We now employ nonstandard analysis to prove the incredible fundamental theorem of calculus, which relates a function's area under its curve to its antiderivative. Before we can do this, however, we must first establish the following result:

> Consider a small change in input δx for some function $f(x)$ that is continuous on some interval. The corresponding small change in the output can be expressed like so:
> $$f(x+\delta x) - f(x) = f'(x) \cdot \delta x + \varepsilon \cdot \delta x,$$
> where $f'(x)$ is the derivative of $f(x)$ and $\varepsilon \approx 0$.

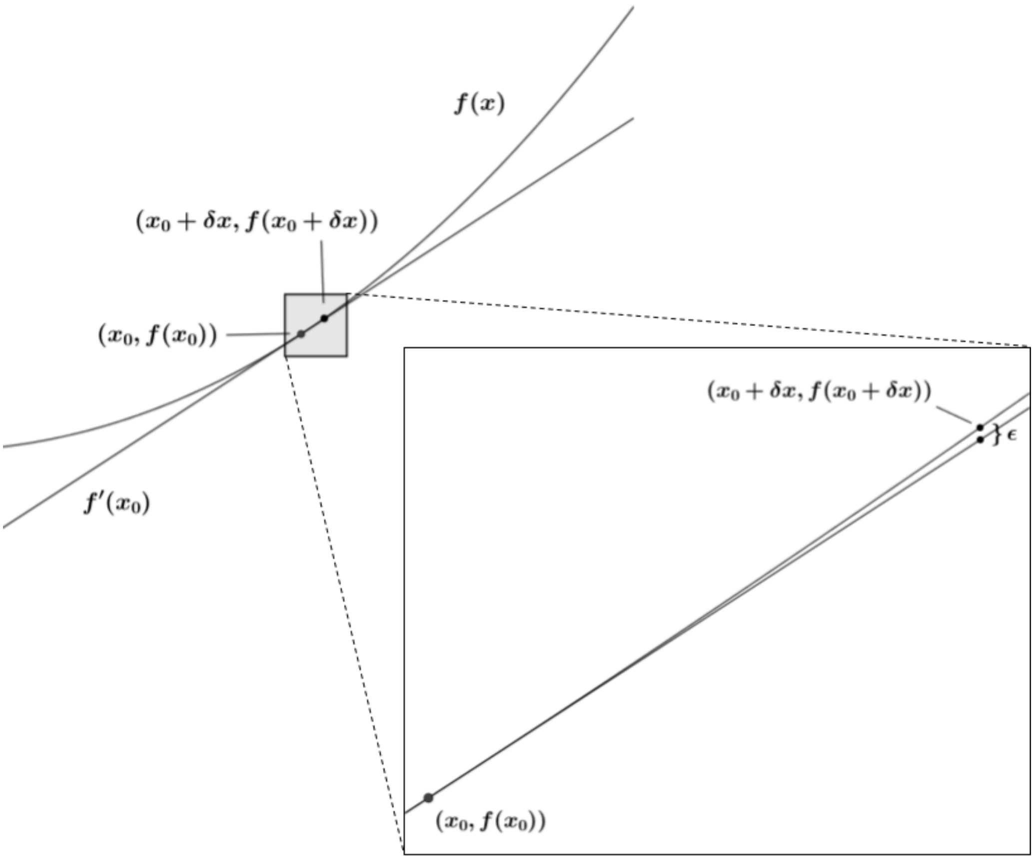

Figure 1: Tangent line visualization

A few small things to note:

1) Δ is typically used represent a small difference between two quantities, while δ is typically used to represent an infinitesimal value. The difference is merely semantical, as both represent the same core idea.

2) In Figure 9, $f'(x_0)$ does *not* represent the actual equation of the tangent line but merely serves as a label showing that the tangent line has a slope of $f'(x_0)$.

3) In Figure 9, ε is not intended to represent the vertical distance between the tangent line and $f(x_0)$. It's meaning will be clarified shortly.

