Analysis of Algorithms

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Linear Programming

Reading: chapter 8

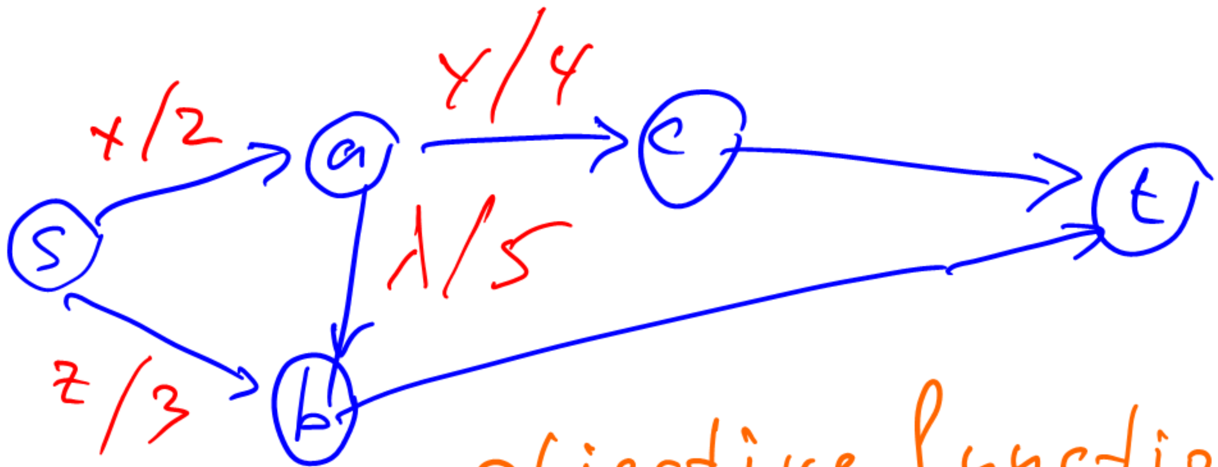
Linear Programming

In this lecture we describe linear programming that is used to express a wide variety of different kinds of problems. LP can solve the max-flow problem and the shortest distance, find optimal strategies in games, and many other things.

We will primarily discuss the setting and how to code up various problems as linear programs.

$Y \leq_p NF$ $\qquad$ $Y \leq_p LP$

NF



max-flow !

objective function :

$$\max(x+z)$$

$$\left.\begin{array}{l} x \le 2 \\ y \le 4 \end{array}\right\} \forall \text{ edges}$$

$$x = y + \lambda \} \quad \forall \text{ vertices}$$

Solving by Reduction

Formally, to reduce a problem Y to a problem X (we write $Y \leq_p X$) we want a function f that maps Y to X such that:

- f is a polynomial time computable
- $\forall$ *instance* $y \in Y$ is solvable if and only if $f(y) \in X$ is solvable.

A Production Problem

A company wishes to produce two types of souvenirs: type-A will result in a profit of $1.00, and type-B in a profit of $1.20.
To manufacture a type-A souvenir requires 2 minutes on machine I and 1 minute on machine II.
A type-B souvenir requires 1 minute on machine I and 3 minutes on machine II.
There are 3 hours available on machine I and 5 hours available on machine II.
How many souvenirs of each type should the company make in order to maximize its profit?

A Production Problem

	Type-A	Type-B	Time Available
Profit/Unit	\$1.00	\$1.20	
Machine I	2 min $2x$	$+$ 1 min y	$\le$ 180 min
Machine II	1 min	3 min	300 min

