

# MaCS Calculus and Vectors Exam Study Guide

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# Unit 1

## Equations of Lines and Planes

## 8.1 Vector and Parametric Equations of a Line in $\mathbb{R}^2$

### A Vector Equation of a Line in $\mathbb{R}^2$

Consider the line  $L$  that passes through the point  $P_0(x_0, y_0)$  and is parallel to the vector  $\vec{u}$ . The point  $P(x, y)$  is a *generic point* on the line.

$$\begin{aligned}\overrightarrow{P_0P} &= t\vec{u} \\ \overrightarrow{OP} - \overrightarrow{OP_0} &= t\vec{u} \\ \vec{r} - \vec{r_0} &= t\vec{u}\end{aligned}$$

The *vector equation* of the line is:

$$\vec{r} = \vec{r_0} + t\vec{u} \mid t \in \mathbb{R}$$

Where:

- $\vec{r} = \overrightarrow{OP}$  is the *position vector* of a *generic point*  $P$  on the line.
- $\vec{r_0} = \overrightarrow{OP_0}$  is the *position vector* of a *specific point*  $P_0$  on the line.
- $\vec{u}$  is a vector parallel to the line called the *direction vector* of the line.
- $t$  is a *real number* corresponding to the generic point  $P$ .

**Note:** The vector equation of a line is *not unique*. It depends on the specific point  $P_0$  and on the direction vector  $\vec{u}$  that are used.

### B Parametric Equations of a Line in $\mathbb{R}^2$

We can rewrite the vector equation of a line:

$$\vec{r} = \vec{r_0} + t\vec{u} \mid t \in \mathbb{R}$$

as:

$$(x, y) = (x_0, y_0) + t(u_x, u_y) \mid t \in \mathbb{R}$$

Split this vector equation into the *parametric equations* of a line in  $\mathbb{R}^2$ :

$$\begin{cases} x = x_0 + tu_x \\ y = y_0 + tu_y \end{cases} \quad t \in \mathbb{R}$$



## C Parallel Lines

Two lines  $L_1$  and  $L_2$  with direction vectors  $\vec{u}_1$  and  $\vec{u}_2$  are *parallel* ( $L_1 \parallel L_2$ ) if:

$$\vec{u}_1 \parallel \vec{u}_2$$

or, there exists  $k \in \mathbb{R}$  such that:

$$\vec{u}_2 = k\vec{u}_1$$

or:

$$\vec{u}_1 \times \vec{u}_2 = \vec{0}$$

or scalar components are *proportional*:

$$\frac{u_{2x}}{u_{1x}} = \frac{u_{2y}}{u_{1y}} = k$$

## D Perpendicular Lines

Two lines  $L_1$  and  $L_2$  with direction vectors  $\vec{u}_1$  and  $\vec{u}_2$  are *perpendicular* ( $L_1 \perp L_2$ ) if:

$$\vec{u}_1 \perp \vec{u}_2$$

or:

$$\vec{u}_1 \cdot \vec{u}_2 = 0$$

or:

$$u_{1x}u_{2x} + u_{1y}u_{2y} = 0$$

## E 2D Perpendicular Vectors

Given a 2D vector  $\vec{u} = (a, b)$ , two 2D vectors perpendicular to  $\vec{u}$  are  $\vec{v} = (-b, a)$  and  $\vec{w} = (b, -a)$ .

Indeed:

$$\vec{u} \cdot \vec{v} = (a, b) \cdot (-b, a) = -ab + ab = 0 \implies \vec{u} \perp \vec{v}$$

## F Special Lines

A line *parallel* to the  $x$ -axis has a direction vector in the form  $\vec{u} = (u_x, 0) \mid u_x \neq 0$ .

A line *parallel* to the  $y$ -axis has a direction vector in the form  $\vec{u} = (0, u_y) \mid u_y \neq 0$ .

## 8.2 Cartesian Equation of a Line

### A Symmetric Equation

The parametric equations of a line in  $\mathbb{R}^2$ :

$$\begin{cases} x = x_0 + tu_x \\ y = y_0 + tu_y \end{cases} \quad t \in \mathbb{R}$$

may be written as:

$$\frac{x - x_0}{u_x} = \frac{y - y_0}{u_y} = t \mid t \in \mathbb{R}$$

The *symmetric equation* of the line is (if it exists):

$$\frac{x - x_0}{u_x} = \frac{y - y_0}{u_y}$$

**Note:** The symmetric equations only exists if  $u_x \neq 0$  and  $u_y \neq 0$ .

### B Normal Equation

Consider a line  $L$  that passes through the specific point  $P_0(x_0, y_0)$  and has the *direction vector*  $\vec{u} = (u_x, u_y)$ .

The vectors  $\vec{n} = (-u_y, u_x) = (A, B)$  or  $\vec{n} = (u_y, -u_x) = (A, B)$  are perpendicular to the vector  $\vec{u}$  and so they are perpendicular to the line  $L$ . These are called *normal vectors* to the line  $L$ .

Let  $P(x, y)$  be a generic point on the line  $L$ . So:

$$\begin{aligned} \overrightarrow{P_0P} \parallel \vec{u} &\implies \overrightarrow{P_0P} \perp \vec{n} \implies \overrightarrow{P_0P} \cdot \vec{n} = 0 \\ (\vec{r} - \vec{r}_0) \cdot \vec{n} &= 0 \end{aligned}$$

The *normal equation* of a line is given by:

$$(\vec{r} - \vec{r}_0) \cdot \vec{n} = 0$$

### C Cartesian Equation

The normal equations can be written as:

$$\begin{aligned} \vec{r} \cdot \vec{n} - \vec{r}_0 \cdot \vec{n} &= 0 \\ (x, y) \cdot (A, B) - (x_0, y_0) \cdot (A, B) &= 0 \\ Ax + By - Ax_0 - By_0 &= 0 \\ Ax + By + C &= 0 \quad \text{where } C = -Ax_0 - By_0 \end{aligned}$$

The *Cartesian equation* of a line is given by:

$$Ax + By + C = 0$$

where  $\vec{n} = (A, B)$  is a *normal vector* and the constant  $C$  depends on a specific point of the line.

## D Slope *y*-intercept Equation

Solve the symmetric equation of a line:

$$\frac{x - x_0}{u_x} = \frac{y - y_0}{u_y} \mid t \in \mathbb{R}$$

for  $y$ :

$$\begin{aligned} y - y_0 &= u_y \frac{x - x_0}{u_x} \\ y &= \frac{u_y}{u_x} x + y_0 - \frac{u_y}{u_x} x_0 \end{aligned}$$

The *slope y-intercept equation* of a line in  $\mathbb{R}^2$  is given by:

$$y = mx + b$$

$$m = \frac{u_y}{u_x}$$

where  $m$  is the *slope* and  $b$  is the *y-intercept* which depends on a specific point of the line.

## E Angle between Two Lines

The *angle* between two lines is determined by the angle between the *direction vectors*:

$$\cos \theta = \frac{\vec{u}_1 \cdot \vec{u}_2}{\|\vec{u}_1\| \|\vec{u}_2\|}$$

**Note:** There are two pairs of equal angles between the two lines. There is a pair of the angle  $\theta_1$ , and a pair of the angle  $\theta_2$ .  $\theta_1 + \theta_2 = 180^\circ$

## 8.3 Vector, Parametric, and Symmetric Equations of a Line in $\mathbb{R}^3$

### A Vector Equation

The vector equation of the line is:

$$\vec{r} = \vec{r}_0 + t\vec{u} \mid t \in \mathbb{R}$$

where:

- $\vec{r} = \overrightarrow{OP}$  is the position vector of a *generic* point  $P$  on the line.
- $\vec{r}_0 = \overrightarrow{OP_0}$  is the position vector of a *specific* point  $P_0$  on the line.
- $\vec{u}$  is a vector parallel to the line called the *direction vector* of the line.
- $t$  is a *real number* corresponding to the generic point  $P$ .

### B Specific Lines

A line is parallel to the  $x$ -axis if  $\vec{u} = (u_x, 0, 0) \mid u_x \neq 0$ . In this case, the line is also *perpendicular to the  $yz$ -plane*.

A line with  $\vec{u} = (0, u_y, u_z) \mid u_y \neq 0 \wedge u_z \neq 0$  is *parallel to the  $yz$ -plane*.

### C Parametric Equations

Rewrite the vector equation of a line:

$$\vec{r} = \vec{r}_0 + t\vec{u} \mid t \in \mathbb{R}$$

as:

$$(x, y, z) = (x_0, y_0, z_0) + t(u_x, u_y, u_z) \mid t \in \mathbb{R}$$

The *parametric equations* of a line in  $\mathbb{R}^3$  are:

$$\begin{cases} x = x_0 + tu_x \\ y = x_0 + tu_y \\ z = x_0 + tu_z \end{cases} \quad t \in \mathbb{R}$$

## D Symmetric Equations

The parametric equations of a line may be written as:

$$\begin{cases} x = x_0 + tu_x \\ y = x_0 + tu_y \\ z = x_0 + tu_z \end{cases} \quad t \in \mathbb{R}$$

From here, the *symmetric equations* of the line are:

$$\frac{x - x_0}{u_x} = \frac{y - y_0}{u_y} = \frac{z - z_0}{u_z}$$
$$u_x \neq 0 \quad u_y \neq 0 \quad u_z \neq 0$$

## E Intersections

A line *intersects the  $x$ -axis* when  $y = z = 0$ .