The result can quickly be shown by starting with the limit definition of the derivative at $(x_0, f(x_0))$,

$$f'(x_0) = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}.$$

As a reminder, this says that the slope of the tangent line at $(x_0, f(x_0))$ is given by the limit of the slope of the **secant line** between $(x_0, f(x_0))$ and $(x_0 + \Delta x, f(x_0 + \Delta x))$ as $\Delta x \to 0$, where a secant line is simply a line that intersects a curve at least twice.

We can see in Figure 9 how the slope of the tangent line slightly differs from the slope of the secant line. Although the secant line is omitted from the figure, this is certain to be the case, because if they were equal, then the gap marked by ε would not exist[1]. Therefore, we can say that

$$\frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} = f'(x_0) + \varepsilon$$
$$f(x + \Delta x) - f(x) = f'(x_0) \cdot \Delta x + \varepsilon \cdot \Delta x,$$

where ε represents the small difference in *slope* between the secant and tangent lines. It can also be thought of as the error inherent in the limit from the slope of the tangent line.

As $\Delta x \to 0$, the slope of the secant line approaches that of the tangent line, so $\varepsilon \to 0$ as well. From an infinitesimal perspective, both values are infinitesimal or approximately zero, and the overarching idea applies broadly to any x where $f(x)$ is continuous, so (1) can be rewritten more generally as

$$f(x + \delta x) - f(x) = f'(x) \cdot \delta x + \varepsilon \cdot \delta x, \tag{3}$$

where ε and δx are both infinitesimal. (4) tells us that a nonlinear change – the left side – is equal to the sum of i) a linear change, $f'(x) \cdot \delta x$, and ii) the product of the error and the change in x. The result has been shown.

A brief review of Riemann sums is warranted as well. A **Riemann sum** is an approximation of the area under a curve via a division into small rectangles. Consider a function $f(x)$ continuous on the interval $[a, b]$. The rectangles are obtained by first partitioning the interval $[a, b]$ into n subintervals, so each partition point x_k is given by

$$x_k = a + \frac{k(b-a)}{n},$$

where k is an integer from 0 to n, and $\dfrac{b-a}{n}$ represents the length of each subinterval. The subintervals act as bases for the rectangles, and the rectangles' heights depend on $f(x)$. For the upcoming proof, we will be focused on the **left Riemann sum**, where the height of each rectangle is determined by the left endpoints of the subintervals. With this concept, the area is approximated by:

$$\sum_{k=1}^{n} f\left[\frac{(k-1)(b-a)}{n}\right] \cdot \Delta x,$$

where $f\left[\dfrac{(k-1)(b-a)}{n}\right]$ is the height of the kth rectangle and Δx is the increment. A more thorough explanation of this summation is not necessary, as the proof that follows relies on a variation of it that will indeed be closely looked at.

We are now ready to prove the fundamental theorem of calculus.

Theorem 5 - The Fundamental Theorem of Calculus

For a function $f(x)$ that is continuous over some closed interval [a,b], if another function $F(x)$ exists where

$$dF(x) = F'(x)dx = f(x)dx$$

for all x in [a,b], then

$$\int_a^b f(x)dx = F(b) - F(a).$$

If $F(x)$ is an **antiderivative** of $f(x)$, or $\frac{d}{dx}F(x) = f(x)$, over the interval $[a, b]$, then the integral of $f(x)$ on said interval is simply the difference between $F(x)$ evaluated at $x = b$ and $x = a$.

[1]It is possible for this gap to not be present, but $f(x)$ would have to be linear. This would result in ε being 0.

