

# MaCS Calculus and Vectors Exam Study Guide

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# Contents

<b>1</b>	<b>Equations of Lines and Planes</b>	<b>1</b>
8.1	Vector and Parametric Equations of a Line in $\mathbb{R}^2$	2
8.1.A	Vector Equation of a Line in $\mathbb{R}^2$	2
8.1.B	Parametric Equations of a Line in $\mathbb{R}^2$	2
8.1.C	Parallel Lines	3
8.1.D	Perpendicular Lines	3
8.1.E	2D Perpendicular Vectors	3
8.1.F	Special Lines	3
8.2	Cartesian Equation of a Line	4
8.2.A	Symmetric Equation	4
8.2.B	Normal Equation	4
8.2.C	Cartesian Equation	4
8.2.D	Slope $y$ -intercept Equation	5
8.2.E	Angle between Two Lines	5
8.3	Vector, Parametric, and Symmetric Equations of a Line in $\mathbb{R}^3$	6
8.3.A	Vector Equation	6
8.3.B	Specific Lines	6
8.3.C	Parametric Equations	6
8.3.D	Symmetric Equations	7
8.3.E	Intersections	7
8.4	Vector and Parametric Equations of a Plane	8
8.4.A	Planes	8
8.4.B	Vector Equation of a Plane	8
8.4.C	Parametric Equations of a Plane	8
8.5	Cartesian Equation of a Plane	9
8.5.A	Normal Equation of a Plane	9
8.5.B	Cartesian Equation of a Plane	9
8.5.C	Angle between Two Planes	9
<b>2</b>	<b>Relationships between Points, Lines, and Planes</b>	<b>10</b>
9.1	Intersection of Two Lines	11
9.1.A	Relative Position of Two Lines	11
9.1.B	Intersection of Two Lines (Algebraic Method)	11
9.1.C	Unique Solution	11

9.1.D	Infinite Number of Solutions . . . . .	11
9.1.E	No Solution (Parallel Lines) . . . . .	12
9.1.F	No Solution (Skew Lines) . . . . .	12
9.1.G	Classifying Lines (Vector Method) . . . . .	12
9.2	Intersection of a Line with a Plane . . . . .	13
9.2.A	Relative Position of a Line and a Plane . . . . .	13
9.2.B	Intersection of a Line and a Plane (Algebraic Method) . . . .	13
9.2.C	Unique Solution (Point Intersection) . . . . .	13
9.2.D	Infinite Number of Solutions (Line Intersection) . . . . .	14
9.2.E	No Solution (No Intersection) . . . . .	14
9.2.F	Classifying Lines . . . . .	14
9.3	Intersection of Two Planes . . . . .	15
9.3.A	Relative Position of Two Planes . . . . .	15
9.3.B	Intersection of Two Planes . . . . .	15
9.3.C	Nonparallel Planes (Line Intersection) . . . . .	15
9.3.D	Coincident Planes (Plane Intersection) . . . . .	16
9.3.E	Parallel and Distinct Planes (No Intersection) . . . . .	16
9.4	Intersection of Three Planes . . . . .	17
9.4.A	Intersection of Three Planes . . . . .	17
9.4.B	Unique Solution (Point Intersection — Noncoplanar Normal Vectors) . . . . .	17
9.4.C	Infinite Number of Solutions (Line Intersection — Nonparallel Planes and Coplanar Normal Vectors) . . . . .	17
9.4.D	Infinite Number of Solutions (Line Intersection — Two Coinci- dent Planes and One Intersecting Plane) . . . . .	18
9.4.E	Infinite Number of Solutions (Plane Intersection — Three Co- incident Planes) . . . . .	18
9.4.F	No Solution (Parallel and Distinct Planes) . . . . .	18
9.4.G	No Solution (H Configuration) . . . . .	19
9.4.H	No Solution (Three Parallel Planes but only Two Coincident Planes) . . . . .	19
9.4.I	No Solution (Delta Configuration) . . . . .	19
9.5	Distance from a Point to a Line . . . . .	21
9.5.A	Distance from a Point to a Line in $\mathbb{R}^2$ . . . . .	21
9.5.B	Distance from a Point to a Line in $\mathbb{R}^3$ . . . . .	21
9.5.C	Distance between Two Parallel Lines . . . . .	22
9.5.D	Perpendicular Line from a Point to a Line . . . . .	22
9.5.E	Shortest Distance between Two Skew Lines . . . . .	22
9.6	Distance from a Point to a Plane . . . . .	23
9.6.A	Distance from a Point to a Plane (I) . . . . .	23
9.6.B	Distance from a Point to a Plane (II) . . . . .	23
9.6.C	Distance between Two Parallel Planes . . . . .	23
1.4	Limit of a Function . . . . .	25
1.4.A	One-Sided Limits . . . . .	25
1.4.B	Limit . . . . .	25

1.4.C	Substitution . . . . .	25
1.4.D	Piecewise defined functions (AP only) . . . . .	26
1.4.E	Limits: Numerical Approach (AP only) . . . . .	26
1.4.F	Limit: Informal Definitions (AP only) . . . . .	26
<b>AP Preparation Differentiability Review</b>		<b>25</b>
1.6	Continuity . . . . .	28
1.6.A	Continuity . . . . .	28
1.6.B	Discontinuity . . . . .	28
1.6.C	Removable Discontinuity . . . . .	28
1.6.D	Jump Discontinuity . . . . .	28
1.6.E	Infinite Discontinuity . . . . .	29
1.6.F	Continuity over an Interval (AP only) . . . . .	29
1.6.G	Elementary Functions (AP only) . . . . .	29
1.6.H	Composition of Functions . . . . .	29
1.6.I	Intermediate Value Theorem (AP only) . . . . .	29
2.1	Derivative Function . . . . .	30
2.1.A	Derivative Function . . . . .	30
2.1.B	Differentiability (AP only) . . . . .	30
2.1.C	Interpretations of Derivative Function . . . . .	30
2.1.D	Notations and Reading . . . . .	30
2.1.E	First Principles . . . . .	31
2.1.F	Differentiability Point . . . . .	31
2.1.G	Non-Differentiability . . . . .	31
2.1.H	Corner Point . . . . .	31
2.1.I	Infinite Slope Point . . . . .	32
<b>A Credit</b>		<b>33</b>

