Time Travelling to Avoid Trump
A Mathematical Analysis

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For Mr. Bouttell. You asked for rigorous. Here it is.

Contents

I The Twin Paradox **1**

1 Stating the Twin Paradox **2**

1.1 The problem . . . 2
1.2 Solution to Trump . . . 2
1.3 The paradox . . . 2
1.4 Approach to resolution of the paradox . . . 3
1.5 Notation and assumptions . . . 3
1.5.1 Notation . . . 3
1.5.2 Assumptions . . . 4
1.5.3 Other terminology . . . 4

2 Setup for the Twin Paradox Resolution **5**

2.1 A naive analysis . . . 5
2.2 Deriving the relativistic Doppler effect . . . 6
2.2.1 More notation . . . 6
2.2.2 The derivation . . . 7
2.3 Minkowski spacetime diagrams and world lines . . . 9

3 Resolving the Twin Paradox **10**

3.1 The relativistic Doppler effect and the twin paradox . . . 10
3.1.1 Recall our assumptions . . . 10
3.1.2 The Doppler analysis . . . 10
3.2 Minkowski spacetime and the twin paradox . . . 13
3.2.1 When does Donald see Alice's turnaround? . . . 13
3.2.2 What does Alice see on Donald's clock at her turnaround? . . . 14
3.2.3 What about Alice's clock? . . . 15
3.2.4 Is it ambiguous? . . . 17
3.3 Length dilation analysis . . . 19
3.3.1 How does this apply to Alice? . . . 19
3.3.2 The problem with the length dilation analysis . . . 20
3.4 Accelerating Alice . . . 21
3.4.1 Magnitude of the Doppler effect . . . 21
3.4.2 Implications of subsection 3.4.1 . . . 23

3.4.3 Conclusion of the accelerating twin paradox resolution 25

II Deriving a Real-World Case 26

4 A real-world analysis 27
4.1 Deriving the formulas 27
4.2 Turning 28
4.2.1 Setup 28
4.2.2 Calculations 28
4.2.3 The total time 30
4.3 The linear case 31

III The Great Trump Escape 32

5 Introducing the Forms of Transportation Used 33
5.1 Running 34
5.2 Flying 35
5.3 Rocketing 36
5.4 Using a Black Hole Starship 37

A Simulation source code 39

Part I

The Twin Paradox

Chapter 1

Stating the Twin Paradox

1.1 The problem

Imagine two people, we will call them Alice[1] and Donald[2].

Donald is currently the president of the United States, and he is doing a bad job. Alice doesn't like this situation, so she is going to to time travel to the future when Donald is no longer president.

1.2 Solution to Trump

Special relativity tells us that time will pass slower for an object in motion than an object at rest (Bruni, Dick, Speijer, & Stewart, 2012; Einstein, 1916). This means that if Alice moves very fast, she will experience less time pass for her than on earth, and will effectively time travel to the future.

1.3 The paradox

The paradox here is that special relativity states that the laws of physics are the same in all inertial frames of reference (Bruni et al., 2012; Einstein, 1916). This means that we could also argue that Alice is not moving, and instead the earth is moving, which would result in Alice aging faster than Donald, which is the opposite of what we want and the opposite of what actually happens (Bruni et al., 2012, pp. 593–594).

This is called the twin paradox.

[1] Of computer security white paper fame.
[2] Of presidential infamy.

1.4 Approach to resolution of the paradox

It is commonly thought that general relativity is needed to resolve this paradox because Alice is in an accelerating frame of reference and special relativity cannot handle accelerating frames of reference. That is not true. Special relativity can indeed handle accelerating frames of reference, but it is more difficult (Gibbs, 1996; Weiss, 2016). However, it is much easier to use special relativity to solve this problem than it is to use general relativity. And in fact, we will later see that the acceleration of Alice is irrelevant to the resolution of the paradox.

1.5 Notation and assumptions

1.5.1 Notation

Our notation

In math, Alice will be referred to as A, and Donald will be referred to as D.

Other physics notation

Following is some common non-trivial physics notation that we will be using:

Velocity

$$v = \|\vec{v}\|$$

We will use v to denote $\|\vec{v}\|$.

Acceleration

$$a = \|\vec{a}\|$$

We will use a to denote $\|\vec{a}\|$.

Speed of light

$$c_0 = 299\,792\,458\,\mathrm{m/s}$$

We will use c_0 as the speed of light in a vacuum. We are using c_0 instead of c because c_0 is the recommended SI notation (BIPM, 2006).

