Time Travelling to Avoid Trump
A Mathematical Analysis

Vincent Macri David White Aviv Haber

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For Mr. Bouttell. You asked for rigorous. Here it is.

Abstract

The purpose of this paper is to show that it is within the laws of physics to time travel one-way into the future, allowing us to avoid having to experience Donald Trump's presidency.

First, Vincent Macri presents part I, in which we solve the twin paradox without invoking general relativity, relying mainly on an analysis of the relativistic Doppler effect.

Next, David White presents II, in which we derive the mathematics required to do a more practical analysis involving changes in direction.

Lastly, Aviv Haber presents part III, in which we explore possible ways to actually travel into the future, ignoring the economic costs of each option, because costs are prohibitive to time travelling in practice.

Contents

Part I

The Twin Paradox

Chapter 1

Stating the Twin Paradox

1.1 The problem

Imagine two people, we will call them Alice[1] and Donald[2].

Donald is currently the president of the United States, and he is doing a bad job. Alice doesn't like this situation, so she is going to to time travel to the future when Donald is no longer president.

1.2 Solution to Trump

Special relativity tells us that time will pass slower for an object in motion than an object at rest (Bruni, Dick, Speijer, & Stewart, 2012; Einstein, 1916). This means that if Alice moves very fast, she will experience less time pass for her than on earth, and will effectively time travel to the future.

1.3 The paradox

The paradox here is that special relativity states that the laws of physics are the same in all inertial frames of reference (Bruni et al., 2012; Einstein, 1916). This means that we could also argue that Alice is not moving, and instead the earth is moving, which would result in Alice aging faster than Donald, which is the opposite of what we want and the opposite of what actually happens (Bruni et al., 2012, pp. 593–594).

This is called the twin paradox.

[1]Of computer security white paper fame.
[2]Of presidential infamy.

1.4 Approach to resolution of the paradox

It is commonly thought that general relativity is needed to resolve this paradox because Alice is in an accelerating frame of reference and special relativity cannot handle accelerating frames of reference. That is not true. Special relativity can indeed handle accelerating frames of reference, but it is more difficult (Gibbs, 1996; Weiss, 2016). However, it is much easier to use special relativity to solve this problem than it is to use general relativity. And in fact, we will later see that the acceleration of Alice is irrelevant to the resolution of the paradox.

1.5 Notation and assumptions

1.5.1 Notation

Our notation

In math, Alice will be referred to as A, and Donald will be referred to as D.

Other physics notation

Following is some common non-trivial physics notation that we will be using:

Velocity

$$v = \|\vec{v}\|$$

We will use v to denote $\|\vec{v}\|$.

Acceleration

$$a = \|\vec{a}\|$$

We will use a to denote $\|\vec{a}\|$.

Speed of light

$$c_0 = 299\,792\,458\,\mathrm{m/s}$$

We will use c_0 as the speed of light in a vacuum. We are using c_0 instead of c because c_0 is the recommended SI notation (BIPM, 2006).

β notation

$$\beta = \frac{v\,\mathrm{m/s}}{c_0\,\mathrm{m/s}}$$

Where v is velocity and c_0 is the speed of light. Also, since nothing can exceed or meet the speed of light, and the direction is not relevant to the amount of time dilation (Bruni et al., 2012; Einstein, 1916), we will say that: $0 \leq \beta < 1$.

1.5.2 Assumptions

Values

We will say that Alice travels a distance of 4 light years at a speed of $0.8\,c_0$ since these are nice numbers to work with.[3] She then turns around and comes back to earth.

Turnaround

We will assume that Alice makes an instantaneous turnaround. Later we will show that this assumption has no effect on the resolution of the paradox.

1.5.3 Other terminology

Alice's trip is split up into two parts. First, she is moving away from earth, which we will call the outbound leg of the trip. After, she is moving towards the earth, which we will call the inbound leg of the trip.

[3] I originally saw these numbers used in Kogut, 2012, p. 35.

Chapter 2

Setup for the Twin Paradox Resolution

2.1 A naive analysis

We have:

$$v = 0.8\,c_0 \implies \beta = 0.8$$
$$d = 4\,\mathrm{ly}$$

And we know this formula from Bruni et al., 2012, p. 583 (modified to use our notation):

$$\Delta t_D = \frac{\Delta t_A}{\sqrt{1-\beta^2}}$$

Which can be rearranged into:

$$\Delta t_A = \Delta t_D \sqrt{1-\beta^2} \tag{2.1}$$

We can trivially calculate how much time should pass for Donald:

$$\Delta t_D = \frac{2d}{v} = \frac{2 \times 4\,\mathrm{ly}}{0.8\,c_0} = \frac{8\,\mathrm{ly}}{0.8\,c_0} = 10\,\mathrm{y}$$

And how much time should pass for Alice follows by simply plugging this into (2.1):

$$\begin{aligned}
\Delta t_A &= 10\,\mathrm{y} \times \sqrt{1-0.8^2} \\
\Delta t_A &= 10\,\mathrm{y} \times \sqrt{\frac{9}{25}} \\
\Delta t_A &= 10\,\mathrm{y} \times \frac{3}{5} \\
\Delta t_A &= 6\,\mathrm{y}
\end{aligned}$$

So, Donald ages by 10 years, and Alice ages by 6 years.

This answer is right, but the problem is that we started by assuming that Donald is stationary and Alice is moving. However, we could have said that Alice is stationary and Donald is moving, and then we would calculate $\Delta t_A = 10$ and $\Delta t_D = 6$, which is wrong. So doing the analysis this way leads to ambiguity. We must develop a more rigorous way to analyze this problem.

2.2 Deriving the relativistic Doppler effect

The simplest way to solve the twin paradox is to use the relativistic Doppler effect to analyze what Donald sees and what Alice sees.

We will have both Alice and Donald flash a light at the other once per second, according to their own proper time. Their lights are infinitely powerful, and can be seen from light years away (once the light has travelled there of course). We will assume the path of the lights are entirely through a perfect vacuum.

2.2.1 More notation

First, let's define some notation specific to this section:

f_s The frequency the emitter (source) flashes their light at. We will make Alice the source.

f_o The frequency the other person (observer) sees the light flashing at. We will make Donald the observer of Alice's flashing light.

t_s The proper time to the next wavefront as seen by the emitting source, Alice.

t_o The proper time to the next wavefront as seen by the person observing the flashes, Donald.

λ The distance to the next wave front of the approaching light wave. λ is calculated as:

$$\lambda = \frac{c_0 \,\mathrm{m/s}}{f_s \,\mathrm{s}^{-1}} = \frac{c_0}{f_s} \,\mathrm{m}$$

2.2.2 The derivation

We start by relating Δt_s to λ and v when Alice is moving away from Donald:

$$\begin{aligned}
\Delta t_s &= \frac{\lambda}{c_0} + \frac{v \times \Delta t_s}{c_0} \\
c_0 \Delta t_s &= \lambda + v \Delta t_s \\
c_0 \Delta t_s - v \Delta t_s &= \lambda \\
\Delta t_s (c_0 - v) &= \lambda \\
\Delta t_s &= \frac{\lambda}{c_0 - v} \\
\Delta t_s &= \frac{1}{c_0 - v} \times \lambda
\end{aligned}$$

Now substitute in $\lambda = \frac{c_0}{f_s}$:

$$\begin{aligned}
\Delta t_s &= \frac{1}{c_0 - v} \times \frac{c_0}{f_s} \\
\Delta t_s &= \frac{c_0}{c_0 - v} \times \frac{1}{f_s} \\
\Delta t_s &= \frac{1}{1 - \frac{v}{c_0}} \times \frac{1}{f_s} \\
\Delta t_s &= \frac{1}{f_s(1 - \beta)} \qquad (2.2)
\end{aligned}$$

Next, we will perform a unit analysis to verify that (2.2) gives us a value in seconds:

$$\begin{aligned}
\Delta t_s &= \frac{1}{\mathrm{s}^{-1}} \\
\Delta t_s &= \mathrm{s}
\end{aligned}$$

So, we have derived (2.2) and verified that it gives us a value in seconds. Now we need to use this formula.

