Sobolev Spaces

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Introduction

In order to continue the study of Sobolev spaces, we will clarify some comments that arose from the previous talk. We will discuss the topology of Sobolev spaces, demonstrate some properties of the derivative in the weak sense, and conclude by showing that Sobolev spaces are Banach spaces for $k = 1, \cdots$ and $1 \leq p \leq \infty$.
Evans (2010),Brezis (2011).

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The Sobolev space

Definition

$$W^{k,p}(U)$$

consist of all locally summable functions $u : U \to \mathbb{R}$ such that for each multiindex α with $|\alpha| \leq k$, $D^{\alpha}u$ exist in the weak sense and belongs to $L^p(U)$.

Definition

If $u \in W^{k,p}(U)$, we define a norm to be:

$$\|u\|_{W^{k,p}(U)} := \begin{cases} \left(\sum_{|\alpha| \leq k} \int_U |D^\alpha u|^p dx\right)^{\frac{1}{p}}, & \text{if } 1 \leq p < \infty, \\ \sum_{|\alpha| \leq k} \sup_U |D^\alpha u|, & \text{if } p = \infty. \end{cases}$$

Definition

1. Let $\{u_m\}_{m=1}^{\infty}$, $u \in W^{k,p}(U)$. We say u_m converges to u in $W^{k,p}(U)$, written:

$$u_m \to u \text{ in } W^{k,p}(U),$$

provided

$$\lim_{m\to\infty} \|u_m - u\|_{W^{k,p}(U)} = 0.$$

2. We writte:

$$u_m \to u \text{ in } W^{k,p}_{loc}(U),$$

to mean:

$$u_m \to u \text{ in } W^{k,p}(V),$$

for each $V \subset\subset U$.

Note

We will see in the exercises that if $n = 1$ and U is an open interval in $\mathbb{R}^1$, then $u \in W^{k,p}(U)$ if and only if u equals a.e. an absolutely continuous function whose ordinary derivative (which exists a.e.) belongs to $L^p(U)$.
Such a simple characterization is however only available for $n = 1$. In general a function can belong to a Sobolev space and yet be discontinuous and/or unbounded.

Lemma

Let $U \subset \mathbb{R}$ open interval (a, b) and $f \in L^1_{loc}((a, b))$ be such that:

$$\int_a^b f \phi' dx = 0 \text{ for all } \phi \in C_c^\infty(U).$$

Then there exist a constant c such that $f = c$ a.e on U.

Proof

Fix a function $\psi \in C_c^1((a,b))$ such that $\int_{(a,b)} \psi dx = 1$. For any function $w \in C_c((a,b))$ there exist $\phi \in C_c^1((a,b))$ such that:

$$\phi' = w - \left(\int_a^b w \right) \psi. \tag{1}$$

Indeed, the function $h = w - \left(\int_a^b w\right)\psi$ is continuous, has compact support in U, and also:

Indeed, the function $h = w - \left(\int_a^b w\right)\psi$ is continuous, has compact support in U, and also:

$$\begin{aligned}\phi(x) &= \int_a^x w(t) - \left(\int_a^b w(y)dy\right)\psi(t)dx \\ &= \int_a^x w(t)dt - \int_a^b \left(\int_a^b w(y)dy\right)\psi(t)dt \\ &= \int_a^x w(t)dt - \left(\int_a^b w(y)dy\right)\int_a^x \psi(t)dt\end{aligned}$$

Note that $supp\,\phi \subset (a,b)$ because the $supp\,w \subset (a,b)$ and $supp\,\psi \subset (a,b)$.

$$
\begin{aligned}
\int_a^b h dx &= \int_a^b w(x) - \left(\int_U w(y)dy\right)\psi(x)dx, \\
&= \int_a^b w(x)dx - \left(\int_U w(x)dx\right)\left(\int_a^b \psi(x)dx\right), \\
&= \int_a^b w(x)dx - \int_a^b w(x)dx, \\
&= 0.
\end{aligned}
$$

