The Classical and Nonstandard Analyses of Limits

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Introduction

A **set** is a collection of elements. These elements can be anything: numbers, shapes, colors, and so on. For instance, consider the set A of the primary colors:

$$A = \{\text{red, blue, yellow}\}$$

A **function** is a mapping between the elements of two sets where each element from one set is assigned to exactly one element from the other. In the realm of single variable calculus, functions predominantly deal with sets of real numbers. For example, the function

$$f(x) = x^2$$

takes a set of inputs (x) and produces a corresponding set of their squares f(x) in the form of **ordered pairs**:

x	f(x)
−3	9
−2	4
−1	1
0	0
1	1
2	4
3	9

Plotting these points on the XY coordinate plane produces the following graph:

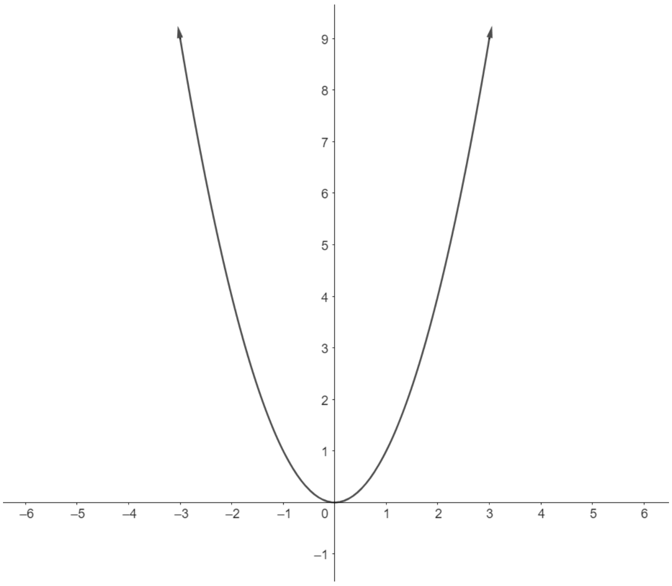

Figure 1: f(x) produces a parabolic curve

Limits analyze how the outputs of such functions behave as their inputs (x) approach either a particular point or ∞. For example, the parabolic function above approaches 4 as x approaches 2 and is denoted like so:

$$\lim_{x\to 2} x^2 = 4$$

As x approaches ∞, x^2 also approaches ∞:

$$\lim_{x\to\infty} x^2 = \infty$$

I intend to explore two approaches of rigorously defining the limit. The first employs the **epsilon-delta** definition of the limit, while the second utilizes the concept of **hyperreal numbers**.

The Classical Approach

The Definition

Real analysis is essentially the rigorous version of calculus. It seeks to strictly define and prove the mechanisms and ideas presented in calculus. For this section, I will be following section A.2 of Appendix A of

> Simmons, G. (1996). *Calculus With Analytic Geometry* (2nd ed.) Mcgraw-Hill Education.

Being mindful of the fact that the absolute value of the difference between two values ($|a-b|$) represents the distance between them will greatly aid in understanding the notation below. The limit is defined in the following manner:

> Let a function $f(x)$ be defined on some interval containing the number c such that there are x's in the domain of $f(x)$ where
>
> $$0 < |x-c| < \delta$$
>
> for every positive number δ. The statement
>
> $$\lim_{x\to c} f(x) = L$$
>
> is then defined like so: For every positive number ε, there exists a positive number δ such that
>
> $$|f(x) - L| < \varepsilon$$
>
> for every x in the domain of $f(x)$ where

$$0 < |x - c| < \delta$$

This definition states that $f(x)$ approaches L as x approaches a certain value if it can be shown that, for any set of outputs that lie within some distance ε from L, there exists a corresponding set of inputs (x's) that lie within some distance δ from c which *guarantees* that $f(x)$ falls within said range of L. In this sense, we can bring the range of outputs as close as we want to L (letting epsilon go to 0) while being absolutely sure that $f(x)$ lies within it. This is known as the **epsilon-delta definition** of the limit.

