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Problem Marcus Chapter 2 Problem 11c. Show that α (as in (b)) must be a root of 1. (Show that its powers are restricted to a finite set.)

My solution to the problem is based on taking the hint quite literally and some results in Chapter 2.

Consider the set $A = \{1, \alpha, \alpha^2, \ldots\}$

Now in Pg.15 of Marcus, Chapter 2, using fact (3) of Theorem 2, we see that all powers of α are indeed algerabic integers, since $\alpha^k \in \mathbb{Z}[\alpha]\ \ \forall k \in \mathbb{N}$ and $\mathbb{Z}[\alpha]$ is finitely generated by $1, \alpha, \ldots, \alpha^{n-1}$ where n is the degree of the monice polynomial in $\mathbb{Z}[x]$ satisfied by α.

Also, using the determinant procedure outlined in the next page, we write

$$\begin{bmatrix} \alpha^k 1 \\ \alpha^k \alpha \\ \vdots \\ \alpha^k \alpha^{n-1} \end{bmatrix} = M_{\mathrm{k}} \begin{bmatrix} 1 \\ \alpha \\ \vdots \\ \alpha^{n-1} \end{bmatrix}$$

where M_{k} is an $n \times n$ matrix in $\mathbb{Z}$ depending on α^k, and see that $\det(\alpha^k I - M)$ is a monic polynomial in $\mathbb{Z}[x]$ satisfied by α^k. But what is important to note here is that the polynomial obtained is of degree n, and is always of degree n, since M_{k} is an $n \times n$ matrix $\forall n$.

What we have proved here is that each power of α is an algebraic integer of fixed degree n, and by part(b) of problem 11, A must be finite.

The set A being finite means that for some $l, m \in \mathbb{Z}$, we must have $\alpha^l = \alpha^m$ which implies $\alpha^{|l-m|} = 1$ and hence α is a root of 1.