We now employ nonstandard analysis to prove the incredible fundamental theorem of calculus, which relates a function's area under its curve to its antiderivative. Before we can do this, however, we must first establish the following result:

> Consider a small change in input δx for some function $f(x)$ that is continuous some interval. The corresponding small change in the output can be expressed like so:
>
> $$f(x+\delta x) - f(x) = f'(x) \cdot \delta x + \varepsilon \cdot \delta x,$$
>
> where $f'(x)$ is the derivative of $f(x)$ and $\varepsilon \approx 0$.

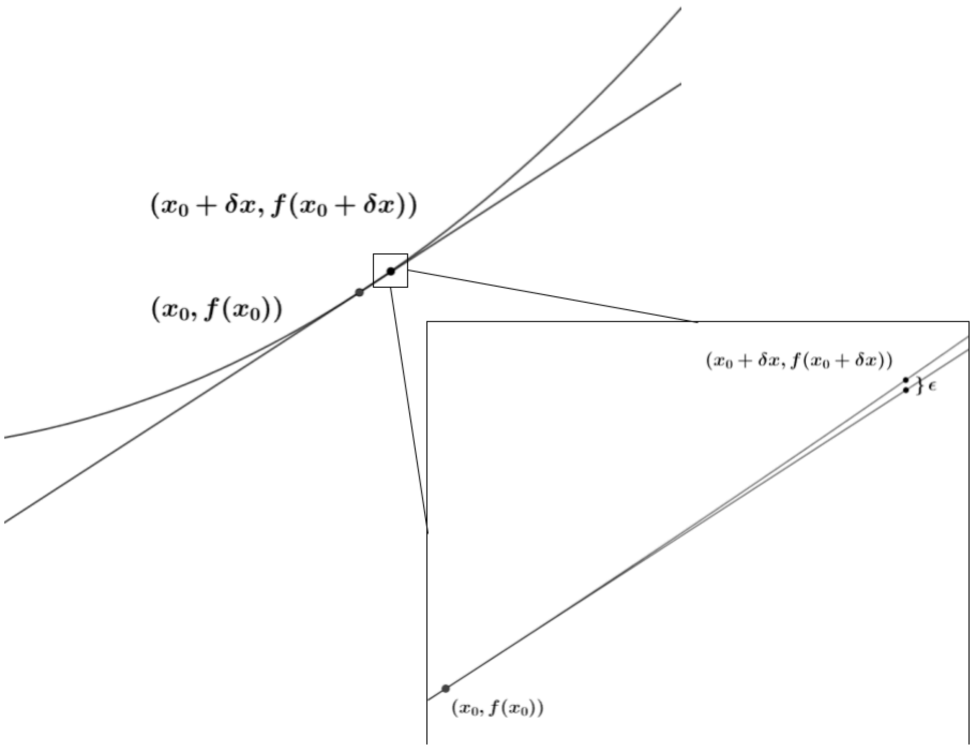

Figure 1: Tangent line visualization

A few small things to note:

1) Δ is typically used represent a small difference between two quantities, while δ is typically used to represent an infinitesimal value. The difference is merely semantical, as both represent the same core idea.

2) In Figure 9, $f'(x_0)$ does not represent the actual equation of the tangent line but merely serves as a label showing that the tangent line has a slope of $f'(x_0)$.

3) In Figure 9, ε is not intended to represent the vertical distance between the tangent line and $f(x_0)$. It's meaning will be clarified shortly.

The result can quickly be shown by starting with the limit definition of the derivative at $(x_0, f(x_0))$,

$$f'(x_0) = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}.$$

As a reminder, this says that the slope of the tangent line at $(x_0, f(x_0))$ is given by the limit of the slope of the secant line between $(x_0, f(x_0))$ and $(x_0 + \Delta x, f(x_0 + \Delta x))$ as $\Delta x \to 0$, where a secant line is simply a line that intersects a curve at least twice.

