MaCS Calculus and Vectors Exam Study Guide

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Unit 1

Equations of Lines and Planes

8.1 Vector and Parametric Equations of a Line in $\mathbb{R}^2$

8.1.A Vector Equation of a Line in $\mathbb{R}^2$

Consider the line L that passes through the point $P_0(x_0, y_0)$ and is parallel to the vector $\vec{u}$. The point $P(x, y)$ is a *generic point* on the line.

$$\overrightarrow{P_0P} = t\vec{u}$$
$$\overrightarrow{OP} - \overrightarrow{OP_0} = t\vec{u}$$
$$\vec{r} - \vec{r_0} = t\vec{u}$$

The *vector equation* of the line is:

$$\vec{r} = \vec{r_0} + t\vec{u} \mid t \in \mathbb{R}$$

Where:

- $\vec{r} = \overrightarrow{OP}$ is the *position vector* of a *generic point* P on the line.
- $\vec{r_0} = \overrightarrow{OP_0}$ is the *position vector* of a *specific point* P_0 on the line.
- $\vec{u}$ is a vector parallel to the line called the *direction vector* of the line.
- t is a *real number* corresponding to the generic point P.

Note: The vector equation of a line is *not unique*. It depends on the specific point P_0 and on the direction vector $\vec{u}$ that are used.

8.1.B Parametric Equations of a Line in $\mathbb{R}^2$

We can rewrite the vector equation of a line:

$$\vec{r} = \vec{r_0} + t\vec{u} \mid t \in \mathbb{R}$$

as:

$$(x, y) = (x_0, y_0) + t(u_x, u_y) \mid t \in \mathbb{R}$$

Split this vector equation into the *parametric equations* of a line in $\mathbb{R}^2$:

$$\begin{cases} x = x_0 + tu_x \\ y = y_0 + yu_y \end{cases} \quad t \in \mathbb{R}$$

8.1.C Parallel Lines

Two lines L_1 and L_2 with direction vectors $\vec{u_1}$ and $\vec{u_2}$ are *parallel* ($L_1 \parallel L_2$) if:

$$\vec{u_1} \parallel \vec{u_2}$$

or, there exists $k \in \mathbb{R}$ such that:

$$\vec{u_2} = k\vec{u_1}$$

or:

$$\vec{u_1} \times \vec{u_2} = \vec{0}$$

or scalar components are *proportional*:

$$\frac{u_{2x}}{u_{1x}} = \frac{u_{2u}}{u_{1u}} = k$$

8.1.D Perpendicular Lines

Two lines L_1 and L_2 with direction vectors $\vec{u_1}$ and $\vec{u_2}$ are *perpendicular* ($L_1 \perp L_2$) if:

$$\vec{u_1} \perp \vec{u_2}$$

or:

$$\vec{u_1} \cdot \vec{u_2} = 0$$

or:

$$u_{1x}u_{2x} + u_{1y}u_{2y} = 0$$

8.1.E 2D Perpendicular Vectors

Given a 2D vector $\vec{u} = (a, b)$, two 2D vectors perpendicular to $\vec{u}$ are $\vec{v} = (-b, a)$ and $\vec{w} = (b, -a)$.

Indeed:

$$\vec{u} \cdot \vec{v} = (a, b) \cdot (-b, a) = -ab + ab = 0 \implies \vec{u} \perp \vec{v}$$

8.1.F Special Lines

A line *parallel* to the x-axis has a direction vector in the form $\vec{u} = (u_x, 0) \mid u_x \neq 0$.

A line *parallel* to the y-axis has a direction vector in the form $\vec{u} = (0, u_y) \mid u_y \neq 0$.

8.2 Cartesian Equation of a Line

8.2.A Symmetric Equation

The parametric equations of a line in $\mathbb{R}^2$:

$$\begin{cases} x = x_0 + tu_x \\ y = y_0 + tu_y \end{cases} \quad t \in \mathbb{R}$$

may be written as:

$$\frac{x - x_0}{u_x} = \frac{y - y_0}{u_y} = t \mid t \in \mathbb{R}$$

The *symmetric equation* of the line is (if it exists):

$$\frac{x - x_0}{u_x} = \frac{y - y_0}{u_y}$$

Note: The symmetric equations only exists if $u_x \neq 0$ and $u_y \neq 0$.

