MaCS Calculus and Vectors Exam Study Guide

Vincent Macri

2017 – 2018 — Semester 1

Contents

Unit 1

Equations of Lines and Planes

8.1 Vector and Parametric Equations of a Line in $\mathbb{R}^2$

8.1.A Vector Equation of a Line in $\mathbb{R}^2$

Consider the line L that passes through the point $P_0(x_0, y_0)$ and is parallel to the vector $\vec{u}$. The point $P(x, y)$ is a *generic point* on the line.

$$\overrightarrow{P_0P} = t\vec{u}$$
$$\overrightarrow{OP} - \overrightarrow{OP_0} = t\vec{u}$$
$$\vec{r} - \vec{r_0} = t\vec{u}$$

The *vector equation* of the line is:

$$\vec{r} = \vec{r_0} + t\vec{u} \mid t \in \mathbb{R}$$

Where:

- $\vec{r} = \overrightarrow{OP}$ is the *position vector* of a *generic point* P on the line.
- $\vec{r_0} = \overrightarrow{OP_0}$ is the *position vector* of a *specific point* P_0 on the line.
- $\vec{u}$ is a vector parallel to the line called the *direction vector* of the line.
- t is a *real number* corresponding to the generic point P.

Note: The vector equation of a line is *not unique*. It depends on the specific point P_0 and on the direction vector $\vec{u}$ that are used.

8.1.B Parametric Equations of a Line in $\mathbb{R}^2$

We can rewrite the vector equation of a line:

$$\vec{r} = \vec{r_0} + t\vec{u} \mid t \in \mathbb{R}$$

as:

$$(x, y) = (x_0, y_0) + t(u_x, u_y) \mid t \in \mathbb{R}$$

Split this vector equation into the *parametric equations* of a line in $\mathbb{R}^2$:

$$\begin{cases} x = x_0 + tu_x \\ y = y_0 + yu_y \end{cases} \quad t \in \mathbb{R}$$

8.1.C Parallel Lines

Two lines L_1 and L_2 with direction vectors $\overrightarrow{u_1}$ and $\overrightarrow{u_2}$ are *parallel* ($L_1 \parallel L_2$) if:

$$\overrightarrow{u_1} \parallel \overrightarrow{u_2}$$

or, there exists $k \in \mathbb{R}$ such that:

$$\overrightarrow{u_2} = k\overrightarrow{u_1}$$

or:

$$\overrightarrow{u_1} \times \overrightarrow{u_2} = \overrightarrow{0}$$

or scalar components are *proportional*:

$$\frac{u_{2x}}{u_{1x}} = \frac{u_{2u}}{u_{1u}} = k$$

8.1.D Perpendicular Lines

Two lines L_1 and L_2 with direction vectors $\overrightarrow{u_1}$ and $\overrightarrow{u_2}$ are *perpendicular* ($L_1 \perp L_2$) if:

$$\overrightarrow{u_1} \perp \overrightarrow{u_2}$$

or:

$$\overrightarrow{u_1} \cdot \overrightarrow{u_2} = 0$$

or:

$$u_{1x}u_{2x} + u_{1y}u_{2y} = 0$$

8.1.E 2D Perpendicular Vectors

Given a 2D vector $\vec{u} = (a, b)$, two 2D vectors perpendicular to $\vec{u}$ are $\vec{v} = (-b, a)$ and $\vec{w} = (b, -a)$.

Indeed:

$$\vec{u} \cdot \vec{v} = (a, b) \cdot (-b, a) = -ab + ab = 0 \implies \vec{u} \perp \vec{v}$$

8.1.F Special Lines

A line *parallel* to the x-axis has a direction vector in the form $\vec{u} = (u_x, 0) \mid u_x \neq 0$.

A line *parallel* to the y-axis has a direction vector in the form $\vec{u} = (0, u_y) \mid u_y \neq 0$.

8.2 Cartesian Equation of a Line

8.2.A Symmetric Equation

The parametric equations of a line in $\mathbb{R}^2$:

$$\begin{cases} x = x_0 + tu_x \\ y = y_0 + tu_y \end{cases} \quad t \in \mathbb{R}$$

may be written as:

$$\frac{x - x_0}{u_x} = \frac{y - y_0}{u_y} = t \mid t \in \mathbb{R}$$

The *symmetric equation* of the line is (if it exists):

$$\frac{x - x_0}{u_x} = \frac{y - y_0}{u_y}$$

Note: The symmetric equations only exists if $u_x \neq 0$ and $u_y \neq 0$.

