Curvatura normal

$$k_n(p,v) = \langle -dN_p(v), v\rangle = k_1\cos^2(\theta) + k_2\sin^2(\theta)$$

Fórmulas de Weingarten

$$e = \langle -dN_p(X_u), X_u\rangle = -\langle N_u, X_u\rangle = \langle N, X_{uu}\rangle$$
$$f = \langle -dN_p(X_u), X_v\rangle = -\langle N_u, X_v\rangle = \langle N, X_{uv}\rangle$$
$$g = \langle -dN_p(X_v), X_v\rangle = -\langle N_v, X_v\rangle = \langle N, X_{vv}\rangle$$

Si $v = aX_u + bX_v$,

$$II_p(v) = a^2e + 2abf + b^2g$$

Operador de forma

$$A_p(X_u) = \frac{eG - fF}{EG - F^2}X_u + \frac{fE - eF}{EG - F^2}X_v$$
$$A_p(X_v) = \frac{fG - Fg}{EG - F^2}X_u + \frac{gE - fF}{EG - F^2}X_v$$

Curvaturas

$$K = \frac{eg - f^2}{EG - F^2} \qquad H = \frac{eG + gE - 2fF}{2(EG - F^2)}$$

si es de revolución y g es de norma 1

$$K = \frac{-f''}{f} \qquad H = \frac{1}{2}\frac{-g' + f(g'f'' - g''f')}{f}$$

si son doblemente ortogonales

$$k_1 = \frac{e}{E} \qquad k_2 = \frac{g}{G}$$

$$p(x) = x^2 - 2xH + K = (x - k_1)(x - k_2)$$

Línea de curvatura

Si $\alpha(t) = X(u(t), v(t))$,

$$\begin{vmatrix} v'^2 & -u'v' & u'^2 \\ E & F & G \\ e & f & g \end{vmatrix} = 0$$

Curvatura y torsión (si α es p.p.a.)

$$k_\alpha^2 = k_n^2 + k_g^2 \qquad \tau_\alpha = \frac{k_n' k_g - k_n k_g'}{k_n^2 + k_g^2} + \tau_g$$

Triedro de Frenet

$$\vec{t}'(s) = k(s)\vec{n}(s)$$
$$\vec{b}'(s) = \tau(s)\vec{n}(s)$$
$$\vec{n}'(s) = -k(s)\vec{t}(s) - \tau(s)\vec{b}(s)$$

Triedro de Darboux

$$\vec{t}'(s) = k_g(s)J\vec{t}(s) + k_n(s)\vec{N}(s)$$
$$(J\vec{t})'(s) = -k_g(s)\vec{t}(s) + \tau_g(s)\vec{N}(s)$$
$$\vec{N}'(s) = -k_n(s)\vec{t}(s) - \tau_g(s)J\vec{t}(s)$$
$$\tau_g(s) = \langle A_{\alpha(s)}\vec{t}(s), J\vec{t}(s)\rangle$$

Símbolos de Christoffel

$$\Gamma_{11}^1E + \Gamma_{11}^2F = E_u \quad \Gamma_{11}^1F + \Gamma_{11}^2G = F_u - \tfrac{1}{2}E_v$$
$$\Gamma_{12}^1E + \Gamma_{12}^2F = \tfrac{1}{2}E_v \quad \Gamma_{12}^1F + \Gamma_{12}^2G = \tfrac{1}{2}G_u$$
$$\Gamma_{22}^1E + \Gamma_{22}^2F = F_v - \tfrac{1}{2}G_u \quad \Gamma_{22}^1F + \Gamma_{22}^2G = G_v$$

$$\begin{pmatrix} \Gamma_{11}^1 & \Gamma_{12}^1 & \Gamma_{22}^1 \\ \Gamma_{11}^2 & \Gamma_{12}^2 & \Gamma_{22}^2 \end{pmatrix} = \frac{1}{EG - F^2}\begin{pmatrix} G & -F \\ -F & E \end{pmatrix}\begin{pmatrix} E_u \\ E_v \\ F_v - G_u \\ F_u - E_v \\ G_u \\ G_v \end{pmatrix}$$

Si es superficie de revolución

$$\Gamma_{11}^1 = 0\,\Gamma_{11}^2 = -\frac{ff'}{(f')^2 + (g')^2}\,\Gamma_{12}^1 = \frac{ff'}{f^2},$$
$$\Gamma_{12}^2 = 0\,\Gamma_{22}^1 = 0\,\Gamma_{22}^2 = \frac{f'f'' + g'g''}{(f')^2 + (g')^2}$$

Si $F = 0$

$$K = -\frac{1}{2\sqrt{EG}}\left[\left(\frac{E_v}{\sqrt{EG}}\right)_v + \left(\frac{G_u}{\sqrt{EG}}\right)_u\right]$$

