# MaCS Calculus and Vectors Exam Study Guide

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# Contents

# Unit 1

# Equations of Lines and Planes

# 8.1 Vector and Parametric Equations of a Line in $\mathbb{R}^2$

## A Vector Equation of a Line in $\mathbb{R}^2$

Consider the line $L$ that passes through the point $P_0(x_0, y_0)$ and is parallel to the vector $\vec{u}$. The point $P(x, y)$ is a *generic point* on the line.

$$\begin{aligned} \overrightarrow{P_0P} &= t\vec{u} \\ \overrightarrow{OP} - \overrightarrow{OP_0} &= t\vec{u} \\ \vec{r} - \vec{r_0} &= t\vec{u} \end{aligned}$$

The *vector equation* of the line is:

$$\vec{r} = \vec{r_0} + t\vec{u} \mid t \in \mathbb{R}$$

Where:

- $\vec{r} = \overrightarrow{OP}$ is the *position vector* of a *generic point* $P$ on the line.
- $\vec{r_0} = \overrightarrow{OP_0}$ is the *position vector* of a *specific point* $P_0$ on the line.
- $\vec{u}$ is a vector parallel to the line called the *direction vector* of the line.
- $t$ is a *real number* corresponding to the generic point $P$.

**Note: The vector equation of a line is *not unique*. It depends on the specific point $P_0$ and on the direction vector $\vec{u}$ that are used.**

## B Parametric Equations of a Line in $\mathbb{R}^2$

We can rewrite the vector equation of a line:

$$\vec{r} = \vec{r_0} + t\vec{u} \mid t \in \mathbb{R}$$

as:

$$(x, y) = (x_0, y_0) + t(u_x, u_y) \mid t \in \mathbb{R}$$

Split this vector equation into the *parametric equations* of a line in $\mathbb{R}^2$:

$$\begin{cases} x = x_0 + tu_x \\ y = y_0 + yu_y \end{cases} \quad t \in \mathbb{R}$$

## C Parallel Lines

Two lines $L_1$ and $L_2$ with direction vectors $\overrightarrow{u_1}$ and $\overrightarrow{u_2}$ are *parallel* ($L_1 \parallel L_2$) if:

$$\overrightarrow{u_1} \parallel \overrightarrow{u_2}$$

or, there exists $k \in \mathbb{R}$ such that:

$$\overrightarrow{u_2} = k\overrightarrow{u_1}$$

or:

$$\overrightarrow{u_1} \times \overrightarrow{u_2} = \overrightarrow{0}$$

or scalar components are *proportional*:

$$\frac{u_{2x}}{u_{1x}} = \frac{u_{2u}}{u_{1u}} = k$$

## D Perpendicular Lines

Two lines $L_1$ and $L_2$ with direction vectors $\overrightarrow{u_1}$ and $\overrightarrow{u_2}$ are *perpendicular* ($L_1 \perp L_2$) if:

$$\overrightarrow{u_1} \perp \overrightarrow{u_2}$$

or:

$$\overrightarrow{u_1} \cdot \overrightarrow{u_2} = 0$$

or:

$$u_{1x}u_{2x} + u_{1y}u_{2y} = 0$$

## E 2D Perpendicular Vectors

Given a 2D vector $\vec{u} = (a, b)$, two 2D vectors perpendicular to $\vec{u}$ are $\vec{v} = (-b, a)$ and $\vec{w} = (b, -a)$.

Indeed:

$$\vec{u} \cdot \vec{v} = (a, b) \cdot (-b, a) = -ab + ab = 0 \implies \vec{u} \perp \vec{v}$$

## F Special Lines

A line *parallel* to the $x$-axis has a direction vector in the form $\vec{u} = (u_x, 0) \mid u_x \neq 0$.

A line *parallel* to the $y$-axis has a direction vector in the form $\vec{u} = (0, u_y) \mid u_y \neq 0$.

## 8.2 Cartesian Equation of a Line

### A Symmetric Equation

The parametric equations of a line in $\mathbb{R}^2$:

$$\begin{cases} x = x_0 + tu_x \\ y = y_0 + tu_y \end{cases} \quad t \in \mathbb{R}$$

may be written as:

$$\frac{x - x_0}{u_x} = \frac{y - y_0}{u_y} = t \mid t \in \mathbb{R}$$

The *symmetric equation* of the line is (if it exists):

$$\frac{x - x_0}{u_x} = \frac{y - y_0}{u_y}$$

**Note: The symmetric equations only exists if $u_x \neq 0$ and $u_y \neq 0$.**

### B Normal Equation

Consider a line $L$ that passes through the specific point $P_0(x_0, y_0)$ and has the *direction vector* $\vec{u} = (u_x, u_y)$.

The vectors $\vec{n} = (-u_y, u_x) = (A, B)$ or $\vec{n} = (u_y, -u_x) = (A, B)$ are perpendicular to the vector $\vec{u}$ and so they are perpendicular to the line $L$. These are called *normal vectors* to the line $L$.

Let $P(x, y)$ be a generic point on the line $L$. So:

$$\overrightarrow{P_0P} \parallel \vec{u} \implies \overrightarrow{P_0P} \perp \vec{n} \implies \overrightarrow{P_0P} \cdot \vec{n} = 0$$

$$(\vec{r} - \vec{r_0}) \cdot \vec{n} = 0$$

The *normal equation* of a line is given by:

$$(\vec{r} - \vec{r_0}) \cdot \vec{n} = 0$$

### C Cartesian Equation

The normal equations can be written as:

$$\begin{aligned} \vec{r} \cdot \vec{n} - \vec{r_0} \cdot \vec{n} &= 0 \\ (x, y) \cdot (A, B) - (x_0, y_0) \cdot (A, B) &= 0 \\ Ax + By - Ax_0 - By_0 &= 0 \\ Ax + By + C &= 0 \qquad \text{where } C = -Ax_0 - By_0 \end{aligned}$$

The *Cartesian equation* of a line is given by:

$$Ax + By + C = 0$$

where $\vec{n} = (A, B)$ is a *normal vector* and the constant $C$ depends on a specific point of the line.

## D Slope $y$-intercept Equation

Solve the symmetric equation of a line:

$$\frac{x - x_0}{u_x} = \frac{y - y_0}{u_y} \mid t \in \mathbb{R}$$

for $y$:

$$\begin{aligned} y - y_0 &= u_y \frac{x - x_0}{u_x} \\ y &= \frac{u_y}{u_x} x + y_0 - \frac{u_y}{u_x} x_0 \end{aligned}$$

The *slope $y$-intercept equation* of a line in $\mathbb{R}^2$ is given by:

$$y = mx + b$$

$$m = \frac{u_y}{u_x}$$

where $m$ is the *slope* and $b$ is the *$y$-intercept* which depends on a specific point of the line.

## E Angle between Two Lines

The *angle* between two lines is determined by the angle between the *direction vectors*:

$$\cos\theta = \frac{\vec{u_1} \cdot \vec{u_2}}{\|\vec{u_1}\| \|\vec{u_2}\|}$$

**Note: There are two pairs of equal angles between the two lines. There is a pair of the angle $\theta_1$, and a pair of the angle $\theta_2$.** $\theta_1 + \theta_2 = 180°$

## 8.3 Vector, Parametric, and Symmetric Equations of a Line in $\mathbb{R}^3$

### A Vector Equation

The vector equation of the line is:

$$\vec{r} = \vec{r_0} + t\vec{u} \mid t \in \mathbb{R}$$

where:

- $\vec{r} = \overrightarrow{OP}$ is the position vector of a *generic* point $P$ on the line.
- $\vec{r_0} = \overrightarrow{OP_0}$ is the position vector of a *specific* point $P_0$ on the line.
- $\vec{u}$ is a vector parallel to the line called the *direction vector* of the line.
- $t$ is a *real number* corresponding to the generic point $P$.

### B Specific Lines

A line is parallel to the $x$-axis if $\vec{u} = (u_x, 0, 0) \mid u_x \neq 0$. In this case, the line is also *perpendicular to the yz-plane.*

A line with $\vec{u} = (0, u_y, u_z) \mid u_y \neq 0 \wedge u_z \neq 0$ is *parallel to the yz-plane.*

### C Parametric Equations

Rewrite the vector equation of a line:

$$\vec{r} = \vec{r_0} + t\vec{u} \mid t \in \mathbb{R}$$

as:

$$(x, y, z) = (x_0, y_0, z_0) + t(u_x, u_y, u_z) \mid t \in \mathbb{R}$$

The *parametric equations* of a line in $\mathbb{R}^3$ are:

$$\begin{cases} x = x_0 + tu_x \\ y = x_0 + tu_y \\ z = x_0 + tu_z \end{cases} \quad t \in \mathbb{R}$$

## D Symmetric Equations

The parametric equations of a line may be written as:

$$\begin{cases} x = x_0 + tu_x \\ y = x_0 + tu_y \\ z = x_0 + tu_z \end{cases} \quad t \in \mathbb{R}$$

From here, the *symmetric equations* of the line are:

$$\frac{x - x_0}{u_x} = \frac{y - y_0}{u_y} = \frac{z - z_0}{u_z}$$

$$u_x \neq 0 \quad u_y \neq 0 \quad u_z \neq 0$$

## E Intersections

A line *intersects the x-axis* when $y = z = 0$.

A line *intersects the xy-plane* when $z = 0$.

## 8.4 Vector and Parametric Equations of a Plane

### A Planes

A plane may be determined by points and lines. There are four main possibilities:

1. Plane determined by three points.
2. Plane determined by two parallel lines.
3. Plane determined by two intersecting lines.
4. Plane determined by a point and a line.

### B Vector Equation of a Plane

Consider a plane $\pi$.

Two vectors $\vec{u}$ and $\vec{v}$, parallel to the plane $\pi$ but not parallel to each other, are called *direction vectors* of the plane $\pi$.

The vector $\overrightarrow{P_0P}$ from a specific point $P_0(x_0, y_0, z_0)$ to a generic point $P(x, y, z)$ of the plane is a *linear combination* of direction vectors $\vec{u}$ and $\vec{v}$:

$$\overrightarrow{P_0P} - s\vec{u} + t\vec{v} \mid s, t \in \mathbb{R}$$

The *vector equation* of the plane is:

$$\pi : \vec{r} = \vec{r_0} + s\vec{u} + t\vec{v} \mid s, t \in \mathbb{R}$$

### C Parametric Equations of a Plane

We write the vector equation of the plane as:

$$(x, y, z) = (x_0, y_0, z_0) + s(u_x, u_y, u_z) + t(v_x, v_y, v_z)$$

or:

$$\begin{cases} x = x_0 + su_x + tv_x \\ y = y_0 + su_y + tv_y \\ z = z_0 + su_z + tv_z \end{cases} \quad s, t \in \mathbb{R}$$

These are the *parametric equations* of a plane.

## 8.5 Cartesian Equation of a Plane

### A Normal Equation of a Plane

A plane may be determined by a *point* $P_0(x_0, y_0, z_0)$ and a *vector* perpendicular to the plane $\vec{n}$ called the *normal vector*.

If $P(x, y, z)$ is a generic point on the plane, then:

$$\overrightarrow{P_0P} \perp \vec{n}$$

and:

$$\overrightarrow{P_0P} \cdot \vec{n} = 0$$

This is the *normal equation* of a plane.

