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第三节 行列式的性质

例：

$$D=\begin{vmatrix}1 & 0 & -2\\3 & 5 & 0\\2 & 4 & 6\end{vmatrix}=26$$

$$\begin{vmatrix}1 & 3 & 2\\0 & 5 & 4\\-2 & 0 & 6\end{vmatrix}=26 \qquad D^T=\begin{vmatrix}1 & 3 & 2\\0 & 5 & 4\\-2 & 0 & 6\end{vmatrix}=26$$

行列式的转置

Transpose of Determinants

性质 1　行列式与它的转置行列式相等

$$D^T = \begin{vmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \cdots & \cdots & & \cdots \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = D$$

证明 记 $D=\det(a_{ij})$的转置行列式

$$D^T=\begin{vmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{vmatrix},$$

即 $b_{ij}=a_{ji}\,(i,j=1,2,\cdots,n)$, 按定义

$$D^T=\sum(-1)^{t(p_1\dots p_n)}b_{1p_1}b_{2p_2}\cdots b_{np_n}=\sum(-1)^{t(p_1\dots p_n)}a_{p_1 1}a_{p_2 2}\cdots a_{p_n n}.$$

而 $\sum(-1)^{t(p_1\dots p_n)}a_{p_1 1}a_{p_2 2}\cdots a_{p_n n}=\sum(-1)^{t(p_1\dots p_n)}a_{1p_1}a_{2p_2}\cdots a_{np_n}$

故 $D=D^T$.

设 $D = \left|a_{ij}\right|$ $D^T = \left|b_{ij}\right|$ 则

$$b_{ij} = a_{ji}$$

例如

$$D = \left|a_{ij}\right| = \begin{vmatrix} 2 & 4 & 7 & -2 \\ 3 & 5 & 6 & 18 \\ -3 & 0 & 9 & -4 \\ -8 & 1 & -7 & 3 \end{vmatrix}$$

$$b_{32} = 6 = a_{23}$$

$$D^T = \left|b_{ij}\right| = \begin{vmatrix} 2 & 3 & -3 & -8 \\ 4 & 5 & 0 & 1 \\ 7 & 6 & 9 & -7 \\ -2 & 18 & -4 & 3 \end{vmatrix}$$

$$b_{13} = -3 = a_{31}$$

性质 1　行列式与它的转置行列式相等

$$D^T = \begin{vmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \cdots & \cdots & & \cdots \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = D$$

因此，行列式中行与列具有同等的地位。

行列式的性质，凡是对行成立的，对列也成立，反之亦然。

性质 2　交换行列式的两行(两列)，行列式变号

$$
\begin{vmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
\cdots & \cdots & \cdots & \cdots \\
a_{i1} & a_{i2} & \cdots & a_{in} \\
\cdots & \cdots & \cdots & \cdots \\
a_{j1} & a_{j2} & \cdots & a_{jn} \\
\cdots & \cdots & \cdots & \cdots \\
a_{n1} & a_{n2} & \cdots & a_{nn}
\end{vmatrix}
\begin{array}{l}
\\ \\ (\text{第 } i \text{ 行}) \\ \\ (\text{第 } j \text{ 行}) \\ \\ \\
\end{array}
= -
\begin{vmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
\cdots & \cdots & \cdots & \cdots \\
a_{j1} & a_{j2} & \cdots & a_{jn} \\
\cdots & \cdots & \cdots & \cdots \\
a_{i1} & a_{i2} & \cdots & a_{in} \\
\cdots & \cdots & \cdots & \cdots \\
a_{n1} & a_{n2} & \cdots & a_{nn}
\end{vmatrix}
$$

例如

$$\begin{vmatrix} 1 & 7 & 5 \\ 6 & 6 & 2 \\ 3 & 5 & 8 \end{vmatrix} = -\begin{vmatrix} 1 & 7 & 5 \\ 3 & 5 & 8 \\ 6 & 6 & 2 \end{vmatrix},$$

$$\begin{vmatrix} 1 & 7 & 5 \\ 6 & 6 & 2 \\ 3 & 5 & 8 \end{vmatrix} = -\begin{vmatrix} 7 & 1 & 5 \\ 6 & 6 & 2 \\ 5 & 3 & 8 \end{vmatrix}.$$

例：$\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1, \begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix} = -1,$

$$\begin{vmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{vmatrix} = -\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 1$$

推论1

如果行列式有两行（两列）完全相同，则行列式等于零

证明 互换相同的两行，有 $\boldsymbol{D = -D}$,

$$\therefore \quad \boldsymbol{D = 0.}$$

$$D = \begin{vmatrix} 2 & 0 & 2 & 0 \\ 5 & -1 & 7 & 2 \\ 2 & 0 & 2 & 0 \\ 8 & 4 & 3 & 5 \end{vmatrix} = -\begin{vmatrix} 2 & 0 & 2 & 0 \\ 5 & -1 & 7 & -2 \\ 2 & 0 & 2 & 0 \\ 8 & 4 & 3 & 5 \end{vmatrix} \Rightarrow D = 0$$

性质 2　互换行列式的两行(两列)，行列式变号

推论2

奇数次互换行列式的两行(两列)，行列式变号。

偶数次互换行列式的两行(两列)，行列式不变。

性质 3

行列式的某一行（列）中所有元素都乘以同一个数 k，等于用数 k 乘以此行列式

即

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \cdots & \cdots & \cdots & \cdots \\ ka_{i1} & ka_{i2} & \cdots & ka_{in} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = k \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = kD$$

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \cdots & \cdots & \cdots & \cdots \\ ka_{i1} & ka_{i2} & \cdots & ka_{in} \\ \cdots & \cdots & & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = k \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