β notation

$$\beta = \frac{v\,\mathrm{m/s}}{c_0\,\mathrm{m/s}}$$

Where v is velocity and c_0 is the speed of light. Also, since nothing can exceed or meet the speed of light, and the direction is not relevant to the amount of time dilation (Bruni et al., 2012; Einstein, 1916), we will say that: $0 \leq \beta < 1$.

1.5.2 Assumptions

Values

We will say that Alice travels a distance of 4 light years at a speed of $0.8\,c_0$ since these are nice numbers to work with (Kogut, 2012, p. 35). She then turns around and comes back to earth.

Turnaround

We will assume that Alice makes an instantaneous turnaround. Later we will show that this assumption has no effect on the resolution of the paradox.

1.5.3 Other terminology

Alice's trip is split up into two parts. First, she is moving away from earth, which we will call the outbound leg of the trip. After, she is moving towards the earth, which we will call the inbound leg of the trip.

Chapter 2

Setup for the Twin Paradox Resolution

2.1 A naive analysis

We have:

$$v = 0.8\, c_0 \implies \beta = 0.8$$
$$d = 4\,\mathrm{ly}$$

And we know this formula from Bruni et al., 2012, p. 583 (modified to use our notation):

$$\Delta t_D = \frac{\Delta t_A}{\sqrt{1-\beta}}$$

Which can be rearranged into:

$$\Delta t_A = \Delta t_D \sqrt{1-\beta^2} \tag{2.1}$$

We can trivially calculate how much time should pass for Donald:

$$\Delta t_D = \frac{2d}{v} = \frac{2 \times 4\,\mathrm{ly}}{0.8\, c_0} = \frac{8\,\mathrm{ly}}{0.8\, c_0} = 10\,\mathrm{y}$$

And how much time should pass for Alice follows by simply plugging this into (2.1):

$$\begin{aligned}
\Delta t_A &= 10\,\mathrm{y} \times \sqrt{1-0.8^2} \\
\Delta t_A &= 10\,\mathrm{y} \times \sqrt{\frac{9}{25}} \\
\Delta t_A &= 10\,\mathrm{y} \times \frac{3}{5} \\
\Delta t_A &= 6\,\mathrm{y}
\end{aligned}$$

So, Donald ages by 10 years, and Alice ages by 6 years.

This answer is right, but the problem is that we started by assuming that Donald is stationary and Alice is moving. However, we could have said that Alice is stationary and Donald is moving, and then we would calculate $\Delta t_A = 10$ and $\Delta t_D = 6$, which is wrong. So doing the analysis this way leads to ambiguity. We must develop a more rigorous way to analyze this problem.

Chapter 3

Resolving the Twin Paradox

3.1 The relativistic Doppler effect and the twin paradox

3.1.1 Recall our assumptions

Recall the assumptions and values we decided to use in subsection 1.5.2:

Distance Alice travels a distance of 4 light years, so $d = 4\,\mathrm{ly}$.

Velocity Alice travels at a speed of $0.8\,c_0$, so $v = 0.8\,c_0$ and $\beta = 0.8$.

The actual values we use do not matter for resolving the paradox. These values were chosen because they give nice numbers when we perform the calculations, which makes the analysis easier to follow.

We also assumed an instantaneous turnaround. This too makes the math easier, but we will show that it does not change the resolution to the paradox. We can assume that Alice is very strong and capable of surviving $479\,667\,932.8\,\mathrm{m/s^2}$ of acceleration.[1] If Alice is not that strong, we can instead say that another person, also named Alice, who is travelling at the same speed as the first Alice but in the opposite direction, passes by the first Alice at the turnaround point and syncs up her clock with the first Alice's clock. This would mean that when the second Alice arrives at earth, her clock will read the same thing as the original Alice's would have if she could survive all of that acceleration. Either way of handling Alice's instantaneous turnaround will work, since we will end up with the same reading on Alice's clock.

3.1.2 The Doppler analysis

Analyzing the twin paradox with the relativistic Doppler effect is helpful because it allows us to calculate what each person sees, and show that they are seeing different

[1] $479\,667\,932.8\,\mathrm{m/s} = 1.6\,c_0$

things, which solves the ambiguity stated in section 2.1.

When we derived the relativistic Doppler equations in section 2.2, we said that only Alice is shining a flashlight once per second. We did this to simplify our notation in that section and make the derivation easier to follow. However, this doesn't work for actually solving the paradox. We must have both Alice and Donald flash their lights once per second, as measured by their own proper time. Both Alice and Donald know that the other person is shining their light once per second.

