MaCS Calculus and Vectors Exam Study Guide

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Contents

Unit 1

Equations of Lines and Planes

8.1 Vector and Parametric Equations of a Line in $\mathbb{R}^2$

8.1 A Vector Equation of a Line in $\mathbb{R}^2$

Consider the line L that passes through the point $P_0(x_0, y_0)$ and is parallel to the vector $\vec{u}$. The point $P(x, y)$ is a *generic point* on the line.

$$\overrightarrow{P_0P} = t\vec{u}$$
$$\overrightarrow{OP} - \overrightarrow{OP_0} = t\vec{u}$$
$$\vec{r} - \vec{r_0} = t\vec{u}$$

The *vector equation* of the line is:

$$\vec{r} = \vec{r_0} + t\vec{u} \mid t \in \mathbb{R}$$

Where:

- $\vec{r} = \overrightarrow{OP}$ is the *position vector* of a *generic point* P on the line.

- $\vec{r_0} = \overrightarrow{OP_0}$ is the *position vector* of a *specific point* P_0 on the line.

- $\vec{u}$ is a vector parallel to the line called the *direction vector* of the line.

- t is a *real number* corresponding to the generic point P.

Note: The vector equation of a line is *not unique*. It depends on the specific point P_0 and on the direction vector $\vec{u}$ that are used.

8.1 B Parametric Equations of a Line in $\mathbb{R}^2$

We can rewrite the vector equation of a line:

$$\vec{r} = \vec{r_0} + t\vec{u} \mid t \in \mathbb{R}$$

as:

$$(x, y) = (x_0, y_0) + t(u_x, u_y) \mid t \in \mathbb{R}$$

Split this vector equation into the *parametric equations* of a line in $\mathbb{R}^2$:

$$\begin{cases} x = x_0 + tu_x \\ y = y_0 + yu_y \end{cases} \quad t \in \mathbb{R}$$

8.1 C   Parallel Lines

Two lines L_1 and L_2 with direction vectors $\vec{u_1}$ and $\vec{u_2}$ are *parallel* $(L_1 \parallel L_2)$ if:

$$\vec{u_1} \parallel \vec{u_2}$$

or, there exists $k \in \mathbb{R}$ such that:
$$\vec{u_2} = k\vec{u_1}$$

or:
$$\vec{u_1} \times \vec{u_2} = \vec{0}$$

or scalar components are *proportional*:

$$\frac{u_{2x}}{u_{1x}} = \frac{u_{2u}}{u_{1u}} = k$$

8.1 D   Perpendicular Lines

Two lines L_1 and L_2 with direction vectors $\vec{u_1}$ and $\vec{u_2}$ are *perpendicular* $(L_1 \perp L_2)$ if:

$$\vec{u_1} \perp \vec{u_2}$$

or:
$$\vec{u_1} \cdot \vec{u_2} = 0$$

or:
$$u_{1x}u_{2x} + u_{1y}u_{2y} = 0$$

8.1 E   2D Perpendicular Vectors

Given a 2D vector $\vec{u} = (a, b)$, two 2D vectors perpendicular to $\vec{u}$ are $\vec{v} = (-b, a)$ and $\vec{w} = (b, -a)$.

Indeed:
$$\vec{u} \cdot \vec{v} = (a, b) \cdot (-b, a) = -ab + ab = 0 \implies \vec{u} \perp \vec{v}$$

8.1 F   Special Lines

A line *parallel* to the x-axis has a direction vector in the form $\vec{u} = (u_x, 0) \mid u_x \neq 0$.

A line *parallel* to the y-axis has a direction vector in the form $\vec{u} = (0, u_y) \mid u_y \neq 0$.

8.2 Cartesian Equation of a Line

8.2 A   Symmetric Equation

The parametric equations of a line in $\mathbb{R}^2$:

$$\begin{cases} x = x_0 + tu_x \\ y = y_0 + tu_y \end{cases} \quad t \in \mathbb{R}$$

may be written as:

$$\frac{x - x_0}{u_x} = \frac{y - y_0}{u_y} = t \mid t \in \mathbb{R}$$

The *symmetric equation* of the line is (if it exists):

$$\frac{x - x_0}{u_x} = \frac{y - y_0}{u_y}$$

Note: The symmetric equations only exists if $u_x \neq 0$ and $u_y \neq 0$.

8.2 B   Normal Equation

Consider a line L that passes through the specific point $P_0(x_0, y_0)$ and has the *direction vector* $\vec{u} = (u_x, u_y)$.

The vectors $\vec{n} = (-u_y, u_x) = (A, B)$ or $\vec{n} = (u_y, -u_x) = (A, B)$ are perpendicular to the vector $\vec{u}$ and so they are perpendicular to the line L. These are called *normal vectors* to the line L.

Let $P(x, y)$ be a generic point on the line L. So:

$$\overrightarrow{P_0P} \parallel \vec{u} \implies \overrightarrow{P_0P} \perp \vec{n} \implies \overrightarrow{P_0P} \cdot \vec{n} = 0$$

$$(\vec{r} - \vec{r_0}) \cdot \vec{n} = 0$$

The *normal equation* of a line is given by:

$$(\vec{r} - \vec{r_0}) \cdot \vec{n} = 0$$

8.2 C   Cartesian Equation

The normal equations can be written as:

$$\vec{r} \cdot \vec{n} - \vec{r_0} \cdot \vec{n} = 0$$
$$(x, y) \cdot (A, B) - (x_0, y_0) \cdot (A, B) = 0$$
$$Ax + By - Ax_0 - By_0 = 0$$
$$Ax + By + C = 0 \qquad \text{where } C = -Ax_0 - By_0$$

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The *Cartesian equation* of a line is given by:

$$Ax + By + C = 0$$

where $\vec{n} = (A, B)$ is a *normal vector* and the constant C depends on a specific point of the line.

8.2 D   Slope y-intercept Equation

Solve the symmetric equation of a line:

$$\frac{x - x_0}{u_x} = \frac{y - y_0}{u_y} \mid t \in \mathbb{R}$$

for y:

$$y - y_0 = u_y \frac{x - x_0}{u_x}$$

$$y = \frac{u_y}{u_x} x + y_0 - \frac{u_y}{u_x} x_0$$

The *slope y-intercept equation* of a line in $\mathbb{R}^2$ is given by:

$$y = mx + b$$

$$m = \frac{u_y}{u_x}$$

where m is the *slope* and b is the *y-intercept* which depends on a specific point of the line.