A line *intersects the  $xy$ -plane* when  $z = 0$ .

## 8.4 Vector and Parametric Equations of a Plane

### A Planes

A plane may be determined by points and lines. There are four main possibilities:

1. Plane determined by three points.
2. Plane determined by two parallel lines.
3. Plane determined by two intersecting lines.
4. Plane determined by a point and a line.

### B Vector Equation of a Plane

Consider a plane  $\pi$ .

Two vectors  $\vec{u}$  and  $\vec{v}$ , parallel to the plane  $\pi$  but not parallel to each other, are called *direction vectors* of the plane  $\pi$ .

The vector  $\overrightarrow{P_0P}$  from a specific point  $P_0(x_0, y_0, z_0)$  to a generic point  $P(x, y, z)$  of the plane is a *linear combination* of direction vectors  $\vec{u}$  and  $\vec{v}$ :

$$\overrightarrow{P_0P} = s\vec{u} + t\vec{v} \mid s, t \in \mathbb{R}$$

The *vector equation* of the plane is:

$$\pi : \vec{r} = \vec{r}_0 + s\vec{u} + t\vec{v} \mid s, t \in \mathbb{R}$$

### C Parametric Equations of a Plane

We write the vector equation of the plane as:

$$(x, y, z) = (x_0, y_0, z_0) + s(u_x, u_y, u_z) + t(v_x, v_y, v_z)$$

or:

$$\begin{cases} x = x_0 + su_x + tv_x \\ y = y_0 + su_y + tv_y \\ z = z_0 + su_z + tv_z \end{cases} \quad s, t \in \mathbb{R}$$

These are the *parametric equations* of a plane.

## 8.5 Cartesian Equation of a Plane

### A Normal Equation of a Plane

A plane may be determined by a *point*  $P_0(x_0, y_0, z_0)$  and a *vector* perpendicular to the plane  $\vec{n}$  called the *normal vector*.

If  $P(x, y, z)$  is a generic point on the plane, then:

$$\overrightarrow{P_0P} \perp \vec{n}$$

and:

$$\overrightarrow{P_0P} \cdot \vec{n} = 0$$

This is the *normal equation* of a plane.

### B Cartesian Equation of a Plane

We write the normal vector of a plane in the form:

$$\vec{n} = (A, B, C)$$

Then, the normal equation may be written as:

$$\begin{aligned}(x - x_0, y - y_0, z - z_0) \cdot (A, B, C) &= 0 \\ Ax + By + Cz - Ax_0 - By_0 - Cz_0 &= 0\end{aligned}$$

or:

$$Ax + By + Cz + D = 0$$

which is called the *Cartesian equation* of a plane.

**Note:** A normal vector to the plane is:

$$\vec{n} = \vec{u} \times \vec{v}$$

where  $\vec{u}$  and  $\vec{v}$  are the direction vectors of the plane.

### C Angle between Two Planes

The *angle* between two planes is defined as the angle between their *normal vectors*:

$$\cos \theta = \frac{\vec{n}_1 \cdot \vec{n}_2}{\|\vec{n}_1\| \|\vec{n}_2\|}$$

**Note:** Using this formula, you may get an *acute* or an *obtuse* angle depending on the normal vectors which are used.

## Unit 2

# Relationships between Points, Lines, and Planes



## 9.1 Intersection of Two Lines

### A Relative Position of Two Lines

Two lines may be:

1. Parallel and distinct.
2. Parallel and coincident.
3. Intersecting (not parallel).
4. Skew (not parallel, not intersecting).

### B Intersection of Two Lines (Algebraic Method)

The point of intersection of two lines  $L_1 : \vec{r} = \vec{r}_{01} + t\vec{u}_1 \mid t \in \mathbb{R}$  and  $L_2 : \vec{r} = \vec{r}_{02} + s\vec{u}_2 \mid s \in \mathbb{R}$  is given by the *solution* of the following system of equations (if it exists):

$$\begin{cases} x_{01} + tu_{x1} = x_{02} + su_{x2} \\ y_{01} + tu_{y1} = y_{02} + su_{y2} \\ z_{01} + tu_{z1} = z_{02} + su_{z2} \end{cases} \quad s, t \in \mathbb{R}$$

**Hint:** Solve by *substitution* or *elimination* the system of two equations and *check* if the third is satisfied.

### C Unique Solution

If by solving the system you end by getting a *unique* value for  $t$  and  $s$  satisfying this system, then the lines have a *unique point of intersection*. To get this point, substitute either the  $t$  value into the line  $L_1$  equation or substitute the  $s$  value into the line  $L_2$  equation.

### D Infinite Number of Solutions

If by solving the system you end by getting two true statements (like  $2 = 2$ ) and one equation in  $s$  and  $t$ , then there exist an *infinite number of solutions* of the system. Therefore the lines intersect at an *infinite number of points*. In this case the lines are parallel and coincident.

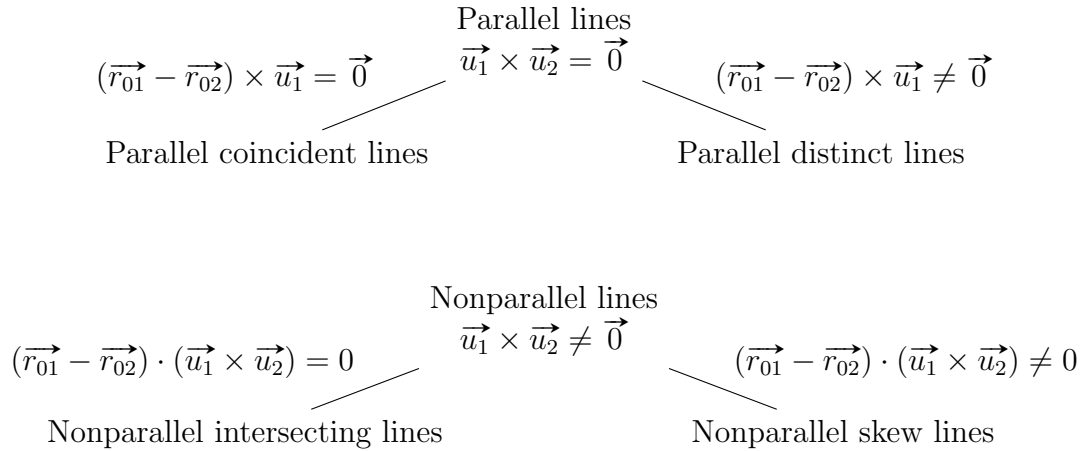
## E No Solution (Parallel Lines)

If by solving the system you get at least one *false* statement (like  $0 = 1$ ) then the system has *no solution*. Therefore, the lines have *no point of intersection*. If, in addition, the lines are parallel ( $\vec{u}_1 \times \vec{u}_2 = \vec{0}$ ), then the lines are *parallel and distinct*.

## F No Solution (Skew Lines)

If by solving the system you get at least one *false* statement (like  $0 = 1$ ) then the system has *no solution*. Therefore, the lines have *no point of intersection*. If, in addition, the lines are *not parallel* ( $\vec{u}_1 \times \vec{u}_2 \neq \vec{0}$ ), then the lines are *skew*.

## G Classifying Lines (Vector Method)



## 9.2 Intersection of a Line with a Plane

### A Relative Position of a Line and a Plane

There are three possible situations:

1. The line *intersects* the plane at a single point.

$$P = L \cap \pi$$

2. The line *lies* on the plane. There are an infinite number of points of intersection.

$$L = L \cap \pi$$

3. The line is *parallel* to the plane but *distinct*. There is no point of intersection.

$$L \cap \pi = \emptyset$$

### B Intersection of a Line and a Plane (Algebraic Method)

To get the intersection between a line  $L$  and a plane  $\pi$ :

1. *Substitute* the parametric equations of the line

$$L : \begin{cases} x = x_0 + tu_x \\ y = y_0 + tu_y \\ z = z_0 + tu_z \end{cases} \quad t \in \mathbb{R}$$

into the Cartesian equation of the plane

$$\pi : Ax + By + Cz + D = 0$$

to get the equation:

$$A(x_0 + tu_x) + B(y_0 + tu_y) + C(z_0 + tu_z) + D = 0 \quad (\text{i})$$

2. *Solve* (if possible) the equation (i) for the parameter  $t$ .
3. *Substitute* the value of the parameter  $t$  into the parametric equations of the line to get the point of intersection.

### C Unique Solution (Point Intersection)

In this case, by solving the equation you get a *unique value* for the parameter  $t$ . Therefore, there is a unique *point of intersection* between the line and the plane.

$$P = L \cap \pi$$

The line *intersects* the plane at a unique point.

## D Infinite Number of Solutions (Line Intersection)

In this case, by solving the equation (i) you get the equation:

$$0t = 0$$

which has an *infinite number of solutions*. Therefore, there are an *infinite number of points of intersection*.

$$L = L \cap \pi$$

The line *lies* on the plane.

## E No Solution (No Intersection)

In this case, by solving the equation (i) you get a false statement like:

$$0t = 1$$

The equation *does not have any solution* and therefore there is *no point of intersection* between the line and the plane.

$$L \cap \pi = \emptyset$$

The line is *parallel* to the plane and *does not lie* on the plane.