Ecuación de Gauss

$$\Gamma_{11}^1\Gamma_{12}^2 + (\Gamma_{11}^2)_v + \Gamma_{11}^2\Gamma_{22}^2 - \Gamma_{12}^1\Gamma_{11}^2 - (\Gamma_{12}^2)_u - \Gamma_{12}^2\Gamma_{12}^2 = EK$$

Ecs. de Mainardi-Codazzi

$$e_v - f_u = e\Gamma_{12}^1 + f(\Gamma_{12}^2 - \Gamma_{11}^1) - g\Gamma_{11}^2$$
$$f_v - g_u = e\Gamma_{22}^1 + f(\Gamma_{22}^2 - \Gamma_{12}^1) - g\Gamma_{12}^2$$

Si es doblemente ortogonal

$$\Gamma_{11}^1 = \frac{1}{2}\frac{E_u}{E},\ \Gamma_{11}^2 = -\frac{1}{2}\frac{E_v}{G},\ \Gamma_{12}^1 = \frac{1}{2}\frac{E_v}{E},$$
$$\Gamma_{12}^2 = \frac{1}{2}\frac{G_u}{G},\ \Gamma_{22}^1 = -\frac{1}{2}\frac{G_u}{E},\ \Gamma_{22}^2 = \frac{1}{2}\frac{G_v}{E}.$$

$$e_v = \frac{E_v}{2}\left(\frac{e}{E} + \frac{g}{G}\right),$$
$$g_u = \frac{G_u}{2}\left(\frac{e}{E} + \frac{g}{G}\right).$$

Campos paralelos

Si $V(t) = a(t)X_u + b(t)X_v$, con V paralelo:

$$a' + au'\Gamma_{11}^1 + (av' + bu')\Gamma_{12}^1 + bv'\Gamma_{12}^1 = 0$$
$$b' + au'\Gamma_{11}^2 + (av' + bu')\Gamma_{12}^2 + bv'\Gamma_{22}^2 = 0$$

Geodésicas

Si $\gamma(t) = X(u(t), v(t))$:

$$u'' + (u')^2\Gamma_{11}^1 + 2u'v'\Gamma_{12}^1 + (v')^2\Gamma_{22}^1 = 0$$
$$v'' + (u')^2\Gamma_{11}^2 + 2u'v'\Gamma_{12}^2 + (v')^2\Gamma_{22}^2 = 0$$

Curvatura geodésica

$$k_\alpha^g(t) = \frac{\langle \alpha''(t), J\alpha'(t)\rangle}{|\alpha'(t)|^3} = \frac{\langle \alpha''(t), N(t)\times\alpha'(t)\rangle}{|\alpha'(t)|^3}$$

Curvatura geodésica

$$k_\alpha^g(t) = \frac{\langle \alpha''(t), J\alpha'(t)\rangle}{|\alpha'(t)|^3} = \frac{\langle \alpha''(t), N(t)\times\alpha'(t)\rangle}{|\alpha'(t)|^3}$$

Curvatura normal

$$k_n(v,p) = II_p(h,k) = \begin{pmatrix} h & k \end{pmatrix}\begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix}\begin{pmatrix} h \\ k \end{pmatrix}$$

Orientación y superficies

- **Cambio de coordenadas**

$$\begin{pmatrix} \overline{X}_{\overline{u}} \\ \overline{X}_{\overline{v}} \end{pmatrix} = \begin{pmatrix} \frac{\partial u}{\partial \overline{u}} & \frac{\partial u}{\partial \overline{v}} \\ \frac{\partial v}{\partial \overline{u}} & \frac{\partial v}{\partial \overline{v}} \end{pmatrix}\begin{pmatrix} X_u \\ X_v \end{pmatrix}$$

- **Grafos**

$$N(u,v) = \frac{(f_u, f_v, 1)}{\|(f_u, f_v, 1)\|}$$

- **Superficie de revolución**

$$N(p) = \frac{\nabla f(p)}{\|\nabla f(p)\|}$$

- **Operador de forma vía derivadas de la normal**

$$A_p = \begin{bmatrix} [-N_u]_{\{X_u, X_v\}} & [-N_v]_{\{X_u, X_v\}} \end{bmatrix}$$

Derivada covariante

$$\frac{DV}{dt}(t) := V'(t)^{\perp} = V'(t) - < V'(t), N(t) > N(t).$$

Tipos de puntos

- Elíptico: $K > 0$ (ej: esfera)
- Hiperbólico: $K < 0$ (ej: silla de montar)
- Parabólico: $K = 0$, $H \neq 0$ (ej: cilindro)
- Plano: $K = 0$, $H = 0$ (ej: plano)
- Umbílico: $k_1 = k_2$ (todos los puntos en esfera)

Tipos de curvas sobre superficies

- **Línea de curvatura:** $A_p(v) \parallel v$
- **Curva asintótica:** $k_n = 0$
- **Geodésica:** $\frac{D}{dt}T = 0$
- **Paralela:** derivada covariante cero