8.2 E   Angle between Two Lines

The *angle* between two lines is determined by the angle between the *direction vectors*:

$$\cos \theta = \frac{\vec{u_1} \cdot \vec{u_2}}{\|\vec{u_1}\| \|\vec{u_2}\|}$$

Note: There are two pairs of equal angles between the two lines. There is a pair of the angle θ_1, and a pair of the angle θ_2. $\theta_1 + \theta_2 = 180°$

8.3 Vector, Parametric, and Symmetric Equations of a Line in $\mathbb{R}^3$

8.3 A Vector Equation

The vector equation of the line is:

$$\vec{r} = \vec{r_0} + t\vec{u} \mid t \in \mathbb{R}$$

where:

- $\vec{r} = \overrightarrow{OP}$ is the position vector of a *generic* point P on the line.
- $\vec{r_0} = \overrightarrow{OP_0}$ is the position vector of a *specific* point P_0 on the line.
- $\vec{u}$ is a vector parallel to the line called the *direction vector* of the line.
- t is a *real number* corresponding to the generic point P.

8.3 B Specific Lines

A line is parallel to the x-axis if $\vec{u} = (u_x, 0, 0) \mid u_x \neq 0$. In this case, the line is also *perpendicular to the yz-plane*.

A line with $\vec{u} = (0, u_y, u_z) \mid u_y \neq 0 \wedge u_z \neq 0$ is *parallel to the yz-plane*.

8.3 C Parametric Equations

Rewrite the vector equation of a line:

$$\vec{r} = \vec{r_0} + t\vec{u} \mid t \in \mathbb{R}$$

as:

$$(x, y, z) = (x_0, y_0, z_0) + t(u_x, u_y, u_z) \mid t \in \mathbb{R}$$

The *parametric equations* of a line in $\mathbb{R}^3$ are:

$$\begin{cases} x = x_0 + tu_x \\ y = x_0 + tu_y \\ z = x_0 + tu_z \end{cases} \quad t \in \mathbb{R}$$

8.3 D   Symmetric Equations

The parametric equations of a line may be written as:

$$\begin{cases} x = x_0 + tu_x \\ y = x_0 + tu_y \\ z = x_0 + tu_z \end{cases} \quad t \in \mathbb{R}$$

From here, the *symmetric equations* of the line are:

$$\frac{x - x_0}{u_x} = \frac{y - y_0}{u_y} = \frac{z - z_0}{u_z}$$

$$u_x \neq 0 \quad u_y \neq 0 \quad u_z \neq 0$$

8.3 E   Intersections

A line *intersects the x-axis* when $y = z = 0$.

A line *intersects the xy-plane* when $z = 0$.

8.4   Vector and Parametric Equations of a Plane

8.4 A   Planes

A plane may be determined by points and lines. There are four main possibilities:

1. Plane determined by three points.

2. Plane determined by two parallel lines.

3. Plane determined by two intersecting lines.

4. Plane determined by a point and a line.

8.4 B   Vector Equation of a Plane

Consider a plane π.

Two vectors $\vec{u}$ and $\vec{v}$, parallel to the plane π but not parallel to each other, are called *direction vectors* of the plane π.

The vector $\overrightarrow{P_0P}$ from a specific point $P_0(x_0, y_0, z_0)$ to a generic point $P(x, y, z)$ of the plane is a *linear combination* of direction vectors $\vec{u}$ and $\vec{v}$:

$$\overrightarrow{P_0P} - s\vec{u} + t\vec{v} \mid s, t \in \mathbb{R}$$

The *vector equation* of the plane is:

$$\pi : \vec{r} = \vec{r_0} + s\vec{u} + t\vec{v} \mid s, t \in \mathbb{R}$$

8.4 C   Parametric Equations of a Plane

We write the vector equation of the plane as:

$$(x, y, z) = (x_0, y_0, z_0) + s(u_x, u_y, u_z) + t(v_x, v_y, v_z)$$

or:

$$\begin{cases} x = x_0 + su_x + tv_x \\ y = y_0 + su_y + tv_y \\ z = z_0 + su_z + tv_z \end{cases} \quad s, t \in \mathbb{R}$$

These are the *parametric equations* of a plane.

8.5 Cartesian Equation of a Plane

8.5 A Normal Equation of a Plane

A plane may be determined by a *point* $P_0(x_0, y_0, z_0)$ and a *vector* perpendicular to the plane $\vec{n}$ called the *normal vector*.

If $P(x, y, z)$ is a generic point on the plane, then:

$$\overrightarrow{P_0P} \perp \vec{n}$$

and:

$$\overrightarrow{P_0P} \cdot \vec{n} = 0$$

This is the *normal equation* of a plane.

8.5 B Cartesian Equation of a Plane

We write the normal vector of a plane in the form:

$$\vec{n} = (A, B, C)$$

Then, the normal equation may be written as:

$$(x - x_0, y - y_0, z - z_0) \cdot (A, B, C) = 0$$
$$Ax + By + Cz - Ax_0 - By_0 - Cz_0 = 0$$

or:

$$Ax + By + Cz + D = 0$$

which is called the *Cartesian equation* of a plane.

Note: A normal vector to the plane is:

$$\vec{n} = \vec{u} \times \vec{v}$$

where $\vec{u}$ and $\vec{v}$ are the direction vectors of the plane.

8.5 C Angle between Two Planes

The *angle* between two planes is defined as the angle between their *normal vectors*:

$$\cos\theta = \frac{\vec{n_1} \cdot \vec{n_2}}{\|\vec{n_1}\|\|\vec{n_2}\|}$$

Note: Using this formula, you may get an *acute* or an *obtuse* angle depending on the normal vectors which are used.

Unit 2

Relationships between Points, Lines, and Planes

9.1 Intersection of Two Lines

9.1 A Relative Position of Two Lines

Two lines may be:

1. Parallel and distinct.

2. Parallel and coincident.

3. Intersecting (not parallel).

4. Skew (not parallel, not intersecting).

9.1 B Intersection of Two Lines (Algebraic Method)

The point of intersection of two lines $L_1 : \vec{r} = \vec{r_{01}} + t\vec{u_1} \mid t \in \mathbb{R}$ and $L_2 : \vec{r} = \vec{r_{02}} + s\vec{u_2} \mid s \in \mathbb{R}$ is given by the *solution* of the following system of equations (if it exists):

$$\begin{cases} x_{01} + tu_{x1} = x_{02} + su_{x2} \\ y_{01} + tu_{y1} = y_{02} + su_{y2} \qquad s, t \in \mathbb{R} \\ z_{01} + tu_{z1} = z_{02} + su_{z2} \end{cases}$$

Hint: Solve by *substitution* or *elimination* the system of two equations and *check* if the third is satisfied.

9.1 C Unique Solution

If by solving the system you end by getting a *unique* value for t and s satisfying this system, then the lines have a *unique point of intersection*. To get this point, substitute either the t value into the line L_1 equation or substitute the s value into the line L_2 equation.