Therefore h has a (unique) primitive with compact support in U. We deduce from 1 that:

$$
\int_U f\left[w - \left(\int_U w\right)\psi\right] = 0 \text{ for all } w \in C_C^\infty(U),
$$

i.e.,

$$\begin{aligned}\int_U f(x)\left[w(x)-\left(\int_U w(y)dy\right)\psi(x)dx\right] &= \int_U f(x)w(x)dx-\left(\int_U w(x)dx\right)\int_U f(x)\psi(x)dx,\\ &= \int_U f(x)w(x)dx-\int_U w(x)\int_U f(y)\psi(y)dydx,\\ &= \int_U\left[f(x)-\int_U f(y)\psi(y)dy\right]w(x)dx.\end{aligned}$$

for all $w \in C_C^\infty(U)$, then:

$$f(x)-\int_U f(y)\psi(y)dy = 0.$$

then:

$$f(x) = \int_U f(x)\psi(x)dx = 0$$

Which allows us to conclude the result.

Lemma

Let $g \in L^1_{loc}(U)$ with $U \subset \mathbb{R}$ interval (a,b); for y_0 fixed in U, set:

$$v(x) = \int_{y_0}^{x} g(t)dt, \text{ with } x \in U.$$

Then $v \in C(U)$ and:

$$\int_U v\phi' = -\int_U g\phi \text{ for all } \phi \in C^1_c(U).$$

Proof

We have:

$$
\begin{aligned}
\int_U v\phi' &= \int_U \left[\int_{y_0}^{x} g(t)dt\right] \phi'(x)dx, \\
&= -\int_a^{y_0}\int_x^{y_0} g(t)\phi'(x)dtdx + \int_{y_0}^{b}\int_{y_0}^{x} g(t)\phi'(x)dtdx, \\
&= -\int_a^{y_0}\int_a^{t} g(t)\phi'(x)dxdt + \int_{y_0}^{b}\int_{t}^{b} g(t)\phi'(x)dxdt, \\
&= -\int_{y_0}^{b} g(t)\phi(t)dt + \int_{y_0}^{b} g(t)(-\phi(t))dt, \\
&= -\int_U g(t)\phi(t)dt.
\end{aligned}
$$

Proof

We have:

$$\begin{aligned}
\int_U v\phi' &= \int_U \left[\int_{y_0}^{x} g(t)dt\right]\phi'(x)dx,\\
&= -\int_a^{y_0}\int_x^{y_0} g(t)\phi'(x)dtdx + \int_{y_0}^{b}\int_{y_0}^{x} g(t)\phi'(x)dtdx,\\
&= -\int_a^{y_0}\int_a^{t} g(t)\phi'(x)dxdt + \int_{y_0}^{b}\int_{t}^{b} g(t)\phi'(x)dxdt,\\
&= -\int_{y_0}^{b} g(t)\phi(t)dt + \int_{y_0}^{b} g(t)(-\phi(t))dt,\\
&= -\int_U g(t)\phi(t)dt.
\end{aligned}$$

Theorem

Let $u \in W^{1,p}(U)$ with $1 \leq p \leq \infty$, and U bounded or unbounded; then there exist a function $\tilde{u} \in C(U)$ such that:

$$u = \tilde{u} \text{ a.e on } U$$

and

$$\tilde{u}(x) - \tilde{u}(y) = \int_x^y u'(t)dt \text{ for all } x, y \in U.$$

Proof

Fix $y_0 \in U$ and set $\tilde{u}(x) = \int_{y_0}^{x} u'(t)dt$. By lemma 16 we have:

$$\int_U \tilde{u}\phi' = -\int_U u'\phi \text{ for all } \phi \in C_c^1(U).$$

On the other hand:

$$\int_U u\phi' = \int_U u'\phi \text{ for all } \phi \in C_c^1(U).$$

Then:

$$\int_U (u - \tilde{u})\phi' = 0 \text{ for all } \phi \in C_c^1(U).$$

then for the lemma 10, we have that $u - \tilde{u} = c$ a.e. On U. In general, the function $\overline{u} = \tilde{u} + C$ has the desired properties.