The next step is to develop the actual Doppler effect formula. We will work off of the special relativity time dilation formula given in Bruni et al., 2012, p. 593, modified to use our notation, and rearrange it into a form more useful for our purposes:

$$\begin{aligned}
\Delta t_s &= \frac{\Delta t_o}{\sqrt{1 - \beta^2}} \\
\Delta t_o &= \Delta t_s \sqrt{1 - \beta^2} \qquad (2.3)
\end{aligned}$$

We will now substitute (2.2) into (2.3) to combine our two equations to develop a third formula:

$$\Delta t_o = \frac{\sqrt{1-\beta^2}}{f_s(1-\beta)} \tag{2.4}$$

Next, we will finish developing the relativistic Doppler shift formula.

Note, by definition:

$$f_o = \frac{1}{\Delta t_o} \tag{2.5}$$

We will now substitute (2.4) into (2.5):

$$\begin{aligned}
f_o &= \frac{1}{\frac{\sqrt{1-\beta^2}}{f_s(1-\beta)}} \\
f_o &= \frac{f_s(1-\beta)}{\sqrt{1-\beta^2}} \\
f_o &= f_s\left(\frac{1-\beta}{\sqrt{1-\beta^2}}\right) \\
f_o &= f_s\left(\frac{\sqrt{(1-\beta)^2}}{\sqrt{1^2-\beta^2}}\right) \\
f_o &= f_s\left(\frac{\sqrt{(1-\beta)(1-\beta)}}{\sqrt{(1-\beta)(1+\beta)}}\right) \\
f_o &= f_s\sqrt{\frac{(1-\beta)(1-\beta)}{(1-\beta)(1+\beta)}} \\
f_o &= f_s\sqrt{\frac{1-\beta}{1+\beta}} \qquad (2.6)
\end{aligned}$$

This is the formula for the relativistic Doppler effect when Alice and Donald are moving away from each other. If Alice is moving towards Donald, then we simply change the sign on β to get:

$$f_o = f_s\sqrt{\frac{1+\beta}{1-\beta}} \tag{2.7}$$

2.3 Minkowski spacetime diagrams and world lines

We must now introduce a new way of analyzing problems in special relativity: the Minkowski spacetime diagrams and world lines.

We will draw the world lines of our situation onto Minkowski spacetime diagrams. This is simply a diagram with time on the vertical axis, and distance on the horizontal axis. A world line is the path an object takes through spacetime.

We will start with a spacetime diagram of what we would expect to see, disregarding special relativity, using Donald as the frame of reference. See figure 2.1.

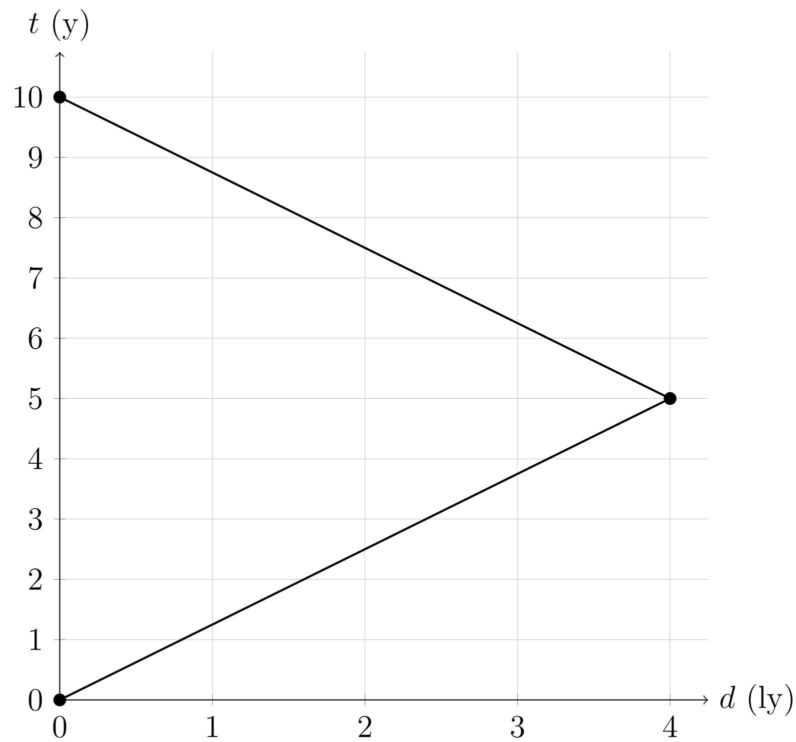


Figure 2.1: Minkowski spacetime diagram of what we would expect to see according to classical physics with Donald as the frame of reference.

Of course, figure 2.1 does not apply, since $\beta = 0.8$ (which is quite large), however, I find it helpful to visualize the problem as well as to introduce the notion of spacetime diagrams.

Chapter 3

Resolving the Twin Paradox

3.1 The relativistic Doppler effect and the twin paradox

3.1.1 Recall our assumptions

Recall the assumptions and values we decided to use in subsection 1.5.2:

Distance Alice travels a distance of 4 light years, so $d = 4\,\mathrm{ly}$.

Velocity Alice travels at a speed of $0.8\,c_0$, so $v = 0.8\,c_0$ and $\beta = 0.8$.

The actual values we use do not matter for resolving the paradox. These values were chosen because they give nice numbers when we perform the calculations, which makes the analysis easier to follow.

We also assumed an instantaneous turnaround. This too makes the math easier, but we will show that it does not change the resolution to the paradox. We can assume that Alice is very strong and capable of surviving $479\,667\,932.8\,\mathrm{m/s^2}$ of acceleration.[1] If Alice is not that strong, we can instead say that another person, also named Alice, who is travelling at the same speed as the first Alice but in the opposite direction, passes by the first Alice at the turnaround point and syncs up her clock with the first Alice's clock. This would mean that when the second Alice arrives at earth, her clock will read the same thing as the original Alice's would have if she could survive all of that acceleration. Either way of handling Alice's instantaneous turnaround will work, since we will end up with the same reading on Alice's clock.

3.1.2 The Doppler analysis

Analyzing the twin paradox with the relativistic Doppler effect is helpful because it allows us to calculate what each person sees, and show that they are seeing different

[1] $479\,667\,932.8\,\mathrm{m/s} = 1.6\,c_0$

things, which solves the ambiguity stated in section 2.1.

When we derived the relativistic Doppler equations in section 2.2, we said that only Alice is shining a flashlight once per second. We did this to simplify our notation in that section and make the derivation easier to follow. However, this doesn't work for actually solving the paradox. We must have both Alice and Donald flash their lights once per second, as measured by their own proper time. Both Alice and Donald know that the other person is shining their light once per second.

This means that both of (2.6) and (2.7) will apply to both Alice and Donald.

We will use the following notation here:

f_A The frequency Alice shines her light at according to her own time.

f_D The frequency Donald shines his light at according to his own time.

f'_A The frequency Alice sees Donald shine his light at according to her own time. This is calculated with (2.6) when Alice is moving away from Donald, and with (2.7) when Alice is moving towards Donald.

f'_D The frequency Donald sees Alice shine her light at according to his own time. This is calculated with (2.6) when Alice is moving away from Donald, and with (2.7) when Alice is moving towards Donald.

Alice moving away from Donald

What Alice sees

$$f'_A = f_D\sqrt{\frac{1-\beta}{1+\beta}}$$

$$f'_A = f_D \times \sqrt{\frac{1}{9}}$$

$$f'_A = \frac{1}{3}f_D$$

So, Alice sees Donald's clock running slowly as she is moving away from him.

What Donald sees

$$f'_D = f_A\sqrt{\frac{1-\beta}{1+\beta}}$$

$$f'_D = f_A \times \sqrt{\frac{1}{9}}$$

$$f'_D = \frac{1}{3}f_A$$

So, Donald sees Alice's clock running slowly as she is moving away from him.

Alice moving towards Donald

What Alice sees

$$f'_A = f_D\sqrt{\frac{1+\beta}{1-\beta}}$$
$$f'_A = f_D \times \sqrt{9}$$
$$f'_A = 3f_D$$

So, Alice sees Donald's clock running quickly as she is moving towards him.

