The Classical and Nonstandard Analyses of Limits

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Introduction

A **set** is a collection of elements. These elements can be anything: numbers, shapes, colors, and so on. For instance, consider the set A of the primary colors:

$$A = \{\text{red, blue, yellow}\}$$

A **function** is a mapping between the elements of two sets where each element from one set is assigned to exactly one element from the other. In the realm of single variable calculus, functions predominantly deal with sets of real numbers. For example, the function

$$f(x) = x^2$$

takes a set of inputs (x) and produces a corresponding set of their squares f(x) in the form of **ordered pairs**:

x	f(x)
-3	9
-2	4
-1	1
0	0
1	1
2	4
3	9

Plotting the points generated by $f(x)$ on the XY coordinate plane produces the following graph:

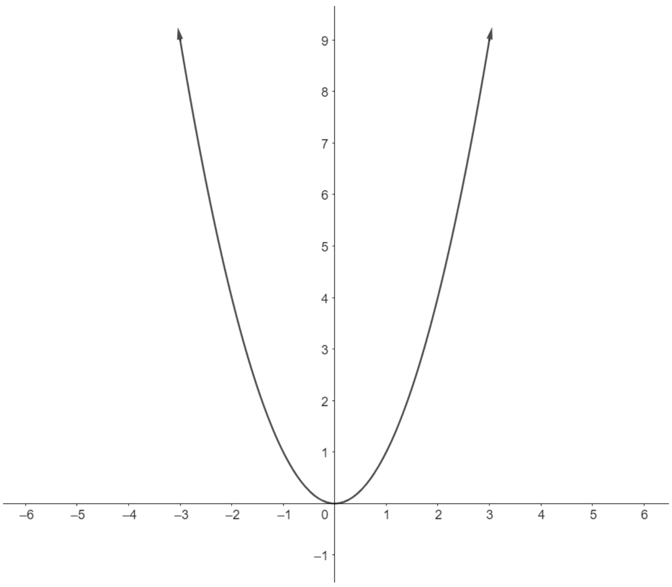

Figure 1: f(x) produces a parabolic curve

Limits analyze how the outputs of such functions behave as their inputs (x) approach either a particular point or ∞. For example, the parabolic function above approaches 4 as x approaches 2 and is denoted like so:

$$\lim_{x\to 2} x^2 = 4$$

As x approaches ∞, x^2 also approaches ∞:

$$\lim_{x\to\infty} x^2 = \infty$$

I intend to explore two approaches of rigorously defining the limit. The first employs the **epsilon-delta** definition of the limit, while the second utilizes the concept of **hyperreal numbers**.

The Classical Approach

The Definition

Real analysis is essentially the rigorous version of calculus. It seeks to strictly define and prove the mechanisms and ideas presented in calculus. For this section, I will be following section A.2 of Appendix A of

> Simmons, G. (1996). *Calculus With Analytic Geometry* (2nd ed.), Mcgraw-Hill Education.

Being mindful of the fact that the absolute value of the difference between two values ($|a-b|$) represents the distance between them will greatly aid in understanding the notation below. The limit is defined in the following manner:

> Let a function $f(x)$ be defined on some interval containing the number c such that there are x's in the domain of $f(x)$ where
>
> $$0 < |x-c| < \delta$$
>
> for every positive number δ. The statement
>
> $$\lim_{x\to c} f(x) = L$$
>
> is then defined like so: For every positive number ε, there exists a positive number δ such that
>
> $$|f(x) - L| < \varepsilon$$
>
> for every x in the domain of $f(x)$ where
>
> $$0 < |x-c| < \delta$$

This definition states that $f(x)$ approaches L as x approaches a certain value c if it can be shown that, for any set of outputs that lie within some distance ε from L, there exists a corresponding set of inputs (x's) that lie within some distance δ from c which *guarantees* that $f(x)$ falls within said range of L. In this sense, we can bring the range of outputs as close as we want to L (letting epsilon go to 0) while being absolutely sure that $f(x)$ lies within it. This is known as the **epsilon-delta definition** of the limit.