8.2.B Normal Equation

Consider a line L that passes through the specific point $P_0(x_0, y_0)$ and has the *direction vector* $\vec{u} = (u_x, u_y)$.

The vectors $\vec{n} = (-u_y, u_x) = (A, B)$ or $\vec{n} = (u_y, -u_x) = (A, B)$ are perpendicular to the vector $\vec{u}$ and so they are perpendicular to the line L. These are called *normal vectors* to the line L.

Let $P(x, y)$ be a generic point on the line L. So:

$$\overrightarrow{P_0P} \parallel \vec{u} \implies \overrightarrow{P_0P} \perp \vec{n} \implies \overrightarrow{P_0P} \cdot \vec{n} = 0$$

$$(\vec{r} - \vec{r_0}) \cdot \vec{n} = 0$$

The *normal equation* of a line is given by:

$$(\vec{r} - \vec{r_0}) \cdot \vec{n} = 0$$

8.2.C Cartesian Equation

The normal equations can be written as:

$$\begin{aligned} \vec{r} \cdot \vec{n} - \vec{r_0} \cdot \vec{n} &= 0 \\ (x, y) \cdot (A, B) - (x_0, y_0) \cdot (A, B) &= 0 \\ Ax + By - Ax_0 - By_0 &= 0 \\ Ax + By + C &= 0 \qquad \text{where } C = -Ax_0 - By_0 \end{aligned}$$

The *Cartesian equation* of a line is given by:

$$Ax + By + C = 0$$

where $\vec{n} = (A, B)$ is a *normal vector* and the constant C depends on a specific point of the line.

8.2.D Slope y-intercept Equation

Solve the symmetric equation of a line:

$$\frac{x - x_0}{u_x} = \frac{y - y_0}{u_y} \mid t \in \mathbb{R}$$

for y:

$$\begin{aligned} y - y_0 &= u_y \frac{x - x_0}{u_x} \\ y &= \frac{u_y}{u_x} x + y_0 - \frac{u_y}{u_x} x_0 \end{aligned}$$

The *slope y-intercept equation* of a line in $\mathbb{R}^2$ is given by:

$$y = mx + b$$

$$m = \frac{u_y}{u_x}$$

where m is the *slope* and b is the *y-intercept* which depends on a specific point of the line.

8.2.E Angle between Two Lines

The *angle* between two lines is determined by the angle between the *direction vectors*:

$$\cos\theta = \frac{\vec{u_1} \cdot \vec{u_2}}{\|\vec{u_1}\|\|\vec{u_2}\|}$$

Note: There are two pairs of equal angles between the two lines. There is a pair of the angle θ_1, and a pair of the angle θ_2. $\theta_1 + \theta_2 = 180°$

8.3 Vector, Parametric, and Symmetric Equations of a Line in $\mathbb{R}^3$

8.3.A Vector Equation

The vector equation of the line is:

$$\vec{r} = \vec{r_0} + t\vec{u} \mid t \in \mathbb{R}$$

where:

- $\vec{r} = \overrightarrow{OP}$ is the position vector of a *generic* point P on the line.
- $\vec{r_0} = \overrightarrow{OP_0}$ is the position vector of a *specific* point P_0 on the line.
- $\vec{u}$ is a vector parallel to the line called the *direction vector* of the line.
- t is a *real number* corresponding to the generic point P.

8.3.B Specific Lines

A line is parallel to the x-axis if $\vec{u} = (u_x, 0, 0) \mid u_x \neq 0$. In this case, the line is also *perpendicular to the yz-plane.*

A line with $\vec{u} = (0, u_y, u_z) \mid u_y \neq 0 \wedge u_z \neq 0$ is *parallel to the yz-plane.*

8.3.C Parametric Equations

Rewrite the vector equation of a line:

$$\vec{r} = \vec{r_0} + t\vec{u} \mid t \in \mathbb{R}$$

as:

$$(x, y, z) = (x_0, y_0, z_0) + t(u_x, u_y, u_z) \mid t \in \mathbb{R}$$

The *parametric equations* of a line in $\mathbb{R}^3$ are:

$$\begin{cases} x = x_0 + tu_x \\ y = x_0 + tu_y \\ z = x_0 + tu_z \end{cases} \quad t \in \mathbb{R}$$

8.3.D Symmetric Equations

The parametric equations of a line may be written as:

$$\begin{cases} x = x_0 + tu_x \\ y = x_0 + tu_y \\ z = x_0 + tu_z \end{cases} \quad t \in \mathbb{R}$$

From here, the *symmetric equations* of the line are:

$$\frac{x - x_0}{u_x} = \frac{y - y_0}{u_y} = \frac{z - z_0}{u_z}$$

$$u_x \neq 0 \quad u_y \neq 0 \quad u_z \neq 0$$

8.3.E Intersections

A line *intersects the* x*-axis* when $y = z = 0$.

