Dirichlet Distribution - Intro

Weather (W) → Cloudy, Rainy, Sunny

$W \sim \text{Cat}(\underline{\theta})$

e.g. $\underline{\theta} = \begin{bmatrix} 0.2 \\ 0.3 \\ 0.5 \end{bmatrix}$

Can we put a distribution over it?

W ... D-state categorical RV

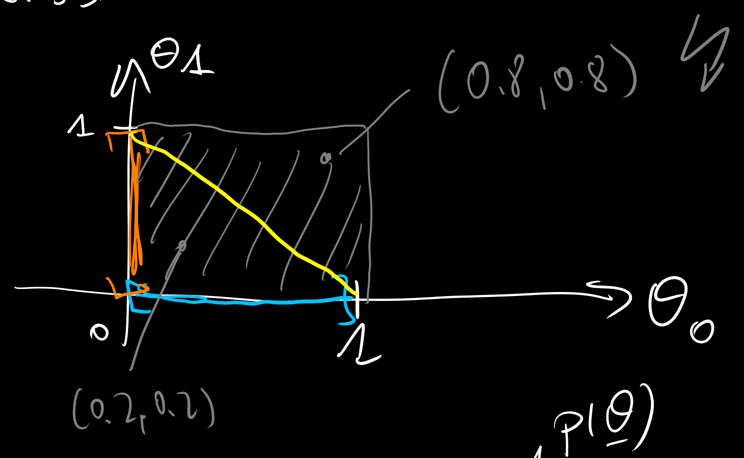
↳ $\underline{\theta} \in \mathbb{R}^D$ ← has D dimensions

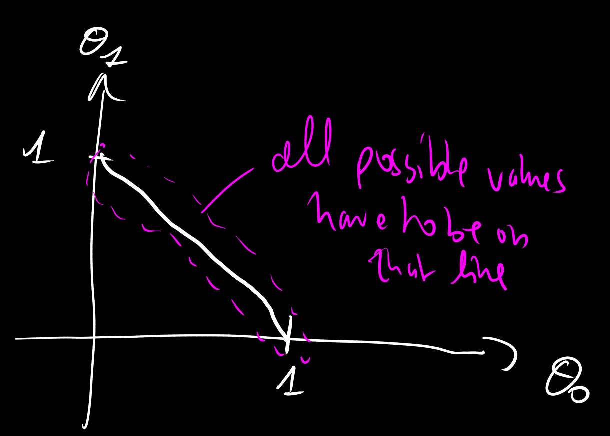
$\theta_d \in [0,1] \quad \forall d$

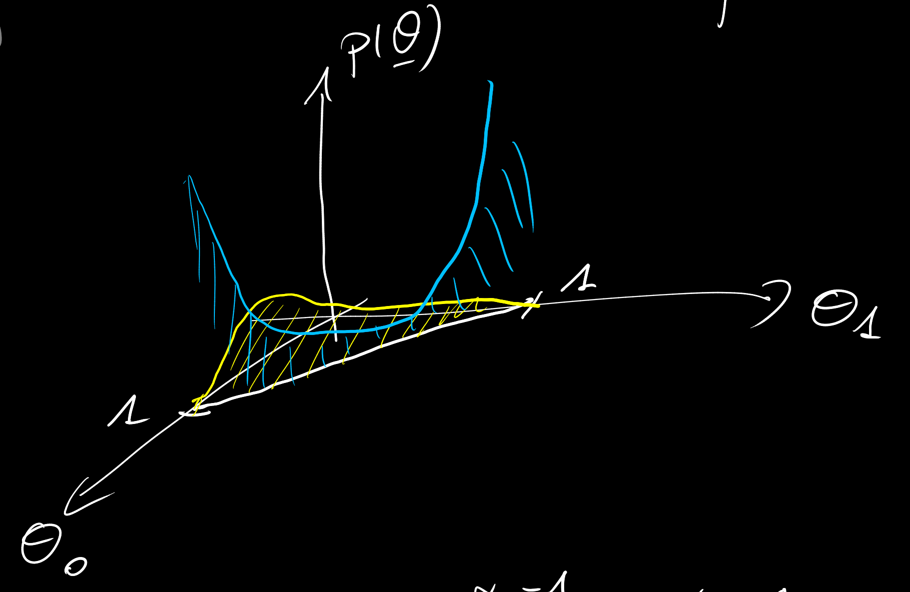
$\sum_{d=0}^{D-1} \theta_d = 1$

How can we encode this?