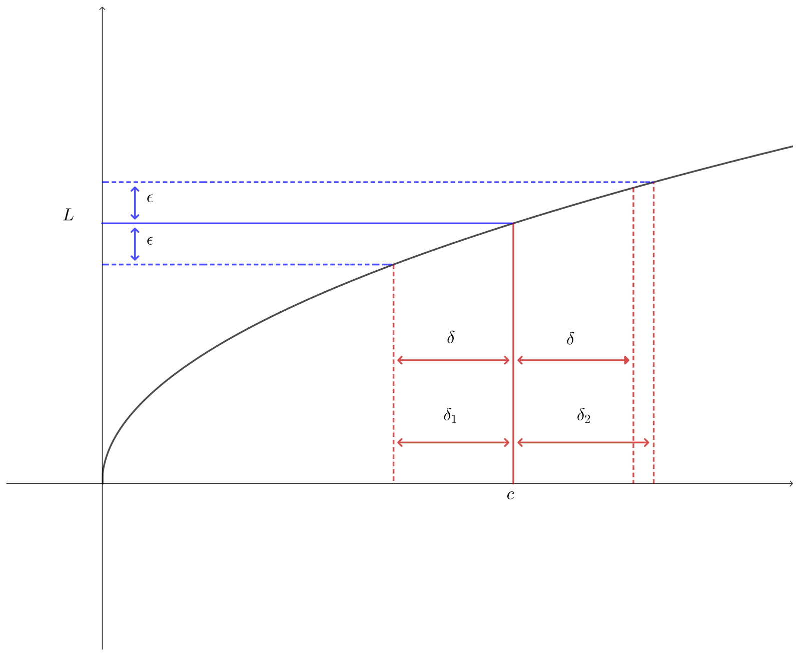

Figure 2: Epsilon-delta definition of the limit visualized

One detail to note in the visualization above (Figure 2) is the presence of δ_1 and δ_2. This is because the x corresponding to $L - \varepsilon$ does not necessarily lie the same distance away from c as the x corresponding to $L + \varepsilon$, since the rate at which $f(x)$ changes may vary as x sweeps from $c - \delta_1$ to $c + \delta_2$. To illustrate this, notice how the curvature in the graph is steeper on the left-hand side of c compared to the curvature on the right-hand side. This means that sweeping through some range of outputs on the left requires a smaller increment of x as opposed to sweeping through that same range of outputs on the right. Therefore, $\delta_1 < \delta_2$ for this particular graph.

This complication can be readily resolved by letting δ equal the smaller of δ_1 and δ_2:

$$\delta = \min(\delta_1, \delta_2)$$

min() is shorthand for taking the smallest value among the set of numbers present between the parentheses. For instance, if $x = \min(1, 2, 3)$, then $x = 1$. Allowing δ to be defined in this way works because of the following reasoning: Assume that $\delta_1 < \delta_2$ and that $|x - c| < \delta_1$ implies $|f(x) - L| < \varepsilon$. If $|x - c| < \delta_1$, then $|x - c| < \delta_2$ since $\delta_1 < \delta_2$. It is therefore assured that $|f(x) - L| < \varepsilon$.

Employing the Definition

The epsilon-delta definition of the limit can now be used to prove various facts and properties. As a basic example, consider the following theorem:

Theorem 1
If f(x) = x, then $\lim_{x\to a} f(x) = a$, or

$$\lim_{x\to a} x = a$$

To prove this, choose some $\varepsilon > 0$, and let $\delta = \varepsilon$. For any x satisfying the inequalities $0 < |x - a| < \delta$, we know that $|f(x) - a| < \varepsilon$. This is because $f(x) = x$, and $\delta = \varepsilon$. The theorem is therefore proven.

We can also prove some essential limit laws: their sums, differences, products, and quotients.

Theorem 2 - Limit Laws
If $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$, then

(i) $\lim_{x\to a}[f(x) + g(x))] = L + M$

(ii) $\lim_{x\to a}[f(x - g(x))] = L - M$

(iii) $\lim_{x\to a} f(x)g(x) = LM$

(iv) $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{L}{M}$

To prove (i), we let $\varepsilon > 0$ be given and allow $\delta_1, \delta_2 > 0$ where

$$0 < |x - a| < \delta_1 \Rightarrow |f(x) - L| < \tfrac{1}{2}\varepsilon$$

and

$$0 < |x - a| < \delta_2 \Rightarrow |f(x) - M| < \tfrac{1}{2}\varepsilon.$$

For those who are unfamiliar with the $\Rightarrow$ symbol, it means that the statement following it is implied (or logically follows) from the statement preceding the symbol.