This means that both of (2.6) and (2.7) will apply to both Alice and Donald.

We will use the following notation here:

f_A The frequency Alice shines her light at according to her own time.

f_D The frequency Donald shines his light at according to his own time.

f'_A The frequency Alice sees Donald shine his light at according to her own time. This is calculated with (2.6) when Alice is moving away from Donald, and with (2.7) when Alice is moving towards Donald.

f'_D The frequency Donald sees Alice shine her light at according to his own time. This is calculated with (2.6) when Alice is moving away from Donald, and with (2.7) when Alice is moving towards Donald.

Alice moving away from Donald

What Alice sees

$$f'_A = f_D\sqrt{\frac{1-\beta}{1+\beta}}$$

$$f'_A = f_D \times \sqrt{\frac{1}{9}}$$

$$f'_A = \frac{1}{3}f_D$$

So, Alice sees Donald's clock running slowly as she is moving away from him.

What Donald sees

$$f'_D = f_A\sqrt{\frac{1-\beta}{1+\beta}}$$

$$f'_D = f_A \times \sqrt{\frac{1}{9}}$$

$$f'_D = \frac{1}{3}f_A$$

So, Donald sees Alice's clock running slowly as she is moving away from him.

Alice moving towards Donald

What Alice sees

$$f'_A = f_D \sqrt{\frac{1+\beta}{1-\beta}}$$
$$f'_A = f_D \times \sqrt{9}$$
$$f'_A = 3f_D$$

So, Alice sees Donald's clock running quickly as she is moving towards him.

What Donald sees

$$f'_D = f_A \sqrt{\frac{1+\beta}{1-\beta}}$$
$$f'_D = f_A \times \sqrt{9}$$
$$f'_D = 3f_A$$

So, Donald sees Alice's clock running quickly as she is moving towards him.

Alice and Donald see the same thing. So why do we end up with Donald aging more if they both see the other age slowly, then they both see the other age quickly?

The answer lies in how long each person sees the other aging at a different speed.

Part II

Deriving a Real-World Case

Chapter 4

A real-world analysis

4.1 Deriving the formulas

Before making any calculations with velocity, we must first redefine our equation for velocity. Given that the speed of light, c_0, is constant and is considered the "universal speed limit", it can be reasoned that $v = at$ will no longer serve as an accurate calculation for velocity since it can exceed c. Therefore we must use $v = c_0 \tanh\left(\frac{at}{c_0}\right)$ as our equation for velocity, as it is the "true acceleration formula" (Gibbs, 1996).

For these calculations to be accurate, we must also assume that the earth is stationary, and that there are no other existing significant gravitational masses other than the earth. Additionally, we must ignore gravitational time dilation.

It should also be noted that the dilation values calculated will be approximations calculated by a written computer simulation. This simulation will sum the time dilation across each 1 second interval for the entirety of the journey, which will not yield a perfect result, but the calculated result will be accurate enough for our purposes.

Part III

The Great Trump Escape

Chapter 5

Introducing the Forms of Transportation Used

In order to apply the mathematics previously developed and test the feasibility of skipping Donalds presidency, we will be using four different forms of transportation. These forms vary from practical to theoretical, and have vastly different potential speeds and accelerations.

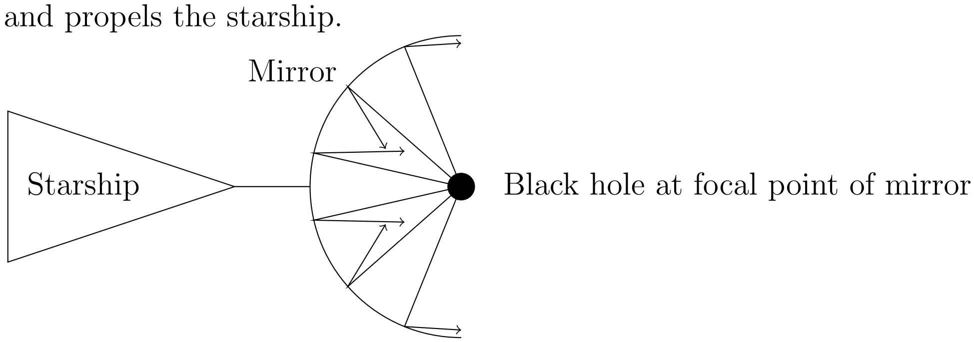
It is important to note that we only care about the maximum speed for forms of transportation on earth, and we only care about acceleration for forms of transportation in space. The reasons for this will be made clear later on.