What Donald sees

$$f'_D = f_A\sqrt{\frac{1+\beta}{1-\beta}}$$
$$f'_D = f_A \times \sqrt{9}$$
$$f'_D = 3f_A$$

So, Donald sees Alice's clock running quickly as she is moving towards him.

Alice and Donald see the same thing. So why do we end up with Donald aging more if they both see the other age slowly, then they both see the other age quickly?

The answer lies in how long each person sees the other aging at a different speed.

3.2 Minkowski spacetime and the twin paradox

Before we go further into the analysis, it is important to note that Donald and Alice do not necessarily need to be sending simple flashes of light once per second. They can also flash an image displaying the current time passed on their clock, and flash that image once per second. This has no effect on what happens, it just means that the other person doesn't need to perform as many calculations. To make this section easier to follow, we will assume that Alice and Donald are flashing an image of their clock. We will also assume that they both have telescopes strong enough to see the flashed image from several light years away.

3.2.1 When does Donald see Alice's turnaround?

When Alice reaches her turnaround point, she is 4 ly away from Donald. This means that it will take 4 years for the light from the turnaround to reach him from this point. Donald can easily calculate when Alice *should* reach the turnaround point though:

$$t = \frac{d}{v}$$
$$t = \frac{4\,\mathrm{ly}}{0.8\,c_0}$$
$$t = 5\,\mathrm{y}$$

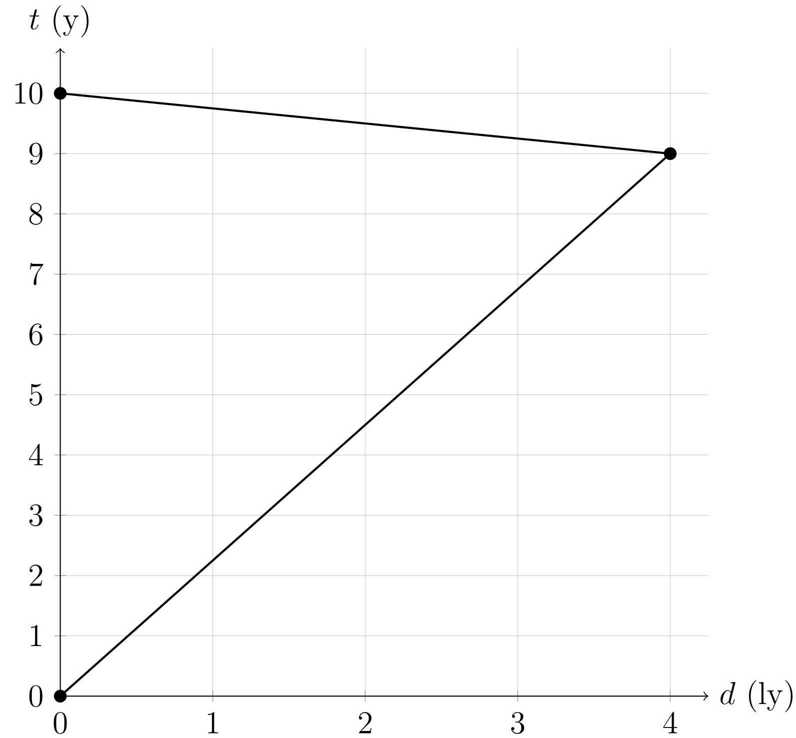
Alice should reach the turnaround in 5 years, but Donald does not see that until 4 years after that, so he will see Alice's turnaround happen 9 years after her departure.

Alice travels towards earth at the same speed she travels away from earth, so Donald can also calculate when Alice should get back:

$$t = \frac{2d}{v}$$
$$t = \frac{8\,\mathrm{ly}}{0.8\,c_0}$$
$$t = 10\,\mathrm{y}$$

The world lines as seen by Donald are shown in figure 3.1 on the following page.

Figure 3.1: Minkowski spacetime diagram of what Donald will see on his own clock.



3.2.2 What does Alice see on Donald's clock at her turnaround?

It Alice's turnaround is 4 ly away, which means it takes 4 y for Donald's light flashes to reach that point.

While Donald does not see Alice's turnaround until 9 years after her departure, he can calculate that it should happen 5 years after her departure. This means that when Alice reaches her turnaround point, she not will see any light flashes sent by Donald after the 1 year mark on his clock.

When Alice is at her turnaround, she will see Donald's clock display 1 year, since the light from after that has not reached her yet. This means that Alice will observe figure 3.2 on the next page as the spacetime diagram according to Donald's clock.

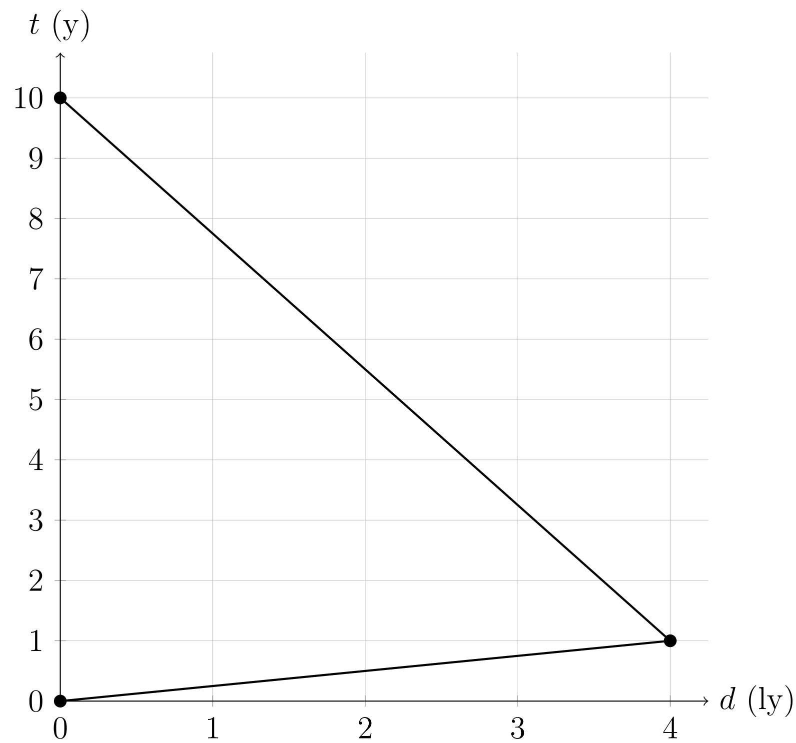


Figure 3.2: Minkowski spacetime diagram of what Alice will see on Donald's clock.

So, 10 years pass for Donald, and Alice sees 10 years pass for him.

3.2.3 What about Alice's clock?

Recall from section 3.1 on page 10 that:

1. Alice's clock runs at one third of the speed of Donald's clock when Alice is on the outbound leg of her trip.
2. Alice's clock runs at three times the speed of Donald's clock when she is on the inbound leg of her trip.

We can use this to translate the passage of time from Donald's clock into Alice's clock.

The outbound leg

Recall from figure 3.1 on page 14 that Donald's clock says that the outbound leg of Alice's trip takes 9 years. To translate from time on Donald's clock into time on Alice's clock during the outbound leg, simply multiply by $\frac{1}{3}$:

$$\begin{aligned}\Delta t_A &= \Delta t_D \times \frac{1}{3} \\ \Delta t_A &= 9\,\mathrm{y} \times \frac{1}{3} \\ \Delta t_A &= 3\,\mathrm{y}\end{aligned}$$

So, 3 years pass for Alice on the outbound leg of her trip.

The inbound leg

Recall from figure 3.1 on page 14 that Donald's clock says that the inbound leg of Alice's trip takes 1 year. To translate from time on Donald's clock into time on Alice's clock during the inbound leg, simply multiply by 3:

$$\begin{aligned}\Delta t_A &= \Delta t_D \times 3 \\ \Delta t_A &= 1\,\mathrm{y} \times 3 \\ \Delta t_A &= 3\,\mathrm{y}\end{aligned}$$

So, 3 years pass for Alice on the inbound leg of her trip.