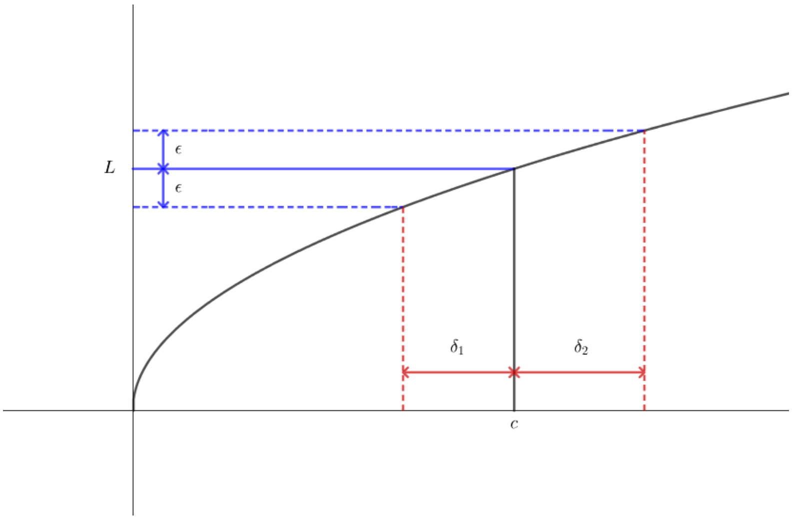

Figure 2: Epsilon-delta definition of the limit visualizaed

Figure 2 is a visualization of the epsilon-delta definition. One minor detail to note is how it shows δ_1 and δ_2 instead of simply δ. This is because the x corresponding to $L - \varepsilon$ does not necessarily lie the same distance away from c as the x corresponding to $L + \varepsilon$, since the rate at which $f(x)$ changes may vary as x sweeps from $c - \delta_1$ to $c + \delta_2$. To illustrate this, notice how the curvature in the graph of Figure 2 is steeper on the left-hand side of c compared to the curvature on the right-hand side. This means that sweeping through some range of outputs on the left requires a smaller range of x's as opposed to sweeping through that same range of outputs on the right. Therefore, $\delta_1 < \delta_2$ for this particular graph.

This complication can be readily resolved by letting δ equal the smaller of δ_1 and δ_2:

$$\delta = \min(\delta_1, \delta_2)$$

min() is shorthand for taking the minimum value of the set of numbers present between the parentheses. Allowing δ to be defined in this way works because of the following reasoning: Assume that $\delta_1 > \delta_2$. If $|x - c| < \delta_1$, then |f(x) - L| <

Employing the Definition

The epsilon-delta definition of the limit can now be used to prove various facts and properties. As a basic example, consider the following theorem:

Theorem 1
If f(x) = x, then $\lim_{x \to a} f(x) = a$, or

$$\lim_{x \to a} x = a$$

To prove this, choose some $\varepsilon > 0$, and let $\delta = \varepsilon$. For any x satisfying the inequalities $0 < |x - a| < \delta$, we know that $|f(x) - a| < \varepsilon$. This is because $f(x) = x$, and $\delta = \varepsilon$. The theorem is therefore proven.

We can also prove some essential limit laws: their sums, differences, products, and quotients.

Theorem 2 - Limit Laws
If $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$, then

(i) $\lim_{x\to a}[f(x+g(x))] = L + M$

(ii) $\lim_{x\to a}[f(x-g(x))] = L - M$

(iii) $\lim_{x\to a} f(x)g(x) = LM$

(iv) $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{L}{M}$

To prove (i), we let $\varepsilon > 0$ be given and allow $\delta_1, \delta_2 > 0$ where

$$0 < |x-a| < \delta_1 \Rightarrow |f(x) - L| < \tfrac{1}{2}\varepsilon$$

and

$$0 < |x-a| < \delta_2 \Rightarrow |f(x) - M| < \tfrac{1}{2}\varepsilon$$

For those who are unfamiliar with the $\Rightarrow$ symbol, it means that the statement following it is implied or logically follows from the statement preceding the symbol.

The $\frac{1}{2}$ in front of the ε's may cause some confusion, but recall that when $\lim_{x\to c} f(x) = L$, the epsilon-delta definition tells us that there exists a set of x's lying within some distance δ from c such that the distance between $f(x)$ and L is always less than ε ($|f(x) - L| < \varepsilon$.) Knowing this, it is then clear that if there exists $\delta > 0$ such that $|f(x) - L| < \frac{1}{2}\varepsilon$ for some $\varepsilon > 0$, then $|f(x) - L| < \varepsilon$ is also implied, because $\frac{1}{2}\varepsilon$ is smaller than ε.

Continuing the proof, we let δ