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The Hydrogen Atom

Understanding Bohr's Model of the Hydrogen Atom and Introduction to Wave Mechanics

Utpal Anand

Science OpensoUrce Software for Teaching Learning
Free/Libre and Open Source Software for Education
IIT Bombay

Pune, December 2024

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Declaration

I, the undersigned, hereby declare that this handbook, entitled “The Hydrogen Atom”, has been compiled to serve as a resource for the study of physics. This work brings together knowledge from various scientific contributions to provide a comprehensive guide for students and learners.

This handbook is intended solely as an educational resource and does not claim originality in its entirety, as it is a compilation of existing scientific knowledge curated for academic purposes.

Pune, December 2024

Utpal Anand

Message to the Readers

Dear Readers,

Welcome to this module, where we delve into the fascinating journey of atomic models—exploring their creation, evolution, and the profound mathematical and philosophical ideas that underpin them.

This module is structured to cater to diverse readers:

- **For Beginners**: Start to understand how the concepts came into existence, ignore mathematical description if seems too advance. Refer Chapter 11 for creating interest.
- **For Intermediate Readers**: Engage with sections that is making to think how and why and try to understamd the mathematical building and experimental designs.
- **For Advanced Learners**: Explore the deeper understanding that led to the foundation of quantum mechanics and QED . Check the The Hydrogen Atom orbitals made by the wave functions using the quantum mechanics.

Some Chapters include exercise , please work out to get some idea about the mathematical and philosophical understanding behind it. I invite you to approach this material with curiosity, as it aims to provide both clarity and inspiration.

Thank you for choosing this module to deepen your understanding of atomic models.

Pune, December 2024

Utpal Anand

Abstract

This module explores Bohr's model of the hydrogen atom, covering its foundational aspects, features, and limitations. The timeline of the development of atomic models is discussed, highlighting the contributions of various scientists and physicists. Many earlier models were rejected, but Bohr's model became the first successful description of the atom. The assumptions, results, and limitations of Bohr's model are presented, along with the model's evolution over time. Bohr's model proposes that electrons orbit the nucleus in discrete, quantized orbits, with angular momentum $L = n\hbar$ (where n is an integer). Although based on classical mechanics and early quantum ideas, the model does not involve the wave properties of electrons. It explains the spectral lines of hydrogen but fails for multi-electron atoms and more complex phenomena.

Keywords: Bohr's Atomic Model, Atomic Models, Quantum Mechanics

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1

Introduction

1.1 Timeline of Atomic Models and Key Developments

1.1.1 Ancient Greek Philosophers (400 BCE)

- **Who:** Leucippus and Democritus.
- **What:** Proposed that matter is made up of small, indivisible particles called "atoms."
- **Significance:** Philosophical idea, not scientific. The concept laid the foundation for later atomic theory.
- **Outcome:** No experimental basis, abandoned until the 18th century.

1.1.2 Dalton's Atomic Theory (1803)

- **Who:** John Dalton.
- **What:** Proposed that atoms are indivisible particles, each element has unique atoms, and atoms combine in simple ratios to form compounds.
- **Significance:** First scientific atomic theory based on experimental evidence.
- **Success:** Explained laws of conservation of mass and definite proportions.
- **Limitation:** Did not explain the internal structure of atoms.

1.1.3 Discovery of the Electron (1897)

- **Who:** J.J. Thomson.
- **What:** Discovered the electron using cathode ray experiments.
- **Significance:** Showed that atoms are not indivisible and contain smaller particles.
- **Success:** Introduced the idea of subatomic particles.
- **Limitation:** Did not explain how electrons were arranged or why atoms were stable.

1.1.4 Thomson's Plum Pudding Model (1904)

- **Who:** J.J. Thomson.
- **What:** Proposed that the atom is a sphere of positive charge with electrons embedded like "raisins in a pudding."
- **Significance:** Early attempt to describe atomic structure.
- **Limitation:** Could not explain atomic stability or spectral lines.

1.1.5 Rutherford's Nuclear Model (1911)

- **Who:** Ernest Rutherford.
- **What:** Gold foil experiment showed that atoms have a dense, positively charged nucleus with electrons orbiting around it.
- **Significance:** Disproved the Plum Pudding Model; introduced the nucleus.
- **Success:** Explained the atom's structure better and introduced the concept of the nucleus.
- **Limitation:** Failed to explain why electrons don't spiral into the nucleus (classical physics predicts instability) or spectral lines.

1.1.6 Bohr's Model of the Hydrogen Atom (1913)

- **Who:** Niels Bohr.
- **What:** Combined Rutherford's model with quantum ideas. Proposed that:
  - Electrons orbit the nucleus in specific quantized energy levels.
  - Angular momentum is quantized ($mvr = n\hbar$).
  - Electrons transition between energy levels by emitting or absorbing photons.
- **Significance:** Explained the hydrogen atom's spectral lines using quantum mechanics.
- **Success:** Correctly predicted the Rydberg formula for hydrogen's spectral lines.
- **Limitation:** Could not explain spectra of atoms with more than one electron or fine spectral lines (spin effects).

1.1.7 Sommerfeld's Extension of Bohr's Model (1916)

- **Who:** Arnold Sommerfeld.
- **What:** Extended Bohr's model by introducing elliptical orbits and additional quantum numbers (azimuthal and magnetic).
- **Significance:** Improved accuracy for explaining fine spectral lines and relativistic effects.
- **Limitation:** Still failed for multi-electron atoms.

1.1.8 Discovery of de Broglie Waves (1924)

- **Who:** Louis de Broglie.
- **What:** Proposed wave-particle duality, where electrons have a wavelength ($\lambda = \frac{h}{p}$).
- **Significance:** Provided a deeper basis for Bohr's quantized orbits as standing waves.
- **Success:** Linked quantum mechanics with atomic structure.
- **Limitation:** Needed experimental validation, which came later.

1.1.9 Schrödinger's Wave Mechanics (1926)

- **Who:** Erwin Schrödinger.
- **What:** Developed the Schrödinger equation, describing electrons as wavefunctions rather than particles in fixed orbits.
- **Significance:** Replaced Bohr's model with the concept of orbitals (regions of probability).
- **Success:** Accurately described hydrogen's energy levels and extended to multi-electron atoms.
- **Limitation:** Did not account for spin or relativistic effects initially.

1.1.10 Discovery of Electron Spin (1925)

- **Who:** George Uhlenbeck and Samuel Goudsmit.
- **What:** Proposed that electrons have intrinsic angular momentum (spin).
- **Significance:** Explained the fine structure of spectral lines.
- **Success:** Incorporated into quantum mechanics.
- **Limitation:** Still needed a complete quantum framework.

1.1.11 Dirac's Relativistic Quantum Mechanics (1928)

- **Who:** Paul Dirac.
- **What:** Combined quantum mechanics with special relativity, predicting electron spin naturally and introducing the concept of antimatter.
- **Significance:** Explained fine structure and spin-orbit coupling for hydrogen.
- **Success:** Highly accurate for hydrogen and hydrogen-like atoms.
- **Limitation:** Complex mathematics and still not fully general.

1.1.12 Development of Quantum Electrodynamics (1940s)

- **Who:** Richard Feynman, Julian Schwinger, Sin-Itiro Tomonaga, and others.
- **What:** Unified quantum mechanics and electromagnetic theory to describe interactions between charged particles and photons.
- **Significance:** Provided precise predictions for the hydrogen atom's energy levels (e.g., Lamb shift).
- **Success:** Matched experimental results to unprecedented accuracy.
- **Limitation:** Extremely complex and computationally intensive.

1.1.13 Experimental Confirmation of the Hydrogen Atom Models

- **Who:** Many physicists over time.
- **What:** Techniques like spectroscopy and particle accelerators have continually tested and confirmed predictions from various models.
- **Significance:** Validated quantum theories and opened doors to modern atomic physics.

Table 1.1: *Summary of Atomic Models*

Model	Key Contributor	Success	Limitation
Dalton's Atomic Theory	John Dalton	Explained chemical combinations.	No internal atomic structure.
Plum Pudding Model	J.J. Thomson	Introduced subatomic particles.	Failed to explain atomic structure.
Rutherford's Nuclear Model	Ernest Rutherford	Introduced the nucleus and electron orbits.	Predicted atomic collapse.
Bohr's Model	Niels Bohr	Explained hydrogen spectral lines.	Failed for multi-electron atoms.
Schrödinger's Model	Erwin Schrödinger	Introduced orbitals and wavefunctions.	Initially neglected spin effects.
Quantum Electrodynamics	Feynman, Schwinger, etc.	Unified quantum mechanics and electromagnetism.	Extremely complex but successful.

