CSCI-570 Fall 2023

Exam 2

INSTRUCTIONS

- The duration of the exam is 140 minutes, closed book and notes.
- No space other than the pages on the exam booklet will be scanned for grading!
- If you require an additional page for a question, you can use the extra page provided at the end of this booklet. However please indicate clearly that you are continuing the solution on the additional page.

1. **True/False Questions (20 points)**

   Mark the following statements as **T** or **F**. No need to provide any justification.

   a) **(T/F)** If all capacities in a flow network are integers, then every maximum flow in the network is such that the flow value on each edge is an integer.

      False. The flow on edges could be fractional.

   b) **(T/F)** In the Ford-Fulkerson algorithm, the choice of augmenting path has no effect on the algorithm's runtime.

      False. In the Ford-Fulkerson algorithm for finding the maximum flow in a network, the choice of augmenting paths can affect the algorithm's runtime, making it sensitive to the path selection strategy

   c) **(T/F)** It is possible for a circulation network to not have a feasible circulation even if all edges have unlimited capacities.

      True. $\sum_v d_v \neq 0$

   d) **(T/F)** If, in a flow network, we multiply all edge capacities by a positive integer 'k'. Only the value of the min-cut gets multiplied by 'k', the rest of all cuts are scaled by some other factor $\geq k$

      True. The value of every cut gets multiplied by k and their relative ordering doesn't change, but it is $\geq$ k (if it was $>$ it would have been True)

   e) **(T/F)** If an LP has unbounded feasible set, then it does not have an optimal solution.

      False. An LP with unbounded feasible set may have an optimal solution.

   f) **(T/F)** An LP could have an optimal solution if its dual problem is unbounded.

False. An LP must be infeasible when its dual problem is unbounded.

g) **(T/F)**.The weak duality theorem still holds when the LP does not have an optimal solution.

True. The weak duality holds for any feasible solutions of the LP and its dual program.

h) **(T/F)** Every problem with a polynomial-time solution is polynomial-time reducible to Hamiltonian Cycle.

True.

i) **(T/F)** Say problem Y is polynomially reducible to Hamiltonian Cycle, and Hamiltonian Cycle is polynomially reducible to problem X, then at least one of the two problems (X or Y) is NP-hard.

True.

j) **(T/F)** Every problem in NP can be solved in polynomial time by a nondeterministic Turing machine.

True.

2. Multiple Choice Questions (10 points)

Please select the most appropriate choice. Each multiple choice question has a single correct answer.

a) For a Linear Programming problem denoted as P and its dual D, which of the following statement is possible?

(a) P is feasible and D is unbounded

(b) P and D have optimal solutions

(c) The optimal value of P is greater than the optimal value of D

(d) None of the above

Answer: b)

Explanation: a) is false due to the strong duality. b) is true, because both P and D could have optimal solutions; c) is false due to the weak duality.

b) Consider the following Linear Programming problem:

$$\begin{aligned} \textbf{maximize } & 3x + 2y \\ \textbf{subject to } & x \geq 0, y \geq 0, \\ & x + 2y \leq 2. \end{aligned}$$

Which of the following statements is false?

(a) This LP has 3 constraints

(b) The feasible set of this LP is unbounded

(c) The optimal solution is $x = 2, y = 0$

(e) The dual program of this LP has only 1 variable

Answer: b)

c) If P != NP, select which of the following is true:

(a) GRAPH 2-COLORING is NP-complete
(b) GRAPH 3-COLORING is NP-complete
(c) Max flow is NP-complete
(d) Linear Programming is NP-Complete

b)

d) Let S be an NP-complete problem and Q and R be two other problems not known to be in NP. Q is polynomial time reducible to S and S is polynomial-time reducible to R. Which one of the following statements is true?

(a) R is NP-hard
(b) R is NP-complete
(c) Q is NP-complete
(d) Q is NP-hard

a), R is NP-hard

e) Assume P != NP. If X is polynomial time reducible to Y then select all true statements.