## F Classifying Lines

Consider the line  $L : \vec{r} = \vec{r}_0 + t\vec{u} \mid t \in \mathbb{R}$ , where  $P_0(x_0, y_0, z_0)$  is a specific point on the line, and the plane  $\pi : Ax + By + Cz + D = 0$ , where  $\vec{n} = (A, B, C)$  is a normal vector to the plane.

1. If  $\vec{n} \cdot \vec{u} \neq 0$  the line *intersects* the plane at a unique point.

$$P = L \cap \pi$$

2. If  $\vec{n} \cdot \vec{u} = 0$  and  $Ax_0 + By_0 + Cz_0 + D = 0$  then the line *lies* on the plane.

$$L = L \cap \pi$$

3. If  $\vec{n} \cdot \vec{u} = 0$  and  $Ax_0 + By_0 + Cz_0 + D \neq 0$  then the line is *parallel* to the plane but *does not lie* on the plane.

$$L \cap \pi = \emptyset$$

**Note.** By solving the equation (i) for  $t$  you will end by getting the same cases and conditions as above.

## 9.3 Intersection of Two Planes

### A Relative Position of Two Planes

Two planes may be:

1. Intersecting (into a line)

$$L = \pi_1 \cap \pi_2$$

2. Coincident

$$\pi_1 = \pi_1 \cap \pi_2 = \pi_2$$

3. Distinct

$$\pi_1 \cap \pi_2 = \emptyset$$

### B Intersection of Two Planes

Consider two planes given by their Cartesian equations:

$$\pi_1 = A_1x + B_1y + C_1z + D_1 = 0$$

$$\pi_2 = A_2x + B_2y + C_2z + D_2 = 0$$

To find the point(s) of intersection between two planes, *solve* the system of equations formed by their Cartesian equations:

$$\begin{cases} \pi_1 = A_1x + B_1y + C_1z + D_1 = 0 \\ \pi_2 = A_2x + B_2y + C_2z + D_2 = 0 \end{cases} \quad (\text{ii})$$

There are *two* equations and *three* unknowns. **Notes:**

1. A normal vector to the plane  $\pi_1$  is  $\vec{n}_1 = (A_1, B_1, C_1)$  and a normal vector to the plane  $\pi_2$  is  $\vec{n}_2 = (A_2, B_2, C_2)$ .
2. If the planes are *parallel* then the coefficients  $A$ ,  $B$ , and  $C$  are *proportional*.
3. If the planes are *coincident* then the coefficients  $A$ ,  $B$ ,  $C$ , and  $D$  are *proportional*.
4. A system of equations is called *compatible* if there is *at least* one solution. A system of equations is called *incompatible* if there is *no solution*.

### C Nonparallel Planes (Line Intersection)

In this case:

$$L = \pi_1 \cap \pi_2$$

- The coefficients  $A$ ,  $B$ , and  $C$  in the scalar equations are *not proportional*.

- The normal vectors are *not parallel*:  $\vec{n}_1 \times \vec{n}_2 \neq \vec{0}$ .
- By solving the system (ii) you will be able to find two variables in terms of the third variable.
- There are an *infinite number of solutions* and therefore an *infinite number of points of intersection*.
- The intersection is a *line* and a *direction vector* for this line is  $\vec{u} = \vec{n}_1 \times \vec{n}_2$ .

## D Coincident Planes (Plane Intersection)

In this case:

$$\pi_1 = \pi_1 \cap \pi_2 = \pi_2$$

- The planes are *parallel* and *coincident*.
- The coefficients  $A$ ,  $B$ ,  $C$ , and  $D$  in the scalar equations are *proportional*.
- One equation in the system (ii) is a *multiple* of the other equation and does not contain additional information (the equations are equivalent).
- By solving the system of equations (ii), you get a *true* statement (like  $0 = 0$ ).
- There are an *infinite number of solutions* and therefore an *infinite number of points of intersection*.
- The intersection is a *plane*.

## E Parallel and Distinct Planes (No Intersection)

In this case:

$$\pi_1 \cap \pi_2 = \emptyset$$

- The planes are *parallel* and *distinct*.
- The coefficients  $A$ ,  $B$ , and  $C$  in the scalar equations are *proportional* but the coefficients  $A$ ,  $B$ ,  $C$ , and  $D$  are *not proportional*.
- By solving the system (ii) you get a *false* statement (like  $0 = 1$ ).
- There is *no solution* and therefore *no point of intersection* between the two planes.

## 9.4 Intersection of Three Planes

### A Intersection of Three Planes

Consider three planes given by their Cartesian equations:

$$\pi_1 : A_1x + B_1y + C_1z + D_1 = 0$$

$$\pi_2 : A_2x + B_2y + C_2z + D_2 = 0$$

$$\pi_3 : A_3x + B_3y + C_3z + D_3 = 0$$

The point(s) of *intersection* of these planes is (are) related by to the *solution(s)* of the following system of equations:

$$\begin{cases} A_1x + B_1y + C_1z + D_1 = 0 \\ A_2x + B_2y + C_2z + D_2 = 0 \\ A_3x + B_3y + C_3z + D_3 = 0 \end{cases} \quad (\text{iii})$$

There are *three* equations and *three* unknowns. You may use *substitution* or *elimination* to solve this system.

### B Unique Solution (Point Intersection — Noncoplanar Normal Vectors)

In this case:

$$P = \pi_1 \cap \pi_2 \cap \pi_3$$

- The planes *intersect* into a *single* point.
- The *normal vectors* are *not coplanar*:

$$\vec{n}_1 \cdot (\vec{n}_2 \times \vec{n}_3) \neq 0$$

- By solving the system (iii), you get a *unique solution* for  $x$ ,  $y$ , and  $z$ .

### C Infinite Number of Solutions (Line Intersection — Non-parallel Planes and Coplanar Normal Vectors)

In this case:

$$L = \pi_1 \cap \pi_2 \cap \pi_3$$

- The planes are *not parallel* but their normal vectors are *coplanar*:

$$\vec{n}_1 \cdot (\vec{n}_2 \times \vec{n}_3) = 0$$

- The intersection is a *line*.
- One scalar equation is a *combination* of the other two equations.
- By solving the system (iii), you may express two variables in terms of the third one using two equations.

## D Infinite Number of Solutions (Line Intersection — Two Coincident Planes and One Intersecting Plane)

In this case:

$$L = \pi_1 \cap \pi_2 \cap \pi_3$$

- Two planes are *coincident* and the third plane is *not parallel* to the coincident planes.
- The intersection is a *line*.
- Two scalar equations are *equivalent*. The *coefficients*  $A$ ,  $B$ ,  $C$ , and  $D$  are *proportional* for these two equations.
- You may express two variables in terms of the third one using two nonequivalent equations.

## E Infinite Number of Solutions (Plane Intersection — Three Coincident Planes)

In this case:

$$\pi_1 \cap \pi_2 \cap \pi_3 = \pi_1 = \pi_2 = \pi_3$$

- The coefficients  $A$ ,  $B$ ,  $C$ , and  $D$  are *proportional* for all three equations.
- Any point of one plane is also a point on the other two planes.
- The intersection is a *plane*.

## F No Solution (Parallel and Distinct Planes)

In this case:

$$\pi_1 \cap \pi_2 \cap \pi_3 = \emptyset$$

- There are three *parallel* and *distinct* planes.
- There is *no point of intersection*.
- There is *no solution* for the system of equations (the system of equations is *incompatible*).



- The coefficients  $A$ ,  $B$ , and  $C$  are *proportional* but the coefficients of  $A$ ,  $B$ ,  $C$ , and  $D$  are *not proportional*.
- By solving the system (iii) you get *false* statements (like  $0 = 1$ ).

## G No Solution (H Configuration)

In this case:

$$\pi_1 \cap \pi_2 \cap \pi_3 = \emptyset$$

- Two planes are *parallel and distinct* and the third plane is *intersecting*.
- There is *no point of intersection*.
- The coefficients  $A$ ,  $B$ , and  $C$  are proportional for two planes.
- There is *no solution* for the system of equations (the system of equations is *incompatible*).
- By solving the system (iii) you get *false* statements (like  $0 = 1$ ).

## H No Solution (Three Parallel Planes but only Two Coincident Planes)

In this case:

$$\pi_1 \cap \pi_2 \cap \pi_3 = \emptyset$$

- Three planes are *parallel* but only two are *coincident*.
- The coefficients of  $A$ ,  $B$ , and  $C$  are *proportional* for all equations but the coefficients  $A$ ,  $B$ ,  $C$ , and  $D$  are *proportional* only for two planes.
- There is *no solution* for the system of equations (the system of equations is *incompatible*).
- By solving the system (iii) you get *false* statements (like  $0 = 1$ ).

## I No Solution (Delta Configuration)

In this case:

$$\pi_1 \cap \pi_2 \cap \pi_3 = \emptyset$$

- The planes are *not parallel* (the coefficients  $A$ ,  $B$ , and  $C$  are not *proportional*).
- The normal vectors are *coplanar* ( $\vec{n}_1 \cdot (\vec{n}_2 \times \vec{n}_3) = 0$ ).
- There is *no point of intersection* between all three planes.

- There is *no solution* for the system of equations (the system of equations is *incompatible*).
- By solving the system (iii) you get *false* statements (like  $0 = 1$ ).

