

MaCS Calculus and Vectors Exam Study Guide

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Unit 1

Equations of Lines and Planes

8.1 Vector and Parametric Equations of a Line in \mathbb{R}^2

8.1.A Vector Equation of a Line in \mathbb{R}^2

Consider the line L that passes through the point $P_0(x_0, y_0)$ and is parallel to the vector \vec{u} . The point $P(x, y)$ is a *generic point* on the line.

$$\begin{aligned}\overrightarrow{P_0P} &= t\vec{u} \\ \overrightarrow{OP} - \overrightarrow{OP_0} &= t\vec{u} \\ \vec{r} - \vec{r_0} &= t\vec{u}\end{aligned}$$

The *vector equation* of the line is:

$$\vec{r} = \vec{r_0} + t\vec{u} \mid t \in \mathbb{R}$$

Where:

- $\vec{r} = \overrightarrow{OP}$ is the *position vector* of a *generic point* P on the line.
- $\vec{r_0} = \overrightarrow{OP_0}$ is the *position vector* of a *specific point* P_0 on the line.
- \vec{u} is a vector parallel to the line called the *direction vector* of the line.
- t is a *real number* corresponding to the generic point P .

Note: The vector equation of a line is *not unique*. It depends on the specific point P_0 and on the direction vector \vec{u} that are used.

8.1.B Parametric Equations of a Line in \mathbb{R}^2

We can rewrite the vector equation of a line:

$$\vec{r} = \vec{r_0} + t\vec{u} \mid t \in \mathbb{R}$$

as:

$$(x, y) = (x_0, y_0) + t(u_x, u_y) \mid t \in \mathbb{R}$$

Split this vector equation into the *parametric equations* of a line in \mathbb{R}^2 :

$$\begin{cases} x = x_0 + tu_x \\ y = y_0 + tu_y \end{cases} \quad t \in \mathbb{R}$$

8.1.C Parallel Lines

Two lines L_1 and L_2 with direction vectors \vec{u}_1 and \vec{u}_2 are *parallel* ($L_1 \parallel L_2$) if:

$$\vec{u}_1 \parallel \vec{u}_2$$

or, there exists $k \in \mathbb{R}$ such that:

$$\vec{u}_2 = k\vec{u}_1$$

or:

$$\vec{u}_1 \times \vec{u}_2 = \vec{0}$$

or scalar components are *proportional*:

$$\frac{u_{2x}}{u_{1x}} = \frac{u_{2y}}{u_{1y}} = k$$

8.1.D Perpendicular Lines

Two lines L_1 and L_2 with direction vectors \vec{u}_1 and \vec{u}_2 are *perpendicular* ($L_1 \perp L_2$) if:

$$\vec{u}_1 \perp \vec{u}_2$$

or:

$$\vec{u}_1 \cdot \vec{u}_2 = 0$$

or:

$$u_{1x}u_{2x} + u_{1y}u_{2y} = 0$$

8.1.E 2D Perpendicular Vectors

Given a 2D vector $\vec{u} = (a, b)$, two 2D vectors perpendicular to \vec{u} are $\vec{v} = (-b, a)$ and $\vec{w} = (b, -a)$.

Indeed:

$$\vec{u} \cdot \vec{v} = (a, b) \cdot (-b, a) = -ab + ab = 0 \implies \vec{u} \perp \vec{v}$$

8.1.F Special Lines

A line *parallel* to the x -axis has a direction vector in the form $\vec{u} = (u_x, 0) \mid u_x \neq 0$.

A line *parallel* to the y -axis has a direction vector in the form $\vec{u} = (0, u_y) \mid u_y \neq 0$.

8.2 Cartesian Equation of a Line

8.2.A Symmetric Equation

The parametric equations of a line in \mathbb{R}^2 :

$$\begin{cases} x = x_0 + tu_x \\ y = y_0 + tu_y \end{cases} \quad t \in \mathbb{R}$$

may be written as:

$$\frac{x - x_0}{u_x} = \frac{y - y_0}{u_y} = t \mid t \in \mathbb{R}$$

The *symmetric equation* of the line is (if it exists):

$$\frac{x - x_0}{u_x} = \frac{y - y_0}{u_y}$$

Note: The symmetric equations only exists if $u_x \neq 0$ and $u_y \neq 0$.

8.2.B Normal Equation

Consider a line L that passes through the specific point $P_0(x_0, y_0)$ and has the *direction vector* $\vec{u} = (u_x, u_y)$.

The vectors $\vec{n} = (-u_y, u_x) = (A, B)$ or $\vec{n} = (u_y, -u_x) = (A, B)$ are perpendicular to the vector \vec{u} and so they are perpendicular to the line L . These are called *normal vectors* to the line L .

