MaCS Advanced Functions Exam Study Guide

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Contents

Unit 1

Functions Characteristics and Properties

Chapter 1

Functions Characteristics and Properties

1.1 Functions

1.1.A Relations

A binary relation is a set of ordered (x, y) pairs.

1.1.B Domain and Range of a Relation

The domain of the relation is the set of all x values such that $x \in$ the set of (x, y) pairs.

The range of the relation is the set of all y values such that $y \in$ the set of (x, y) pairs.

1.1.C Functions

A function from a set X (the domain) to a set Y (the range) is a rule that assigns to each element $x \in X$ exactly one element $y \in Y$ ($f : X \to Y$).

1.1.D Graph

The graph of a function f is the graph of the (x, y) pairs, where $y = f(x)$.

1.1.E The Vertical Line Test

All functions are relations, but not all relations are functions.

A relation is a function is there exist no two distinct y values with the same x value.

1.1.F Domain and Range

The domain D of a function f is the set $\{x \mid x \in \mathbb{R} \wedge y = f(x) \text{ is defined}\}$.

The range R of a function f is the set $\{y \mid y \in \mathbb{R} \wedge y = f(x) \text{ is defined}\}$.

1.1.G Restrictions

Division by 0 kills kittens. So for:

$$\frac{a}{b} \qquad b \neq 0$$

Square root is not defined for negative numbers. So for:

$$\sqrt{a} \qquad a \geq 0$$

A square is not negative. So for:

$$x^2 \qquad x^2 \geq 0$$

A square root is not negative. So for:

$$\sqrt{x} \qquad x \geq 0$$

1.2 Exploring Absolute Value

1.2.A Absolute Value

The absolute value $|x|$ of a real number x is the distance between x and 0.

1.2.B Definition of Absolute Value

$$|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$$

1.2.C Properties of Absolute Value

The absolute value has the following properties:

- $|a| = |-a|$
- $|a| = 0 \iff a = 0$
- $|ab| = |a||b|$
- $|\frac{a}{b}| = \frac{|a|}{|b|}$
- $|a + b| \leq |a| + |b|$ (triangle inequality)

1.2.D Distance Between Two Numbers

If $A(a)$ and $B(b)$ are two points on the number line, corresponding to the two numbers a and b respectively, the distance between the points can be expressed using the absolute value as:

$$d(A, B) = |b - a|$$

1.2.E Equations

Consider $E(x)$, an algebraic expression containing the variable x. The equation

$$|E(x)| = a \qquad a \geq 0$$

can be solved by isolating x from the equation

$$E(x) = \pm a$$

1.2.F Absolute Value Function

The absolute value function is defined by:

$$y = f(x) = |x|$$

1.2.G Inequalities

The comparison operators are used to create inequalities.

The comparison operators are:

$$< \qquad \leq \qquad = \qquad \neq \qquad > \qquad \geq$$

1.2.H Interval Notation

The following notations are equivalent:

- $a < x \leq b$ (inequality notation)
- $x \in [a, b)$ (interval notation)
- $\{x \in \mathbb{R} \mid a < x \leq b\}$ (set notation)

1.2.I Transformations

Given a parent function $f(x)$, we can create new functions using transformations:

$$g(x) = af(b(x - c)) + d$$

Transformations involving a

If $|a| > 1$, there is a vertical stretch by a factor of $|a|$.

If $|a| < 1$, there is a vertical compression by a factor of $|a|$.

If $a < 0$, there is a reflection in the x axis.

Transformations involving b

If $|b| > 1$, there is a horizontal compression by a factor of $\frac{1}{|b|}$.

If $|b| < 1$, there is a horizontal stretch by a factor of $\frac{1}{|b|}$.

If $b < 0$, there is a reflection in the y axis.

Transformations involving c

If $c > 0$, there is a horizontal translation c units to the right.

If $c < 0$, there is a horizontal translation c units to the left.

Transformations involving d

If $d > 0$, there is a vertical translation d units up.

If $d < 0$, there is a vertical translation d units down.

1.3 Properties of Graphs of Functions

1.3.A Domain and Range

The domain of a function is the set of all x values where the function is defined.

The range of a function is the set of all y values such that $y = f(x)$.

1.3.B x-intercepts and y-intercepts

The x-intercepts are the x_{int} values such that $f(x_{int}) = 0$

The y-intercept is the y_{int} value such that $y_{int} = f(0)$ (if it exists).

1.3.C Intervals of Increase or Decrease (Turning Points)

The function increases if the slope of the tangent line is positive.

The function decreases if the slope of the tangent line is negative.

A turning point is a point where the function changes from increasing to decreasing or vice versa.

1.3.D Maximum and Minimum Points

The point $(a, f(a))$ is a maximum point is $f(a) \geq f(x)$ in the neighbourhood of $x = a$.

The point $(a, f(a))$ is a minimum point is $f(a) \leq f(x)$ in the neighbourhood of $x = a$.

1.3.E Odd and Even Functions

The function f is even if $f(-x) = f(x)$ (the graph is symmetric about the y-axis).

The function f is odd if $f(-x) = -f(x)$ (the graph is symmetric about the origin).

1.3.F Continuous and Discontinuous Functions

A continuous function has no holes, finite gaps (jumps), or infinite breaks.

1.3.G Vertical and Horizontal Asymptotes

The vertical line $x = a$ is a vertical asymptote if the y values approach $\pm\infty$ in the neighbourhood of $x = a$.

The horizontal line $y = a$ is a horizontal asymptote if the values of y approach a as x approaches ∞.

1.3.H Horizontal and Vertical Tangent Lines

When the tangent line is horizontal, the slope is 0.

When the tangent line is vertical, the slope is unbounded (approaches ∞).

1.3.I End Behaviour

The end behaviour is related to the y values as x becomes unbounded.

1.3.J Concavity Upward and Downward

The graph of a function is concave upwards if the graph lies above all its tangents.

The graph of a function is concave downwards if the graph lies below all its tangents.

1.3.K Corner, Cusp, and Infinite Slope Points

A corner point is a point with 2 distinct tangent lines.

A cusp point is a turning point with a vertical tangent line.

An infinite slope point is a non-turning point with a vertical tangent line.

1.3.L Periodic Functions

A function is periodic if there exists T such that $f(x + T) = f(x)$.

1.3.M Axis of Symmetry and Axis

The graph of a function has an axis of symmetry $x = a$ if the graph is symmetric about the vertical line $x = a$.

1.4 Sketching Graphs of Functions

1.4.A Parent Functions

The parent functions are functions in their simplest form. We use parent functions and transformations to create more complex functions.

For example:

$$g(x) = -2\frac{1}{(x-1)^2} + 3$$

is a transformation of the parent function:

$$f(x) = \frac{1}{x^2}$$

To graph a parent function, use key points.

1.4.B Transformations

Given a parent function f, we can create new functions using transformations.

See 1.2.I.

1.4.C Mapping Formulas

By comparing the original (parent or old function):

$$y_{old} = f(x_{old})$$

and the image (new) function:

$$y_{new} = ay_{old} + d$$

or:

$$\begin{cases} y_{new} & = & ay_{old} + d \\ x_{new} & = & \frac{x_{old}}{b} + c \end{cases}$$

A point (x_{old}, y_{old}) on the original (parent or old) function corresponds to the point (x_{new}, y_{new}) on the image (new) function.

1.4.D Domain and Range

After transformations, the domain and the range may be changed. Use the mapping formulas to find the new ones.

1.5 Inverse Relations

1.5.A Inverse Relation

For any relation there is an inverse relation obtained by interchanging (switching) x and y for all ordered pairs in the original relation.

1.5.B Symmetry

The graph of a relation and the graph of its inverse relation are symmetrical about the line $y = x$.

1.5.C Corresponding Key Points

A point $P(x, y)$ on the relation r corresponds to the point $P'(y, x)$ on the inverse relation r^{-1}

The points P and P' are symmetrical about the line $y = x$.

1.5.D Domain and Range

The domain of the inverse relation r^{-1} is the same as the range of the relation r. Formally:

$$D_{r^{-1}} = R_r$$

The range of the inverse relation r^{-1} is the same as the range of the domain r. Formally:

$$R_{r^{-1}} = D_r$$

1.5.E Inverse Relation of a Function

Any function is a relation.

So any function f has an inverse relation f^{-1}.

The inverse relation of a function may or may not be a function.

1.5.F Algebraic Method

To find the inverse of a function:

1. Write the original function in the form $y = f(x)$
2. Swap the variables x and y
3. Solve the last expression for y
4. Replace y with $f^{-1}(x)$

1.5.G One-to-One Functions

If the inverse relation of f is also a function, then f is a one-to-one function. Formally:

$$y = f(x) \iff x = f^{-1}(y)$$

1.5.H Horizontal Line Test

One-to-one functions pass the horizontal line test:

Any horizontal line intersects the graph at no more than one point.

1.5.I Restricted Domains

By restricting the domain of a function (which is not one-to-one), we may obtain a one-to-one function.

1.6 Piecewise Functions

1.6.A Piecewise-Defined Functions

A piecewise-defined function requires more than one formula to define the function. Each formula is defined on a different interval.

The domain of a piecewise function is the union ($\cup$) of all intervals used to define the function.

Heaviside Function

The Heaviside function is defined by:

$$H(x) = \begin{cases} 1, & \text{if } x \geq 0 \\ 0, & \text{if } x < 0 \end{cases}$$

1.6.B Continuity

A continuous function can be drawn without lifting your pencil from the paper. See 1.3.F.

1.6.C Absolute Value of a Function

See 1.2.

1.7 Exploring Operations with Functions

1.7.A Arithmetic Combinations

$$(f+g)(x) = f(x) + g(x)$$
$$(f-g)(x) = f(x) - g(x)$$
$$(fg)(x) = f(x)g(x)$$
$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} \quad | \quad g(x) \neq 0$$

1.7.B Domain

The domain of $f+g$, $f-g$, and fg is the intersection of the domain of f and the domain of g.

$$D_{f+g} = D_f \cap D_g$$
$$D_{f-g} = D_f \cap D_g$$
$$D_{fg} = D_f \cap D_g$$
$$D_{f \div g} = D_f \cap D_g \quad | \quad g(x) \neq 0$$

1.8 Composition of Functions

1.8.A Composition of Functions

$$(f \circ g)(x) = f(g(x))$$

1.8.B Domain and Range

The domain of $f \circ g$ is a subset of the domain of g:

$$D_{f \circ g} = \{x \in D_g \mid g(x) \in D_f\}$$

The range of $f \circ g$ is a subset of the domain of f.

Unit 2

Polynomial functions

Chapter 3

Polynomial Functions

3.1 Exploring Polynomial Functions

3.1.A Polynomial Functions

A polynomial function $y = f(x)$ is defined by:

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$$

- a_n, a_{n-1}, $\ldots$, a_1, a_0 are real numbers, called the coefficients of the polynomial function
- a_n is called the leading coefficient
- $a_n x^n$ is called the leading term
- a_0 is called the constant term
- n is a non-negative integer that gives the degree of the polynomial function
- The degree of the polynomial function is the largest exponent of x.

3.1.B Order

The terms of a polynomial function can be written in any order because addition is commutative.

3.1.C Specific Polynomials

For $n = 0$, $f(x) = a_0$ is called a constant function.

For $n = 1$, $f(x) = a_1 x + a_0$ is called a linear function.

For $n = 2,\ f(x) = a_2x^2 + a_1x + a_0$ is called a quadratic function.

For $n = 3,\ f(x) = a_3x^3 + a_2x^2 + a_1x + a_0$ is called a cubic function.

For $n = 4,\ f(x) = a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$ is called a quartic function.

For $n = 5,\ f(x) = a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$ is called a quintic function.

3.1.D Operations with Polynomial Functions

All the four operations ($+$, $-$, $\times$, and $\div$) are defined for polynomial functions.