Total time for Alice

If 3 years pass for Alice on her outbound leg, and 3 years pass for her on her inbound leg, then we can trivially calculate how much time passes for her in total:

$$3\,\mathrm{y} + 3\,\mathrm{y} = 6\,\mathrm{y}$$

So, 6 years pass for Alice, and Donald sees 6 years pass for her.

At this point, we have shown that Alice and Donald clearly experience different passages of time, and here we can conclude our resolution to the twin paradox. Just to be completely sure that we did this correctly though, we will repeat the calculations done in this section from the perspective of Alice, to show that there is no ambiguity in performing the analysis this way.

3.2.4 Is it ambiguous?

In subsection 3.2.3 on page 15, we calculated the years passed based on what Donald experiences. Does it work if we calculate it from Alice's experience as well? Let's try.

The outbound leg

Alice sees 1 year pass on Donald's clock during the outbound leg of her trip. However, Alice knows that she is seeing Donald's clock running at one third of its actual speed. So, by reversing the effects of the relativistic Doppler shift, she can calculate how long her outbound trip takes. To do this, she simply divides by $\frac{1}{3}$:

$$
\begin{aligned}
\Delta t_A &= \Delta t_D \div \frac{1}{3} \\
\Delta t_A &= 1\,\mathrm{y} \div \frac{1}{3} \\
\Delta t_A &= 3\,\mathrm{y}
\end{aligned}
$$

So, Alice calculates that the outbound leg of her trip takes 3 years.

The inbound leg

Alice sees 9 years pass on Donald's clock during the inbound leg of her trip. However, Alice knows that she is seeing Donald's clock running at three times its actual speed. So, by reversing the effects of the relativistic Doppler shift, she can calculate how long her inbound trip takes. To do this, she simply divides by 3:

$$
\begin{aligned}
\Delta t_A &= \Delta t_D \div 3 \\
\Delta t_A &= 9\,\mathrm{y} \div \frac{1}{3} \\
\Delta t_A &= 3\,\mathrm{y}
\end{aligned}
$$

So, Alice calculates that the inbound leg of her trip takes 3 years.

Total time for Alice

Again, if 3 years pass for Alice on the outbound leg, and 3 years pass for her on the inbound leg, then we can trivially calculate how much time passes for her in total:

$$3\,\mathrm{y} + 3\,\mathrm{y} = 6\,\mathrm{y}$$

So, 6 years pass for Alice, and Donald sees 6 years pass for her.

At this point, our resolution of the twin paradox is pretty solid. If you have been paying close attention, you may have noticed that our solution also works for a non-instantaneous turnaround involving acceleration. It is okay if you do not see this yet. It will be demonstrated in section 3.4 on page 21. However, first we will show one final way to demonstrate that Alice experiences 6 years pass for herself in section 3.3 on the next page.

3.3 Length dilation analysis

It seems quite unfair to Alice to say that she can only calculate how much time passes for her by looking at Donald's clock and reversing the effect through some calculations. After all, Alice does have a clock on her ship. Why can't she use that clock?

She can, but I've saved this approach it until now because it involves another way of looking at the problem and introduces a new concept: length dilation.

Just as time dilates for objects moving quickly, so does length, and this is represented by the following formula (Bruni et al., 2012, pp. 588–594):

$$L_m = L_s\sqrt{1 - \frac{v^2}{{c_0}^2}} \tag{3.1}$$

Here, L_m represents the length of an object as measured by an observer moving relative to it at a speed v. The object has a length of L_s when measured by someone at rest relative to the object.

3.3.1 How does this apply to Alice?

Alice is in two different inertial frames of reference during her trip. She is in one inertial frame of reference during the outbound leg of her trip, and she is in another one on the inbound leg of her trip. Recall that special relativity states that the laws of physics are the same in all inertial frames of reference (Bruni et al., 2012; Einstein, 1916).

So, we can say that Alice is at rest, and the entire universe is moving relative to her. This allows us to apply length dilation to the universe.

We will call L_U the length of the portion of the universe Alice is travelling through. The universe measures this length, L_U, as 4 ly. However, since the universe is moving relative to Alice, she will measure a different length, which we will call L_A.

We replace L_m from (3.1) with L_A, L_s with L_U, and $\frac{v^2}{c_0}$ with β, then calculate L_A:

$$\begin{aligned}
L_A &= L_U\sqrt{1-\beta^2} \\
L_A &= 4\,\mathrm{ly}\sqrt{1-0.8^2} \\
L_A &= 4\,\mathrm{ly}\sqrt{\frac{9}{25}} \\
L_A &= 4\,\mathrm{ly}\times\frac{3}{5} \\
L_A &= 2.4\,\mathrm{ly}
\end{aligned}$$

Alice measures the length of each leg of her trip as $2.4\,\mathrm{ly}$. She knows her speed is $0.8\,c_0$, and so she can easily calculate how long one leg of her journey will take:

$$\begin{aligned}
\Delta t_A &= \frac{d}{v} \\
\Delta t_A &= \frac{2.4\,\mathrm{ly}}{0.8\,c_0} \\
\Delta t_A &= 3\,\mathrm{y}
\end{aligned}$$

So, Alice calculates that each leg of her journey takes 3 years.

Again, there are two legs of her journey, and $3+3=6$, so her entire journey will take 6 years.

3.3.2 The problem with the length dilation analysis

While this analysis was a lot simpler than our previous analysis with Minkowski spacetime and the relativistic Doppler effect, it is not as robust. This analysis only works in an inertial frame of reference. In practice, Alice will undergo acceleration, and the length dilation analysis no longer holds for that.

Next, we will show that our original analysis also works if Alice undergoes acceleration.

3.4 Accelerating Alice

Recall how we performed the analysis in section 3.2 on page 13 using Minkowski space-time diagrams.

We used the fact that the relativistic Doppler effect causes both Alice and Donald to see the other person's clock running slowly when they are moving away from each other. We also used the fact that both Alice and Donald see the other person's clock running quickly when they are moving towards each other.

Recall the formulas (2.6) and (2.7) on page 8:

Moving away from each other

$$f_o = f_s\sqrt{\frac{1-\beta}{1+\beta}}$$

Moving towards each other

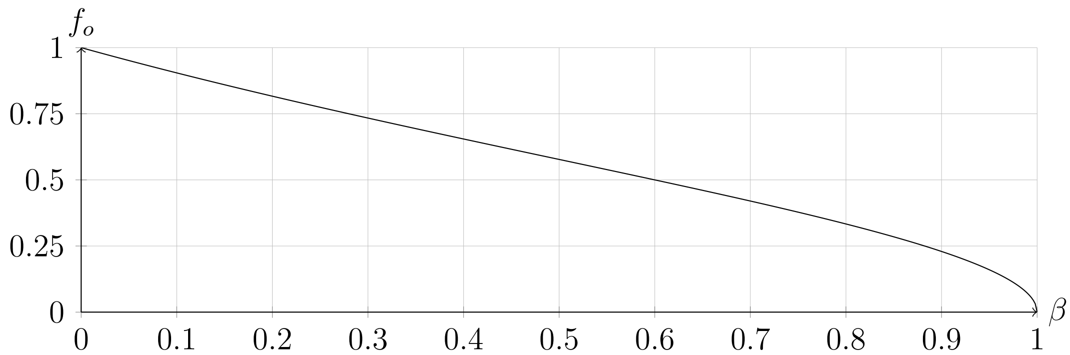
$$f_o = f_s\sqrt{\frac{1+\beta}{1-\beta}}$$

3.4.1 Magnitude of the Doppler effect

Remember when we defined our notation in subsection 1.5.1 on page 3 that $0 \leq \beta < 1$. Also, since division by 0 is not allowed, and a speed of 0 is not relevant, we will also place the restriction that $\beta \neq 0$, resulting in $0 < \beta < 1$. Next, we will analyze the magnitude of both formulas by simply graphing all β values. See figure 3.3 and 3.4 on the following page.

