Creating a maximally entangled 2-qubit state $|00\rangle \rightarrow \frac{|00\rangle + |11\rangle}{\sqrt{2}}$

Initial State

$$|\psi\rangle = |0\rangle_1 \otimes |0\rangle_2 \longrightarrow \begin{pmatrix}1\\0\end{pmatrix}_1 \otimes \begin{pmatrix}1\\0\end{pmatrix}_2 = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}_{12}$$

Apply Hadamard to first qubit

$$(\hat{H}_1 \otimes \mathbb{1}_2)|0\rangle_1 \otimes |0\rangle_2 = \frac{1}{\sqrt{2}}(|0\rangle_1 + |1\rangle_1) \otimes |0\rangle_2$$

$$\longrightarrow \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}_{H_1} \otimes \begin{pmatrix}1\\0\end{pmatrix}_{\mathbb{1}_1} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\1\\0\end{pmatrix}$$

Apply CNOT Gate to both qubits

$$CX_{12}\frac{1}{\sqrt{2}}(|0\rangle_1 + |1\rangle_1) \otimes |0\rangle_2$$

$$= \frac{1}{\sqrt{2}}\left(|0\rangle_1 \otimes |0\rangle_2 + |1\rangle_1 \otimes |1\rangle_2\right)$$

$$= \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)$$

$$\longrightarrow \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\1\\0\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\0\\1\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\0\\0\end{pmatrix} + \frac{1}{\sqrt{2}}\begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

$(H|0\rangle_1) \otimes (\mathbb{1}|0\rangle_2)$:

$$\frac{1}{\sqrt{2}}\begin{pmatrix}1&1\\1&-1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

$$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\1\\0\end{pmatrix}$$

$(\hat{H}_1 \otimes \mathbb{1})(|0\rangle_1 \otimes |0\rangle_2)$:

$$H \otimes \mathbb{1} = \frac{1}{\sqrt{2}}\begin{pmatrix}1&1\\1&-1\end{pmatrix}\begin{pmatrix}1&0\\0&1\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1&0&1&0\\0&1&0&1\\1&0&-1&0\\0&1&0&-1\end{pmatrix}$$

$$\Rightarrow \frac{1}{\sqrt{2}}\begin{pmatrix}1&0&1&0\\0&1&0&1\\1&0&-1&0\\0&1&0&-1\end{pmatrix}\begin{pmatrix}1\\0\\0\\0\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\1\\0\end{pmatrix} \checkmark$$

CNOT gate: $|0\rangle\langle 0| \otimes \mathbb{1} + |1\rangle\langle 1| \otimes \sigma_x$

$$|0\rangle\langle 0| \otimes \mathbb{1} \longrightarrow \begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}1&0\end{pmatrix} \otimes \begin{pmatrix}1&0\\0&1\end{pmatrix}$$

$$= \begin{pmatrix}1&0\\0&0\end{pmatrix} \otimes \begin{pmatrix}1&0\\0&1\end{pmatrix} = \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}$$

$$|1\rangle\langle 1| \otimes \sigma_x \longrightarrow \begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}0&1\end{pmatrix} \otimes \begin{pmatrix}0&1\\1&0\end{pmatrix}$$

$$= \begin{pmatrix}0&0\\0&1\end{pmatrix} \otimes \begin{pmatrix}0&1\\1&0\end{pmatrix} = \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}$$

$$\text{CNOT} \rightarrow \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix} + \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix} = \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix} \checkmark$$

Is $\left[|0\rangle\langle 0| \otimes \mathbb{1} + |1\rangle\langle 1| \otimes \sigma_x\right]\left[\frac{|0\rangle \otimes |0\rangle + |1\rangle \otimes |0\rangle}{\sqrt{2}}\right]$ equivalent to $\frac{1}{\sqrt{2}}\left[|0\rangle\langle 0|0\rangle + \mathbb{1}|1\rangle\right] \otimes \left[|1\rangle\langle 1|0\rangle + \sigma_x|0\rangle\right]$?

$$\frac{1}{\sqrt{2}}(|0\rangle\langle 0|0\rangle + \mathbb{1}|1\rangle) = \frac{1}{\sqrt{2}}\left[\begin{pmatrix}1&0\\0&0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} + \begin{pmatrix}1&0\\0&1\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}\right] = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} \; ; \; |1\rangle\langle 1|0\rangle + \sigma_x|0\rangle = \begin{pmatrix}0&0\\0&1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} + \begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix}$$

$$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\0\\1\end{pmatrix}$$

no, doesn't appear so

Creating A 3 Qubit GHZ state : $|000\rangle \longrightarrow \frac{|000\rangle + |111\rangle}{\sqrt{2}}$