Let $x \geq 0$ be the number of type-A

$y \geq 0$ — " — " type-B

Objective function: $\max\limits_{x,y} (x + 1.2y)$

A Linear Program

We want to maximize the objective function

$$\max(x + 1.2y)$$

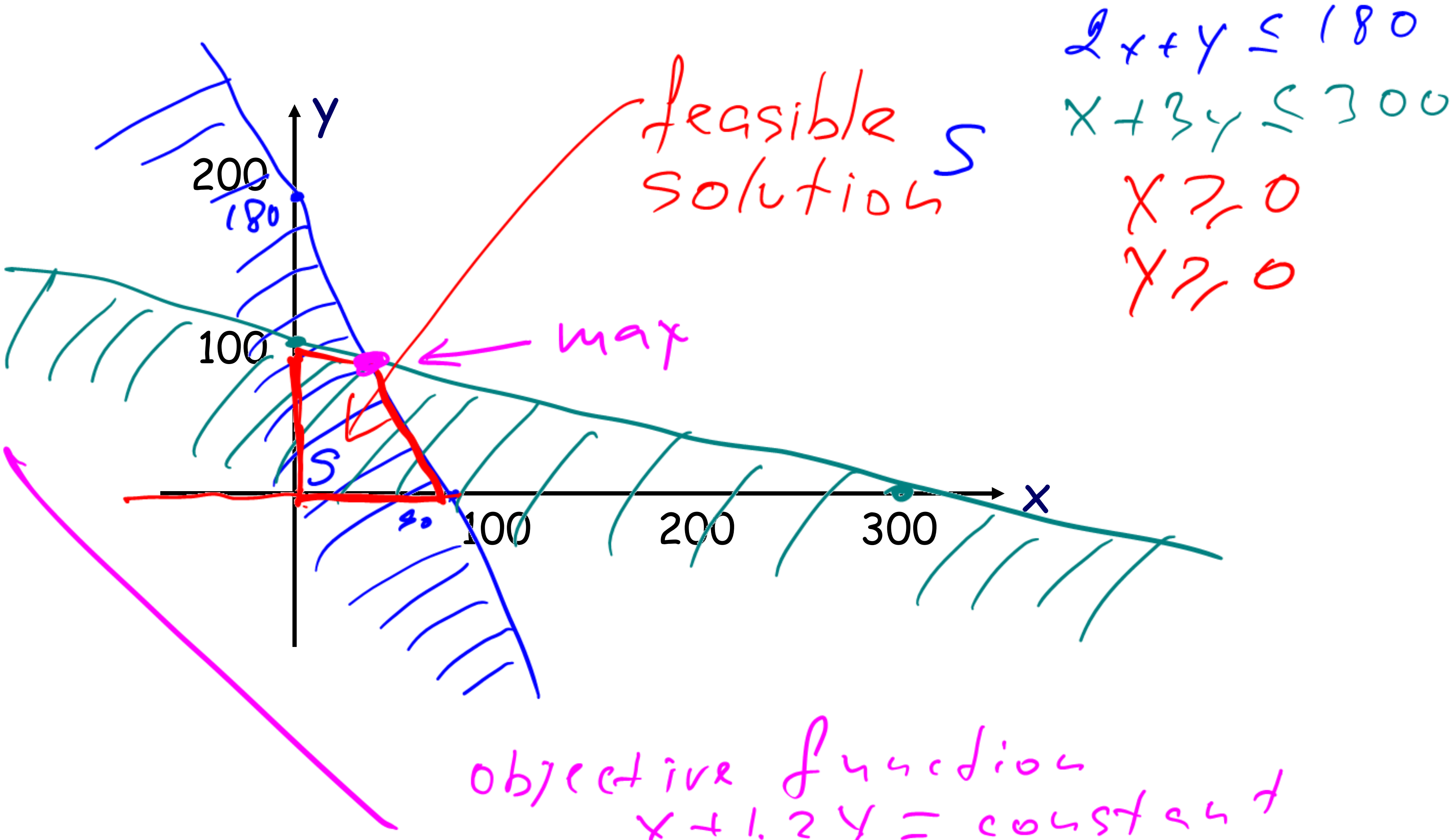
subject to the system of inequalities:

$$2x + y \le 180$$

$$x + 3y \le 300$$

$$x \ge 0, \; y \ge 0$$

A Production Problem



We need to find the feasible point that is farthest in the “objective” direction

Fundamental Theorem

If a linear programming problem has a solution, then it must occur at a vertex, or corner point, of the feasible set S associated with the problem.

If the objective function P is optimized at *two* adjacent vertices of S, then it is optimized at *every point* on the line segment joining these vertices, in which case there are infinitely many solutions to the problem.

Existence of Solution

Suppose we are given a LP problem with a feasible set S and an objective function P. There are 3 cases to consider

① S is empty, LP has no solution

$\max(x)$
$x \le -1$
$x \ge 0$

② S is unbounded, LP may or may not have a solution

$\max(x)$
$x \ge 0$

$\max(x)$
$x \le 0$

③ S is bounded
LP has a solution(s)

Standard LP form

We say that a maximization linear program with n variables is in standard form if for every variable x_k we have the inequality $x_k \geq 0$ and all other m linear inequalities.