2

Black-Body Radiation

2.1 Black-Body Radiation

Black-body radiation presents an intriguing problem in physics. A black body is an idealized object that absorbs all radiation, irrespective of frequency. When heated, it emits radiation across all frequencies. We can conceptualize such a body as a cavity within a solid material maintained at a constant temperature, T. The radiation inside the cavity reaches thermal equilibrium at this temperature.The fundamental question is: What is the energy density of radiation at a given frequency ν emitted by a black body?

2.1.1 Modelling the Cavity

To study this, imagine the cavity as a cubical space with side length L inside a conductor. The electromagnetic waves within this cavity form standing waves, as these waves must vanish at the cavity walls to maintain equilibrium. For simplicity, consider one-dimensional waves that vanish at $x = 0$ and $x = L$. In this setup, the wave completes an integral number n of half-periods within the cavity:

$$L = \frac{n\lambda}{2}$$

2.1.2 Wave Vector and Frequency

The wave number corresponding to these waves is $k = \frac{\pi}{L}$, and since n is positive, the standing wave remains unchanged under a sign change of n. Extending this idea to three dimensions, the wave number k becomes a vector $\vec{k}$ with components:

$$k_x, k_y, k_z = \frac{\pi}{L}(n_x, n_y, n_z)$$

Here, n_x, n_y, n_z are integers greater than or equal to 1. Consequently, the magnitude of the wave vector is:

$$|\vec{k}|^2 = \frac{\pi^2}{L^2}(n_x^2 + n_y^2 + n_z^2)$$

The frequency ν relates to $|\vec{k}|$ by $\nu = \frac{c}{2\pi}|\vec{k}|$, leading to:

$$n_x^2 + n_y^2 + n_z^2 = \frac{4L^2}{c^2}\nu^2$$

2.1.3 Boundary Conditions and the Wave Equation

To derive these relationships more formally, consider the wave equation for the electric field:

$$\nabla^2 \vec{E} = \frac{\partial^2 \vec{E}}{\partial t^2}$$

The boundary conditions for the cubical cavity require the electric field $\vec{E}$ to vanish at the cavity walls:

$$\vec{E}(x=0,y,z) = \vec{E}(x=L,y,z) = \vec{E}(x,y=0,z) = \vec{E}(x,y=L,z) = \vec{E}(x,y,z=0) = \vec{E}(x,y,z=L) = 0$$

These conditions are satisfied by:

$$\vec{E}(\vec{x},t) = \vec{E}_0 \sin\left(\frac{n_x \pi x}{L}\right) \sin\left(\frac{n_y \pi y}{L}\right) \sin\left(\frac{n_z \pi z}{L}\right) \sin\left(\frac{2\pi c t}{\lambda}\right)$$

2.1.4 Energy Density and Radiation Law

From these conditions, we derive the number of allowed modes $N(\nu)d\nu$ in the frequency range ν to $\nu + d\nu$. Using the shell volume in (n_x, n_y, n_z)-space, adjusted for one octant and two light polarizations, we obtain:

$$N_\nu d\nu = 8\pi \frac{L^3}{c^3} \nu^2 d\nu$$

Dividing by the cavity volume L^3, the mode density per unit volume is:

$$n_\nu = \frac{N_\nu}{L^3} = 8\pi \frac{\nu^2}{c^3} d\nu$$

To find the energy density, we multiply the mode density by the average energy of each mode, derived using the Boltzmann distribution. In classical physics, this average energy is $\langle E \rangle = k_B T$, leading to the Rayleigh-Jeans law:

$$u_\nu(T) = 8\pi k_B T \frac{\nu^2}{c^3}$$

While accurate at low frequencies, this formula predicts infinite energy density at high frequencies, a contradiction known as the "ultraviolet catastrophe."

2.1.5 Planck's Solution

In 1900, Planck resolved this by proposing that radiation energy is quantized, with energy $h\nu$ per photon. The possible energies are $nh\nu$ (where n is an integer). This leads to a modified average energy:

$$\langle E \rangle = \frac{h\nu}{e^{h\nu/k_B T} - 1}$$

Substituting this into the energy density gives Planck's radiation law:

$$u_\nu(T) = 8\pi \frac{h\nu^3}{c^3} \frac{1}{e^{h\nu/k_B T} - 1}$$

For $h\nu \ll k_B T$, Planck's law reduces to the Rayleigh-Jeans law. However, at high frequencies ($h\nu > k_B T$), the energy density decreases rapidly, avoiding the ultraviolet catastrophe and aligning perfectly with experimental results.

Planck's quantization marked a pivotal shift in physics, bridging classical and quantum theories.

2.2 Exercise

2.2.1 Conceptual Questions

1. Define a black body. What is its significance in the study of radiation?
2. Write the formula relating wavelength λ to the side length L of the cavity for standing waves in one dimension.
3. What are the boundary conditions for the electric field $\vec{E}$ in the cubical cavity?
4. Explain why the Rayleigh-Jeans law fails at high frequencies.
5. What modification did Planck introduce to the classical theory of black-body radiation?

2.2.2 Derivations and Applications

1. Derive the expression for the wave number k in terms of n and L for a one-dimensional wave in the cavity.
2. Extend the relationship for k to three dimensions and show how $|\vec{k}|^2 = \frac{\pi^2}{L^2}(n_x^2 + n_y^2 + n_z^2)$.
3. Calculate the number of allowed modes N_ν in the frequency range ν to $\nu + d\nu$ for a cubic cavity of side length $L = 1\,\text{m}$. Assume $c = 3 \times 10^8\,\text{m/s}$.
4. Derive the expression for mode density per unit volume, $n_\nu = \frac{N_\nu}{L^3}$.
5. Using the Rayleigh-Jeans law, show that energy density $u_\nu(T)$ diverges as $\nu \to \infty$.

2.2.3 Planck's Radiation Law

1. Show how the quantization of energy $E = nh\nu$ leads to the average energy per mode $\langle E \rangle = \frac{h\nu}{e^{h\nu/k_BT}-1}$ using Boltzmann statistics.
2. Derive Planck's radiation law:

$$u_\nu(T) = 8\pi \frac{h\nu^3}{c^3} \frac{1}{e^{h\nu/k_BT} - 1}$$

   starting from the mode density and quantized energy distribution.
3. For a black body at $T = 5000\,\text{K}$, calculate the energy density $u_\nu(T)$ at $\nu = 6 \times 10^{14}\,\text{Hz}$. Use $h = 6.63 \times 10^{-34}\,\text{J·s}$, $k_B = 1.38 \times 10^{-23}\,\text{J/K}$, and $c = 3 \times 10^8\,\text{m/s}$.
4. Verify that Planck's radiation law reduces to the Rayleigh-Jeans law when $h\nu \ll k_BT$.
5. Explain mathematically why Planck's law avoids the ultraviolet catastrophe, focusing on the behavior of the exponential term $e^{h\nu/k_BT}$.

3

Planck's Quantum Hypothesis

3.1 Planck's Quantization and Blackbody Radiation

Background

Max Planck deduced the relationship $E = hf$ as part of his work on blackbody radiation in 1900. His deduction was revolutionary, as it introduced the concept of quantization in physics, laying the foundation for quantum mechanics.

The Problem: Blackbody Radiation

Physicists were trying to understand the spectrum of electromagnetic radiation emitted by a perfect blackbody (an idealized object that absorbs and emits all radiation).

- Classical physics, specifically the Rayleigh-Jeans law, predicted that the energy emitted at shorter wavelengths (higher frequencies) would diverge to infinity—a problem known as the *ultraviolet catastrophe*.
- Experimental results, however, showed that blackbody radiation reached a peak at a specific wavelength and then dropped off.

Planck's Hypothesis

Planck sought to resolve this discrepancy by proposing a radical assumption:

1. **Energy Quantization:** Energy is not continuous but quantized, meaning it can only be emitted or absorbed in discrete packets (later called *quanta*).

$$E = hf$$

   where:

   - E = energy of the quantum,
   - f = frequency of radiation,
   - h = Planck's constant (6.626×10^{-34} J·s).

Derivation of $E = hf$

Planck derived $E = hf$ while trying to fit the blackbody radiation curve to experimental data. Here is a simplified outline:

1. **Energy Distribution Assumption:** Planck assumed that the electromagnetic radiation inside the blackbody cavity is due to oscillators (now interpreted as vibrating atoms or molecules). Each oscillator can only have discrete energy levels given by:

$$E_n = nhf \quad \text{where } n = 0, 1, 2, \ldots$$

2. **Probability of Energy States:** Using statistical mechanics, he calculated the probability of an oscillator being in a specific energy state E_n:

$$P_n \propto e^{-\frac{E_n}{k_B T}},$$

where k_B is the Boltzmann constant and T is the temperature.