9.1 D Infinite Number of Solutions

If by solving the system you end by getting two true statements (like $2 = 2$) and one equation in s and t, then there exist an *infinite number of solutions* of the system. Therefore the lines intersect at an *infinite number of points*. In this case the lines are parallel and coincident.

9.1 E   No Solution (Parallel Lines)

If by solving the system you get at least one *false* statement (like $0 = 1$) then the system has *no solution.* Therefore, the lines have *no point of intersection.* If, in addition, the lines are parallel ($\vec{u_1} \times \vec{u_2} = \vec{0}$), then the lines are *parallel and distinct.*

9.1 F   No Solution (Skew Lines)

If by solving the system you get at least one *false* statement (like $0 = 1$) then the system has *no solution.* Therefore, the lines have *no point of intersection.* If, in addition, the lines are *not parallel* ($\vec{u_1} \times \vec{u_2} \neq \vec{0}$), then the lines are *skew.*

9.1 G   Classifying Lines (Vector Method)

$$\text{Parallel lines}$$
$$\vec{u_1} \times \vec{u_2} = \vec{0}$$

$(\vec{r_{01}} - \vec{r_{02}}) \times \vec{u_1} = \vec{0}$ $(\vec{r_{01}} - \vec{r_{02}}) \times \vec{u_1} \neq \vec{0}$

Parallel coincident lines   Parallel distinct lines

$$\text{Nonparallel lines}$$
$$\vec{u_1} \times \vec{u_2} \neq \vec{0}$$

$(\vec{r_{01}} - \vec{r_{02}}) \cdot (\vec{u_1} \times \vec{u_2}) = 0$ $(\vec{r_{01}} - \vec{r_{02}}) \cdot (\vec{u_1} \times \vec{u_2}) \neq 0$

Nonparallel intersecting lines   Nonparallel skew lines
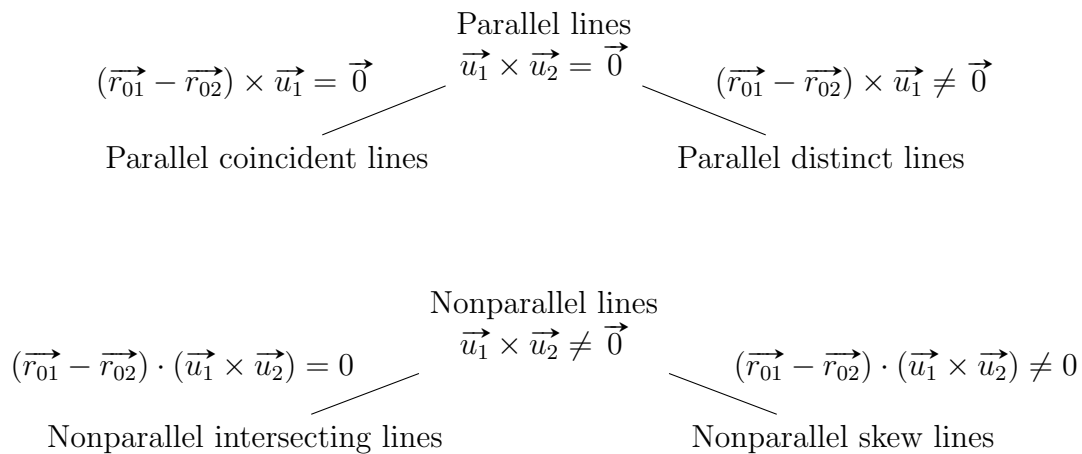
9.2   Intersection of a Line with a Plane

9.2 A   Relative Position of a Line and a Plane

There are three possible situations:

1. The line *intersects* the plane at a single point.
$$P = L \cap \pi$$

2. The line *lies* on the plane. There are an infinite number of points of intersection.
$$L = L \cap \pi$$

3. The line is *parallel* to the plane but *distinct*. There is no point of intersection.
$$L \cap \pi = \varnothing$$

9.2 B   Intersection of a Line and a Plane (Algebraic Method)

To get the intersection between a line L and a plane π:

1. *Substitute* the parametric equations of the line
$$L : \begin{cases} x = x_0 + tu_x \\ y = y_0 + tu_y \quad t \in \mathbb{R} \\ z = z_0 + tu_z \end{cases}$$

   into the Cartesian equation of the plane
$$\pi : Ax + By + Cz + D = 0$$

   to get the equation:
$$A(x_0 + tu_x) + B(y_0 + tu_y) + C(z_0 + tu_z) + D = 0 \qquad \text{(i)}$$

2. *Solve* (if possible) the equation (i) for the parameter t.

3. *Substitute* the value of the parameter t into the parametric equations of the line to get the point of intersection.

9.2 C   Unique Solution (Point Intersection)

In this case, by solving the equation you get a *unique value* for the parameter t. Therefore, there is a unique *point of intersection* between the line and the plane.
$$P = L \cap \pi$$

The line *intersects* the plane at a unique point.

9.2 D   Infinite Number of Solutions (Line Intersection)

In this case, by solving the equation (i) you get the equation:

$$0t = 0$$

which has an *infinite number of solutions*. Therefore, there are an *infinite number of points of intersection*.

$$L = L \cap \pi$$

The line *lies* on the plane.

9.2 E   No Solution (No Intersection)

In this case, by solving the equation (i) you get a false statement like:

$$0t = 1$$

The equation *does not have any solution* and therefore there is *no point of intersection* between the line and the plane.

$$L \cap \pi = \varnothing$$

The line is *parallel* to the plane and *does not lie* on the plane.

9.2 F   Classifying Lines

Consider the line $L : \vec{r} = \vec{r_0} + t\vec{u} \mid t \in \mathbb{R}$, where $P_0(x_0, y_0, z_0)$ is a specific point on the line, and the plane $\pi : Ax + By + Cz + D = 0$, where $\vec{n} = (A, B, C)$ is a normal vector to the plane.

1. If $\vec{n} \cdot \vec{u} \neq 0$ the line *intersects* the plane at a unique point.

$$P = L \cap \pi$$

2. If $\vec{n} \cdot \vec{u} = 0$ and $Ax_0 + By_0 + Cz_0 + D = 0$ then the line *lies* on the plane.

$$L = L \cap \pi$$

3. If $\vec{n} \cdot \vec{u} = 0$ and $Ax_0 + By_0 + Cz_0 + D \neq 0$ then the line is *parallel* to the plane but *does not lie* on the plane.

$$L \cap \pi = \varnothing$$

Note. By solving the equation (i) for t you will end by getting the same cases and conditions as above.