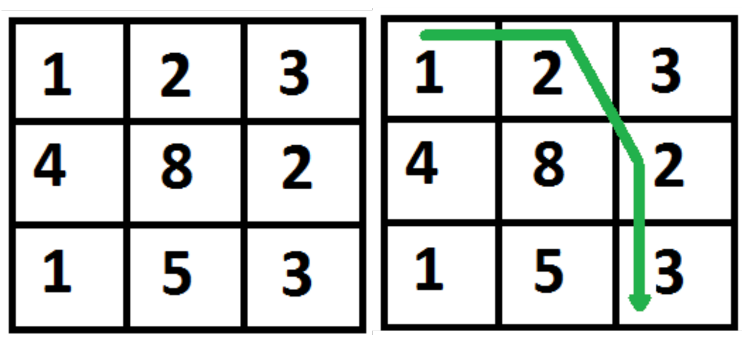
(a) If Y belongs to NP but not NPC then 3SAT is not polynomial time reducible to X
(b) If X belongs to P then Y necessarily belongs to NP
(c) If Y is the decision version of the vertex cover problem then X can be any problem

a), If Y belongs to NP but not NPC then 3SAT is not polynomial time reducible to X

3. Dynamic Programming (20 points)

Given a two-dimensional matrix where each cell represents a cost to traverse through that cell, determine the minimum cost path from the top-left corner (0,0) to a specific position (M, N) in the matrix. The total cost of a path is calculated as the sum of the costs of all cells on that path, including the starting cell (0,0) and the destination cell (M, N). Movement from a cell (i, j) is restricted to only three possible next cells: directly below $(i + 1, j)$, directly to the right $(i, j + 1)$, and diagonally lower right $(i + 1, j + 1)$.

Note. Assume that all costs in the matrix are positive integers.



For example, The path with minimum cost is highlighted in the following figure. The path is $(0, 0) \to (0, 1) \to (1, 2) \to (2, 2)$. The cost of the path is $8 = (1 + 2 + 2 + 3)$.

a) Define (in plain English) subproblems to be solved. (3 points)

Let $minCost[m][n]$ represent the minimum cost of starting from $(0, 0)$ and reaching (m, n).

Rubrics (3 points):

- Completely irrelevant definition: -3 points

- Any alternate definition is eligible for full credits.

b) Write a recurrence relation for the subproblems and provide an explanation for it. (10 points)

$minCost(m, n) = min(minCost(m-1, n-1), minCost(m-1, n), minCost(m, n-1)) + cost[m][n]$

Explanation: This problem has the optimal substructure property. The path to reach (m, n) must be through one of the 3 cells: (m-1, n-1) or (m-1, n) or (m, n-1). So minimum cost to reach (m, n) can be written as "minimum of the 3 cells plus cost[m][n]".

Rubrics :

- Completely irrelevant recurrence relation: -10 points
- Any alternate recurrence relation is eligible for full credits.

c) Make sure you specify base cases and their values; where the final answer can be found. (3 points)

There are 2 possible base cases:

- If n is less than zero or m is less than zero then return Inf
- If m is equal to zero and n is equal to zero then return cost[m][n](Base Case)

the final answer - minCost[M][N]

Rubrics :

- Either of the two base cases can get full points
- Wrong/missing base values: -2 point
- Wrong/missing final answer: -1 point

d) What is the space and time complexity of your solution? Explain your answer. (4 point)

Space complexity: $O(MN)$. Time complexity: $O(MN)$.

Rubrics (4 point):

- Wrong/missing time complexity or explanation: -2 point
- Wrong/missing space complexity or explanation: -2 point

4. Linear Programming (20 points)

A factory produces two products: Product A and Product B, using three types of resources: R1, R2, and R3. The objective is to maximize the factory's profits while adhering to resource constraints. The available information and constraints are summarized as follows:

- Each month, the factory buys a limited amount of resources: 90 units of R1, 80 units of R2 and 45 units of R3.
- To produce one unit of Product A, it requires 1 unit of R1, 2 units of R2, and 1 unit of R3.
- To produce one unit of Product B, it requires 3 units of R1, 1 unit of R2, and 1 unit of R3.
- The profit per unit of A is 5, the profit per unit of B is 4.