5.4 Using a Black Hole Starship

Up until now, the modes of transport have at least been somewhat realistic. Now, we enter the area of theoretical transportation. The details of how a black hole starship functions are outside the scope of this paper (and would add roughly 20 more pages), so we will only cover the basics.

In order for a black hole starship to function, a very small black hole must be created. Specifically, the Schwarzschild radius of the black hole (distance from the centre at which light cannot escape the gravitational force) should be about 9×10^{-19} m (Crane & Westmoreland, 2009). This small black hole will be placed at the focal point of a parabolic reflector, which is attached to the starship (Crane & Westmoreland, 2009). The black hole will emit a type of energy called Hawking radiation which reflects off the parabolic reflector and propels the ship forward with immense energy. According to Crane and Westmoreland, 2009, a proposed system with a black hole of mass 6.06×10^{8} kg should be able to accelerate to $0.1\, c_0$ in 20 days.

Figure 5.1: Diagram of a black hole starship. The Hawking radiation from the black hole reflects and propels the starship.



As with the space shuttle, we will be calculating acceleration since the black hole starship will be travelling in space. In addition, since the top speed is reasonably close to the speed of light, we must account for time dilation and cannot simply use the kinematics formula for acceleration. Recall from Gibbs, 1996 that:

$$v = c_0 \tanh\left(\frac{aT}{c_0}\right)$$

This can be rearranged to solve for acceleration:

$$a = \frac{c_0 \operatorname{arctanh}\left(\frac{v}{c_0}\right)}{T}$$

$$a = \frac{299\,792\,458\,\mathrm{m/s} \times \operatorname{arctanh}\left(\frac{0.1(299\,792\,458\,\mathrm{m/s})}{299\,792\,458\,\mathrm{m/s}}\right)}{20 \times 24 \times 60 \times 60}$$