# Unit 1

## Equations of Lines and Planes

## 8.1 Vector and Parametric Equations of a Line in $\mathbb{R}^2$

### 8.1.A Vector Equation of a Line in $\mathbb{R}^2$

Consider the line  $L$  that passes through the point  $P_0(x_0, y_0)$  and is parallel to the vector  $\vec{u}$ . The point  $P(x, y)$  is a *generic point* on the line.

$$\begin{aligned}\overrightarrow{P_0P} &= t\vec{u} \\ \overrightarrow{OP} - \overrightarrow{OP_0} &= t\vec{u} \\ \vec{r} - \vec{r_0} &= t\vec{u}\end{aligned}$$

The *vector equation* of the line is:

$$\vec{r} = \vec{r_0} + t\vec{u} \mid t \in \mathbb{R}$$

Where:

- $\vec{r} = \overrightarrow{OP}$  is the *position vector* of a *generic point*  $P$  on the line.
- $\vec{r_0} = \overrightarrow{OP_0}$  is the *position vector* of a *specific point*  $P_0$  on the line.
- $\vec{u}$  is a vector parallel to the line called the *direction vector* of the line.
- $t$  is a *real number* corresponding to the generic point  $P$ .

**Note:** The vector equation of a line is *not unique*. It depends on the specific point  $P_0$  and on the direction vector  $\vec{u}$  that are used.

### 8.1.B Parametric Equations of a Line in $\mathbb{R}^2$

We can rewrite the vector equation of a line:

$$\vec{r} = \vec{r_0} + t\vec{u} \mid t \in \mathbb{R}$$

as:

$$(x, y) = (x_0, y_0) + t(u_x, u_y) \mid t \in \mathbb{R}$$

Split this vector equation into the *parametric equations* of a line in  $\mathbb{R}^2$ :

$$\begin{cases} x = x_0 + tu_x \\ y = y_0 + tu_y \end{cases} \quad t \in \mathbb{R}$$

### 8.1.C Parallel Lines

Two lines  $L_1$  and  $L_2$  with direction vectors  $\vec{u}_1$  and  $\vec{u}_2$  are *parallel* ( $L_1 \parallel L_2$ ) if:

$$\vec{u}_1 \parallel \vec{u}_2$$

or, there exists  $k \in \mathbb{R}$  such that:

$$\vec{u}_2 = k\vec{u}_1$$

or:

$$\vec{u}_1 \times \vec{u}_2 = \vec{0}$$

or scalar components are *proportional*:

$$\frac{u_{2x}}{u_{1x}} = \frac{u_{2y}}{u_{1y}} = k$$

### 8.1.D Perpendicular Lines

Two lines  $L_1$  and  $L_2$  with direction vectors  $\vec{u}_1$  and  $\vec{u}_2$  are *perpendicular* ( $L_1 \perp L_2$ ) if:

$$\vec{u}_1 \perp \vec{u}_2$$

or:

$$\vec{u}_1 \cdot \vec{u}_2 = 0$$

or:

$$u_{1x}u_{2x} + u_{1y}u_{2y} = 0$$

### 8.1.E 2D Perpendicular Vectors

Given a 2D vector  $\vec{u} = (a, b)$ , two 2D vectors perpendicular to  $\vec{u}$  are  $\vec{v} = (-b, a)$  and  $\vec{w} = (b, -a)$ .

Indeed:

$$\vec{u} \cdot \vec{v} = (a, b) \cdot (-b, a) = -ab + ab = 0 \implies \vec{u} \perp \vec{v}$$

### 8.1.F Special Lines

A line *parallel* to the  $x$ -axis has a direction vector in the form  $\vec{u} = (u_x, 0) \mid u_x \neq 0$ .

A line *parallel* to the  $y$ -axis has a direction vector in the form  $\vec{u} = (0, u_y) \mid u_y \neq 0$ .

## 8.2 Cartesian Equation of a Line

### 8.2.A Symmetric Equation

The parametric equations of a line in  $\mathbb{R}^2$ :

$$\begin{cases} x = x_0 + tu_x \\ y = y_0 + tu_y \end{cases} \quad t \in \mathbb{R}$$

may be written as:

$$\frac{x - x_0}{u_x} = \frac{y - y_0}{u_y} = t \mid t \in \mathbb{R}$$

The *symmetric equation* of the line is (if it exists):

$$\frac{x - x_0}{u_x} = \frac{y - y_0}{u_y}$$

**Note:** The symmetric equations only exists if  $u_x \neq 0$  and  $u_y \neq 0$ .

### 8.2.B Normal Equation

Consider a line  $L$  that passes through the specific point  $P_0(x_0, y_0)$  and has the *direction vector*  $\vec{u} = (u_x, u_y)$ .

The vectors  $\vec{n} = (-u_y, u_x) = (A, B)$  or  $\vec{n} = (u_y, -u_x) = (A, B)$  are perpendicular to the vector  $\vec{u}$  and so they are perpendicular to the line  $L$ . These are called *normal vectors* to the line  $L$ .

Let  $P(x, y)$  be a generic point on the line  $L$ . So:

$$\begin{aligned} \overrightarrow{P_0P} \parallel \vec{u} &\implies \overrightarrow{P_0P} \perp \vec{n} \implies \overrightarrow{P_0P} \cdot \vec{n} = 0 \\ (\vec{r} - \vec{r}_0) \cdot \vec{n} &= 0 \end{aligned}$$

The *normal equation* of a line is given by:

$$(\vec{r} - \vec{r}_0) \cdot \vec{n} = 0$$

### 8.2.C Cartesian Equation

The normal equations can be written as:

$$\begin{aligned} \vec{r} \cdot \vec{n} - \vec{r}_0 \cdot \vec{n} &= 0 \\ (x, y) \cdot (A, B) - (x_0, y_0) \cdot (A, B) &= 0 \\ Ax + By - Ax_0 - By_0 &= 0 \\ Ax + By + C &= 0 \quad \text{where } C = -Ax_0 - By_0 \end{aligned}$$



The *Cartesian equation* of a line is given by:

$$Ax + By + C = 0$$

where  $\vec{n} = (A, B)$  is a *normal vector* and the constant  $C$  depends on a specific point of the line.