Moving away from each other

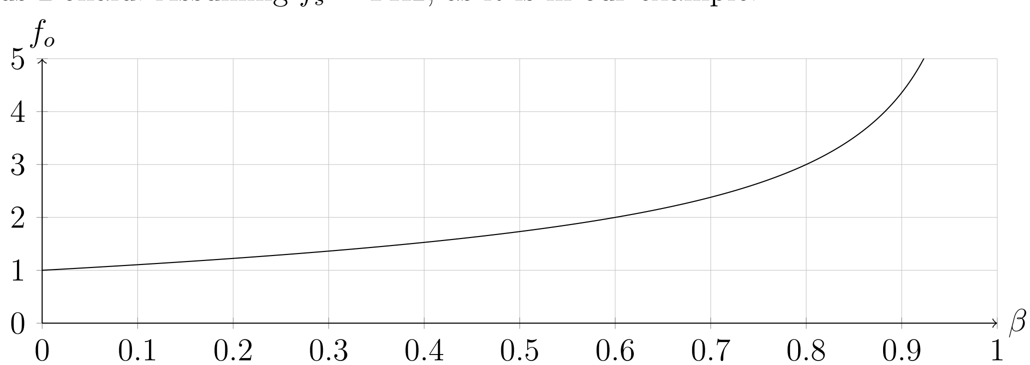
Figure 3.3: Graph of f_o vs β for the relativistic Doppler effect when Alice is moving away from Donald. Assuming $f_s = 1\,\text{Hz}$, as it is in our example.



From this graph, we can see that both Alice and Donald will always see each other's clocks running slowly when they are moving away from each other.

Moving towards each other

Figure 3.4: Graph of f_o vs β for the relativistic Doppler effect when Alice is moving towards Donald. Assuming $f_s = 1\,\text{Hz}$, as it is in our example.



From this graph, we can see that both Alice and Donald will always see each other's clocks running quickly when they are moving towards each other. This graph goes to infinity in the portion that was cutoff because it could not fit on the page.

3.4.2 Implications of subsection 3.4.1

From 3.3 and 3.4 on the previous page we see that no matter the value of β, Alice and Donald will observe time speed up and slow down in the same way as in our example with an inertial frame of reference, just by different amounts. However, as they move away from each other, they will both always see a slow clock for the other person. And as they move towards each other, they will both always see a fast clock for the other person.

Remember that while doing our calculations, what mattered to get the results we did was that both Alice and Donald saw the other person's clock running slowly on the outbound leg, and quickly on the inbound leg. A different value of β would change how much dilation occurs, but would not change the fact that dilation is occurring.

We can draw out the world lines for what would happen if Alice was accelerating. See figure 3.5 to help with visualization, and figures 3.6 and 3.7 on the next page for what actually happens.

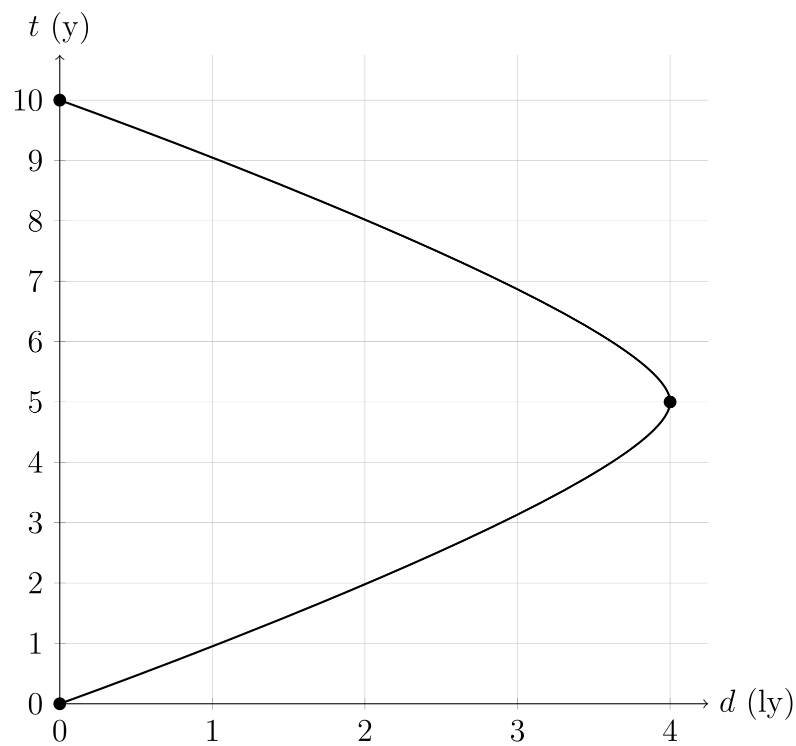


Figure 3.5: Minkowski spacetime diagram of what we would expect to see according to classical physics with Donald as the frame of reference, if Alice is accelerating for her whole journey.

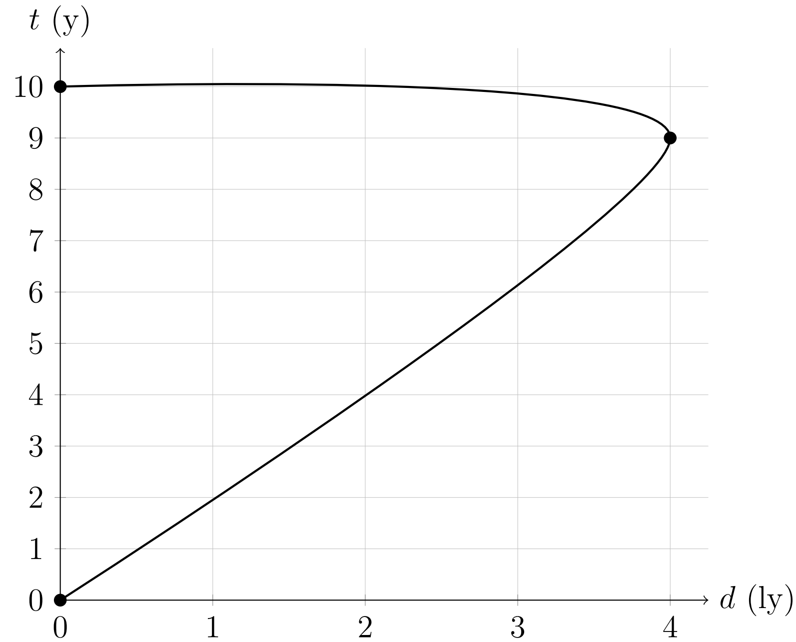


Figure 3.6: Minkowski spacetime diagram of what Donald will see on his own clock, if Alice is accelerating for her whole journey.

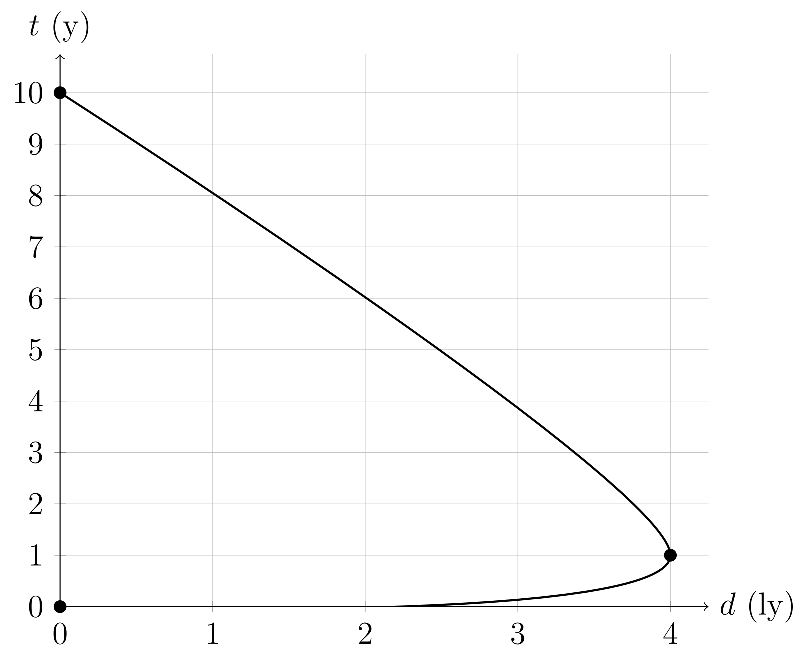


Figure 3.7: Minkowski spacetime diagram of what Alice will see on Donald's clock, if Alice is accelerating for her whole journey.