3.1.E y-intercept

The y-intercept of a polynomial function is equal to the constant term.

$$f(0) = a_0$$

3.1.F Finite Differences

The nth finite differences of a polynomial function of degree n are constant. This constant c is related to a_n and n by:

$$c = n!a_n$$

Given the nth difference, and n, you can solve for the leading coefficient of the polynomial:

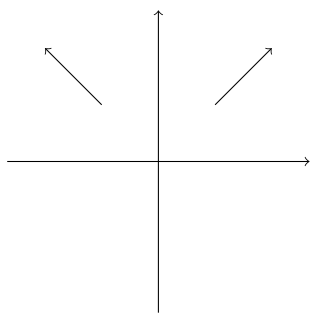
$$a_n = \frac{\Delta^n y}{n!(\Delta x)^n}$$

3.2 Characteristics of Polynomial Functions

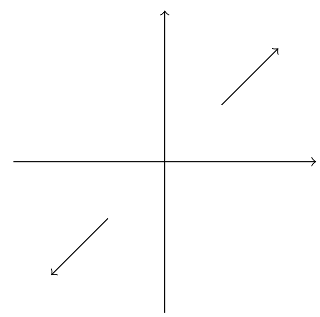
3.2.A End Behaviour

For $x \to \pm\infty$ the graph of the polynomial function resembles the graph of the leading term $y = a_n x^n$.

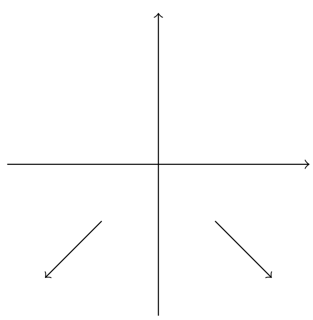
For $a_n > 0$, n is even



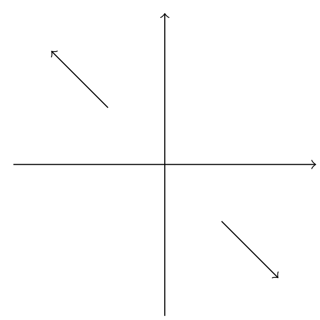
For $a_n > 0$, n is odd



For $a_n < 0$, n is even



For $a_n < 0$, n is odd



3.2.B Symmetry

A polynomial function is even ($f(-x) = f(x)$) if all the powers of x are even.

A polynomial function is odd ($f(-x) = -f(x)$) if all the powers of x are odd.

3.2.C Zeros and the Fundamental Theorem of Algebra

- A polynomial function $P(x)$ of degree n has n zeros (real or complex).
- If the coefficients of the polynomial function are real numbers, the complex zeros come in conjugate pairs.
- The number of complex zeros must 0 or a multiple of 2.

- The number of real zeros is at most n. These zeros may be distinct or coincident.
- A polynomial function of even degree may have no real zero (all may be complex).
- A polynomial function of odd degree must have at least one real zero.

3.2.D Turning Points

A turning point is where a function changes from increasing to decreasing or decreasing to increasing.

A polynomial function of degree n has at most $n - 1$ turning points.

3.2.E Extrema Points

- Extremum is either a minimum or a maximum.
- An extremum may be local or global.
- Each turning point is a local extremum.
- A polynomial function of even degree has either a global minimum or a global maximum point.
- A polynomial function of odd degree has neither a global minimum nor a global maximum point.

3.3 Polynomial Functions in Factored Form

3.3.A Simple Zeroes

Some polynomial functions can be factored in the form:

$$f(x) = a_n(x - x_1)(x - x_2)\dots(x - x_{n-1})(x - x_n)$$

x_1, x_2, $\dots$, x_{n-1}, and x_n are n distinct real numbers and the zeros of the polynomial function.

- The function changes sign at each x-intercept.
- The tangent line at each x-intercept is not horizontal.

3.3.B Repeated Zeros

Some polynomial functions can be factored in the form:

$$f(x) = a_n(x - x_1)^{m_1}(x - x_2)^{m_2}\dots(x - x_k)^{m_k}$$

x_1 is a zero of multiplicity m_1, x_2 is a zero of multiplicity m_2, and so on.

The polynomial function has

$$\sum_{x=1}^{k}(m_x) = n$$

real zeros.

- If m_1 is odd, the function changes sign at $x = x_1$ and the graph crosses the x-axis.
- If m_1 is even, the function does not change the sign at $x = x_1$ and the graph touches the x-axis.
- If the multiplicity $m_1 > 1$ then the tangent line at $x = x_1$ is horizontal.

3.3.C Non-real Zeros

A polynomial function with non-real zeros can be factored as:

$$f(x) = (a_1x^2 + b_1x + c)^{m_1} \times \dots$$

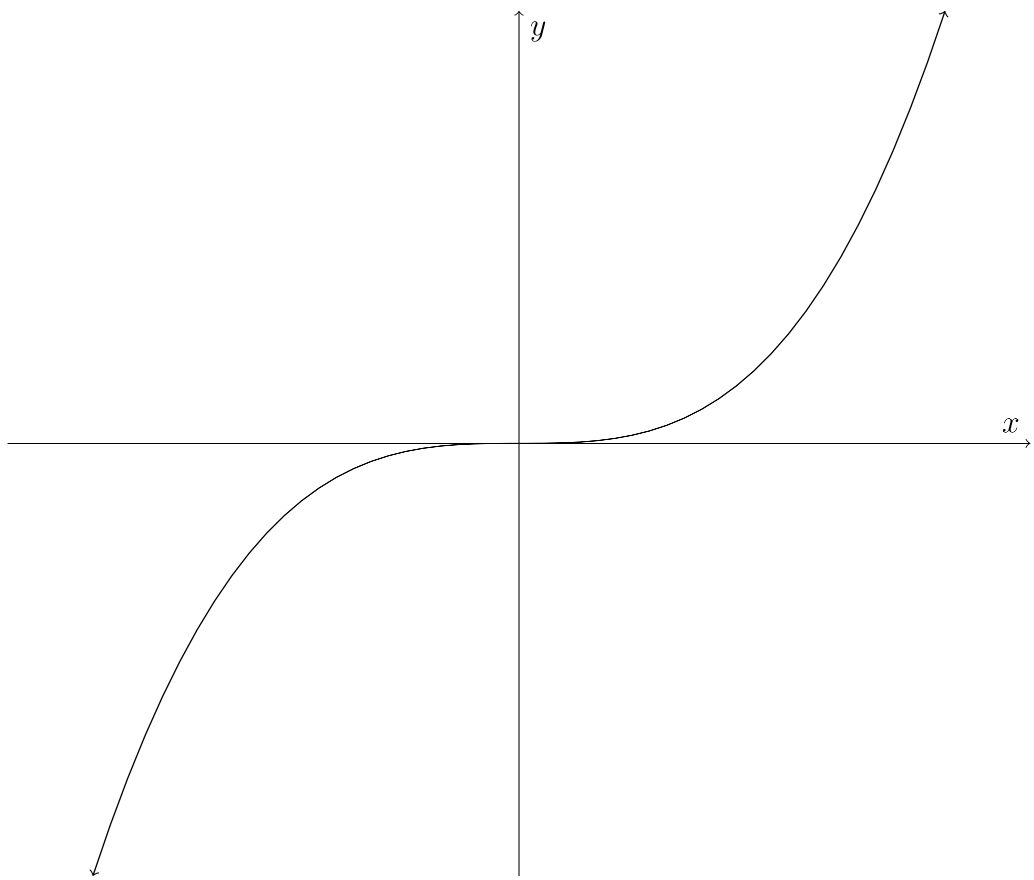
Each trinomial $a_1x^2 + b_1x + c$ has the same sign for all real numbers x.

3.4 Transformations of Power Functions (Cubic, Quartic, and Other)

3.4.A Cubic Function

The cubic function has the parent $f(x) = x^3$ and after transformations may be written as:

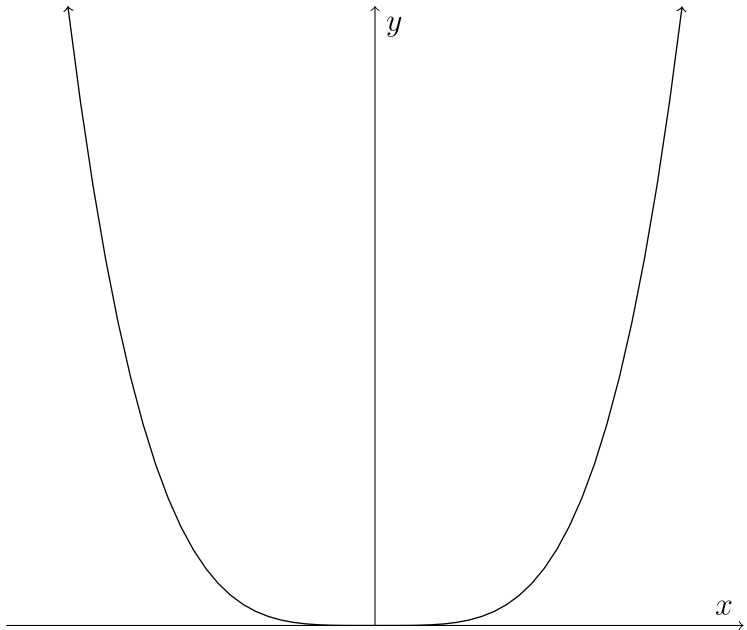
$$f(x) = a[b(x - c)^3] + d$$



3.4.B Quartic Function

The quartic function has the parent $f(x) = x^4$ and after transformations may be written as:

$$f(x) = a[b(x - c)^3] + d$$



3.4.C Power Function (Real Exponent)

The power function with a real exponent is defined by:

$$f(x) = x^b \quad | \quad b \in \mathbb{R}$$

3.4.D Power Function (Rational Exponent)

The power function with a rational exponent is defined by:

$$f(x) = x^{\frac{m}{n}} \quad | \quad m \in \mathbb{Z} \land n \in \mathbb{Z}$$

3.5 Dividing Polynomials

3.5.A Division of Natural Numbers

If D and $d \neq 0$ are two natural numbers, then there are unique numbers q and r such that the following is true:

$$\frac{D}{d} = q + \frac{r}{d} \qquad \text{or} \qquad D = dq + r$$

$$0 \leq r < d$$

where:

- D is the dividend
- d is the divisor
- q is the quotient
- r is the remainder

If $r = 0$ then D is divisible by d.

3.5.B Division of Polynomials

If $D(x)$ and $d(x) \neq 0$ are two polynomial functions, then there are two unique polynomials $q(x)$ and $r(x)$ such that the following relation (called division statement) is true:

$$\frac{D(x)}{d(x)} = q(x) + \frac{r(x)}{d(x)} \qquad \text{or} \qquad D(x) = d(x)q(x) + r(x)$$

$$0 \leq \text{degree}(r) < \text{degree}(d)$$

Same rules as above.

3.5.C Synthetic Division Algorithm

Synthetic division is a shorthand for dividing a polynomial $P(x)$ by a linear divisor $x - b$.

Example

$$\frac{-2x^3 + 3x^2 - 4x + 5}{x - 2}$$

$$\begin{array}{r|rrrr} & -2 & 3 & -4 & 5 \\ 2 & & -4 & -2 & -12 \\ \hline & -2 & -1 & -6 & -7 \end{array}$$

Here, the remainder is -7.

3.6 Factoring Polynomials

3.6.A The Remainder Theorem

If a polynomial $P(x)$ is divided by $x - b$ then the remainder is $r = P(b)$.

Proof.

$$\begin{aligned} P(x) &= q(x)(x-b) + r(x) \\ P(b) &= q(b)(b-b) + r(b) \\ P(b) &= r \end{aligned}$$

$\square$

3.6.B The Remainder Theorem (II)

If a polynomial $P(x)$ is divided by $ax - b$ then the remainder is $r = P\left(\frac{b}{a}\right)$.