$|\psi\rangle = |0\rangle_1 \otimes |0\rangle_2 \otimes |0\rangle_3$

Apply a Hadamard gate to the first qubit

$|\psi'\rangle = H_1 \otimes \mathbb{1}_2 |\psi\rangle = (H_1 \otimes \mathbb{1}_2 \otimes \mathbb{1}_3)(|0\rangle_1 \otimes |0\rangle_2 \otimes |0\rangle_3) = \left(\frac{|0\rangle_1 + |1\rangle_1}{\sqrt{2}}\right) \otimes |0\rangle_2 \otimes |0\rangle_3$

Apply a C_x NOT between the 1st and 2nd qubit

$$|\psi''\rangle = CX_{12} \otimes \mathbb{1}_3 |\psi'\rangle$$

$$= \left\{ \Big[|0\rangle\langle 0| \otimes \mathbb{1} + |1\rangle\langle 1| \otimes \sigma_x \Big]_{12} \otimes \mathbb{1}_3 \right\} \left[\left(\frac{|0\rangle_1 + |1\rangle_1}{\sqrt{2}}\right) \otimes |0\rangle_2 \otimes |0\rangle_3 \right]$$

$$= \frac{|0\rangle_1 \otimes |0\rangle_2 + |1\rangle_1 \otimes |1\rangle_2}{\sqrt{2}} \otimes |0\rangle_3 = \frac{1}{\sqrt{2}} \Big[|0\rangle_1 \otimes |0\rangle_2 \otimes |0\rangle_3 + |1\rangle_1 \otimes |1\rangle_2 \otimes |0\rangle_3 \Big]$$

- Our next step is to apply a C_xNOT gate to the 1st and 3rd qubits.
- But how do we do this, given our matrix representation of $CX \rightarrow \begin{pmatrix} 1&0&0&0 \\ 0&1&0&0 \\ 0&0&0&1 \\ 0&0&1&0 \end{pmatrix}$?
- Previously when doing a CNOT on 1 and 2, we did $CNOT_{12} \otimes \mathbb{1}_3$. But how would this work for $CNOT_{13}$?
- The key is to note that we have been working in the basis $\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3$. If we first change our basis to $\mathcal{H}_1 \otimes \mathcal{H}_3 \otimes \mathcal{H}_2$ or $\mathcal{H}_2 \otimes \mathcal{H}_1 \otimes \mathcal{H}_3$, we can apply the operators $CNOT_{13} \otimes \mathbb{1}_2$ or $\mathbb{1}_2 \otimes CNOT_{13}$ respectively

Therefore, Change basis : $\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3 \longrightarrow \mathcal{H}_1 \otimes \mathcal{H}_3 \otimes \mathcal{H}_2$, so

$$\frac{1}{\sqrt{2}} \Big[|0\rangle_1 \otimes |0\rangle_2 \otimes |0\rangle_3 + |1\rangle_1 \otimes |1\rangle_2 \otimes |0\rangle_3 \Big] \longrightarrow \frac{1}{\sqrt{2}} \Big[|0\rangle_1 \otimes |0\rangle_3 \otimes |0\rangle_2 + |1\rangle_1 \otimes |0\rangle_3 \otimes |1\rangle_2 \Big]$$

- Now apply CNOT to qubits $\in \mathcal{H}_1$ and $\mathcal{H}_3$

$$|\psi'''\rangle = CNOT_{13} \otimes \mathbb{1}_2 |\psi''\rangle = \Big[CNOT_{13} \otimes \mathbb{1}_2 \Big] \frac{1}{\sqrt{2}} \Big[|0\rangle_1 \otimes |0\rangle_3 \otimes |0\rangle_2 + |1\rangle_1 \otimes |0\rangle_3 \otimes |1\rangle_2 \Big]$$

$$= \Big[CNOT_{13} |0\rangle_1 \otimes |0\rangle_3 \otimes \mathbb{1}_2 |0\rangle_2 + CNOT_{13} |1\rangle_1 \otimes |0\rangle_3 \otimes \mathbb{1}_2 |1\rangle_2 \Big] = \frac{1}{\sqrt{2}} \Big[|0\rangle_1 \otimes |0\rangle_3 \otimes |0\rangle_2 + |1\rangle_1 \otimes |1\rangle_3 \otimes |1\rangle_2 \Big]$$

Now change basis again, back to where we were before : $\mathcal{H}_1 \otimes \mathcal{H}_3 \otimes \mathcal{H}_2 \longrightarrow \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3$

$$\frac{1}{\sqrt{2}} \Big[|0\rangle_1 \otimes |0\rangle_3 \otimes |0\rangle_2 + |1\rangle_1 \otimes |1\rangle_3 \otimes |1\rangle_2 \Big] \longrightarrow \frac{1}{\sqrt{2}} \Big[|0\rangle_1 \otimes |0\rangle_2 \otimes |0\rangle_3 + |1\rangle_1 \otimes |1\rangle_2 \otimes |1\rangle_3 \Big]$$

$$= \frac{|000\rangle + |111\rangle}{\sqrt{2}}$$

https://quantumcomputing.stackexchange.com/questions/4252/how-to-derive-the-cnot-matrix-for-a-3-qubit-system-where-the-control-target-qu