	Resource R1	Resource R2	Resource R3	Profit
Product A	1	2	1	5
Product B	3	1	1	4
Limit	90	80	45	

Formulate a linear programming model to help the company maximize profits. You do not have to solve the resulting LP.

a) Describe what your LP variables represent (4 points).

b) Define your objective function for maximizing profits (4 points).

c) Show your constraints (4 points).

d) Formulate the dual program of the described LP (8 points).

Solution:

a) Let x be the number of units of Product A to produce, y be the number of units of Product B to produce.

b) The objective function to maximize profits can be expressed as:

$$Maximize \qquad 5x + 4y$$

c) The constraints are as follows:

$$\begin{gathered} x + 3y \leq 90 \text{ (R1 constraint)} \\ 2x + y \leq 80 \text{ (R2 constraint)} \\ x + y \leq 45 \text{ (R3 constraint)} \\ x, y \geq 0 \text{ (non-negativity constraint)} \end{gathered}$$

d) The dual program has three variables u,v,w for the constraints of R1, R2 and R3. The objective function is to minimize $90u + 80v + 45w$. The constraints are:

$$\begin{gathered} u + 2v + w \geq 5 \text{ (Product A constraint)} \\ 3u + v + w \geq 4 \text{ (Product B constraint)} \\ u, v, w \geq 0 \text{ (non-negativity constraint)} \end{gathered}$$

Rubrics:

a) 4 points: LP variables represent. -2 point for more than two variables.

b) 4 points: for the objective function. At most 2 points for more than two variables (even the objective function is correct, only 2 points due to more than two variables).

c) 4 points: 1 point for each correct constraint. At most 2 points for more than two variables.

d) 8 points: 1 point for the variables (u, v, w), 1 point for the correct objective, and 2 points for each correct constraint. At most 4 points for more than three variables or two in (a).

5. **Network Circulation (15 points)**

The USC Leavey Library is always busy. You are hired to solve their room assignment problem. You are given n students and m rooms. Each student can only be assigned to one room, and each room has the capacity to hold one to ten students. Every student has a preference list containing a subset of rooms they would like to be assigned to. Moreover, every room must have at least one student assigned to it.

Devise a polynomial-time algorithm to determine whether a feasible assignment of students to rooms, that meets all the above constraints, is possible. Describe how your solution can identify which student is assigned to which room.

a) Construct a network flow-circulation graph. (4 points)

b) Write your claim. (3 points)

c) Provide a forward proof of your claim.(4 points)

d) Provide a backward proof of your claim.(4 points)

Solution:
Note:

- We mentioned that the preference lists of students can not be empty. That implies if a room can not be assigned to a student (as per their preference list) then there is no feasible circulation. However, we would still award points if you've run flow-circulation/max-flow and assigned rooms to students with some students left without any rooms. In such a case you'll be awarded a maximum of 12 points.

a) Construct a flow network as follows:

i. For each student i create a vertex a_i, for each room j create a vertex b_j, if room j is one of the student i's possible choices, add an edge from a_i to b_j with lower bound 0 and upper bound 1.

ii. Create a super source s, connect s to all a_i with an edge of lower bound and upper bound 1.

iii. Create a super sink t, connect all b_j to t with an edge of lower bound 1 and upper bound 10.

iv. Connect t to s with an edge of lower bound n and upper bound n.

v. If there is a flow through an edge between student a_i and room b_j, then the student a_i is assigned to room b_j.

vi. We can convert this problem into a max flow problem by removing lower bounds and adding a super source and super sink node which can be solved in polynomial time.

b) Claim: Every student can be assigned to a room successfully, where each room has atleast one student and a max of ten students, if and only if there exists a feasible circulation in the network. i.e. max flow is n.

c) Forward Proof: If a feasible assignment exists, then max flow is n.

