# پردازش سیگنال گرافی

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دانشگاه صنعتی شریف

مهندسی برق


برنا خدابنده ۴۰۰۱۰۹۸۹۸


تمرین شماره 3

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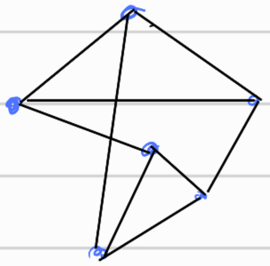


$r=3$ $\quad$ $d=2$ $\quad H(L)=\sum_{i=0}^{M-1}\alpha_i L^i \Rightarrow h(\lambda)=\sum_{i=0}^{M-1}\alpha_i \lambda^i$ $\quad$ (سوال ۲)

$h(\lambda)$: nonzero for $k$ values of $\lambda$: $\exists \lambda_1,\dots,\lambda_k : h(\lambda_i)\neq 0$

$A=W,\ d_i=r \Rightarrow L=rI-A \Rightarrow \lambda_i(L)=r-\lambda_i(A),\ \Phi_i(L)=\Phi_i(A)$

minimal polynomial: $Q(A)=\sum_{i=0}^{s}\beta_i A^i=0,\ \forall \lambda\in eig(A): Q(\lambda_A)=0$

$Q(A)=\sum_{i=0}^{s}\beta_i A^i=\sum_{i=0}^{s}\beta_i (rI-L)^i=\sum_{i=0}^{s} q_i L^i=0=Q'(L) : Q'(\lambda_L)=0$

$deg(Q'(\lambda))=deg(Q(\lambda))=s\geq d+1 : \exists (i,j): \forall k<d: A^k_{(i,j)}=0$
$A^d(i,j)=1$

$\Rightarrow deg(Q)\geq d+1,\ \exists \lambda_1,\dots,\lambda_n : h(\lambda_i)\neq 0,\ \lambda_{k+1},\dots,\lambda_n : h(\lambda_i)=0$

only distinct terms

let: $f(\lambda)=h(\lambda)\times(\lambda-\lambda_1)\times(\lambda-\lambda_2)\times\dots\times(\lambda-\lambda_k) : h(\lambda_i)\neq 0$

$\Rightarrow \forall \lambda\in eig(L): f(\lambda)=0,\ deg(f(\lambda))=k+deg(h(\lambda))$

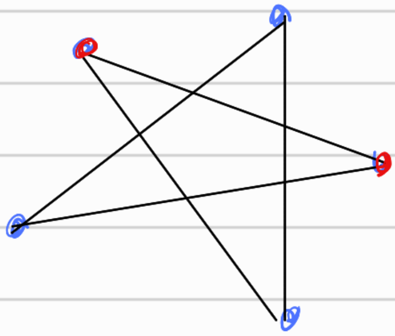
$\Rightarrow deg(f(\lambda))=M-1+k$

$Q(\lambda)$: minimal polynomial $\Rightarrow deg(f)\geq deg(Q)\geq d+1$

$\Rightarrow M-1+k\geq d+1 \Rightarrow \boxed{M\geq d-k+2}$

$k=2,\ r=0,\ s=1$

(سوال



$\forall (i,j)\in E:\ A_{ij}=1,\ A^2_{ij}=$ length 2 between neighbor $=r$

$\forall (i,j)\notin E\ (i\neq j):\ A_{ij}=0,\ A^2_{ij}=$ 〃 〃 〃 non-neighbor $=s$

$\forall\ i=j:\ A_{ii}=0,\ A^2_{ii}=k$ (الف

$A^2-rA+sA:\ \forall i\neq j:\ (A^2+(s-r)A)_{(i,j)}=s,\ (i=j):k$

$\Rightarrow$ let: $P(A)=A^2+(s-r)A+(s-k)I \Rightarrow P(A)_{(i,j)}=s$

$\Rightarrow P(A)=s\mathbb{1}\mathbb{1}^T,\ \text{rank}(P(A))=1 \Rightarrow 1$ non zero eigenvalue

$P(A)\mathbb{1}=s\mathbb{1}\mathbb{1}^T\mathbb{1}=ns\mathbb{1} \Rightarrow \mathbb{1}\in eig(P(A))$

