Deriving the Gaussian from a Maximum Entropy Principle

Normal / Gaussian $\quad X \sim \mathcal{N}(X; \mu, \sigma^2) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2\sigma^2}(x-\mu)^2\right)$

Normalization Constant

Where does this come from?

→ Gaussian / Normal is maximizing the entropy under prescribed mean & variance

→ Differential Entropy

$$H(p_{(x)}) = -\int_{-\infty}^{\infty} p_{(x)} \ln(p_{(x)})\, dx$$

Entropy is a Functional

$$\left(= \underset{p(x)}{\arg\min}\ -H(p)\right)$$

$$p^*(x) = \underset{p(x)}{\arg\max}\ H(p)$$

s.t.: $\int_{-\infty}^{\infty} p(x)\, dx = 1$ (normalization)

$\int_{-\infty}^{\infty} p(x) \cdot x\, dx = \mu$ (prescribed mean first order moment)

$\int_{-\infty}^{\infty} p(x) \cdot (x-\mu)^2\, dx = \sigma^2$ (prescribed variance second order central moment)

Build a Lagrangian

$$\mathcal{L}(p, \lambda_0, \lambda_1, \lambda_2) = \int_{-\infty}^{\infty} p \ln p\, dx + \lambda_0 \cdot \left(\int_{-\infty}^{\infty} p(x)\, dx - 1\right)$$

$$+ \lambda_1 \left(\int_{-\infty}^{\infty} p(x) \cdot x\, dx - \mu\right) + \lambda_2 \cdot \left(\int_{-\infty}^{\infty} p(x) \cdot (x-\mu)^2\, dx - \sigma^2\right)$$

$$= \int_{-\infty}^{\infty} \underbrace{p \ln p + \lambda_0 p + \lambda_1 p \cdot x + \lambda_2 p (x-\mu)^2}_{=:\ell}\, dx - \lambda_0 - \lambda_1 \mu - \lambda_2 \sigma^2$$

Another functional

Some scalars

→ Take derivative and set to 0

$\frac{\partial \mathcal{L}}{\partial p} \stackrel{!}{=} 0 \quad$ I $\quad \rightsquigarrow \frac{\partial \ell}{\partial p} = \ln p + \frac{p}{p} + \lambda_0 + \lambda_1 x + \lambda_2 (x-\mu)^2$

$\frac{\partial \mathcal{L}}{\partial \lambda_0} = \int_{-\infty}^{\infty} p\, dx - 1 \stackrel{!}{=} 0 \quad$ II

$\frac{\partial \mathcal{L}}{\partial \lambda_1} = \int_{-\infty}^{\infty} p \cdot x\, dx - \mu \stackrel{!}{=} 0 \quad$ III

$\frac{\partial \mathcal{L}}{\partial \lambda_2} = \int_{-\infty}^{\infty} p \cdot (x-\mu)^2\, dx - \sigma^2 \stackrel{!}{=} 0 \quad$ IV

I rearrange for p

$$\ln p_{(x)} = -1 - \lambda_0 - \lambda_1 x - \lambda_2 (x-\mu)^2$$

$$= -1 - \lambda_0 - \lambda_1 x - \lambda_2 (x^2 - 2x\mu + \mu^2)$$

I forgot this in the video

$$= -1 - \lambda_0 - \lambda_1 x - \lambda_2 x^2 + \lambda_2 2\mu x - \lambda_2 \mu^2$$

$$= \underbrace{-\lambda_2}_{a} x^2 + \underbrace{(2\lambda_2 \mu - \lambda_1)}_{b} x \underbrace{- 1 - \lambda_0 - \lambda_2 \mu^2}_{c}$$

Completing the square

$a = -\lambda_2 \quad b = 2\lambda_2\mu - \lambda_1 \quad c = -1 - \lambda_0 - \lambda_2\mu^2$

$$= ax^2 + bx + c$$

$$= a(x-d)^2 + e$$

$a = a \quad d = \frac{-b}{2a} \quad e = c - \frac{b^2}{4a}$

$a = -\lambda_2 \quad d = -\frac{(2\lambda_2\mu - \lambda_1)}{2 \cdot (-\lambda_2)} \quad = -1 - \lambda_0 - \lambda_2\mu^2 - \frac{(2\lambda_2\mu - \lambda_1)^2}{4 \cdot (-\lambda_2)}$

$= \frac{2\lambda_2\mu - \lambda_1}{2\lambda_2} \quad = -1 - \lambda_0 - \lambda_2\mu^2 + \lambda_2 \cdot \left(\frac{2\lambda_2\mu - \lambda_1}{2\lambda_2}\right)^2$