Proof.

$$\begin{aligned} P(x) &= q(x)(ax-b) + r(x) \\ P\left(\frac{b}{a}\right) &= q\left(\frac{b}{a}\right)\left(\frac{ab}{a} - b\right) + r\left(\frac{b}{a}\right) \\ P(b) &= r \end{aligned}$$

$\square$

3.6.C The Factor Theorem

A polynomial $P(x)$ has $x - b$ as a factor if and only if $P(b) = 0$.

In this case b is a zero of the polynomial function $P(x)$.

3.6.D Integral Zero Theorem

If $x = b$ is an integral zero of the polynomial $P(x)$ with integral coefficients, then b is a factor (divisor) of the constant term a_0 of the polynomial.

3.7 Factoring a Sum or Difference of Cubes (Powers)

3.7.A Difference of Cubes

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$

3.7.B Sum of Cubes

$$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$$

3.7.C Difference of Two Powers

For any natural number n, the following identity is true:

$$a^n - b^n = (a - b)(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \cdots + a^2b^{n-3} + ab^{n-2} + b^{n-1})$$

3.7.D Sum of Two Powers

If n is an odd natural number, the following identity is true:

$$a^n + b^n = (a + b)(a^{n-1} - a^{n-2}b + a^{n-3}b^2 - \cdots \pm a^2b^{n-3} \mp ab^{n-2} \pm b^{n-1}$$

Chapter 4

Solving Polynomial and Linear Equations and Inequalities

4.1 Solving Polynomial Equations

4.1.A The Fundamental Theorem of Algebra

A polynomial function $P(x)$ of degree n has n zeros (real or complex).

The complete factorization of the polynomial function $P(x)$ is:

$$P(x) = a_n(x - x_1)(x - x_2) \ldots (x - x_{n-1})(x - x_n)$$

where x_1, x_2, ..., x_{n-1}, and x_n are the zeros (real or complex, distinct or coincident).

The complex zeros come in conjugate pairs. So:

$$n = r + 2m \quad | \quad m \geq 0$$

where r is the number of real zeros and $2m$ is the number of complex zeros.

4.1.B Integral Zero Theorem

If $x = b$ is an integral zero of the polynomial $P(x)$ with integral coefficients, then b is a factor of the constant term a_0 of the polynomial.

4.1.C Rational Zero Theorem

If $x = \frac{b}{a}$ is a rational zero of the polynomial $P(x)$ with rational coefficients then b is a factor (divisor) of the constant term a_0 and a is a factor of the leading coefficient a_n.

4.1.D Real Zeros

If $x = a + b\sqrt{c}$ (a, b, and c are rational numbers, $c > 0$) is a zero of a polynomial with integral coefficients $P(x)$, then $x = a - b\sqrt{c}$ is also a zero of this polynomial.

4.1.E Technology

Is some cases the real zeros may only be found by using technology.

4.2 Solving Linear Inequalities

4.2.A Inequalities

The inequality symbols are used to create inequalities. They are:

$$< \qquad \leq \qquad \geq \qquad >$$

The solution set is all numbers that make the inequality a true statement.

4.2.B Inequality Properties

The inequality $a < b$ is equivalent to:

$$\begin{aligned} a + c &< b + c & & \\ ac &< bc & &\text{for } c > 0 \\ ac &> bc & &\text{for } c < 0 \end{aligned}$$

4.2.C Simultaneous (Double) Inequality

The simultaneous inequality $a < x \leq b$ is equivalent to:

$$a < x \wedge x \leq b$$

$$x \in (a, b]$$

4.2.D Inequations (I)

The inequation

$$|E(x)| < a \quad | \quad a \geq 0$$

is equivalent to:

$$-a < E(x) < a \qquad \text{or} \qquad -a < E(x) \wedge E(x) < a$$

Never write $E(x) < \pm a$ or you will go deaf.

4.2.E Inequations (II)

The inequation

$$|E(x)| > a \quad | \quad a \geq 0$$

is equivalent to:

$$E(x) < -a \vee E(x) > a$$

4.3 Solving Polynomial Inequalities

4.3.A Sign Chart Method

Use a sign chart to specify the sign of each factor and then combine them to find the sign of the whole factored polynomial.

4.3.B Graphical Method

Graph the factored polynomial and then conclude about its sign.

4.3.C Algorithm to Solve Polynomial Inequalities

In order to solve an inequality involving a polynomial expression:

- Move all terms to one side of the inequality
- Factor the polynomial
- Use the sign chart or graphical method to find the sign of the polynomial
- Write the solution set

4.3.D Technology

When the polynomial is not factored, use technology to find the solution.

Unit 3

Rational Functions

Chapter 5

Rational Functions

5.1 Graphs of Reciprocal Functions

5.1.A General Rules

Consider a function $y = f(x)$ and its reciprocal $g(x) = \frac{1}{f(x)}$. Here are some general rules:

- If $f(x)$ has a zero at $x = a$, then $g(x)$ has a vertical asymptote $x = a$
- If $f(x)$ has a vertical asymptote $x = a$ ($y \to \pm\infty$ as $x \to a$), then $g(x)$ has a zero at $x = a$
- If $f(x)$ is unbounded as x becomes unbounded ($y \to \pm\infty$ as $x \to \pm\infty$) then $g(x)$ has the horizontal asymptote $y = 0$
- If $f(x)$ has a horizontal asymptote $y = a$ ($y \to a$ as $x \to \pm\infty$), then $g(x)$ has a horizontal asymptote $y = \frac{1}{a}$
- If $f(x)$ is increasing/decreasing over an interval, then $g(x)$ is decreasing/increasing over the same interval
- If $f(x)$ has a local minimum/maximum at $(a, f(a))$, then $g(x)$ has a local maximum/minimum at $(a, g(a))$
- If $f(x)$ is even/odd/neither, then so is $g(x)$

5.2 Exploring Quotients of Polynomial Functions (Rational Functions)

5.2.A Rational Functions

A rational function is a function of the form:

$$f(x) = \frac{P(x)}{Q(x)}$$

where $P(x)$ and $Q(x)$ are polynomial functions.

5.2.B Domain

The domain of a rational function is determined by the restriction $Q(x) \neq 0$.

5.2.C y-intercept

The y-intercept for a rational function $f(x)$ is the point $(0, f(0))$ if 0 is in the domain of f.

5.2.D Holes

The x values where $Q(x) = 0$ do not belong to the domain of the function and must not appear on the graph of a rational function. These values might create holes in the graph of the rational function.

5.2.E Vertical Asymptotes

If $x = a$ is a vertical asymptote, then the value of the function becomes unbounded as $x \to a$ from the left or right.
In short, $x = a$ is a vertical asymptote if:

$$\lim_{x \to a-} f(x) = f(a - h) = \pm\infty$$

or

$$\lim_{x \to a+} f(x) = f(a + h) = \pm\infty$$

where $h > 0 \wedge h$ is very small

If $x = a$ is a vertical asymptote for a rational function $f(x) = \frac{P(x)}{Q(x)}$, then a is a zero of $Q(x)$ and not a zero of $P(x)$.

5.2.F Behaviour Near the Vertical Asymptote

The value of the rational function is unbounded as $x \to a$ from the left and the right. To find the unbounded value of the rational function, use substitution, $x = a \pm h$ and the following types of limits:

$$\frac{1}{h} = \infty \qquad \frac{1}{-h} = -\infty \qquad \frac{1}{h^2} = \infty \qquad \frac{1}{(-h)^2} = \infty \qquad \frac{1}{(-h)^3} = -\infty$$

and so on.

5.2.G Horizontal Asymptotes

If $y = c$ is a horizontal asymptote, then $f(x) \to c$ as $x \to \pm\infty$.
In short, $y = c$ is a horizontal asymptote if:

$$\lim_{x \to \infty} f(x) = c$$

or

$$\lim_{x \to -\infty} f(x) = c$$

where c is a finite number.

Some functions may have two different horizontal asymptotes (one as $x \to \infty$ and one as $x \to -\infty$). Rational functions have at most one horizontal asymptote.

Consider the case of the rational function:

$$f(x) = \frac{P(x)}{Q(x)} = \frac{a_n x^n + \cdots + a_1 x + a_0}{b_m x^m + \cdots + b_1 x + a_0}$$

- If $P(x)$ and $Q(x)$ have the same degree $(n = m)$, then the equation of the horizontal asymptote is $y = \frac{a_n}{b_m}$.
- If the degree of $P(x)$ is less than the degree of $Q(x)$, then the equation of the horizontal asymptote is $y = 0$.
- If the degree of $P(x)$ is greater than the degree of $Q(x)$ then the rational function does not have a horizontal asymptote.

5.2.H Oblique Asymptotes

A rational function has an oblique asymptote if the degree of $P(x)$ is one unit greater than the degree of $Q(x)$. To find the equation of the oblique asymptote use long division and express the rational function in the form:

$$f(x) = mx + b + \frac{c}{R(x)}$$

As $|x| \to \pm\infty$, the term $\frac{c}{R(x)}$ approaches 0 and $f(x)$ approaches $y = mx + b$.

5.2.I Graph Sketching

We have these tools to help us sketch the graphs of functions:

- x-intercepts and y-intercepts
- symmetry
- vertical asymptotes
- horizontal asymptotes
- oblique asymptotes
- sign charts

5.3 Graphs of Rational Functions

5.3.A Characteristics of the Rational Function

$$f(x) = \frac{ax+b}{cx+d} \quad | \quad a, c \neq 0$$

Case 1: $cx + d$ **is not a factor of** $ax + b$

Domain $\mathbb{R} \setminus \{-\frac{d}{c}\}$

Range $\mathbb{R} \setminus \{\frac{a}{c}\}$

x-intercept $-\frac{b}{a}$

y-intercept $\frac{b}{d}$ if $d \neq 0$

Symmetry neither even nor odd

Vertical asymptote $x = -\frac{d}{c}$

Horizontal asymptote $y = ac$

Continuity There exists an infinite break at $x = -\frac{d}{c}$

Case 2: $cx + d$ **is a factor of** $ax + b$

Domain $\mathbb{R} \setminus \{-\frac{d}{c}\}$

Range $\{\frac{a}{c}\}$

x-intercept none

y-intercept $\frac{b}{d}$ if $d \neq 0$

Symmetry neither even nor odd

Vertical asymptote none

Horizontal asymptote $y = \frac{a}{c}$

Continuity There exists a hole at $x = -\frac{d}{c}$

5.4 Solving Rational Equations

5.4.A Rational Equations

To solve a rational equation:

1. State restrictions
2. Multiply by the least common denominator
3. Solve the polynomial equation
4. Verify restrictions
5. Verify solution by using substitution

5.4.B Cross Multiplication

A rational equation of the form: $\frac{P(x)}{Q(x)} = \frac{R(x)}{S(x)}$, where $P(x)$, $Q(x)$, $R(x)$, and $S(x)$ are polynomial functions may be solved with cross-multiplication.

$$\frac{P(x)}{Q(x)} = \frac{R(x)}{Q(x)} \qquad \equiv \qquad P(x)S(s) = Q(x)R(x)$$

5.4.C Shortcut

For a rational equation of the form $\frac{P(x)}{Q(x)} = 0$:

$$\frac{P(x)}{Q(x)} = 0 \quad \equiv \quad P(x) = 0 \qquad \text{if restrictions are satisfied}$$

5.4.D No Solution

When c is a constant:

$$\frac{\text{c}}{P(x)} = 0 \quad \text{has no solution} \qquad \Longleftrightarrow \qquad c \neq 0$$

5.5 Solving Rational Inequalities

5.5.A Rational Inequalities

In order to solve a nonlinear rational inequality:

1. State restrictions
2. Move all the terms to one side
3. Multiply the expression by the least common denominator
4. Simplify and factor both numerator and denominator
5. Create a sign chart or graph to find the solution set
6. Verify if restrictions are satisfied

5.6 Component Partial Fraction Decomposition

5.6.A Partial Fractions

The process of breaking down a rational expression such as $\frac{3x+7}{(x+5)(x+1)}$ into partial fractions such as $\frac{2}{x+5} + \frac{1}{x+1}$ is called partial fraction decomposition.