An LP in standard form is written as

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad c = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix},$$

$$c^T x = x^T c$$

$$\max\ (c_1x_1 + \ldots + c_nx_n)$$

subject to

$$\left.\begin{aligned} a_{11}x_1 + \ldots + a_{1n}x_n &\leq b_1 \\ &\vdots \\ a_{m1}x_1 + \ldots + a_{mn}x_n &\leq b_m \end{aligned}\right\}$$

$$Ax \leq b$$

$$A = \begin{pmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & & \\ a_{m1} & \ldots & a_{mn} \end{pmatrix}, \quad b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

$$x_1 \geq 0, \ldots, x_n \geq 0$$

Standard LP in Matrix Form

The vector c is the column vector $(c_1, \ldots, c_n)$.
The vector x is the column vector $(x_1, \ldots, x_n)$.
The matrix A is the $n \times m$ matrix of coefficients of the left-hand sides of the inequalities, and
$b = (b_1, \ldots, b_m)$ is the vector of right-hand sides of the inequalities.

$$\max\ (c^T x)$$

subject to

$$A x \le b$$
$$\underbrace{x \ge 0}$$

$x \in \mathbb{R}$

Exercise: Convert to Matrix Form

$\max(x_1 + 1.2\, x_2)$
$2x_1 + x_2 \le 180$
$x_1 + 3x_2 \le 300$
$x_1 \ge 0$
$x_2 \ge 0$

b

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$c = \begin{pmatrix} 1 \\ 1.2 \end{pmatrix}$$

$$b = \begin{pmatrix} 180 \\ 300 \end{pmatrix}$$

$$(Ab)^T = b^T A^T$$

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$$

Algorithms for LP

The standard algorithm for solving LPs is the Simplex Algorithm, due to Dantzig, 1947.

This algorithm starts by finding a vertex of the polytope, and then moving to a neighbor with increased cost as long as this is possible. By linearity and convexity, once it gets stuck it has found the optimal solution.

Unfortunately, simplex does not run in polynomial time it does well in practice, but poorly in theory.

Algorithms for LP

In 1974 Khachian has shown that LP could be done in polynomial time by something called the Ellipsoid Algorithm (but it tends to be fairly slow in practice).

In 1984, Karmarkal discovered a faster polynomial-time algorithm called "interior-point". While simplex only moves along the outer faces of the polytope, "interior-point" algorithm moves inside the polytope.

MATLAB

https://www.mathworks.com/help/optim/ug/linprog.html

linprog

Linear programming solver

Finds the minimum of a problem specified by

$$\min_x f^T x \text{ such that } \begin{cases} A \cdot x \leq b, \\ Aeq \cdot x = beq, \\ lb \leq x \leq ub. \end{cases}$$

f, *x*, *b*, *beq*, *lb*, and *ub* are vectors, and *A* and *Aeq* are matrices.

Description

`x = linprog(f,A,b)` solves min `f'*x` such that `A*x ≤ b`.

`x = linprog(f,A,b,Aeq,beq)` includes equality constraints `Aeq*x = beq`. Set `A = []` and `b = []` if no inequalities exist.

`x = linprog(f,A,b,Aeq,beq,lb,ub)` defines a set of lower and upper bounds on the design variables, `x`, so that the solution is always in the range `lb ≤ x ≤ ub`. Set `Aeq = []` and `beq = []` if no equalities exist.

Break.

Discussion Problem 1

A cargo plane can carry a maximum weight of 100 tons and a maximum volume of 60 cubic meters. There are three materials to be transported, and the cargo company may choose to carry any amount of each, up to the maximum available limits given below.

	Density	Volume	Price
Material 1	2 tons/m^3	40 m^3	\$1,000 per m^3
Material 2	1 tons/m^3	30 m^3	\$2,000 per m^3
Material 3	3 tons/m^3	20 m^3	\$12,000 per m^3

Write a linear program that optimizes revenue within the constraints.