$= \mu - \frac{\lambda_1}{2\lambda_2} \quad = -1 - \lambda_0 - \lambda_2\mu^2 + \lambda_2\left(\mu - \frac{\lambda_1}{2\lambda_2}\right)^2$

$$\ln(p) = a \cdot (x-d)^2 + e$$

$$= -\lambda_2 \cdot \left(x - \left(\mu - \frac{\lambda_1}{2\lambda_2}\right)\right)^2 - 1 - \lambda_0 - \lambda_2\mu^2 + \lambda_2\left(\mu - \frac{\lambda_1}{2\lambda_2}\right)^2$$

$$= -\lambda_2 \cdot \left(x - \mu + \frac{\lambda_1}{2\lambda_2}\right)^2 - 1 - \lambda_0 - \lambda_2\mu^2 + \lambda_2\left(\mu - \frac{\lambda_1}{2\lambda_2}\right)^2$$

$y := x - \mu + \frac{\lambda_1}{2\lambda_2} \quad \Leftrightarrow \quad x = y + \mu - \frac{\lambda_1}{2\lambda_2}$

$$= -\lambda_2 y^2 - 1 - \lambda_0 - \lambda_2\mu^2 + \lambda_2\left(\mu - \frac{\lambda_1}{2\lambda_2}\right)^2 \quad | \exp(\ldots)$$

$$p(y) = \exp\left(-\lambda_2 y^2 - 1 - \lambda_0 - \lambda_2\mu^2 + \lambda_2\left(\mu - \frac{\lambda_1}{2\lambda_2}\right)^2\right)$$

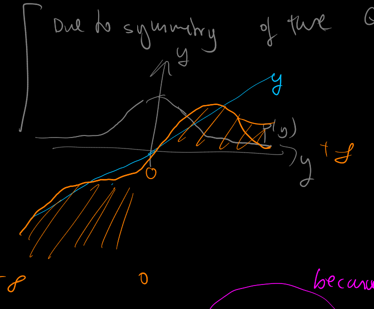
plug into III

$$\int_{-\infty}^{\infty} p(x) \cdot x\, dx = \mu$$

$$\int_{-\infty}^{\infty} p(y) \cdot \left(y + \mu - \frac{\lambda_1}{2\lambda_2}\right) dy = \mu$$

$$\underbrace{\int_{-\infty}^{\infty} p(y) \cdot y\, dy}_{=0} + \int_{-\infty}^{\infty} p(y) \cdot \left(\mu - \frac{\lambda_1}{2\lambda_2}\right) dy = \mu$$

Due to symmetry of the Gaussian



⇒ vanishes

because constant w.r.t. y

$$0 + \int_{-\infty}^{\infty} p(y) \cdot \left(\mu - \frac{\lambda_1}{2\lambda_2}\right) dy = \mu$$

$$\left(\mu - \frac{\lambda_1}{2\lambda_2}\right) \cdot \underbrace{\int_{-\infty}^{\infty} p(y)\, dy}_{=1 \text{ because of II}} = \mu$$

$$\cancel{\mu} - \frac{\lambda_1}{2\lambda_2} = \cancel{\mu}$$

$$-\frac{\lambda_1}{2\lambda_2} = 0$$

$$\boxed{\lambda_1 = 0}$$

plug $\lambda_1 = 0$ into $p(y)$

$$p(y) = \exp\left(-\lambda_2 y^2 - 1 - \lambda_0 - \lambda_2\mu^2 + \lambda_2 \cdot \left(\underbrace{\frac{\lambda_1}{2\lambda_2}}_{=0} - \mu\right)^2\right)$$

$$p(y) = \exp\left(-\lambda_2 y^2 - 1 - \lambda_0 \cancel{- \lambda_2\mu^2} \cancel{+ \lambda_2\mu^2}\right)$$

$$y = x - \mu + \underbrace{\frac{\lambda_1}{2\lambda_2}}_{=0}$$

$$p(x) = \exp\left(-\lambda_2 \cdot (x-\mu)^2 - 1 - \lambda_0\right)$$

Subst.: $z := x - \mu \quad \Leftrightarrow \quad x = z + \mu$

$$p(z) = \exp\left(-\lambda_2 \cdot z^2 - 1 - \lambda_0\right)$$

plug into II $\quad \int_{-\infty}^{\infty} p(x)\, dx = 1$

$$\Rightarrow \int_{-\infty}^{\infty} p(z)\, dz = 1$$

$$\int_{-\infty}^{\infty} \exp\left(-\lambda_2 z^2 - 1 - \lambda_0\right) dz = 1$$

$$\exp(-1 - \lambda_0) \underbrace{\int_{-\infty}^{\infty} \exp(-\lambda_2 z^2)\, dz}_{\text{Similar to normalization derivation: } \sqrt{\frac{\pi}{\lambda_2}}} = 1$$

$$e^{-1-\lambda_0} \cdot \sqrt{\frac{\pi}{\lambda_2}} = 1 \qquad (*)$$

plug $p(z)$ into IV $\quad \int_{-\infty}^{\infty} p(x) \cdot (x-\mu)^2\, dx = \sigma^2$