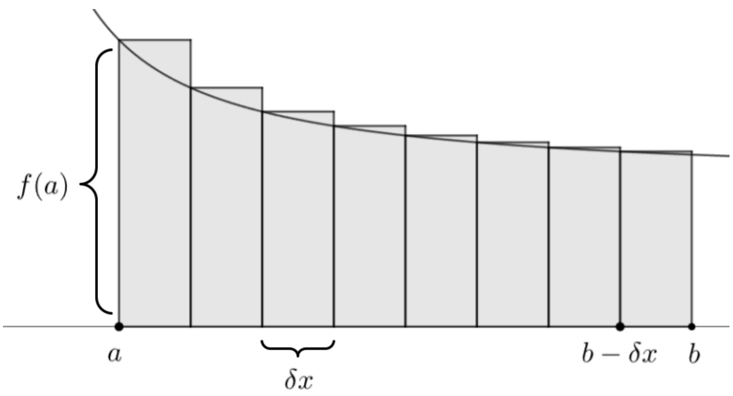

Figure 2: Riemann sum visualized

The integral is approximated by the left Riemann sum of $f(x)$ from a to b where the increment is an infinitesimal δx:

$$\int_a^b f(x)dx \approx \sum_{\substack{x=a \\ \text{step } \delta x}}^{b-\delta x} f(x) \cdot \delta x, \text{ where } \delta x \approx 0. \tag{3}$$

The notation of the summation in (3) may perplex readers who are familiar with

$$\sum_{k=1}^{n} a_k$$

where k is the **index** and a_k is the kth term in the sum. The same thing is happening in (3), but instead of the index incrementing by 1, it instead increments by δx – hence the "step δx".

(3) can be conceptualized with the visual shown in Figure 10 but instead on a far smaller scale with the area divided up into an infinitely large number of rectangles. Notice how the sum only increments from a to $b - \delta x$, as $f(b - \delta x)$ corresponds to the height of the final rectangle toward the right of Figure 10.

Proceeding with the proof, since $F'(x) = f(x)$, (2) tells us that

$$F(x + \delta x) - F(x) = f(x) \cdot \delta x + \varepsilon \cdot \delta x.$$

We can now sum both sides

$$\sum_{\substack{x=a \\ \text{step } \delta x}}^{b-\delta x} [F(x + \delta x) - F(x)] = \sum_{\substack{x=a \\ \text{step } \delta x}}^{b-\delta x} f(x) \cdot \delta x + \sum_{\substack{x=a \\ \text{step } \delta x}}^{b-\delta x} \varepsilon \cdot \delta x, \tag{4}$$

and this produces a telescoping sum on the left. This is because

$$\sum_{\substack{x=a \\ \text{step } \delta x}}^{b-\delta x} [F(x + \delta x) - F(x)]$$

$$= [F(a + \delta x) - F(a)] - [F(a + 2\delta x) - F(a + \delta x)] - [F(a + 3\delta x) - F(a + 2\delta x)] - \cdots - [F(b - \delta x + \delta x) - F(b - \delta x)]$$

$$= F(b') - F(a),$$

where b' is a number that is close to b but not exactly equal. The reason for this is that the δx's in the quantity $F(b - \delta x + \delta x)$ are not necessarily the same. The rationale is subtle, but it essentially boils down to the fact that adding a long sequence of infinitesimals to a will not land us exactly on b but on a value very close to it. In particular, the value $b - \delta x$ *conceptualizes* the idea of a number infinitely close to b but does not represent a definite value where δx cancels out to zero. It's best to think of the quantity $F(b - \delta x + \delta x)$ as one that lies infinitely close to $F(b)$.

From here, we can now say that

$$\int_a^b f(x)dx \approx \sum_{\substack{x=a \\ \text{step } \delta x}}^{b-\delta x} f(x)dx = F(b') - F(a) - \sum_{\substack{x=a \\ \text{step } \delta x}}^{b-\delta x} \varepsilon \cdot \delta x$$

which is obtained by replacing the summation on the left of (4) with $F(b') - F(a)$ and subtracting $\sum_{\substack{x=a \\ \text{step } \delta x}}^{b-\delta x} \varepsilon \cdot \delta x$ from both sides. Subtracting again gives

$$\sum_{\substack{x=a \\ \text{step } \delta x}}^{b-\delta x} f(x)dx - [F(b') - F(a)] = - \sum_{\substack{x=a \\ \text{step } \delta x}}^{b-\delta x} \varepsilon \cdot \delta x,$$

and taking the absolute value of both sides results in the inequalities

$$\left| \sum_{\substack{x=a \\ \text{step } \delta x}}^{b-\delta x} f(x)dx - [F(b') - F(a)] \right| \leq \left| \sum_{\substack{x=a \\ \text{step } \delta x}}^{b-\delta x} \varepsilon \cdot \delta x \right| \leq \sum_{\substack{x=a \\ \text{step } \delta x}}^{b-\delta x} |\varepsilon| \cdot \delta x.$$