## 9.5 Distance from a Point to a Line

### A Distance from a Point to a Line in $\mathbb{R}^2$

Let  $L : Ax + By + C = 0$  be a line in  $\mathbb{R}^2$ ,  $P_1(x_1, y_1)$  be a *generic point* on the  $xy$ -plane and  $P_0(x_0, y_0)$  be a specific point on this line, so:  $Ax_0 + By_0 + C = 0$ .

The *distance*  $d$  between the point  $P_1(x_1, y_1)$  to the line  $L$  is given by (*scalar projection* of  $\overrightarrow{P_0P_1}$  onto the normal vector  $\vec{n}$ ):

$$d = \frac{|\overrightarrow{P_0P_1} \cdot \vec{n}|}{\|\vec{n}\|} \quad (\text{iv})$$

Using  $\vec{n} = (A, B)$ ,  $\|\vec{n}\| = \sqrt{A^2 + B^2}$  and:

$$\begin{aligned} \overrightarrow{P_0P_1} \cdot \vec{n} &= (x_1 - x_0, y_1 - y_0) \cdot (A, B) \\ &= A(x_1 - x_0) + B(y_1 - y_0) \\ &= Ax_1 + By_1 - Ax_0 - By_0 \\ &= Ax_1 + By_1 + C \end{aligned}$$

the formula (iv) may be written as:

$$d = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}} \quad (\text{v})$$

### B Distance from a Point to a Line in $\mathbb{R}^3$

Let  $L : \vec{r} = \vec{r}_0 + t\vec{u} \mid t \in \mathbb{R}$  be a line defined by its vector equation and  $P_0(x_0, y_0, z_0)$  be a specific point on this line.

The distance  $d$  from a point  $P_1(x_1, y_1, z_1)$  to the line  $L$  may be found using:

$$d = \left\| \overrightarrow{P_0P_1} \right\| \sin \alpha \quad (\text{vi})$$

where  $\alpha$  is the angle formed by the intersection of  $\overrightarrow{P_0P_1}$  and  $\vec{u}$ .

Because  $\left\| \overrightarrow{P_0P_1} \times \vec{u} \right\| = \left\| \overrightarrow{P_0P_1} \right\| \|\vec{u}\| \sin \alpha$ , the formula (vi) can also be written as:

$$d = \frac{\left\| \overrightarrow{P_0P_1} \times \vec{u} \right\|}{\|\vec{u}\|} \quad (\text{vii})$$

**Note:** The formula (vii) may be applied also in  $\mathbb{R}^2$  by considering the third component  $z = 0$ .

## C Distance between Two Parallel Lines

To find the *distance* between two parallel lines:

1. Find a *specific point* on one of these lines.
2. Find the distance from that specific point to the other line using one of the relations above.

## D Perpendicular Line from a Point to a Line

Let  $L : \vec{r} = \vec{r}_0 + t\vec{u} \mid t \in \mathbb{R}$  be a line defined by its vector equation and  $P(x, y, z)$  be a generic point in  $\mathbb{R}^3$ .

The line perpendicular to the line  $L$  that passes through the point  $P$  is called the *perpendicular line* and intersects the line  $L$  at a point  $F$  called the *foot* of the perpendicular line.

The foot  $F$  of the perpendicular line may be found from the equation (because  $\overrightarrow{PF} \perp \vec{u}$ ):

$$\overrightarrow{PF} \cdot \vec{u} = 0$$

A *vector equation* of the perpendicular line is:

$$\vec{r} = \overrightarrow{OP} + s\overrightarrow{PF} \mid s \in \mathbb{R}$$

## E Shortest Distance between Two Skew Lines

Two skew lines lie into *two parallel planes*. The vector  $\vec{u}_1 \times \vec{u}_2$  is *perpendicular* to both lines and therefore perpendicular to parallel planes the lines lie on.

The *shortest distance* between two skew lines  $L_1 : \vec{r} = \vec{r}_{01} + t\vec{u}_1 \mid t \in \mathbb{R}$  and  $L_2 : \vec{r} = \vec{r}_{02} + s\vec{u}_2 \mid s \in \mathbb{R}$  is given by the *scalar projection* of the vector  $\vec{r}_{01} - \vec{r}_{02}$  onto the vector  $\vec{u}_1 \times \vec{u}_2$ :

$$d = \frac{|(\vec{r}_{01} - \vec{r}_{02}) \cdot (\vec{u}_1 \times \vec{u}_2)|}{\|\vec{u}_1 \times \vec{u}_2\|} \quad (\text{viii})$$

## 9.6 Distance from a Point to a Plane

### A Distance from a Point to a Plane (I)

Consider a plane  $\pi$  with a *normal vector*  $\vec{n}$  and a point  $P_0(x_0, y_0, z_0)$  on this plane. The *distance* from a point  $P_1(x_1, y_1, z_1)$  to the plane  $\pi$  is given by the *scalar projection* of the vector  $\overrightarrow{P_0P_1}$  onto the normal vector  $\vec{n}$ :

$$d = \frac{|\overrightarrow{P_0P_1} \cdot \vec{n}|}{\|\vec{n}\|} \quad (\text{ix})$$

### B Distance from a Point to a Plane (II)

If the plane  $\pi$  is given by the *Cartesian equation*  $\pi : Ax + By + Cz + D = 0$ , then the *distance* from a point  $P_1(x_1, y_1, z_1)$  to the plane is given by:

$$d = \frac{|Ax_1 + By_1 + Cz_1 + D|}{\sqrt{A^2 + B^2 + C^2}} \quad (\text{x})$$

Indeed,

$$P_0 \in \pi \implies Ax_0 + By_0 + Cz_0 + D = 0$$

$$\begin{aligned} \overrightarrow{P_0P_1} \cdot \vec{n} &= (x_1 - x_0, y_1 - y_0, z_1 - z_0) \cdot (A, B, C) \\ &= Ax_1 + By_1 + Cz_1 - Ax_0 - By_0 - Cz_0 \\ &= Ax_1 + By_1 + Cz_1 + D \end{aligned}$$

### C Distance between Two Parallel Planes

To get the *distance* between *two parallel planes*:

1. Find a specific point into one of these planes.
2. Find the distance between that specific point and the other plane using one of the formulas above.

# AP Preparation Differentiability Review

## 1.4 Limit of a Function

### A One-Sided Limits

The behaviour of the function  $y = f(x)$  near  $x = a$  is described by three numbers:

1. The left hand limit:

$$L = \lim_{x \rightarrow a^-} f(x)$$

the limit of the function  $f(x)$  as  $x$  approaches  $a$  from the left.

2. The value of the function at  $x = a$ :

$$f(a)$$

3. The right hand limit:

$$R = \lim_{x \rightarrow a^+} f(x)$$

the limit of the function  $f(x)$  as  $x$  approaches  $a$  from the right.

#### Notes:

1. In order to exist, both the left and right hand limits must be numbers.
2. If either the left or right hand limit is not a number, then the limit does not exist (DNE).
3. Infinite limits (like  $\infty$  or  $-\infty$ ) are not considered numbers but they are used to give information about the behaviour of a function near the number  $x = a$ .

### B Limit

The limit of a function  $y = f(x)$  exists at  $x = a$  if:

$$L \text{ and } R \text{ exist and } L = R$$

In this case we write:

$$\lim_{x \rightarrow a} f(x)$$

the limit of the function  $f(x)$  as  $x$  approaches  $a$ .

**Note:** The function may or may not be defined at  $x = a$ .

## C Substitution

If the function is defined by a formula (algebraic expression) then the limit of the function at a number  $x = a$  may be determined by substitution:

$$\lim_{x \rightarrow a} f(x) = f(a)$$

### Notes:

1. In order to use substitution, the function must be defined on both sides of the number  $x = a$ .
2. Substitution does not work if you get one of the following 7 indeterminate cases:

$$\infty - \infty \quad 0 \times \infty \quad \frac{0}{0} \quad \frac{\infty}{\infty} \quad 1^\infty \quad \infty^0 \quad 0^0$$

## D Piecewise defined functions (AP only)

If the function changes the formula at  $x = a$  then:

1. Use the appropriate formula to find the left-hand and right-hand limits.
2. Compare the left-hand and right-hand limits to conclude about the limit of the function at  $x = a$ .

Example:

$$f(x) = \begin{cases} f_1(x) & | \ x < a \\ f_2(x) & | \ x > a \end{cases}$$

At  $x = a$ :

$$L = f_1(a) \quad R = f_2(a)$$

## E Limits: Numerical Approach (AP only)

The limit of a function  $y = f(x)$  at a number  $x = a$  may be estimated numerically. To do that:

1. Use a sequence of numbers  $x$  approaching  $x = a$  from the left and from the right.
2. Find the value of the function at each number  $x$ .
3. Analyze the values and make a conclusion (guess the limit).
4. Be careful at the “difference catastrophe”.