3. **Average Energy of an Oscillator:** The average energy of an oscillator is:

$$\langle E \rangle = \frac{\sum_{n=0}^{\infty} E_n e^{-\frac{E_n}{k_B T}}}{\sum_{n=0}^{\infty} e^{-\frac{E_n}{k_B T}}}.$$

Substituting $E_n = nhf$, the sums can be evaluated to yield:

$$\langle E \rangle = \frac{hf}{e^{\frac{hf}{k_B T}} - 1}.$$

4. **Planck's Radiation Law:** Using the average energy $\langle E \rangle$, Planck derived the energy density $u(f, T)$ of radiation at a given frequency f:

$$u(f, T) = \frac{8\pi f^2}{c^3} \cdot \frac{hf}{e^{\frac{hf}{k_B T}} - 1}.$$

This law matched experimental data perfectly.

Significance of Quantization

Planck's introduction of $E = hf$ as a mathematical tool to solve the blackbody radiation problem turned out to have deeper implications:

- It contradicted classical physics, where energy was assumed to be continuous.
- It introduced Planck's constant h, which became a cornerstone of quantum theory.
- It paved the way for future developments in atomic theory, quantum mechanics, and modern physics.
- Albert Einstein later extended this idea to explain the *photoelectric effect*, solidifying the foundation of quantum mechanics.

How Discretization Helped Match Experimental Data

Planck's quantization assumption resolved the ultraviolet catastrophe by modifying the way energy states contribute to radiation:

1. In classical physics, energy levels are continuous, and high-frequency modes ($f \to \infty$) contribute infinitely to energy density, leading to a divergence.

2. With quantization ($E_n = nhf$), high-frequency modes require larger energy quanta, making them statistically improbable at typical thermal energies ($k_B T$). This exponentially suppresses their contribution:
$$u(f, T) \propto \frac{hf}{e^{hf/k_B T} - 1}.$$
3. At low frequencies ($hf \ll k_B T$), the law approximates the Rayleigh-Jeans result:
$$u(f, T) \propto f^2 k_B T.$$
4. At high frequencies ($hf \gg k_B T$), the exponential term dominates, suppressing $u(f, T)$ and avoiding divergence.

Thus, discretization accurately modeled the experimental curve, avoiding the ultraviolet catastrophe and matching observed radiation spectra across all wavelengths.

3.2 Exercise

1. Explain the ultraviolet catastrophe and how Planck's hypothesis resolves it.
2. Derive the Planck radiation law starting from the assumption of quantized energy levels $E_n = nhf$.
3. Calculate the energy density $u(f, T)$ for a blackbody at temperature $T = 3000\,\mathrm{K}$ and frequency $f = 10^{14}\,\mathrm{Hz}$ using Planck's radiation law. Use the constants $h = 6.626 \times 10^{-34}\,\mathrm{J{\cdot}s}$, $k_B = 1.38 \times 10^{-23}\,\mathrm{J/K}$, and $c = 3 \times 10^8\,\mathrm{m/s}$.
4. Show that Planck's law reduces to the Rayleigh-Jeans law at low frequencies ($hf \ll k_B T$).
5. Discuss the physical significance of the factor $e^{hf/k_B T} - 1$ in Planck's law.
6. Explain the role of Planck's constant h in the development of quantum mechanics and its significance in $E = hf$.
7. How does the quantization of energy suppress contributions from high-frequency modes to the energy density in Planck's law?
8. What are the implications of Planck's hypothesis for the concept of energy in classical physics?
9. Explain how Planck's work on blackbody radiation influenced Einstein's explanation of the photoelectric effect.
10. Why was the assumption of energy quantization necessary for matching the blackbody radiation curve to experimental data?

4

Disagreement with Classical Physics

4.1 Larmor Formula and Its Derivation

Preliminaries

Setup:

- A charged particle of charge q undergoes acceleration $\mathbf{a}$.
- The particle emits electromagnetic radiation due to the changing electric field it creates.

Electromagnetic Radiation Field: The electric field at a distance r from a moving charge has two components:

- A *near-field* component $(1/r^2)$ associated with the electrostatic field.
- A *radiation field* $(1/r)$ associated with the changing dipole moment of the charge, which is relevant for power radiation.

Radiated Power: The key to finding the radiated power is calculating the energy flux through a spherical surface (Poynting vector), where the energy is carried by the radiation field.

—

Electromagnetic Fields of an Accelerated Charge

Retarded Potentials

The electric and magnetic fields generated by a moving charge are derived from the retarded potentials:

$$\phi(\mathbf{r}, t) = \frac{q}{4\pi\epsilon_0 |\mathbf{r} - \mathbf{r}'(t_r)|}, \quad \mathbf{A}(\mathbf{r}, t) = \frac{\mu_0 q \mathbf{v}(t_r)}{4\pi |\mathbf{r} - \mathbf{r}'(t_r)|}$$

where:

- ϕ is the scalar potential.
- $\mathbf{A}$ is the vector potential.
- $t_r = t - |\mathbf{r} - \mathbf{r}'|/c$ is the retarded time.

Radiation Fields

For a non-relativistic charge $(v \ll c)$, the dominant contributions to the electric $(\mathbf{E})$ and magnetic $(\mathbf{B})$ fields far from the charge are:

$$\mathbf{E}_{\text{rad}} = \frac{q}{4\pi\epsilon_0 c^2 r} \hat{\mathbf{n}} \times [\hat{\mathbf{n}} \times \mathbf{a}],$$

$$\mathbf{B}_{\text{rad}} = \frac{\mu_0 q}{4\pi r}\hat{\mathbf{n}} \times \mathbf{E}_{\text{rad}},$$

where:

- $\mathbf{a}$ is the instantaneous acceleration of the charge.
- $\hat{\mathbf{n}}$ is the unit vector pointing from the charge to the observation point.

—

Energy Flux (Poynting Vector)

The Poynting vector ($\mathbf{S}$) represents the energy flux per unit area due to the electromagnetic field:

$$\mathbf{S} = \frac{1}{\mu_0}\mathbf{E} \times \mathbf{B}.$$

Substitute the radiation fields into the Poynting vector:

$$\mathbf{S} = \frac{1}{\mu_0}\left(\mathbf{E}_{\text{rad}} \times \mathbf{B}_{\text{rad}}\right).$$

Using the relations for $\mathbf{E}_{\text{rad}}$ and $\mathbf{B}_{\text{rad}}$:

$$\mathbf{S} = \frac{q^2|\mathbf{a}|^2}{16\pi^2\epsilon_0 c^3 r^2}\sin^2\theta\,\hat{\mathbf{n}},$$

where θ is the angle between $\mathbf{a}$ and $\hat{\mathbf{n}}$.

—

Total Power Radiated

The total power radiated is the integral of the Poynting vector's radial component over a spherical surface of radius r:

$$P = \int \mathbf{S} \cdot d\mathbf{A}.$$

For a sphere of radius r:

$$P = \frac{q^2|\mathbf{a}|^2}{16\pi^2\epsilon_0 c^3 r^2}\int_0^{2\pi}\int_0^{\pi}\sin^2\theta\, r^2\sin\theta\, d\theta\, d\phi.$$

Evaluate the angular integrals:

$$\int_0^{2\pi} d\phi = 2\pi, \quad \int_0^{\pi}\sin^3\theta\, d\theta = \frac{4}{3}.$$

Substitute these results:

$$P = \frac{q^2|\mathbf{a}|^2}{16\pi^2\epsilon_0 c^3}\cdot 2\pi \cdot \frac{4}{3} = \frac{2}{3}\frac{q^2|\mathbf{a}|^2}{4\pi\epsilon_0 c^3}.$$

—

Final Result (Larmor Formula)

The radiated power is:

$$P = \frac{2}{3}\frac{q^2|\mathbf{a}|^2}{4\pi\epsilon_0 c^3}.$$

This is the **Larmor formula**, valid for non-relativistic charges ($v \ll c$).
—

Significance

- The Larmor formula explains why a classically accelerating electron emits electromagnetic radiation, causing energy loss.
- This was central to the instability of classical atomic models, as the radiated energy causes the electron to spiral into the nucleus.
- Bohr's quantized orbits resolve this issue by preventing radiation in stable states.