$A\mathbb{1}=k\mathbb{1} \Rightarrow \mathbb{1}\in eig(A),\ \lambda_1=k,\ \mathbb{1}=v_1$

$\forall v\in R^n\perp\mathbb{1}:\ P(A)v=0 \Rightarrow \forall v\in eig(A)/dc:\ P(A)v=0$

$\Rightarrow [A^2+(s-r)A+(s-k)I]v=0 \Rightarrow \lambda^2+(s-r)\lambda+(s-k)=0$

$\Rightarrow \lambda_A=\dfrac{r-s\pm\sqrt{(s-r)^2+4(k-s)}}{2},\quad L=D-A=kI-A$

$\Rightarrow \lambda_L=k-\lambda_A,\quad \lambda_A\in\left\{k,\dfrac{r-s+\sqrt{(r-s)^2+4(k-s)}}{2},\dfrac{r-s-\sqrt{(r-s)^2+4(k-s)}}{2}\right\}$

$\Rightarrow \lambda_L=k-\lambda_A \Rightarrow \boxed{\lambda_L\in\left\{0,\dfrac{2k-r+s+\sqrt{(r-s)^2+4(k-s)}}{2},\dfrac{2k-r+s-\sqrt{(r-s)^2+4(k-s)}}{2}\right\}}$

(ب

$\sum(x)_{(i)}=0 \Rightarrow x\in R^n\perp\mathbb{1} \Rightarrow x=\sum_{v\in eig(L)\perp\mathbb{1}}\gamma_i v_i=\gamma_2 v_2+\gamma_3 v_3$

$f(L)=\sum_{i=0}^{m-1}\alpha_i L^i,\ Lv_{2,3}=\theta_{2,3}v_{2,3}$

$$y = f(L)x = \sum_{i=0}^{M-1} \alpha_i L^i x = \sum_{i=0}^{M-1} \alpha_i L^i (r_2 v_2 + r_3 v_3)$$

$$= r_2 \sum \alpha_i L^i v_2 + r_3 \sum \alpha_i L^i v_3 = r_2 v_2 \sum \alpha_i \theta_2^i + r_3 v_3 \sum \alpha_i \theta_3^i$$

$$\Rightarrow y = r_2 f(\theta_2) v_2 + r_3 f(\theta_3) v_3$$

$$y^T x = 0 \Rightarrow r_2^2 f(\theta_2) + r_3^2 f(\theta_3) = 0 \quad , \quad \|x\|_2 = 1 \Rightarrow r_2^2 + r_3^3 = 1$$

$$\Rightarrow r_2^2 (f(\theta_2) - f(\theta_3)) + f(\theta_3) = 0 \Rightarrow r_2^2 = \frac{f(\theta_3)}{f(\theta_3) - f(\theta_2)} , \; r_3^2 = \frac{f(\theta_2)}{f(\theta_2) - f(\theta_3)}$$

$$\|y\|_2^2 = r_2^2 f(\theta_2)^2 + r_3^2 f(\theta_3)^2$$

$$\|y\|_2^2 = \frac{1}{f(\theta_2) - f(\theta_3)} \cdot \left[ f(\theta_3) f(\theta_2)^2 - f(\theta_2) f(\theta_3)^2 \right]$$

$$\|y\|_2^2 = f(\theta_2) f(\theta_3) \Rightarrow \boxed{\|y\|_2 = \sqrt{f(\theta_2) f(\theta_3)}}$$

(سوال ۳

$$y_i = \sqrt{n}\, y_{i-1} \circledast_c \delta_i \Rightarrow \hat{y}_i = \sqrt{n}\, \hat{y}_{i-1} \odot \hat{\delta}_i$$

$$\hat{y}_i(k) = \sqrt{n}\, \hat{y}_{i-1}(k)\, \hat{\delta}_i(k), \quad \hat{\delta}_i = U^H \delta_i = U_i^H$$