let's start with D=2







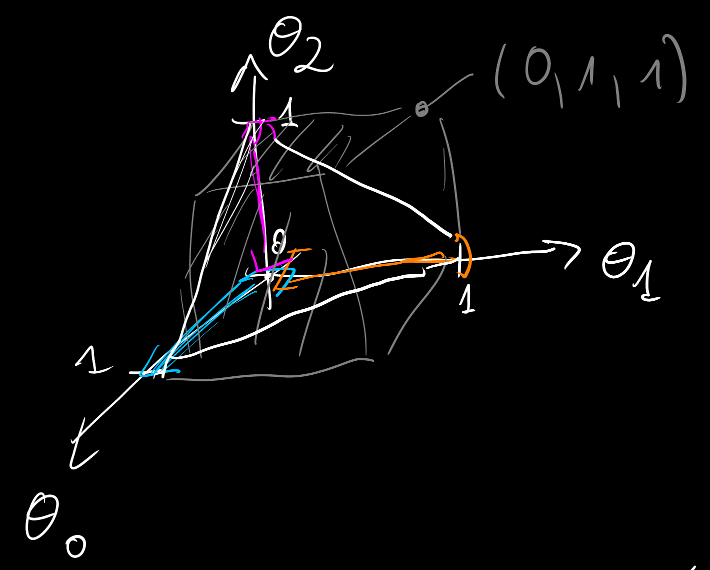
$p(\underline{\theta}) \sim \theta_0^{\alpha_0 - 1} \cdot \theta_1^{\alpha_1 - 1} \quad , \alpha_0, \alpha_1 > 0$

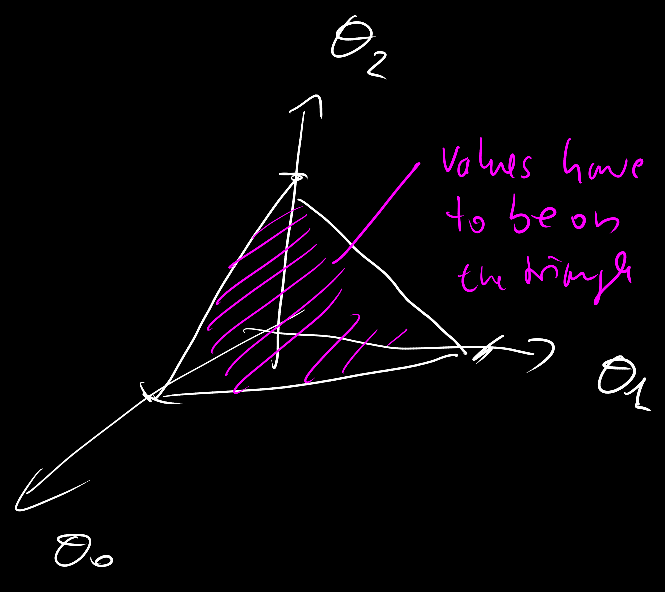
two-state categorical ≙ Bernoulli

Recall: Beta-Distribution

$p(\theta) \sim \theta^{\alpha - 1} \cdot (1-\theta)^{\beta - 1}$

Say D=3





$p(\underline{\theta}) \sim \theta_0^{\alpha_0 - 1} \, \theta_1^{\alpha_1 - 1} \, \theta_2^{\alpha_2 - 1} \quad , \alpha_0, \alpha_1, \alpha_2 > 0$

in general, we have D states, so $\underline{\theta} \in \mathbb{R}^D$

→ the values $\underline{\theta}$ have to be on a simplex of (D-1) dimensions with vertices at

$\begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ \vdots \end{bmatrix}$

pdf: $p(\underline{\theta}) \sim \prod_{d=0}^{D-1} \theta_d^{\alpha_d - 1}$

normalization?

$$p(\underline{\theta}) = \frac{\Gamma\left(\sum_{d=0}^{D-1} \alpha_d\right)}{\prod_{d=0}^{D-1} \Gamma(\alpha_d)} \cdot \prod_{d=0}^{D-1} \theta_d^{\alpha_d - 1} =: \text{Dir}(\underline{\theta}; \underline{\alpha})$$

Parameters

$\underline{\alpha} \in \mathbb{R}^D \quad , \underline{\alpha} > \underline{0}$