A line *intersects the* xy*-plane* when $z = 0$.

8.4 Vector and Parametric Equations of a Plane

8.4.A Planes

A plane may be determined by points and lines. There are four main possibilities:

1. Plane determined by three points.
2. Plane determined by two parallel lines.
3. Plane determined by two intersecting lines.
4. Plane determined by a point and a line.

8.4.B Vector Equation of a Plane

Consider a plane π.

Two vectors $\vec{u}$ and $\vec{v}$, parallel to the plane π but not parallel to each other, are called *direction vectors* of the plane π.

The vector $\overrightarrow{P_0P}$ from a specific point $P_0(x_0, y_0, z_0)$ to a generic point $P(x, y, z)$ of the plane is a *linear combination* of direction vectors $\vec{u}$ and $\vec{v}$:

$$\overrightarrow{P_0P} - s\vec{u} + t\vec{v} \mid s, t \in \mathbb{R}$$

The *vector equation* of the plane is:

$$\pi : \vec{r} = \vec{r_0} + s\vec{u} + t\vec{v} \mid s, t \in \mathbb{R}$$

8.4.C Parametric Equations of a Plane

We write the vector equation of the plane as:

$$(x, y, z) = (x_0, y_0, z_0) + s(u_x, u_y, u_z) + t(v_x, v_y, v_z)$$

or:

$$\begin{cases} x = x_0 + su_x + tv_x \\ y = y_0 + su_y + tv_y \\ z = z_0 + su_z + tv_z \end{cases} \quad s, t \in \mathbb{R}$$

These are the *parametric equations* of a plane.

8.5 Cartesian Equation of a Plane

8.5.A Normal Equation of a Plane

A plane may be determined by a *point* $P_0(x_0, y_0, z_0)$ and a *vector* perpendicular to the plane $\vec{n}$ called the *normal vector*.

If $P(x, y, z)$ is a generic point on the plane, then:

$$\overrightarrow{P_0P} \perp \vec{n}$$

and:

$$\overrightarrow{P_0P} \cdot \vec{n} = 0$$

This is the *normal equation* of a plane.

8.5.B Cartesian Equation of a Plane

We write the normal vector of a plane in the form:

$$\vec{n} = (A, B, C)$$

Then, the normal equation may be written as:

$$\begin{aligned} (x - x_0, y - y_0, z - z_0) \cdot (A, B, C) = 0 \\ Ax + By + Cz - Ax_0 - By_0 - Cz_0 = 0 \end{aligned}$$

or:

$$Ax + By + Cz + D = 0$$

which is called the *Cartesian equation* of a plane.

Note: A normal vector to the plane is:

$$\vec{n} = \vec{u} \times \vec{v}$$

where $\vec{u}$ and $\vec{v}$ are the direction vectors of the plane.

8.5.C Angle between Two Planes

The *angle* between two planes is defined as the angle between their *normal vectors*:

$$\cos\theta = \frac{\vec{n_1} \cdot \vec{n_2}}{\|\vec{n_1}\|\|\vec{n_2}\|}$$

Note: Using this formula, you may get an *acute* or an *obtuse* angle depending on the normal vectors which are used.

Unit 2

Relationships between Points, Lines, and Planes

9.1 Intersection of Two Lines

9.1.A Relative Position of Two Lines

Two lines may be:

1. Parallel and distinct.
2. Parallel and coincident.
3. Intersecting (not parallel).
4. Skew (not parallel, not intersecting).

9.1.B Intersection of Two Lines (Algebraic Method)

The point of intersection of two lines $L_1 : \overrightarrow{r} = \overrightarrow{r_{01}} + t\overrightarrow{u_1} \mid t \in \mathbb{R}$ and $L_2 : \overrightarrow{r} = \overrightarrow{r_{02}} + s\overrightarrow{u_2} \mid s \in \mathbb{R}$ is given by the *solution* of the following system of equations (if it exists):

$$\begin{cases} x_{01} + tu_{x1} = x_{02} + su_{x2} \\ y_{01} + tu_{y1} = y_{02} + su_{y2} \\ z_{01} + tu_{z1} = z_{02} + su_{z2} \end{cases} \quad s, t \in \mathbb{R}$$

Hint: Solve by *substitution* or *elimination* the system of two equations and *check* if the third is satisfied.