The rational function $R(x) = \frac{P(x)}{Q(x)} \mid Q(x) \neq 0$ is called a proper fraction when the degree of $P(x)$ is equal to the degree of $Q(x)$. We assume that $P(x)$ and $Q(x)$ have no common zeros.

Partial fraction decomposition depends on the factors in the denominator $Q(x)$.

When $Q(x)$ is factored as a product of $(ax + b)^n$ and $(ax^2 + bx + c)^m$:

$$n \in \mathbb{N}^* \qquad m \in \mathbb{N}^* \qquad a \in \mathbb{R} \qquad b \in \mathbb{R} \qquad c \in \mathbb{R}$$

and $(ax^2 + bx + c)$ is irreducible over the real numbers.

Case 1: $Q(x)$ contains only distinct linear factors

$$Q(x) = (a_1x + b_1)(a_2x + b_2)\dots(a_nx + b_n)$$

In this case, unique real constants C_1, C_2, ..., C_n could be found such that:

$$\frac{P(x)}{Q(x)} = \frac{C_1}{a_1x + b_1} + \frac{C_2}{a_2x + b_2} + \dots + \frac{C_n}{a_nx + b_n}$$

Example Decompose $\frac{2x+1}{(x-1)(x+3)}$ into partial fractions.

Step 1

$$\frac{2x + 1}{(x - 1)(x + 3)} = \frac{A}{x - 1} + \frac{B}{x + 3}$$

Step 2 Find the LCD of the right side and equate the numerators:

$$2x + 1 = A(x + 3) + B(x - 1)$$

Step 3 Find A and B by equating all coefficients of all powers of x:

For the coefficient of x^0:

$$1 = 3A - B$$

For the coefficient of x^1:

$$2 = A + B$$

Solve the system to find that:

$$A = \frac{3}{4} \qquad B = \frac{5}{4}$$

Rewrite the rational expression as a sum of two partial fractions

$$\frac{2x + 1}{(x - 1)(x + 3)} = \frac{3}{4(x - 1)} + \frac{5}{4(x + 1)}$$

Case 2: $Q(x)$ **contains only one repeated linear factor**

$Q(x)$ contains a repeated factor of $(ax+b)$ such that $Q(x) = (ax+b)^n$.

In this case, the partial fraction decomposition of $R(x)$ could be written as:

$$R(x) = \frac{C_1}{(ax+b)} + \frac{C_2}{(ax+b)^2)} + \cdots + \frac{C_n}{(ax+b)^n}$$

Example Decompose $\frac{6x-1}{x^3(2x-1)}$ into partial fractions.

Step 1 Write $R(x)$ according to case 1 and case 2 as:

$$\frac{6x-1}{x^3(2x-1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{2x-1}$$

Step 2 Multiply the expression by $x^3(2x-1)$ and equate the numerators from both sides:

$$6x - 1 = Ax^2(2x-1) + Bx(2x-1) + C(2x-1) + Dx^3$$

Step 3 Solve for A, B, C, and D by equating all coefficients of all powers of x:

For the coefficient of x^0:

$$-1 = -C$$

For the coefficient of x^1:

$$6 = -B + 2C$$

For the coefficient of x^2:

$$0 = -A + 2B$$

For the coefficient of x^3:

$$0 = 2A + D$$

Now solve the system:

$$\begin{aligned} -1 &= C \\ C &= 1 \end{aligned}$$

$$\begin{aligned} 0 &= -A - 8 \quad \text{(Substitute in } B = -4\text{)} \\ A &= -8 \end{aligned}$$

$$\begin{aligned} 6 &= -B + 2 \quad \text{(Substitute in } C = 1\text{)} \\ B &= -4 \end{aligned}$$

$$\begin{aligned} 0 &= -16 + D \quad \text{(Substitute in } A = -8\text{)} \\ D &= 16 \end{aligned}$$

Step 4 Rewrite the rational expression as a sum of the partial fractions:

$$\frac{6x-1}{x^3(2x-1)} = -\frac{8}{x} - \frac{4}{x^2} + \frac{1}{x^3} + \frac{16}{2x-1}$$

Case 3: $Q(x)$ **contains only nonrepeated irreducible quadratic factors**

$(a_i x + b_i x + c)$ are among the factors of $Q(x)$.

In this case, the partial fraction decomposition of $R(x)$ could be written as:

$$R(x) = \frac{A_1x + B_1}{(a_1x^2 + b_1x + c_1)} + \frac{A_2x + B_2}{(a_2x^2 + b_2x + c_2)} + \cdots + \frac{A_nx + B_n}{a_nx^2 + b_nx + c_n}$$

Example Decompose $\frac{4x}{(x^2+1)(x^2+2x+3)}$ into partial fractions.

Step 1 Write $R(x)$ according to case 3 as:

$$\frac{4x}{(x^2+1)(x^2+2x+3)} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{x^2+2x+3}$$

Step 2 Multiply the expression by $(x^2+1)(x^2+2x+3)$ and equate the numerators from both sides:

$$4x = (Ax+B)(x^2+2x+3) + (Cx+D)(x^2+1)$$

Step 3 Solve for A, B, C, and D by equating all coefficients of all powers of x:

For the coefficient of x^0:

$$0 = 3B + D$$

For the coefficient of x^1:

$$4 = 3A + 2B + C$$

For the coefficient of x^2:

$$0 = 2A + B + D$$

For the coefficient of x^3:

$$0 = A + C$$

Solve the system:

$$\begin{aligned} 3B + D &= 2A + B + D \\ B &= A \end{aligned}$$

$$\begin{array}{rl} 4 & = 5A + C \\ - \quad 0 & = A + C \\ \hline 4 & = 4A \end{array}$$

$$A = 1 \quad B = 1$$

$$\begin{aligned} 0 &= A + C \\ C &= -1 \end{aligned} \qquad \text{(Substitute in } A = 1\text{)}$$

$$\begin{aligned} 0 &= 2A + B + D \\ D &= -3 \end{aligned} \qquad \text{(Substitute in } A = 1 \text{ and } B = 1\text{)}$$

Step 4 Rewrite the rational expression as a sum of the partial fractions:

$$\frac{4x}{(x^2+1)(x^2+2x+3)} = \frac{x+1}{x^2+1} - \frac{x+3}{x^2+2x+3}$$

Unit 4

Exponential and Logarithmic Functions

Chapter 8

Exponential and Logarithmic Functions

8.1 Exploring the Logarithmic Function

8.1.A Logarithmic Functions

The logarithmic function is defined as the inverse function of the exponential function.

So if $f(x) = b^x$, then $f^{-1}(x) = \log_b x$.

$$y = b^x \quad \equiv \quad x = \log_b y$$

$$b \in \{\mathbb{N}^* \setminus 1\}$$

8.1.B Domain, Range, and Other Restrictions

The domain and the range of the exponential function are defined by:

$$b^x : (-\infty, +\infty) \to (0, +\infty)$$

The domain and range of the logarithmic function are defined by:

$$\log_b x : (0, +\infty) \to (-\infty, +\infty)$$

$$b > 0, b \neq 1$$

8.1.C Basic Formulas

Prove: $\log_b 1 = 0$

Proof.

$$y = b^x \quad \equiv \quad x = \log_b y$$

Here, $y = 1$ and $x = 0$.

$$b^0 = 1$$

$\square$

Prove: $\log_b b = 1$

Proof.

$$y = b^x \quad \equiv \quad x = \log_b y$$

Here, $y = b$ and $x = 1$.

$$b = b^1$$

$\square$

Prove: $\log_b \frac{1}{b} = -1$

Proof.

$$y = b^x \quad \equiv \quad x = \log_b y$$

Here, $y = \frac{1}{b}$ and $x = -1$.

$$b^{-1} = \frac{1}{b}$$

$\square$

Prove: $\log_{\frac{1}{b}} b = -1$

Proof.

$$y = b^x \quad \equiv \quad x = \log_b y$$

Here, $y = b$ and $x = -1$.

$$b^{-1} = \frac{1}{b}$$

$\square$

Prove: $\log_b b^n = n$

Proof.

$$y = b^x \quad \equiv \quad x = \log_b y$$

Here, $y = b^n$ and $x = n$.

$$b^n = b^n$$

$\square$

8.1.D Basic Equations

Solve each equation by converting it to the exponential form.

Solve: $x = \log_5 25$

$$\begin{aligned} 5^x &= 25 \\ 5^x &= 5^2 \\ x &= 2 \end{aligned}$$

Solve: $x = \log_4 1$

$$\begin{aligned} b^x &= 1 \\ x &= 0 \end{aligned}$$

Solve: $\log_x 16 = 2$

$$\begin{aligned} 16 &= x^2 \\ x &= \pm 4 \\ x > 0, x \neq 1 \implies x &= 4 \end{aligned}$$

Solve: $\log_x 3 = \frac{1}{2}$

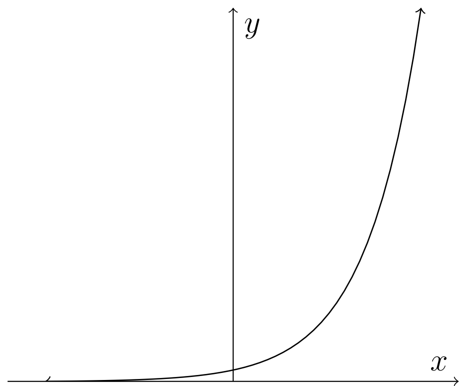
$$\begin{aligned} x^{\frac{1}{2}} &= 3 \\ x &= 9 \end{aligned}$$

Solve: $\log_2 x = -2$

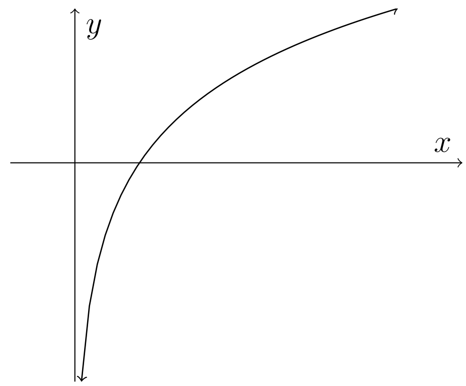
$$\begin{aligned} x &= 2^{-2} \\ x &= \frac{1}{4} \end{aligned}$$

8.1.E Graph of the Logarithmic Function

$f(x) = 2^x$



$f^{-1}(x) = \log_2 x$



8.1.F Characteristics of the Logarithmic Function

Domain $(0, +\infty)$

Range $(-\infty, +\infty)$

x-intercept $(1, 0)$

y-intercept None

Increase/Decreasing Increasing if $b > 1$. Decreasing if $0 < b < 1$.

Horizontal asymptote None

Vertical asymptote $x = 1$

Continuity Continuous

One-to-one Yes

8.2 Transformations of Logarithmic Functions

8.2.A Transformations of Logarithmic Functions

The function:

$$g(x) = A \log_b B(x - C) + D$$

is a transformation of the parent function $f(x) = \log_b x$.

Features of $g(x)$:

Domain if $C > 0$**:** $(C, +\infty)$

Domain if $C < 0$**:** $(-\infty, C)$

Range: $\mathbb{R}$

Vertical Asymptote: $x = C$

8.3 Evaluating Logarithms

8.3.A Specific Logarithms

The logarithm to base 10 is called the decimal or common logarithm. We use the shortcut:

$$\log_{10} x = \lg x$$

The logarithm to the base e is called the natural logarithm. We use the shortcut:

$$\log_{\mathrm{e}} x = \ln x$$

8.3.B Evaluating Logarithms

You can use the exponential-logarithmic conversion to evaluate logarithms.