## 8.2.D Slope $y$ -intercept Equation

Solve the symmetric equation of a line:

$$\frac{x - x_0}{u_x} = \frac{y - y_0}{u_y} \mid t \in \mathbb{R}$$

for  $y$ :

$$\begin{aligned} y - y_0 &= u_y \frac{x - x_0}{u_x} \\ y &= \frac{u_y}{u_x} x + y_0 - \frac{u_y}{u_x} x_0 \end{aligned}$$

The *slope  $y$ -intercept equation* of a line in  $\mathbb{R}^2$  is given by:

$$y = mx + b$$

$$m = \frac{u_y}{u_x}$$

where  $m$  is the *slope* and  $b$  is the  *$y$ -intercept* which depends on a specific point of the line.

## 8.2.E Angle between Two Lines

The *angle* between two lines is determined by the angle between the *direction vectors*:

$$\cos \theta = \frac{\vec{u}_1 \cdot \vec{u}_2}{\|\vec{u}_1\| \|\vec{u}_2\|}$$

**Note:** There are two pairs of equal angles between the two lines. There is a pair of the angle  $\theta_1$ , and a pair of the angle  $\theta_2$ .  $\theta_1 + \theta_2 = 180^\circ$

## 8.3 Vector, Parametric, and Symmetric Equations of a Line in $\mathbb{R}^3$

### 8.3.A Vector Equation

The vector equation of the line is:

$$\vec{r} = \vec{r}_0 + t\vec{u} \mid t \in \mathbb{R}$$

where:

- $\vec{r} = \overrightarrow{OP}$  is the position vector of a *generic* point  $P$  on the line.
- $\vec{r}_0 = \overrightarrow{OP_0}$  is the position vector of a *specific* point  $P_0$  on the line.
- $\vec{u}$  is a vector parallel to the line called the *direction vector* of the line.
- $t$  is a *real number* corresponding to the generic point  $P$ .

### 8.3.B Specific Lines

A line is parallel to the  $x$ -axis if  $\vec{u} = (u_x, 0, 0) \mid u_x \neq 0$ . In this case, the line is also *perpendicular to the  $yz$ -plane*.

A line with  $\vec{u} = (0, u_y, u_z) \mid u_y \neq 0 \wedge u_z \neq 0$  is *parallel to the  $yz$ -plane*.

### 8.3.C Parametric Equations

Rewrite the vector equation of a line:

$$\vec{r} = \vec{r}_0 + t\vec{u} \mid t \in \mathbb{R}$$

as:

$$(x, y, z) = (x_0, y_0, z_0) + t(u_x, u_y, u_z) \mid t \in \mathbb{R}$$

The *parametric equations* of a line in  $\mathbb{R}^3$  are:

$$\begin{cases} x = x_0 + tu_x \\ y = x_0 + tu_y \\ z = x_0 + tu_z \end{cases} \quad t \in \mathbb{R}$$

### 8.3.D Symmetric Equations

The parametric equations of a line may be written as:

$$\begin{cases} x = x_0 + tu_x \\ y = x_0 + tu_y \\ z = x_0 + tu_z \end{cases} \quad t \in \mathbb{R}$$

From here, the *symmetric equations* of the line are:

$$\frac{x - x_0}{u_x} = \frac{y - y_0}{u_y} = \frac{z - z_0}{u_z}$$
$$u_x \neq 0 \quad u_y \neq 0 \quad u_z \neq 0$$

### 8.3.E Intersections

A line *intersects the  $x$ -axis* when  $y = z = 0$ .

A line *intersects the  $xy$ -plane* when  $z = 0$ .

## 8.4 Vector and Parametric Equations of a Plane

### 8.4.A Planes

A plane may be determined by points and lines. There are four main possibilities:

1. Plane determined by three points.
2. Plane determined by two parallel lines.
3. Plane determined by two intersecting lines.
4. Plane determined by a point and a line.

### 8.4.B Vector Equation of a Plane

Consider a plane  $\pi$ .

Two vectors  $\vec{u}$  and  $\vec{v}$ , parallel to the plane  $\pi$  but not parallel to each other, are called *direction vectors* of the plane  $\pi$ .

The vector  $\overrightarrow{P_0P}$  from a specific point  $P_0(x_0, y_0, z_0)$  to a generic point  $P(x, y, z)$  of the plane is a *linear combination* of direction vectors  $\vec{u}$  and  $\vec{v}$ :

$$\overrightarrow{P_0P} = s\vec{u} + t\vec{v} \mid s, t \in \mathbb{R}$$

The *vector equation* of the plane is:

$$\pi : \vec{r} = \vec{r}_0 + s\vec{u} + t\vec{v} \mid s, t \in \mathbb{R}$$

### 8.4.C Parametric Equations of a Plane

We write the vector equation of the plane as:

$$(x, y, z) = (x_0, y_0, z_0) + s(u_x, u_y, u_z) + t(v_x, v_y, v_z)$$

or:

$$\begin{cases} x = x_0 + su_x + tv_x \\ y = y_0 + su_y + tv_y \\ z = z_0 + su_z + tv_z \end{cases} \quad s, t \in \mathbb{R}$$

These are the *parametric equations* of a plane.

## 8.5 Cartesian Equation of a Plane

### 8.5.A Normal Equation of a Plane

A plane may be determined by a *point*  $P_0(x_0, y_0, z_0)$  and a *vector* perpendicular to the plane  $\vec{n}$  called the *normal vector*.

If  $P(x, y, z)$  is a generic point on the plane, then:

$$\overrightarrow{P_0P} \perp \vec{n}$$

and:

$$\overrightarrow{P_0P} \cdot \vec{n} = 0$$

This is the *normal equation* of a plane.

### 8.5.B Cartesian Equation of a Plane

We write the normal vector of a plane in the form:

$$\vec{n} = (A, B, C)$$

Then, the normal equation may be written as:

$$\begin{aligned}(x - x_0, y - y_0, z - z_0) \cdot (A, B, C) &= 0 \\ Ax + By + Cz - Ax_0 - By_0 - Cz_0 &= 0\end{aligned}$$

or:

$$Ax + By + Cz + D = 0$$

which is called the *Cartesian equation* of a plane.

**Note:** A normal vector to the plane is:

$$\vec{n} = \vec{u} \times \vec{v}$$

where  $\vec{u}$  and  $\vec{v}$  are the direction vectors of the plane.

### 8.5.C Angle between Two Planes

The *angle* between two planes is defined as the angle between their *normal vectors*:

$$\cos \theta = \frac{\vec{n}_1 \cdot \vec{n}_2}{\|\vec{n}_1\| \|\vec{n}_2\|}$$

**Note:** Using this formula, you may get an *acute* or an *obtuse* angle depending on the normal vectors which are used.