## F Limit: Informal Definitions (AP only)

**Left-Hand Limit** If the values of  $y = f(x)$  can be made arbitrarily close to  $L$  by taking  $x$  sufficiently close to  $a$  with  $x < a$ , then:

$$\lim_{x \rightarrow a^-} f(x) = L$$

**Right-Hand Limit** If the values of  $y = f(x)$  can be made arbitrarily close to  $R$  by taking  $x$  sufficiently close to  $a$  with  $x > a$ , then:

$$\lim_{x \rightarrow a^+} f(x) = R$$

**Limit** If the values of  $y = f(x)$  can be made arbitrarily close to  $l$  by taking  $x$  sufficiently close to  $a$  from both sides, then:

$$\lim_{x \rightarrow a} f(x) = l$$

**Infinite Limit** If the values of  $y = f(x)$  can be made arbitrarily large by taking  $x$  sufficiently close to  $a$  from both sides, then:

$$\lim_{x \rightarrow a} f(x) = \infty$$

## 1.6 Continuity

### A Continuity

A function  $y = f(x)$  is continuous at a number  $x = a$  if

$$L = R = f(a)$$

where:

$L = \lim_{x \rightarrow a^-} f(x)$  is the left-hand limit at  $x = a$ .

$R = \lim_{x \rightarrow a^+} f(x)$  is the right-hand limit at  $x = a$ .

$f(a)$  is the value of the function at  $x = a$ .

**Note: A function is continuous if it can be drawn without lifting your pencil from the paper.**

### B Discontinuity

If  $y = f(x)$  is not continuous at  $x = a$  then we say: “ $y = f(x)$  is discontinuous at  $x = a$ ” or “ $y = f(x)$  has a discontinuity at  $x = a$ ”.

### C Removable Discontinuity

A function  $y = f(x)$  has a removable discontinuity at  $x = a$  if:

1.  $L = R = \lim_{x \rightarrow a} f(x)$  exists
2.  $f(a)$  DNE or  $\lim_{x \rightarrow a} f(x) \neq f(a)$

**Note: A removable discontinuity can be removed by redefining the function  $x = a$  as  $f(a) \stackrel{\text{def}}{=} \lim_{x \rightarrow a} f(x)$ .**

### D Jump Discontinuity

A function  $y = f(x)$  has a jump discontinuity at  $x = a$  if:

$$L = \lim_{x \rightarrow a^-} f(x) \neq \lim_{x \rightarrow a^+} f(x) = R$$

## **E Infinite Discontinuity**

A function  $y = f(x)$  has an infinite discontinuity at  $x = a$  if at least one side of the limit is unbounded (approaches  $\infty$  or  $-\infty$ ).

## **F Continuity over an Interval (AP only)**

A function  $y = f(x)$  is continuous over an open interval  $(a, b)$  if the function is continuous at every number in that interval.

A function is continuous from the right at  $x = a$  if  $R = f(a)$ .

A function is continuous from the left at  $x = a$  if  $L = f(a)$ .

## **G Elementary Functions (AP only)**

Elementary functions (polynomial, power, rational, trigonometric, exponential, and logarithmic) are continuous over their domain.

## **H Composition of Functions**

If  $g$  is continuous at  $x = a$  and  $f$  is continuous at  $g(a)$  then  $f(g(x))$  is continuous at  $x = a$ .

## **I Intermediate Value Theorem (AP only)**

If  $y = f(x)$  is a continuous function over the interval  $[a, b]$  with  $f(a) \neq f(b)$ , then for any number  $N$  between  $f(a)$  and  $f(b)$  there exist a number  $c \in (a, b)$  such that  $f(c) = N$ .

## 2.1 Derivative Function

### A Derivative Function

Given a function  $y = f(x)$ , the *derivative function* of  $f$  is a *new function* called  $f'$  ( $f$  prime), defined at  $x$  by:

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

A function  $y = f(x)$  is *differentiable* at  $x$  if  $f'(x)$  exists.

### B Differentiability (AP only)

A function  $y = f(x)$  is differentiable over an open interval  $(a, b)$  if the function is differentiable at every number in that interval.

The domain of derivative function  $f'(x)$  is a subset of the domain of the original function  $f$  ( $D_{f'} \subseteq D_f$ ). So a function is defined over  $D_f$  but is differentiable over  $D_{f'}$ .

### C Interpretations of Derivative Function

1. The *slope of the tangent line* to the graph of  $y = f(x)$  at the point  $P(a, f(a))$  is given by  $m_T = f'(a)$ .
2. The *instantaneous rate of change* in the variable  $y$  with respect to the variable  $x$ , where  $y = f(x)$ , at  $x = a$  is given by  $IRC = f'(a)$ .

### D Notations and Reading

#### Lagrange or prime notation

$$y' = f'(a)$$

Reading: “y prime” or “f prime of (at) x”.

#### Leibnitze notation

$$\frac{dy}{dx} = \frac{d}{dx} f(x) = Df(x) = D_x f(x)$$
$$\frac{dy}{dx}$$

Reading: “dee y by dee x”.

## Evaluating

$$f'(a) = \left. \frac{dy}{dx} \right|_{x=a}$$

Reading: “dee y by dee x at x equals a”.

## E First Principles

*Differentiation* is the process to find the derivative function for a given function.

*First Principles* is the process of differentiation by computing any of the following limits:

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$
$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x-h)}{2h}$$
$$f'(x) = \lim_{u \rightarrow x} \frac{f(u) - f(x)}{u - x}$$

## F Differentiability Point

A function  $y = f(x)$  is *differentiable* at  $x$  if  $f'(x)$  exists.

If the function  $y = f(x)$  is *differentiable* at  $x = a$  then the tangent line at  $P(a, f(a))$  is *unique* and *not vertical* (the slope of the tangent line is not  $\infty$  or  $-\infty$ ).

## G Non-Differentiability

A function is *not differentiable* at  $x = a$  if  $f'(a)$  *does not exist*.

### Notes:

- If a function  $f$  is *not continuous* at  $x = a$  then the function  $f$  is *not differentiable* at  $x = a$ .
- If a function is differentiable at  $x = a$  then the function is continuous at  $x = a$ .
- If a function  $f$  is *continuous* at  $x = a$  then the function  $f$  *may or may not be* differentiable at  $x = a$ .

## H Corner Point

$P(a, f(a))$  is a *corner point* if there are *two* distinct tangent lines at  $P$ , one for the left-hand branch and one for the right-hand branch.

## I Infinite Slope Point

$P(a, f(a))$  is an *infinite slope point* if the tangent line at  $P$  is vertical and the function is increasing or decreasing in the neighbourhood of the point  $P$ .

$$f'(a) = \infty \quad \vee \quad f'(a) = -\infty$$

## J Cusp Point

$P(a, f(a))$  is a *cusp point* if the tangent line at  $P$  is vertical and the function is increasing on one side of the point  $P$  and decreasing on the other side.

$$f'(a) = DNE$$

## 2.2 Derivative of Polynomial Functions

### A Power Rule

If  $y = f(x) = x^n \mid x, n \in \mathbb{R}$  is the *power* function then:

$$y' = f'(x) = (x^n)' = nx^{n-1}$$

Some useful specific case:

$$(1)' = 0$$

$$(x)' = 1$$

$$(\sqrt{x})' = \frac{1}{2\sqrt{x}}$$

### B Constant Function Rule

If  $y = f(x) = c \mid c \in \mathbb{R}$  is the *constant* function then:

$$f'(x) = (c)' = 0$$

### C Constant Multiple Rule

If  $g(x) = cf(x)$  then:

$$g'(x) = (cf(x))' = cf'(x)$$

$$\frac{d}{dx}g(x) = \frac{d}{dx}(cf(x)) = c\frac{d}{dx}f(x)$$

### D Sum and Difference Rules

$$(f(x) \pm g(x))' = f'(x) \pm g'(x)$$

### E Tangent Line

The *equation of the tangent line* at the point  $P(a, f(a))$  to the curve  $y = f(x)$  is:

$$y = f'(a)(x - a) + f(a) \tag{xi}$$

## F Normal Line (AP only)

If  $m_T = f'(a)$  is the slope of the tangent line at  $P(a, f(a))$ , the slope of the normal line  $m_N$  is given by:

$$m_N = -\frac{1}{m_T}$$

## G Differentiability for Piecewise Defined Function (AP only)

Consider the piecewise defined function:

$$f(x) = \begin{cases} f_1(x) & x < a \\ c & x = a \\ f_2(x) & x > a \end{cases}$$

The function  $f$  is *differentiable* at  $x = a$  if:

1. The function is continuous at  $x = a$ .
2.  $f'_1(a) = f'_2(a)$  (the slope of the tangent line for the left branch is equal to the slope of the tangent line for the right branch).