The $\frac{1}{2}$'s in front of the ε's may cause some confusion, but recall that when $\lim_{x\to c} f(x) = L$, the epsilon-delta definition tells us that there exists a set of x's lying within some distance δ from c such that the distance between $f(x)$ and L is always less than ε. Knowing this, it is then clear that if there exists $\delta > 0$ such that $|f(x) - L| < \frac{1}{2}\varepsilon$ for some $\varepsilon > 0$, then $|f(x) - L| < \varepsilon$ is also implied, because $\frac{1}{2}\varepsilon$ is smaller than ε.

Continuing the proof, we let $\delta = \min(\delta_1, \delta_2)$. If $0 < |x - a| < \delta$, then

$$\begin{aligned} |[f(x) + g(x)] - (L + M)| &= |[f(x) - L] + [g(x) - M]| && (1)\\ &\le |f(x) - L| + |g(x) - M| && (2)\\ &< \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon = \varepsilon, && (3) \end{aligned}$$

proving (i). Analyzing the three-step sequence above in further detail: (1) simply takes advantage of the associative property of addition and moves terms around. (2) is a subtle application of the distances variation of the **triangle inequality** which states that, for real numbers x and y,

$$|x + y| \le |x| + |y|.$$

It essentially says that the distance between the sum of two numbers and 0 can be no more than the combined distances of x to 0 and y to 0. For example, let $x = 2$ and $y = 3$. Therefore, $|x + y| = 5 = |x| + |y|$. In general, $|x + y| = |x| + |y|$ when either a) both numbers are of the same sign or b) at least one of the numbers is 0. Let $x = -2$ and $y = 3$. We then have $|x + y| = 1 < |x| + |y| = 5$. In general, $|x + y| < |x| + |y|$ if $x, y \neq 0$ and are of opposite sign. It can ultimately be seen that $|x + y|$ is in fact less than or equal to $|x| + |y|$. Regarding the limit proof above, the two values involved in the triangle inequality are $f(x) - L$ and $g(x) - M$.

(3) substitutes both $|f(x) - L|$ and $|g(x) - M|$ for $\frac{1}{2}\varepsilon$. Since $|f(x) - L| < \frac{1}{2}\varepsilon$ and $|g(x) - M| < \frac{1}{2}\varepsilon$, it follows that $|f(x) - L| + |g(x) - M| < \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon$. The theorem is ultimately proven, because the difference between the function, $f(x) + g(x)$, and the desired limit, $L + M$, was shown to be less than any $\varepsilon > 0$ given an appropriate δ.

The proof of (ii) is similar to that of (i). We once again allow $\delta_1, \delta_2 > 0$ where

$$0 < |x - a| < \delta_1 \Rightarrow |f(x) - L| < \tfrac{1}{2}\varepsilon$$

and

$$0 < |x - a| < \delta_2 \Rightarrow |f(x) - M| < \tfrac{1}{2}\varepsilon.$$

Let $\delta = \min(\delta_1, \delta_2)$. If $0 < |x - c| < \delta$, then

$$\begin{aligned} |[f(x) - g(x)| - (L - M)]| &= |[f(x) - L] + [M - g(x)]|\\ &\le |f(x) - L| + |M - g(x)|\\ &= |f(x) - L| + |g(x) - M|\\ &< \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon = \varepsilon, \end{aligned}$$

proving (ii).

To prove (iii), we add and subtract $f(x)M$ to help relate the quantity $f(x)g(x) - LM$ to the differences $f(x) - L$ and $g(x) - M$:

$$\begin{aligned} |f(x)g(x) - LM| &= |[f(x)g(x) - f(x)M] + [f(x)M - LM]| \\ &\leq |f(x)g(x) - f(x)M| + |f(x)M - LM| \\ &= |f(x)||g(x) - M| + |M||f(x) - L| \\ &\leq |f(x)||g(x) - M| + (|M| + 1)|f(x) - L|. \end{aligned} \quad (4)$$

Provided some $\varepsilon > 0$, it is certain that $\delta_1, \delta_2, \delta_3 > 0$ all exist where

$$0 < |x - a| < \delta_1 \Rightarrow |f(x) - L| < 1 \Rightarrow |f(x)| < |L| + 1; \quad (5)$$

$$0 < |x - a| < \delta_2 \Rightarrow |g(x) - M| < \tfrac{1}{2}\varepsilon\left(\tfrac{1}{|L| + 1}\right); \quad (6)$$

$$0 < |x - a| < \delta_3 \Rightarrow |f(x) - L| < \tfrac{1}{2}\varepsilon\left(\tfrac{1}{|M| + 1}\right). \quad (7)$$