8.3.C Technology

You can use your calculator to evaluate logarithmic functions,

8.4 Laws of Logarithms

8.4.A Power Law

$$\log_b x^n = n \log_b x; \quad x > 0$$

Proof.
Let $a = \log_b x \implies x = b^a$

$$\begin{aligned} &\log_b(x^n) \\ =&\log_b(b^{an}) \\ =&an \\ =&n \log_b(x) \end{aligned}$$

$\square$

8.4.B Product Law

$$\log_b(xy) = \log_b x + \log_b y; \quad x, y > 0$$

Proof.
Let $c = \log_b x \implies x = b^c$
Let $d = \log_b y \implies y = b^d$

$$\begin{aligned} &\log_b(xy) \\ =&\log_b(b^c \times b^d) \\ =&\log_b(b^{c+d}) \\ =&c + d \\ =&\log_b x + \log_b y \end{aligned}$$

$\square$

8.4.C Quotient Law

$$\log_b \frac{x}{y} = \log_b x - \log_b y; \quad x, y > 0$$

Proof.
Let $c = \log_b x \implies x = b^c$
Let $d = \log_b y \implies y = b^d$

$$\begin{aligned}
&\log_b \left(\frac{x}{y}\right) \\
=&\log_b \left(\frac{b^c}{b^d}\right) \\
=&\log_b (b^{c-d}) \\
=&c - d \\
=&\log_b x - \log_b y
\end{aligned}$$

$\square$

8.4.D Change of Base Law

$$\log_a (x) = \frac{\log_b x}{\log_b a}$$

Proof.
Let $c = \log_b x \implies x = b^c$
Let $d = \log_b a \implies a = b^d$

$$\begin{aligned}
&\log_a x \\
=&\log_{b^d} b^c \\
=&\log_{b^d} (b^{d \times \frac{c}{d}}) \\
=&\frac{c}{d} \\
=&\frac{\log_b x}{\log_b a}
\end{aligned}$$

$\square$

8.4.E Change of Base Formula for Exponential Function

$$a^x = b^{x \log_b a}$$

Proof.

$$\begin{aligned}
\log_b a^x &= \log_b a^x \\
b^{\log_b a^x} &= a^x \\
b^{x \log_b a} &= a^x
\end{aligned}$$

$\square$

8.5 Solving Exponential Equations

8.5.A One-to-one Property

The exponential function is a one-to-one function. So:

$$a^x = a^y \iff x = y$$

$$a > 0,\ a \neq 1 \qquad x \in \mathbb{R},\ y \in \mathbb{R}$$

8.5.B Change of Variable

Sometimes, the change of the variable may help in solving the exponential equation. For example:

$$a^x = y; \quad y > 0$$

8.5.C Logarithms

Sometimes, logarithms are needed in order to solve exponential equations.

8.5.D Applications

Exponential equations are used for applications such as population growth and half-life.

Population Growth

$$P(t) = P_0 \times \mathrm{e}^{kt}$$

Where:

- $P(t)$ is the population at time t
- P_0 is the population at $t = 0$
- e is Euler's number
- k is the growth constant
- t is the time

Half-Life

$$P(t) = P_0 \left(\frac{1}{2}\right)^{t \div k} = P_0 \left(\mathrm{e}^{(-t \ln 2) \div k}\right)$$

Where:

- $P(t)$ is the amount at time t
- P_0 is the amount at $t = 0$
- e is Euler's number
- k is the half-life
- t is the time

8.6 Solving Logarithmic Equations

8.6.A Exponential-Logarithmic Conversion

$$b^x = y \quad \equiv \quad x = \log_b y$$
$$b > 0,\ b \neq 1; \quad y > 0,\ x \in \mathbb{R}$$

8.6.B One-to-one Property

The logarithmic function is a one-to-one function. So:

$$\log_b x = \log_b y \iff x = y$$

$$b > 0,\ b \neq 1,\ x \in \mathbb{R}^*,\ y \in \mathbb{R}^*$$

8.6.C Technology

You can also use a scientific calculator to solve logarithmic equations.

Example

$$\ln x + \log x = 5$$
$$x \doteq 32.656$$

8.6.D Inequalities and Logarithms

If $b > 1$ then:

$$\log_b x > \log_b y \iff x > y$$

If $b < 1$ then:

$$\log_b x > \log_b y \iff x < y$$

Unit 5

Calculus

Chapter 2

Finding the Derivative

2.1 Derivative Function

2.1.A Derivative Function

Given a function $y = f(x)$, the derivative function of f is a new function called f', defined by:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

2.1.B Differentiability

$f(x)$ is differentiable at x if $f'(x)$ exists.

2.1.C Interpretations of Derivative Function

The slope of the tangent line to the graph of $y = f(x)$ at the point $P(a, f(a))$ is given by: $m = f'(a)$.

The IRC in the variable y with respect to x where $y = f(x)$ and $x = a$ is given by: $IRC = f'(a)$.

2.1.D Notations and Reading

Lagrange Notation $y' = f'(x)$

Leibnitz Notation $\frac{dy}{dx} = \frac{d}{dx} f(x)$

2.1.E First Principles

Differentiation: the process to find the derivative of a function.

First Principles: the process of differentiation by computing the limit:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

2.1.F Non-Differentiability

A function is not differentiable at $x = a$ if $f'(a)$ does not exist.

- If a function f is not continuous at $x = a$ then the function f is not differentiable at $x = a$.
- If a function f is continuous at $x = a$ then the function f may or may not be differentiable at $x = a$.

2.1.G Differentiability Point

If the function $y = f(x)$ is differentiable at $x = a$ then there is only one tangent line at $P(a, f(a))$, and it is not vertical.

2.1.H Corner Point

$P(a, f(a))$ is a corner point if there are two distinct tangent lines at P, one for the left-hand branch and one for the right-hand branch.

For example:

$$f(x) = \begin{cases} f_1(x), & x < a \\ f_2(x), & x > a \end{cases} \qquad \text{and } f_1'(a) \neq f_2'(a)$$

2.1.I Infinite Slope Point

$P(a, f(a))$ is an infinite slope point if the tangent line at P is vertical and the function is increasing or decreasing in the neighborhood of P.

$$f'(a) = \infty \qquad \text{or} \qquad f'(a) = -\infty$$

2.1.J Cusp Point

$P(a, f(a))$ is a cusp point if the tangent line at P is vertical and the function is increasing on one side of the point P and decreasing on the other side. In this case, $\nexists\ f'(a)$.

2.2 Derivative of Polynomial Functions

2.2.A Power Rule

Consider the power function

$$y = x^n; \qquad x,\ n \in \mathbb{R}$$

Then:

$$(x^n)' = nx^{n-1}$$
$$\frac{d}{dx}x^n = nx^{n-1}$$

Some useful specific cases:

$$\left(\frac{1}{x^n}\right)' = -\frac{n}{x^{n+1}}$$
$$\left(\frac{a}{x^n}\right)' = -\frac{na}{x^{n+1}}$$
$$(1)' = 0$$
$$(x)' = 1$$
$$(\sqrt{x})' = \frac{1}{2\sqrt{x}}$$

2.2.B Constant Function Rule

Consider a constant function:

$$f(x) = c \qquad c \in \mathbb{R}$$

Then:

$$(c)' = 0$$
$$\frac{d}{dx}c = 0$$

2.2.C Constant Multiple Rule

Consider the function:

$$g(x) = cf(x)$$

Then:

$$[cf(x)]' = cf'(x)$$
$$\frac{d}{dx}[cf(x)] = c\frac{d}{dx}f(x)$$
$$(cf)' = cf'$$

2.2.D Sum and Difference Rules

$$[f(x) \pm g(x)]' = f'(x) \pm g'(x)$$
$$\frac{d}{dx}[f(x) \pm g(x)] = \frac{d}{dx}f(x) \pm \frac{d}{dx}g(x)$$
$$(f \pm g)' = f' \pm g'$$

2.2.E Tangent Line

To find the equation of the tangent line at the point $P(a, f(a))$:

1. Find derivative function $f'(x)$
2. Find the slope of the tangent using $m = f'(a)$
3. Use the slope-point formula to get the equation of the tangent line:

$$y - f(a) = m(x - a) \qquad \equiv \qquad y = m(x - a) + f(a)$$

2.2.F Normal Line

If m_T is the slope of the tangent line, then the slope of the normal line is given by:

$$m_N = -\frac{1}{m_T}$$

2.2.G Differentiability for Piecewise Defined Functions

Consider the piecewise defined function:

$$f(x) = \begin{cases} f_1(x), & x < a \\ c, & x = a \\ f_2(x), & x > a \end{cases}$$

$f(x)$ is differentiable at $x = a$ if and only if both:

1. the function is continuous at $x = a$
2. $f_1'(a) = f_2'(a)$

2.3 Product Rule

2.3.A Product Rule

If f and g are differentiable at x then so is fg and:

$$(fg)'(x) = f'(x)g(x) + f(x)g'(x)$$

$$(fg)' = f'g + fg'$$

2.3.B Product of Three Functions

If f, g, and h are differentiable at x then so is fgh and:

$$(fgh)' = f'gh + fg'h + fgh'$$

Proof.

$$\begin{aligned}
&(fgh)' \\
=&(fg)'h + fgh' \\
=&(f'g + fg')h + fgh' \\
=&f'gh + fg'h + fgh'
\end{aligned}$$

□

2.3.C Generalized Power Rule

If f is differentiable at x, then so if f^n and:

$$([f(x)]^n)' = n[f(x)]^{n-1}f'(x)$$

$$(f^n)' = nf^{n-1}f'$$

2.4 Quotient Rule

2.4.A Quotient Rule

If f and g are differentiable at x and $g(x) \neq 0$ then so is $\frac{f}{g}$ and:

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$$

"Low dee high minus high dee low. Square the bottom and off we go."

$$\left(\frac{f}{g}\right)' = \frac{gf' - fg'}{g^2}$$

2.5 Chain Rule

2.5.A Composition of Functions

If $u = g(x)$ and $v = f(u)$ then:

$$x \underset{u=g(x)}{\longrightarrow} u \underset{v=f(u)}{\longrightarrow} v$$

and

$$v = f(u) = f(g(x)) = (f \circ g)(x)$$

2.5.B Chain Rule (Leibniz Notation)

$$\Delta x \underset{u=g(x)}{\longrightarrow} \Delta u \underset{v=f(u)}{\longrightarrow} \Delta v$$

and

$$\frac{dv}{dx} = \frac{dv}{du}\frac{du}{dx}$$

2.5.C Composition of Three Functions

$$x \underset{u=h(x)}{\longrightarrow} u \underset{v=g(u)}{\longrightarrow} v \underset{w=f(v)}{\longrightarrow} w$$

$$\frac{dw}{dx} = \frac{dw}{dv}\frac{dv}{du}\frac{du}{dx}$$

2.5.D Chain Rule (Lagrange Notation)

$$v = f(u) = f(g(x)) = (f \circ g)(x)$$

$$\frac{dv}{dx} \to [f(g(x))]'$$

$$\frac{dv}{du} \to f'(u) = f'(g(x))$$

$$\frac{du}{dx} \to g'(x)$$

$$\frac{dv}{dx} = \frac{dv}{du}\frac{du}{dx} \to (f \circ g)'(x) = [f(g(x))]' = f'(g(x))g'(x)$$

If g is differentiable at x and f is differentiable at $f(x)$ then the composition $(f \circ g)(x) = f(g(x))$ is differentiable at x and

So, the derivative of $f(g(x))$ is the derivative of the outside function f evaluated of the inside function g times the derivative of the inside function g.

Chapter 5

Derivatives of Exponential, Trigonometric, and Logarithmic Functions

5.1 Derivative of Exponential Function

5.1.A Review of Exponential Functions

The exponential function is defined as:

$$y = f(x) = b^x; \quad b > 0,\ b \neq 1$$

The x-axis is a horizontal asymptote.