### B Cartesian Equation of a Plane

We write the normal vector of a plane in the form:

$$\vec{n} = (A, B, C)$$

Then, the normal equation may be written as:

$$(x - x_0, y - y_0, z - z_0) \cdot (A, B, C) = 0$$
$$Ax + By + Cz - Ax_0 - By_0 - Cz_0 = 0$$

or:

$$Ax + By + Cz + D = 0$$

which is called the *Cartesian equation* of a plane.

**Note: A normal vector to the plane is:**

$$\vec{n} = \vec{u} \times \vec{v}$$

**where $\vec{u}$ and $\vec{v}$ are the direction vectors of the plane.**

### C Angle between Two Planes

The *angle* between two planes is defined as the angle between their *normal vectors*:

$$\cos\theta = \frac{\vec{n_1} \cdot \vec{n_2}}{\|\vec{n_1}\| \|\vec{n_2}\|}$$

**Note: Using this formula, you may get an *acute* or an *obtuse* angle depending on the normal vectors which are used.**

# Unit 2

# Relationships between Points, Lines, and Planes

## 9.1 Intersection of Two Lines

### A Relative Position of Two Lines

Two lines may be:

1. Parallel and distinct.
2. Parallel and coincident.
3. Intersecting (not parallel).
4. Skew (not parallel, not intersecting).

### B Intersection of Two Lines (Algebraic Method)

The point of intersection of two lines $L_1 : \overrightarrow{r} = \overrightarrow{r_{01}} + t\overrightarrow{u_1} \mid t \in \mathbb{R}$ and $L_2 : \overrightarrow{r} = \overrightarrow{r_{02}} + s\overrightarrow{u_2} \mid s \in \mathbb{R}$ is given by the *solution* of the following system of equations (if it exists):

$$\begin{cases} x_{01} + tu_{x1} = x_{02} + su_{x2} \\ y_{01} + tu_{y1} = y_{02} + su_{y2} \qquad s, t \in \mathbb{R} \\ z_{01} + tu_{z1} = z_{02} + su_{z2} \end{cases}$$

**Hint: Solve by *substitution* or *elimination* the system of two equations and *check* if the third is satisfied.**

### C Unique Solution

If by solving the system you end by getting a *unique* value for $t$ and $s$ satisfying this system, then the lines have a *unique point of intersection*. To get this point, substitute either the $t$ value into the line $L_1$ equation or substitute the $s$ value into the line $L_2$ equation.

### D Infinite Number of Solutions

If by solving the system you end by getting two true statements (like $2 = 2$) and one equation in $s$ and $t$, then there exist an *infinite number of solutions* of the system. Therefore the lines intersect at an *infinite number of points*. In this case the lines are parallel and coincident.

## E No Solution (Parallel Lines)

If by solving the system you get at least one *false* statement (like $0 = 1$) then the system has *no solution*. Therefore, the lines have *no point of intersection*. If, in addition, the lines are parallel ($\overrightarrow{u_1} \times \overrightarrow{u_2} = \overrightarrow{0}$), then the lines are *parallel and distinct*.

## F No Solution (Skew Lines)

If by solving the system you get at least one *false* statement (like $0 = 1$) then the system has *no solution*. Therefore, the lines have *no point of intersection*. If, in addition, the lines are *not parallel* ($\overrightarrow{u_1} \times \overrightarrow{u_2} \neq \overrightarrow{0}$), then the lines are *skew*.

## G Classifying Lines (Vector Method)

Parallel lines

$\overrightarrow{u_1} \times \overrightarrow{u_2} = \overrightarrow{0}$

$(\overrightarrow{r_{01}} - \overrightarrow{r_{02}}) \times \overrightarrow{u_1} = \overrightarrow{0}$ — Parallel coincident lines

$(\overrightarrow{r_{01}} - \overrightarrow{r_{02}}) \times \overrightarrow{u_1} \neq \overrightarrow{0}$ — Parallel distinct lines

Nonparallel lines

$\overrightarrow{u_1} \times \overrightarrow{u_2} \neq \overrightarrow{0}$

$(\overrightarrow{r_{01}} - \overrightarrow{r_{02}}) \cdot (\overrightarrow{u_1} \times \overrightarrow{u_2}) = 0$ — Nonparallel intersecting lines

$(\overrightarrow{r_{01}} - \overrightarrow{r_{02}}) \cdot (\overrightarrow{u_1} \times \overrightarrow{u_2}) \neq 0$ — Nonparallel skew lines

## 9.2 Intersection of a Line with a Plane

### A Relative Position of a Line and a Plane

There are three possible situations:

1. The line *intersects* the plane at a single point.
$$P = L \cap \pi$$
2. The line *lies* on the plane. There are an infinite number of points of intersection.
$$L = L \cap \pi$$
3. The line is *parallel* to the plane but *distinct*. There is no point of intersection.
$$L \cap \pi = \varnothing$$

### B Intersection of a Line and a Plane (Algebraic Method)

To get the intersection between a line $L$ and a plane $\pi$:

1. *Substitute* the parametric equations of the line
$$L : \begin{cases} x = x_0 + tu_x \\ y = y_0 + tu_y \\ z = z_0 + tu_z \end{cases} \quad t \in \mathbb{R}$$
into the Cartesian equation of the plane
$$\pi : Ax + By + Cz + D = 0$$
to get the equation:
$$A(x_0 + tu_x) + B(y_0 + tu_y) + C(z_0 + tu_z) + D = 0 \tag{i}$$
2. *Solve* (if possible) the equation (i) for the parameter $t$.
3. *Substitute* the value of the parameter $t$ into the parametric equations of the line to get the point of intersection.

### C Unique Solution (Point Intersection)

In this case, by solving the equation you get a *unique value* for the parameter $t$. Therefore, there is a unique *point of intersection* between the line and the plane.

$$P = L \cap \pi$$

The line *intersects* the plane at a unique point.

## D Infinite Number of Solutions (Line Intersection)

In this case, by solving the equation (i) you get the equation:

$$0t = 0$$

which has an *infinite number of solutions*. Therefore, there are an *infinite number of points of intersection*.

$$L = L \cap \pi$$

The line *lies* on the plane.

## E No Solution (No Intersection)

In this case, by solving the equation (i) you get a false statement like:

$$0t = 1$$

The equation *does not have any solution* and therefore there is *no point of intersection* between the line and the plane.

$$L \cap \pi = \varnothing$$

The line is *parallel* to the plane and *does not lie* on the plane.

## F Classifying Lines

Consider the line $L : \vec{r} = \vec{r_0} + t\vec{u} \mid t \in \mathbb{R}$, where $P_0(x_0, y_0, z_0)$ is a specific point on the line, and the plane $\pi : Ax + By + Cz + D = 0$, where $\vec{n} = (A, B, C)$ is a normal vector to the plane.

1. If $\vec{n} \cdot \vec{u} \neq 0$ the line *intersects* the plane at a unique point.

$$P = L \cap \pi$$

2. If $\vec{n} \cdot \vec{u} = 0$ and $Ax_0 + By_0 + Cz_0 + D = 0$ then the line *lies* on the plane.

$$L = L \cap \pi$$

3. If $\vec{n} \cdot \vec{u} = 0$ and $Ax_0 + By_0 + Cz_0 + D \neq 0$ then the line is *parallel* to the plane but *does not lie* on the plane.

$$L \cap \pi = \varnothing$$

**Note. By solving the equation (i) for $t$ you will end by getting the same cases and conditions as above.**

## 9.3 Intersection of Two Planes

### A Relative Position of Two Planes

Two planes may be:

1. Intersecting (into a line)
$$L = \pi_1 \cap \pi_2$$
2. Coincident
$$\pi_1 = \pi_1 \cap \pi_2 = \pi_2$$
3. Distinct
$$\pi_1 \cap \pi_2 = \varnothing$$

### B Intersection of Two Planes

Consider two planes given by their Cartesian equations:

$$\pi_1 = A_1x + B_1y + C_1z + D_1 = 0$$

$$\pi_2 = A_2x + B_2y + C_2z + D_2 = 0$$

To find the point(s) of intersection between two planes, *solve* the system of equations formed by their Cartesian equations:

$$\begin{cases}\pi_1 = A_1x + B_1y + C_1z + D_1 = 0\pi_2 = A_2x + B_2y + C_2z + D_2 = 0\end{cases} \tag{ii}$$

There are *two* equations and *three* unknowns. **Notes:**

1. A normal vector to the plane $\pi_1$ is $\overrightarrow{n_1} = (A_1, B_1, C_1)$ and a normal vector to the plane $\pi_2$ is $\overrightarrow{n_2} = (A_2, B_2, C_2)$.
2. If the planes are *parallel* then the coefficients $A$, $B$, and $C$ are *proportional.*
3. If the planes are *coincident* then the coefficients $A$, $B$, $C$, and $D$ are *proportional.*
4. A system of equations is called *compatible* if there is *at least* one solution. A system of equations is called *incompatible* if there is *no solution.*

### C Nonparallel Planes (Line Intersection)

In this case:
$$L = \pi_1 \cap \pi_2$$

- The coefficients $A$, $B$, and $C$ in the scalar equations are *not proportional.*

- The normal vectors are *not parallel*: $\overrightarrow{n_1} \times \overrightarrow{n_2} \neq \overrightarrow{0}$.
- By solving the system (ii) you will be able to find two variables in terms of the third variable.
- There are an *infinite number of solutions* and therefore an *infinite number of points of intersection*.
- The intersection is a *line* and a *direction vector* for this line is $\overrightarrow{u} = \overrightarrow{n_1} \times \overrightarrow{n_2}$.

## D Coincident Planes (Plane Intersection)

In this case:

$$\pi_1 = \pi_1 \cap \pi_2 = \pi_2$$

- The planes are *parallel* and *coincident*.
- The coefficients $A$, $B$, $C$, and $D$ in the scalar equations are *proportional*.
- One equation in the system (ii) is a *multiple* of the other equation and does not contain additional information (the equations are equivalent).
- By solving the system of equations (ii), you get a *true* statement (like $0 = 0$).
- There are an *infinite number of solutions* and therefore an *infinite number of points of intersection*.
- The intersection is a *plane*.

## E Parallel and Distinct Planes (No Intersection)

In this case:

$$\pi_1 \cap \pi_2 = \varnothing$$

- The planes are *parallel* and *distinct*.
- The coefficients $A$, $B$, and $C$ in the scalar equations are *proportional* but the coefficients $A$, $B$, $C$, and $D$ are *not proportional*.
- By solving the system (ii) you get a *false* statement (like $0 = 1$).
- There is *no solution* and therefore *no point of intersection* between the two planes.

## 9.4 Intersection of Three Planes

### A Intersection of Three Planes

Consider three planes given by their Cartesian equations:

$$\pi_1 : A_1x + B_1y + C_1z + D_1 = 0$$

$$\pi_2 : A_2x + B_2y + C_2z + D_2 = 0$$

$$\pi_3 : A_3x + B_3y + C_3z + D_3 = 0$$

The point(s) of *intersection* of these planes is (are) related by to the *solution(s)* of the following system of equations:

$$\begin{cases} A_1x + B_1y + C_1z + D_1 = 0 \\ A_2x + B_2y + C_2z + D_2 = 0 \\ A_3x + B_3y + C_3z + D_3 = 0 \end{cases} \tag{iii}$$

There are *three* equations and *three* unknowns. You may use *substitution* or *elimination* to solve this system.