推论 行列式某一行(某一列)的公因子可以提到行列式外

也可以将行列式外的乘积因子乘到行列式的某一行（某一列）

例如

$$\begin{vmatrix} 2 & 4 & 7 & -2 \\ 3 & 12 & 6 & 18 \\ -3 & 0 & 9 & -4 \\ -8 & 1 & -7 & 8 \end{vmatrix} = 3\begin{vmatrix} 2 & 4 & 7 & -2 \\ 1 & 4 & 2 & 6 \\ -3 & 0 & 9 & -4 \\ -8 & 1 & -7 & 8 \end{vmatrix}$$

$$= 3 \cdot 2\begin{vmatrix} 2 & 4 & 7 & -1 \\ 1 & 4 & 2 & 3 \\ -3 & 0 & 9 & -2 \\ -8 & 1 & -7 & 4 \end{vmatrix}$$

又如

$$k\begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix} = \begin{vmatrix} a & b & c \\ p & q & r \\ kx & ky & kz \end{vmatrix} = \begin{vmatrix} a & kb & c \\ p & kq & r \\ x & ky & z \end{vmatrix}$$

思考?

$$k\begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix} \neq \begin{vmatrix} ka & kb & kc \\ kp & kq & kr \\ kx & ky & kz \end{vmatrix}$$

$$\begin{vmatrix} ka & kb & kc \\ kp & kq & kr \\ kx & ky & kz \end{vmatrix} = k^3 \begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix}$$

设 ***D*** 是 ***n*** 阶行列式：$D = \left| a_{ij} \right|$

则 $\left| ka_{ij} \right| = k^n \left| a_{ij} \right| = k^n D$

注意：$\left| ka_{ij} \right| \neq k \left| a_{ij} \right|$

推论 1

如果行列式有一行（列）元素全为零，则该行列式等于零

$$D=\begin{vmatrix} 8 & 7 & 0 & 5 \\ 5 & -1 & 0 & -2 \\ 1 & 10 & 0 & 3 \\ 8 & 9 & 0 & 5 \end{vmatrix}=0 \qquad D=\begin{vmatrix} 8 & -2 & 6 & 5 \\ 0 & 0 & 0 & 0 \\ -1 & 10 & 9 & 7 \\ 8 & 6 & 3 & -5 \end{vmatrix}=0$$

推论 2

如果行列式有两行（两列）元素成比例，则行列式等于零

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ 8 & 0 & 18 & 10 \\ -1 & 7 & 6 & 2 \\ 4 & 0 & 9 & 5 \end{vmatrix} = 0$$

性质 4 若行列式的某一行（列）中所有元素 都是两个数之和，例如:

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \cdots & \cdots & \cdots & \cdots \\ b_{i1}+c_{i1} & b_{i2}+c_{i2} & \cdots & b_{in}+c_{in} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

则可按该行（列），用以下方式分解成两个行列式的和:

$$D = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \cdots & \cdots & \cdots & \cdots \\ b_{i1} + c_{i1} & b_{i2} + c_{i2} & \cdots & b_{in} + c_{in} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

$$= \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \cdots & \cdots & \cdots & \cdots \\ b_{i1} & b_{i2} & \cdots & b_{in} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \cdots & \cdots & \cdots & \cdots \\ c_{i1} & c_{i2} & \cdots & c_{in} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = D_1 + D_2$$

例如

$$\begin{vmatrix} a & b & c \\ p & q & r \\ x+u & y+v & z+w \end{vmatrix} = \begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix} + \begin{vmatrix} a & b & c \\ p & q & r \\ u & v & w \end{vmatrix}$$

$$\begin{vmatrix} a & b+u & c \\ p & q+v & r \\ x & y+w & z \end{vmatrix} = \begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix} + \begin{vmatrix} a & u & c \\ p & v & r \\ x & w & z \end{vmatrix}$$

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \cdots & \cdots & \cdots & \cdots \\ kb_{i1}+lc_{i1} & kb_{i2}+lc_{i2} & \cdots & kb_{in}+lc_{in} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

性质3和5叫做行列式关于一行(一列)的**线性性质**

$$= k\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \cdots & \cdots & \cdots & \cdots \\ b_{i1} & b_{i2} & \cdots & b_{in} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} + l\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \cdots & \cdots & \cdots & \cdots \\ c_{i1} & c_{i2} & \cdots & c_{in} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

性质 5　把行列式的某一行(列)的各个元素乘以同 一个数 k，然后加到另一行(列)，行列式不变

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \cdots & \cdots & \cdots & \cdots \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{i1}+ka_{j1} & a_{i2}+ka_{j2} & \cdots & a_{in}+ka_{jn} \\ \cdots & \cdots & \cdots & \cdots \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

证

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{i1}+ka_{j1} & a_{i2}+ka_{j2} & \cdots & a_{in}+ka_{jn} \\ \cdots & \cdots & \cdots & \cdots \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

由性质4

$$= \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \cdots & \cdots & \cdots & \cdots \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \cdots & \cdots & \cdots & \cdots \\ ka_{j1} & ka_{j2} & \cdots & ka_{jn} \\ \cdots & \cdots & \cdots & \cdots \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \cdots & \cdots & \cdots & \cdots \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

$=0$

例如

$$\begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix} = \begin{vmatrix} a & b & c \\ p & q & r \\ x+2p & y+2q & z+2r \end{vmatrix}$$

$$= \begin{vmatrix} a & b-c & c \\ p & q-r & r \\ x+2p & y+2q-(z+2r) & z+2r \end{vmatrix}$$

注意：不能将某一行(列)的倍数加到同一行(列)