5.1.B Number e

The number e is defined by:

$$\mathrm{e} = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n$$

which can also be written as:

$$\mathrm{e} = \lim_{u\to 0}(1 + u)^{\frac{1}{u}}$$

5.1.C Exponential Function

The exponential function e^x may be evaluated using the limit:

$$\mathrm{e}^x = \lim_{n\to\infty}\left(1 + \frac{x}{n}\right)^n$$

5.1.D Derivative of e^x

$$(\mathrm{e}^x)' = \mathrm{e}^x$$

5.1.E Derivative of $\mathrm{e}^{f(x)}$

$$\left(\mathrm{e}^{f(x)}\right)' = \mathrm{e}^{f(x)} f'(x)$$

5.1.F Derivative of $b^x,\ b > 0,\ b \neq 1$

$$(b^x)' = (\ln b) b^x$$

5.1.G Derivative of $b^{f(x)}$

$$\left(b^{f(x)}\right)' = (\ln b) b^{f(x)} f'(x)$$

5.4 Derivative of Trigonometric Functions

5.4.A Review of Trigonometric Functions

$$\sin x : \mathbb{R} \mapsto [-1,1] \qquad \sin(x+2\pi) = \sin x$$
$$\cos x : \mathbb{R} \mapsto [-1,1] \qquad \cos(x+2\pi) = \cos x$$
$$\sin\left(x + \frac{\pi}{2}\right) = \cos x \qquad \sin(x+\pi) = -\sin x$$
$$\sin(2x) = 2\sin x \cos x \qquad \cos(2x) = \cos^2 x - \sin^2 x$$
$$\tan x = \frac{\sin x}{\cos x} \qquad \tan x : \mathbb{R} \setminus \left\{\frac{\pi}{2} + n\pi,\ n \in \mathbb{Z}\right\} \mapsto \mathbb{R}$$

Fundamental Limits

$$\lim_{h\to 0} \frac{\sin h}{h} = 1$$
$$\lim_{h\to 0} \frac{\cos h - 1}{h} = 0$$
$$\lim_{h\to 0} \frac{\mathrm{e}^h - 1}{h} = 1$$

5.4.B Derivative of $\sin x$

$$(\sin x)' = \cos x$$

5.4.C Derivative of $\sin f(x)$

$$[\sin f(x)]' = (\cos f(x)) f'(x)$$

5.4.D Derivative of $\cos x$

$$(\cos x)' = -\sin x$$

5.4.E Derivative of $\cos f(x)$

$$[\cos f(x)]' = -[\sin f(x)] f'(x)$$

5.4.F Derivative of $\tan x$

$$(\tan x)' = \frac{1}{\cos^2 x} = \sec^2 x$$

$$[\tan f(x)]' = \frac{f'(x)}{\cos^2 f(x)} = \sec^2 f(x) f'(x)$$

5.5 Derivative of Logarithmic Function

5.5.A Review of Logarithmic Function

$$y = b^x \quad \equiv \quad x = \log_b y$$

$$y = f(x) = \log_b x, \qquad b > 0,\ b \neq 1, x > 0$$
$$\log_b(xy) = \log_b x + \log_b y$$
$$\log_b \frac{x}{y} = \log_b x - \log_b y$$
$$\log_b x^n = n \log_b x$$
$$\log_a x = \frac{\log_b x}{\log_b a}$$
$$\log_b 1 = 0$$
$$\log_b b = 1$$

5.5.B Derivative of $\ln x$

$$(\ln x)' = \frac{1}{x}$$

5.5.C Derivative of $\ln f(x)$

$$[\ln f(x)]' = \frac{f'(x)}{f(x)}$$

5.5.D Derivative of $\log_b x$

$$(\log_b x)' = \frac{1}{(\ln b)x}$$

5.5.E Derivative of $\log_b f(x)$

$$[\log_b f(x)]' = \frac{f'(x)}{(\ln b)f(x)}$$

Unit 6

Introduction To Vectors

Chapter 6

Introduction to Vectors

6.1 An Introduction to Vectors

6.1.A Scalars and Vectors

Scalars are described with a number.

Vectors are described with magnitude and direction.

6.1.B Geometric Vectors

Geometric vectors are vectors not related to any coordinate system.

For example: $\overrightarrow{AB}$

6.1.C Algebraic Vectors

Algebraic vectors are vectors related to a coordinate system.

For example: $\vec{v} = (2, 3, -1)$

6.1.D Position Vector

The position vector is the directed line segment $\overrightarrow{OP}$ from the origin of the coordinate system O to a generic point P.

6.1.E Displacement Vector

The displacement vector $\overrightarrow{AB}$ is the directed line segment from the point A to the point B.

6.1.F Pythagorean Theorem

In a right triangle ABC with $\angle C = 90°$, the following relation is true:

$$c^2 = a^2 + b^2$$

6.1.G Magnitude

The magnitude of $\vec{v}$ is denoted by $|\vec{v}|$, $\|\vec{v}\|$, or v.

6.1.H 3D Pythagorean Theorem

$$d^2 = a^2 + b^2 + c^2$$

6.1.I Equivalent or Equal Vectors

Two vectors are equivalent or equal if they have the same magnitude and direction.

6.1.J Opposite Vectors

Two vectors are called opposite if they have the same magnitude and opposite direction.

The vector opposite $\vec{v}$ is denoted by $-\vec{v}$. The vector opposite $\overrightarrow{AB}$ is $\overrightarrow{BA}$.

Also, $\overrightarrow{AB} = -\overrightarrow{BA}$.

6.1.K Parallel Vectors

Two vectors are parallel if their directions are either the same or opposite.

If $\vec{v_1}$ and $\vec{v_2}$ are parallel, then we write $\vec{v_1} \parallel \vec{v_2}$.

6.1.L Direction

True bearing is measured clockwise from North.

Quadrant bearing is given by the angle between the North-South line and the vector.

6.2 Addition and Subtraction of Geometric Vectors

6.2.A Addition of Two Vectors

The vector addition $\vec{s}$ of two vectors $\vec{a}$ and $\vec{b}$ is denoted by $\vec{a} + \vec{b}$ and is called the sum or resultant of the two vectors. So:

$$\vec{s} = \vec{a} + \vec{b}$$

6.2.B Triangle Rule (Tail to Tip Rule)

In order to find the resultant of two geometric vectors:

1. Place the second vector with its tail on the tip of the first vector.
2. The resultant is a vector with the tail at the tail of the first vector and the head at the head of the second vector.

6.2.C Polygon Rule

In order to find the resultant of n geometric vectors:

1. Place the next vector with its tail on the tip of the previous vector.
2. The resultant is a vector with the tail at the tail of the fist vector and the head at the head of the last vector.

6.2.D Parallelogram Rule (Tail to Tail Rule)

To add two geometric vectors, the following rule can also be used:

1. Position both vectors with their tails at the same point.
2. Build a parallelogram using the vectors as two sides.
3. The resultant is given by the diagonal of the parallelogram starting from the common tail point.

6.2.E Sine Law

For any triangle $\triangle ABC$, the following relation is true:

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} \qquad \equiv \qquad \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

6.2.F Cosine Law

For any triangle $\triangle ABC$, the following relations are true:

$$a^2 = b^2 + c^2 - 2bc\cos\angle A$$
$$b^2 = a^2 + c^2 - 2ac\cos\angle B$$
$$c^2 = a^2 + b^2 - 2ab\cos\angle C$$

6.2.G Magnitude and Direction for Vector Sum

Let $\theta = \angle(\vec{a}, \vec{b})$ be the angle between the vectors $\vec{a}$ and $\vec{b}$ when they are placed tail to tail.

6.2.H Vector Subtraction

The subtraction operation between two vectors $\vec{a} - \vec{b}$ can be understood as a vector addition between the first vector and the opposite of the second vector:

$$\vec{d} = \vec{a} - \vec{b} = \vec{a} + (-\vec{b})$$

6.2.I Inverse Operation

The vector subtraction operation is the inverse operation of the vector addition:

$$\vec{d} = \vec{a} - \vec{b} \quad \equiv \quad \vec{a} = \vec{b} + \vec{d}$$

6.2.J Magnitude and Direction for Vector Difference

Let $\vec{a}$ and $\vec{b}$ be two vectors and $\vec{d} = \vec{a} - \vec{b}$ be the vector difference. Let θ be the angle between the vectors $\vec{a}$ and $\vec{b}$ when they are placed tail to tail:

$$\vec{d} = \vec{a} - \vec{b}$$

The magnitude of the vector difference is given by:

$$\|\vec{a} - \vec{b}\|^2 = \|\vec{a}\|^2 + \|\vec{b}\|^2 - 2\|\vec{a}\|\,\|\vec{b}\|\cos\theta$$

The direction of $\vec{d}$ is given by the angles α and β formed by the vector sum and the vectors $\vec{b}$ and $\vec{a}$ respectively:

$$\frac{\|\vec{a}\|}{\sin\alpha} = \frac{\|\vec{b}\|}{\sin\beta} = \frac{\|\vec{a} - \vec{b}\|}{\sin\theta}$$

6.3 Multiplication of a Vector by a Scalar

6.3.A Multiplication of a Vector by a Scalar

By multiplying a vector $\vec{v}$ by a scalar k, we obtain a new vector noted $k\vec{v}$ with the following properties:

- $k\vec{v}$ has the same direction as $\vec{v}$ if $k > 0$ and the opposite direction if $k < 0$
- $\|k\vec{v}\| = |k| \times \|\vec{v}\|$

6.3.B Properties

The following properties apply for multiplication of a vector by a scalar:

$$k(\vec{a} + \vec{b}) = k\vec{a} + k\vec{b}$$

$$k(m\vec{a}) = (km)\vec{a} = km\vec{a}$$

$$(k + m)\vec{a} = k\vec{a} + m\vec{a}$$

6.3.C Vector Unit

A unit vector is a vector having a magnitude of 1. For any vector $\vec{v}$, a unit vector parallel to $\vec{v}$ is given by:

$$\hat{v} = \frac{\vec{v}}{\|\vec{v}\|}$$

6.4 Properties of Vectors

6.4.A Properties of Vectors

$$\vec{a} + \vec{b} = \vec{b} + \vec{a}$$

$$\vec{a} + \vec{0} = \vec{0} + \vec{a} + \vec{a}$$

$$\vec{a} + (-\vec{a}) = (-\vec{a}) + \vec{a} = \vec{0}$$

$$(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$$

$$\|k\vec{a}\| = |k|\,\|\vec{a}\|$$

$$k(\vec{a} + \vec{b}) = k\vec{a} + k\vec{b}$$

$$(kl)\vec{a} = k(l\vec{a}) = l(k\vec{a})$$

$$(k+l)\vec{a} = k\vec{a} + l\vec{a}$$

$$1\vec{a} = \vec{a}$$

$$(-1)\vec{a} = -\vec{a}$$

$$0\vec{a} = \vec{0}$$

$$\|\vec{0}\| = 0$$

6.5 Vectors in R^2 and R^3

6.5.A Polar Coordinates

Given a Cartesian system of coordinates, 2D vector $\vec{v}$ may be defined by its magnitude $\|\vec{v}\|$ and the counter-clockwise angle θ between the positive direction of the x-axis and the vector.

The pair $(\|\vec{v}\|, \theta)$ determines the polar coordinates of the 2D vector and $\vec{v} = (\|\vec{v}\|, \theta)$.

6.5.B Scalar Components for a 2D Vector

Consider a 2D vector with the tail in the origin of the Cartesian system. Parallels through its tip to the coordinate axes intersect the x-axis at v_x and the y-axis at v_y.

The pair (v_x, v_y) determines the scalar coordinates of the 2D vector and $\vec{v} = (v_x, v_y)$.