Let $P(x, y)$ be a generic point on the line L . So:

$$\begin{aligned} \overrightarrow{P_0P} \parallel \vec{u} &\implies \overrightarrow{P_0P} \perp \vec{n} \implies \overrightarrow{P_0P} \cdot \vec{n} = 0 \\ (\vec{r} - \vec{r}_0) \cdot \vec{n} &= 0 \end{aligned}$$

The *normal equation* of a line is given by:

$$(\vec{r} - \vec{r}_0) \cdot \vec{n} = 0$$

8.2.C Cartesian Equation

The normal equations can be written as:

$$\begin{aligned} \vec{r} \cdot \vec{n} - \vec{r}_0 \cdot \vec{n} &= 0 \\ (x, y) \cdot (A, B) - (x_0, y_0) \cdot (A, B) &= 0 \\ Ax + By - Ax_0 - By_0 &= 0 \\ Ax + By + C &= 0 \quad \text{where } C = -Ax_0 - By_0 \end{aligned}$$

The *Cartesian equation* of a line is given by:

$$Ax + By + C = 0$$

where $\vec{n} = (A, B)$ is a *normal vector* and the constant C depends on a specific point of the line.

8.2.D Slope y -intercept Equation

Solve the symmetric equation of a line:

$$\frac{x - x_0}{u_x} = \frac{y - y_0}{u_y} \mid t \in \mathbb{R}$$

for y :

$$\begin{aligned} y - y_0 &= u_y \frac{x - x_0}{u_x} \\ y &= \frac{u_y}{u_x} x + y_0 - \frac{u_y}{u_x} x_0 \end{aligned}$$

The *slope y -intercept equation* of a line in \mathbb{R}^2 is given by:

$$y = mx + b$$

$$m = \frac{u_y}{u_x}$$

where m is the *slope* and b is the *y -intercept* which depends on a specific point of the line.

8.2.E Angle Between two Lines

The *angle* between two lines is determined by the angle between the *direction vectors*:

$$\cos \theta = \frac{\vec{u}_1 \cdot \vec{u}_2}{\|\vec{u}_1\| \|\vec{u}_2\|}$$

Note: There are two pairs of equal angles between the two lines. There is a pair of the angle θ_1 , and a pair of the angle θ_2 . $\theta_1 + \theta_2 = 180^\circ$

8.3 Vector, Parametric, and Symmetric Equations of a Line in \mathbb{R}^3

8.3.A Vector Equation

The vector equation of the line is:

$$\vec{r} = \vec{r}_0 + t\vec{u} \mid t \in \mathbb{R}$$

where:

- $\vec{r} = \overrightarrow{OP}$ is the position vector of a *generic* point P on the line.
- $\vec{r}_0 = \overrightarrow{OP_0}$ is the position vector of a *specific* point P_0 on the line.
- \vec{u} is a vector parallel to the line called the *direction vector* of the line.
- t is a *real number* corresponding to the generic point P .

8.3.B Specific Lines

A line is parallel to the x -axis if $\vec{u} = (u_x, 0, 0) \mid u_x \neq 0$. In this case, the line is also *perpendicular to the yz -plane*.

A line with $\vec{u} = (0, u_y, u_z) \mid u_y \neq 0 \wedge u_z \neq 0$ is *parallel to the yz -plane*.

8.3.C Parametric Equations

Rewrite the vector equation of a line:

$$\vec{r} = \vec{r}_0 + t\vec{u} \mid t \in \mathbb{R}$$

as:

$$(x, y, z) = (x_0, y_0, z_0) + t(u_x, u_y, u_z) \mid t \in \mathbb{R}$$

The *parametric equations* of a line in \mathbb{R}^3 are:

$$\begin{cases} x = x_0 + tu_x \\ y = x_0 + tu_y \\ z = x_0 + tu_z \end{cases} \quad t \in \mathbb{R}$$

8.3.D Symmetric Equations

The parametric equations of a line may be written as:

$$\begin{cases} x = x_0 + tu_x \\ y = x_0 + tu_y \\ z = x_0 + tu_z \end{cases} \quad t \in \mathbb{R}$$

From here, the *symmetric equations* of the line are:

$$\frac{x - x_0}{u_x} = \frac{y - y_0}{u_y} = \frac{z - z_0}{u_z}$$
$$u_x \neq 0 \quad u_y \neq 0 \quad u_z \neq 0$$

8.3.E Intersections

A line *intersects the x -axis* when $y = z = 0$.

A line *intersects the xy -plane* when $z = 0$.

Appendix A

Credit

Thank you for everything, Mrs. Gugoiu.