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9.3   Intersection of Two Planes

9.3 A   Relative Position of Two Planes

Two planes may be:

1. Intersecting (into a line)
$$L = \pi_1 \cap \pi_2$$

2. Coincident
$$\pi_1 = \pi_1 \cap \pi_2 = \pi_2$$

3. Distinct
$$\pi_1 \cap \pi_2 = \varnothing$$

9.3 B   Intersection of Two Planes

Consider two planes given by their Cartesian equations:

$$\pi_1 = A_1 x + B_1 y + C_1 z + D_1 = 0$$

$$\pi_2 = A_2 x + B_2 y + C_2 z + D_2 = 0$$

To find the point(s) of intersection between two planes, *solve* the system of equations formed by their Cartesian equations:

$$\left\{ \pi_1 = A_1 x + B_1 y + C_1 z + D_1 = 0 \quad \pi_2 = A_2 x + B_2 y + C_2 z + D_2 = 0 \right. \qquad \text{(ii)}$$

There are *two* equations and *three* unknowns. **Notes:**

1. A normal vector to the plane π_1 is $\vec{n_1} = (A_1, B_1, C_1)$ and a normal vector to the plane π_2 is $\vec{n_2} = (A_2, B_2, C_2)$.

2. If the planes are *parallel* then the coefficients A, B, and C are *proportional*.

3. If the planes are *coincident* then the coefficients A, B, C, and D are *proportional*.

4. A system of equations is called *compatible* if there is *at least* one solution. A system of equations is called *incompatible* if there is *no solution*.

9.3 C   Nonparallel Planes (Line Intersection)

In this case:
$$L = \pi_1 \cap \pi_2$$

- The coefficients A, B, and C in the scalar equations are *not proportional*.

- The normal vectors are *not parallel*: $\vec{n_1} \times \vec{n_2} \neq \vec{0}$.

- By solving the system (ii) you will be able to find two variables in terms of the third variable.

- There are an *infinite number of solutions* and therefore an *infinite number of points of intersection*.

- The intersection is a *line* and a *direction vector* for this line is $\vec{u} = \vec{n_1} \times \vec{n_2}$.

9.3 D Coincident Planes (Plane Intersection)

In this case:
$$\pi_1 = \pi_1 \cap \pi_2 = \pi_2$$

- The planes are *parallel* and *coincident*.

- The coefficients A, B, C, and D in the scalar equations are *proportional*.

- One equation in the system (ii) is a *multiple* of the other equation and does not contain additional information (the equations are equivalent).

- By solving the system of equations (ii), you get a *true* statement (like $0 = 0$).

- There are an *infinite number of solutions* and therefore an *infinite number of points of intersection*.

- The intersection is a *plane*.

9.3 E Parallel and Distinct Planes (No Intersection)

In this case:
$$\pi_1 \cap \pi_2 = \varnothing$$

- The planes are *parallel* and *distinct*.

- The coefficients A, B, and C in the scalar equations are *proportional* but the coefficients A, B, C, and D are *not proportional*.

- By solving the system (ii) you get a *false* statement (like $0 = 1$).

- There is *no solution* and therefore *no point of intersection* between the two planes.

9.4   Intersection of Three Planes

9.4 A   Intersection of Three Planes

Consider three planes given by their Cartesian equations:

$$\pi_1 : A_1 x + B_1 y + C_1 z + D_1 = 0$$

$$\pi_2 : A_2 x + B_2 y + C_2 z + D_2 = 0$$

$$\pi_3 : A_3 x + B_3 y + C_3 z + D_3 = 0$$

The point(s) of *intersection* of these planes is (are) related by to the *solution(s)* of the following system of equations:

$$\begin{cases} A_1 x + B_1 y + C_1 z + D_1 = 0 \\ A_2 x + B_2 y + C_2 z + D_2 = 0 \\ A_3 x + B_3 y + C_3 z + D_3 = 0 \end{cases} \qquad \text{(iii)}$$

There are *three* equations and *three* unknowns. You may use *substitution* or *elimination* to solve this system.

9.4 B   Unique Solution (Point Intersection — Noncoplanar Normal Vectors)

In this case:

$$P = \pi_1 \cap \pi_2 \cap \pi_3$$

- The planes *intersect* into a *single* point.

- The *normal vectors* are *not coplanar*:

$$\vec{n_1} \cdot (\vec{n_2} \times \vec{n_3}) \neq 0$$

- By solving the system (iii), you get a *unique solution* for x, y, and z.

9.4 C   Infinite Number of Solutions (Line Intersection — Non-parallel Planes and Coplanar Normal Vectors)

In this case:

$$L = \pi_1 \cap \pi_2 \cap \pi_3$$

- The planes are *not parallel* but their normal vectors are *coplanar*:

$$\vec{n_1} \cdot (\vec{n_2} \times \vec{n_3}) = 0$$

- The intersection is a *line.*

- One scalar equation is a *combination* of the other two equations.

- By solving the system (iii), you may express two variables in terms of the third one using two equations.

9.4 D Infinite Number of Solutions (Line Intersection — Two Coincident Planes and One Intersecting Plane)

In this case:
$$L = \pi_1 \cap \pi_2 \cap \pi_3$$

- *Two* planes are *coincident* and the third plane is *not parallel* to the coincident planes.

- The intersection is a *line.*

- Two scalar equations are *equivalent.* The *coefficients* A, B, C, and D are *proportional* for these two equations.

- You may express two variables in terms of the third one using two nonequivalent equations.

9.4 E Infinite Number of Solutions (Plane Intersection — Three Coincident Planes)

In this case:
$$\pi_1 \cap \pi_2 \cap \pi_3 = \pi_1 = \pi_2 = \pi_3$$

- The coefficients A, B, C, and D are *proportional* for all three equations.

- Any point of one plane is also a point on the other two planes.

- The intersection is a *plane.*

9.4 F No Solution (Parallel and Distinct Planes)

In this case:
$$\pi_1 \cap \pi_2 \cap \pi_3 = \varnothing$$

- There are three *parallel* and *distinct* planes.

- There is *no point of intersection.*

- There is *no solution* for the system of equations (the system of equations is *incompatible*).

- The coefficients A, B, and C are *proportional* but the coefficients of A, B, C, and D are *not proportional.*

- By solving the system (iii) you get *false* statements (like $0 = 1$).

9.4 G   No Solution (H Configuration)

In this case:
$$\pi_1 \cap \pi_2 \cap \pi_3 = \varnothing$$

- Two planes are *parallel and distinct* and the third plane is *intersecting.*

- There is *no point of intersection.*

- The coefficients A, B, and C are proportional for two planes.

- There is *no solution* for the system of equations (the system of equations is *incompatible*).