Example

Let $\{r_k\}_{k=1}^{\infty}$ be a countable set dense subset of $U = B_1(0)$.
Let us consider

$$u(x) = \sum_{k=1}^{\infty} \frac{1}{2^k} |x - r_k|^{-\alpha} \ (x \in B_1(0)).$$

We note that

$$\begin{aligned}
\left\| \frac{x_i - r_{k_i}}{|x - r_k|^{\alpha+2}} \right\|_p^p &\leq \int_{B_1(0)} \left| \frac{x_i - r_{k_i}}{|x - r_k|^{\alpha+2}} \right|^p dx, \\
&\leq \int_{B_1(0)} \frac{1}{|x - r_k|^{(\alpha+1)p}} dx \qquad w = x - r_k, dw = dx \\
&\leq \int_{B_1(r_k)} \frac{1}{|w|^{(\alpha+1)p}} dx, \\
&\leq \int_{B_2(0)} \frac{1}{|w|^{(\alpha+1)p}} dx, \\
&\leq M,
\end{aligned}$$

where $M > 0$ is a finite constant that do not depend on r_k.

We claim that

$$\sum_{k=0}^{\infty} \frac{1}{2^k} \frac{1}{|x - r_k|^{\alpha}}, \quad \sum_{k=0}^{\infty} -\frac{\alpha}{2^k} \frac{(x_i - r_{k_i})}{|x - r_k|^{\alpha+2}} \in L^p(B_1(0)),$$

for all $i = 1, 2, \cdots, n$. We focus on proving that the second term is in L^p as the first one follows by similar considerations.

Given that $\alpha < \frac{n-p}{p}$, when $N_1 < N_2 \in \mathbb{Z}^+$, we obtain

$$\begin{aligned}
\left\| \sum_{k=1}^{N_2} \frac{\alpha}{2^k} \frac{(x_i - r_{k_i})}{|x - r_{k_i}|^{\alpha+2}} - \sum_{k=0}^{N_1} \frac{\alpha}{2^k} \frac{(x_i - r_{k_i})}{|x - r_k|^{\alpha+2}} \right\|_p &\leq \left\| \sum_{k=N_1}^{N_2} \frac{\alpha}{2^k} \frac{(x_i - r_{k_i})}{|x - r_{k_i}|^{\alpha+2}} \right\|_p, \\
&\leq |\alpha| \sum_{k=N_1}^{N_2} \frac{1}{2^k} \left\| \frac{(x_i - r_{k_i})}{|x - r_k|^{\alpha+2}} \right\|_p, \\
&\leq |\alpha| M \sum_{k=N_1}^{N_2} \frac{1}{2^k}.
\end{aligned}$$

Hence, take it $N_1, N_2 \to \infty$, we deduce

$$\left\| \sum_{k=1}^{N_2} \frac{\alpha}{2^k} \frac{(x_i - r_{k_i})}{|x - r_{k_i}|^{\alpha+2}} - \sum_{k=0}^{N_1} \frac{\alpha}{2^k} \frac{(x_i - r_{k_i})}{|x - r_k|^{\alpha+2}} \right\|_p \to 0.$$

Consequenly, the partial sum is a Cauchy secuence in $L^p(B_1(0))$, thus by completeness of the space $L^p(B_1(0))$ we obtain that

$$v(x) = \sum_{k=0}^{\infty} -\frac{\alpha}{2^k} \frac{(x_i - r_{k_i})}{|x - r_k|^{\alpha+2}} \in L^p(B_1(0)).$$

Note that:

$$
\begin{aligned}
\int_U u(x)\phi_{x_i}(x)dx &= \int_U \sum_{k=1}^{\infty} \frac{1}{2^k}|x-r_k|^{-\alpha}\phi_{x_i}(x)dx,\\
&= \lim_{N\to\infty}\int_{B_1(0)} \sum_{k=1}^{N} \frac{1}{2^k}\frac{1}{|x-r_k|^{\alpha}}\phi_{x_i}(x)dx\\
&= \lim_{N\to\infty}\int_{B_1(0)} \sum_{k=1}^{N} -\frac{\alpha}{2}\frac{(x_i-r_{k_i})}{|x-r_k|^{\alpha+2}}\phi(x)dx,\\
&= \int_{B_1(0)} \sum_{k=1}^{\infty} -\frac{\alpha}{2}\frac{(x_i-r_{k_i})}{|x-r_k|^{\alpha+2}}\phi(x)dx,\\
&= \int_{B_1(0)} v(x)\phi(x)dx.
\end{aligned}
$$

We conclude that $u_{x_i} = v$ in the weak sense, i.e., $u \in W^{1,p}(B_1(0))$.