$U_i^H$ : ستون iام $U^H$

$$\hat{y}_i(k) = \sqrt{n}\, \hat{y}_{i-1}(k)\, U^H_{ki} = \sqrt{n}\, U^H_{ki}\, \hat{y}_{i-1}(k) = \sqrt{n}\, \bar{U}_{ik}\, \hat{y}_{i-1}(k) = \hat{y}_i(k)$$

$$\langle y_i \rangle = \frac{1}{n} \langle y_i, \mathbb{1} \rangle = \frac{1}{n} \langle \hat{y}_i, \hat{\mathbb{1}} \rangle = \langle \hat{y}_i, \frac{1}{n} \hat{\mathbb{1}} \rangle$$

we assume $\mathbb{1} \in$ eig(shift), like if we were using $L$ as shift

$\Rightarrow \exists\, i_0 : U_{1 i_0} = U_{2 i_0} = \dots = U_{n i_0} = \frac{1}{\sqrt{n}} \quad (U^H U = I)$ $\rightsquigarrow$ $i_0$ : index of column of $\mathbb{1}$ in $U$

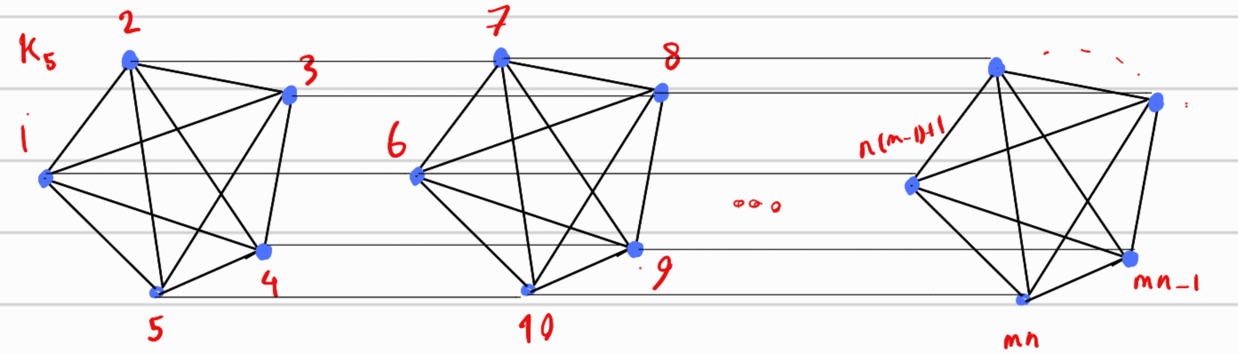
$$\Rightarrow \hat{\mathbb{1}} = U^H \mathbb{1} = \delta_{i_0} \times \sum \frac{1}{\sqrt{n}} = \sqrt{n}\, \delta_{i_0} \Rightarrow \frac{1}{n} \hat{\mathbb{1}} = \frac{1}{\sqrt{n}} \delta_{i_0}$$

$$\Rightarrow \langle y_i \rangle = \langle \hat{y}_i, \frac{1}{n} \hat{\mathbb{1}} \rangle = \langle \hat{y}_i, \frac{1}{\sqrt{n}} \delta_{i_0} \rangle = \frac{1}{\sqrt{n}} \hat{y}_i(i_0) = m_i$$

$$m_i = \frac{1}{\sqrt{n}} \hat{y}_i(i_0) = \frac{1}{\cancel{\sqrt{n}}} \cdot \cancel{\sqrt{n}}\, \overset{\frac{1}{\sqrt{n}}}{\cancel{\bar{U}_{i i_0}}}\, \hat{y}_{i-1}(i_0) = \frac{1}{\sqrt{n}} \hat{y}_{i-1}(i_0) = m_{i-1} = m_i$$

$$\Rightarrow m_n = m_0 \Rightarrow \boxed{\langle y_n \rangle = \langle y_0 \rangle}$$

(سوال ۴




we can easily see: $H = K_n \times K_m$, $\times$: cartesian product.

$$\mathcal{L}_H = D_H - W_H = I_n \otimes D_{K_m} + D_{K_n} \otimes I_m - W_H = I_n \otimes L_{K_m} + L_{K_n} \otimes I_m$$

$\Rightarrow$ let: $\beta_{ij} = \psi_i \otimes \Phi_j$, $L_{K_m}\Phi_j = \gamma_j \Phi_j$, $L_{K_n}\psi_i = \lambda_i \psi_i$

$$L_H \beta_{ij} = (L_{K_n} \otimes I_m + I_n \otimes L_{K_m})\, \psi_i \otimes \Phi_j = \overset{\lambda_i \psi_i}{\cancel{(L_{K_n}\psi_i)}} \otimes \Phi_j + \psi_i \otimes \overset{\gamma_i \Phi_j}{\cancel{(L_{K_m}\Phi_j)}}$$