- By solving the system (iii) you get *false* statements (like $0 = 1$).

9.4 H   No Solution (Three Parallel Planes but only Two Coincident Planes)

In this case:
$$\pi_1 \cap \pi_2 \cap \pi_3 = \varnothing$$

- Three planes are *parallel* but only two are *coincident.*

- The coefficients of A, B, and C are *proportional* for all equations but the coefficients A, B, C, and D are *proportional* only for two planes.

- There is *no solution* for the system of equations (the system of equations is *incompatible*).

- By solving the system (iii) you get *false* statements (like $0 = 1$).

9.4 I   No Solution (Delta Configuration)

In this case:
$$\pi_1 \cap \pi_2 \cap \pi_3 = \varnothing$$

- The planes are *not parallel* (the coefficients A, B, and C are not *proportional*).

- The normal vectors are *coplanar* $(\vec{n_1} \cdot (\vec{n_2} \times \vec{n_3}) = 0)$.

- There is *no point of intersection* between all three planes.

- There is *no solution* for the system of equations (the system of equations is *incompatible*).

- By solving the system (iii) you get *false* statements (like $0 = 1$).

9.5 Distance from a Point to a Line

9.5 A Distance from a Point to a Line in $\mathbb{R}^2$

Let $L : Ax + By + C = 0$ be a line in $\mathbb{R}^2$, $P_1(x_1, y_1)$ be a *generic point* on the xy-plane and $P_0(x_0, y_0)$ be a specific point on this line, so: $Ax_0 + By_0 + C = 0$.

The *distance d* between the point $P_1(x_1, y_1)$ to the line L is given by (*scalar projection* of $\overrightarrow{P_0P_1}$ onto the normal vector $\vec{n}$):

$$d = \frac{\left| \overrightarrow{P_0P_1} \cdot \vec{n} \right|}{\|\vec{n}\|} \tag{iv}$$

Using $\vec{n} = (A, B), \|\vec{n}\| = \sqrt{A^2 + B^2}$ and:

$$\begin{aligned}
\overrightarrow{P_0P_1} \cdot \vec{n} &= (x_1 - x_0, y_1 - y_0) \cdot (A, B) \\
&= A(x_1 - x_0) + B(y_1 - y_0) \\
&= Ax_1 + By_1 - Ax_0 - By_0 \\
&= Ax_1 + By_1 + C
\end{aligned}$$

the formula (iv) may be written as:

$$d = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}} \tag{v}$$

9.5 B Distance from a Point to a Line in $\mathbb{R}^3$

Let $L : \vec{r} = \vec{r_0} + t\vec{u} \mid t \in \mathbb{R}$ be a line defined by its vector equation and $P_0(x_0, y_0, z_0)$ be a specific point on this line.

The distance d from a point $P_1(x_1, y_1, z_1)$ to the line L may be found using:

$$d = \left\| \overrightarrow{P_0P_1} \right\| \sin \alpha \tag{vi}$$

where α is the angle formed by the intersection of $\overrightarrow{P_0P_1}$ and $\vec{u}$.

Because $\left\| \overrightarrow{P_0P_1} \times \vec{u} \right\| = \left\| \overrightarrow{P_0P_1} \right\| \|\vec{u}\| \sin \alpha$, the formula (vi) can also be written as:

$$d = \frac{\left\| \overrightarrow{P_0P_1} \times \vec{u} \right\|}{\|\vec{u}\|} \tag{vii}$$

Note: The formula (vii) may be applied also in $\mathbb{R}^2$ by considering the third component $z = 0$.

9.5 C   Distance between Two Parallel Lines

To find the *distance* between two parallel lines:

1. Find a *specific point* on one of these lines.

2. Find the distance from that specific point to the other line using one of the relations above.

9.5 D   Perpendicular Line from a Point to a Line

Let $L : \vec{r} = \vec{r_0} + t\vec{u} \mid t \in \mathbb{R}$ be a line defined by its vector equation and $P(x, y, z)$ be a generic point in $\mathbb{R}^3$.

The line perpendicular to the line L that passes through the point P is called the *perpendicular line* and intersects the line L at a point F called the *foot* of the perpendicular line.

The foot F of the perpendicular line may be found from the equation (because $\vec{PF} \perp \vec{u}$):

$$\vec{PF} \cdot \vec{u} = 0$$

A *vector equation* of the perpendicular line is:

$$\vec{r} = \vec{OP} + s\vec{PF} \mid s \in \mathbb{R}$$

9.5 E   Shortest Distance between Two Skew Lines

Two skew lines lie into *two parallel planes*. The vector $\vec{u_1} \times \vec{u_2}$ is *perpendicular* to both lines and therefore perpendicular to parallel planes the lines lie on.

The *shortest distance* between two skew lines $L_1 : \vec{r} = \vec{r_{01}} + t\vec{u_1} \mid t \in \mathbb{R}$ and $L_2 : \vec{r} = \vec{r_{02}} + s\vec{u_2} \mid s \in \mathbb{R}$ is given by the *scalar projection* of the vector $\vec{r_{01}} - \vec{r_{02}}$ onto the vector $\vec{u_1} \times \vec{u_2}$:

$$d = \frac{\left| (\vec{r_{01}} - \vec{r_{02}}) \cdot (\vec{u_1} \times \vec{u_2}) \right|}{\| \vec{u_1} \times \vec{u_2} \|} \tag{viii}$$

9.6   Distance from a Point to a Plane

9.6 A   Distance from a Point to a Plane (I)

Consider a plane π with a *normal vector* $\vec{n}$ and a point $P_0(x_0, y_0, z_0)$ on this plane. The *distance* from a point $P_1(x_1, y_1, z_1)$ to the plane π is given by the *scalar projection* of the vector $\overrightarrow{P_0P_1}$ onto the normal vector $\vec{n}$:

$$d = \frac{\left| \overrightarrow{P_0P_1} \cdot \vec{n} \right|}{\| \vec{n} \|} \tag{ix}$$

9.6 B   Distance from a Point to a Plane (II)

If the plane π is given by the *Cartesian equation* $\pi : Ax + By + Cz + D = 0$, then the *distance* from a point $P_1(x_1, y_1, z_1)$ to the plane is given by:

$$d = \frac{|Ax_1 + By_1 + C_z + D|}{\sqrt{A^2 + B^2 + C^2}} \tag{x}$$

Indeed,

$$P_0 \in \pi \implies Ax_0 + By_0 + Cz_0 + D = 0$$

$$\begin{aligned} \overrightarrow{P_0P_1} \cdot \vec{n} &= (x_1 - x_0, y_1 - y_0, z_1 - z_0) \cdot (A, B, C) \\ &= Ax_1 + By_1 + Cz_1 - Ax_0 - By_0 - Cz_0 \\ &= Ax_1 + By_1 + Cz_1 + D \end{aligned}$$

9.6 C   Distance between Two Parallel Planes

To get the *distance* between *two parallel planes*:

1. Find a specific point into one of these planes.

2. Find the distance between that specific point and the other plane using one of the formulas above.

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AP Preparation Differentiability Review

1.4   Limit of a Function

1.4 A   One-Sided Limits

The behaviour of the function $y = f(x)$ near $x = a$ is described by three numbers:

1. The left hand limit:
$$L = \lim_{x \to a^-} f(x)$$
   the limit of the function $f(x)$ as x approaches a from the left.