4.2 Energy Radiation and the Instability of Classical Atoms

Classical electrodynamics states that a charged particle undergoing acceleration emits electromagnetic radiation. The power radiated is given by the **Larmor formula**:

$$P = \frac{2}{3}\frac{e^2 a^2}{4\pi\epsilon_0 c^3},$$

where:

- P: Power radiated,
- $a = \frac{v^2}{r}$: Centripetal acceleration of the electron.

Rate of Energy Loss

As the electron radiates energy, its total energy E decreases, causing the radius r of the orbit to shrink over time. The rate of energy loss can be written as:

$$\frac{dE}{dt} = -P.$$

Collapse Time

The total time t for the electron to spiral into the nucleus can be estimated by integrating the rate of change of the orbital radius r. The collapse time t is approximately proportional to:

$$t \propto \frac{r_0^2 v_0}{c^3} \sim 10^{-8}\,\text{seconds},$$

where:

- r_0: Initial orbital radius,
- v_0: Initial orbital velocity,
- c: Speed of light.

Conclusion

This result implies that, according to classical physics, atoms would be unstable and collapse almost instantaneously, which contradicts experimental observations. This problem highlights the failure of classical physics in explaining atomic stability and led to the development of quantum mechanics.

4.3 Exercise

1. Derive the Larmor formula for the total power radiated by a non-relativistic accelerating charge.
2. Explain the role of the Poynting vector in calculating the power radiated by an accelerated charge. How is it related to the energy flux?
3. Discuss the significance of retarded potentials in the derivation of the radiation fields from an accelerating charge.
4. The radiation power is proportional to $\sin^2\theta$. Explain the physical meaning of this angular dependence and its implications for the energy distribution of radiation.
5. Using the Larmor formula, explain why classical physics predicts that atoms would collapse almost instantaneously.
6. A charge $q = 1.6\times10^{-19}$ C accelerates at $a = 10^{16}\,\text{m/s}^2$. Calculate the total power radiated using the Larmor formula. Use $c = 3\times10^8\,\text{m/s}$ and $\epsilon_0 = 8.85\times10^{-12}\,\text{F/m}$.
7. Derive an expression for the collapse time of a classical electron spiraling into the nucleus, assuming a circular orbit.
8. Classical physics predicts the instability of atoms due to radiation loss. Discuss how quantum mechanics resolves this problem.
9. Derive the expressions for the radiation fields $\mathbf{E}_{\text{rad}}$ and $\mathbf{B}_{\text{rad}}$ for a charge undergoing non-relativistic acceleration.
10. List the assumptions in the classical derivation of the Larmor formula. Under what conditions do these assumptions break down, and how does this lead to the need for quantum mechanics?

5

The Rydberg Formula

The Rydberg Formula(1888)

5.1 Johannes Rydberg and the Development of His Formula

1. Analyzing Spectral Lines

Scientists observed that heated elements emit light at specific, discrete wavelengths. These emissions, known as *atomic spectra*, form series of lines unique to each element.

2. Focus on Wavenumbers

Rydberg simplified the analysis by working with *wavenumbers* ($\tilde{\nu} = 1/\lambda$), which represent the number of waves per unit length. This approach made it easier to identify mathematical relationships.

3. Integer Relationships

By carefully plotting the wavenumbers of spectral lines within a series against integers, Rydberg discovered a consistent pattern. He expressed the wavenumbers as a difference of two terms, each involving the square of integers:

$$\frac{1}{\lambda} = R\left(\frac{1}{n_1^2} - \frac{1}{n_2^2}\right), \quad n_2 > n_1. \tag{5.1}$$

Here:

- λ is the wavelength of the spectral line,
- R is the Rydberg constant,
- n_1 and n_2 are positive integers (with $n_2 > n_1$).

4. Empirical Formula

Rydberg's formula was initially *empirical*, meaning it was derived purely from observational data without a theoretical foundation. It accurately described the spectral lines for hydrogen and other hydrogen-like atoms.

5. Bohr's Contribution

Niels Bohr later provided a theoretical explanation for Rydberg's formula using his *quantized model of the atom*. Bohr demonstrated that the integers n_1 and n_2 correspond to transitions between quantized electron energy levels, linking Rydberg's formula to early quantum mechanics.

Summary

Johannes Rydberg's meticulous analysis of atomic spectra led to the discovery of a mathematical relationship between wavenumbers of spectral lines. His empirical formula:

$$\frac{1}{\lambda} = R\left(\frac{1}{n_1^2} - \frac{1}{n_2^2}\right)$$

laid the foundation for Bohr's atomic model and contributed significantly to the development of quantum mechanics.

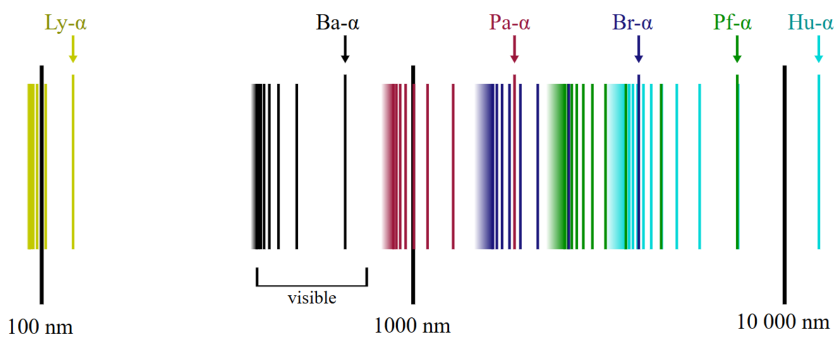

Figure 5.1: *The spectral series of hydrogen, on a logarithmic scale.*

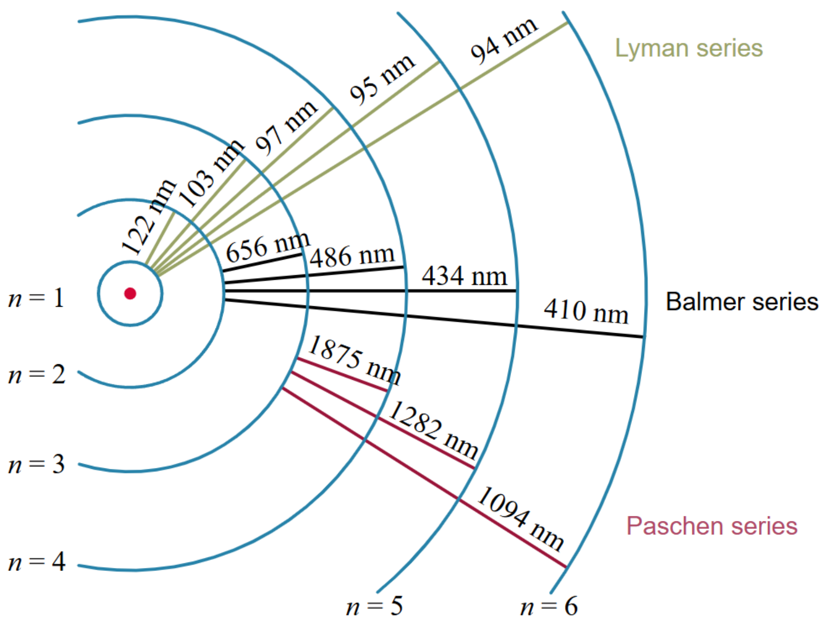

Figure 5.2: *Electron transitions and their resulting wavelengths for hydrogen. Energy levels are not to scale.*

5.2 Exercise

1. Derive an expression for the energy difference between two energy levels in a hydrogen atom using the Rydberg formula. Show how this relates to the frequency of the emitted photon.
2. The Rydberg formula is often applied to hydrogen-like ions (e.g., He^+). Derive the modified Rydberg constant for such ions and explain the role of the nuclear charge Z.
3. For the Balmer series $(n_1 = 2)$, calculate the limit of the series as $n_2 \to \infty$. Interpret the result physically.
4. Using the Rydberg formula, find the ratio of wavelengths between the first two lines of the Paschen series $(n_1 = 3)$.
5. Consider a spectral line with wavenumber $\tilde{\nu} = 1.522 \times 10^6\,\mathrm{m}^{-1}$. Determine the possible quantum numbers n_1 and n_2 for this transition, assuming it belongs to the hydrogen spectrum.
6. The transition $n_2 = 4 \to n_1 = 3$ in a hydrogen-like ion emits a photon with wavelength $820\,\mathrm{nm}$. Determine the nuclear charge Z of the ion.
7. Show that the Rydberg formula predicts a decreasing spacing between spectral lines as n_2 increases for a given n_1. Illustrate this behavior mathematically for the Lyman series $(n_1 = 1)$.
8. Verify that the Rydberg constant R_H has dimensions of length^{-1}. Use dimensional analysis to confirm the correctness of the formula.
9. Given the experimental value $R_H = 1.097{\times}10^7\,\mathrm{m}^{-1}$, calculate the ionization energy of hydrogen in electron volts (eV).
10. Mathematically derive the relationship between the wavelength of a spectral line and the energy of the corresponding photon. Use this to explain why shorter wavelengths correspond to higher-energy transitions.