Now that we have the different world lines, we need to figure out what changes in our calculations.

In our calculations in section 3.2 on page 13, we used the distance from earth to the turnaround, the time it takes to get there, and the effects of the relativistic Doppler effect. Velocity was only used to calculate the relativistic Doppler effect, and to calculate how long the trip takes.

3.4.3 Conclusion of the accelerating twin paradox resolution

In our new world lines, we kept the distance and time the same. We also showed before that the Doppler effect will act in the same direction, but with a different magnitude as velocity changes. The only other thing that has changed at any point of the world lines is velocity, since acceleration is simply the derivative of velocity. But, again, velocity was only used to initially calculate the length of the trip. Leaving the length of the trip the same, and varying velocity over time (applying acceleration), we see that nothing will change our solution to the paradox. All that will change is the amount of time dilation[2], but the paradox remains solved.

Finally, we can consider the twin paradox to be completely resolved for both infinite and finite accelerations.

[2] Doing the actual calculations for this is quite complicated. In the second half of this paper, we will approach these calculation with a computer simulation.

Part II

Deriving a Real-World Case

Chapter 4

A real-world analysis

4.1 Deriving the formulas

Before making any calculations with velocity, we must first redefine our equation for velocity. Given that the speed of light, c_0, is constant and is considered the "universal speed limit", it can be reasoned that $v = at$ will no longer serve as an accurate calculation for velocity since it can exceed c_0. Therefore we must use $v = c_0 \tanh\left(\frac{at}{c_0}\right)$ as our equation for velocity, as it is the "true acceleration formula" (Gibbs, 1996).

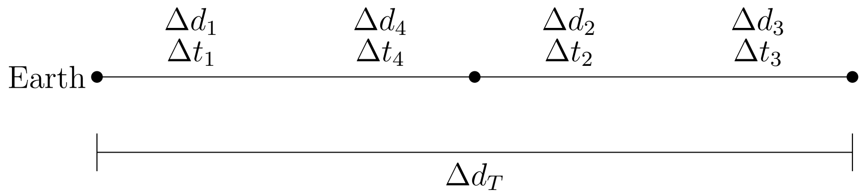
For these calculations to be accurate, we must also assume that the earth is stationary, and that there are no other existing significant gravitational masses other than the earth. Additionally, we must ignore gravitational time dilation.

It should also be noted that the dilation values calculated will be approximations calculated by a written computer simulation. This simulation will sum the time dilation across each 1 second interval for the entirety of the journey, which will not yield a perfect result, but the calculated result will be accurate enough for our purposes.

4.2 The linear case

When travelling through space with acceleration, there are two possible basic paths that the travelling ship can take. One is a linear shape as shown below.

Figure 4.1: Diagram of the spaceship taking a linear path.



In this path, the ship will accelerate forward (away from the earth) in a straight line for a time Δt_1, then it will begin to decelerate at the same rate opposite to it's velocity for a time Δt_2 until it reaches a halt. The ship will then accelerate backwards (towards the earth) in a straight line for a time Δt_3, and finally, it will decelerate for a time Δt_4 until it reaches a halt at it's starting position (position of the earth).

Given that the acceleration of the ship is constant, it is sufficient to say that $\Delta t_1 = \Delta t_2$ since the ship must decelerate at the same rate at which it accelerates. Given this, it is also sufficient to say that $\Delta t_3 = \Delta t_4$. We can also say that $\Delta d_1 + \Delta d_2 = \Delta d_3 + \Delta d_4$ given that the distance travelled to and from the turning point must be equal. This means that $\Delta t_1 + \Delta t_2 = \Delta t_3 + \Delta t_4$ meaning: $Delt_1 = \Delta t_2 = \Delta t_3 = \Delta t_4$.

Since the acceleration and deceleration times are all equal: $\Delta T_T = 4\Delta t_1$.

Since the simulator will calculate the values using a summation technique, the time dilation over an accelerating time frame will be equal to the time dilation over an equal time frame decelerating with the same acceleration magnitude. This means that the time dilation over the time frame Δt_1 is equal to the dilation over the other three time frames, thus the simulator will only calculate the dilation for the first time frame, and will multiply the result by 4 for the final dilation.

$$\Delta d_1 + \Delta d_2 + \Delta d_3 + \Delta d_4 = \Delta d_T$$
$$\Delta d_1 = \Delta d_2 = \Delta d_3 = \Delta d_4$$
$$\Delta d_T = 4\Delta d_1$$

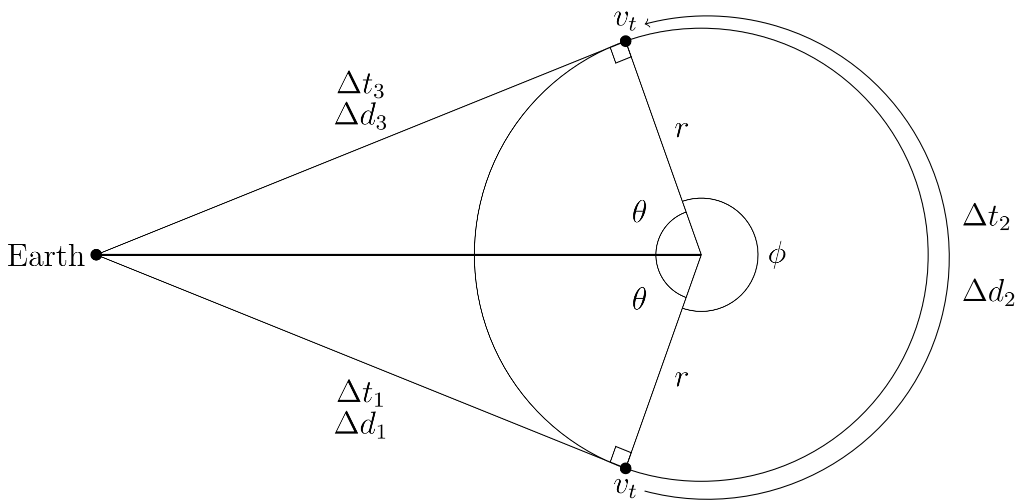
$$\Delta t_1 + \Delta t_2 + \Delta t_3 + \Delta t_4 = \Delta t_T$$
$$\Delta t_1 = \Delta t_2 = \Delta t_3 = \Delta t_4$$
$$\Delta t_T = 4\Delta t_1$$

4.3 Turning

4.3.1 Setup

The other option is for the ship to complete a circular turn to change directions. This can be modelled by the diagram below.

Figure 4.2: Diagram of what happens when Alice turns.



In this path, the ship will accelerate forward (away from the earth) in a straight line for a time Δt_1.

Using this diagram we can set up the following equation: $\Delta T_T = \Delta t_1 + \Delta t_2 + \Delta t_3$.

As shown in the diagram, both Δd_1 and Δd_3 are tangent to the circle which models the turn, and pass through the same point, being the position of the earth. As such, we can say that $\Delta t_1 = \Delta t_3$ and $\Delta d_1 = \Delta d_3$.

We then get the following equation: $\Delta T_T = 2\Delta t_1 + \Delta t_2$.

We must solve for Δt_2 in terms of Δt_1 in order to get a workable equation, so we must set up the following set of equations:

4.3.2 Equations

Known values

a The acceleration.

c_0 The speed of light.

T_T The total time.