$$\int_{-\infty}^{\infty} p(z) \cdot (z + \cancel{\mu} - \cancel{\mu})^2\, dz = \sigma^2$$

$$\int_{-\infty}^{\infty} p(z)\, z^2\, dz = \sigma^2$$

$$\int_{-\infty}^{\infty} \exp\left(-\lambda_2 z^2 - 1 - \lambda_0\right) \cdot z^2\, dz = \sigma^2$$

$$e^{-1-\lambda_0} \cdot \int_{-\infty}^{\infty} \exp(-\lambda_2 z^2) \cdot z^2\, dz = \sigma^2$$

$$F(\lambda_2) = \int_{-\infty}^{\infty} \exp(-\lambda_2 z^2)\, dz = \sqrt{\frac{\pi}{\lambda_2}}$$

$$-\frac{dF}{d\lambda_2} = \int_{-\infty}^{\infty} (-1)(-z^2) \cdot \exp(-\lambda_2 z^2)\, dz$$

$$e^{-1-\lambda_0} \cdot \left(-\underbrace{\frac{d}{d\lambda_2}\left(\sqrt{\frac{\pi}{\lambda_2}}\right)}\right) = \sigma^2$$

$$\frac{d}{d\lambda_2}\left(\pi^{1/2} \cdot \lambda_2^{-1/2}\right) = \pi^{1/2} \cdot \left(-\frac{1}{2}\right) \cdot \lambda_2^{-3/2} = -\frac{1}{2}\sqrt{\frac{\pi}{\lambda_2^3}}$$

$$e^{-1-\lambda_0} \cdot (-1) \cdot \left(-\frac{1}{2}\sqrt{\frac{\pi}{\lambda_2^3}}\right) = \sigma^2$$

$$e^{-1-\lambda_0} \frac{1}{2}\sqrt{\frac{\pi}{\lambda_2^3}} = \sigma^2 \qquad (**)$$

Solving a system of nonlinear equations

$(*) \quad e^{-1-\lambda_0} \cdot \sqrt{\frac{\pi}{\lambda_2}} = 1 \quad \rightarrow e^{-1-\lambda_0} = \sqrt{\frac{\lambda_2}{\pi}}$

$(**) \quad e^{-1-\lambda_0} \cdot \frac{1}{2}\sqrt{\frac{\pi}{\lambda_2^3}} = \sigma^2 \quad \rightarrow e^{-1-\lambda_0} = 2\sigma^2 \sqrt{\frac{\lambda_2^3}{\pi}}$

$$\sqrt{\frac{\lambda_2}{\pi}} = 2\sigma^2 \sqrt{\frac{\lambda_2^3}{\pi}} \qquad |(\ )^2$$

$$\frac{\cancel{\lambda_2}}{\cancel{\pi}} = 4\sigma^4 \cdot \frac{\lambda_2^{\cancel{3}\,2}}{\cancel{\pi}}$$

$$\lambda_2^2 = \frac{1}{4\sigma^4}$$

$$\lambda_2 = \pm \frac{1}{2\sigma^2} \qquad (\text{we need "+"})$$

$$\boxed{\lambda_2 = \frac{1}{2\sigma^2}}$$

plug into $(*)$

$$e^{-1-\lambda_0} \cdot \sqrt{\frac{\pi}{1/2\sigma^2}} = 1$$

$$e^{-1-\lambda_0} \cdot \sigma\sqrt{2\pi} = 1$$

$$e^{-1-\lambda_0} = \frac{1}{\sigma\sqrt{2\pi}} \qquad | \ln(\ldots)$$

$$-1 - \lambda_0 = -\ln(\sigma\sqrt{2\pi})$$

$$\boxed{\lambda_0 = \ln(\sigma\sqrt{2\pi}) - 1}$$

plug into $p(x)$

$$p(x) = \exp\left(-\lambda_2 (x-\mu)^2 - 1 - \lambda_0\right)$$

$$p(x) = \exp\left(-\frac{1}{2\sigma^2}(x-\mu)^2 \cancel{-1} - \left(\ln(\sigma\sqrt{2\pi}) \cancel{-1}\right)\right)$$

$$= \exp\left(-\frac{1}{2\sigma^2}(x-\mu)^2 - \ln(\sigma\sqrt{2\pi})\right)$$

$$= \exp\left(-\ln(\sigma\sqrt{2\pi})\right) \cdot \exp\left(-\frac{1}{2\sigma^2}(x-\mu)^2\right)$$

$$\boxed{p(x) = \frac{1}{\sigma\sqrt{2\pi}} \cdot \exp\left(-\frac{1}{2\sigma^2}(x-\mu)^2\right)}$$

Gaussian / Normal Distribution