9.1.C Unique Solution

If by solving the system you end by getting a *unique* value for t and s satisfying this system, then the lines have a *unique point of intersection*. To get this point, substitute either the t value into the line L_1 equation or substitute the s value into the line L_2 equation.

9.1.D Infinite Number of Solutions

If by solving the system you end by getting two true statements (like $2 = 2$) and one equation in s and t, then there exist an *infinite number of solutions* of the system. Therefore the lines intersect at an *infinite number of points*. In this case the lines are parallel and coincident.

9.1.E No Solution (Parallel Lines)

If by solving the system you get at least one *false* statement (like $0 = 1$) then the system has *no solution*. Therefore, the lines have *no point of intersection*. If, in addition, the lines are parallel ($\overrightarrow{u_1} \times \overrightarrow{u_2} = \overrightarrow{0}$), then the lines are *parallel and distinct*.

9.1.F No Solution (Skew Lines)

If by solving the system you get at least one *false* statement (like $0 = 1$) then the system has *no solution*. Therefore, the lines have *no point of intersection*. If, in addition, the lines are *not parallel* ($\overrightarrow{u_1} \times \overrightarrow{u_2} \neq \overrightarrow{0}$), then the lines are *skew*.

9.1.G Classifying Lines (Vector Method)

Parallel lines

$\overrightarrow{u_1} \times \overrightarrow{u_2} = \overrightarrow{0}$

$(\overrightarrow{r_{01}} - \overrightarrow{r_{02}}) \times \overrightarrow{u_1} = \overrightarrow{0}$ Parallel coincident lines

$(\overrightarrow{r_{01}} - \overrightarrow{r_{02}}) \times \overrightarrow{u_1} \neq \overrightarrow{0}$ Parallel distinct lines

Nonparallel lines

$\overrightarrow{u_1} \times \overrightarrow{u_2} \neq \overrightarrow{0}$

$(\overrightarrow{r_{01}} - \overrightarrow{r_{02}}) \cdot (\overrightarrow{u_1} \times \overrightarrow{u_2}) = 0$ Nonparallel intersecting lines

$(\overrightarrow{r_{01}} - \overrightarrow{r_{02}}) \cdot (\overrightarrow{u_1} \times \overrightarrow{u_2}) \neq 0$ Nonparallel skew lines

9.2 Intersection of a Line with a Plane

9.2.A Relative Position of a Line and a Plane

There are three possible situations:

1. The line *intersects* the plane at a single point.
$$P = L \cap \pi$$
2. The line *lies* on the plane. There are an infinite number of points of intersection.
$$L = L \cap \pi$$
3. The line is *parallel* to the plane but *distinct*. There is no point of intersection.
$$L \cap \pi = \varnothing$$

9.2.B Intersection of a Line and a Plane (Algebraic Method)

To get the intersection between a line L and a plane π:

1. *Substitute* the parametric equations of the line
$$L : \begin{cases} x = x_0 + tu_x \\ y = y_0 + tu_y \\ z = z_0 + tu_z \end{cases} \quad t \in \mathbb{R}$$
into the Cartesian equation of the plane
$$\pi : Ax + By + Cz + D = 0$$
to get the equation:
$$A(x_0 + tu_x) + B(y_0 + tu_y) + C(z_0 + tu_z) + D = 0 \tag{i}$$
2. *Solve* (if possible) the equation (i) for the parameter t.
3. *Substitute* the value of the parameter t into the parametric equations of the line to get the point of intersection.

9.2.C Unique Solution (Point Intersection)

In this case, by solving the equation you get a *unique value* for the parameter t. Therefore, there is a unique *point of intersection* between the line and the plane.

$$P = L \cap \pi$$

The line *intersects* the plane at a unique point.

9.2.D Infinite Number of Solutions (Line Intersection)

In this case, by solving the equation (i) you get the equation:

$$0t = 0$$

which has an *infinite number of solutions*. Therefore, there are an *infinite number of points of intersection*.

$$L = L \cap \pi$$

The line *lies* on the plane.