Acceleration distance

$$v = c_0 \tanh\left(\frac{at}{c_0}\right)$$
$$v_t = c_0 \tanh\left(\frac{a\Delta t_1}{c_0}\right)$$
$$\Delta d_1 = \int c_0 \tanh\left(\frac{a\Delta t_1}{c_0}\right) \mathrm{d}\Delta t_1$$
$$\Delta d_1 = \int \frac{c_0}{a} \times \frac{\sinh\left(\frac{a\Delta t_1}{c_0}\right)}{\cosh\left(\frac{a\Delta t_1}{c_0}\right)} \mathrm{d}\Delta t_1$$
$$\Delta d_1 = \frac{{c_0}^2}{a} \int \frac{1}{\frac{\sinh\left(\frac{a\Delta t_1}{c_0}\right)}{\cosh\left(\frac{a\Delta t_1}{c_0}\right)}} \mathrm{d}\Delta t_1$$
$$\Delta d_1 = \frac{{c_0}^2}{a} \ln\left|\cosh\left(\frac{c_0\Delta t_1}{c_0}\right)\right|$$
$$\frac{a\Delta t_1}{c_0} > 0 \implies \cosh > 0$$
$$\Delta d_1 = \frac{{c_0}^2}{a} \ln\left(\cosh\left(\frac{c_0\Delta t_1}{c_0}\right)\right)$$

Turn distance

$$\tan\theta = \frac{\Delta d_1}{r}$$
$$\theta = \arctan\left(\frac{\Delta d_1}{r}\right)$$

$$\phi = 2\pi - 2\theta$$

$$\Delta d_2 = \phi r$$

Turn time

$$\Delta t_2 = \frac{\Delta d_2}{v_t}$$

Turn radius

$$a = \frac{v^2}{r} \qquad r = \frac{v^2}{a}$$

Using these equations we can substitute into $\Delta T_T = 2\Delta t_1 + \Delta t_2$ to get our final equation in subsection 4.3.3 on the next page.

4.3.3 The total time

$$\Delta t_1 = \Delta t_3$$

$$\Delta T_T = \Delta t_1 + \Delta t_2 + \Delta t_s$$
$$\Delta T_T = 2\Delta t_1 + \Delta t_2$$
$$\Delta T_T = 2\Delta t_1 + \frac{\left(2\pi - 2\arctan\left(\frac{\Delta d_1}{r}\right)\right)\frac{v^2}{a}}{v}$$
$$\Delta T_T = 2\Delta t_1 + \frac{\left(2\pi - 2\arctan\left(\frac{\frac{{c_0}^2}{a}\ln\left(\cosh\left(\frac{a\Delta t_1}{c_0}\right)\right)}{\frac{v^2}{a}}\right)\right)v}{a}$$
$$\Delta T_T = 2\Delta t_1 + \frac{\left(2\pi - 2\arctan\left(\frac{{c_0}^2\ln\left(\cosh\left(\frac{a\Delta t_1}{c_0}\right)\right)}{\left(c_0\tanh\left(\frac{a\Delta t_1}{c_0}\right)\right)^2}\right)\right)\times c_0\tanh\left(\frac{a\Delta t_1}{c_0}\right)}{a}$$
$$\Delta T_T = 2\Delta t_1 + \frac{\left(2\pi - 2\arctan\left(\frac{\ln\left(\cosh\left(\frac{a\Delta t_1}{c_0}\right)\right)}{\left(\tanh\left(\frac{a\Delta t_1}{c_0}\right)\right)^2}\right)\right)\times c_0\tanh\left(\frac{a\Delta t_1}{c_0}\right)}{a} \tag{4.1}$$

This gives us the final equation (4.1).

It is not possible to isolate for Δt_1 in this equation, so we will have the computer approximate the value and calculate based off that.

For the same reasons stated above, the time dilation over the time frame Δt_1 is equal to the time dilation over the time frame Δt_3. Given that the magnitude of velocity over the turn will be equal, we can calculate the time dilation over the time frame Δt_2 without summation. Thus, the simulator will only calculate the time dilation initial time step, then multiply by 2, then it will sum the result with the dilation over the time step for the turn.

The program to solve it was written by David White, and the source code is in appendix A on page 38.

The simulator will use the above approximations for time dilation in each of the scenarios listed above to approximate the optimal condition for each scenario; being the trip time (the time relative to Alice) which yields an arrival within minutes of the end of Trump's presidency.

The simulator will provide the following information:

- The number of seconds between the end of trumps presidency and the arrival of the space ship.
- The trip time, the total time that the astronaut spends in the ship (relative to him).
- The total time that the astronaut spends in the ship (relative to people on earth).
- The total time skipped (the discrepancy between the time the astronaut experiences and the time people on earth experience).

Part III

The Great Trump Escape

Chapter 5

Introducing the Forms of Transportation Used

In order to apply the mathematics previously developed and test the feasibility of skipping Donalds presidency, we will be using four different forms of transportation. These forms vary from practical to theoretical, and have vastly different potential speeds and accelerations.

It is important to note that we only care about the maximum speed for forms of transportation on earth, and we only care about acceleration for forms of transportation in space. The reasons for this will be made clear later on.

5.1 Running

This first example of transport is the simplest and also the slowest. We will assume that Alice can run at a very high speed for an indefinite period of time. To find the maximum speed, we will be using the upper limit of human potential in sprinting, Usain Bolt. In Berlin in 2009, Bolt ran the 100m sprint in a world record time of 9.58 s (IAAF, n.d.). Using the kinematics equation $v = \frac{\Delta d}{\Delta t}$, we can calculate Bolt's speed to be $\frac{100\,\mathrm{m}}{9.58\,\mathrm{s}}$, or about 10.44 m/s.

5.2 Flying

Our next form of transportation is significantly faster than running, but still relatively slow in the grand scheme of things. We will assume that Alice is flying around the earth, so we still only need to find maximum speed. We will also assume that Alice is able to maintain a very high speed indefinitely. The fastest aircraft in the world, the Lockheed Martin SR-71 Blackbird, can reach a maximum speed just over Mach 3 (Lockheed Martin, n.d.), or about 1029 m/s.

5.3 Rocketing

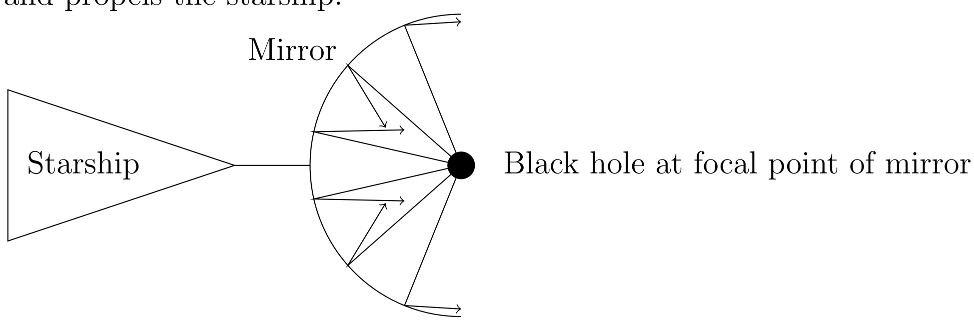
The next form of transport breaks away from the barriers of earth and into the realm of space travel. Thus, we will be needing to find acceleration rather than maximum speed. In "Calculating rocket acceleration," 2011, the acceleration of the Space Shuttle *Discovery* is calculated by dividing the net force by the total mass of the rocket, which is two million kilograms. "Calculating rocket acceleration," 2011 calculates the net force by subtracting the thrust force of 30.5 meganewtons by the gravitational force on the space shuttle. Since the space shuttle will be flying through space, which we will assume is empty, the gravitational force can be ignored. This means the net force is $30.5\,\mathrm{MN}$. We can use a rearranged version Newton's second law, $a = \frac{F}{m}$, to solve for acceleration. This gives us a value of $\frac{3.05 \times 10^{7}\,\mathrm{N}}{2.00 \times 10^{6}\,\mathrm{kg}}$, or $15.25\,\mathrm{m/s^2}$.

5.4 Using a Black Hole Starship

Up until now, the modes of transport have at least been somewhat realistic. Now, we enter the area of theoretical transportation. The details of how a black hole starship functions are outside the scope of this paper (and would add roughly 20 more pages), so we will only cover the basics.