### B Unique Solution (Point Intersection — Noncoplanar Normal Vectors)

In this case:

$$P = \pi_1 \cap \pi_2 \cap \pi_3$$

- The planes *intersect* into a *single* point.
- The *normal vectors* are *not coplanar*:

$$\overrightarrow{n_1} \cdot (\overrightarrow{n_2} \times \overrightarrow{n_3}) \neq 0$$

- By solving the system (iii), you get a *unique solution* for $x$, $y$, and $z$.

### C Infinite Number of Solutions (Line Intersection — Nonparallel Planes and Coplanar Normal Vectors)

In this case:

$$L = \pi_1 \cap \pi_2 \cap \pi_3$$

- The planes are *not parallel* but their normal vectors are *coplanar*:

$$\overrightarrow{n_1} \cdot (\overrightarrow{n_2} \times \overrightarrow{n_3}) = 0$$

- The intersection is a *line*.
- One scalar equation is a *combination* of the other two equations.
- By solving the system (iii), you may express two variables in terms of the third one using two equations.

## D Infinite Number of Solutions (Line Intersection — Two Coincident Planes and One Intersecting Plane)

In this case:

$$L = \pi_1 \cap \pi_2 \cap \pi_3$$

- *Two* planes are *coincident* and the third plane is *not parallel* to the coincident planes.
- The intersection is a *line*.
- Two scalar equations are *equivalent*. The *coefficients* $A$, $B$, $C$, and $D$ are *proportional* for these two equations.
- You may express two variables in terms of the third one using two nonequivalent equations.

## E Infinite Number of Solutions (Plane Intersection — Three Coincident Planes)

In this case:

$$\pi_1 \cap \pi_2 \cap \pi_3 = \pi_1 = \pi_2 = \pi_3$$

- The coefficients $A$, $B$, $C$, and $D$ are *proportional* for all three equations.
- Any point of one plane is also a point on the other two planes.
- The intersection is a *plane*.

## F No Solution (Parallel and Distinct Planes)

In this case:

$$\pi_1 \cap \pi_2 \cap \pi_3 = \varnothing$$

- There are three *parallel* and *distinct* planes.
- There is *no point of intersection*.
- There is *no solution* for the system of equations (the system of equations is *incompatible*).

- The coefficients $A$, $B$, and $C$ are *proportional* but the coefficients of $A$, $B$, $C$, and $D$ are *not proportional*.
- By solving the system (iii) you get *false* statements (like $0 = 1$).

## G No Solution (H Configuration)

In this case:

$$\pi_1 \cap \pi_2 \cap \pi_3 = \varnothing$$

- Two planes are *parallel and distinct* and the third plane is *intersecting*.
- There is *no point of intersection*.
- The coefficients $A$, $B$, and $C$ are proportional for two planes.
- There is *no solution* for the system of equations (the system of equations is *incompatible*).
- By solving the system (iii) you get *false* statements (like $0 = 1$).

## H No Solution (Three Parallel Planes but only Two Coincident Planes)

In this case:

$$\pi_1 \cap \pi_2 \cap \pi_3 = \varnothing$$

- Three planes are *parallel* but only two are *coincident*.
- The coefficients of $A$, $B$, and $C$ are *proportional* for all equations but the coefficients $A$, $B$, $C$, and $D$ are *proportional* only for two planes.
- There is *no solution* for the system of equations (the system of equations is *incompatible*).
- By solving the system (iii) you get *false* statements (like $0 = 1$).

## I No Solution (Delta Configuration)

In this case:

$$\pi_1 \cap \pi_2 \cap \pi_3 = \varnothing$$

- The planes are *not parallel* (the coefficients $A$, $B$, and $C$ are not *proportional*).
- The normal vectors are *coplanar* ($\overrightarrow{n_1} \cdot (\overrightarrow{n_2} \times \overrightarrow{n_3}) = 0$).
- There is *no point of intersection* between all three planes.

- There is *no solution* for the system of equations (the system of equations is *incompatible*).
- By solving the system (iii) you get *false* statements (like $0 = 1$).

## 9.5 Distance from a Point to a Line

### A Distance from a Point to a Line in $\mathbb{R}^2$

Let $L : Ax + By + C = 0$ be a line in $\mathbb{R}^2$, $P_1(x_1, y_1)$ be a *generic point* on the $xy$-plane and $P_0(x_0, y_0)$ be a specific point on this line, so: $Ax_0 + By_0 + C = 0$.

The *distance* $d$ between the point $P_1(x_1, y_1)$ to the line $L$ is given by (*scalar projection* of $\overrightarrow{P_0P_1}$ onto the normal vector $\vec{n}$):

$$d = \frac{\left|\overrightarrow{P_0P_1} \cdot \vec{n}\right|}{\|\vec{n}\|} \tag{iv}$$

Using $\vec{n} = (A, B)$, $\|\vec{n}\| = \sqrt{A^2 + B^2}$ and:

$$\begin{aligned}
\overrightarrow{P_0P_1} \cdot \vec{n} &= (x_1 - x_0, y_1 - y_0) \cdot (A, B) \\
&= A(x_1 - x_0) + B(y_1 - y_0) \\
&= Ax_1 + By_1 - Ax_0 - By_0 \\
&= Ax_1 + By_1 + C
\end{aligned}$$

the formula (iv) may be written as:

$$d = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}} \tag{v}$$

### B Distance from a Point to a Line in $\mathbb{R}^3$

Let $L : \vec{r} = \vec{r_0} + t\vec{u} \mid t \in \mathbb{R}$ be a line defined by its vector equation and $P_0(x_0, y_0, z_0)$ be a specific point on this line.

The distance $d$ from a point $P_1(x_1, y_1, z_1)$ to the line $L$ may be found using:

$$d = \left\|\overrightarrow{P_0P_1}\right\| \sin\alpha \tag{vi}$$

where $\alpha$ is the angle formed by the intersection of $\overrightarrow{P_0P_1}$ and $\vec{u}$.

Because $\left\|\overrightarrow{P_0P_1} \times \vec{u}\right\| = \left\|\overrightarrow{P_0P_1}\right\| \|\vec{u}\| \sin\alpha$, the formula (vi) can also be written as:

$$d = \frac{\left\|\overrightarrow{P_0P_1} \times \vec{u}\right\|}{\|\vec{u}\|} \tag{vii}$$

**Note: The formula** (vii) **may be applied also in $\mathbb{R}^2$ by considering the third component $z = 0$.**

## C Distance between Two Parallel Lines

To find the *distance* between two parallel lines:

1. Find a *specific point* on one of these lines.
2. Find the distance from that specific point to the other line using one of the relations above.

## D Perpendicular Line from a Point to a Line

Let $L : \vec{r} = \vec{r_0} + t\vec{u} \mid t \in \mathbb{R}$ be a line defined by its vector equation and $P(x, y, z)$ be a generic point in $\mathbb{R}^3$.

The line perpendicular to the line $L$ that passes through the point $P$ is called the *perpendicular line* and intersects the line $L$ at a point $F$ called the *foot* of the perpendicular line.

The foot $F$ of the perpendicular line may be found from the equation (because $\overrightarrow{PF} \perp \vec{u}$):

$$\overrightarrow{PF} \cdot \vec{u} = 0$$

A *vector equation* of the perpendicular line is:

$$\vec{r} = \overrightarrow{OP} + s\overrightarrow{PF} \mid s \in \mathbb{R}$$

## E Shortest Distance between Two Skew Lines

Two skew lines lie into *two parallel planes*. The vector $\vec{u_1} \times \vec{u_2}$ is *perpendicular* to both lines and therefore perpendicular to parallel planes the lines lie on.

The *shortest distance* between two skew lines $L_1 : \vec{r} = \overrightarrow{r_{01}} + t\vec{u_1} \mid t \in \mathbb{R}$ and $L_2 : \vec{r} = \overrightarrow{r_{02}} + s\vec{u_2} \mid s \in \mathbb{R}$ is given by the *scalar projection* of the vector $\overrightarrow{r_{01}} - \overrightarrow{r_{02}}$ onto the vector $\vec{u_1} \times \vec{u_2}$:

$$d = \frac{\left|(\overrightarrow{r_{01}} - \overrightarrow{r_{02}}) \cdot (\vec{u_1} \times \vec{u_2})\right|}{\|\vec{u_1} \times \vec{u_2}\|} \tag{viii}$$

## 9.6 Distance from a Point to a Plane

### A Distance from a Point to a Plane (I)

Consider a plane $\pi$ with a *normal vector* $\vec{n}$ and a point $P_0(x_0, y_0, z_0)$ on this plane. The *distance* from a point $P_1(x_1, y_1, z_1)$ to the plane $\pi$ is given by the *scalar projection* of the vector $\overrightarrow{P_0P_1}$ onto the normal vector $\vec{n}$:

$$d = \frac{\left|\overrightarrow{P_0P_1} \cdot \vec{n}\right|}{\|\vec{n}\|} \tag{ix}$$

### B Distance from a Point to a Plane (II)

If the plane $\pi$ is given by the *Cartesian equation* $\pi : Ax + By + Cz + D = 0$, then the *distance* from a point $P_1(x_1, y_1, z_1)$ to the plane is given by:

$$d = \frac{|Ax_1 + By_1 + C_z + D|}{\sqrt{A^2 + B^2 + C^2}} \tag{x}$$

Indeed,

$$P_0 \in \pi \implies Ax_0 + By_0 + Cz_0 + D = 0$$

$$\begin{aligned} \overrightarrow{P_0P_1} \cdot \vec{n} &= (x_1 - x_0, y_1 - y_0, z_1 - z_0) \cdot (A, B, C) \\ &= Ax_1 + By_1 + Cz_1 - Ax_0 - By_0 - Cz_0 \\ &= Ax_1 + By_1 + Cz_1 + D \end{aligned}$$

### C Distance between Two Parallel Planes

To get the *distance* between *two parallel planes*:

1. Find a specific point into one of these planes.
2. Find the distance between that specific point and the other plane using one of the formulas above.

# AP Preparation Differentiability Review

## 1.4 Limit of a Function

### A One-Sided Limits

The behaviour of the function $y = f(x)$ near $x = a$ is described by three numbers:

1. The left hand limit:
$$L = \lim_{x \to a^-} f(x)$$
the limit of the function $f(x)$ as $x$ approaches $a$ from the left.
2. The value of the function at $x = a$:
$$f(a)$$
3. The right hand limit:
$$R = \lim_{x \to a^+} f(x)$$
the limit of the function $f(x)$ as $x$ approaches $a$ from the right.

**Notes:**

1. In order to exist, both the left and right hand limits must be numbers.
2. If either the left or right hand limit is not a number, then the limit does not exist (DNE).
3. Infinite limits (like $\infty$ or $-\infty$) are not considered numbers but they are used to give information about the behaviour of a function near the number $x = a$.

