Models of the universe from differential geometry.

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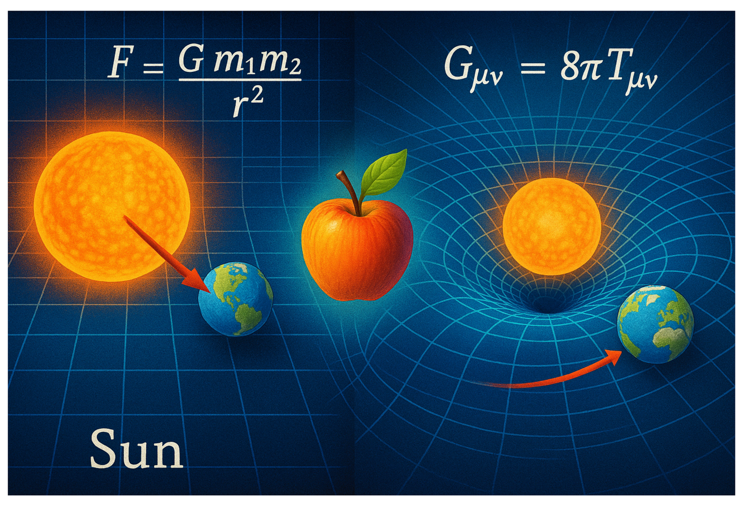
Abstract

The document provides a clear and accessible introduction to the idea that gravity is a manifestation of spacetime curvature, as proposed by Einstein in his theory of general relativity.

1. Geodesics and curvature: It explains that the natural path of objects in curved spacetime is a geodesic, which generalizes the concept of a straight line.
2. Gravity as geometry: Instead of viewing gravity as an instantaneous force (as in Newtonian theory), Einstein proposed that objects follow the curvature of spacetime shaped by the presence of matter.
3. Models of the universe: Three types of universe are presented depending on their curvature:
   (a) Flat (zero curvature).
   (b) Spherical (positive curvature, closed universe).
   (c) Hyperbolic (negative curvature, open universe).
   The sum of the angles in a triangle varies with the type of curvature.
4. Escape velocity and black holes: The concept of escape velocity is introduced as the minimum needed to break free from a gravitational field. If this velocity equals the speed of light, a black hole forms. The Schwarzschild radius is derived from this idea, showing its dependence on the object's mass.

Introduction

Since ancient times, humans have questioned the nature of the universe. For centuries, Newton's gravity explained the motion of celestial bodies as an attractive force. But Einstein redefined gravity as the curvature of spacetime caused by matter and energy.
This insight explained phenomena like black holes and cosmic expansion, showing that objects follow curved paths —geodesics— through a deformed spacetime. Understanding the universe requires studying its curvature and concepts like escape velocity.



Models of the Universe from the Perspective of Differential Geometry

Since general relativity, the geometric structure of the universe has become essential to understanding its behavior. Spacetime is modeled as a four-dimensional Lorentzian manifold, where curvature—determined by mass and energy—governs the motion of matter and light.

Curvature and Global Structure

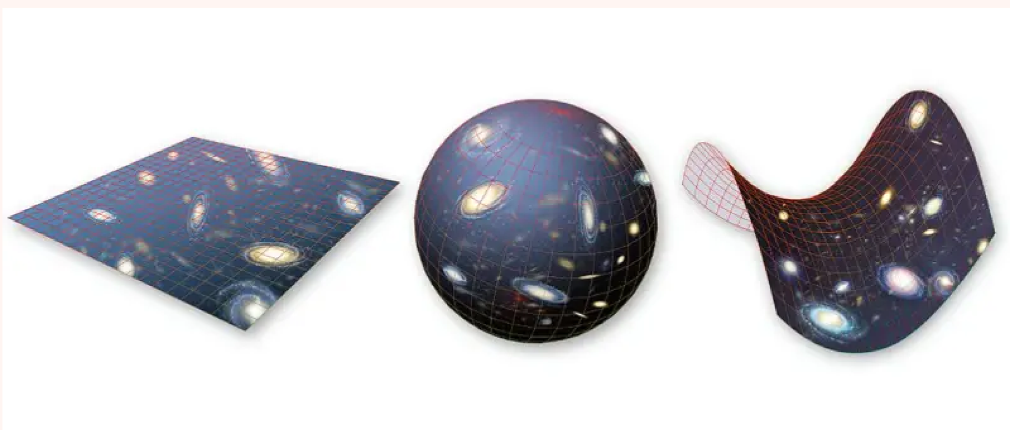
Curvature is a tool that characterizes the local geometry of a manifold. There are several notions of curvature (sectional, scalar, Ricci), but in cosmology, the case of manifolds with *constant curvature* is especially relevant. Each of these geometries gives rise to a different model of the universe:

- **Closed universe:** positive curvature (e.g., 3-sphere), triangle angle sum $> 180°$.
- **Flat universe:** zero curvature ($\mathbb{R}^3$), triangle angle sum = $180°$.
- **Open universe:** negative curvature (hyperboloid), triangle angle sum $< 180°$.