6

Atomic Models

6.1 Dalton's Atomic Model (Early 19th Century)

- **Postulates:**
  1. All matter is made of indivisible and indestructible atoms.
  2. Atoms of a given element are identical in mass and properties.
  3. Atoms of different elements have different masses and properties.
  4. Chemical reactions involve the rearrangement of atoms. No atoms are created or destroyed in a chemical reaction.
  5. Compounds are formed by a combination of two or more different kinds of atoms.
- **Briefly:** Dalton envisioned atoms as solid, indivisible spheres, like tiny billiard balls.
- **Logical framework:** Dalton's model was primarily conceptual. It provided a logical framework for understanding chemical observations at the time, such as:
  - **Law of Conservation of Mass:** In a chemical reaction, the total mass of reactants equals the total mass of products. This is explained by the fact that atoms are neither created nor destroyed.
  - **Law of Definite Proportions:** A given chemical compound always contains its component elements in fixed ratio (by mass). This is explained by the idea that compounds are formed by combining specific numbers of atoms of different elements.

6.2 J.J. Thomson's Atomic Model (Late 19th Century)

- **Postulates:**
  1. Atoms are not indivisible but contain subatomic particles called electrons.
  2. Electrons are negatively charged particles with a small mass.
  3. The atom is a sphere of positive charge, with electrons embedded in it to balance the charge.
- **Briefly:** Thomson's model, often called the "plum pudding" model, pictured the atom as a positively charged sphere with negatively charged electrons scattered throughout, like plums in a pudding or raisins in a cake.
- **Basis of the model:** Thomson's model was based on his experiments with cathode rays, which demonstrated the existence of electrons.

- **Cathode Ray Experiments:** Thomson showed that cathode rays were streams of negatively charged particles (electrons) with a specific charge-to-mass ratio (e/m). This suggested that electrons were constituents of all atoms.
- **Logical Leap:** Since atoms are neutral overall, Thomson reasoned that there must be a positive charge within the atom to balance the negative charge of the electrons. He proposed a uniform sphere of positive charge as the simplest way to achieve this.

Key Differences and Advancements

- **Divisibility of Atoms:** Dalton considered atoms to be indivisible, while Thomson showed that atoms contain subatomic particles (electrons). This was a major shift in understanding atomic structure.
- **Internal Structure:** Dalton's model had no internal structure; atoms were simply featureless spheres. Thomson's model introduced the idea of internal structure, with electrons embedded in a positive charge.
- **Experimental Basis:** Dalton's model was based on laws of chemical combination, while Thomson's model was based on experimental evidence from cathode ray experiments.

Thomson's model was a significant step forward, but it was later superseded by Rutherford's nuclear model, which introduced the concept of a dense, positively charged nucleus at the center of the atom.

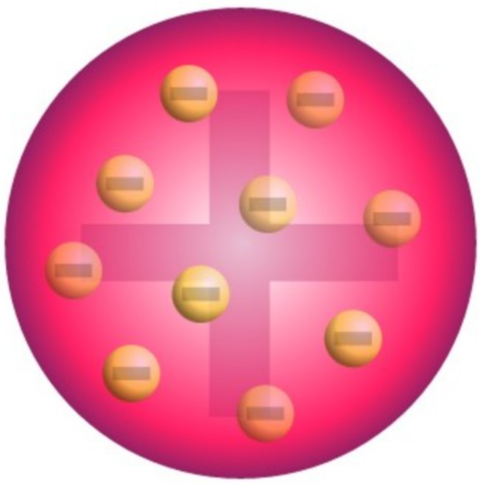

Figure 6.1: *JJ Thomson's Atomic Model*

6.3 Rutherford's Atomic Model (1911)

Gold Foil Experiment

- Conducted by **Ernest Rutherford**, assisted by *Geiger and Marsden*.
- A narrow beam of high-energy alpha particles (α) from a radioactive source was directed at an extremely thin gold foil (about $0.0004\,\text{cm}$ thick).
- A fluorescent screen coated with zinc sulfide was used to detect the scattered alpha particles.

Observations and Statistical Results

1. **Majority Passed Undeflected:**
   - About 99% of the α particles passed straight through the gold foil without any deflection.
2. **Small Deflections:**
   - A small fraction (about 1%) of α particles were deflected by small angles (a few degrees).
3. **Large Angle Deflections:**
   - An extremely small number (about 1 in 8000) of α particles were deflected at angles greater than 90°.

Conclusions

1. **Presence of a Nucleus:**
   - The large-angle deflections indicated the presence of a small, dense, and positively charged region in the atom, called the **nucleus**.
2. **Empty Space:**
   - Since most of the α particles passed through the foil undeflected, Rutherford concluded that atoms are mostly **empty space**.
3. **Nuclear Size:**
   - The nucleus is extremely small compared to the overall size of the atom.
   - Rutherford estimated the nucleus to be about 10^{-14} m in size, while the atom itself has a size of about 10^{-10} m.

Rutherford's Atomic Model

- The atom consists of:
  - A tiny, dense, positively charged **nucleus** at the center, containing most of the atom's mass.
  - Negatively charged **electrons** that revolve around the nucleus in the empty space.
- The atom is electrically neutral because the positive charge of the nucleus is balanced by the negative charge of electrons.

Limitations of Rutherford's Model

- Rutherford's model could not explain the **stability of the atom**.
- According to classical electrodynamics:
  - Electrons revolving around the nucleus would continuously emit radiation.
  - This loss of energy would cause electrons to spiral into the nucleus.
  - Thus, the atom would collapse, which does not happen in reality.

Summary

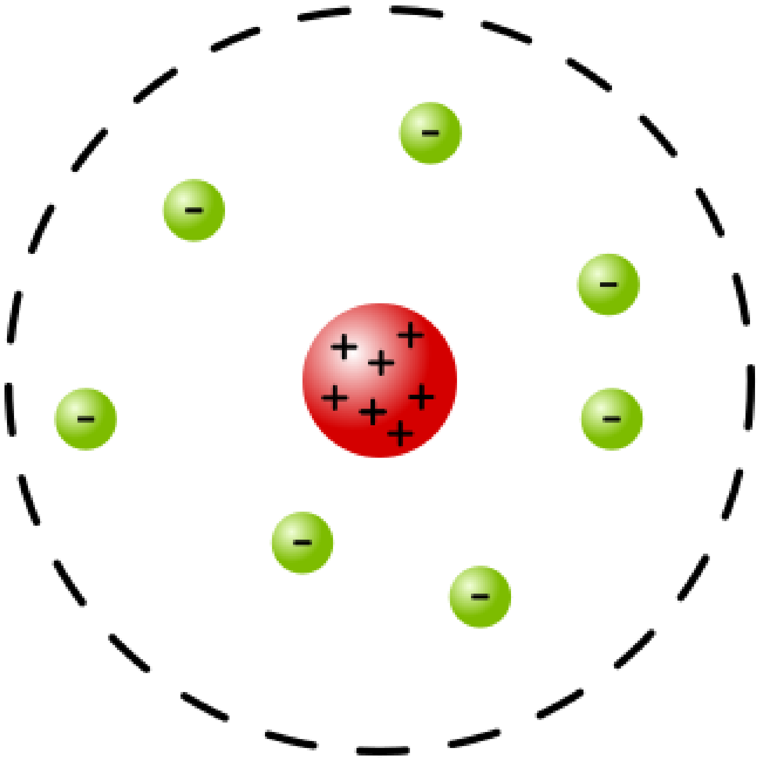

Figure 6.2: *Rutherford's Atomic Model*

Table 6.1: *Results of the Gold Foil Experiment*

Observation Type	**Percentage (%)**	**Conclusion**
No Deflection	99	Atoms are mostly empty space.
Small Angle Deflections	1	Presence of a positive nucleus.
Large Angle Deflections ($> 90^{\circ}$)	~ 0.01	Nucleus is dense and small.