### B Limit

The limit of a function $y = f(x)$ exists at $x = a$ if:

$$L \text{ and } R \text{ exist and } L = R$$

In this case we write:

$$\lim_{x \to a} f(x)$$

the limit of the function $f(x)$ as $x$ approaches $a$.

**Note: The function may or may not be defined at $x = a$.**

## C Substitution

If the function is defined by a formula (algebraic expression) then the limit of the function at a number $x = a$ may be determined by substitution:

$$\lim_{x \to a} f(x) = f(a)$$

**Notes:**

1. In order to use substitution, the function must be defined on both sides of the number $x = a$.
2. Substitution does not work if you get one of the following 7 indeterminate cases:

$$\infty - \infty \qquad 0 \times \infty \qquad \frac{0}{0} \qquad \frac{\infty}{\infty} \qquad 1^{\infty} \qquad \infty^{0} \qquad 0^{0}$$

## D Piecewise defined functions (AP only)

If the function changes the formula at $x = a$ then:

1. Use the appropriate formula to find the left-hand and right-hand limits.
2. Compare the left-hand and right-hand limits to conclude about the limit of the function at $x = a$.

Example:

$$f(x) = \begin{cases} f_1(x) \mid x < a \\ f_2(x) \mid x > a \end{cases}$$

At $x = a$:

$$L = f_1(a) \qquad R = f_2(a)$$

## E Limits: Numerical Approach (AP only)

The limit of a function $y = f(x)$ at a number $x = a$ may be estimated numerically. To do that:

1. Use a sequence of numbers $x$ approaching $x = a$ from the left and from the right.
2. Find the value of the function at each number $x$.
3. Analyze the values and make a conclusion (guess the limit).
4. Be careful at the “difference catastrophe”.

## F Limit: Informal Definitions (AP only)

**Left-Hand Limit** If the values of $y = f(x)$ can be made arbitrarily close to $L$ by taking $x$ sufficiently close to $a$ with $x < a$, then:

$$\lim_{x \to a^-} f(x) = L$$

**Right-Hand Limit** If the values of $y = f(x)$ can be made arbitrarily close to $R$ by taking $x$ sufficiently close to $a$ with $x > a$, then:

$$\lim_{x \to a^+} f(x) = R$$

**Limit** If the values of $y = f(x)$ can be made arbitrarily close to $l$ by taking $x$ sufficiently close to $a$ from both sides, then:

$$\lim_{x \to a} f(x) = l$$

**Infinite Limit** If the values of $y = f(x)$ can be made arbitrarily large by taking $x$ sufficiently close to $a$ from both sides, then:

$$\lim_{x \to a} f(x) = \infty$$

## 1.6 Continuity

### A Continuity

A function $y = f(x)$ is continuous at a number $x = a$ if

$$L = R = f(a)$$

where:

$L = \lim_{x \to a^-} f(x)$ is the left-hand limit at $x = a$.

$R = \lim_{x \to a^+} f(x)$ is the right-hand limit at $x = a$.

$f(a)$ is the value of the function at $x = a$.

**Note: A function is continuous if it can be drawn without lifting your pencil from the paper.**

### B Discontinuity

If $y = f(x)$ is not continuous at $x = a$ then we say: "$y = f(x)$ is discontinuous at $x = a$" or "$y = f(x)$ has a discontinuity at $x = a$".

### C Removable Discontinuity

A function $y = f(x)$ has a removable discontinuity at $x = a$ if:

1. $L = R = \lim_{x \to a} f(x)$ exists
2. $f(a)$ DNE or $\lim_{x \to a} f(x) \neq f(a)$

**Note: A removable discontinuity can be removed be redefining the function $x = a$ as $f(a) \stackrel{def}{=} \lim_{x \to a} f(x)$.**

### D Jump Discontinuity

A function $y = f(x)$ has a jump discontinuity at $x = a$ if:

$$L = \lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x) = R$$

## E Infinite Discontinuity

A function $y = f(x)$ has an infinite discontinuity at $x = a$ if at least one side of the limit is unbounded (approaches $\infty$ or $-\infty$).

## F Continuity over an Interval (AP only)

A function $y = f(x)$ is continuous over an open interval $(a, b)$ if the function is continuous at every number in that interval.

A function is continuous from the right at $x = a$ if $R = f(a)$.

A function is continuous from the left at $x = a$ if $L = f(a)$.

## G Elementary Functions (AP only)

Elementary functions (polynomial, power, rational, trigonometric, exponential, and logarithmic) are continuous over their domain.

## H Composition of Functions

If $g$ is continuous at $x = a$ and $f$ is continuous at $g(a)$ then $f(g(x))$ is continuous at $x = a$.

## I Intermediate Value Theorem (AP only)

If $y = f(x)$ is a continuous function over the interval $[a, b]$ with $f(a) \neq f(b)$, then for any number $N$ between $f(a)$ and $f(b)$ there exist a number $c \in (a, b)$ such that $f(c) = N$.

## 2.1 Derivative Function

### A Derivative Function

Given a function $y = f(x)$, the *derivative function* of $f$ is a *new function* called $f'$ ($f$ prime), defined at $x$ by:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

A function $y = f(x)$ is *differentiable* at $x$ if $f'(x)$ exists.

### B Differentiability (AP only)

A function $y = f(x)$ is differentiable over an open interval $(a, b)$ if the function is differentiable at every number in that interval.

The domain of derivative function $f'(x)$ is a subset of the domain of the original function $f$ ($D_{f'} \subseteq D_f$). So a function is defined over $D_f$ but is differentiable over $D_{f'}$.

### C Interpretations of Derivative Function

1. The *slope of the tangent line* to the graph of $y = f(x)$ at the point $P(a, f(a))$ is given by $m_T = f'(a)$.
2. The *instantaneous rate of change* in the variable $y$ with respect to the variable $x$, where $y = f(x)$, at $x = a$ is given by $IRC = f'(a)$.

### D Notations and Reading

**Lagrange or prime notation**

$$y' = f'(a)$$

Reading: "y prime" or "f prime of (at) x".

**Leibnitze notation**

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x} f(x) = \mathrm{D} f(x) = \mathrm{D}_x\, f(x)$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}$$

Reading: "dee y by dee x".

**Evaluating**

$$f'(a) = \left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=a}$$

Reading: "dee y by dee x at x equals a".

## E First Principles

*Differentiation* is the process to find the derivative function for a given function.

*First Principles* is the process of differentiation by computing any of the following limits:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x-h)}{2h}$$

$$f'(x) = \lim_{u\to x} \frac{f(u) - f(x)}{u - x}$$

## F Differentiability Point

A function $y = f(x)$ is *differentiable* at $x$ if $f'(x)$ exists.

If the function $y = f(x)$ is *differentiable* at $x = a$ then the tangent line at $P(a, f(a))$ is *unique* and *not vertical* (the slope of the tangent line is not $\infty$ or $-\infty$).

## G Non-Differentiability

A function is *not differentiable* at $x = a$ if $f'(a)$ *does not exist.*

**Notes:**

- If a function $f$ is *not continuous* at $x = a$ then the function $f$ is *not differentiable* at $x = a$.
- If a function is differentiable at $x = a$ then the function is continuous at $x = a$.
- If a function $f$ is *continuous* at $x = a$ then the function $f$ *may or may not be* differentiable at $x = a$.

## H Corner Point

$P(a, f(a))$ is a *corner point* if there are *two* distinct tangent lines at $P$, one for the left-hand branch and one for the right-hand branch.

## I Infinite Slope Point

$P(a, f(a))$ is an *infinite slope point* if the tangent line at $P$ is vertical and the function is increasing or decreasing in the neighbourhood of the point $P$.

$$f'(a) = \infty \quad \vee \quad f'(a) = -\infty$$

## J Cusp Point

$P(a, f(a))$ is a *cusp point* if the tangent line at $P$ is vertical and the function is increasing on one side of the point $P$ and decreasing on the other side.

$$f'(a) = DNE$$

## 2.2 Derivative of Polynomial Functions

### A Power Rule

If $y = f(x) = x^n \mid x, n \in \mathbb{R}$ is the *power* function then:

$$y' = f'(x) = (x^n)' = nx^{n-1}$$

Some useful specific case:

$$(1)' = 0$$
$$(x)' = 1$$
$$\left(\sqrt{x}\right)' = \frac{1}{2\sqrt{x}}$$

### B Constant Function Rule

If $y = f(x) = c \mid c \in \mathbb{R}$ is the *constant* function then:

$$f'(x) = (c)' = 0$$

### C Constant Multiple Rule

If $g(x) = cf(x)$ then:

$$g'(x) = \big(cf(x)\big)' = cf'(x)$$
$$\frac{\mathrm{d}}{\mathrm{d}x}g(x) = \frac{\mathrm{d}}{\mathrm{d}x}(cf(x)) = c\frac{\mathrm{d}}{\mathrm{d}x}f(x)$$

### D Sum and Difference Rules

$$\big(f(x) \pm g(x)\big)' = f'(x) \pm g'(x)$$

### E Tangent Line

The *equation of the tangent line* at the point $P(a, f(a))$ to the curve $y = f(x)$ is:

$$y = f'(a)(x - a) + f(a) \tag{xi}$$

## F Normal Line (AP only)

If $m_T = f'(a)$ is the slope of the tangent line at $P(a, f(a))$, the slope of the normal line $m_N$ is given by:

$$m_N = -\frac{1}{m_T}$$

## G Differentiability for Piecewise Defined Function (AP only)

Consider the piecewise defined function:

$$f(x) = \begin{cases} f_1(x) & x < a \\ c & x = a \\ f_2(x) & x > a \end{cases}$$

The function $f$ is *differentiable* at $x = a$ if:

1. The function is continuous at $x = a$.
2. $f_1'(a) = f_2'(a)$ (the slope of the tangent line for the left branch is equal to the slope of the tangent line for the right branch).

## 2.3 Product Rule

### A Product Rule

If $f$ and $g$ are differentiable at $x$ then so is $fg$ and:

$$\big(f(x)g(x)\big)' = f'(x)g(x) + f(x)g'(x)$$

$$(fg)' = f'g + fg'$$

### B Product of Three Functions

If $f$, $g$, and $h$ are differentiable at $x$ then so is $fgh$ and:

$$(fgh)' = f'gh + fg'h + fgh'$$

### C Generalized Power Rule

If $f$ is differentiable at $x$, then so is $f^n$ and:

$$\Big(\big(f(x)\big)^n\Big) = n\left(f(x)\right)^{n-1} f'(x) \tag{xii}$$

$$(f^n)' = nf^{n-1}f'$$

## 2.4 Quotient Rule

### A Quotient Rule

If $f$ and $g$ are differentiable at $x$ and $g(x) \neq 0$ then so is $\frac{f}{g}$ and:

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{\big(f(x)\big)^2} \tag{xiii}$$

## 2.5 Chain Rule

### A Composition of Functions

If $u = g(x)$ and $v = f(u)$ then:

$$x \xrightarrow[u=g(x)]{} u \xrightarrow[v=f(u)]{} v$$

and

$$v = f(u) = f(g(x)) = (f \circ g)(x)$$

### B Chain Rule (Leibniz Notation)

$$\Delta x \xrightarrow[u=g(x)]{} \Delta u \xrightarrow[v=f(u)]{} \Delta v$$

and

$$\frac{\Delta v}{\Delta x} = \frac{\Delta v}{\Delta u}\frac{\Delta u}{\Delta x} \to \frac{\mathrm{d}v}{\mathrm{d}x} = \frac{\mathrm{d}v}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x}$$

Therefore:

$$\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{\mathrm{d}v}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x}$$

### C Composition of Three Functions

$$x \xrightarrow[u=h(x)]{} u \xrightarrow[v=g(u)]{} v \xrightarrow[w=f(v)]{} w$$

$$\frac{\mathrm{d}w}{\mathrm{d}x} = \frac{\mathrm{d}w}{\mathrm{d}v}\frac{\mathrm{d}v}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x}$$

### D Chain Rule (Prime Notation)

$$\big(f(g(x))\big)' = f'(g(x))g'(x)$$

If $g$ is differentiable at $x$ and $f$ is differentiable at $g(x)$ then the composition $(f \circ g)(x) = f(g(x))$ is differentiable at $x$ and:

$$(f \circ g)'(x) = \big(f(g(x))\big)' = f'(g(x))g'(x)$$

So, the derivative of $f(g(x))$ is the derivative of the *outside* function $f$ evaluated at the inside function $g(x)$ times the derivative of the inside function $g$ at $x$.