6.5.C Link Between the Polar Coordinates and Scalar Components

To convert a vector from the polar coordinates $\vec{v} = (\|\vec{v}\|, \theta)$ to the scalar components $\vec{v} = (v_x, v_y)$, use the formulas:

$$v_x = \|\vec{v}\| \cos\theta$$

$$v_y = \|\vec{v}\| \sin\theta$$

To convert a vector from the scalar components $\vec{v} = (v_x, v_y)$ to the polar coordinates $\vec{v} = (\|\vec{v}\|, \theta)$, use the formulas:

$$\|\vec{v}\| = \sqrt{{v_x}^2 + {v_y}^2}$$

$$\tan\theta = \frac{v_y}{v_x} \qquad \equiv \qquad \theta = \tan^{-1}\left(\frac{v_y}{v_x}\right)$$

6.5.D Magnitude of a 2D Algebraic Vector

The magnitude of a 2D algebraic vector $\vec{v} = (v_x, v_y)$ is given by:

$$\|\vec{v}\| = \sqrt{{v_x}^2 + {v_y}^2}$$

6.5.E Standard Unit Vectors

$$\vec{i} = (1, 0)$$
$$\vec{j} = (0, 1)$$

6.5.F Vector Components for a 2D Vector

Any vector $\vec{v}$ may be decomposed into two $\perp$ vector components $\vec{v}_x$ and $\vec{v}_y$, where:

$$\vec{v}_x \parallel \vec{i} \qquad \vec{v}_y \parallel \vec{j}$$
$$\vec{v} = \vec{v}_x + \vec{v}_y$$

The link between the scalar components and the vector components is given by:

$$\vec{v}_x = v_x \vec{i} \qquad \vec{v}_y = v_y \vec{j}$$

A 2D vector may also be written in algebraic form as:

$$\vec{v} = \vec{v}_x + \vec{v}_y = v_x \vec{i} + v_y \vec{j} = (v_x, v_y)$$

6.5.G Position 2D Vector

The directed line segment $\overrightarrow{OP}$, from the origin O to a generic point $P(x, y)$ determines a vector called the position vector and:

$$\overrightarrow{OP} = (x, y) = x \vec{i} + y \vec{j}$$

6.5.H Displacement 2D Vector

The directed line segment $\overrightarrow{AB}$ from the point $A(x_A, y_a)$ to the point $B(x_B, y_B)$ determines a vector called the displacement vector and:

$$\overrightarrow{AB} = (x_B - x_A, y_B - y_A) = (x_B - x_A) \vec{i} + (y_B - y_A) \vec{j}$$

6.5.I Direction Angles

Consider a 3D coordinate system and a 3D vector $\vec{v}$ with the tail in the origin O. Direction angles are the angles α, β, and γ between the vector and the positive directions of the coordinate axes.

α: the angle between the vector and the x-axis.

β: the angle between the vector and the y-axis.

γ: the angle between the vector and the z-axis.

6.5.J Scalar Components of a 3D Vector

Consider a 3D coordinate system and a 3D vector $\vec{v}$ with the tail in the origin O. Parallel planes through its tip to the coordinate planes intersect the x-axis at v_x, the y-axis at v_y, and the z-axis at v_z.

The triple (v_x, v_y, v_z) determines the scalar components of the 3D vector and $\vec{v} = (v_x, v_y, v_z)$.

6.5.K Link Between the Direction Angles and the 3D Scalar Coordinates

The link between the direction angles (α, β, and γ) and the scalar components of a vector (v_x, v_y, and v_z) is given by:

$$v_x = \|\vec{v}\| \cos\alpha$$
$$v_y = \|\vec{v}\| \cos\beta$$
$$v_z = \|\vec{v}\| \cos\gamma$$

and by:

$$\|\vec{v}\| = \sqrt{{v_x}^2 + {v_y}^2 + {v_z}^2}$$
$$\cos\alpha = \frac{v_x}{\|\vec{v}\|}$$
$$\cos\beta = \frac{v_y}{\|\vec{v}\|}$$
$$\cos\gamma = \frac{v_z}{\|\vec{v}\|}$$

Note that:

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

6.5.L Magnitude of a 3D Algebraic Vector

The magnitude of a 3D algebraic vector $\vec{v} = (v_x, v_y, v_z)$ is given by:

$$\|\vec{v}\| = \sqrt{{v_x}^2 + {v_y}^2 + {v_z}^2}$$

6.5.M 3D Standard Unit Vectors

$$\vec{i} = (1, 0, 0)$$
$$\vec{j} = (0, 1, 0)$$
$$\vec{k} = (0, 0, 1)$$

6.5.N Vector Components for a 3D Vector

Any 3D vector $\vec{v}$ may be decomposed into three $\perp$ vector components $\vec{v}_x$, $\vec{v}_y$, and $\vec{v}_z$ where:

$$\vec{v}_x \parallel \vec{i} \qquad \vec{v}_y \parallel \vec{j} \qquad \vec{v}_z \parallel \vec{k}$$

$$\vec{v} = \vec{v}_x + \vec{v}_y + \vec{v}_z$$

The link between the scalar components and the vector components is given by:

$$\vec{v}_x = v_x \vec{i} \qquad \vec{v}_y = v_y \vec{j} \qquad \vec{v}_z = v_z \vec{k}$$

A 3D vector may be written in algebraic form as:

$$\vec{v} = \vec{v}_x + \vec{v}_y + \vec{v}_z = v_x \vec{i} + v_y \vec{j} + v_z \vec{k} = (v_x, v_y, v_z)$$

6.5.O Position 3D Vector

The directed line segment $\overrightarrow{OP}$ from the origin O to a generic point $P(x, y, z)$ determines a vector called the position vector and:

$$\overrightarrow{OP} = (x, y, z) = x \vec{i} + y \vec{j} + z \vec{k}$$

6.5.P Displacement 3D Vector

The directed line segment $\overrightarrow{AB}$ from the point $A(x_A, y_A, z_A)$ to the point $B(x_B, y_B, z_B)$ determines a vector called the displacement vector and:

$$\overrightarrow{AB} = (x_B - x_A, y_B - y_A, z_B - z_A)$$

$$\overrightarrow{AB} = (x_B - x_A)\vec{i} + (y_B - y_A)\vec{j} + (z_B - z_A)\vec{k}$$

6.6 Operations with Algebraic Vectors in R^2

6.6.A 2D Algebraic Vectors

A 2D algebraic vector may be written in components form as:

$$\vec{v} = (v_x, v_y)$$

or in terms of unit vectors as:

$$\vec{v} = v_x \vec{i} + v_y \vec{j}$$

and has a magnitude given by:

$$\|\vec{v}\| = \sqrt{{v_x}^2 + {v_y}^2}$$

6.6.B Addition of 2D Algebraic Vectors

$$\vec{a} + \vec{b} = (a_x + b_x, a_y + b_y)$$

6.6.C Substitute of 2D Algebraic Vectors

$$\vec{a} - \vec{b} = (a_x - b_x, a_y - b_y)$$

6.6.D Multiplication of 2D Algebraic Vector by a Scalar

$$\lambda\vec{a} = (\lambda a_x, \lambda a_y)$$

6.6.E Vector Equations

Use backward operations to solve equations using vectors:

$$\vec{x} + \vec{a} = \vec{b} \implies \vec{x} = \vec{b} - \vec{a}$$

$$\vec{a} - \vec{x} = \vec{b} \implies \vec{x} = \vec{a} - \vec{b}$$

$$\lambda\vec{x} = \vec{a} \implies \vec{x} = \frac{\vec{a}}{\lambda}$$

6.7 Operations with Algebraic Vectors in R^3

6.7.A 3D Algebraic Vectors

A 3D algebraic vector may be written as:

$$\vec{v} = (v_x, v_y, v_z) \qquad \equiv \qquad \vec{v} = v_x \vec{i} + v_y \vec{j} + v_z \vec{k}$$

and has a magnitude given by:

$$\|\vec{v}\| = \sqrt{{v_x}^2 + {v_y}^2 + {v_z}^2}$$

6.7.B Addition of 3D Algebraic Vectors

$$\vec{a} + \vec{b} = (a_x + b_x, a_y + b_y, a_z + b_z)$$

6.7.C Subtraction of 3D Algebraic Vectors

$$\vec{a} - \vec{b} = (a_x - b_x, a_y - b_y, a_z - b_z)$$

6.7.D Multiplication of 3D Algebraic Vector by a Scalar

$$\lambda \vec{a} = (\lambda a_x, \lambda a_y, \lambda a_z)$$

6.7.E Midpoint

The midpoint of the segment line $\overrightarrow{AB}$ is the point M such that $\overrightarrow{MA} + \overrightarrow{MB} = \vec{0}$

6.7.F Centroid

The centroid of a system of points $P_1, P_2, \ldots, P_n$ is the point C defined by:

$$\overrightarrow{OC} = \frac{\overrightarrow{OP}_1 + \overrightarrow{OP}_2 + \cdots + \overrightarrow{OP}_n}{n}$$

6.7.G Parallelism

$\vec{a} \parallel \vec{b}$ if there exists λ such that $\vec{a} = \lambda \vec{b}$.

$\vec{a}$ and $\vec{b}$ may have the same direction or the opposite direction.

6.7.H Co-linearity

Three points A, B, and C are collinear if $\overrightarrow{AB} \parallel \overrightarrow{BC}$.

6.7.I Linear Dependency

$\vec{a}$, $\vec{b}$, and $\vec{c}$ are linear dependent if there exists λ and μ such that $\vec{c} = \lambda\vec{a} + \mu\vec{b}$. The vectors must be coplanar for this to ever be true.

Unit 7

Applications of Vectors

Chapter 7

Applications of Vectors

7.1 Vectors as Forces

7.1.A Vector Force

The force is a vector and the measurement unit is N.

7.1.B Resultant Force

The vector sum of a system of forces is called the resultant.

$$\vec{R} = \vec{F}_1 + \vec{F}_2 + \cdots + \vec{F}_n$$

7.1.C Algebraic Resultant Force

The scalar components of the resultant force are given by:

$$R_x = F_{1x} + F_{2x} + \cdots + F_{nx}$$

$$R_y = F_{1y} + F_{2y} + \cdots + F_{ny}$$

The magnitude and the direction of the resultant force are given by:

$$\|\vec{R}\| = \sqrt{{R_x}^2 + {R_y}^2}$$

$$\tan\theta = \frac{R_y}{R_x} \quad \equiv \quad \theta = \tan^{-1}\left(\frac{R_y}{R_x}\right)$$

7.1.D Equilibrium

A system of forces is in a state of equilibrium if $\vec{R} = \vec{0}$.

Three forces in equilibrium form a triangle.

Consequently:

- the forces are coplanar
- the largest magnitude is less than or equal to the sum of the other two magnitudes

7.1.E Equilibrant Force

The equilibrant force is the force $\vec{E}$ required to be added to a system of forces with a resultant force $\vec{R}$ such that the new force is at equilibrium.

$$\vec{R} + \vec{E} = \vec{0}$$

$$\vec{E} = -\vec{R}$$

$$\vec{E} = (-R_x, -R_y)$$

7.2 Velocity

7.2.A Velocity

Velocity is a vector and the measurement unit is m/s or km/h.

7.2.B Relative Velocity

The relative velocity of the object B travelling at $\vec{v}_B$ relative to the object A travelling at $\vec{v}_A$ is given by:

$$\vec{v}_{BA} = \vec{v}_B - \vec{v}_A$$

If A is at rest ($\vec{v}_A = \vec{0}$), then $\vec{v}_{BA} = \vec{v}_B$.

7.2.C Boat Velocity

The boat velocity relative to the ground is the vector sum between the boat velocity relative to the water ($\vec{v}_{bw}$) and the water velocity relative to the ground ($\vec{b}_{wg}$):

$$\vec{v}_{bg} = \vec{v}_{bw} + \vec{v}_{bg}$$

7.2.D Plane Velocity

The plane velocity relative to the ground is the vector sum between the plane velocity relative to the air ($\vec{v}_{pa}$) and the air velocity relative to the ground ($\vec{v}_{ag}$):

$$\vec{v}_{pg} = \vec{v}_{pa} + \vec{v}_{ag}$$

7.3 Dot Product of two Geometric Vectors

7.3.A Definition

The dot product of two geometric vectors $\vec{a}$ and $\vec{b}$ with an angle $\theta = \angle(\vec{a}, \vec{b})$ between them (when positioned tail to tail) is a scalar defined by:

$$\vec{a} \cdot \vec{b} = \|\vec{a}\| \|\vec{b}\| \cos\theta$$

By convention $0° \leq \theta \leq 180°$.

