Bernoulli Distribution
Maximum Likelihood Estimation

Weather $\left\{ \begin{array}{l} \to \text{good} \\ \to \text{bad} \end{array} \right.$ (W) $\qquad W \in \{\overset{0}{\text{Bad}}, \overset{1}{\text{Good}}\}$

$$p(W) = \theta^{W}(1-\theta)^{(1-W)}$$

given: dataset: $D = \{\overset{w^{[0]}}{B}, G, G, B, B, G, \ldots \overset{w^{[N-1]}}{\}}$ $\quad$ (N - samples)

but we don't know θ

$\hookrightarrow \theta$??? $\to$ MLE

$$p(D) = \prod_{i=0}^{N-1} p(W = w^{[i]}) =: \mathcal{L}(D;\theta)$$

$\to$ log likelihood

$$\ell(D;\theta) = \log \mathcal{L}(D;\theta) = \log \prod_{i=0}^{N-1} p(W = w^{[i]})$$

$$= \sum_{i=0}^{N-1} \underbrace{\log p(W = w^{[i]})}_{\text{log-prob}}$$

$$\log p(W) = \log\left(\theta^{W} \cdot (1-\theta)^{(1-W)}\right)$$

$$= \log(\theta^{W}) + \log\left((1-\theta)^{(1-W)}\right)$$

$$= W \log(\theta) + (1-W)\log(1-\theta)$$

$$\ell(D;\theta) = \sum_{i=0}^{N-1}\left(w^{[i]}\log(\theta) + (1-w^{[i]})\log(1-\theta)\right)$$

$$\theta^{*} = \underset{\theta \in [0,1]}{\operatorname{argmax}} \left(\ell(D;\theta)\right)$$ $\to$ Take derivative and set to zero

$$\frac{\partial \ell}{\partial \theta} = \frac{\partial}{\partial \theta} \sum_{i=0}^{N-1} (\ldots) = \sum_{i=0}^{N-1} \frac{\partial}{\partial \theta}\left(w^{[i]}\log\theta + (1-w^{[i]})\log(1-\theta)\right)$$

$$= \sum_{i=0}^{N-1}\left(\frac{w^{[i]}}{\theta} \cancel{+} - \frac{1-w^{[i]}}{1-\theta}\right)$$

$$= \sum_{i=0}^{N-1} \frac{w^{[i]}(1-\theta) - \theta(1-w^{[i]})}{\theta(1-\theta)}$$

$$= \sum_{i=0}^{N-1} \frac{w^{[i]} - \cancel{w^{[i]}\theta} - \theta + \cancel{\theta w^{[i]}}}{\theta(1-\theta)}$$

$$= \sum_{i=0}^{N-1} \frac{w^{[i]} - \theta}{\cancel{\theta(1-\theta)}} \overset{!}{=} 0$$

$$\Leftrightarrow \sum_{i=0}^{N-1}(w^{[i]} - \theta) = 0$$

$$\sum_{i=0}^{N-1} w^{[i]} - \sum_{i=0}^{N-1}\theta = 0$$

$$\sum_{i=0}^{N-1} w^{[i]} = \sum_{i=0}^{N-1}\theta$$

$$\sum_{i=0}^{N-1} w^{[i]} = \theta \cdot N$$

$$\Leftrightarrow \theta = \underbrace{\frac{1}{N}\sum_{i=0}^{N-1} w^{[i]}}_{\text{mean of the dataset}} = \theta^{*}_{MLE}$$