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Next we verify certain properties of weak derivatives. Note very carefully that whereas these various rules are obviously true for smooth functions, functions in Sobolev space are not necessarily smooth: we must always rely solely upon the definition of weak derivatives.

Theorem (Properties of weak derivaties)

Assume $u, v \in W^{k,p}(U)$, and $|\alpha| \leq k$. Then:

1. $D^\alpha u \in W^{k-|\alpha|,p}(U)$ and $D^\beta(D^\alpha u) = D^\alpha(D^\beta u) = D^{\alpha+\beta} u$ for all multiindex α, β with $|\alpha| + |\beta| \leq k$.
2. For each $\lambda, \mu \in \mathbb{R}, \lambda u + \mu v \in W^{k,p}(U)$ and
$$D^\alpha(\lambda u + \mu v) = \lambda D^\alpha u + \mu D^\alpha v, |\alpha| \leq k.$$
3. If V is an open subset of U, then $u \in W^{k,p}(V)$.
4. If $\zeta \in C_c^\infty(U)$, then $\zeta u \in W^{k,p}(U)$ and:
$$D^\alpha(\zeta u) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} D^\beta \zeta D^{\alpha-\beta} u \qquad \text{Leibniz's formula,}$$
where $\binom{\alpha}{\beta} = \frac{\alpha!}{\beta!(\alpha-\beta)!}$.

Proof

1. firts fix $\phi \in C_c^\infty(U)$. Then $\partial^\beta \phi \in C_c^\infty(U)$, and so:

$$\begin{aligned}\int_U \partial^\alpha u \partial^\beta \phi dx &= (-1)^{|\alpha|} \int_U u \partial^{\alpha+\beta} \phi dx, \\ &= (-1)^{|\alpha|}(-1)^{|\alpha+\beta|} \int_U \partial^{\alpha+\beta} u \phi dx, \\ &= (-1)^{|\beta|} \int_U \partial^{\alpha+\beta} u \phi dx.\end{aligned}$$

   Thus $\partial^\beta \left(\partial^\alpha u\right) = \partial^{\alpha+\beta} u$ in a weak sense, analogously $\partial^\alpha \left(\partial^\beta u\right) = \partial^{\alpha+\beta} u$.
2. The property is true because of the linearity of the integral.
3. The property is obviously trivial.

1. We prove by inducction on $|\alpha|$. Suppose firts $|\alpha| = 1$. Choose any $\phi \in C_c^\infty(U)$. Then:

$$\begin{aligned}\int_U \zeta u \partial^\alpha \phi dx &= (-1)^{|\alpha|} \int_U \partial^\alpha(\zeta u)\phi dx \\ &= -\int_U \left(\partial^\alpha(\zeta)u + \zeta\partial^\alpha(\zeta)\right)\phi dx \\ &= -\int_U \left(\sum_{\beta \leq |\alpha|} \binom{\alpha}{\beta} \partial^\beta \zeta \partial^{\alpha-\beta} u\right)\phi dx\end{aligned}$$

Thus $\partial^\alpha(\zeta u) = \zeta\partial^\alpha u + \partial^\alpha \zeta u$ as required.

Assume that $l < k$ and that the proposition holds for all $|\alpha| \leq l$ and all functions ζ.
Let α be a multi-index with $|\alpha| = l + 1$. Then $\alpha = \beta + \gamma$ for some $|\beta| = l$, $|\gamma| = 1$. For a test function ϕ, we have:

$$\begin{aligned}\int_U \zeta u \partial^\alpha \phi dx &= \int_U \zeta u \partial^\beta (\partial^\gamma \phi) dx \\ &= (-1)^{|\beta|} \int_U \sum_{\sigma \leq \beta} \binom{\beta}{\sigma} \partial^\sigma \zeta \partial^{\beta - \sigma} u \partial^\gamma \phi dx\end{aligned}$$