If there is a feasible assignment, then this means each student is assigned to one room in the preference list. This means a flow of unit 1 flows from source to student node ai. From student, it goes to one of the rooms bj in their preference list. Then this flow goes to the sink. Since there is a feasible assignment, this means all the constaints are met. Since this is true for all students, a total of n unit of flow flows from source to sink and the max flow is n.

d) Backward Proof: If there is a feasible circulation, or max flow is n, then there exists a feasible assignment of students to rooms.

   If the max flow is n, this means all the edges from source to student nodes ai have a flow of unit 1. Since a flow of 1 is coming into ai, then a flow of unit 1 will go out of ai to bj (one of their preferred rooms). This means every student will be assigned to one room. A capacity of 10 in the edge from room to sink will ensure that no room is assigned more that 10 students.

Rubrics:

- (+4 points) For correct construction of the graph
- (-1 point) For every mistake in edge capacity/lower bound assignments
- (-2 points) For not converting into max-flow problem
- (+3 points) Writing the correct claim.
- (+4 points) Writing the correct forward proof.
- (+4 points) Writing the correct backward proof.
- (-3 points) If the student's solution might have some students left without a room. i.e their claim says feasible circulation is possible even if max-flow is not n
- (No deduction) For proofs/claims based on either max-flow or flow circulation.

6. NP-Completeness (15 points)

In a country, there are N cities connected by some undirected roads. Each city has an associated integer value, which can be positive, negative, or zero. You want to know whether there exists a cycle in this network of cities (a cycle is a path that starts and ends at the same city and does not visit any city or road more than once) such that the sum of the values of the cities on the cycle is exactly zero. Note that a single vertex does not constitute a cycle.. Prove that this problem is NP-Complete. Use a reduction from the Hamiltonian cycle problem. Complete the following five steps:

a) Show that the problem belongs to NP (2 points).

The solution of the problem can be easily verified in polynomial time, just check if the sum of the values on the cycle is equal to 0, and the cycle doesn't visit a city or a road twice. Thus it is in NP.

b) Show a polynomial time construction using a reduction from Hamiltonian cycle (5 points).

Given a Hamiltonian cycle problem. It asks if the graph exists a cycle go through all the vertices of the graph and doesn't visit a vertex twice. Let the number of vertices be N. Then we construct an instance of the zero-cycle problem (our problem), as, we set the value of one vertex to be $N - 1$, and the value of any other vertices ($N - 1$ vertices) to be -1.

c) Write down the claim that the Hamiltonian cycle problem is polynomially reducible to the original problem. (2 points)

The given graph G has a Hamiltonian cycle if and only if the there is a cycle in the zero-cycle problem's graph G' and the sum of values of the cities on the cycle is equal to 0.

d) Prove the claim in the direction from the Hamiltonian cycle problem to the reduced problem. (3 points)

If we have a solution of the Hamiltonian cycle problem, then the total value of the cycle will be $(N-1)+(-1)\times(N-1)=0$, so the zero-cycle problem can also find a solution.

e) Prove the claim in the direction from the reduced problem to the Hamiltonian cycle problem. (3 points)

If we have a solution of the zero-cycle problem, we want to prove that it's indeed a solution of the Hamiltonian cycle problem, i.e. we want to prove it will visit all vertices. If the vertex doesn't visit the vertex that has value $N-1$, then all the value on the cycle will be -1, so the total value will be negative, will not be 0, so it's impossible. So the cycle must visit the vertex that has value $N-1$. Then, in order to make the total value to be 0, it must visit all the rest of the vertices ($N-1$ vertices), since the value of them are all -1. So it will visit all the vertices. So it is a solution of the Hamiltonian cycle problem.

Rubrics:

a) 2 pts for the correct solution
b) There may have various value assignment plans for the vertices. If the plan can make sure that, any cycle that sums up to 0 must pass all the vertices, it's correct.
c) 2 pts if the claim is correct.
d) There can be multiple ways of proving this. As long as the proof supports the direction from Hamiltonian path problem to reduced problem, award 3 pts.
e) Same as (d). There can be multiple ways of proving this. As long as the proof supports the direction from the reduced problem to Hamiltonian problem award 3 pts.

Additional space

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