2. The value of the function at $x = a$:

$$f(a)$$

3. The right hand limit:
$$R = \lim_{x \to a^+} f(x)$$
   the limit of the function $f(x)$ as x approaches a from the right.

Notes:

1. In order to exist, both the left and right hand limits must be numbers.

2. If either the left or right hand limit is not a number, then the limit does not exist (DNE).

3. Infinite limits (like ∞ or $-\infty$) are not considered numbers but they are used to give information about the behaviour of a function near the number $x = a$.

1.4 B   Limit

The limit of a function $y = f(x)$ exists at $x = a$ if:

$$L \text{ and } R \text{ exist and } L = R$$

In this case we write:
$$\lim_{x \to a} f(x)$$
the limit of the function $f(x)$ as x approaches a.

Note: The function may or may not be defined at $x = a$.

1.4 C   Substitution

If the function is defined by a formula (algebraic expression) then the limit of the function at a number $x = a$ may be determined by substitution:

$$\lim_{x \to a} f(x) = f(a)$$

Notes:

1. In order to use substitution, the function must be defined on both sides of the number $x = a$.

2. Substitution does not work if you get one of the following 7 indeterminate cases:

$$\infty - \infty \qquad 0 \times \infty \qquad \frac{0}{0} \qquad \frac{\infty}{\infty} \qquad 1^{\infty} \qquad \infty^0 \qquad 0^0$$

1.4 D   Piecewise defined functions (AP only)

If the function changes the formula at $x = a$ then:

1. Use the appropriate formula to find the left-hand and right-hand limits.

2. Compare the left-hand and right-hand limits to conclude about the limit of the function at $x = a$.

Example:

$$f(x) = \begin{cases} f_1(x) \mid x < a \\ f_2(x) \mid x > a \end{cases}$$

At $x = a$:

$$L = f_1(a) \qquad R = f_2(a)$$

1.4 E   Limits: Numerical Approach (AP only)

The limit of a function $y = f(x)$ at a number $x = a$ may be estimated numerically. To do that:

1. Use a sequence of numbers x approaching $x = a$ from the left and from the right.

2. Find the value of the function at each number x.

3. Analyze the values and make a conclusion (guess the limit).

4. Be careful at the "difference catastrophe".

1.4 F   Limit: Informal Definitions (AP only)

Left-Hand Limit If the values of $y = f(x)$ can be made arbitrarily close to L by taking x sufficiently close to a with $x < a$, then:

$$\lim_{x \to a^-} f(x) = L$$

Right-Hand Limit If the values of $y = f(x)$ can be made arbitrarily close to R by taking x sufficiently close to a with $x > a$, then:

$$\lim_{x \to a^+} f(x) = R$$

Limit If the values of $y = f(x)$ can be made arbitrarily close to l by taking x sufficiently close to a from both sides, then:

$$\lim_{x \to a} f(x) = l$$

Infinite Limit If the values of $y = f(x)$ can be made arbitrarily large by taking x sufficiently close to a from both sides, then:

$$\lim_{x \to a} f(x) = \infty$$

1.6　Continuity

1.6 A　Continuity

A function $y = f(x)$ is continuous at a number $x = a$ if

$$L = R = f(a)$$

where:

$L = \lim\limits_{x \to a^-} f(x)$ is the left-hand limit at $x = a$.

$R = \lim\limits_{x \to a^+} f(x)$ is the right-hand limit at $x = a$.

$f(a)$ is the value of the function at $x = a$.

Note: A function is continuous if it can be drawn without lifting your pencil from the paper.

1.6 B　Discontinuity

If $y = f(x)$ is not continuous at $x = a$ then we say: "$y = f(x)$ is discontinuous at $x = a$" or "$y = f(x)$ has a discontinuity at $x = a$".

1.6 C　Removable Discontinuity

A function $y = f(x)$ has a removable discontinuity at $x = a$ if:

1. $L = R = \lim\limits_{x \to a} f(x)$ exists

2. $f(a)$ DNE or $\lim\limits_{x \to a} f(x) \neq f(a)$

Note: A removable discontinuity can be removed be redefining the function $x = a$ as $f(a) \overset{def}{=} \lim\limits_{x \to a} f(x)$.

1.6 D　Jump Discontinuity

A function $y = f(x)$ has a jump discontinuity at $x = a$ if:

$$L = \lim\limits_{x \to a^-} f(x) \neq \lim\limits_{x \to a^+} f(x) = R$$

1.6 E   Infinite Discontinuity

A function $y = f(x)$ has an infinite discontinuity at $x = a$ if at least one side of the limit is unbounded (approaches ∞ or $-\infty$).

1.6 F   Continuity over an Interval (AP only)

A function $y = f(x)$ is continuous over an open interval (a, b) if the function is continuous at every number in that interval.

A function is continuous from the right at $x = a$ if $R = f(a)$.

A function is continuous from the left at $x = a$ if $L = f(a)$.

1.6 G   Elementary Functions (AP only)

Elementary functions (polynomial, power, rational, trigonometric, exponential, and logarithmic) are continuous over their domain.

1.6 H   Composition of Functions

If g is continuous at $x = a$ and f is continuous at $g(a)$ then $f(g(x))$ is continuous at $x = a$.

1.6 I   Intermediate Value Theorem (AP only)

If $y = f(x)$ is a continuous function over the interval $[a, b]$ with $f(a) \neq f(b)$, then for any number N between $f(a)$ and $f(b)$ there exist a number $c \in (a, b)$ such that $f(c) = N$.

2.1   Derivative Function

2.1 A   Derivative Function

Given a function $y = f(x)$, the *derivative function* of f is a *new function* called f' (f prime), defined at x by:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

A function $y = f(x)$ is *differentiable* at x if $f'(x)$ exists.

2.1 B   Differentiability (AP only)

A function $y = f(x)$ is differentiable over an open interval (a, b) if the function is differentiable at every number in that interval.