The first inequality is just a way of saying that if two quantities are equal, then it is technically true that the absolute value of the first quantity cannot be larger than that of the second. It is an important inequality in the proof, as the quanitty on the left represents the distance between the approximated integral and the difference of the antiderivative evaluated at (almost) b and a, so we want to show that it goes infinitely close to 0. Now why does the second inequality hold? Recall that the triangle inequality states that $|a + b| \leq |a| + |b|$ for any numbers a and b. In this case, the second inequality is merely an extended triangle inequality. The reader is encouraged to reconsider the analogy utilized back when the triangle inequality was first introduced, extend the logic to this case, and convince themselves of this fact.

If $\max(|\varepsilon|)$ is the largest term in the summation $\sum_{\substack{x=a \\ \text{step } \delta x}}^{b-\delta x} |\varepsilon| \cdot \delta x$, then

$$\sum_{\substack{x=a \\ \text{step } \delta x}}^{b-\delta x} |\varepsilon| \cdot \delta x \leq \max(|\varepsilon|) \cdot \sum_{\substack{x=a \\ \text{step } \delta x}}^{b-\delta x} \delta x = \max(|\varepsilon|) \cdot (b' - a) \approx 0.$$

The first inequality can be understood by thinking of the sum

$$1 \cdot 2 + 2 \cdot 2 + 3 \cdot 2 + \cdots + 10 \cdot 2,$$

where each term is given by $a \cdot 2$ and a is a whole number from 1 through 10. If we now take the maximum value of a, 10, and replace every a in the sum with 10, then the sum clearly grows larger. The equality following the inequality replaces $\sum_{\substack{x=a \\ \text{step } \delta x}}^{b-\delta x} \delta x$ with $(b' - a)$. Remember that these calculations all stem from applying (2) to the fact that $\frac{d}{dx}F(x) = f(x)$, so $\varepsilon \approx 0$, which means

$$\max(|\varepsilon|) \cdot (b' - a) \approx 0,$$

and

$$\int_a^b f(x)dx \approx \sum_{\substack{x=a \\ \text{step } \delta x}}^{b-\delta x} f(x)\delta x \approx F(b') - F(a).$$

Lastly, since $F(x)$ is continuous, $F(b') \approx F(b)$ (remember that b' is off from b by merely an infinitesimal amount), we reach the conclusion that

$$\int_a^b f(x)dx = F(b) - F(a).$$

A few minutiae are left out for the sake of brevity, and the proof as a whole is far from airtight, but it succeeds in giving us a solid insight as to how the fundamental theorem of calculus can be argued from the angle of nonstandard analysis.

Reflection

The advent of nonstandard analysis lead to proofs of various results that were not unable to be practically proven with real analysis (proving them with real analysis was not impossible, though it would have been exceedingly difficult). What this entails back then was that the ease of notation with nonstandard analysis helped mathematicians see things that were more hidden from a real analysis perspective. It has been shown, however, that a result proven using one system can readily be proven with the other, and as of today the applications of nonstandard analysis seem to steer toward an area of mathematics that is not particularly connected with other areas; nonstandard analysis as a whole has managed to produce its own set of unique questions and problems. In general, real analysis (and more broadly standard analysis) and nonstandard analysis by no means replace each other, but rather serve as two distinct approaches toward analysis, and a preference toward either approach ultimately comes down to what one is first exposed to in their mathematical endeavors.