This classification is formalized in the FLRW metric:

$$ds^2 = -c^2dt^2 + a(t)^2\left(\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right),$$

where $k \in \{-1, 0, 1\}$ represents spatial curvature.



Light, Curvature, and Black Holes

In relativity, light follows null geodesics, satisfying

$$g(\dot\gamma, \dot\gamma) = 0,$$

meaning it moves along the "straightest path" in curved spacetime. Massive objects can bend its path (gravitational lensing). If an object is dense enough that the escape velocity equals the speed of light, a black hole forms, with Schwarzschild radius

$$r_s = \frac{2GM}{c^2},$$

which defines the event horizon.

Black Holes from the Perspective of Differential Geometry

Pseudo-Riemannian Manifolds

A **pseudo-Riemannian manifold** (M, g) is a differentiable manifold where the metric tensor g is symmetric, non-degenerate, and not positive-definite. Unlike a Riemannian metric (which is positive-definite), the pseudo-Riemannian metric allows intervals that can be positive, negative, or zero, depending on the direction of the vector.

Spacetime as a Lorentzian Manifold

In general relativity, spacetime is modeled as a **Lorentzian manifold** of dimension 4, with metric signature $(-+++)$. Given this structure, the squared interval between two events is:

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu$$

Where:

- $ds^2 < 0$: timelike trajectory.
- $ds^2 = 0$: lightlike (null) trajectory.
- $ds^2 > 0$: spacelike trajectory.

Curvature of Spacetime

The curvature is described by the **Riemann tensor**:

$$R^\rho{}_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma}$$

This tensor measures how a vector changes when parallel transported around a closed loop on the manifold.

Two important contractions of the Riemann tensor are:

- **Ricci tensor**:
$$R_{\mu\nu} = R^\lambda{}_{\mu\lambda\nu}$$
- **Scalar curvature**:
$$R = g^{\mu\nu}R_{\mu\nu}$$

Einstein Field Equations

The Einstein field equations relate the geometry of spacetime to its matter and energy content:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}$$

- The left-hand side describes the curvature of spacetime.
- The right-hand side encodes matter and energy via the stress-energy tensor $T_{\mu\nu}$.

Schwarzschild Solution

In the case of spherical symmetry in vacuum ($T_{\mu\nu} = 0$), the solution to the Einstein equations is the **Schwarzschild metric**:

$$ds^2 = -\left(1 - \frac{2GM}{c^2 r}\right)c^2dt^2 + \left(1 - \frac{2GM}{c^2 r}\right)^{-1}dr^2 + r^2 d\Omega^2$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta\, d\phi^2$ represents the area element of a 2-sphere.

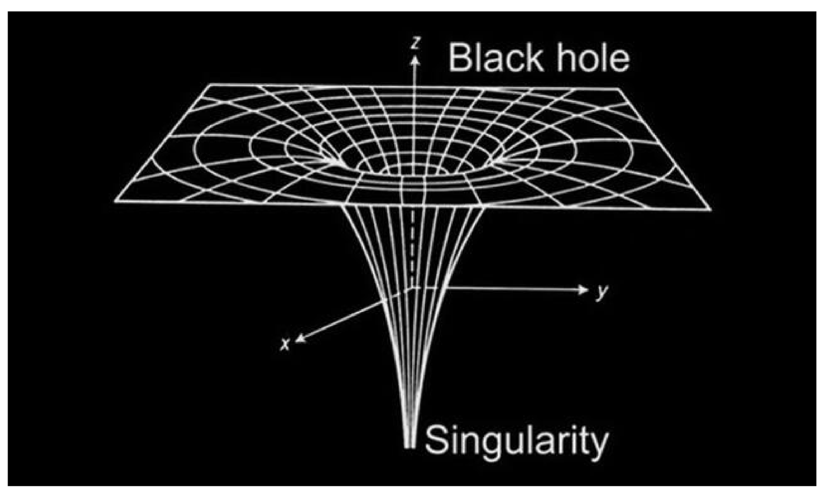
The quantity:

$$r_s = \frac{2GM}{c^2}$$

is called the **Schwarzschild radius**. If a body collapses within this radius, a **black hole** forms.

Event Horizon and Singularity

- The **event horizon** is located at $r = r_s$. Beyond this point, no signal or particle can escape to infinity.
- At $r = 0$, there is a **singularity** where geometric quantities such as curvature become infinite.



Geometric Interpretation

From the viewpoint of differential geometry, a **black hole** is a region of spacetime where curvature is so strong that it alters the causal structure. Timelike geodesics inside the horizon are all directed toward the singularity. This means that, once the horizon is crossed, the "future" of any particle inevitably leads to the center, which is not a choice but a geometric consequence.

Thus, black holes are not "physical objects" in the classical sense, but regions defined by the **geometric structure** of spacetime, as determined by Einstein's equations.

Referencias

[1] Robert M. Wald. General relativity. Chicago-London: The University of Chicago Press. XIII, 491 p. $ 34.50 (1984)., 1984.

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