Let x_1, x_2, x_3 be the volumes

Objective function:

$$\max(1000x_1 + 2000x_2 + 12000x_3)$$

Subject to:

$$2x_1 + x_2 + 3x_3 \leq 100$$

$$x_1 + x_2 + x_3 \leq 60$$

$$0 \leq x_1 \underbrace{\leq 40}_{?}, \quad 0 \leq x_2 \underbrace{\leq 30}_{?}, \quad 0 \leq x_3 \underbrace{\leq 20}_{?}$$

In matrix form:

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad c = \begin{pmatrix} 1000 \\ 2000 \\ 12000 \end{pmatrix}, \quad b = \begin{pmatrix} 100 \\ 60 \\ 40 \\ 30 \\ 20 \end{pmatrix}$$

$$A = \begin{pmatrix} 2 & 1 & 3 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0$$

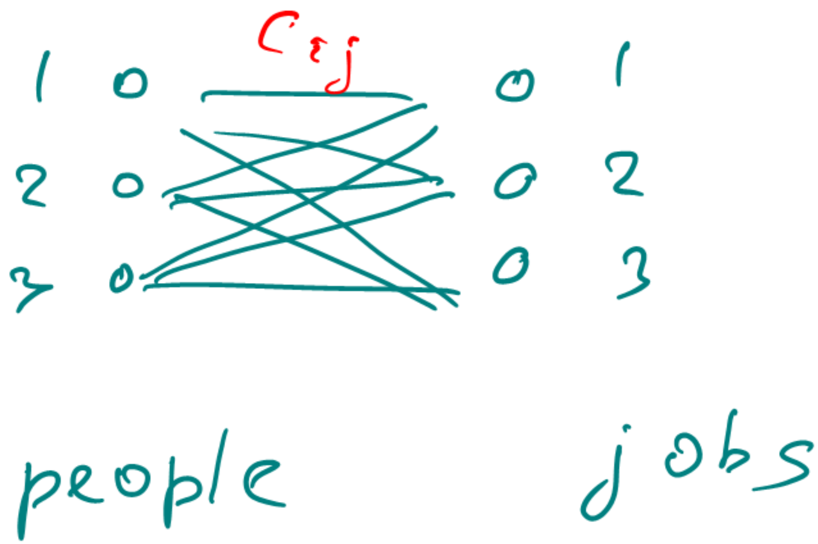
$$\left.\begin{aligned} x_1 \leq 40 &\rightarrow 1 \cdot x_1 + 0 \cdot x_2 + 0 \cdot x_3 \leq 40 \\ x_2 \leq 30 &\rightarrow 0 \cdot x_1 + 1 \cdot x_2 + 0 \cdot x_3 \leq 30 \end{aligned}\right\} A$$

Discussion Problem 2

There are n people and n jobs. You are given a cost matrix, C, where c_{ij} represents the cost of assigning person i to do job j. You need to assign all the jobs to people and also only one job to a person. You also need to minimize the total cost of your assignment. Write a linear program that minimizes the total cost of your assignment.

~~max~~ matching

min



① Define variables

let x_{ij} be an assignment between i-th person and j-th job.

② Objective function

$$\min \sum_{i,j} x_{ij} \cdot c_{ij}$$

③ Constraints

ⓐ pick a person, $i = 1, 2, \ldots, n$

$$x_{i1} + x_{i2} + \ldots + x_{in} = 1$$

ⓑ pick a job, $j = 1, 2, \ldots, n$

$$x_{1j} + x_{2j} + \ldots + x_{nj} = 1$$

$$x_{ij} = \begin{cases} 1, \\ 0 \end{cases}$$

Integer LP (ILP),

all variables $\in \mathbb{N}$

we do not know how to solve ILP in polynomia time.

similar to 0-1 knapsack problem

Discussion Problem 3

Convert the following LP to standard form

$$\max\ (5x_1 - 2x_2 + 9x_3)$$

$$3x_1 + x_2 + 4x_3 = 8$$
$$2x_1 + 7x_2 - 6x_3 \le 4$$
$$x_1 \le 0,\ x_3 \ge 1$$

!!

$3x_1 + x_2 + 4x_3 \ge 8$

$3x_1 + x_2 + 4x_3 \le 8$

$z_1 = -x_1, \quad z_1 \ge 0$

$z_3 = x_3 - 1, \quad z_3 \ge 0$

$-\infty < x_2 < \infty$

free variable

$z_4 = x_2^2 \ge 0$ non-linear

$z_4 = |x_2| \ge 0$

$x_2 = z_5 - z_6, \quad z_5 \ge 0, z_6 \ge 0$

Write your LP in new variables: z, z_3, z_5, z_6

DIY

Discussion Problem 4

Explain why LP cannot contain constrains in the form of *strong* inequalities.