## Unit 2

# Relationships between Points, Lines, and Planes

## 9.1 Intersection of Two Lines

### 9.1.A Relative Position of Two Lines

Two lines may be:

1. Parallel and distinct.
2. Parallel and coincident.
3. Intersecting (not parallel).
4. Skew (not parallel, not intersecting).

### 9.1.B Intersection of Two Lines (Algebraic Method)

The point of intersection of two lines  $L_1 : \vec{r} = \vec{r}_{01} + t\vec{u}_1 \mid t \in \mathbb{R}$  and  $L_2 : \vec{r} = \vec{r}_{02} + s\vec{u}_2 \mid s \in \mathbb{R}$  is given by the *solution* of the following system of equations (if it exists):

$$\begin{cases} x_{01} + tu_{x1} = x_{02} + su_{x2} \\ y_{01} + tu_{y1} = y_{02} + su_{y2} \\ z_{01} + tu_{z1} = z_{02} + su_{z2} \end{cases} \quad s, t \in \mathbb{R}$$

**Hint:** Solve by *substitution* or *elimination* the system of two equations and *check* if the third is satisfied.

### 9.1.C Unique Solution

If by solving the system you end by getting a *unique* value for  $t$  and  $s$  satisfying this system, then the lines have a *unique point of intersection*. To get this point, substitute either the  $t$  value into the line  $L_1$  equation or substitute the  $s$  value into the line  $L_2$  equation.

### 9.1.D Infinite Number of Solutions

If by solving the system you end by getting two true statements (like  $2 = 2$ ) and one equation in  $s$  and  $t$ , then there exist an *infinite number of solutions* of the system. Therefore the lines intersect at an *infinite number of points*. In this case the lines are parallel and coincident.

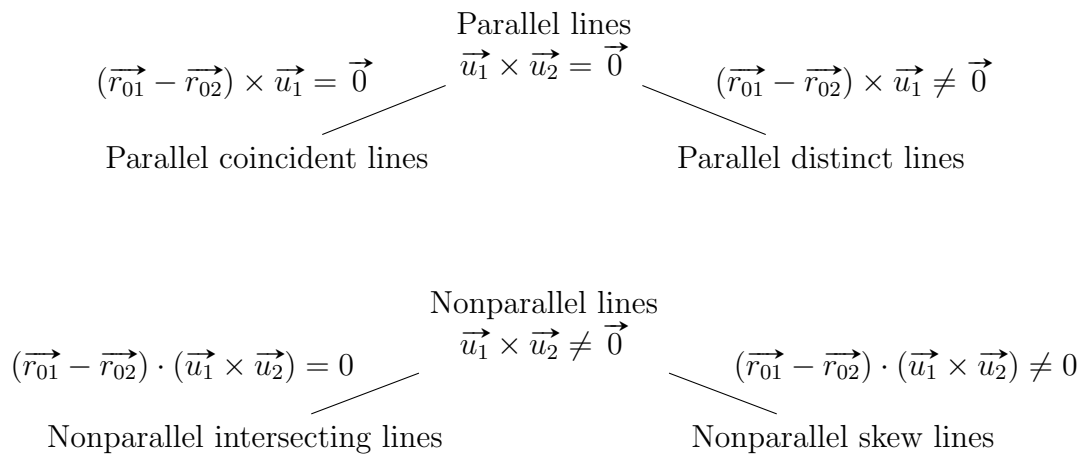
### 9.1.E No Solution (Parallel Lines)

If by solving the system you get at least one *false* statement (like  $0 = 1$ ) then the system has *no solution*. Therefore, the lines have *no point of intersection*. If, in addition, the lines are parallel ( $\vec{u}_1 \times \vec{u}_2 = \vec{0}$ ), then the lines are *parallel and distinct*.

### 9.1.F No Solution (Skew Lines)

If by solving the system you get at least one *false* statement (like  $0 = 1$ ) then the system has *no solution*. Therefore, the lines have *no point of intersection*. If, in addition, the lines are *not parallel* ( $\vec{u}_1 \times \vec{u}_2 \neq \vec{0}$ ), then the lines are *skew*.

### 9.1.G Classifying Lines (Vector Method)





## 9.2 Intersection of a Line with a Plane

### 9.2.A Relative Position of a Line and a Plane

There are three possible situations:

1. The line *intersects* the plane at a single point.

$$P = L \cap \pi$$

2. The line *lies* on the plane. There are an infinite number of points of intersection.

$$L = L \cap \pi$$

3. The line is *parallel* to the plane but *distinct*. There is no point of intersection.

$$L \cap \pi = \emptyset$$

### 9.2.B Intersection of a Line and a Plane (Algebraic Method)

To get the intersection between a line  $L$  and a plane  $\pi$ :

1. *Substitute* the parametric equations of the line

$$L : \begin{cases} x = x_0 + tu_x \\ y = y_0 + tu_y \\ z = z_0 + tu_z \end{cases} \quad t \in \mathbb{R}$$

into the Cartesian equation of the plane

$$\pi : Ax + By + Cz + D = 0$$

to get the equation:

$$A(x_0 + tu_x) + B(y_0 + tu_y) + C(z_0 + tu_z) + D = 0 \quad (\text{i})$$

2. *Solve* (if possible) the equation (i) for the parameter  $t$ .
3. *Substitute* the value of the parameter  $t$  into the parametric equations of the line to get the point of intersection.

### 9.2.C Unique Solution (Point Intersection)

In this case, by solving the equation you get a *unique value* for the parameter  $t$ . Therefore, there is a unique *point of intersection* between the line and the plane.

$$P = L \cap \pi$$

The line *intersects* the plane at a unique point.

### 9.2.D Infinite Number of Solutions (Line Intersection)

In this case, by solving the equation (i) you get the equation:

$$0t = 0$$

which has an *infinite number of solutions*. Therefore, there are an *infinite number of points of intersection*.

$$L = L \cap \pi$$

The line *lies* on the plane.

### 9.2.E No Solution (No Intersection)

In this case, by solving the equation (i) you get a false statement like:

$$0t = 1$$

The equation *does not have any solution* and therefore there is *no point of intersection* between the line and the plane.

$$L \cap \pi = \emptyset$$

The line is *parallel* to the plane and *does not lie* on the plane.

### 9.2.F Classifying Lines

Consider the line  $L : \vec{r} = \vec{r}_0 + t\vec{u} \mid t \in \mathbb{R}$ , where  $P_0(x_0, y_0, z_0)$  is a specific point on the line, and the plane  $\pi : Ax + By + Cz + D = 0$ , where  $\vec{n} = (A, B, C)$  is a normal vector to the plane.