**Note: If the outside function is the power function, then the chain rule is equivalent to the generalized power rule** (xii)**.**

## 5.4 Derivative of Trigonometric Functions

### A Review of Trigonometric Functions

$$\sin(x)\colon \mathbb{R} \to [-1, 1]$$
$$\cos(x)\colon \mathbb{R} \to [-1, 1]$$
$$\tan(x)\colon \left\{\mathbb{R} \setminus \left\{\frac{\pi}{2} + n\pi \mid n \in \mathbb{Z}\right\}\right\} \to \mathbb{R}$$

$$\sin(x + 2\pi) = \sin(x)$$
$$\cos(x + 2\pi) = \cos(x)$$

$$\sin^2(x) + \cos^2(x) = 1$$

$$\sin(a + b) = \sin(a)\cos(b) + \cos(a)\sin(b)$$
$$\cos(a + b) = \cos(a)\cos(b) - \sin(a)\sin(b)$$

$$\sin\left(x + \frac{\pi}{2}\right) = \cos(x)$$
$$\sin(2x) = 2\sin(x)\cos(x)$$
$$\cos(2x) = \cos^2(x) - \sin^2(x)$$
$$\tan(x) = \frac{\sin(x)}{\cos(x)}$$

$$\lim_{x\to 0} \frac{\sin x}{x} = 1 \tag{xiv}$$
$$\lim_{x\to 0} \frac{\cos x - 1}{x} = 0 \tag{xv}$$

## B Derivative of $\sin(x)$

$$(\sin x)' = \cos x$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\sin x = \cos x$$

*Proof.*

$$(\sin x)' = \lim_{h\to 0}\frac{\sin(x+h)-\sin(x)}{h}$$
$$(\sin x)' = \lim_{h\to 0}\frac{\sin(x)\cos(h)+\cos(x)\sin(h)-\sin(x)}{h}$$
$$(\sin x)' = \lim_{h\to 0}\frac{\sin(x)\cos(h)-\sin(x)}{h}+\lim_{h\to 0}\frac{\cos(x)\sin(h)}{h}$$
$$(\sin x)' = \sin(x)\lim_{h\to 0}\frac{\cos(h)-1}{h}+\cos(x)\lim_{h\to 0}\frac{\sin(h)}{h}$$

Now, using the limits (xiv) and (xv):

$$(\sin x)' = sin(x)\times 0+\cos(x)\times 1$$
$$(\sin x)' = \cos(x)$$

$\square$

## C Derivative of $\sin\big(f(x)\big)$

By using the chain rule:

$$\Big(\sin\big(f(x)\big)\Big)' = \Big(\cos\big(f(x)\big)\Big)\, f'(x)$$

## D Derivative of $\cos x$

$$(\cos x)' = -\sin x$$

## E Derivative of $\cos\big(f(x)\big)$

By using the chain rule:

$$\Big(\cos\big(f(x)\big)\Big)' = -\Big(\sin\big(f(x)\big)\Big)\, f'(x)$$

# F Derivative of $\tan x$

$$(\tan x)' = \frac{1}{\cos^2 x} = \sec^2 x$$

## 5.1 Derivative of Exponential Function

### A Review of Exponential Functions

The exponential function is defined as:

$$y = f(x) = b^x \mid b > 0 \land b \neq 1$$

The $x$-axis ($y = 0$) is a horizontal asymptote.

### B Number $e$

The number $e$ is defined by:

$$e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$$

which can be written also as:

$$e = \lim_{u \to 0} (1 + u)^{\frac{1}{u}}$$

### C Derivative of $e^x$

$$(e^x)' = e^x$$

The proof of this is based on the fact that:

$$\lim_{h \to 0} \frac{e^h - 1}{h} = 1$$

### D Derivative of $e^{f(x)}$

By using the chain rule:

$$\left(e^{f(x)}\right)' = e^{f(x)} f'(x)$$

### E Derivative of $b^x \mid b > 0 \land b \neq 1$

$$(b^x)' = (\ln b) b^x$$

*Proof.*

$$(b^x)' = \left(e^{x \ln b}\right)' = e^{x \ln b} (\ln b) = (\ln b) b^x$$

□

## F Derivative of $b^{f(x)}$

By using the chain rule:

$$\left(b^{f(x)}\right)' = (\ln b)b^{f(x)}f'(x)$$

## 5.1 Derivative of Logarithmic Function

### A Review of Logarithmic Function

$$y = b^x \equiv x = \log_b y$$
$$y = f(x) = \log_b x \mid b > 0 \land b \neq 1 \land x > 0$$
$$\log_b(xy) = \log_b(x) + \log_b(y)$$
$$\log_b\left(\frac{x}{y}\right) = \log_b(x) - \log_b(y)$$
$$\log_b x^n = n \log_b x$$
$$\log_b x = \frac{\log_a x}{\log_a b}$$
$$\log_b 1 = 0$$
$$\log_b b = 1$$

### B Derivative of $\ln x$

$$(\ln x)' = \frac{1}{x}$$

*Proof.*

$$y = \ln x \implies x = e^y \implies x' = (e^y)'$$
$$x' = (e^y)' \implies 1 = e^y y' \implies y' = \frac{1}{e^y} \implies y' = \frac{1}{x}$$
$$\therefore (\ln x)' = \frac{1}{x}$$

□

### C Derivative of $\ln(f(x))$

By using the chain rule:

$$\big(\ln f(x)\big)' = \frac{f'(x)}{f(x)}$$

## D Derivative of $\log_b x$

$$(\log_b x)' = \frac{1}{(\ln b)x}$$

*Proof.*

$$(\log_b x)' = \left(\frac{\ln x}{\ln b}\right)' = \frac{1}{\ln b}(\ln x)' = \frac{1}{(\ln b)x}$$

□

## E Derivative of $\log_b f(x)$

By using the chain rule:

$$\Big(\log_b \big(f(x)\big)\Big)' = \frac{f'(x)}{(\ln b)f(x)}$$

# Logarithmic Differentiation (AP)

## A Logarithmic Differentiation

If the function formula contains many factors, then logarithmic differentiation is a fast method to differentiate.

Use the following algorithm:

1. Take natural logarithms of both sides of $y = f(x)$.
2. Differentiate with respect to $x$.
3. Isolate $y' = \frac{\mathrm{d}y}{\mathrm{d}x}$.

## B Function Raise to a Function

To differentiate a function $f(x)$ raised to another function $g(x)$, use the formula:

$$\left(f(x)^{g(x)}\right)' = g(x)f(x)^{g(x)-1}f'(x) + \ln(f(x))f(x)^{g(x)}g'(x)$$

**Notes:**

1. The first part $g(x)f(x)^{g(x)-1}$ comes from using the power rule and chain rule and by considering $g(x)$ constant.
2. The second part $\ln(f(x))f(x)^{g(x)}g'(x)$ comes from using the exponential rule and chain rule and by considering $f(x)$ constant.

# Inverse Trigonometric Functions and Their Derivatives

## A Inverse Sine Function

The inverse of the sine function:

$$f(x) = \sin x\colon \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to [-1, 1]$$

is:

$$f'(x) = \arcsin x = \sin^{-1} x\colon [-1, 1] \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

## B Trigonometric Identities with Inverse Sine

$$\arcsin x = \theta \equiv \sin\theta = x$$

## C Inverse Cosine Function

The inverse of the cosine function:

$$f(x) = \cos x\colon [0, \pi] \to [-1, 1]$$

is:

$$f'(x) = \arccos x = \cos^{-1} x\colon [-1, 1] \to [0, \pi]$$

## D Trigonometric Identities with Inverse Cosine

$$\arccos x = \theta \equiv \cos\theta = x$$

## E Inverse Tangent Function

The inverse of the tangent function:

$$f(x) = \tan x\colon \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to [-\infty, \infty]$$

is:

$$f'(x) = \arctan x = \tan^{-1} x\colon [-\infty, \infty] \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

## F Trigonometric Identities with Inverse Tangent

$$\arctan x = \theta \equiv \tan\theta = x$$

## G Derivative of the Inverse Function

If $f^{-1}$ is the inverse function of the function $f$ then:

$$y = f^{-1}(x) \equiv x = f(y)$$

If derivative rule of a function is known, then the derivative of the inverse of that function may be found using:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\frac{\mathrm{d}x}{\mathrm{d}y}}$$

## H Derivative of Inverse Trigonometric Functions

Differentiation rules for the inverse trigonometric functions are:

$$\frac{\mathrm{d}}{\mathrm{d}x}\arcsin x = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\arccos x = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\arctan x = \frac{1}{1+x^2}$$

## I Reciprocal of Trigonometric Functions

Reciprocal of trigonometric functions are defined by:

$$\sec x = \frac{1}{\cos x}$$

$$\csc x = \frac{1}{\sin x}$$

$$\cot x = \frac{1}{\tan x}$$

Their inverses may be computed by using the following formulas:

$$\operatorname{arcsec} x = \arccos\frac{1}{x}$$

$$\operatorname{arccsc} x = \arcsin\frac{1}{x}$$

$$\operatorname{arccot} x = \arctan\frac{1}{x}$$

# Unit 3

# Applications of Differentiation

# Implicit Differentiation (AP)

## A Relations Defined Implicitly

A relation between two variables $x$ and $y$ is defined implicitly by an equation like:

$$f(x, y) = 0$$

**Notes:**

1. One variable may be considered dependant on the other variable or both may be considered dependant on the third one like $t$.
2. The equation may be solved with respect to the variables $x$ or $y$ or may not be solved.
3. The graph of the relation may or may not pass the vertical or horizontal line tests.

## B Terminology

Let $(x, y)$ and $(x + \Delta x, y + \Delta y)$ be two points satisfying $f(x, y) = 0$. Then:

$$\frac{\Delta y}{\Delta x} = \frac{1}{\frac{\Delta x}{\Delta y}}$$

And as $\Delta x \to 0$, $\Delta y \to 0$:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\frac{\mathrm{d}x}{\mathrm{d}y}}$$

**Notes:**

- $\frac{\mathrm{d}y}{\mathrm{d}x}$ means differentiation of the variable $y$ with respect to the variable $x$.
- $\frac{\mathrm{d}x}{\mathrm{d}y}$ means differentiation of the variables $x$ with respect to the variable $y$.
- The tangent line is horizontal when $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$.
- The tangent line is vertical when $\frac{\mathrm{d}x}{\mathrm{d}y} = 0$.