(5) comes from the fact that $\lim_{x \to a} f(x) = L$, so a $\delta_1 > 0$ exists for every $\varepsilon > 0$. In the case of (5), $\varepsilon = 1$. While (6) and (7) may look confusing, they once again stem from the definition of the limit. For instance, since $\lim_{x \to a} g(x) = M$, a $\delta_2 > 0$ exists for every $\varepsilon > 0$. ε is fundamentally a positive value, which makes $\frac{1}{2}\varepsilon\left(\frac{1}{|L| + 1}\right)$ also positive, justifying (6).

Resuming the proof, we let $\delta = \min(\delta_1, \delta_2, \delta_3)$. Then

$$0 < |x - a| < \delta \Rightarrow |f(x)g(x) - LM| < \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon = \varepsilon,$$

proving (iii). This final step is justified as so: (4) showed that

$$|f(x)g(x) - LM| \leq |f(x)||g(x) - M| + (|M| + 1)|f(x) - L|.$$

Since $|f(x)| < |L| + 1$ from (5), and $|g(x) - M| < \frac{1}{2}\varepsilon\left(\frac{1}{|L| + 1}\right)$ from (6),

$$|f(x)||g(x) - M| < (|L| + 1)\left[\tfrac{1}{2}\varepsilon\left(\tfrac{1}{|L| + 1}\right)\right] = \tfrac{1}{2}\varepsilon.$$

Since $|f(x) - L| < \frac{1}{2}\varepsilon\left(\frac{1}{|M| + 1}\right)$,

$$(|M| + 1)|f(x) - L| < (|M| + 1)\left[\tfrac{1}{2}\varepsilon\left(\tfrac{1}{|M| + 1}\right)\right] = \tfrac{1}{2}\varepsilon,$$

justifying the final step.

To prove (iv), we take advantage of the fact that

$$\lim_{x \to a}\left[\tfrac{f(x)}{g(x)}\right] = \lim_{x \to a}\left[f(x) \cdot \tfrac{1}{g(x)}\right]$$

due to (iii), so all that is needed is to show that

$$\lim_{x \to a}\left[\tfrac{1}{g(x)}\right] = \left[\tfrac{1}{M}\right].$$

If $g(x) \neq 0$, then

$$\left|\frac{1}{g(x)} - \frac{1}{M}\right| = \frac{|g(x) - M|}{|M||g(x)|} \quad (1)$$

Let $\delta_1 > 0$ where

$$0 < |x - a| < \delta_1 \Rightarrow |g(x) - M| < \tfrac{1}{2}\ |M| \quad (2)$$

so that

$$|g(x)| > \tfrac{1}{2}\ |M|$$

$$\frac{1}{|g(x)|} < \frac{2}{|M|}$$

which means

$$\left|\tfrac{1}{g(x)} - \tfrac{1}{M}\right| < \tfrac{2}{|M|^2}|g(x) - M|. \quad (3)$$

From (2), we know that the distance between $g(x)$ and M is smaller than $\frac{1}{2}\,|M|$. This means that $\frac{1}{2}|M| < |g(x)| < \frac{3}{2}|M|$, so $|g(x)| > \frac{1}{2}|M|$. (3) is achieved by substituting $\frac{1}{|g(x)|}$ on the right hand side of (1) with $\frac{2}{|M|}$.

Let $\varepsilon > 0$ be provided and $\delta_2 > 0$ such that

$$0 < |x - a| < \delta_2 \Rightarrow |g(x) - M| < \frac{|M|^2}{2}\varepsilon.$$

If $\delta = \min(\delta_1, \delta_2)$, then

$$0 < |x - a| < \delta \Rightarrow \left|\frac{1}{g(x)} - \frac{1}{M}\right| < \frac{2}{|M|^2} \cdot \frac{|M|^2}{2}\varepsilon = \varepsilon,$$

the final step coming from substituting $|g(x) - M|$ in (3) with $\frac{|M|^2}{2}\varepsilon$. This concludes the proof of (iv) and ultimately Theorem 2 in its entirety. It is interesting seeing how the limit laws are, in essence, results of brief sequences of subtle algebraic manipulations.