## 2.3 Product Rule

### A Product Rule

If  $f$  and  $g$  are differentiable at  $x$  then so is  $fg$  and:

$$(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$$

$$(fg)' = f'g + fg'$$

### B Product of Three Functions

If  $f$ ,  $g$ , and  $h$  are differentiable at  $x$  then so is  $fgh$  and:

$$(fgh)' = f'gh + fg'h + fgh'$$

### C Generalized Power Rule

If  $f$  is differentiable at  $x$ , then so is  $f^n$  and:

$$\left( (f(x))^n \right)' = n (f(x))^{n-1} f'(x) \tag{xii}$$

$$(f^n)' = n f^{n-1} f'$$

## 2.4 Quotient Rule

### A Quotient Rule

If  $f$  and  $g$  are differentiable at  $x$  and  $g(x) \neq 0$  then so is  $\frac{f}{g}$  and:

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2} \quad (\text{xiii})$$

## 2.5 Chain Rule

### A Composition of Functions

If  $u = g(x)$  and  $v = f(u)$  then:

$$x \xrightarrow[u=g(x)]{} u \xrightarrow[v=f(u)]{} v$$

and

$$v = f(u) = f(g(x)) = (f \circ g)(x)$$

### B Chain Rule (Leibniz Notation)

$$\Delta x \xrightarrow[u=g(x)]{} \Delta u \xrightarrow[v=f(u)]{} \Delta v$$

and

$$\frac{\Delta v}{\Delta x} = \frac{\Delta v}{\Delta u} \frac{\Delta u}{\Delta x} \rightarrow \frac{dv}{dx} = \frac{dv}{du} \frac{du}{dx}$$

Therefore:

$$\frac{dv}{dx} = \frac{dv}{du} \frac{du}{dx}$$

### C Composition of Three Functions

$$x \xrightarrow[u=h(x)]{} u \xrightarrow[v=g(u)]{} v \xrightarrow[w=f(v)]{} w$$
$$\frac{dw}{dx} = \frac{dw}{dv} \frac{dv}{du} \frac{du}{dx}$$

### D Chain Rule (Prime Notation)

$$(f(g(x)))' = f'(g(x))g'(x)$$

If  $g$  is differentiable at  $x$  and  $f$  is differentiable at  $g(x)$  then the composition  $(f \circ g)(x) = f(g(x))$  is differentiable at  $x$  and:

$$(f \circ g)'(x) = (f(g(x)))' = f'(g(x))g'(x)$$

So, the derivative of  $f(g(x))$  is the derivative of the *outside* function  $f$  evaluated at the inside function  $g(x)$  times the derivative of the inside function  $g$  at  $x$ .

**Note:** If the outside function is the power function, then the chain rule is equivalent to the generalized power rule (xii).

## 5.4 Derivative of Trigonometric Functions

### A Review of Trigonometric Functions

$$\sin(x): \mathbb{R} \rightarrow [-1, 1]$$

$$\cos(x): \mathbb{R} \rightarrow [-1, 1]$$

$$\tan(x): \left\{ \mathbb{R} \setminus \left\{ \frac{\pi}{2} + n\pi \mid n \in \mathbb{Z} \right\} \right\} \rightarrow \mathbb{R}$$

$$\sin(x + 2\pi) = \sin(x)$$

$$\cos(x + 2\pi) = \cos(x)$$

$$\sin^2(x) + \cos^2(x) = 1$$

$$\sin(a + b) = \sin(a) \cos(b) + \cos(a) \sin(b)$$

$$\cos(a + b) = \cos(a) \cos(b) - \sin(a) \sin(b)$$

$$\sin\left(x + \frac{\pi}{2}\right) = \cos(x)$$

$$\sin(2x) = 2 \sin(x) \cos(x)$$

$$\cos(2x) = \cos^2(x) - \sin^2(x)$$

$$\tan(x) = \frac{\sin(x)}{\cos(x)}$$

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1 \tag{xiv}$$

$$\lim_{x \rightarrow 0} \frac{\cos x - 1}{x} = 0 \tag{xv}$$

## B Derivative of $\sin(x)$

$$(\sin x)' = \cos x$$

$$\frac{d}{dx} \sin x = \cos x$$

*Proof.*

$$\begin{aligned}(\sin x)' &= \lim_{h \rightarrow 0} \frac{\sin(x+h) - \sin(x)}{h} \\(\sin x)' &= \lim_{h \rightarrow 0} \frac{\sin(x) \cos(h) + \cos(x) \sin(h) - \sin(x)}{h} \\(\sin x)' &= \lim_{h \rightarrow 0} \frac{\sin(x) \cos(h) - \sin(x)}{h} + \lim_{h \rightarrow 0} \frac{\cos(x) \sin(h)}{h} \\(\sin x)' &= \sin(x) \lim_{h \rightarrow 0} \frac{\cos(h) - 1}{h} + \cos(x) \lim_{h \rightarrow 0} \frac{\sin(h)}{h}\end{aligned}$$

Now, using the limits (xiv) and (xv):

$$\begin{aligned}(\sin x)' &= \sin(x) \times 0 + \cos(x) \times 1 \\(\sin x)' &= \cos(x)\end{aligned}$$

□

## C Derivative of $\sin(f(x))$

By using the chain rule:

$$\left( \sin(f(x)) \right)' = \left( \cos(f(x)) \right) f'(x)$$

## D Derivative of $\cos x$

$$(\cos x)' = -\sin x$$

## E Derivative of $\cos(f(x))$

By using the chain rule:

$$\left( \cos(f(x)) \right)' = - \left( \sin(f(x)) \right) f'(x)$$

## **F   Derivative of $\tan x$**

$$(\tan x)' = \frac{1}{\cos^2 x} = \sec^2 x$$

## 5.1 Derivative of Exponential Function

### A Review of Exponential Functions

The exponential function is defined as:

$$y = f(x) = b^x \mid b > 0 \wedge b \neq 1$$

The  $x$ -axis ( $y = 0$ ) is a horizontal asymptote.

### B Number $e$

The number  $e$  is defined by:

$$e = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n$$

which can be written also as:

$$e = \lim_{u \rightarrow 0} (1 + u)^{\frac{1}{u}}$$

### C Derivative of $e^x$

$$(e^x)' = e^x$$

The proof of this is based on the fact that:

$$\lim_{h \rightarrow 0} \frac{e^h - 1}{h} = 1$$

### D Derivative of $e^{f(x)}$

By using the chain rule:

$$\left(e^{f(x)}\right)' = e^{f(x)} f'(x)$$

### E Derivative of $b^x \mid b > 0 \wedge b \neq 1$

$$(b^x)' = (\ln b)b^x$$

*Proof.*

$$(b^x)' = \left(e^{x \ln b}\right)' = e^{x \ln b} (\ln b) = (\ln b)b^x$$

□

## **F   Derivative of $b^{f(x)}$**

By using the chain rule:

$$\left(b^{f(x)}\right)' = (\ln b)b^{f(x)}f'(x)$$



## 5.1 Derivative of Logarithmic Function

### A Review of Logarithmic Function

$$y = b^x \equiv x = \log_b y$$

$$y = f(x) = \log_b x \mid b > 0 \wedge b \neq 1 \wedge x > 0$$

$$\log_b(xy) = \log_b(x) + \log_b(y)$$

$$\log_b\left(\frac{x}{y}\right) = \log_b(x) - \log_b(y)$$

$$\log_b x^n = n \log_b x$$

$$\log_b x = \frac{\log_a x}{\log_a b}$$

$$\log_b 1 = 0$$

$$\log_b b = 1$$

### B Derivative of $\ln x$

$$(\ln x)' = \frac{1}{x}$$

*Proof.*

$$y = \ln x \implies x = e^y \implies x' = (e^y)'$$

$$x' = (e^y)' \implies 1 = e^y y' \implies y' = \frac{1}{e^y} \implies y' = \frac{1}{x}$$

$$\therefore (\ln x)' = \frac{1}{x}$$

□

### C Derivative of $\ln(f(x))$

By using the chain rule:

$$(\ln f(x))' = \frac{f'(x)}{f(x)}$$

## D Derivative of $\log_b x$

$$(\log_b x)' = \frac{1}{(\ln b)x}$$

*Proof.*

$$(\log_b x)' = \left( \frac{\ln x}{\ln b} \right)' = \frac{1}{\ln b} (\ln x)' = \frac{1}{(\ln b)x}$$

□

## E Derivative of $\log_b f(x)$

By using the chain rule:

$$\left( \log_b (f(x)) \right)' = \frac{f'(x)}{(\ln b)f(x)}$$

# Logarithmic Differentiation (AP)

## A Logarithmic Differentiation

If the function formula contains many factors, then logarithmic differentiation is a fast method to differentiate.

Use the following algorithm:

1. Take natural logarithms of both sides of  $y = f(x)$ .
2. Differentiate with respect to  $x$ .
3. Isolate  $y' = \frac{dy}{dx}$ .

## B Function Raise to a Function

To differentiate a function  $f(x)$  raised to another function  $g(x)$ , use the formula:

$$\left(f(x)^{g(x)}\right)' = g(x)f(x)^{g(x)-1}f'(x) + \ln(f(x))f(x)^{g(x)}g'(x)$$

**Notes:**

1. The first part  $g(x)f(x)^{g(x)-1}$  comes from using the power rule and chain rule and by considering  $g(x)$  constant.
2. The second part  $\ln(f(x))f(x)^{g(x)}g'(x)$  comes from using the exponential rule and chain rule and by considering  $f(x)$  constant.