$\max(7x_1 - x_2 + 5x_3)$

$x_1 + x_2 + 4x_3 < 8$

$3x_1 - x_2 + 2x_3 > 3$

$2x_1 + 5x_2 - x_3 \le -7$

$x_1, x_2, x_3 \ge 0$

Example:

$\max(x)$

$x < 1$

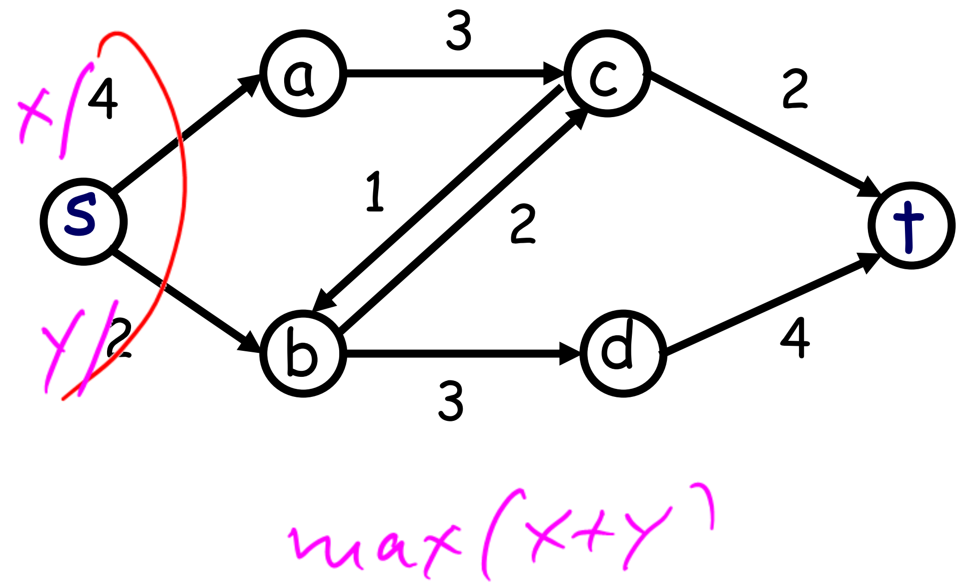
$x \geq 0$

$x = 1$ is not a solution

$1 < 1$ False

Exercise: Max-Flow as LP

Write a max-flow problem as a linear program.



$NF \leq_p LP$

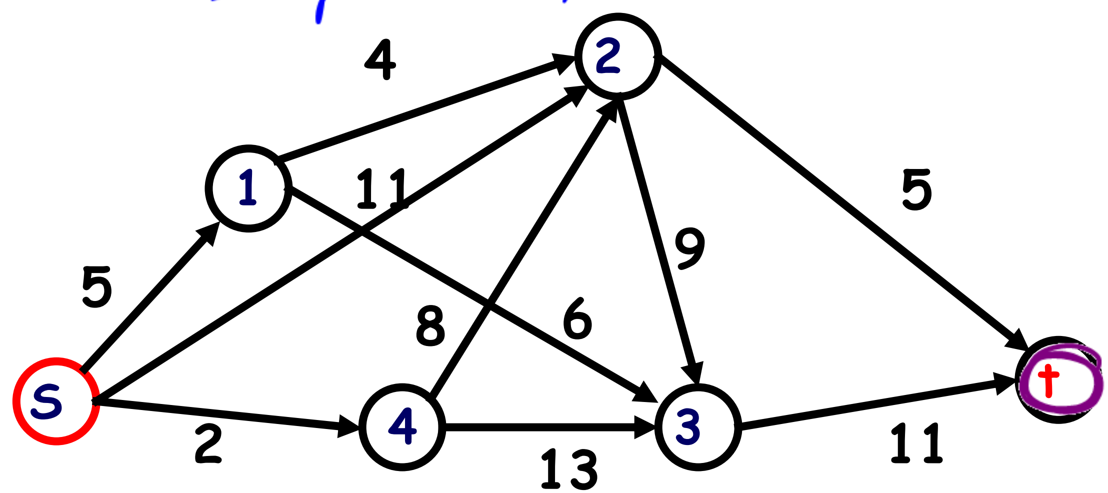
see the first slide

Exercise: Shortest Path as LP

Write a shortest st-path problem as a linear program.

Let $d(v)$ be the shortest path from s to v.