(by the induction hypothesis)

$$= (-1)^{|\beta| + |\gamma|} \int_U \sum_{\sigma \leq \beta} \binom{\beta}{\sigma} \partial^\gamma (\partial^\sigma \zeta \, \partial^{\beta - \sigma} u) \phi \, dx$$

(by the induction hypothesis again)

$$= (-1)^{|\alpha|} \int_U \sum_{\sigma \leq \beta} \binom{\beta}{\sigma} \left[\partial^\rho \zeta \, \partial^{\alpha-\rho} u + \partial^\sigma \zeta \, \partial^{\alpha-\sigma} u\right] \phi \, dx$$

(where $\rho = \sigma + \gamma$)

$$= (-1)^{|\alpha|} \int_U \left[\sum_{\sigma \leq \alpha} \binom{\alpha}{\sigma} \partial^\sigma \zeta \, \partial^{\alpha-\sigma} u\right] \phi \, dx,$$

since

$$\binom{\beta}{\sigma - \gamma} + \binom{\beta}{\sigma} = \binom{\alpha}{\sigma}.$$

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Theorem (Sobolev spaces as function spaces)

For each $k = 1, \dots$ and $1 \le p \le \infty$, the Sobolev space $W^{k,p}(U)$ is a Banach space.

Proof

1) First, conditions 1 and 2 of being a norm are met by being a sum of norms, let's look at the case of the triangular inequality.
Given $u, v \in W^{k,p}(U)$ and using Minkowski inequality,

$$
\begin{aligned}
\|u+v\|_{W^{k,p}(U)} &= \left(\sum_{|\alpha|\leq k} \|\partial^\alpha u + \partial^\alpha v\|_{L^p(U)}^p \right)^{1/p} \\
&\leq \left(\sum_{|\alpha|\leq k} \left(\|\partial^\alpha u\|_{L^p(U)} + \|\partial^\alpha v\|_{L^p(U)} \right)^p \right)^{1/p} \\
&\leq \left(\sum_{|\alpha|\leq k} \|\partial^\alpha u\|_{L^p(U)}^p \right)^{1/p} + \left(\sum_{|\alpha|\leq k} \|\partial^\alpha v\|_{L^p(U)}^p \right)^{1/p} \\
&= \|u\|_{W^{k,p}(U)} + \|v\|_{W^{k,p}(U)}.
\end{aligned}
$$

Thus, $\|u\|_{W^{k,p}(U)}$ satisfies the norm properties.

2) Now, we show that $W^{k,p}(U)$ is complete. Assume $\{u_m\}_{m=1}^{\infty}$ is a Cauchy sequence in $W^{k,p}(U)$. Then, for each $|\alpha| \leq k$, the sequence $\{\partial^\alpha u_m\}_{m=1}^{\infty}$ is Cauchy in $L^p(U)$. Since $L^p(U)$ is complete, there exist functions $u_\alpha \in L^p(U)$ such that

$$\partial^\alpha u_m \to u_\alpha \quad \text{in} \quad L^p(U),$$

for each $|\alpha| \leq k$. In particular,

$$u_m \to u_{(0,\dots,0)} =: u \quad \text{in} \quad L^p(U).$$

3) We claim that

$$u \in W^{k,p}(U), \quad \partial^\alpha u = u_\alpha \quad (|\alpha| \leq k).$$

To verify this, fix $\varphi \in C_c^\infty(U)$. Then,

$$\begin{aligned}
\int_U u \partial^\alpha \varphi \, dx &= \lim_{m\to\infty} \int_U u_m \partial^\alpha \varphi \, dx \\
&= \lim_{m\to\infty} (-1)^{|\alpha|} \int_U \partial^\alpha u_m \varphi \, dx \\
&= (-1)^{|\alpha|} \int_U u_\alpha \varphi \, dx.
\end{aligned}$$

Thus, $\partial^\alpha u_m \to \partial^\alpha u$ in $L^p(U)$ for all $|\alpha| \leq k$, proving that $u_m \to u$ in $W^{k,p}(U)$. Hence, $W^{k,p}(U)$ is complete, and thus a Banach space.

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Thanks for your attention.

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