## C Differentiation Revised

Consider the expression $E(x, y) = 2xy^2$.

If $x$ is considered independent:

$$\frac{\mathrm{d}}{\mathrm{d}x}E(x, y) = \frac{\mathrm{d}}{\mathrm{d}x}(2xy^2) = y^2\frac{\mathrm{d}}{\mathrm{d}x}(2x) + (2x)\frac{\mathrm{d}}{\mathrm{d}x}y^2 = 2y^2\frac{\mathrm{d}x}{\mathrm{d}x} + 4xy\frac{\mathrm{d}y}{\mathrm{d}x} = 2y^2 + 4xy\frac{\mathrm{d}y}{\mathrm{d}x}$$

If $y$ is considered independent:

$$\frac{\mathrm{d}}{\mathrm{d}x}E(x,y) = \frac{\mathrm{d}}{\mathrm{d}y}(2xy^2) = y^2\frac{\mathrm{d}}{\mathrm{d}y}(2x) + (2x)\frac{\mathrm{d}}{\mathrm{d}y}y^2 = 2y^2\frac{\mathrm{d}x}{\mathrm{d}y} + 4xy\frac{\mathrm{d}y}{\mathrm{d}y} = 2y^2\frac{\mathrm{d}x}{\mathrm{d}y} + 4xy$$

If $t$ is considered independent:

$$\frac{\mathrm{d}}{\mathrm{d}x}E(x,y) = \frac{\mathrm{d}}{\mathrm{d}t}(2xy^2) = y^2\frac{\mathrm{d}}{\mathrm{d}t}(2x) + (2x)\frac{\mathrm{d}}{\mathrm{d}t}y^2 = 2y^2\frac{\mathrm{d}x}{\mathrm{d}t} + 4xy\frac{\mathrm{d}y}{\mathrm{d}t}$$

## D Implicit Differentiation

To differentiation with respect to the variable $x$ in a relation given implicitly by $f(x,y) = 0$:

1. Apply the operator $\frac{\mathrm{d}}{\mathrm{d}x}$ to both sides:

$$\frac{\mathrm{d}}{\mathrm{d}x}f(x,y) = \frac{\mathrm{d}}{\mathrm{d}x}0$$

2. Use the chain rule and differentiate by keeping in mind that $\frac{\mathrm{d}x}{\mathrm{d}x} = 1$.
3. Solve for $\frac{\mathrm{d}y}{\mathrm{d}x} = IRC = m_T$ or $\frac{\mathrm{d}x}{\mathrm{d}y} = \frac{1}{\frac{\mathrm{d}y}{\mathrm{d}x}}$.
4. Substitute $x$ and $y$ with given values (if necessary).

**Note: The following differentiations are also possible:**

$$\frac{\mathrm{d}}{\mathrm{d}y}f(x,y) = \frac{\mathrm{d}}{\mathrm{d}y}0$$

$$\frac{\mathrm{d}}{\mathrm{d}t}f(x,y) = \frac{\mathrm{d}}{\mathrm{d}t}0$$

## 3.9 Related Rates

### A Algorithm to Solve Related Rates Applications

1. Assign variables $x$, $y$, $z$, $\ldots$ to quantities involved in application.
2. Discover relations (constraints) between these quantities and write down their restrictions. A diagram or geometry formulas may help.
3. Use these relations to eliminate variables which are not essential to application. At this step, related variables are part of an explicit equation:

$$x = f(y, z, \ldots) \tag{xvi}$$

   or are part of an implicit equation:

$$f(y, z, \ldots) = 0 \tag{xvii}$$

4. Identify the independent quantity and assign a variable to it (usually this is the time $t$).
5. Use the chain rule to differentiate with respect to the independent variable $t$ the equation (xvi) or (xvii):

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t} f(y, z, \ldots) \quad \text{or} \quad \frac{\mathrm{d}}{\mathrm{d}t} f(y, z, \ldots) = 0 \tag{xviii}$$

6. Substitute all given data or other data obtained from (xvi) or (xvii) equations.
7. Solve for the remaining unknown rate of change.

**Note.** $\frac{\mathrm{d}x}{\mathrm{d}y}$, $\frac{\mathrm{d}y}{\mathrm{d}t}$, **$\ldots$ are instantaneous rates of change are they are related by** (xviii)**.**

## 3.10 Linear Approximation and Differentials

### A Linear Approximation

The definition of derivative function at $x = a$:

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

may be written:

$$f(x) \approx f(a) + f'(a)(x - a)\ ,\ \ x \to a$$

which is called the *linear or tangent line approximation* of the function $y = f(x)$ near $x = a$. **Note. Linear approximation is only possible if $f'(a)$ exists.**

### B Approximate Formulas

The definition of derivative function at $x = a$:

$$f'(a) = \lim_{x \to 0} \frac{f(a + x) - f(a)}{x}$$

written in the form:

$$f(a + x) \approx f(a) + f'(a)x\ ,\ \ x \to 0$$

permits generation of *approximate formulas*.

### C Numerical Approximation

Numerical approximation is based on the formula:

$$f(a + x) \approx f(a) + f'(a)x\ ,\ \ x \to 0$$

### D Differentials

Derivative function may be written in the form:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f'(x)$$

$\mathrm{d}x$ and $\mathrm{d}y$ are called *differentials* and they are related by the formula:

$$\mathrm{d}y = f'(x)\,\mathrm{d}x$$

This formula is called the *differential form* of the function $y = f(x)$.

If $\mathrm{d}x, \mathrm{d}y \to 0$ then the previous formula is exact.

If $\mathrm{d}x$ and $\mathrm{d}y$ are finite, we replace them by $\Delta x$ and $\Delta y$ and the previous formula becomes approximately:

$$\Delta y \approx f'(x)\Delta x$$

## E Error Propagation

If the variable $x$ is measured with a finite error $\Delta x$, then the real value is $x + \Delta x$.

The *absolute error* in computing the value of the function $y = f(x)$ is approximately given by:

$$\Delta y \approx f'(x)\Delta x$$

and its *relative error* $\frac{\Delta y}{y}$ may be approximated by:

$$\frac{\Delta y}{y} = \frac{\Delta y}{f(x)} \approx \frac{f'(x)\Delta x}{f(x)}$$

## 3.2 Maximum and Minimum on an Interval: Extreme Values

### A Global Maximum

A function $f$ has a *global (absolute) maximum* at $x = c$ is $f(x) \leq f(c)$ for all $x \in D_f$.

$f(c)$ is called the *global (absolute) maximum value*.

$(c, f(c))$ is called the *global (absolute) maximum point*.

**Note. An *extremum* is either a minimum or maximum (value, point, local, or global).**

### B Global Minimum

A function $f$ has a *global (absolute) minimum* at $x = c$ is $f(x) \geq f(c)$ for all $x \in D_f$.

$f(c)$ is called the *global (absolute) minimum value*.

$(c, f(c))$ is called the *global (absolute) minimum point*.

**Notes:**

**Extrema** The plural of extremum.

**Minima** The plural of minimum.

**Maxima** The plural of maximum.

### C Global (Absolute) Extrema Algorithm

To find the global (absolute) extrema for a *continuous* function $f$ over a close interval $[a, b]$:

1. Identify all *critical* numbers over $(a, b)$.
2. Find the *values* of the function $f(c)$ at each critical number $c$ in $(a, b)$.
3. Find the *values* $f(a)$ and $f(b)$.
4. From the values obtained at part 2 and 3:
    - The *largest* represents the *global (absolute) maximum* value.
    - The *smallest* represents the *global (absolute) minimum* value.

**Note. A *critical number* $c$ is a number such that $f'(c) = 0$ or $f'(c)$ DNE.**

## 4.1 Increasing and Decreasing Functions. Critical Points: Local Maxima and Minima

### A Increasing or Decreasing Functions

Let $y = f(x)$ be a differentiable function over $(a, b)$. Then:

1. $f$ is *increasing* over $(a, b)$ if:
   - $ARC = m_S = \frac{f(x_2)-f(x_1)}{x_2-x_1} > 0$ over any interval $[x_1, x_2 \subseteq (a, b)$.
   - $IRC = m_T = f'(x) > 0$ for all $x \in (a, b)$.
2. $f$ is *decreasing* over $(a, b)$ if:
   - $ARC = m_S = \frac{f(x_2)-f(x_1)}{x_2-x_1} < 0$ over any interval $[x_1, x_2 \subseteq (a, b)$.
   - $IRC = m_T = f'(x) < 0$ for all $x \in (a, b)$.
3. $f$ is *constant* over $(a, b)$ if:
   - $ARC = m_S = \frac{f(x_2)-f(x_1)}{x_2-x_1} = 0$ over any interval $[x_1, x_2 \subseteq (a, b)$.
   - $IRC = m_T = f'(x) = 0$ for all $x \in (a, b)$.

### B Local Maximum

A function $f$ has a *local (relative) maximum* at $x = c$ if:

- $f(x) \leq f(c)$ when $x$ is sufficiently close to $c$ (from both sides).
- $f'(x)$ changes sign from positive to negative at $c$.

### C Local Minimum

A function $f$ has a *local (relative) minimum* at $x = c$ if::

- $f(x) \geq f(c)$ when $x$ is sufficiently close to $c$ (from both sides).
- $f'(x)$ changes sign from negative to positive at $c$.

### D Critical Numbers and Critical Points

The number $c \in D_f$ is a *critical number* if:

$$f'(c) = 0 \quad \text{or} \quad f'(c) \text{ DNE}$$

The point $P(c, f(c))$ is called a *critical point*.

**Notes:**

1. A local extremum happens always at a critical point (Fermat's theorem).
2. At a critical number a function may or may not have a local extremum.

## 4.2 The Mean Value Theorem (AP)

### A Rolle's Theorem

Let $y = f(x)$ be a function continuous on $[a, b]$ and differentiable on $(a, b)$.

If $f(a) = f(b)$ then there is a number $c \in (a, b)$ such that $f'(c) = 0$.

**Note. Tangent line is horizontal at $P(c, f(c))$.**

### B Mean Value Theorem

Let $y = f(x)$ be a function continuous on $[a, b]$ and differentiable on $(a, b)$. Then there is a number $c \in (a, b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.

**Note. Slope of tangent line at $P(c, f(c))$ is equal to slope of secant line.**

### C Theorem

If $f'(x) = 0$ for all $x \in (a, b)$, then $f$ is constant on $(a, b)$.

### D Theorem

If $f'(x) = g'(x)$ for all $x \in (a, b)$, then $f - g$ is constant on $(a, b)$ and $f(x) = g(x) + c$ where $c$ is a constant.

## 4.3 Asymptotes

### A Vertical Asymptote

If the value of $f(x)$ can be made *arbitrarily large* by taking $x$ *sufficiently close* to $a$ with $x < a$ then $\lim\limits_{x \to a^-} f(x) = \infty$. The line $x = a$ is called a *vertical asymptote* to the graph of $y = f(x)$.