The domain of derivative function $f'(x)$ is a subset of the domain of the original function f ($D_{f'} \subseteq D_f$). So a function is defined over D_f but is differentiable over $D_{f'}$.

2.1 C   Interpretations of Derivative Function

1. The *slope of the tangent line* to the graph of $y = f(x)$ at the point $P(a, f(a))$ is given by $m_T = f'(a)$.

2. The *instantaneous rate of change* in the variable y with respect to the variable x, where $y = f(x)$, at $x = a$ is given by $IRC = f'(a)$.

2.1 D   Notations and Reading

Lagrange or prime notation
$$y' = f'(a)$$

   Reading: "y prime" or "f prime of (at) x".

Leibnitze notation
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x} f(x) = \mathrm{D}f(x) = \mathrm{D}_x\, f(x)$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}$$

   Reading: "dee y by dee x".

Evaluating

$$f'(a) = \left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=a}$$

Reading: "dee y by dee x at x equals a".

2.1 E First Principles

Differentiation is the process to find the derivative function for a given function.

First Principles is the process of differentiation by computing any of the following limits:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x-h)}{2h}$$

$$f'(x) = \lim_{u \to x} \frac{f(u) - f(x)}{u - x}$$

2.1 F Differentiability Point

A function $y = f(x)$ is *differentiable* at x if $f'(x)$ exists.

If the function $y = f(x)$ is *differentiable* at $x = a$ then the tangent line at $P(a, f(a))$ is *unique* and *not vertical* (the slope of the tangent line is not ∞ or $-\infty$).

2.1 G Non-Differentiability

A function is *not differentiable* at $x = a$ if $f'(a)$ *does not exist*.

Notes:

- If a function f is *not continuous* at $x = a$ then the function f is *not differentiable* at $x = a$.

- If a function is differentiable at $x = a$ then the function is continuous at $x = a$.

- If a function f is *continuous* at $x = a$ then the function f *may or may not be* differentiable at $x = a$.

2.1 H Corner Point

$P(a, f(a))$ is a *corner point* if there are *two* distinct tangent lines at P, one for the left-hand branch and one for the right-hand branch.

2.1 I   Infinite Slope Point

$P(a, f(a))$ is an *infinite slope point* if the tangent line at P is vertical and the function is increasing or decreasing in the neighbourhood of the point P.

$$f'(a) = \infty \quad \vee \quad f'(a) = -\infty$$

2.1 J   Cusp Point

$P(a, f(a))$ is a *cusp point* if the tangent line at P is vertical and the function is increasing on one side of the point P and decreasing on the other side.

$$f'(a) = DNE$$

2.2　Derivative of Polynomial Functions

2.2 A　Power Rule

If $y = f(x) = x^n \mid x, n \in \mathbb{R}$ is the *power* function then:

$$y' = f'(x) = (x^n)' = nx^{n-1}$$

Some useful specific case:

$$(1)' = 0$$
$$(x)' = 1$$
$$(\sqrt{x})' = \frac{1}{2\sqrt{x}}$$

2.2 B　Constant Function Rule

If $y = f(x) = c \mid c \in \mathbb{R}$ is the *constant* function then:

$$f'(x) = (c)' = 0$$

2.2 C　Constant Multiple Rule

If $g(x) = cf(x)$ then:

$$g'(x) = \big(cf(x)\big)' = cf'(x)$$
$$\frac{\mathrm{d}}{\mathrm{d}x}g(x) = \frac{\mathrm{d}}{\mathrm{d}x}(cf(x)) = c\frac{\mathrm{d}}{\mathrm{d}x}f(x)$$

2.2 D　Sum and Difference Rules

$$\big(f(x) \pm g(x)\big)' = f'(x) \pm g'(x)$$

2.2 E　Tangent Line

The *equation of the tangent line* at the point $P(a, f(a))$ to the curve $y = f(x)$ is:

$$y = f'(a)(x - a) + f(a) \tag{xi}$$

2.2 F   Normal Line (AP only)

If $m_T = f'(a)$ is the slope of the tangent line at $P(a, f(a))$, the slope of the normal line m_N is given by:

$$m_N = -\frac{1}{m_T}$$

2.2 G   Differentiability for Piecewise Defined Function (AP only)

Consider the piecewise defined function:

$$f(x) = \begin{cases} f_1(x) & x < a \\ c & x = a \\ f_2(x) & x > a \end{cases}$$

The function f is *differentiable* at $x = a$ if:

1. The function is continuous at $x = a$.

2. $f_1'(a) = f_2'(a)$ (the slope of the tangent line for the left branch is equal to the slope of the tangent line for the right branch).

2.3   Product Rule

2.3 A   Product Rule

If f and g are differentiable at x then so is fg and:

$$\big(f(x)g(x)\big)' = f'(x)g(x) + f(x)g'(x)$$

$$(fg)' = f'g + fg'$$

2.3 B   Product of Three Functions

If f, g, and h are differentiable at x then so is fgh and:

$$(fgh)' = f'gh + fg'h + fgh'$$

2.3 C   Generalized Power Rule

If f is differentiable at x, then so is f^n and:

$$\Big((f(x))^n\Big) = n\,(f(x))^{n-1}\,f'(x) \tag{xii}$$

$$(f^n)' = nf^{n-1}f'$$

2.4 Quotient Rule

2.4 A Quotient Rule

If f and g are differentiable at x and $g(x) \neq 0$ then so is $\frac{f}{g}$ and:

$$\left(\frac{f(x)}{g(x)} \right)' = \frac{f'(x)g(x) - f(x)g'(x)}{\left(f(x) \right)^2} \tag{xiii}$$

2.5 Chain Rule

2.5 A Composition of Functions

If $u = g(x)$ and $v = f(u)$ then:

$$x \xrightarrow[u=g(x)]{} u \xrightarrow[v=f(u)]{} v$$

and

$$v = f(u) = f(g(x)) = (f \circ g)(x)$$

2.5 B Chain Rule (Leibniz Notation)

$$\Delta x \xrightarrow[u=g(x)]{} \Delta u \xrightarrow[v=f(u)]{} \Delta v$$

and

$$\frac{\Delta v}{\Delta x} = \frac{\Delta v}{\Delta u}\frac{\Delta u}{\Delta x} \rightarrow \frac{\mathrm{d}v}{\mathrm{d}x} = \frac{\mathrm{d}v}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x}$$

Therefore:

$$\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{\mathrm{d}v}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x}$$

2.5 C Composition of Three Functions

$$x \xrightarrow[u=h(x)]{} u \xrightarrow[v=g(u)]{} v \xrightarrow[w=f(v)]{} w$$

$$\frac{\mathrm{d}w}{\mathrm{d}x} = \frac{\mathrm{d}w}{\mathrm{d}v}\frac{\mathrm{d}v}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x}$$

2.5 D Chain Rule (Prime Notation)

$$\big(f(g(x))\big)' = f'(g(x))g'(x)$$

If g is differentiable at x and f is differentiable at $g(x)$ then the composition $(f \circ g)(x) = f(g(x))$ is differentiable at x and:

$$(f \circ g)'(x) = \big(f(g(x))\big)' = f'(g(x))g'(x)$$

So, the derivative of $f(g(x))$ is the derivative of the *outside* function f evaluated at the inside function $g(x)$ times the derivative of the inside function g at x.

