[x^{[i]2}] - 2\,\mathbb{E}[x^{[i]}\mu] + \cancel{\mathbb{E}[\mu^2]}\right)$$

$$- \left(\cancel{\mathbb{E}[\mu^2]} - 2\,\mathbb{E}[\mu\,\mu_{MLE}] + \mathbb{E}[\mu_{MLE}^2]\right)$$

$$\mathbb{E}[x^{[i]}\mathbb{E}[X]] = \mathbb{E}[X]\,\mathbb{E}[x^{[i]}]$$

$$= \frac{1}{N}\sum_{i=0}^{N-1}\left(\mathbb{E}[x^{[i]2}] \cancel{- 2\mu\,\underbrace{\mathbb{E}[x^{[i]}]}_{=\mathbb{E}[X]}}\right)$$

$$- \left(\cancel{-2\mu\,\underbrace{\mathbb{E}[\mu_{MLE}]}_{=\mathbb{E}[X]}} + \mathbb{E}[\mu_{MLE}^2]\right)$$

$$= \frac{1}{N}\sum_{i=0}^{N-1}\mathbb{E}[x^{[i]2}] - \mathbb{E}[\mu_{MLE}^2]$$

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in general

$$V[X] = \mathbb{E}\left[\left(X - \mathbb{E}[X]\right)^2\right]$$

$$= \mathbb{E}\left[X^2 - 2X\,\mathbb{E}[X] + \mathbb{E}[X]^2\right]$$

$$= \mathbb{E}[X^2] - 2\,\mathbb{E}[X\,\mathbb{E}[X]] + \mathbb{E}[\mathbb{E}[X]^2]$$

$$= \mathbb{E}[X^2] - 2\,\mathbb{E}[X]\,\mathbb{E}[X] + \mathbb{E}[X]^2$$

$$= \mathbb{E}[X^2] - \mathbb{E}[X]^2$$

$$\Leftrightarrow \quad \mathbb{E}[X^2] = V[X] + \mathbb{E}[X]^2$$

$$= \frac{1}{N}\sum_{i=0}^{N-1}\left(\underbrace{V[x^{[i]}]}_{=V[X]} \cancel{+ \underbrace{\mathbb{E}[x^{[i]}]}_{=\mathbb{E}[X]}{}^2}\right)$$

$$- \left(V[\mu_{MLE}] \cancel{+ \underbrace{\mathbb{E}[\mu_{MLE}]}_{=\mathbb{E}[X]}{}^2}\right)$$

$$= V[X] - V[\mu_{MLE}]$$

$$V[\mu_{MLE}] = V\left[\frac{1}{N}\sum_{i=0}^{N-1}x^{[i]}\right]$$

$$= \frac{1}{N^2}V\left[\sum_{i=0}^{N-1}x^{[i]}\right]$$

Variance is not linear

$$\stackrel{\text{i.i.d.}}{=} \frac{1}{N^2}\sum_{i=0}^{N-1}\underbrace{V[x^{[i]}]}_{V[X]}$$

$$= \frac{1}{N^2}V[X]\underbrace{\sum_{i=0}^{N-1}1}_{N}$$

$$= \frac{N}{N^2}V[X]$$

$$= \frac{V[X]}{N}$$

$$= V[X] - \frac{V[X]}{N}$$

$$= \frac{N-1}{N}V[X] \qquad = \mathbb{E}[\sigma^2_{MLE}]$$

$\frac{1}{\text{Bessel's correction}}$ → it is biased

Define a new estimator

$$\sigma^2_{ub} = \frac{N}{N-1}\sigma^2_{MLE}$$

$$= \frac{\cancel{N}}{N-1}\cdot\frac{1}{\cancel{N}}\sum_{i=0}^{N-1}\left(x^{[i]}-\mu_{MLE}\right)^2$$

$$= \frac{1}{N-1}\sum_{i=0}^{N-1}\left(x^{[i]}-\mu_{MLE}\right)^2$$