**Notes:**

1. A function of the form $f(x) = \frac{p(x)}{q(x)}$ has a vertical asymptote at $x = a$ if $p(a) \neq 0 \wedge q(a) = 0$.
2. A function of the form $f(x) = p(x) \log_b q(x)$ has a vertical asymptote $x = a$ if $p(a) \neq 0 \wedge q(a) = 0$.

### B Horizontal Asymptote

A horizontal line $y = b$ is called a horizontal asymptote to the graph of $y = f(x)$ if $\lim\limits_{x \to \pm\infty} f(x) = b$.

**Notes:**

1. A horizontal asymptote may be crossed or touched by the graph of the function.
2. The graph of a function may have at most two horizontal asymptotes (one as $x \to -\infty$ and one as $x \to +\infty$).

### C Limits at Infinity

If $a > 0$, then:

$$\lim_{x \to \pm\infty} x^a = (\pm\infty)^a$$

$$\lim_{x \to \pm\infty} \frac{1}{x^a} = \frac{1}{(\pm\infty)^a} = 0$$

$$\lim_{x \to \pm\infty} (a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0) = \lim_{x \to \pm\infty} a_n x^n$$

$$\lim_{x \to \pm\infty} \frac{a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0}{b_m x^m + b_{m-1} x^{m-1} + \cdots + b_2 x^2 + b_1 x + b_0} = \lim_{x \to \pm\infty} \frac{a_n x^n}{b_m x^m}$$

## D Horizontal Asymptotes for Rational Functions

A rational function of the form:

$$f(x) = \frac{P_n(x)}{Q_m(x)} = \frac{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0}{b_m x^m + b_{m-1}x^{m-1} + \cdots + b_2x^2 + b_1x + b_0}$$

has:

- A horizontal asymptote $y = 0$ if $m > n$.
- A horizontal asymptote $y = \frac{a_n}{m_n}$ if $m = n$.
- No horizontal asymptote if $m < n$.

**Note. A rational function may have at most one horizontal asymptote.**

## E Oblique (Slant) Asymptote

The line $y = ax + b$ is an oblique (slant) asymptote for the curve $y = f(x)$ if:

$$\lim_{x \to \pm\infty} \big(f(x) - (ax + b)\big) = 0$$

**Notes:**

1. An oblique asymptote may be crossed or touched by the graph of the function.
2. The graph of a function may have at most two oblique asymptotes (one as $x \to -\infty$ and one as $x \to +\infty$).
3. The graph of a function may have one horizontal asymptote and one oblique asymptote (one as $x \to -\infty$ and the other as $x \to +\infty$).

## F Oblique Asymptotes for Rational Functions

A rational function of the form:

$$f(x) = \frac{P_n(x)}{Q_m(x)} = \frac{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0}{b_m x^m + b_{m-1}x^{m-1} + \cdots + b_2x^2 + b_1x + b_0}$$

has an oblique (slant) asymptote if $n = m + 1$.

**Note. To get the equation of the oblique (slant) asymptote, use the *long division algorithm* to write the rational function in the form:**

$$f(x) = \frac{P_n(x)}{Q_m(x)} = ax + b + \frac{R(x)}{Q_m(x)}$$

**where** $0 \leq \text{degree}(R) < \text{degree}(Q_m)$**.**

## G Oblique Asymptotes for any Functions

The oblique asymptote $y = ax + b$ may be obtained by computing the following two limits at infinity:

$$a = \lim_{x\to\pm\infty} \frac{f(x)}{x}$$

$$b = \lim_{x\to\pm\infty} \big(f(x) - ax\big)$$

**Note. In order for the oblique asymptote to be defined, both limits above must exist (must be finite numbers).**

## 4.4 Concavity and Points of Inflection

### A Concavity Upward

The graph of the function $f$ has a *concavity upward* if:

- Graph lies above all its tangents.
- Tangents rotate counter-clockwise.
- $m_T = IRC = f'(x)$ increases
- $f''(x) > 0$

### B Concavity Downward

The graph of the function $f$ has a *concavity downward* if:

- Graph lies below all its tangents.
- Tangents rotate clockwise.
- $m_T = IRC = f'(x)$ decreases
- $f''(x) < 0$

### C Test for Concavity

Let $f$ be a function twice differentiable ($f''(x)$ exits) over $(a, b)$.

1. If $f''(x) > 0$ for all $x \in (a, b)$, then the function $f$ (or its graph) is *concave upward* over $(a, b)$.
2. If $f''(x) < 0$ for all $x \in (a, b)$, then the function $f$ (or its graph) is *concave downward* over $(a, b)$.
3. If $f''(x) = 0$ for all $x \in (a, b)$, then the function $f$ (or its graph) has *no concavity* over $(a, b)$. In this case, $f'(m) = m$, and $f(x) = mx + b$. The function is linear and its graph is a straight line.

### D Point of Inflection

A point $P(i, f(i))$ on the graph of $y = f(x)$ is called a point of inflection if the concavity of the graph *changes* at $P$ from concave upward to concave downward or from concave downward to concave upward.

## E Second Derivative Test

Let $f$ be a twice differentiable function over an open interval containing the critical number $c$ and $f'(c) = 0$.

1. If $f''(c) > 0$ then $f$ has a local minimum at $x = c$.
2. If $f''(c) < 0$ then $f$ has a local maximum at $x = c$.
3. If $f''(c) = 0$ then $f$ may have a local minimum, maximum, or neither (inconclusive case). Use the first derivative test to conclude.

## 4.5 An Algorithm for Curve Sketching

### A Algorithm for Curve Sketching

1. Domain
   - denominator $\neq 0$
   - radicand $\geq 0$ (even roots)
   - logarithmic argument $> 0$ (logarithmic functions)
2. Intercepts
   - $f(x) = 0$ ($x$-intercepts are real zeros)
   - numerator $= 0$
   - $y$-int $= f(0)$ (if exists)
3. Symmetry
   - $f(-x) = f(x)$ (even functions are symmetric about the $y$-axis)
   - $f(-x) = -f(x)$ (odd functions are symmetric about the origin)
   - $f(x + T) = f(x)$ (periodic functions have cycles)
4. Asymptotes
   - Compute $\lim\limits_{x\to\pm\infty} f(x)$ (horizontal asymptote)
   - Compute $\lim\limits_{x\to a} f(x)$ (vertical asymptote where $a$ is a zero of the denominator but not of the numerator).
   - $\log_b f(x)$ may generate vertical asymptotes when $\lim\limits_{x\to a} f(x) = 0$
   - Compute long division (to find the oblique asymptotes for rational functions)
5. First Derivative
   - Compute $f'(x)$
   - Find critical numbers ($f'(x) = 0$ or $f'(x)$ DNE)
   - Create the sign chart for $f'(x)$ (first derivative test)
   - Find intervals of increase/decrease
   - Find all extrema
6. Second Derivative
   - Compute $f''(x)$

- Find points where $f''(x) = 0$ or $f''(x)$ DNE
- Create the sign chart for $f''(x)$
- Find points of inflection
- Find intervals of concavity upward/downward
- Find local extrema by using the second derivative test (if necessary)

7. Curve Sketching
    - Use broken lines to draw the asymptotes
    - Plot $x$ and $y$-intercepts, extrema, and inflection points
    - Draw the curve near the asymptotes
    - Sketch the curve

## B Link between a Function and its Derivative

If the function $y = f(x)$ is double differentiable then $f'(x)$ and $f''(x)$ exist. The following statements show the link between $f$, $f'$, and $f''$:

1. $y = f(x)$ is increasing $\iff f'(x) > 0$
2. $y = f(x)$ is decreasing $\iff f'(x) < 0$
3. $y = f(x)$ has a local extrema at $x = a \implies$ tangent line at $P(a, f(a))$ is horizontal $\implies f'(a) = 0$
4. $y' = f'(x)$ is increasing $\iff f''(x) > 0 \iff y = f(x)$ is concave up
5. $y' = f'(x)$ is decreasing $\iff f''(x) < 0 \iff y = f(x)$ is concave down
6. $y' = f'(x)$ has a local extrema at $x = b \implies$ tangent line on the graph of $y' = f'(x)$ is horizontal $\implies f''(b) = 0$, $y = f(x)$ has an inflection point at $P(b, f(b))$.

## C Symmetry and Derivatives Functions

If the function $y = f(x)$ is even/odd then the derivative function $y' = f'(x)$ is odd/even.

## 3.3 Optimization

### A Algorithm for Solving Optimization Applications

1. Read and understand the situation.
2. Draw a diagram (if necessary).
3. Assign variables to quantities involved and state restrictions.
4. Find relations (constraints) between these variables.
5. Write the dependent variable (the one which is minimized or maximized) as a function of one single variable (the independent variable) by eliminating all unnecessary variables.
6. Find extrema (maximum or minimum) for the dependant variable (using global extrema algorithm, first derivative test, or the second derivative test).
7. Check if extrema satisfy the conditions of the application.
8. Find the value of other variables at extrema (if necessary).
9. Write the conclusion statement.

### B Cost, Revenue, and Profit

Let $C(x)$ be the *cost* of manufacturing $x$ units.

The *average cost per unit* is $u(x) = \frac{C(x)}{x}$.

The *marginal cost* is the cost of manufacturing the unit number $x$: $C(x) - C(x-1) \approx C'(x)$.

Let $p(x)$ be the *demand* function (*price per unit* when selling $x$ units).

The *revenue* function is $R(x) = xp(x)$.

The *profit* function is $P(x) = R(x) - C(x)$.

# Unit 4

# Integrals

## 4.9 Antiderivatives and Indefinite Integrals

### A Integration

Differentiation is the process of finding $f'(x)$ given $f(x)$. *Integration* is the inverse process of differentiation. It means finding $f(x)$ given $f'(x)$.

$$f(x) \xrightarrow{\text{Differentiation}} f'(x)$$

$$f(x) \xleftarrow{\text{Integration}} f'(x)$$

### B Antiderivative

The *antiderivative* of the function $f(x)$ is a function $F(x)$ such that $F'(x) = f(x)$.