# Inverse Trigonometric Functions and Their Derivatives

## A Inverse Sine Function

The inverse of the sine function:

$$f(x) = \sin x: \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \rightarrow [-1, 1]$$

is:

$$f'(x) = \arcsin x = \sin^{-1} x: [-1, 1] \rightarrow \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

## B Trigonometric Identities with Inverse Sine

$$\arcsin x = \theta \equiv \sin \theta = x$$

## C Inverse Cosine Function

The inverse of the cosine function:

$$f(x) = \cos x: [0, \pi] \rightarrow [-1, 1]$$

is:

$$f'(x) = \arccos x = \cos^{-1} x: [-1, 1] \rightarrow [0, \pi]$$

## D Trigonometric Identities with Inverse Cosine

$$\arccos x = \theta \equiv \cos \theta = x$$

## E Inverse Tangent Function

The inverse of the tangent function:

$$f(x) = \tan x: \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \rightarrow [-\infty, \infty]$$

is:

$$f'(x) = \arctan x = \tan^{-1} x: [-\infty, \infty] \rightarrow \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

## F Trigonometric Identities with Inverse Tangent

$$\arctan x = \theta \equiv \tan \theta = x$$

## G Derivative of the Inverse Function

If  $f^{-1}$  is the inverse function of the function  $f$  then:

$$y = f^{-1}(x) \equiv x = f(y)$$

If derivative rule of a function is known, then the derivative of the inverse of that function may be found using:

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$$

## H Derivative of Inverse Trigonometric Functions

Differentiation rules for the inverse trigonometric functions are:

$$\begin{aligned}\frac{d}{dx} \arcsin x &= \frac{1}{\sqrt{1-x^2}} \\ \frac{d}{dx} \arccos x &= -\frac{1}{\sqrt{1-x^2}} \\ \frac{d}{dx} \arctan x &= \frac{1}{1+x^2}\end{aligned}$$

## I Reciprocal of Trigonometric Functions

Reciprocal of trigonometric functions are defined by:

$$\begin{aligned}\sec x &= \frac{1}{\cos x} \\ \csc x &= \frac{1}{\sin x} \\ \cot x &= \frac{1}{\tan x}\end{aligned}$$

Their inverses may be computed by using the following formulas:

$$\begin{aligned}\operatorname{arcsec} x &= \arccos \frac{1}{x} \\ \operatorname{arccsc} x &= \arcsin \frac{1}{x} \\ \operatorname{arccot} x &= \arctan \frac{1}{x}\end{aligned}$$

## Unit 3

# Applications of Differentiation

# Implicit Differentiation (AP)

## A Relations Defined Implicitly

A relation between two variables  $x$  and  $y$  is defined implicitly by an equation like:

$$f(x, y) = 0$$

**Notes:**

1. One variable may be considered dependant on the other variable or both may be considered dependant on the third one like  $t$ .
2. The equation may be solved with respect to the variables  $x$  or  $y$  or may not be solved.
3. The graph of the relation may or may not pass the vertical or horizontal line tests.

## B Terminology

Let  $(x, y)$  and  $(x + \Delta x, y + \Delta y)$  be two points satisfying  $f(x, y) = 0$ . Then:

$$\frac{\Delta y}{\Delta x} = \frac{1}{\frac{\Delta x}{\Delta y}}$$

And as  $\Delta x \rightarrow 0$ ,  $\Delta y \rightarrow 0$ :

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$$

**Notes:**

- $\frac{dy}{dx}$  means differentiation of the variable  $y$  with respect to the variable  $x$ .
- $\frac{dx}{dy}$  means differentiation of the variables  $x$  with respect to the variable  $y$ .
- The tangent line is horizontal when  $\frac{dy}{dx} = 0$ .
- The tangent line is vertical when  $\frac{dx}{dy} = 0$ .

## C Differentiation Revised

Consider the expression  $E(x, y) = 2xy^2$ .

If  $x$  is considered independent:

$$\frac{d}{dx}E(x, y) = \frac{d}{dx}(2xy^2) = y^2 \frac{d}{dx}(2x) + (2x) \frac{d}{dx}y^2 = 2y^2 \frac{dx}{dx} + 4xy \frac{dy}{dx} = 2y^2 + 4xy \frac{dy}{dx}$$

If  $y$  is considered independent:

$$\frac{d}{dx}E(x, y) = \frac{d}{dy}(2xy^2) = y^2 \frac{d}{dy}(2x) + (2x) \frac{d}{dy}y^2 = 2y^2 \frac{dx}{dy} + 4xy \frac{dy}{dy} = 2y^2 \frac{dx}{dy} + 4xy$$

If  $t$  is considered independent:

$$\frac{d}{dx}E(x, y) = \frac{d}{dt}(2xy^2) = y^2 \frac{d}{dt}(2x) + (2x) \frac{d}{dt}y^2 = 2y^2 \frac{dx}{dt} + 4xy \frac{dy}{dt}$$

## D Implicit Differentiation

To differentiation with respect to the variable  $x$  in a relation given implicitly by  $f(x, y) = 0$ :

1. Apply the operator  $\frac{d}{dx}$  to both sides:

$$\frac{d}{dx}f(x, y) = \frac{d}{dx}0$$

2. Use the chain rule and differentiate by keeping in mind that  $\frac{dx}{dx} = 1$ .
3. Solve for  $\frac{dy}{dx} = IRC = m_T$  or  $\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}$ .
4. Substitute  $x$  and  $y$  with given values (if necessary).

**Note: The following differentiations are also possible:**

$$\frac{d}{dy}f(x, y) = \frac{d}{dy}0$$

$$\frac{d}{dt}f(x, y) = \frac{d}{dt}0$$



## 3.9 Related Rates

### A Algorithm to Solve Related Rates Applications

1. Assign variables  $x, y, z, \dots$  to quantities involved in application.
2. Discover relations (constraints) between these quantities and write down their restrictions. A diagram or geometry formulas may help.
3. Use these relations to eliminate variables which are not essential to application. At this step, related variables are part of an explicit equation:

$$x = f(y, z, \dots) \quad (\text{xvi})$$

or are part of an implicit equation:

$$f(y, z, \dots) = 0 \quad (\text{xvii})$$

4. Identify the independent quantity and assign a variable to it (usually this is the time  $t$ ).
5. Use the chain rule to differentiate with respect to the independent variable  $t$  the equation (xvi) or (xvii):

$$\frac{dx}{dt} = \frac{d}{dt} f(y, z, \dots) \quad \text{or} \quad \frac{d}{dt} f(y, z, \dots) = 0 \quad (\text{xviii})$$

6. Substitute all given data or other data obtained from (xvi) or (xvii) equations.
7. Solve for the remaining unknown rate of change.

**Note.**  $\frac{dx}{dy}, \frac{dy}{dt}, \dots$  are instantaneous rates of change are they are related by (xviii).

## 3.10 Linear Approximation and Differentials

### A Linear Approximation

The definition of derivative function at  $x = a$ :

$$f'(a) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$$

may be written:

$$f(x) \approx f(a) + f'(a)(x - a) , \quad x \rightarrow a$$

which is called the *linear or tangent line approximation* of the function  $y = f(x)$  near  $x = a$ . **Note. Linear approximation is only possible if  $f'(a)$  exists.**

### B Approximate Formulas

The definition of derivative function at  $x = a$ :

$$f'(a) = \lim_{x \rightarrow 0} \frac{f(a + x) - f(a)}{x}$$

written in the form:

$$f(a + x) \approx f(a) + f'(a)x , \quad x \rightarrow 0$$

permits generation of *approximate formulas*.

### C Numerical Approximation

Numerical approximation is based on the formula:

$$f(a + x) \approx f(a) + f'(a)x , \quad x \rightarrow 0$$

### D Differentials

Derivative function may be written in the form:

$$\frac{dy}{dx} = f'(x)$$

$dx$  and  $dy$  are called *differentials* and they are related by the formula:

$$dy = f'(x) dx$$

This formula is called the *differential form* of the function  $y = f(x)$ .

If  $dx, dy \rightarrow 0$  then the previous formula is exact.

If  $dx$  and  $dy$  are finite, we replace them by  $\Delta x$  and  $\Delta y$  and the previous formula becomes approximately:

$$\Delta y \approx f'(x) \Delta x$$

## E Error Propagation

If the variable  $x$  is measured with a finite error  $\Delta x$ , then the real value is  $x + \Delta x$ .

The *absolute error* in computing the value of the function  $y = f(x)$  is approximately given by:

$$\Delta y \approx f'(x)\Delta x$$

and its *relative error*  $\frac{\Delta y}{y}$  may be approximated by:

$$\frac{\Delta y}{y} = \frac{\Delta y}{f(x)} \approx \frac{f'(x)\Delta x}{f(x)}$$

## 3.2 Maximum and Minimum on an Interval: Extreme Values

### A Global Maximum

A function  $f$  has a *global (absolute) maximum* at  $x = c$  is  $f(x) \leq f(c)$  for all  $x \in D_f$ .

$f(c)$  is called the *global (absolute) maximum value*.

$(c, f(c))$  is called the *global (absolute) maximum point*.

**Note.** An *extremum* is either a minimum or maximum (value, point, local, or global).

### B Global Minimum

A function  $f$  has a *global (absolute) minimum* at  $x = c$  is  $f(x) \geq f(c)$  for all  $x \in D_f$ .

$f(c)$  is called the *global (absolute) minimum value*.

$(c, f(c))$  is called the *global (absolute) minimum point*.

**Notes:**

**Extrema** The plural of extremum.

**Minima** The plural of minimum.

**Maxima** The plural of maximum.

### C Global (Absolute) Extrema Algorithm

To find the global (absolute) extrema for a *continuous* function  $f$  over a close interval  $[a, b]$ :

1. Identify all *critical* numbers over  $(a, b)$ .
2. Find the *values* of the function  $f(c)$  at each critical number  $c$  in  $(a, b)$ .
3. Find the *values*  $f(a)$  and  $f(b)$ .
4. From the values obtained at part 2 and 3:
  - The *largest* represents the *global (absolute) maximum* value.
  - The *smallest* represents the *global (absolute) minimum* value.