$d(s) = 0$



$d(v) \leq d(u) + c(u,v)$

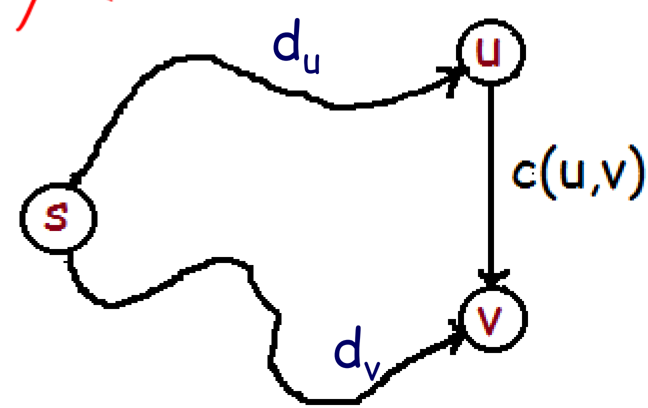
$d(1) \leq d(s) + 5$

$d(4) \leq d(s) + 2$

$d(2) \leq d(1) + 4$

$d(2) \leq d(4) + 8$

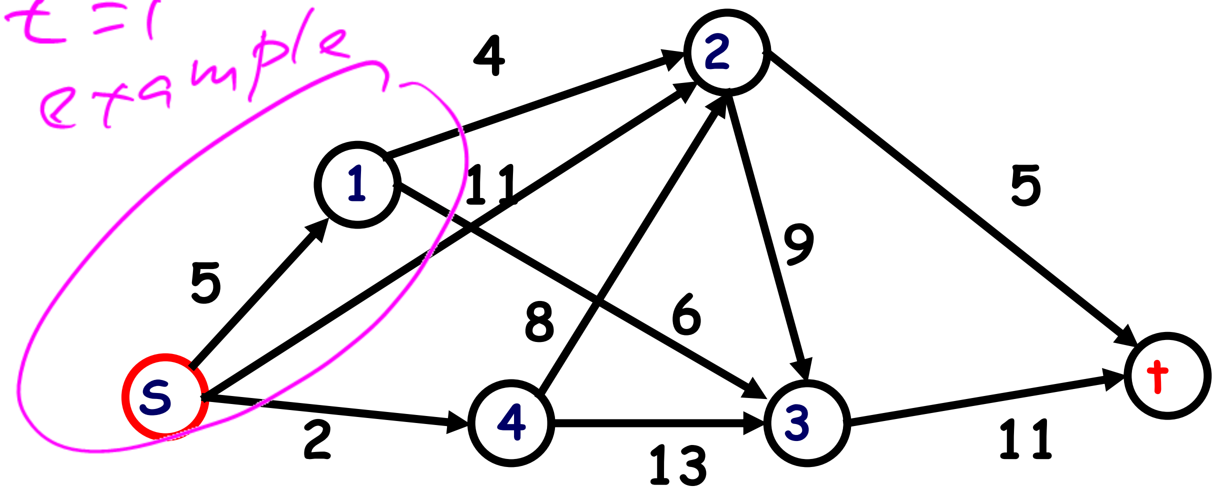
for each edge



Objective function:

$\min d(t)$

$t=1$ example



constraints!

$d(1) \le d(s)+5 = 5$

$d(1) \ge 0$

objective function

$\min d(1) \rightarrow d(1) = 0$

$$\boxed{\begin{aligned} &\max d(1) \\ &d(1) \le 5 \\ &d(1) \ge 0 \end{aligned}} \rightarrow d(1) = 5$$

Discussion Problem 5

Write a 0-1 Knapsack Problem as a linear program.

0-1 Knapsack $\leq_p$ LP

Given n items with weights $w_1, w_2, \ldots, w_n$ and values $v_1, v_2, \ldots, v_n$. Put these items in a knapsack of capacity W to get the maximum total value in the knapsack.