Note: If the outside function is the power function, then the chain rule is equivalent to the generalized power rule (xii).

5.4 Derivative of Trigonometric Functions

5.4 A Review of Trigonometric Functions

$$\sin(x) \colon \mathbb{R} \to [-1, 1]$$
$$\cos(x) \colon \mathbb{R} \to [-1, 1]$$
$$\tan(x) \colon \left\{ \mathbb{R} \setminus \left\{ \frac{\pi}{2} + n\pi \mid n \in \mathbb{Z} \right\} \right\} \to \mathbb{R}$$

$$\sin(x + 2\pi) = \sin(x)$$
$$\cos(x + 2\pi) = \cos(x)$$

$$\sin^2(x) + \cos^2(x) = 1$$

$$\sin(a + b) = \sin(a)\cos(b) + \cos(a)\sin(b)$$
$$\cos(a + b) = \cos(a)\cos(b) - \sin(a)\sin(b)$$

$$\sin\left(x + \frac{\pi}{2}\right) = \cos(x)$$
$$\sin(2x) = 2\sin(x)\cos(x)$$
$$\cos(2x) = \cos^2(x) - \sin^2(x)$$
$$\tan(x) = \frac{\sin(x)}{\cos(x)}$$

$$\lim_{x \to 0} \frac{\sin x}{x} = 1 \tag{xiv}$$

$$\lim_{x \to 0} \frac{\cos x - 1}{x} = 0 \tag{xv}$$

5.4 B   Derivative of $\sin(x)$

$$(\sin x)' = \cos x$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\sin x = \cos x$$

Proof.

$$(\sin x)' = \lim_{h \to 0} \frac{\sin(x+h) - \sin(x)}{h}$$

$$(\sin x)' = \lim_{h \to 0} \frac{\sin(x)\cos(h) + \cos(x)\sin(h) - \sin(x)}{h}$$

$$(\sin x)' = \lim_{h \to 0} \frac{\sin(x)\cos(h) - \sin(x)}{h} + \lim_{h \to 0} \frac{\cos(x)\sin(h)}{h}$$

$$(\sin x)' = \sin(x)\lim_{h \to 0} \frac{\cos(h) - 1}{h} + \cos(x)\lim_{h \to 0} \frac{\sin(h)}{h}$$

Now, using the limits (xiv) and (xv):

$$(\sin x)' = sin(x) \times 0 + \cos(x) \times 1$$

$$(\sin x)' = \cos(x)$$

$\square$

5.4 C   Derivative of $\sin\left(f(x)\right)$

By using the chain rule:

$$\left(\sin\left(f(x)\right)\right)' = \left(\cos\left(f(x)\right)\right) f'(x)$$

5.4 D   Derivative of $\cos x$

$$(\cos x)' = -\sin x$$

5.4 E   Derivative of $\cos\left(f(x)\right)$

By using the chain rule:

$$\left(\cos\left(f(x)\right)\right)' = -\left(\sin\left(f(x)\right)\right) f'(x)$$

5.4 F   Derivative of $\tan x$

$$(\tan x)' = \frac{1}{\cos^2 x} = \sec^2 x$$

5.1   Derivative of Exponential Function

5.1 A   Review of Exponential Functions

The exponential function is defined as:

$$y = f(x) = b^x \mid b > 0 \wedge b \neq 1$$

The x-axis $(y = 0)$ is a horizontal asymptote.

5.1 B   Number e

The number e is defined by:

$$e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$$

which can be written also as:

$$e = \lim_{u \to 0} (1 + u)^{\frac{1}{u}}$$

5.1 C   Derivative of e^x

$$(e^x)' = e^x$$

The proof of this is based on the fact that:

$$\lim_{h \to 0} \frac{e^h - 1}{h} = 1$$

5.1 D   Derivative of $e^{f(x)}$

By using the chain rule:

$$\left(e^{f(x)}\right)' = e^{f(x)} f'(x)$$

5.1 E   Derivative of $b^x \mid b > 0 \wedge b \neq 1$

$$(b^x)' = (\ln b) b^x$$

Proof.

$$(b^x)' = \left(e^{x \ln b}\right)' = e^{x \ln b} (\ln b) = (\ln b) b^x$$

$\square$

5.1 F   Derivative of $b^{f(x)}$

By using the chain rule:

$$\left(b^{f(x)}\right)' = (\ln b)b^{f(x)}f'(x)$$

5A   Derivative of Logarithmic Function

5A A   Review of Logarithmic Function

$$y = b^x \equiv x = \log_b y$$
$$y = f(x) = \log_b x \mid b > 0 \land b \neq 1 \land x > 0$$
$$\log_b(xy) = \log_b(x) + \log_b(y)$$
$$\log_b\left(\frac{x}{y}\right) = \log_b(x) - \log_b(y)$$
$$\log_b x^n = n \log_b x$$
$$\log_b x = \frac{\log_a x}{\log_a b}$$
$$\log_b 1 = 0$$
$$\log_b b = 1$$

5A B   Derivative of $\ln x$

$$(\ln x)' = \frac{1}{x}$$

Proof.

$$y = \ln x \implies x = e^y \implies x' = (e^y)'$$
$$x' = (e^y)' \implies 1 = e^y y' \implies y' = \frac{1}{e^y} \implies y' = \frac{1}{x}$$
$$\therefore (\ln x)' = \frac{1}{x}$$

$\square$

5A C   Derivative of $\ln(f(x))$

By using the chain rule:
$$\bigl(\ln f(x)\bigr)' = \frac{f'(x)}{f(x)}$$

5A D   Derivative of $\log_b x$

$$(\log_b x)' = \frac{1}{(\ln b)x}$$

Proof.

$$(\log_b x)' = \left(\frac{\ln x}{\ln b}\right)' = \frac{1}{\ln b}(\ln x)' = \frac{1}{(\ln b)x}$$

$\square$

5A E   Derivative of $\log_b f(x)$

By using the chain rule:

$$\left(\log_b \big(f(x)\big)\right)' = \frac{f'(x)}{(\ln b)f(x)}$$