8.2.B Normal Equation

Consider a line L that passes through the specific point $P_0(x_0, y_0)$ and has the *direction vector* $\vec{u} = (u_x, u_y)$.

The vectors $\vec{n} = (-u_y, u_x) = (A, B)$ or $\vec{n} = (u_y, -u_x) = (A, B)$ are perpendicular to the vector $\vec{u}$ and so they are perpendicular to the line L. These are called *normal vectors* to the line L.

Let $P(x, y)$ be a generic point on the line L. So:

$$\overrightarrow{P_0P} \parallel \vec{u} \implies \overrightarrow{P_0P} \perp \vec{n} \implies \overrightarrow{P_0P} \cdot \vec{n} = 0$$

$$(\vec{r} - \vec{r_0}) \cdot \vec{n} = 0$$

The *normal equation* of a line is given by:

$$(\vec{r} - \vec{r_0}) \cdot \vec{n} = 0$$

8.2.C Cartesian Equation

The normal equations can be written as:

$$\begin{aligned} \vec{r} \cdot \vec{n} - \vec{r_0} \cdot \vec{n} &= 0 \\ (x, y) \cdot (A, B) - (x_0, y_0) \cdot (A, B) &= 0 \\ Ax + By - Ax_0 - By_0 &= 0 \\ Ax + By + C &= 0 \qquad \text{where } C = -Ax_0 - By_0 \end{aligned}$$

The *Cartesian equation* of a line is given by:

$$Ax + By + C = 0$$

where $\vec{n} = (A, B)$ is a *normal vector* and the constant C depends on a specific point of the line.

8.2.D Slope y-intercept Equation

Solve the symmetric equation of a line:

$$\frac{x - x_0}{u_x} = \frac{y - y_0}{u_y} \mid t \in \mathbb{R}$$

for y:

$$\begin{aligned} y - y_0 &= u_y \frac{x - x_0}{u_x} \\ y &= \frac{u_y}{u_x} x + y_0 - \frac{u_y}{u_x} x_0 \end{aligned}$$

The *slope y-intercept equation* of a line in $\mathbb{R}^2$ is given by:

$$y = mx + b$$

$$m = \frac{u_y}{u_x}$$

where m is the *slope* and b is the *y-intercept* which depends on a specific point of the line.

8.2.E Angle between Two Lines

The *angle* between two lines is determined by the angle between the *direction vectors*:

$$\cos\theta = \frac{\vec{u_1} \cdot \vec{u_2}}{\|\vec{u_1}\| \|\vec{u_2}\|}$$

Note: There are two pairs of equal angles between the two lines. There is a pair of the angle θ_1, and a pair of the angle θ_2. $\theta_1 + \theta_2 = 180^\circ$

8.3 Vector, Parametric, and Symmetric Equations of a Line in $\mathbb{R}^3$

8.3.A Vector Equation

The vector equation of the line is:

$$\vec{r} = \vec{r_0} + t\vec{u} \mid t \in \mathbb{R}$$

where:

- $\vec{r} = \overrightarrow{OP}$ is the position vector of a *generic* point P on the line.
- $\vec{r_0} = \overrightarrow{OP_0}$ is the position vector of a *specific* point P_0 on the line.
- $\vec{u}$ is a vector parallel to the line called the *direction vector* of the line.
- t is a *real number* corresponding to the generic point P.

8.3.B Specific Lines

A line is parallel to the x-axis if $\vec{u} = (u_x, 0, 0) \mid u_x \neq 0$. In this case, the line is also *perpendicular to the yz-plane.*

A line with $\vec{u} = (0, u_y, u_z) \mid u_y \neq 0 \wedge u_z \neq 0$ is *parallel to the yz-plane.*