### C Families of Antiderivatives

If $F(x)$ is an antiderivative of $f(x)$, so is $F(x) + C$, where $C$ is a constant called the *constant of integration.*

### D Initial Condition

An antiderivative of a function may be uniquely identified by an initial condition:

$$F(a) = b$$

### E Indefinite Integrals

The *indefinite integral* is the most commonly used notation for antiderivatives. If $F(x)$ is an antiderivative of $f(x)$, then we write:

$$F(x) = \int f(x)\,\mathrm{d}x$$

By definition:

$$\frac{\mathrm{d}}{\mathrm{d}x} \int f(x)\,\mathrm{d}x = f(x)$$

## F List of Indefinite Integrals

$$\int x^n \,\mathrm{d}x = \frac{x^{n+1}}{n+1} + C \mid n \neq -1$$
$$\int \frac{1}{x} \,\mathrm{d}x = \ln|x| + C$$
$$\int e^x \,\mathrm{d}x = e^x + C$$
$$\int a^x \,\mathrm{d}x = \frac{a^x}{\ln a} + C$$
$$\int \mathrm{d}x = x + C$$
$$\int \sin(x) \,\mathrm{d}x = -\cos(x) + C$$
$$\int \cos(x) \,\mathrm{d}x = \sin(x) + C$$
$$\int \sec^2(x) \,\mathrm{d}x = \tan(x) + C$$
$$\int \csc^2(x) \,\mathrm{d}x = -\cot(x) + C$$
$$\int \sec(x)\tan(x) \,\mathrm{d}x = \sec(x) + C$$
$$\int \csc(x)\cot(x) \,\mathrm{d}x = -\csc(x) + C$$
$$\int \frac{1}{x^2+1} \,\mathrm{d}x = \arctan(x) + C$$
$$\int \frac{1}{\sqrt{1-x^2}} \,\mathrm{d}x = \arcsin(x) + C$$

## G Properties of Antiderivatives or Indefinite Integrals

$$\int cf(x) \,\mathrm{d}x = c \int f(x) \,\mathrm{d}x$$
$$\int \big(f(x) \pm g(x)\big) \,\mathrm{d}x = \int f(x) \,\mathrm{d}x \pm \int g(x) \,\mathrm{d}x$$

## 5.1 Riemann Sums

### A Riemann Sums

Riemann sums are a way of approximating the area under a function.

### B Finite Riemann Sums

The area under the function $f(x)$ over the interval $[a, b]$ can approximated using $n$ rectangles as:

$$A \approx \sum_{i=1}^{n} f(x_i^*)\Delta x \qquad \text{where } \Delta x = \frac{b-a}{n}$$

### C Infinite Riemann Sums

The area under $f(x)$ can be found exactly by taking the limit of the Riemann sum as $n \to \infty$:

$$A = \int_a^b f(x)\,\mathrm{d}x = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x \qquad \text{where } \Delta x = \frac{b-a}{n}$$

## 5.2 The Definite Integral

### A Riemann Sum

Let $y = f(x)$ be a function defined on $[a, b]$.

The sequence:

$$x_0 = a < x_1 < x_2 < \cdots < x_{n-1} M x_n = b$$

defines a *partition* of the interval $[a, b]$ in $n$ subintervals of widths:

$$\Delta x_1 = x_1 - x_0, \Delta x_2 = x_2 - x_1, \ldots, \Delta x_n = x_n - x_{n-1}$$

Let $x_1^* \in [x_0, x_1], x_2^* \in [x_1, x_2], \ldots, x_n^* \in [x_{n-1}, x_n]$ be a sequence of any numbers in each interval.

The *Riemann sum* is defined by:

$$f(x_1^* \Delta x_1 + f(x_2^*) \Delta x_2 + \cdots + f(x_n^*) \Delta x_n = \sum_{i=1}^{n} f(x_i^*) \Delta x_i$$

**Note. The Riemann sum approximates the *area* of the region bounded by the graph of $y = f(x)$, the $x$-axis, and the vertical lines $x = a$ and $x = b$.**

### B Definite Integral

The function $y = f(x)$ is *integrable* on $[a, b]$ if the following limit exists:

$$\lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*) \Delta x_i$$

This limit may be written symbolically:

$$\int_a^b f(x) \, \mathrm{d}x$$

and is called the *definite integral*.

### C Theorem

If $y = f(x)$ is *continuous* on $[a, b]$ (or $y = f(x)$ has only a finite number of jump discontinuities) then $y = f(x)$ is *integrable* on $[a, b]$.

That means that the definite integral $\displaystyle\int_a^b f(x) \, \mathrm{d}x$ exists.

## D Fundamental Theorem of Calculus (Part 1)

If $F(x)$ is one antiderivative of $f(x)$ then:

$$\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a) = F(x)\Big|_a^b$$

## E Properties of Definite Integrals

$$\int_a^a f(x)\,\mathrm{d}x = 0$$

$$\int_a^b \mathrm{d}x = b - a$$

$$\int_a^b f(x)\,\mathrm{d}x = -\int_b^a f(x)\,\mathrm{d}x$$

$$\int_a^b f(x)\,\mathrm{d}x = \int_a^c f(x)\,\mathrm{d}x + \int_c^b f(x)\,\mathrm{d}x$$

$$\int_a^b cf(x)\,\mathrm{d}x = c\int_a^b f(x)\,\mathrm{d}x$$

$$\int_a^b \big(f(x) \pm g(x)\big)\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x \pm \int_a^b g(x)\,\mathrm{d}x$$

## F More Properties of Definite Integrals

If $f(x) \geq 0$ on $[a, b]$ then $\int_a^b f(x)\,\mathrm{d}x \geq 0$.

If $f(x) \geq g(x)$ on $[a, b]$ then $\int_a^b f(x)\,\mathrm{d}x \geq \int_a^b g(x)\,\mathrm{d}x$.

If $m \leq f(x) \leq M$ on $[a, b]$ then:

$$m(a-b) \leq \int_a^b f(x)\,\mathrm{d}x \leq M(a-b)$$

## G Fundamental Theorem of Calculus (Part 2)

If $y = f(x)$ is continuous on $[a, b]$, then the function defined by:

$$F(x) = \int_a^x f(t)\,\mathrm{d}t$$

is continuous on $[a, b]$ and differentiable on $(a, b)$, and:

$$F'(x) = f(x)$$

So:

$$\frac{\mathrm{d}}{\mathrm{d}x} \int_a^x f(t)\,\mathrm{d}t = f(x)$$

Note that:

$$\frac{\mathrm{d}}{\mathrm{d}x} \int_a^{u(x)} f(t)\,\mathrm{d}t = \left( \frac{\mathrm{d}}{\mathrm{d}u} \int_a^u f(t)\,\mathrm{d}t \right) \frac{\mathrm{d}u}{\mathrm{d}x} = f(u)\frac{\mathrm{d}u}{\mathrm{d}x} = f(u(x))u'(x)$$

## 5.5 The Substitution Rule

### A Substitution Rule for Indefinite Integrals

If $u = g(x)$ is differentiable on the range of a continuous function $y = f(x)$ then $\mathrm{d}u = g'(x)\,\mathrm{d}x$ and

$$\int f(u(x))g'(x)\,\mathrm{d}x = \int f(u)\,\mathrm{d}u$$

### B Linear Substitution

In this case:

$$u = g(x) = ax + b$$
$$\mathrm{d}u = a\,\mathrm{d}x$$
$$x = \frac{u-b}{a}$$

### C More about Linear Substitution

Integrals of the forms:

$$\int P_n(x)\sqrt[m]{ax+b}\,\mathrm{d}x$$
$$\int \frac{P_n(x)}{\sqrt[m]{ax+b}}\,\mathrm{d}x$$

where $P_n(x)$ is a polynomial function of degree $n$ may be solved by the linear substitution:

$$u = ax + b$$

### D Substitution Rule and Power Rule

If $u = g(x)$ then $\mathrm{d}u = g'(x)\,\mathrm{d}x$ and:

$$\int f^n(g(x))g'(x)\,\mathrm{d}x = \int f^n(u)\,\mathrm{d}u = \frac{f^{n+1}(u)}{n+1} + C = \frac{f^{n+1}(g(x))}{n+1} + C$$

## E Trigonometric Identities

$$\sin^2 x + \cos^2 x = 1$$

$$\sin(2x) = 2\sin(x)\cos(x)$$

$$\begin{aligned}\cos(2x) &= \cos^2(x) - \sin^2(x)\\ \cos(2x) &= 2\cos^2(x) - 1\\ \cos(2x) &= 1 - 2\sin^2(x)\end{aligned}$$

## F Substitution Rule for Definite Integrals

If $g'(x)$ is continuous on $[a, b]$ and $y = f(x)$ is continuous on the range of $u = g(x)$, then:

$$\mathrm{d}u = g'(x)\,\mathrm{d}x$$

and

$$\int_a^b f(g(x))g'(x)\,\mathrm{d}x = \int_{g(a)}^{g(b)} f(u)\,\mathrm{d}u$$

## G Symmetry

Let $y = f(x)$ be a continuous function on $[a, b]$.

If $y = f(x)$ is even, then $f(-x) = f(x)$ and

$$\int_{-a}^{a} f(x)\,\mathrm{d}x = 2\int_0^a f(x)\,\mathrm{d}x$$

If $y = f(x)$ is odd, then $f(-x) = -f(x)$ and

$$\int_{-a}^{a} f(x)\,\mathrm{d}x = 0$$

# Unit 5

# Applications of Integration

# 5.1 Applications of Integration

## A Area Under a Curve

The area under the curve $y = f(x)$ and the $x$-axis from $x = a$ to $x = b$ may be evaluated by:

$$A = \int_a^b f(x)\,\mathrm{d}x$$

**Notes:**

- On the intervals where $f(x) \geq 0$ the area $A \geq 0$.
- On the intervals where $f(x) \leq 0$ the area $A \leq 0$.
- If $y = f(x)$ does not change its sign on $[a, b]$, then the formula above represents the net (or algebraic) area under the curve.

## B Total Positive Area

The total positive area between $y = f(x)$ and the $x$-axis may be computed by:

$$A = \int_a^b \big|f(x)\big|\,\mathrm{d}x$$

## C Area between a Curve and the $y$-axis

The area under the curve $x = f(y)$ and the $y$-axis from $y = a$ to $y = b$ may be evaluated by:

$$A = \int_a^b f(y)\,\mathrm{d}y$$

## D Area between Two Curves

Let $f(x) \geq g(x)$ on $[a, b]$. The area between these curves may be evaluated by:

$$A = \int_a^b \big(f(x) - g(x)\big)\,\mathrm{d}x$$

If $f(x) - g(x)$ does not change sign on $[a, b]$, the use the following formula to evaluate the total area between these curves:

$$A = \int_a^b \big|f(x) - g(x)\big|\,\mathrm{d}x$$

## E Net Change

If $f(x)$ represents the instantaneous rate of change of a function $F(x)$, then $f(x) = F'(x)$ and the *net change* of the function $F(x)$ on the interval $[a, b]$ is given by the definite integral:

$$\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a)$$

## F Displacement

If $v(t)$ represents the instantaneous velocity of a position function $s(t)$, then $v(t) = s'(t) = \frac{\mathrm{d}s}{\mathrm{d}t}$ and the *net change* of the position function $s(t)$ on the interval $[a, b]$ is called *displacement* and is given by the definite integral:

$$\int_a^b v(t)\,\mathrm{d}t = s(b) - s(a)$$

## G Distance Travelled

Speed is defined by *speed* $= \big|v(t)\big|$ and the total distance travelled is defined by:

$$\int_a^b \big|v(t)\big|\,\mathrm{d}t$$

## H Velocity Change

If $a(t)$ represents the acceleration function of a velocity function $v(t)$, then $a(t) = v'(t) = \frac{\mathrm{d}v}{\mathrm{d}t}$ and the *net change* of the velocity function $v(t)$ on the interval $[a, b]$ is given by the definite integral:

$$\int_a^b a(t)\,\mathrm{d}t = v(b) - v(a)$$

## I Cost

If $C(x)$ is the cost of producing $x$ units, then the marginal cost is $C'(x)$ and the increase in cost when production is increase from $x = a$ units to $x = b$ units is given by the definite integral:

$$\int_a^b C'(x)\,\mathrm{d}x = C(b) - C(a)$$