$= (\lambda_i + \gamma_j)\, \psi_i \otimes \Phi_j = \boxed{(\lambda_i + \gamma_j)\beta_{ij} = L_Q \beta_{ij}}$, there are $n \times m$ distinct $\beta_{ij}$ values $\Rightarrow$ all eigenvectors

$\lambda_i$ : eig$(K_n) = \{0, n, n, \dots, n\} \Rightarrow \lambda_i \in \{0, n\}$

$\gamma_i$ : eig$(K_m) = \{0, m, m, \dots, m\} \Rightarrow \gamma_i \in \{0, m\}$

eigenvalues: $\boxed{\Lambda_H = \{\overset{dc}{0}, \underbrace{n, n, \dots, n}_{\# n-1}, \underbrace{m, m, \dots, m}_{\# m-1}, \underbrace{mn, mn, \dots, mn}_{\# (m-1)(n-1)}\}}$

eigenvectors: $\beta_{ij} = \psi_i \otimes \Phi_j$, $\alpha \to \alpha_1, \alpha_2$ : $\alpha_2 = \alpha \bmod n$
$\forall \alpha \in \mathbb{R}$ $\qquad \alpha_1 = \text{floor}(\alpha / n)$

$\psi_i$ : we need $\psi_0$ to be constant and all other to have zero mean.

arbitarely we choose the columns of the DFT matrix

$\Rightarrow U_{K_n} = DFT_n$, $U_{K_m} = DFT_m$ : $DFT_K(l, m) = \frac{1}{\sqrt{K}} e^{\frac{2\pi i}{K} lm}$

we use $D^n \equiv DFT_n$

assuming nodes start from 0 instead

$$\beta_{ij} = \psi_i \otimes \Phi_j \Rightarrow U_H = U_{K_n} \otimes U_{K_m} = D^n \otimes D^m = U_H$$

$$y = \delta_K \otimes \delta_\ell = U(\delta_K \hat{\otimes} \delta_\ell) = U(\hat{\delta}_K \odot \hat{\delta}_\ell) = U((U^H\delta_n) \odot (U^H\delta_\ell))$$

$$\Rightarrow y = U(u_\ell \odot u_K) \quad , \quad u_K = \psi_{K_1} \otimes \Phi_{K_2} = D^n_{K_1} \otimes D^m_{K_2}$$

$$u_K(a) = D^n(a_1, k_1)\, D^m(a_2, k_2)$$

$$= \frac{1}{\sqrt{mn}} \exp\Big(\underbrace{\frac{2\pi i}{n} a_1 k_1 \quad \frac{2\pi i}{m} a_2 k_2}_{\theta_{aK}}\Big)$$

$$(u_\ell \odot u_K)(a) = u_\ell(a)\, u_K(a) = \frac{1}{mn} \exp\Big(\underbrace{\theta_{aK} + \theta_{a\ell}}_{K_1+\ell_1,\; K_2+\ell_2}\Big)$$

$$\theta_{xK} : \text{linear with } \begin{matrix} x_1 - 1 \\ x_2 - 1 \end{matrix} \Rightarrow \theta_{aK} + \theta_{a\ell} = \frac{2\pi i}{n}(a_1 - 1)(K_1 + \ell_1 - 2) + \frac{2\pi i}{m}(a_2 - 1)(K_2 + \ell_2 - 2)$$

$$\Rightarrow \left.\begin{matrix} k_1 + \ell_1 \\ K_2 + \ell_2 \end{matrix}\right| \Rightarrow \text{we want to find } t \text{, such that } \theta_{aK} + \theta_{a\ell} = \theta_{at} + 2k\pi$$

$$\Rightarrow t_1 = (K_1 + \ell_1) \bmod n, \quad t_2 = (K_2 + \ell_2) \bmod n$$

$$t = nt_1 + t_2 \Rightarrow t = n(K_1 + \ell_1) \bmod n + \underbrace{(K_2 + \ell_2) \bmod n}_{(K \bmod n + \ell \bmod n) \bmod n}$$