8.3.C Parametric Equations

Rewrite the vector equation of a line:

$$\vec{r} = \vec{r_0} + t\vec{u} \mid t \in \mathbb{R}$$

as:

$$(x, y, z) = (x_0, y_0, z_0) + t(u_x, u_y, u_z) \mid t \in \mathbb{R}$$

The *parametric equations* of a line in $\mathbb{R}^3$ are:

$$\begin{cases} x = x_0 + tu_x \\ y = x_0 + tu_y \\ z = x_0 + tu_z \end{cases} \quad t \in \mathbb{R}$$

8.3.D Symmetric Equations

The parametric equations of a line may be written as:

$$\begin{cases} x = x_0 + tu_x \\ y = x_0 + tu_y \\ z = x_0 + tu_z \end{cases} \quad t \in \mathbb{R}$$

From here, the *symmetric equations* of the line are:

$$\frac{x - x_0}{u_x} = \frac{y - y_0}{u_y} = \frac{z - z_0}{u_z}$$

$$u_x \neq 0 \quad u_y \neq 0 \quad u_z \neq 0$$

8.3.E Intersections

A line *intersects the x-axis* when $y = z = 0$.

A line *intersects the xy-plane* when $z = 0$.

8.4 Vector and Parametric Equations of a Plane

8.4.A Planes

A plane may be determined by points and lines. There are four main possibilities:

1. Plane determined by three points.
2. Plane determined by two parallel lines.
3. Plane determined by two intersecting lines.
4. Plane determined by a point and a line.

8.4.B Vector Equation of a Plane

Consider a plane π.

Two vectors $\vec{u}$ and $\vec{v}$, parallel to the plane π but not parallel to each other, are called *direction vectors* of the plane π.

The vector $\overrightarrow{P_0P}$ from a specific point $P_0(x_0, y_0, z_0)$ to a generic point $P(x, y, z)$ of the plane is a *linear combination* of direction vectors $\vec{u}$ and $\vec{v}$:

$$\overrightarrow{P_0P} - s\vec{u} + t\vec{v} \mid s, t \in \mathbb{R}$$

The *vector equation* of the plane is:

$$\pi : \vec{r} = \vec{r_0} + s\vec{u} + t\vec{v} \mid s, t \in \mathbb{R}$$

8.4.C Parametric Equations of a Plane

We write the vector equation of the plane as:

$$(x, y, z) = (x_0, y_0, z_0) + s(u_x, u_y, u_z) + t(v_x, v_y, v_z)$$

or:

$$\begin{cases} x = x_0 + su_x + tv_x \\ y = y_0 + su_y + tv_y \\ z = z_0 + su_z + tv_z \end{cases} \quad s, t \in \mathbb{R}$$

These are the *parametric equations* of a plane.

8.5 Cartesian Equation of a Plane

8.5.A Normal Equation of a Plane

A plane may be determined by a *point* $P_0(x_0, y_0, z_0)$ and a *vector* perpendicular to the plane $\vec{n}$ called the *normal vector*.

If $P(x, y, z)$ is a generic point on the plane, then:

$$\overrightarrow{P_0P} \perp \vec{n}$$

and:

$$\overrightarrow{P_0P} \cdot \vec{n} = 0$$

This is the *normal equation* of a plane.

8.5.B Cartesian Equation of a Plane

We write the normal vector of a plane in the form:

$$\vec{n} = (A, B, C)$$

Then, the normal equation may be written as:

$$\begin{aligned}(x - x_0, y - y_0, z - z_0) \cdot (A, B, C) &= 0 \\ Ax + By + Cz - Ax_0 - By_0 - Cz_0 &= 0\end{aligned}$$

or:

$$Ax + By + Cz + D = 0$$

which is called the *Cartesian equation* of a plane.

Note: A normal vector to the plane is:

$$\vec{n} = \vec{u} \times \vec{v}$$

where $\vec{u}$ and $\vec{v}$ are the direction vectors of the plane.

8.5.C Angle between Two Planes

The *angle* between two planes is defined as the angle between their *normal vectors*:

$$\cos\theta = \frac{\vec{n_1} \cdot \vec{n_2}}{\|\vec{n_1}\|\|\vec{n_2}\|}$$

Note: Using this formula, you may get an *acute* or an *obtuse* angle depending on the normal vectors which are used.

Appendix A

Credit

Thank you for everything, Mrs. Gugoiu.