7.3.B Properties of Dot Product

- $\vec{a} \cdot \vec{b}$ is a scalar and a real number
- If $\vec{a} \perp \vec{b}$ then $\vec{a} \cdot \vec{b} = 0$ because $\cos 90° = 0$
- If $\vec{a} \cdot \vec{b} = 0$ then $\|\vec{a}\| = 0$ or $\|\vec{b}\| = 0$ or $\vec{a} \perp \vec{b}$
- If $0° < \theta < 90°$ then $\cos\theta > 0$ and $\vec{a} \cdot \vec{b} > 0$

7.4 Dot Product of Algebraic Vectors

7.4.A Dot Product for Standard Unit Vectors

The dot product of the standard unit vectors is given by:

$$\vec{i} \cdot \vec{i} = 1 \qquad \vec{j} \cdot \vec{j} = 1 \qquad \vec{k} \cdot \vec{k} = 1$$

$$\vec{i} \cdot \vec{j} = 0 \qquad \vec{j} \cdot \vec{k} = 0 \qquad \vec{k} \cdot \vec{i} = 0$$

7.4.B Dot Product for Two Algebraic Vectors

$$\vec{a} \cdot \vec{b} = a_x b_x + a_y b_y + a_z b_z$$

7.4.C Angle Between Two Vectors

The angle $\theta = \angle(\vec{a}, \vec{b})$ between two vectors $\vec{a}$ and $\vec{b}$ when positioned tail to tail is given by:

$$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{\|\vec{a}\|\|\vec{b}\|} = \frac{a_x b_x + a_y b_x + a_z b_z}{\sqrt{{a_x}^2 + {a_y}^2 + {a_z}^2}\sqrt{{b_x}^2 + {b_y}^2 + {b_z}^2}}$$

Notes:

- If $\cos\theta = 1$ then $\vec{a}$ and $\vec{b}$ are parallel and have the same direction
- If $\cos\theta = -1$ then $\vec{a}$ and $\vec{b}$ are parallel but have opposite directions
- If $\cos\theta = 0$ then $\vec{a} \perp \vec{b}$
- If $\cos\theta > 0$ then $0° < \theta < 90°$
- If $\cos\theta < 0$ then $90° < \theta < 180°$

7.5 Scalar and Vector Projections

7.5.A Scalar Projection

The scalar projection of the vector $\vec{a}$ onto the vector $\vec{b}$ is a scalar defined as:

$$SProj(\vec{a} \text{ onto } \vec{b}) = \|\vec{a}\| \cos\theta \qquad \text{where } \theta = \angle(\vec{a}, \vec{b})$$

$$SProj(\vec{a} \text{ onto } \vec{b}) \equiv SProj_{\vec{b}}\vec{a}$$

7.5.B Special Cases

Consider two vectors $\vec{a}$ and $\vec{b}$:

- If $\vec{a} \parallel \vec{b}$ and $\vec{a}$ has the same direction as $\vec{b}$ then $SProj_{\vec{b}}\vec{a} = \|\vec{a}\|$
- If $\vec{a} \parallel \vec{b}$ and $\vec{a}$ has the opposite direction of $\vec{b}$ then $SProj_{\vec{b}}\vec{a} = -\|\vec{a}\|$
- If $\vec{a} \perp \vec{b}$ then $SProj_{\vec{b}}\vec{a} = 0$

7.5.C Dot Product and Scalar Projection

$$SProj_{\vec{b}}\vec{a} = \frac{\vec{a} \cdot \vec{b}}{\|\vec{b}\|}$$

$$SProj_{\vec{i}}\vec{a} = a_x$$
$$SProj_{\vec{j}}\vec{a} = a_y$$
$$SProj_{\vec{k}}\vec{a} = a_z$$

7.5.D Vector Projection

The vector projection of the vector $\vec{a}$ onto the vector $\vec{b}$ is a vector defined as:

$$VProj_{\vec{b}}\vec{a} = \|\vec{a}\| \cos\theta \frac{\vec{b}}{\|\vec{b}\|}$$

$$VProj_{\vec{b}}\vec{a} = (SProj_{\vec{b}}\vec{a})\hat{b}$$

7.5.E Dot Product and Vector Projection

The vector projection of the vector $\vec{a}$ onto the vector $\vec{b}$ can be written using the dot product as:

$$VProj_{\vec{b}}\vec{a} = \frac{(\vec{a} \cdot \vec{b})\vec{b}}{\| \vec{b} \|^2}$$

$$VProj_{\vec{i}}\vec{a} = a_x \vec{i}$$
$$VProj_{\vec{j}}\vec{a} = a_y \vec{j}$$
$$VProj_{\vec{k}}\vec{a} = a_z \vec{j}$$

7.6 Cross Product

7.6.A Right Hand System

Also known as the Gugoiu Nation gang symbol, the first three fingers (thumb, index, and middle) on the right hand correspond to the x, y, and z axes respectively.

7.6.B Cork-Screw Rule

The cork-screw rule describes a right hand system based on the cork-screw property: if you rotate the x-axis towards the y-axis using the shortest path, the screw goes in the positive direction of the z-axis.

7.6.C Cross Product

The cross product between two vectors $\vec{a}$ and $\vec{b}$ is a vector quantity denoted by $\vec{a} \times \vec{b}$ having the following properties:

- $\|\vec{a} \times \vec{b}\| = \|\vec{a}\|\|\vec{b}\| \sin \alpha \qquad \text{where } \alpha = \angle(\vec{a}, \vec{b})$
- $\vec{a} \times \vec{b}$ is $\perp$ to both $\vec{a}$ and $\vec{b}$
- the vectors $\vec{a}$, $\vec{b}$, and $\vec{a} \times \vec{b}$ form a right-handed system with $\vec{a}$ as the thumb, $\vec{b}$ as the index finger, and $\vec{a} \times \vec{b}$ as the middle finger.

7.6.D Specific Cases

- If $\vec{a} \parallel \vec{b}$ then $\vec{a} \times \vec{b} = \vec{0}$
- If $\vec{a} \perp \vec{b}$ then $\|\vec{a} \times \vec{b}\|$ is at a maximum for the given magnitudes
- If $\vec{a} = \vec{b}$ then $\vec{a} \times \vec{b} = \vec{0}$

7.6.E Cross Product of Unit Vectors

The cross product of the standard unit vectors are given by:

$$
\begin{array}{ccc}
\vec{i} \times \vec{i} = \vec{0} & \vec{j} \times \vec{j} = \vec{0} & \vec{k} \times \vec{k} = \vec{0} \\
\vec{i} \times \vec{j} = \vec{k} & \vec{j} \times \vec{k} = \vec{i} & \vec{k} \times \vec{i} = \vec{j} \\
\vec{i} \times \vec{k} = \overrightarrow{-j} & \vec{j} \times \vec{i} = \overrightarrow{-k} & \vec{k} \times \vec{j} = \overrightarrow{-i}
\end{array}
$$

7.6.F Cross Product of Two Algebraic Vectors

$$\vec{a} \times \vec{b} = \vec{i}(a_y b_z - a_z b_y) - \vec{j}(a_x b_z - a_z b_x) + \vec{k}(a_x b_y - a_y b_x)$$

7.6.G Properties of Cross Product

- $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$ (anti-commutative property)
- $\lambda(\vec{a} \times \vec{b}) = (\lambda \vec{a}) \times \vec{b} = \vec{a} \times (\lambda \vec{b})$
- $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$ (distributive property)
- $\vec{a} \times \vec{b} = \vec{0} \iff ((\vec{a} = \vec{0}) \vee (\vec{b} = \vec{0}) \vee (\vec{a} \parallel \vec{b}))$
- $\vec{a} \times \vec{0} = \vec{0}$
- $\vec{a} \times \vec{a} = \vec{0}$
- $\vec{a} \cdot (\vec{b} \times \vec{c}) = \vec{b} \cdot (\vec{c} \times \vec{a}) = \vec{c} \cdot (\vec{a} \times \vec{b})$ (mixed product)
- $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{c} \cdot \vec{a})\vec{b} - (\vec{b} \cdot \vec{a})\vec{c}$ (triple cross product)

7.7 Applications of the Dot and Cross Product

7.7.A Work

The work W done by a constant force $\vec{F}$ acting on an object during a displacement $\vec{d}$ is given by:

$$W = \vec{F} \cdot \vec{d} = \|\vec{F}\|\|\vec{d}\| \cos \alpha$$

where

$$\alpha = \angle(\vec{F}, \vec{d})$$

7.7.B Torque

The torque (rotational or turning effect) about the point A, created by a force $\vec{F}$ acting on an object located at the point B is given by:

$$\vec{\tau} = \overrightarrow{AB} \times \vec{F} = \vec{r} \times \vec{F}$$

$$\|\vec{\tau}\| = rF \sin \alpha$$

where

$$\alpha = \angle\left(\vec{F}, \vec{r}\right)$$

$\vec{r}$ is in m

$\vec{F}$ is in N

$\vec{\tau}$ is in N m

7.7.C Parallelogram Area

$$A = \|\vec{a} \times \vec{b}\| = \|\vec{a}\|\|\vec{b}\| \sin \alpha$$
$$\alpha = \angle(\vec{a}, \vec{b})$$

7.7.D Triangle Area

The area of a triangle defined by the vectors $\vec{a}$ and $\vec{b}$ is given by:

$$A = \frac{\|\vec{a} \times \vec{b}\|}{2} = \frac{\|\vec{a}\|\|\vec{b}\| \sin \alpha}{2}$$

$$\alpha = \angle(\vec{a}, \vec{b})$$

7.7.E Parallelepiped Volume

The volume of a parallelepiped defined by the vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ is given by:

$$V = |\vec{c} \cdot (\vec{a} \times \vec{b})| = |\vec{a} \cdot (\vec{b} \times \vec{c})| = |\vec{b} \cdot (\vec{c} \times \vec{a})|$$

Appendix

Appendix A

Equations

Fundamental Limits

$$\lim_{h\to 0}\frac{\sin h}{h}=1$$
$$\lim_{h\to 0}\frac{\cos h-1}{h}=0$$
$$\lim_{h\to 0}\frac{\mathrm{e}^h-1}{h}=1$$

Deriving Rules

$$(x^n)'=nx^{n-1}$$
$$\left(\frac{a}{x^n}\right)'=-\frac{na}{x^{n+1}}$$
$$(\sqrt{x})'=\frac{1}{2\sqrt{x}}$$
$$[f(x)\pm g(x)]'=f'(x)\pm g'(x)$$

$$(fg)'=f'g+fg'$$
$$(f^n)'=nf^{n-1}f'$$

$$\left(\frac{f}{g}\right)'=\frac{f'g-fg'}{g^2}$$

$$(f\circ g)'(x)=f'(g(x))g'(x)$$

Tangent Line

$$m = f'(a)$$

$$y - f(a) = m(x - a) \quad \equiv \quad y = m(x - a) + f(a)$$

Logarithms

$$b^x = y \quad \equiv \quad x = \log_b y$$

$$\log_b x^n = n \log_b x$$

$$\log_b(xy) = \log_b x + \log_b y$$

$$\log_b \left(\frac{x}{y}\right) = \log_b x - \log_b y$$

$$\log_a(x) = \frac{\log_b x}{\log_b a}$$

$$a^x = b^{x \log_b a}$$

Vectors

$$\vec{a} \cdot \vec{b} = \|\vec{a}\|\|\vec{b}\| \cos\theta$$

$$\vec{a} \times \vec{b} = \|\vec{a}\|\|\vec{b}\| \sin\theta$$

$$SProj_{\vec{b}}\vec{a} = \|\vec{a}\| \cos\theta = \frac{\vec{a} \cdot \vec{b}}{\|\vec{b}\|}$$

$$VProj_{\vec{b}}\vec{a} = SProj_{\vec{b}}\vec{a}\hat{b} = \frac{(\vec{a} \cdot \vec{b})\vec{b}}{\|\vec{b}\|^2}$$

$$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{\|\vec{a}\|\|\vec{b}\|}$$

Pray to Gugoiu and it will be okay!