1. If  $\vec{n} \cdot \vec{u} \neq 0$  the line *intersects* the plane at a unique point.

$$P = L \cap \pi$$

2. If  $\vec{n} \cdot \vec{u} = 0$  and  $Ax_0 + By_0 + Cz_0 + D = 0$  then the line *lies* on the plane.

$$L = L \cap \pi$$

3. If  $\vec{n} \cdot \vec{u} = 0$  and  $Ax_0 + By_0 + Cz_0 + D \neq 0$  then the line is *parallel* to the plane but *does not lie* on the plane.

$$L \cap \pi = \emptyset$$

**Note.** By solving the equation (i) for  $t$  you will end by getting the same cases and conditions as above.

## 9.3 Intersection of Two Planes

### 9.3.A Relative Position of Two Planes

Two planes may be:

1. Intersecting (into a line)

$$L = \pi_1 \cap \pi_2$$

2. Coincident

$$\pi_1 = \pi_1 \cap \pi_2 = \pi_2$$

3. Distinct

$$\pi_1 \cap \pi_2 = \emptyset$$

### 9.3.B Intersection of Two Planes

Consider two planes given by their Cartesian equations:

$$\pi_1 = A_1x + B_1y + C_1z + D_1 = 0$$

$$\pi_2 = A_2x + B_2y + C_2z + D_2 = 0$$

To find the point(s) of intersection between two planes, *solve* the system of equations formed by their Cartesian equations:

$$\begin{cases} \pi_1 = A_1x + B_1y + C_1z + D_1 = 0 \\ \pi_2 = A_2x + B_2y + C_2z + D_2 = 0 \end{cases} \quad (\text{ii})$$

There are *two* equations and *three* unknowns. **Notes:**

1. A normal vector to the plane  $\pi_1$  is  $\vec{n}_1 = (A_1, B_1, C_1)$  and a normal vector to the plane  $\pi_2$  is  $\vec{n}_2 = (A_2, B_2, C_2)$ .
2. If the planes are *parallel* then the coefficients  $A$ ,  $B$ , and  $C$  are *proportional*.
3. If the planes are *coincident* then the coefficients  $A$ ,  $B$ ,  $C$ , and  $D$  are *proportional*.
4. A system of equations is called *compatible* if there is *at least* one solution. A system of equations is called *incompatible* if there is *no solution*.

### 9.3.C Nonparallel Planes (Line Intersection)

In this case:

$$L = \pi_1 \cap \pi_2$$

- The coefficients  $A$ ,  $B$ , and  $C$  in the scalar equations are *not proportional*.

- The normal vectors are *not parallel*:  $\vec{n}_1 \times \vec{n}_2 \neq \vec{0}$ .
- By solving the system (ii) you will be able to find two variables in terms of the third variable.
- There are an *infinite number of solutions* and therefore an *infinite number of points of intersection*.
- The intersection is a *line* and a *direction vector* for this line is  $\vec{u} = \vec{n}_1 \times \vec{n}_2$ .

### 9.3.D Coincident Planes (Plane Intersection)

In this case:

$$\pi_1 = \pi_1 \cap \pi_2 = \pi_2$$

- The planes are *parallel* and *coincident*.
- The coefficients  $A$ ,  $B$ ,  $C$ , and  $D$  in the scalar equations are *proportional*.
- One equation in the system (ii) is a *multiple* of the other equation and does not contain additional information (the equations are equivalent).
- By solving the system of equations (ii), you get a *true* statement (like  $0 = 0$ ).
- There are an *infinite number of solutions* and therefore an *infinite number of points of intersection*.
- The intersection is a *plane*.

### 9.3.E Parallel and Distinct Planes (No Intersection)

In this case:

$$\pi_1 \cap \pi_2 = \emptyset$$

- The planes are *parallel* and *distinct*.
- The coefficients  $A$ ,  $B$ , and  $C$  in the scalar equations are *proportional* but the coefficients  $A$ ,  $B$ ,  $C$ , and  $D$  are *not proportional*.
- By solving the system (ii) you get a *false* statement (like  $0 = 1$ ).
- There is *no solution* and therefore *no point of intersection* between the two planes.

## 9.4 Intersection of Three Planes

### 9.4.A Intersection of Three Planes

Consider three planes given by their Cartesian equations:

$$\pi_1 : A_1x + B_1y + C_1z + D_1 = 0$$

$$\pi_2 : A_2x + B_2y + C_2z + D_2 = 0$$

$$\pi_3 : A_3x + B_3y + C_3z + D_3 = 0$$

The point(s) of *intersection* of these planes is (are) related by to the *solution(s)* of the following system of equations:

$$\begin{cases} A_1x + B_1y + C_1z + D_1 = 0 \\ A_2x + B_2y + C_2z + D_2 = 0 \\ A_3x + B_3y + C_3z + D_3 = 0 \end{cases} \quad (\text{iii})$$

There are *three* equations and *three* unknowns. You may use *substitution* or *elimination* to solve this system.

### 9.4.B Unique Solution (Point Intersection — Noncoplanar Normal Vectors)

In this case:

$$P = \pi_1 \cap \pi_2 \cap \pi_3$$

- The planes *intersect* into a *single* point.
- The *normal vectors* are *not coplanar*:

$$\vec{n}_1 \cdot (\vec{n}_2 \times \vec{n}_3) \neq 0$$

- By solving the system (iii), you get a *unique solution* for  $x$ ,  $y$ , and  $z$ .

### 9.4.C Infinite Number of Solutions (Line Intersection — Non-parallel Planes and Coplanar Normal Vectors)

In this case:

$$L = \pi_1 \cap \pi_2 \cap \pi_3$$

- The planes are *not parallel* but their normal vectors are *coplanar*:

$$\vec{n}_1 \cdot (\vec{n}_2 \times \vec{n}_3) = 0$$

- The intersection is a *line*.
- One scalar equation is a *combination* of the other two equations.
- By solving the system (iii), you may express two variables in terms of the third one using two equations.

#### 9.4.D Infinite Number of Solutions (Line Intersection — Two Coincident Planes and One Intersecting Plane)

In this case:

$$L = \pi_1 \cap \pi_2 \cap \pi_3$$

- Two planes are *coincident* and the third plane is *not parallel* to the coincident planes.
- The intersection is a *line*.
- Two scalar equations are *equivalent*. The *coefficients*  $A$ ,  $B$ ,  $C$ , and  $D$  are *proportional* for these two equations.
- You may express two variables in terms of the third one using two nonequivalent equations.