The final application of the epsilon-delta limit that will be analyzed is the classic **squeeze theorem**.

Theorem 3 - Squeeze Theorem
If there exists a $p > 0$ where

$$g(x) \leq f(x) \leq h(x)$$

for all x satisfying the inequalities $0 < |x - a| < p$, and if $\lim_{x\to a} g(x) = L$ and $\lim_{x\to a} h(x) = L$, then

$$\lim_{x\to a} f(x) = L$$

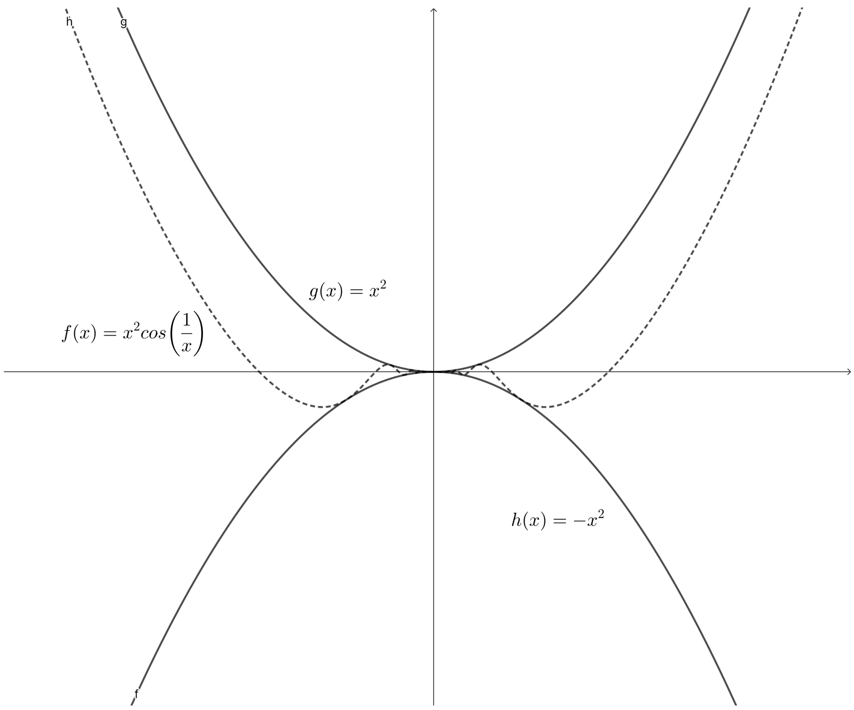

Figure 3: Squeeze theorem visualized

If a function $f(x)$ is bounded between two other functions $g(x)$ and $h(x)$, and $g(x)$ and $h(x)$ both approach the same limit L as x goes to a, then $f(x)$ is "squeezed" into the same limit L. For example, consider the functions displayed in Figure 3. We can show that $x^2\cos(\frac{1}{x})$ lies in between x^2 and $-x^2$ like so:

$$-1 \leq \cos(x) \leq 1$$
$$-x^2 \leq x^2\cos(x) \leq x^2$$

Since $\lim_{x\to 0} -x^2 = \lim_{x\to 0} x^2 = 0$, the squeeze theorem tells us that $\lim_{x\to 0} x^2\cos(\frac{1}{x}) = 0$.

To prove the theorem, let $\varepsilon > 0$ be provided, and choose $\delta_1, \delta_2 > 0$ where

$$0 < |x-a| < \delta_1 \Rightarrow L - \varepsilon < g(x) < L + \varepsilon$$

and

$$0 < |x-a| < \delta_2 \Rightarrow L - \varepsilon < h(x) < L + \varepsilon$$

Let $\delta = \min(p, \delta_1, \delta_2)$. Then

$$\begin{gathered} 0 < |x-a| < \delta \Rightarrow L - \varepsilon < g(x) \leq f(x) \leq h(x) < L + \varepsilon \\ L - \varepsilon < f(x) < L + \varepsilon \\ |f(x) - L| < \varepsilon \end{gathered}$$

and thus the theorem is proven.

The Nonstandard Approach

Introduction

The introduction follows sections 1.4 and 1.5 of

Keisler, H. (1976). *Foundations of Infinitesimal Calculus*, Prindle Weber & Schmidt.