7

Bohr's Contribution

7.1 Bohr's Quantization Hypothesis and Atomic Model

The Problem: Instability in Rutherford's Atomic Model

1. **Electron Radiation and Energy Loss:**
   - In Rutherford's model, electrons revolve around the nucleus in circular orbits, resulting in centripetal acceleration.
   - According to classical electromagnetism (Larmor formula), an accelerating charge emits electromagnetic radiation.
   - This radiation leads to energy loss, causing the electron to spiral inward and eventually collapse into the nucleus.
2. **Consequences of the Classical Problem:**
   - Atoms would collapse within a fraction of a second, making the existence of stable matter impossible.
   - The model could not explain the discrete spectral lines observed in atomic spectra (e.g., hydrogen spectrum).

Bohr's Hypotheses

To resolve these issues, **Niels Bohr** proposed two revolutionary postulates:

1. **Quantized Orbits:**
   - Electrons can occupy only specific orbits where the angular momentum is quantized:

$$L = m_e v r = n\hbar, \quad \text{where } n = 1, 2, 3, \ldots.$$

   - These orbits correspond to discrete energy levels, given by:

$$E_n = -\frac{R_H c}{n^2} = -\frac{13.6\,\text{eV}}{n^2}.$$

2. **No Radiation in Stable Orbits:**
   - An electron in a quantized orbit does not radiate energy, even though it is accelerating.

- Radiation occurs only when the electron transitions between two energy levels, emitting or absorbing a photon with energy:

$$\Delta E = E_f - E_i = h\nu.$$

How Quantization Solved the Problem

1. **Stability of Atoms:**
   - Electrons are restricted to quantized orbits, preventing continuous energy loss.
   - The lowest energy level ($n = 1$) corresponds to the most stable state, ensuring the electron does not spiral into the nucleus.
2. **Explanation of Spectral Lines:**
   - The discrete energy levels explain the observed spectral lines in hydrogen and other atoms.
   - Each spectral line corresponds to a transition between two quantized energy levels:

$$\nu = \frac{E_i - E_f}{h}.$$

Bohr's Atomic Model

- The atom consists of:
  - A dense, positively charged nucleus at the center.
  - Electrons revolving around the nucleus in quantized orbits without radiating energy.
- The energy levels are indexed by the principal quantum number n, with each level having a fixed energy:

$$E_n = -\frac{13.6\,\text{eV}}{n^2}.$$

Impact and Legacy

1. **Solved Classical Problems:**
   - Introduced stability to atomic structure.
   - Provided a theoretical explanation for the discrete spectral lines observed experimentally.
2. **Foundation for Quantum Mechanics:**
   - While later superseded by quantum mechanics, Bohr's model was a pivotal step in understanding atomic structure.

7.2 Mathematical Derivations in Bohr's Atomic Model

1. Centripetal Force and Coulomb's Law

In Bohr's model, the electron revolves around the nucleus under the influence of the Coulomb force, which provides the necessary centripetal force for circular motion.

$$F_c = F_e \quad \Longrightarrow \quad \frac{m_e v^2}{r} = \frac{1}{4\pi\epsilon_0}\frac{e^2}{r^2},$$

where:

- m_e is the mass of the electron,
- v is the velocity of the electron,
- r is the radius of the orbit,
- e is the elementary charge,
- ϵ_0 is the permittivity of free space.

Rearranging gives the relationship between v and r:

$$v^2 = \frac{1}{4\pi\epsilon_0}\frac{e^2}{m_e r}.$$

2. Quantization of Angular Momentum

Bohr postulated that the angular momentum of the electron is quantized and given by:

$$L = m_e v r = n\hbar, \quad n = 1, 2, 3, \ldots,$$

where $\hbar = \frac{h}{2\pi}$ is the reduced Planck's constant.

From this, the velocity v can be expressed as:

$$v = \frac{n\hbar}{m_e r}.$$

3. Radius of the Orbit

Substituting the expression for v from the angular momentum quantization into the centripetal force equation:

$$\left(\frac{n\hbar}{m_e r}\right)^2 = \frac{1}{4\pi\epsilon_0}\frac{e^2}{m_e r}.$$

Simplify to find the radius of the n-th orbit:

$$r_n = \frac{4\pi\epsilon_0 n^2\hbar^2}{m_e e^2}.$$

This shows that the radius depends on n^2, meaning higher energy levels correspond to larger orbits.

4. Total Energy of the Electron

The total energy E_n is the sum of the kinetic energy (K) and potential energy (U):

$$K = \frac{1}{2}m_e v^2, \quad U = -\frac{1}{4\pi\epsilon_0}\frac{e^2}{r}.$$

From the centripetal force equation, $m_e v^2 = \frac{1}{4\pi\epsilon_0}\frac{e^2}{r}$, so:

$$K = \frac{1}{2}\left(\frac{1}{4\pi\epsilon_0}\frac{e^2}{r}\right) = \frac{1}{8\pi\epsilon_0}\frac{e^2}{r}.$$

Thus, the total energy becomes:

$$E_n = K + U = \frac{1}{8\pi\epsilon_0}\frac{e^2}{r} - \frac{1}{4\pi\epsilon_0}\frac{e^2}{r}.$$

Simplify:

$$E_n = -\frac{1}{8\pi\epsilon_0}\frac{e^2}{r}.$$

Substitute r_n into this expression:

$$E_n = -\frac{1}{8\pi\epsilon_0}\frac{e^2}{\frac{4\pi\epsilon_0 n^2\hbar^2}{m_e e^2}}.$$

Simplify further:

$$E_n = -\frac{m_e e^4}{8\epsilon_0^2 h^2}\frac{1}{n^2}.$$

Let $R_H = \frac{m_e e^4}{8\epsilon_0^2 h^3 c}$, the Rydberg constant, and express E_n as:

$$E_n = -\frac{R_H c}{n^2}.$$

Note

Strictly speaking, it is necessary to make a small correction to this formula. The mass of the nucleus is certainly much bigger than that of the electron, but not infinite. Thus, the electron and atomic nucleus revolve about their common center of gravity, which is not exactly identical with the center of the atom. If we take this into consideration, we have to replace the mass of the electron (m) by the so-called reduced mass m' in the formula above:

Reduced mass of the electron:

$$m' = \frac{m_N m}{m_N + m}$$

Where:

- m ...mass of the electron
- m_N ...mass of the nucleus

5. Explanation of Spectral Lines

When an electron transitions between two energy levels n_i and n_f, a photon of energy ΔE is emitted or absorbed:

$$\Delta E = E_{n_f} - E_{n_i} = -R_H c\left(\frac{1}{n_f^2} - \frac{1}{n_i^2}\right).$$

The corresponding frequency of the emitted or absorbed photon is:

$$\nu = \frac{\Delta E}{h} = R_H\left(\frac{1}{n_i^2} - \frac{1}{n_f^2}\right).$$

This matches the experimentally observed spectral lines, providing strong support for Bohr's model.