#### 9.4.E Infinite Number of Solutions (Plane Intersection — Three Coincident Planes)

In this case:

$$\pi_1 \cap \pi_2 \cap \pi_3 = \pi_1 = \pi_2 = \pi_3$$

- The coefficients  $A$ ,  $B$ ,  $C$ , and  $D$  are *proportional* for all three equations.
- Any point of one plane is also a point on the other two planes.
- The intersection is a *plane*.

#### 9.4.F No Solution (Parallel and Distinct Planes)

In this case:

$$\pi_1 \cap \pi_2 \cap \pi_3 = \emptyset$$

- There are three *parallel* and *distinct* planes.
- There is *no point of intersection*.
- There is *no solution* for the system of equations (the system of equations is *incompatible*).

- The coefficients  $A$ ,  $B$ , and  $C$  are *proportional* but the coefficients of  $A$ ,  $B$ ,  $C$ , and  $D$  are *not proportional*.
- By solving the system (iii) you get *false* statements (like  $0 = 1$ ).

#### 9.4.G No Solution (H Configuration)

In this case:

$$\pi_1 \cap \pi_2 \cap \pi_3 = \emptyset$$

- Two planes are *parallel and distinct* and the third plane is *intersecting*.
- There is *no point of intersection*.
- The coefficients  $A$ ,  $B$ , and  $C$  are proportional for two planes.
- There is *no solution* for the system of equations (the system of equations is *incompatible*).
- By solving the system (iii) you get *false* statements (like  $0 = 1$ ).

#### 9.4.H No Solution (Three Parallel Planes but only Two Coincident Planes)

In this case:

$$\pi_1 \cap \pi_2 \cap \pi_3 = \emptyset$$

- Three planes are *parallel* but only two are *coincident*.
- The coefficients of  $A$ ,  $B$ , and  $C$  are *proportional* for all equations but the coefficients  $A$ ,  $B$ ,  $C$ , and  $D$  are *proportional* only for two planes.
- There is *no solution* for the system of equations (the system of equations is *incompatible*).
- By solving the system (iii) you get *false* statements (like  $0 = 1$ ).

#### 9.4.I No Solution (Delta Configuration)

In this case:

$$\pi_1 \cap \pi_2 \cap \pi_3 = \emptyset$$

- The planes are *not parallel* (the coefficients  $A$ ,  $B$ , and  $C$  are not *proportional*).
- The normal vectors are *coplanar* ( $\vec{n}_1 \cdot (\vec{n}_2 \times \vec{n}_3) = 0$ ).
- There is *no point of intersection* between all three planes.

- There is *no solution* for the system of equations (the system of equations is *incompatible*).
- By solving the system (iii) you get *false* statements (like  $0 = 1$ ).



## 9.5 Distance from a Point to a Line

### 9.5.A Distance from a Point to a Line in $\mathbb{R}^2$

Let  $L : Ax + By + C = 0$  be a line in  $\mathbb{R}^2$ ,  $P_1(x_1, y_1)$  be a *generic point* on the  $xy$ -plane and  $P_0(x_0, y_0)$  be a specific point on this line, so:  $Ax_0 + By_0 + C = 0$ .

The *distance*  $d$  between the point  $P_1(x_1, y_1)$  to the line  $L$  is given by (*scalar projection* of  $\overrightarrow{P_0P_1}$  onto the normal vector  $\vec{n}$ ):

$$d = \frac{|\overrightarrow{P_0P_1} \cdot \vec{n}|}{\|\vec{n}\|} \quad (\text{iv})$$

Using  $\vec{n} = (A, B)$ ,  $\|\vec{n}\| = \sqrt{A^2 + B^2}$  and:

$$\begin{aligned} \overrightarrow{P_0P_1} \cdot \vec{n} &= (x_1 - x_0, y_1 - y_0) \cdot (A, B) \\ &= A(x_1 - x_0) + B(y_1 - y_0) \\ &= Ax_1 + By_1 - Ax_0 - By_0 \\ &= Ax_1 + By_1 + C \end{aligned}$$

the formula (iv) may be written as:

$$d = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}} \quad (\text{v})$$

### 9.5.B Distance from a Point to a Line in $\mathbb{R}^3$

Let  $L : \vec{r} = \vec{r}_0 + t\vec{u} \mid t \in \mathbb{R}$  be a line defined by its vector equation and  $P_0(x_0, y_0, z_0)$  be a specific point on this line.

The distance  $d$  from a point  $P_1(x_1, y_1, z_1)$  to the line  $L$  may be found using:

$$d = \left\| \overrightarrow{P_0P_1} \right\| \sin \alpha \quad (\text{vi})$$

where  $\alpha$  is the angle formed by the intersection of  $\overrightarrow{P_0P_1}$  and  $\vec{u}$ .

Because  $\left\| \overrightarrow{P_0P_1} \times \vec{u} \right\| = \left\| \overrightarrow{P_0P_1} \right\| \|\vec{u}\| \sin \alpha$ , the formula (vi) can also be written as:

$$d = \frac{\left\| \overrightarrow{P_0P_1} \times \vec{u} \right\|}{\|\vec{u}\|} \quad (\text{vii})$$

**Note:** The formula (vii) may be applied also in  $\mathbb{R}^2$  by considering the third component  $z = 0$ .

### 9.5.C Distance between Two Parallel Lines

To find the *distance* between two parallel lines:

1. Find a *specific point* on one of these lines.
2. Find the distance from that specific point to the other line using one of the relations above.

### 9.5.D Perpendicular Line from a Point to a Line

Let  $L : \vec{r} = \vec{r}_0 + t\vec{u} \mid t \in \mathbb{R}$  be a line defined by its vector equation and  $P(x, y, z)$  be a generic point in  $\mathbb{R}^3$ .

The line perpendicular to the line  $L$  that passes through the point  $P$  is called the *perpendicular line* and intersects the line  $L$  at a point  $F$  called the *foot* of the perpendicular line.