In order for a black hole starship to function, a very small black hole must be created. Specifically, the Schwarzschild radius of the black hole (distance from the centre at which light cannot escape the gravitational force) should be about 9×10^{-19} m (Crane & Westmoreland, 2009). This small black hole will be placed at the focal point of a parabolic reflector, which is attached to the starship (Crane & Westmoreland, 2009). The black hole will emit a type of energy called Hawking radiation which reflects off the parabolic reflector and propels the ship forward with immense energy. According to Crane and Westmoreland, 2009, a proposed system with a black hole of mass 6.06×10^{8} kg should be able to accelerate to $0.1\, c_0$ in 20 days.

Figure 5.1: Diagram of a black hole starship. The Hawking radiation from the black hole reflects and propels the starship.



As with the space shuttle, we will be calculating acceleration since the black hole starship will be travelling in space. In addition, since the top speed is reasonably close to the speed of light, we must account for time dilation and cannot simply use the kinematics formula for acceleration. Recall from Gibbs, 1996 that:

$$v = c_0 \tanh\left(\frac{aT}{c_0}\right)$$

This can be rearranged to solve for acceleration:

$$
\begin{aligned}
a &= \frac{c_0 \operatorname{arctanh}\left(\frac{v}{c_0}\right)}{T} \\
a &= \frac{299\,792\,458\,\mathrm{m/s} \times \operatorname{arctanh}\left(\frac{0.1(299\,792\,458\,\mathrm{m/s})}{299\,792\,458\,\mathrm{m/s}}\right)}{20 \times 24 \times 60 \times 60} \\
a &\doteq 17.407
\end{aligned}
$$

Appendix A

Simulation source code

```java
import java.util.Scanner;

//Java class which approximates time dilation for certain special relativity cases using a summation technique
public class TimeDilationCalc
{
    final int TOTAL_TIME = 99779400; //The desired time elapsed for the observer
    final int C = 299792458; //The speed of light, constant
    final long C_SQ = C*C; //The speed of light squared, used to save calculations while simulating
    final double PI2 = Math.PI*2; //2 times the value of PI, usedd to save calculations while simulating
    final int acceptableError = 5; //How much error is acceptable when approximating delta t 1 for turning case

    double acceleration;//The acceleration of the ship inputted by the user
    double cSqOverAccel;//Speed of light squared divided by acceleration, pre calculated to save time when simulating

    Scanner s;//Scanner class to read input

    public TimeDilationCalc ()
    {
        s = new Scanner (System.in);
    }

    //User input to get acceleration + calculate sSqOverAccel
    public void getSimData ()
    {
        System.out.print("Enter ship acceleration: ");
        this.acceleration = s.nextDouble();
        cSqOverAccel = C_SQ/acceleration;
    }

    //Calculates velocity for a given acceleration time using the proper acceleration formula
    public double velocity (int accelTime)
```

```
    {
        return C*Math.tanh(acceleration*accelTime/C);
    }

    //Approximates how much time the ship will spend turning around for a given time spend accelerating
    public double turnTime (int accelTime)
    {
        double accelTimeOverC = acceleration*accelTime/C;
        double velocity = Math.tanh(accelTimeOverC);
        double theta = accelTimeOverC;
        theta = Math.cosh(theta);
        theta = Math.log(theta);
        theta *= 1/(velocity*velocity);
        theta = Math.atan(theta);
        double turnTime = PI2 - 2*theta;
        turnTime *= velocity * C;
        turnTime /= acceleration;
        return turnTime;
    }

    //Approximates how much time the ship should spend accelerating so that the trip lasts for the number of seconds given by tripTime
    public int findAccelTime (int tripTime)
    {
        int accelTime = 0;
        for (int i = 0; i < tripTime; i++)
        {
            if (Math.abs(turnTime (i) + 2*i - tripTime) < acceptableError)
            {
                accelTime = i;
                break;
            }
        }
        return accelTime;
    }

    //Approximates the amount of time elapsed for an observer given a specified trip time for a turned path.
    public double getObserverTimeTurning (int tripTime)
    {
        double observerTime = 0;
        int accelTime = findAccelTime(tripTime);
        for (int i = 0; i <= accelTime; i++)
        {
```

```
            observerTime += getDilatedTime (1, velocity(i));
        }
        observerTime *= 2;
        double topSpeed = velocity(accelTime);
        observerTime += getDilatedTime(tripTime - 2*accelTime, topSpeed);
        return observerTime;
    }

    //Approximates the amount of time elapsed for an observer given a specified trip time for a linear path.
    public double getObserverTimeStraight(int tripTime)
    {
        double observerTime = 0;
        int accelTime = tripTime/4;
        for (int i = 0; i <= accelTime; i++)
        {
            observerTime += getDilatedTime (1, velocity(i));
        }
        observerTime *= 4;
        return observerTime;
    }

    //Calculates time elapsed for an observer for the given velocity over the given time
    public double getDilatedTime (double passengerTime, double passengerVelocity)
    {
        double dilatedTime = Math.pow(passengerVelocity, 2);
        dilatedTime /= C_SQ;
        dilatedTime = 1 - dilatedTime;
        dilatedTime = Math.sqrt(dilatedTime);
        dilatedTime = passengerTime/dilatedTime;
        return dilatedTime;
    }

    //Approximates how much time must pass for the passenger so that the oberver experiences the time specified by TOTAL_TIME. This is for a turned path.
    public void optimumTurnTime ()
    {
        System.out.println();
        System.out.println("-------------");
        System.out.println("|TURNED PATH|");
        System.out.println("-------------");
        System.out.println();
```

```
        int tripTime = TOTAL_TIME;
        double dilatedTime = getObserverTimeTurning(tripTime);
        double diff = dilatedTime - TOTAL_TIME;
        int errorMag = (int) Math.ceil(Math.log10(Math.abs(tripTime)));
        for (int i = errorMag - 1; i >= 0; i --)
        {
            double sign = Math.signum(diff);
            int modif = (int) (sign*Math.pow(10, i));
            System.out.print(i);
            while (sign == Math.signum(diff) && Math.round(diff) != 0)
            {
                tripTime -= modif;
                dilatedTime = getObserverTimeTurning(tripTime);
                diff = dilatedTime - TOTAL_TIME;
                System.out.print(".");
            }
        }
        System.out.println();
        System.out.println();
        System.out.println("Time after presidency end (arrival): " + Math.round(dilatedTime - TOTAL_TIME));
        System.out.println("Trip Time Elapsed: " + tripTime);
        System.out.println("Earth Time Elapsed: " + dilatedTime);
        System.out.println("Time Skipped: " + (dilatedTime - tripTime));
    }

    //Approximates how much time must pass for the passenger so that the oberver experiences the time specified by TOTAL_TIME. This is for a straight path.
    public void optimumStraightTime ()
    {
        System.out.println();
        System.out.println("---------------");
        System.out.println("|STRAIGHT PATH|");
        System.out.println("---------------");
        System.out.println();
        int tripTime = TOTAL_TIME;
        double dilatedTime = getObserverTimeStraight(tripTime);
        double diff = dilatedTime - TOTAL_TIME;
        int errorMag = (int) Math.ceil(Math.log10(Math.abs(tripTime)));
        for (int i = errorMag - 1; i >= 0; i --)
        {
            double sign = Math.signum(diff);
            int modif = (int) (sign*Math.pow(10, i));
            System.out.print(i);
```

```
            while (sign == Math.signum(diff) && Math.round(diff) != 0)
            {
                tripTime -= modif;
                dilatedTime = getObserverTimeStraight(tripTime);
                diff = dilatedTime - TOTAL_TIME;
                System.out.print(".");
            }
        }
        System.out.println();
        System.out.println();
        System.out.println("Time after presidency end (arrival): " + Math.round(dilatedTime - TOTAL_TIME));
        System.out.println("Trip Time Elapsed: " + tripTime);
        System.out.println("Earth Time Elapsed: " + dilatedTime);
        System.out.println("Time Skipped: " + (dilatedTime - tripTime));
    }

    public static void main(String[] args)
    {
        TimeDilationCalc t = new TimeDilationCalc();
        t.getSimData();
        t.optimumTurnTime();
        t.optimumStraightTime();
        System.exit(0);
    }
}
```

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