$$t_1 \to t_1 + n \equiv t \to t + mn, \; t \in [1, mn] \Rightarrow \boxed{t = t \bmod mn} \quad = (\ell + K) \bmod n$$

$$\Rightarrow (u_\ell \odot u_K)(a) = \frac{1}{mn} \exp(\theta_{at}) = \frac{1}{\sqrt{mn}} u_t(a) \; : \; \theta_{aK} + \theta_{a\ell} = \theta_{at}$$

$$\Rightarrow u_\ell \odot u_K = \frac{1}{\sqrt{mn}} u_t$$

$$y = U(u_\ell \odot u_K) = \frac{1}{\sqrt{mn}} U u_t = \boxed{\frac{1}{\sqrt{mn}} \delta_t = \delta_\ell \circledast_H \delta_K}$$

in the end we fix for modes starting from 1, checked with matlab

$$\Rightarrow \boxed{t = 1 + \Big\{\Big(n \times \Big[\Big(\Big\lfloor \frac{K-1}{n} \Big\rfloor + \Big\lfloor \frac{\ell - 1}{n} \Big\rfloor\Big) \bmod n\Big] + (\ell + K - 2) \bmod n\Big) \bmod mn\Big\}}$$

(5)سوال

$\omega(x)$: largest eigenvalue of $x$ : $\hat{x}(\omega(x)) \neq 0$

$PW_\omega(G) = \text{span}(u_i : \lambda_i \leq \omega) \Rightarrow$ all signals with bandwidth $\omega$

$V$: edges of $G$, $S \subset V$ : a "good sampling" if for two signals

$f, g$ : $f(S) = g(S), S \subset V \Rightarrow f(V) = g(V)$ : (I)

$\omega_c(S)$: largest $\omega$ such that (I) holds for $f, g \in PW_\omega(G)$

$L_2(S)$ : if $f \in L_2(S) \Rightarrow \forall i \notin S : f(i) = 0$

(a

$\forall f, g \in PW_\omega(G)$ : $f = \sum_{\lambda_i \leq \omega} \alpha_i u_i = U\hat{f}$ , $g = \sum_{\lambda_i \leq \omega} \beta_i u_i = U\hat{g}$

: $\hat{f}(\lambda \geq \omega) = \hat{g}(\lambda \geq \omega) = 0$ , $f(S) = I_{\downarrow S} f$ , $g(S) = I_{\downarrow S} g$

let : $U_\omega = \begin{bmatrix} U_{1,1} & \cdots & U_{1,\omega} \\ U_{2,1} & \cdots & U_{2,\omega} \\ \vdots & & \\ U_{n,1} & \cdots & U_{n,\omega} \end{bmatrix}$ , $\hat{f}_\omega = \begin{bmatrix} \hat{f}_1 \\ \hat{f}_2 \\ \vdots \\ \hat{f}_\omega \end{bmatrix} \Rightarrow f = \sum \hat{f}_i u_i = \sum \hat{f}_{\omega(i)} U_{\omega(i)}$

$\Rightarrow f = U_\omega \hat{f}_\omega$ , $g = U_\omega \hat{g}_\omega$

$f(S) = I_{\downarrow S} U_\omega \hat{f}_\omega$ , $g(S) = I_{\downarrow S} U_\omega \hat{g}_\omega \Rightarrow$ if this leads to $\hat{f}_\omega = \hat{g}_\omega \Rightarrow S$ is sufficient

$f(S) = g(S) \Rightarrow I_{S\downarrow} U_\omega \hat{f}_\omega = I_{\downarrow S} U_\omega \hat{g}_\omega \Rightarrow I_{\downarrow S} U_\omega (\hat{f}_\omega - \hat{g}_\omega) = 0$