We can see in Figure 9 how the slope of the tangent line slightly differs from the slope of the secant line. Although the secant line is omitted from the figure, this is certain to be the case, because if they were equal, then the gap marked by ε would not exist[1]. Therefore, we can say that

$$\frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} = f'(x_0) + \varepsilon$$
$$f(x + \Delta x) - f(x) = f'(x_0) \cdot \Delta x + \varepsilon \cdot \Delta x, \tag{1}$$

where ε represents the small difference in slope between the secant and tangent lines. It can also be thought of as the error inherent in the limit from the slope of the tangent line.

As $\Delta x \to 0$, the slope of the secant line approaches that of the tangent line, so $\varepsilon \to 0$ as well. From an infinitesimal perspective, both values are infinitesimal or approximately zero, and the overarching idea applies broadly to any x where $f(x)$ is continuous, so (1) can be rewritten more generally as

$$f(x + \delta x) - f(x) = f'(x) \cdot \delta x + \varepsilon \cdot \delta x, \tag{2}$$

where ε and δx are both infinitesimal. (2) tells us that a nonlinear change – the left side – is equal to the sum of i) a linear change, $f'(x) \cdot \delta x$, and ii) the product of the error and the change in x, and this product is even smaller than δx.

We are now ready to prove the first fundamental theorem of calculus.

Theorem 5 - The Fundamental Theorem of Calculus

For a function $f(x)$ that is continuous over some closed interval [a,b], if another function $F(x)$ exists where

$$dF(x) = F'(x)dx = f(x)dx$$

for all x in [a,b], then

$$\int_a^b f(x)dx = F(b) - F(a).$$

If $F(x)$ is an **antiderivative** of $f(x)$, or $\frac{d}{dx}F(x) = f(x)$, over the interval $[a, b]$, then the integral of $f(x)$ on said interval is simply the difference between $F(x)$ evaluated at $x = b$ and $x = a$.

As a reminder, a **Riemann sum** is an approximation of the area under a curve via a division into small rectangles. These rectangles are obtained by first partitioning the interval $[a, b]$ into n subintervals where each partition point x_k is given by

$$x_k = a + \frac{k(b-a)}{n},$$

where $0 \leq k \leq n$ and k is a positive integer.Each subinterval acts as a base for a rectangle, and the rectangles' heights depend on $f(x)$. For example, if the interval ()Riemann sums are essentially summations ($\sum$), so n acts as an **increment**. For the proof, we will be focused on a variation of this concept known as the **left Riemann sum**, where the height of each rectangle is determined by the left endpoints of the subintervals.

[1]It is possible for this gap to not be present, but $f(x)$ would have to be linear. This would result in ε being 0.

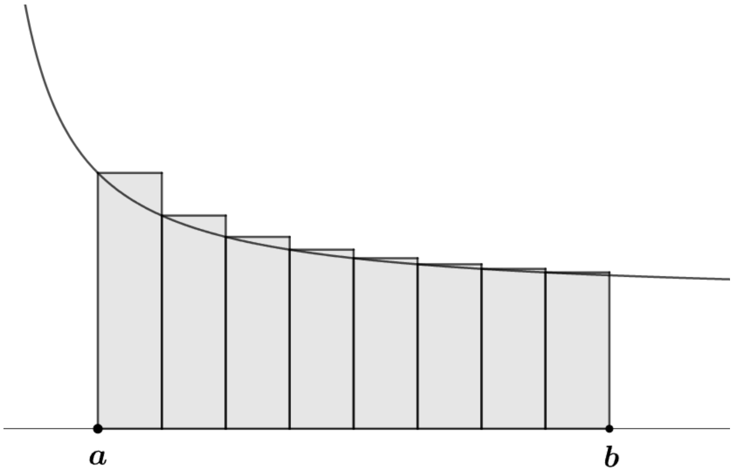

Figure 2: Riemann sum visualized

The integral is approximated by the left Riemann sum of $f(x)$ from a to b where the increment is an infinitesimal δx:

$$\int_a^b f(x)dx \approx \sum_{\substack{x=a \\ \text{step } \delta x}}^{b-\delta x} f(x) \cdot \delta x, \text{ where } \delta x \approx 0.$$

In this case, one can imagine the visual shown in Figure 10 but instead with the area divided up into an infinitely large number of rectangles. Notice how the summation only increments from a to $b - \delta x$, as $f(b - \delta x)$ corresponds to the height of the final rectangle toward the right of Figure 10.