Bohr's Model depicted on Apps on physics

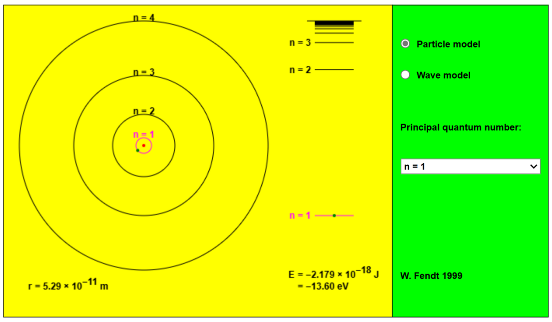

Figure 7.1: *Electron depicted as a particle in Bohr's Model (n=1)*

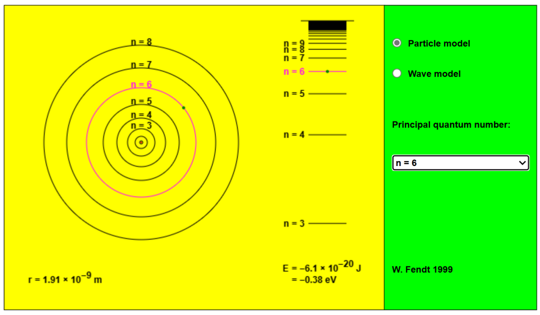

Figure 7.2: *Electron depicted as a particle in Bohr's Model (n=6)*

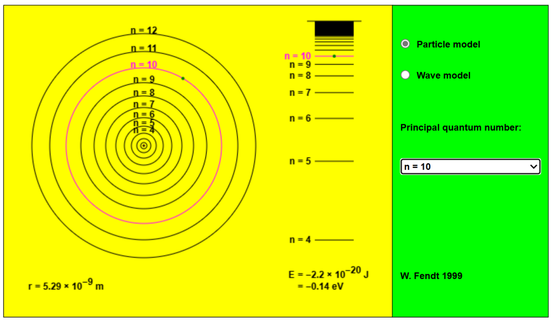

Figure 7.3: *Electron depicted as a particle in Bohr's Model (n=10)*

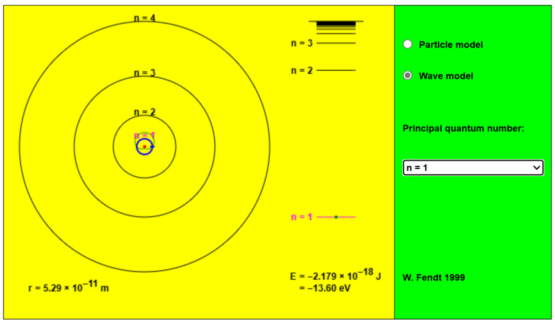

Figure 7.4: *Electron depicted as a wave in Bohr's Model (n=1)*

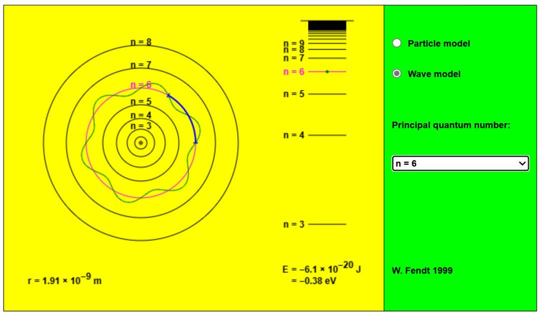

Figure 7.5: *Electron depicted as a wave in Bohr's Model (n=6)*

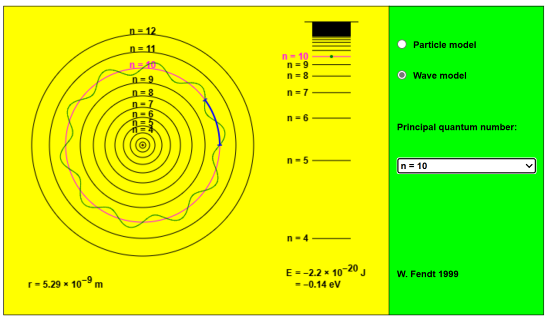

Figure 7.6: *Electron depicted as a wave in Bohr's Model (n=10)*

7.3 Exercise

1. Derive the expression for the radius of the n-th orbit in Bohr's atomic model and show its dependence on the principal quantum number n.
2. Using the concept of reduced mass, derive the modified formula for the energy levels in Bohr's atomic model.
3. Explain why the angular momentum quantization condition $L = n\hbar$ leads to discrete energy levels. Derive the expression for energy levels $E_n = -\frac{13.6\,\text{eV}}{n^2}$.
4. For a hydrogen atom, calculate the wavelength of the photon emitted during the transition from $n = 3$ to $n = 2$.
5. Show that the total energy of the electron in the n-th orbit is half of its potential energy in magnitude but negative in sign. Verify this relation mathematically.
6. Demonstrate that as $n \longrightarrow \infty$, the energy difference between adjacent levels $E_{n+1} - E_n$ approaches zero.
7. In a hydrogen-like ion with $Z = 3$, calculate the energy difference between the first and second energy levels. Compare this value to that of a hydrogen atom.
8. Using the Bohr model, determine the frequency of the spectral line corresponding to the $n = 5$ to $n = 3$ transition in a hydrogen atom. Express your answer in Hz.
9. Derive the expression for the velocity of the electron in the n-th orbit and show its proportionality to $1/n$.
10. Explain the role of the Coulomb force in balancing the centripetal force in Bohr's model. Derive the mathematical relationship between the two forces.

8

De Broglie's Wave-Particle Duality

8.1 De Broglie's Wave-Particle Duality

1. Wave Nature of Matter

Louis de Broglie proposed that every moving particle is associated with a wave, known as a "matter wave," with a wavelength given by:

$$\lambda = \frac{h}{p},$$

where:

- λ is the wavelength of the matter wave,
- h is Planck's constant,
- $p = mv$ is the momentum of the particle (non-relativistic case).

2. Generalization for Relativistic Particles

For particles with relativistic momentum, the wavelength is expressed as:

$$p = \frac{\sqrt{E^2 - (mc^2)^2}}{c}, \quad \lambda = \frac{h}{p},$$

where:

- E is the total energy,
- m is the mass of the particle,
- c is the speed of light.

3. Supporting Experiments

1. **Davisson-Germer Experiment:** Confirmed electron diffraction, proving the wave nature of electrons.
2. **Double-Slit Experiment:** Demonstrated interference patterns for particles, consistent with wave behavior.

4. Implications for Quantum Mechanics

De Broglie's hypothesis laid the foundation for quantum mechanics by:

- Explaining the quantization of electron orbits in Bohr's model as standing waves,
- Introducing the concept of matter waves, leading to Schrödinger's wave equation.

Why He Proposed It (His Thinking)

De Broglie was inspired by the following ideas:

1. **Einstein's Explanation of Photons (1905):** Light exhibits particle-like behavior, as demonstrated by photons with:
$$E = h\nu \quad \text{(energy)} \quad \text{and} \quad p = \frac{h}{\lambda} \quad \text{(momentum)}.$$
2. **Symmetry in Nature:** De Broglie believed that symmetry was a fundamental principle of nature. If waves (such as light) could exhibit particle-like behavior, then particles should also exhibit wave-like properties.
3. **Quantized Orbits in Bohr's Model:** He reasoned that the quantized orbits of electrons in atoms might correspond to standing waves, providing a natural explanation for the stability of these orbits.

8.2 Interference Experiments

Why Matter Wave?

Thomas Young's Double-Slit Experiment (1801):
Thomas Young demonstrated the wave nature of light by conducting the famous double-slit experiment. When light passed through two closely spaced slits, it produced alternating bright and dark fringes on a screen, an effect explained by constructive and destructive interference of light waves. This experiment provided strong evidence that light behaves as a wave.
Fresnel's Contribution (1815–1820):
Building on Young's work, Augustin-Jean Fresnel refined the mathematical description of interference using wave theory. His contributions further solidified the understanding of light as a wave and explained many optical phenomena with remarkable precision.
Significance for Wave-Particle Duality:
These experiments highlighted the wave nature of light, but the later discovery of phenomena like the photoelectric effect suggested that light also exhibits particle-like behavior. Together, these findings laid the foundation for the concept of wave-particle duality, where light behaves as both a wave and a particle depending on the context.

8.3 Concept of Duality in Physics

How Duality conept came into existence?

The timeline of wave-particle duality begins with 1905, when Albert Einstein extended Planck's quantum hypothesis to light, proposing that light behaves both as a wave and as particles (photons) in his explanation of the photoelectric effect. This idea laid the foundation for understanding the dual nature of light. Nearly two decades later, in 1924, Louis de Broglie extended the concept of wave-particle duality to matter, proposing that particles like electrons also exhibit wave-like properties, with their wavelength related to their momentum. This hypothesis was experimentally confirmed in 1927 by the electron diffraction experiments conducted by Davisson and Germer, providing direct evidence of the wave-like behavior of particles and marking a pivotal moment in the development of quantum mechanics.

9

Franck-Hertz and Stern-Gerlach Experiment

9.1 Franck-Hertz Experiment (1914)

Objective

The objective of the Franck-Hertz experiment was to demonstrate the quantization of energy levels in atoms, providing experimental support for the Bohr model of the atom.

Experimental Setup

The apparatus consisted of:

- A vacuum tube filled with mercury vapor.
- A heated cathode to emit electrons through thermionic emission.
- A grid to accelerate electrons towards an anode.
- A retarding potential to prevent low-energy electrons from reaching the anode.
- An ammeter to measure the current at the anode.