9.2.E No Solution (No Intersection)

In this case, by solving the equation (i) you get a false statement like:

$$0t = 1$$

The equation *does not have any solution* and therefore there is *no point of intersection* between the line and the plane.

$$L \cap \pi = \varnothing$$

The line is *parallel* to the plane and *does not lie* on the plane.

9.2.F Classifying Lines

Consider the line $L : \vec{r} = \vec{r_0} + t\vec{u} \mid t \in \mathbb{R}$, where $P_0(x_0, y_0, z_0)$ is a specific point on the line, and the plane $\pi : Ax + By + Cz + D = 0$, where $\vec{n} = (A, B, C)$ is a normal vector to the plane.

1. If $\vec{n} \cdot \vec{u} \neq 0$ the line *intersects* the plane at a unique point.

$$P = L \cap \pi$$

2. If $\vec{n} \cdot \vec{u} = 0$ and $Ax_0 + By_0 + Cz_0 + D = 0$ then the line *lies* on the plane.

$$L = L \cap \pi$$

3. If $\vec{n} \cdot \vec{u} = 0$ and $Ax_0 + By_0 + Cz_0 + D \neq 0$ then the line is *parallel* to the plane but *does not lie* on the plane.

$$L \cap \pi = \varnothing$$

Note. By solving the equation (i) **for** t **you will end by getting the same cases and conditions as above.**

9.3 Intersection of Two Planes

9.3.A Relative Position of Two Planes

Two planes may be:

1. Intersecting (into a line)
$$L = \pi_1 \cap \pi_2$$
2. Coincident
$$\pi_1 = \pi_1 \cap \pi_2 = \pi_2$$
3. Distinct
$$\pi_1 \cap \pi_2 = \varnothing$$

9.3.B Intersection of Two Planes

Consider two planes given by their Cartesian equations:

$$\pi_1 = A_1x + B_1y + C_1z + D_1 = 0$$

$$\pi_2 = A_2x + B_2y + C_2z + D_2 = 0$$

To find the point(s) of intersection between two planes, *solve* the system of equations formed by their Cartesian equations:

$$\begin{cases}\pi_1 = A_1x + B_1y + C_1z + D_1 = 0\pi_2 = A_2x + B_2y + C_2z + D_2 = 0\end{cases} \tag{ii}$$

There are *two* equations and *three* unknowns. **Notes:**

1. A normal vector to the plane π_1 is $\overrightarrow{n_1} = (A_1, B_1, C_1)$ and a normal vector to the plane π_2 is $\overrightarrow{n_2} = (A_2, B_2, C_2)$.
2. If the planes are *parallel* then the coefficients A, B, and C are *proportional.*
3. If the planes are *coincident* then the coefficients A, B, C, and D are *proportional.*
4. A system of equations is called *compatible* if there is *at least* one solution. A system of equations is called *incompatible* if there is *no solution.*

9.3.C Nonparallel Planes (Line Intersection)

In this case:

$$L = \pi_1 \cap \pi_2$$

- The coefficients A, B, and C in the scalar equations are *not proportional.*

- The normal vectors are *not parallel*: $\vec{n_1} \times \vec{n_2} \neq \vec{0}$.
- By solving the system (ii) you will be able to find two variables in terms of the third variable.
- There are an *infinite number of solutions* and therefore an *infinite number of points of intersection*.
- The intersection is a *line* and a *direction vector* for this line is $\vec{u} = \vec{n_1} \times \vec{n_2}$.

9.3.D Coincident Planes (Plane Intersection)

In this case:

$$\pi_1 = \pi_1 \cap \pi_2 = \pi_2$$

- The planes are *parallel* and *coincident*.
- The coefficients A, B, C, and D in the scalar equations are *proportional*.
- One equation in the system (ii) is a *multiple* of the other equation and does not contain additional information (the equations are equivalent).
- By solving the system of equations (ii), you get a *true* statement (like $0 = 0$).
- There are an *infinite number of solutions* and therefore an *infinite number of points of intersection*.
- The intersection is a *plane*.

9.3.E Parallel and Distinct Planes (No Intersection)

In this case:

$$\pi_1 \cap \pi_2 = \varnothing$$

- The planes are *parallel* and *distinct*.
- The coefficients A, B, and C in the scalar equations are *proportional* but the coefficients A, B, C, and D are *not proportional*.
- By solving the system (ii) you get a *false* statement (like $0 = 1$).
- There is *no solution* and therefore *no point of intersection* between the two planes.

Appendix A

Credit

Thank you for everything, Mrs. Gugoiu.