The foot  $F$  of the perpendicular line may be found from the equation (because  $\overrightarrow{PF} \perp \vec{u}$ ):

$$\overrightarrow{PF} \cdot \vec{u} = 0$$

A *vector equation* of the perpendicular line is:

$$\vec{r} = \overrightarrow{OP} + s\overrightarrow{PF} \mid s \in \mathbb{R}$$

### 9.5.E Shortest Distance between Two Skew Lines

Two skew lines lie into *two parallel planes*. The vector  $\vec{u}_1 \times \vec{u}_2$  is *perpendicular* to both lines and therefore perpendicular to parallel planes the lines lie on.

The *shortest distance* between two skew lines  $L_1 : \vec{r} = \vec{r}_{01} + t\vec{u}_1 \mid t \in \mathbb{R}$  and  $L_2 : \vec{r} = \vec{r}_{02} + s\vec{u}_2 \mid s \in \mathbb{R}$  is given by the *scalar projection* of the vector  $\vec{r}_{01} - \vec{r}_{02}$  onto the vector  $\vec{u}_1 \times \vec{u}_2$ :

$$d = \frac{|(\vec{r}_{01} - \vec{r}_{02}) \cdot (\vec{u}_1 \times \vec{u}_2)|}{\|\vec{u}_1 \times \vec{u}_2\|} \quad (\text{viii})$$

## 9.6 Distance from a Point to a Plane

### 9.6.A Distance from a Point to a Plane (I)

Consider a plane  $\pi$  with a *normal vector*  $\vec{n}$  and a point  $P_0(x_0, y_0, z_0)$  on this plane. The *distance* from a point  $P_1(x_1, y_1, z_1)$  to the plane  $\pi$  is given by the *scalar projection* of the vector  $\overrightarrow{P_0P_1}$  onto the normal vector  $\vec{n}$ :

$$d = \frac{|\overrightarrow{P_0P_1} \cdot \vec{n}|}{\|\vec{n}\|} \quad (\text{ix})$$

### 9.6.B Distance from a Point to a Plane (II)

If the plane  $\pi$  is given by the *Cartesian equation*  $\pi : Ax + By + Cz + D = 0$ , then the *distance* from a point  $P_1(x_1, y_1, z_1)$  to the plane is given by:

$$d = \frac{|Ax_1 + By_1 + Cz_1 + D|}{\sqrt{A^2 + B^2 + C^2}} \quad (\text{x})$$

Indeed,

$$P_0 \in \pi \implies Ax_0 + By_0 + Cz_0 + D = 0$$

$$\begin{aligned} \overrightarrow{P_0P_1} \cdot \vec{n} &= (x_1 - x_0, y_1 - y_0, z_1 - z_0) \cdot (A, B, C) \\ &= Ax_1 + By_1 + Cz_1 - Ax_0 - By_0 - Cz_0 \\ &= Ax_1 + By_1 + Cz_1 + D \end{aligned}$$

### 9.6.C Distance between Two Parallel Planes

To get the *distance* between *two parallel planes*:

1. Find a specific point into one of these planes.
2. Find the distance between that specific point and the other plane using one of the formulas above.

# AP Preparation Differentiability Review

## 1.4 Limit of a Function

### 1.4.A One-Sided Limits

The behaviour of the function  $y = f(x)$  near  $x = a$  is described by three numbers:

1. The left hand limit:

$$L = \lim_{x \rightarrow a^-} f(x)$$

the limit of the function  $f(x)$  as  $x$  approaches  $a$  from the left.

2. The value of the function at  $x = a$ :

$$f(a)$$

3. The right hand limit:

$$R = \lim_{x \rightarrow a^+} f(x)$$

the limit of the function  $f(x)$  as  $x$  approaches  $a$  from the right.

#### Notes:

1. In order to exist, both the left and right hand limits must be numbers.
2. If either the left or right hand limit is not a number, then the limit does not exist (DNE).
3. Infinite limits (like  $\infty$  or  $-\infty$ ) are not considered numbers but they are used to give information about the behaviour of a function near the number  $x = a$ .

### 1.4.B Limit

The limit of a function  $y = f(x)$  exists at  $x = a$  if:

$$L \text{ and } R \text{ exist and } L = R$$

In this case we write:

$$\lim_{x \rightarrow a} f(x)$$

the limit of the function  $f(x)$  as  $x$  approaches  $a$ .

**Note:** The function may or may not be defined at  $x = a$ .

### 1.4.C Substitution

If the function is defined by a formula (algebraic expression) then the limit of the function at a number  $x = a$  may be determined by substitution:

$$\lim_{x \rightarrow a} f(x) = f(a)$$

**Notes:**

1. In order to use substitution, the function must be defined on both sides of the number  $x = a$ .
2. Substitution does not work if you get one of the following 7 indeterminate cases:

$$\infty - \infty \quad 0 \times \infty \quad \frac{0}{0} \quad \frac{\infty}{\infty} \quad 1^\infty \quad \infty^0 \quad 0^0$$

### 1.4.D Piecewise defined functions (AP only)

If the function changes the formula at  $x = a$  then:

1. Use the appropriate formula to find the left-hand and right-hand limits.
2. Compare the left-hand and right-hand limits to conclude about the limit of the function at  $x = a$ .

Example:

$$f(x) = \begin{cases} f_1(x) & | \ x < a \\ f_2(x) & | \ x > a \end{cases}$$

At  $x = a$ :

$$L = f_1(a) \quad R = f_2(a)$$

### 1.4.E Limits: Numerical Approach (AP only)

The limit of a function  $y = f(x)$  at a number  $x = a$  may be estimated numerically. To do that:

1. Use a sequence of numbers  $x$  approaching  $x = a$  from the left and from the right.
2. Find the value of the function at each number  $x$ .
3. Analyze the values and make a conclusion (guess the limit).
4. Be careful at the “difference catastrophe”.

### 1.4.F Limit: Informal Definitions (AP only)

**Left-Hand Limit** If the values of  $y = f(x)$  can be made arbitrarily close to  $L$  by taking  $x$  sufficiently close to  $a$  with  $x < a$ , then:

$$\lim_{x \rightarrow a^-} f(x) = L$$

**Right-Hand Limit** If the values of  $y = f(x)$  can be made arbitrarily close to  $R$  by taking  $x$  sufficiently close to  $a$  with  $x > a$ , then:

$$\lim_{x \rightarrow a^+} f(x) = R$$

**Limit** If the values of  $y = f(x)$  can be made arbitrarily close to  $l$  by taking  $x$  sufficiently close to  $a$  from both sides, then:

$$\lim_{x \rightarrow a} f(x) = l$$

**Infinite Limit** If the values of  $y = f(x)$  can be made arbitrarily large by taking  $x$  sufficiently close to  $a$  from both sides, then:

$$\lim_{x \rightarrow a} f(x) = \infty$$

## 1.6 Continuity

### 1.6.A Continuity

A function  $y = f(x)$  is continuous at a number  $x = a$  if

$$L = R = f(a)$$

where:

$L = \lim_{x \rightarrow a^-} f(x)$  is the left-hand limit at  $x = a$ .