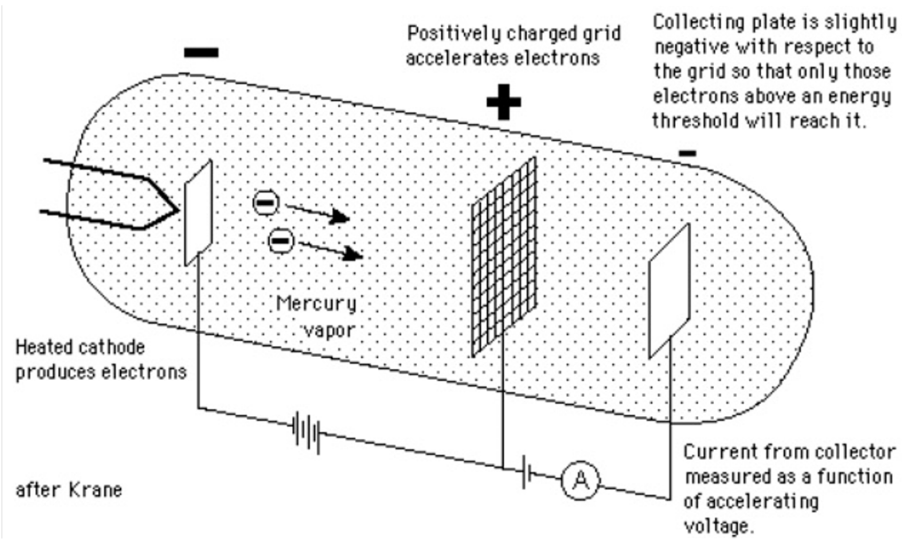

Figure 9.1: *Experimental setup of the Franck-Hertz experiment.*

Procedure

- Electrons are emitted from the cathode and accelerated through mercury vapor.

- As the electrons gain kinetic energy by traveling through the accelerating potential V, their energy is given by:

$$K.E. = eV,$$

where e is the electron charge and V is the accelerating voltage.
- When the electron's kinetic energy matches the energy difference between two quantized energy levels of the mercury atom, it transfers its energy to the atom, exciting it to a higher energy state.
- This inelastic collision results in a sharp drop in the measured current at the anode, as the electrons lose energy and can no longer overcome the retarding potential.

Mathematical Basis

The energy difference between two quantized energy levels of a mercury atom is:

$$\Delta E = h\nu,$$

where h is Planck's constant and ν is the frequency of the emitted photon when the atom de-excites. The energy of the accelerated electrons must equal this difference:

$$eV = \Delta E.$$

Results and Conclusions

- The experiment showed periodic drops in the current as the accelerating voltage increased, corresponding to quantized energy absorption by the mercury atoms.
- The energy required to excite a mercury atom was measured as approximately 4.9 eV, consistent with spectroscopic data.
- This experiment confirmed the quantized nature of atomic energy levels and supported the Bohr model.

—

9.2 Stern-Gerlach Experiment (1922)

Objective

The objective of the Stern-Gerlach experiment was to demonstrate the quantization of angular momentum and provide evidence for intrinsic spin in particles.

Experimental Setup

The apparatus consisted of:

- A source of silver atoms heated in an oven to create a beam of neutral particles.
- A collimator to narrow the beam.
- A non-uniform magnetic field to exert a spatially varying force on the magnetic moments of the atoms.
- A detector (photographic plate) to observe the deflected beam.

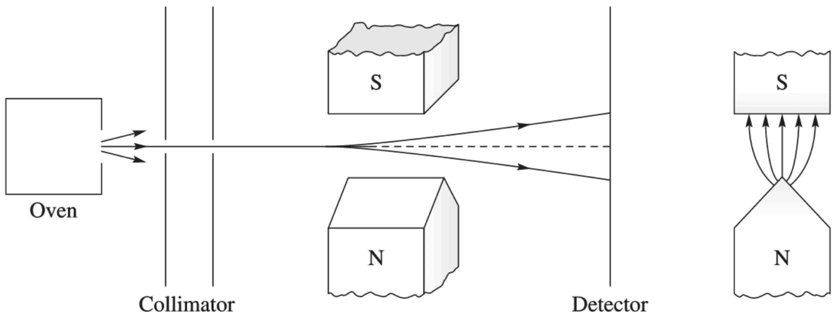

Figure 9.2: *Experimental setup of the Stern-Gerlach experiment.*

Procedure

- A beam of silver atoms was passed through a non-uniform magnetic field.
- The magnetic field gradient produced a force on the magnetic moment μ of the atoms:

$$F = \mu \frac{\partial B}{\partial z},$$

  where $\frac{\partial B}{\partial z}$ is the magnetic field gradient along the z-axis.
- The force caused the atoms to be deflected in discrete directions depending on their quantized magnetic moment.

Mathematical Basis

The quantized magnetic moment is related to the spin quantum number m_s as:

$$\mu_z = -g_s \mu_B m_s,$$

where:

- g_s is the gyromagnetic ratio,
- μ_B is the Bohr magneton,
- m_s is the spin quantum number ($m_s = \pm\frac{1}{2}$ for electrons).

The deflection Δz on the detector is proportional to the force F and the time of flight of the atoms.

Results and Conclusions

- The silver atom beam split into two distinct spots on the detector, corresponding to $m_s = +\frac{1}{2}$ and $m_s = -\frac{1}{2}$.
- This provided direct evidence for the quantization of angular momentum and the existence of intrinsic spin.
- The experiment also confirmed the quantum mechanical concept that spin is a fundamental property of particles.

10

Wave Mechanics

10.1 Wave Mechanics: A Fundamental View of the Quantum World

Wave mechanics, a branch of quantum mechanics, describes the behavior of matter at atomic and subatomic scales, blending wave-like and particle-like properties.

10.1.1 Core Principles

1. Wave-Particle Duality: - Particles such as electrons and photons exhibit both wave-like behaviors (interference, diffraction) and particle-like behaviors (definite positions and momenta).

2. Wave Function (Ψ): - A particle's state is described by a wave function Ψ, with $|\Psi|^2$ representing the probability density of finding the particle:

$$P(x,t) = |\Psi(x,t)|^2.$$

3. Schrödinger Equation: - Governs the evolution of the wave function: - Time-dependent form:

$$i\hbar\frac{\partial\Psi}{\partial t} = \hat{H}\Psi.$$

- Time-independent form:

$$\hat{H}\Psi = E\Psi,$$

used to find quantized energy levels.

4. Quantization: - Physical properties like energy and angular momentum take discrete values, contrasting with classical mechanics.

5. Uncertainty Principle: - Certain properties, like position (x) and momentum (p), have a measurement limit:

$$\Delta x \Delta p \geq \frac{\hbar}{2}.$$

6. Superposition: - Particles can exist in multiple states simultaneously, collapsing into a definite state upon measurement.

7. Quantum Tunneling: - Particles can penetrate potential barriers even without sufficient classical energy, with the probability determined by Ψ.

10.1.2 Summary of Key Insights

Wave mechanics provides a probabilistic framework to describe phenomena like: - Wave-particle duality, - Energy quantization, - Superposition, - Quantum tunneling, - The uncertainty principle.

These principles explain interference, diffraction, and other phenomena beyond classical mechanics.

10.1.3 Mathematical Representation

The time-independent Schrödinger equation:

$$\hat{H}\Psi = E\Psi,$$

where: - $\hat{H}$: Hamiltonian operator (total energy), - E: Quantized energy level, - Ψ: Wave function encoding the quantum state.

This equation forms the foundation of wave mechanics, predicting quantum-scale behavior efficiently.

Note

Chapter 10 Wave Mechanics is very introductory and shallow part of "The Quantum Mechanics". Please refer to any of the recommended books for further deep understanding (intended for undergraduates).

Introductory Quantum Mechanics Books

1. **Introduction to Quantum Mechanics** – *David J. Griffiths & Darrell F. Schroeter*
2. **Quantum Mechanics: Concepts and Applications** – *Nouredine Zettili*
3. **The Principles of Quantum Mechanics** – *P. A. M. Dirac*
4. **Quantum Mechanics: The Theoretical Minimum** – *Leonard Susskind & Art Friedman*
5. **Quantum Physics: An Introduction** – *Michel Le Bellac*

11

Further Advancements

11.1 Historical Development and Insights in Quantum Mechanics

Werner Heisenberg, Max Born and Pascual Jordan formulated Matrix Mechanics in 1925,while Schrodinger formulated Wave Mechanics in 1926. In 1927, Heisenberg came up with the idea of the uncertainty principle. After which, further advancements took place, like

1. Dirac and Quantum Electrodynamics, Relativistic QM and the prediction of antimatter
2. Energy Corrections (Fine and Hyperfine structure), Lamb shift & QED refinements
3. Significance and Applications:
   (a) Role in developing lasers, spectroscopy and Quantum Computing
   (b) Insights into other atoms and complex quantum systems

Interesting Insight of Quantum Computing

Quantum Chip called Willow, took just five minutes to solve a computational problem so hard it would have taken today's Super-Computer around 10 septillion (or 10^{25}) years to crack. 10^{25} years, a number that vastly exceeds the age of universe. And it can reduce errors exponentially as we scale up using more qubits.

– By Hartmut Neven, Founder and Lead, Google Quantum AI

For more details, visit the official blog: Google Willow Quantum Chip Blog.

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