$$a \doteq 17.407$$

Appendix A

Simulation source code

```
import java.util.ArrayList;
import java.util.Scanner;

public class TimeDilationCalc
{
    final int TOTAL_TIME = 99779400;
    //final int TOTAL_TIME = 8*365*24*60*60;
    final double C = 299792458;
    final double C_SQ = C*C;
    final double PI2 = Math.PI*2;
    final int acceptableError = 5;

    double acceleration;
    double cSqOverAccel;
    int arrivalTimeMargin = 10;

    Scanner s;

    public TimeDilationCalc ()
    {
        s = new Scanner (System.in);
    }

    public void getSimData ()
    {
        System.out.print("Enter ship acceleration: ");
        this.acceleration = s.nextDouble();
        cSqOverAccel = C_SQ/acceleration;
    }

    public double velocity (int accelTime)
    {
        return C*Math.tanh(acceleration*accelTime/C);
    }

    public double turnTime (int accelTime)
    {
        double accelTimeOverC = acceleration*accelTime/C;
        double velocity = Math.tanh(accelTimeOverC);
```

```
        double theta = accelTimeOverC;
        theta = Math.cosh(theta);
        theta = Math.log(theta);
        theta *= 1/(velocity*velocity);
        theta = Math.atan(theta);
        double turnTime = PI2 - 2*theta;
        turnTime *= velocity * C;
        turnTime /= acceleration;
        return turnTime;
    }

    public int findAccelTime (int tripTime)
    {
        int accelTime = 0;
        for (int i = 0; i < tripTime; i++)
        {
            if (Math.abs(turnTime (i) + 2*i - tripTime) < acceptableError)
            {
                accelTime = i;
                break;
            }
        }
        return accelTime;
    }

    public double getObserverTimeTurning (int tripTime)
    {
        double observerTime = 0;
        int accelTime = findAccelTime(tripTime);
        for (int i = 0; i <= accelTime; i++)
        {
            observerTime += getDilatedTime (1, velocity(i));
        }
        observerTime *= 2;
        double topSpeed = velocity(accelTime);
        observerTime += getDilatedTime(tripTime - 2*accelTime, topSpeed);
        return observerTime;
    }

    public double getObserverTimeStraight(int tripTime)
    {
        double observerTime = 0;
        int accelTime = tripTime/4;
        for (int i = 0; i <= accelTime; i++)
```

```
        {
            observerTime += getDilatedTime (1, velocity(i));
        }
        observerTime *= 4;
        return observerTime;
    }

    public double getDilatedTime (double passengerTime, double passengerVelocity)
    {
        double dilatedTime = Math.pow(passengerVelocity, 2);
        dilatedTime /= C_SQ;
        dilatedTime = 1 - dilatedTime;
        dilatedTime = Math.sqrt(dilatedTime);
        dilatedTime = passengerTime/dilatedTime;
        return dilatedTime;
    }

    public void optimumTurnTime ()
    {
        int tripTime = TOTAL_TIME;
        double dilatedTime = getObserverTimeTurning(tripTime);
        double diff = dilatedTime - TOTAL_TIME;
        while (Math.abs(diff) > arrivalTimeMargin || diff <= 0)
        {
           tripTime -= diff;
           dilatedTime = getObserverTimeTurning(tripTime);
           diff = dilatedTime - TOTAL_TIME;
           System.out.println("difference: " + Math.round(diff));
        }
        System.out.println("difference: " + Math.round(dilatedTime - TOTAL_TIME));
        System.out.println(tripTime);
        System.out.println(dilatedTime);
    }

    public void optimumTurnTime2 ()
    {
        System.out.println();
        System.out.println("-------------");
        System.out.println("|TURNED PATH|");
        System.out.println("-------------");
        System.out.println();
        int tripTime = TOTAL_TIME;
        double dilatedTime = getObserverTimeTurning(tripTime);
        double diff = dilatedTime - TOTAL_TIME;
```

```java
        int errorMag = (int) Math.ceil(Math.log10(Math.abs(tripTime)));
        for (int i = errorMag - 1; i >= 0; i --)
        {
            double sign = Math.signum(diff);
            int modif = (int) (sign*Math.pow(10, i));
            System.out.print(i);
            while (sign == Math.signum(diff) && Math.round(diff) != 0)
            {
                tripTime -= modif;
                dilatedTime = getObserverTimeTurning(tripTime);
                diff = dilatedTime - TOTAL_TIME;
                System.out.print(".");
                //System.out.println("difference: " + Math.round(diff));
            }
        }
        System.out.println();
        System.out.println();
        System.out.println("Time after presidency end (arrival): " + Math.round(dilatedTime - TOTAL_TIME));
        System.out.println("Trip Time Elapsed: " + tripTime);
        System.out.println("Earth Time Elapsed: " + dilatedTime);
        System.out.println("Time Skipped: " + (dilatedTime - tripTime));
    }

    public void optimumStraightTime ()
    {
        int tripTime = TOTAL_TIME;
        double dilatedTime = getObserverTimeTurning(tripTime);
        double diff = dilatedTime - TOTAL_TIME;
        while (Math.abs(diff) > arrivalTimeMargin || diff <= 0)
        {
           tripTime -= diff;
           dilatedTime = getObserverTimeStraight(tripTime);
           diff = dilatedTime - TOTAL_TIME;
           System.out.println("difference: " + Math.round(diff));
        }
        System.out.println("difference: " + Math.round(dilatedTime - TOTAL_TIME));
        System.out.println(tripTime);
        System.out.println(dilatedTime);
    }

    public void optimumStraightTime2 ()
    {
```

```java
        System.out.println();
        System.out.println("---------------");
        System.out.println("|STRAIGHT PATH|");
        System.out.println("---------------");
        System.out.println();
        int tripTime = TOTAL_TIME;
        double dilatedTime = getObserverTimeStraight(tripTime);
        double diff = dilatedTime - TOTAL_TIME;
        int errorMag = (int) Math.ceil(Math.log10(Math.abs(tripTime)));
        for (int i = errorMag - 1; i >= 0; i --)
        {
            double sign = Math.signum(diff);
            int modif = (int) (sign*Math.pow(10, i));
            System.out.print(i);
            while (sign == Math.signum(diff) && Math.round(diff) != 0)
            {
                tripTime -= modif;
                dilatedTime = getObserverTimeStraight(tripTime);
                diff = dilatedTime - TOTAL_TIME;
                System.out.print(".");
                //System.out.println("difference: " + Math.round(diff));
            }
        }
        System.out.println();
        System.out.println();
        System.out.println("Time after presidency end (arrival): " + Math.round(dilatedTime - TOTAL_TIME));
        System.out.println("Trip Time Elapsed: " + tripTime);
        System.out.println("Earth Time Elapsed: " + dilatedTime);
        System.out.println("Time Skipped: " + (dilatedTime - tripTime));
    }

    public void thing ()
    {
        double num = s.nextDouble();
        num = Math.log10(num);
        num = Math.ceil(num);
        System.out.println(num);
    }

    public static void main(String[] args)
    {
        TimeDilationCalc t = new TimeDilationCalc();
        //t.thing();
```

```
        t.getSimData();
        t.optimumTurnTime2();
        t.optimumStraightTime2();
        System.exit(0);
    }
}
```

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