$R = \lim_{x \rightarrow a^+} f(x)$  is the right-hand limit at  $x = a$ .

$f(a)$  is the value of the function at  $x = a$ .

**Note:** A function is continuous if it can be drawn without lifting your pencil from the paper.

### 1.6.B Discontinuity

If  $y = f(x)$  is not continuous at  $x = a$  then we say: “ $y = f(x)$  is discontinuous at  $x = a$ ” or “ $y = f(x)$  has a discontinuity at  $x = a$ ”.

### 1.6.C Removable Discontinuity

A function  $y = f(x)$  has a removable discontinuity at  $x = a$  if:

1.  $L = R = \lim_{x \rightarrow a} f(x)$  exists
2.  $f(a)$  DNE or  $\lim_{x \rightarrow a} f(x) \neq f(a)$

**Note:** A removable discontinuity can be removed by redefining the function  $x = a$  as  $f(a) \stackrel{\text{def}}{=} \lim_{x \rightarrow a} f(x)$ .

### 1.6.D Jump Discontinuity

A function  $y = f(x)$  has a jump discontinuity at  $x = a$  if:

$$L = \lim_{x \rightarrow a^-} f(x) \neq \lim_{x \rightarrow a^+} f(x) = R$$



### 1.6.E Infinite Discontinuity

A function  $y = f(x)$  has an infinite discontinuity at  $x = a$  if at least one side of the limit is unbounded (approaches  $\infty$  or  $-\infty$ ).

### 1.6.F Continuity over an Interval (AP only)

A function  $y = f(x)$  is continuous over an open interval  $(a, b)$  if the function is continuous at every number in that interval.

A function is continuous from the right at  $x = a$  if  $R = f(a)$ .

A function is continuous from the left at  $x = a$  if  $L = f(a)$ .

### 1.6.G Elementary Functions (AP only)

Elementary functions (polynomial, power, rational, trigonometric, exponential, and logarithmic) are continuous over their domain.

### 1.6.H Composition of Functions

If  $g$  is continuous at  $x = a$  and  $f$  is continuous at  $g(a)$  then  $f(g(x))$  is continuous at  $x = a$ .

### 1.6.I Intermediate Value Theorem (AP only)

If  $y = f(x)$  is a continuous function over the interval  $[a, b]$  with  $f(a) \neq f(b)$ , then for any number  $N$  between  $f(a)$  and  $f(b)$  there exist a number  $c \in (a, b)$  such that  $f(c) = N$ .

## 2.1 Derivative Function

### 2.1.A Derivative Function

Given a function  $y = f(x)$ , the *derivative function* of  $f$  is a *new function* called  $f'$  ( $f$  prime), defined at  $x$  by:

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

A function  $y = f(x)$  is *differentiable* at  $x$  if  $f'(x)$  exists.

### 2.1.B Differentiability (AP only)

A function  $y = f(x)$  is differentiable over an open interval  $(a, b)$  if the function is differentiable at every number in that interval.

The domain of derivative function  $f'(x)$  is a subset of the domain of the original function  $f$  ( $D_{f'} \subseteq D_f$ ). So a function is defined over  $D_f$  but is differentiable over  $D_{f'}$ .

### 2.1.C Interpretations of Derivative Function

1. The *slope of the tangent line* to the graph of  $y = f(x)$  at the point  $P(a, f(a))$  is given by  $m_T = f'(a)$ .
2. The *instantaneous rate of change* in the variable  $y$  with respect to the variable  $x$ , where  $y = f(x)$ , at  $x = a$  is given by  $IRC = f'(a)$ .

### 2.1.D Notations and Reading

**Lagrange or prime notation**

$$y' = f'(a)$$

Reading: “y prime” or “f prime of (at) x”.

**Leibnitze notation**

$$\frac{dy}{dx} = \frac{d}{dx} f(x) = Df(x) = D_x f(x)$$

$$\frac{dy}{dx}$$

Reading: “dee y by dee x”.

## Evaluating

$$f'(a) = \left. \frac{dy}{dx} \right|_{x=a}$$

Reading: “dee y by dee x at x equals a”.

## 2.1.E First Principles

*Differentiation* is the process to find the derivative function for a given function.

*First Principles* is the process of differentiation by computing any of the following limits:

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$
$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x-h)}{2h}$$
$$f'(x) = \lim_{u \rightarrow x} \frac{f(u) - f(x)}{u - x}$$

## 2.1.F Differentiability Point

A function  $y = f(x)$  is *differentiable* at  $x$  if  $f'(x)$  exists.

If the function  $y = f(x)$  is *differentiable* at  $x = a$  then the tangent line at  $P(a, f(a))$  is *unique* and *not vertical* (the slope of the tangent line is not  $\infty$  or  $-\infty$ ).

## 2.1.G Non-Differentiability

A function is *not differentiable* at  $x = a$  if  $f'(a)$  *does not exist*.

### Notes:

- If a function  $f$  is *not continuous* at  $x = a$  then the function  $f$  is *not differentiable* at  $x = a$ .
- If a function is differentiable at  $x = a$  then the function is continuous at  $x = a$ .
- If a function  $f$  is *continuous* at  $x = a$  then the function  $f$  *may or may not be* differentiable at  $x = a$ .

## 2.1.H Corner Point

$P(a, f(a))$  is a *corner point* if there are *two* distinct tangent lines at  $P$ , one for the left-hand branch and one for the right-hand branch.

### 2.1.I Infinite Slope Point

$P(a, f(a))$  is an *infinite slope point* if the tangent line at  $P$  is vertical and the function is increasing or decreasing in the neighbourhood of the point  $P$ .

$$f'(a) = \infty \quad \vee \quad f'(a) = -\infty$$

### 2.1.J Cusp Point

$P(a, f(a))$  is a *cusp point* if the tangent line at  $P$  is vertical and the function is increasing on one side of the point  $P$  and decreasing on the other side.

$$f'(a) = DNE$$

# Appendix A

## Credit

Thank you for everything, Mrs. Gugoiu.