$$\text{Given } \sum_{k=1}^{m} w_k \leq W$$

$$\text{optimize } \sum_{k=1}^{m} v_k \rightarrow \max$$

Knapsack as LP

① variables

$$x_k = \begin{cases} 1, & k\text{-th item is taken} \\ 0 \end{cases}$$

② Objective function

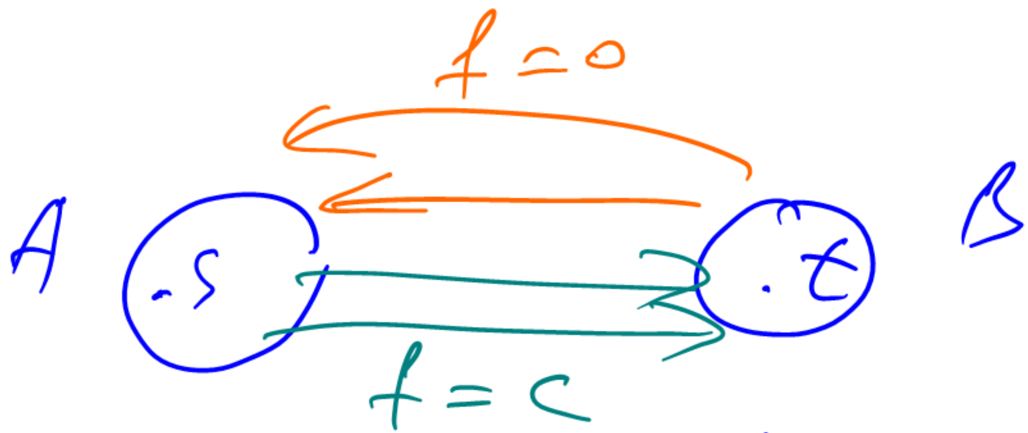
$$\max \sum_{i=1}^{n} x_i \cdot v_i$$

③ Constraints

$$\sum_{i=1}^{n} x_i \cdot w_i \leq W$$

Dual LP

To every linear program there is a dual linear program



① ∀flow, ∀cut : flow ≤ cut

② ∃f, ∃c : flow = cut

Duality

<u>Definition.</u> The dual of the standard (primal) maximum problem

$\max c^T x$

$Ax \leq b$ and $x \geq 0$

primal

is defined to be the standard minimum problem

$\min b^T y$

$A^T y \geq c$ and $y \geq 0$

dual

Exercise: duality

Consider the LP:

$$\max(7x_1 - x_2 + 5x_3)$$
$$x_1 + x_2 + 4x_3 \leq 8$$
$$3x_1 - x_2 + 2x_3 \leq 3$$
$$2x_1 + 5x_2 - x_3 \leq -7$$
$$x_1, x_2, x_3 \geq 0$$

primal

↓

matrix form

↓

dual

Write the dual problem.

primal LP		dual LP
$\max(c^T x)$ $A x \leq b$ $x \geq 0$	⟷	$\min(b^T y)$ $A^T y \geq c$ $y \geq 0$

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad c = \begin{pmatrix} 7 \\ -1 \\ 5 \end{pmatrix}, \quad b = \begin{pmatrix} 8 \\ 3 \\ -7 \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 1 & 4 \\ 3 & -1 & 2 \\ 2 & 5 & -1 \end{pmatrix} \qquad y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

$$A^T = \begin{pmatrix} 1 & 3 & 2 \\ 1 & -1 & 5 \\ 4 & 2 & -1 \end{pmatrix}$$

dual:

$$\min_y (8y_1 + 3y_2 - 7y_3)$$

$$y_1 + 3y_2 + 2y_3 \geq 7$$

$$y_1 - y_2 + 5y_3 \geq -1$$

$$4y_1 + 2y_2 - y_3 \geq 5$$

From Primal to Dual

Consider the max LP constrains

$$y_1 \left(a_{11}x_1+\ldots+a_{1n}x_n\right) \le b_1 \, y_1$$

$$+ \quad \vdots$$

$$y_m \left(a_{m1}x_1+\ldots+a_{mn}x_n\right) \le b_m \, y_m$$

$y_i \ge 0$ new variables

1) Multiply each equation by a new variable $y_k \geq 0$.
2) Add up those m equations.
3) Collect terms wrt to x_k.
4) Choose y_k in a way such that $A^T y \geq c$.

$$x_1(\underbrace{y_1 \cdot a_{11} + y_2 \cdot a_{21} + \dots + y_m \cdot a_{m1}}_{\geq c_1})$$

$$+ \dots +$$

$$x_n(\underbrace{y_1 a_{1n} + y_2 a_{2n} + \dots + y_m a_{mn}}_{\geq c_n})$$

$$\leq y_1 b_1 + y_2 b_2 + \dots + y_m b_m$$

New constraints

$$\left.\begin{aligned} y_1 a_{11} + \dots + y_m a_{m1} &\geq c_1 \\ &\dots \\ y_1 a_{1n} + \dots + y_m a_{mn} &\geq c_n \end{aligned}\right\} A^T y \geq c$$