**Note.** A *critical number*  $c$  is a number such that  $f'(c) = 0$  or  $f'(c)$  DNE.

## 4.1 Increasing and Decreasing Functions. Critical Points: Local Maxima and Minima

### A Increasing or Decreasing Functions

Let  $y = f(x)$  be a differentiable function over  $(a, b)$ . Then:

1.  $f$  is *increasing* over  $(a, b)$  if:
  - $ARC = m_S = \frac{f(x_2) - f(x_1)}{x_2 - x_1} > 0$  over any interval  $[x_1, x_2] \subseteq (a, b)$ .
  - $IRC = m_T = f'(x) > 0$  for all  $x \in (a, b)$ .
2.  $f$  is *decreasing* over  $(a, b)$  if:
  - $ARC = m_S = \frac{f(x_2) - f(x_1)}{x_2 - x_1} < 0$  over any interval  $[x_1, x_2] \subseteq (a, b)$ .
  - $IRC = m_T = f'(x) < 0$  for all  $x \in (a, b)$ .
3.  $f$  is *constant* over  $(a, b)$  if:
  - $ARC = m_S = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = 0$  over any interval  $[x_1, x_2] \subseteq (a, b)$ .
  - $IRC = m_T = f'(x) = 0$  for all  $x \in (a, b)$ .

### B Local Maximum

A function  $f$  has a *local (relative) maximum* at  $x = c$  if:

- $f(x) \leq f(c)$  when  $x$  is sufficiently close to  $c$  (from both sides).
- $f'(x)$  changes sign from positive to negative at  $c$ .

### C Local Minimum

A function  $f$  has a *local (relative) minimum* at  $x = c$  if:

- $f(x) \geq f(c)$  when  $x$  is sufficiently close to  $c$  (from both sides).
- $f'(x)$  changes sign from negative to positive at  $c$ .

### D Critical Numbers and Critical Points

The number  $c \in D_f$  is a *critical number* if:

$$f'(c) = 0 \quad \text{or} \quad f'(c) \text{ DNE}$$

The point  $P(c, f(c))$  is called a *critical point*.

**Notes:**

1. A local extremum happens always at a critical point (Fermat's theorem).
2. At a critical number a function may or may not have a local extremum.

## 4.2 The Mean Value Theorem (AP)

### A Rolle's Theorem

Let  $y = f(x)$  be a function continuous on  $[a, b]$  and differentiable on  $(a, b)$ .

If  $f(a) = f(b)$  then there is a number  $c \in (a, b)$  such that  $f'(c) = 0$ .

**Note.** Tangent line is horizontal at  $P(c, f(c))$ .

### B Mean Value Theorem

Let  $y = f(x)$  be a function continuous on  $[a, b]$  and differentiable on  $(a, b)$ . Then there is a number  $c \in (a, b)$  such that  $f'(c) = \frac{f(b)-f(a)}{b-a}$ .

**Note.** Slope of tangent line at  $P(c, f(c))$  is equal to slope of secant line.

### C Theorem

If  $f'(x) = 0$  for all  $x \in (a, b)$ , then  $f$  is constant on  $(a, b)$ .

### D Theorem

If  $f'(x) = g'(x)$  for all  $x \in (a, b)$ , then  $f - g$  is constant on  $(a, b)$  and  $f(x) = g(x) + c$  where  $c$  is a constant.

## 4.3 Asymptotes

### A Vertical Asymptote

If the value of  $f(x)$  can be made *arbitrarily large* by taking  $x$  *sufficiently close* to  $a$  with  $x < a$  then  $\lim_{x \rightarrow a^-} f(x) = \infty$ . The line  $x = a$  is called a *vertical asymptote* to the graph of  $y = f(x)$ .

**Notes:**

1. A function of the form  $f(x) = \frac{p(x)}{q(x)}$  has a vertical asymptote at  $x = a$  if  $p(a) \neq 0 \wedge q(a) = 0$ .
2. A function of the form  $f(x) = p(x) \log_b q(x)$  has a vertical asymptote  $x = a$  if  $p(a) \neq 0 \wedge q(a) = 0$ .

### B Horizontal Asymptote

A horizontal line  $y = b$  is called a horizontal asymptote to the graph of  $y = f(x)$  if  $\lim_{x \rightarrow \pm\infty} f(x) = b$ .

**Notes:**

1. A horizontal asymptote may be crossed or touched by the graph of the function.
2. The graph of a function may have at most two horizontal asymptotes (one as  $x \rightarrow -\infty$  and one as  $x \rightarrow +\infty$ ).

### C Limits at Infinity

If  $a > 0$ , then:

$$\lim_{x \rightarrow \pm\infty} x^a = (\pm\infty)^a$$

$$\lim_{x \rightarrow \pm\infty} \frac{1}{x^a} = \frac{1}{(\pm\infty)^a} = 0$$

$$\lim_{x \rightarrow \pm\infty} (a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0) = \lim_{x \rightarrow \pm\infty} a_n x^n$$

$$\lim_{x \rightarrow \pm\infty} \frac{a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0}{b_m x^m + b_{m-1} x^{m-1} + \cdots + b_2 x^2 + b_1 x + b_0} = \lim_{x \rightarrow \pm\infty} \frac{a_n x^n}{b_m x^m}$$



## D Horizontal Asymptotes for Rational Functions

A rational function of the form:

$$f(x) = \frac{P_n(x)}{Q_m(x)} = \frac{a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0}{b_m x^m + b_{m-1} x^{m-1} + \cdots + b_2 x^2 + b_1 x + b_0}$$

has:

- A horizontal asymptote  $y = 0$  if  $m > n$ .
- A horizontal asymptote  $y = \frac{a_n}{b_m}$  if  $m = n$ .
- No horizontal asymptote if  $m < n$ .

**Note.** A rational function may have at most one horizontal asymptote.

## E Oblique (Slant) Asymptote

The line  $y = ax + b$  is an oblique (slant) asymptote for the curve  $y = f(x)$  if:

$$\lim_{x \rightarrow \pm\infty} (f(x) - (ax + b)) = 0$$

**Notes:**

1. An oblique asymptote may be crossed or touched by the graph of the function.
2. The graph of a function may have at most two oblique asymptotes (one as  $x \rightarrow -\infty$  and one as  $x \rightarrow +\infty$ ).
3. The graph of a function may have one horizontal asymptote and one oblique asymptote (one as  $x \rightarrow -\infty$  and the other as  $x \rightarrow +\infty$ ).

## F Oblique Asymptotes for Rational Functions

A rational function of the form:

$$f(x) = \frac{P_n(x)}{Q_m(x)} = \frac{a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0}{b_m x^m + b_{m-1} x^{m-1} + \cdots + b_2 x^2 + b_1 x + b_0}$$

has an oblique (slant) asymptote if  $n = m + 1$ .

**Note.** To get the equation of the oblique (slant) asymptote, use the *long division algorithm* to write the rational function in the form:

$$f(x) = \frac{P_n(x)}{Q_m(x)} = ax + b + \frac{R(x)}{Q_m(x)}$$

where  $0 \leq \text{degree}(R) < \text{degree}(Q_m)$ .

## G Oblique Asymptotes for any Functions

The oblique asymptote  $y = ax + b$  may be obtained by computing the following two limits at infinity:

$$a = \lim_{x \rightarrow \pm\infty} \frac{f(x)}{x}$$

$$b = \lim_{x \rightarrow \pm\infty} (f(x) - ax)$$

**Note.** In order for the oblique asymptote to be defined, both limits above must exist (must be finite numbers).

## 4.4 Concavity and Points of Inflection

### A Concavity Upward

The graph of the function  $f$  has a *concavity upward* if:

- Graph lies above all its tangents.
- Tangents rotate counter-clockwise.
- $m_T = IRC = f'(x)$  increases
- $f''(x) > 0$

### B Concavity Downward

The graph of the function  $f$  has a *concavity downward* if:

- Graph lies below all its tangents.
- Tangents rotate clockwise.
- $m_T = IRC = f'(x)$  decreases
- $f''(x) < 0$

### C Test for Concavity

Let  $f$  be a function twice differentiable ( $f''(x)$  exists) over  $(a, b)$ .

1. If  $f''(x) > 0$  for all  $x \in (a, b)$ , then the function  $f$  (or its graph) is *concave upward* over  $(a, b)$ .
2. If  $f''(x) < 0$  for all  $x \in (a, b)$ , then the function  $f$  (or its graph) is *concave downward* over  $(a, b)$ .
3. If  $f''(x) = 0$  for all  $x \in (a, b)$ , then the function  $f$  (or its graph) has *no concavity* over  $(a, b)$ . In this case,  $f'(m) = m$ , and  $f(x) = mx + b$ . The function is linear and its graph is a straight line.

### D Point of Inflection

A point  $P(i, f(i))$  on the graph of  $y = f(x)$  is called a point of inflection if the concavity of the graph *changes* at  $P$  from concave upward to concave downward or from concave downward to concave upward.

## E Second Derivative Test

Let  $f$  be a twice differentiable function over an open interval containing the critical number  $c$  and  $f'(c) = 0$ .

1. If  $f''(c) > 0$  then  $f$  has a local minimum at  $x = c$ .
2. If  $f''(c) < 0$  then  $f$  has a local maximum at  $x = c$ .
3. If  $f''(c) = 0$  then  $f$  may have a local minimum, maximum, or neither (inconclusive case). Use the first derivative test to conclude.