Objective function

$$x_1 \cdot c_1 + x_2 \cdot c_2 \leq b_1 \cdot y_1 + b_2 y_2 + \dots + b_m y_m$$

$$+ \dots + x_n c_n$$

in matrix form

max $\rightarrow \boxed{c^T x \le b^T y} \leftarrow$ min

primal dual

Weak Duality

$\max (c^T x)$ $A x \le b$ $x \ge 0$	$\Longleftrightarrow$	$\min (b^T y)$ $A^T y \ge c$ $y \ge 0$
primal linear program		dual linear program

Weak Duality. The optimum of the dual is an upper bound to the optimum of the primal.

$$\text{opt(primal)} \le \text{opt(dual)}$$

Weak Duality

$(Ax)^T = x^T A^T$

$\max(c^T x)$ $A x \le b$ $x \ge 0$	⟺	$\min(b^T y)$ $A^T y \ge c$ $y \ge 0$

Theorem (The weak duality).
Let P and D be primal and dual LP correspondingly.
If x is a feasible solution for P and y is a feasible solution for D, then
$c^T x \le b^T y$.

Proof (in matrix form).

$c^T x = x^T c \le x^T (A^T y) = ((Ax))^T y \le b^T y$

Weak Duality: opt(primal) ≤ opt(dual)

<u>Corollary 1.</u> If a standard problem and its dual are both feasible, then both are feasible bounded.

Suppose x is feasible,

$\underbrace{c^T x}_{\text{number}} \le b^T y$ → min ≠ −∞

Suppose y is feasible,

+∞ ≠ max ← $c^T x \le \underbrace{b^T y}_{\text{number}}$

<u>Corollary 2.</u> If one problem has an unbounded solution, then the dual of that problem is infeasible.

$\underbrace{c^T x}_{\text{unbounded}} \le b^T y$

$x = +\infty$

$\Rightarrow b^T y \geq +\infty$

y has no solution

Strong Duality

$\max(c^T x)$ $Ax \le b$ $x \ge 0$	$\Longleftrightarrow$	$\min(b^T y)$ $A^T y \ge c$ $y \ge 0$

Theorem (The strong duality).
Let P and D be primal and dual LP correspondingly.
If P and D are feasible, then $c^T x = b^T y$.

The proof of this theorem is beyond the scope of this course.

567 $\to$ SVM

Possibilities for the Feasibility

$\max(c^T x)$

$Ax \leq b$
$x \geq 0$

$\min(b^T y)$

$A^T y \geq c$
$y \geq 0$

P\D	F.B.	F.U.	I.
F.B.	Yes cor. 1	NO cor. 1	? NO
F.U.	NO cor. 1	? NO	Yes cor. 2
I.	? NO	Yes cor. 2	? Yes

example

example below

feasible bounded - F.B.
feasible unbounded - F.U.
infeasible - I.

$\max(2x_1 + x_2)$

$x_1 - x_2 \geq 4$

$x_1 - x_2 \leq 2$

infeasible

$\xrightarrow{\text{dual}}$

$\min(-4y_1 + 2y_2)$

$-y_1 + y_2 \geq 2$

$y_1 - y_2 \geq 1$

$y_1 - y_2 \leq -2$

infeasible

Convex

Nonlinear Optimization

$$\min f(x_1, x_2, \ldots, x_n)$$

$$h_i(x_1, x_2, \ldots, x_n) \leq 0,\ i = 1,, \ldots, m.$$

$$x_k \geq 0,\ k = 1, , \ldots, n.$$

Here f and/or h are nonlinear functions.

The problem is solved using Lagrange multipliers λ_k.

Lagrange Duality (KKT-1951)

Primal in x:

$\min f(x)$
subject to
$h_k(x) \le 0$

Dual in λ:

$\max g(\lambda)$
subject to
$\lambda_k \ge 0$

The Lagrangian: $$L(x,\lambda) = f(x) + \sum_k \lambda_k h_k(x)$$

The dual: $$g(\lambda) = \min_x L(x,\lambda)$$

Weak Duality: Let P and D be the optimum of primal and dual problems respectively. Then opt(P) ≤ opt(D).

Equality (strong duality) holds for convex functions under some conditions.