$\Rightarrow \hat{f}_\omega - \hat{g}_\omega \in N(I_{S\downarrow} U_\omega) \Rightarrow$ if $N(I_{S\downarrow} U_\omega) = \phi \Rightarrow \hat{f}_\omega = \hat{g}_\omega \Rightarrow f(V) = g(V)$

$$I_{S\downarrow} U_m = \begin{array}{c c} & i_k \\ k & \begin{bmatrix} & \vdots & \\ \cdots & 1 & \cdots \\ & \vdots & \end{bmatrix} \end{array} \begin{bmatrix} U_{1,1} & \cdots & U_{1,\omega} \\ \vdots & \ddots & \vdots \\ U_{n,1} & \cdots & U_{n,\omega} \end{bmatrix} = \begin{bmatrix} U_{i_1,1} & \cdots & U_{i_1,\omega} \\ U_{i_2,1} & \cdots & U_{i_2,\omega} \\ \vdots & \ddots & \vdots \\ U_{i_k,1} & \cdots & U_{i_k,\omega} \end{bmatrix} = U_{S,\omega}$$

we need: $\text{rank}(U_{S,\omega}) = |S|$ ; if so $\Rightarrow N(U_{S,\omega}) = \phi$

$f(S) = U_{S,\omega} \hat{f}_\omega$ , $N(U_{S,\omega}) = \phi \Rightarrow \nexists x \in \mathbb{R}^\omega : U_{S,\omega} x = 0$

$\Rightarrow f(S) \underset{\hat{f}_\omega \neq 0}{\neq} 0 \Rightarrow f \notin L_2(S^c)$ * same for $g$*, except for $f=0$ on $g=0$

$\Rightarrow \forall f, g \in PW_\omega(G) \wedge f(S) = g(S) \Leftrightarrow f(V) = g(V) \Rightarrow f, g \notin L_2(S^c)$

$\Rightarrow$ $\boxed{PW_\omega(G) \cap L_2(S^c) = \{0\}}$

$x \in PW_\omega(G) \Rightarrow x = \sum_{\lambda_i \leq \omega} \alpha_i u_i = U_\omega \hat{f}_\omega : PW_\omega(G) = C(U_\omega)$ (same as before)

$L_2(S^c) \cap PW_\omega(G) = \{0\} \Rightarrow \forall \Phi \in L_2(S^c) : \Phi \notin PW_\omega(G) = C(U_\omega)$

$\Rightarrow \Phi = U_\omega \Phi_\omega + e : e \in (C(U_\omega))^\perp = \mathbb{R}^n - C(U_\omega) = C(U) - C(U_\omega) = \mathrm{span}\{u_i : \lambda_i > \omega\}$

$\Rightarrow \Phi \in C(U_\omega) \cup \mathrm{span}(u_i : \lambda_i > \omega) \Rightarrow \hat{\Phi} : \begin{bmatrix} \\ - \\ 0 \end{bmatrix}^{\omega}$ $\rightarrow$ has atleast one nonzero term

$\Rightarrow \omega(\Phi) > \omega \Rightarrow$ $\boxed{\omega < \inf_{\substack{\Phi \in L_2(S^c) \\ \Phi \neq 0}} \omega(\Phi)}$ $\rightarrow$ in short, $\forall \Phi \in L_2(S) : \Phi \notin PW_\omega(G)$

so it must have higher frequencies than $\omega$

$\inf_{\substack{\Phi_2(S^c) \\ \Phi \neq 0}} \omega(\Phi) \triangleq \omega_c(S) \Rightarrow$ $\boxed{\omega < \inf_{\substack{\Phi \in L_2(S^c) \\ \Phi \neq 0}} \omega(\Phi) \triangleq \omega_c(S)}$

$\inf_{\substack{\Phi \in L_2(S^c) \\ \Phi \neq 0}} \omega(\Phi) = \omega_c(S)$ : since choosing any $\omega$ higher than this

would result in $S$ not being a good choice.

$$D = [\overbrace{I_{N/M}\ 0_{N/M} \cdots 0_{N/M}}^{M \text{ times}}] \in \mathbb{C}^{(N/M)\times N}, \quad Dx: \text{sampled}$$ (6 שאלה

$$B^M = \begin{bmatrix} (A^M)_{1,1} & 0 \\ (A^M)_{2,1} & (A^M)_{2,2} \end{bmatrix}, \quad (A^M)_{1,1} \in \mathbb{C}^{(N/M)\times(N/M)} : \text{first } (N/M) \text{ rows/cols}$$

$$\bar{A} = DB^M D^T$$

(a

$$H(A) = \sum_{k=0}^{L} h_k A^k, \quad D^T D = \begin{bmatrix} I_{N/M} \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} [I_{N/M}\ 0 \cdots 0] = \begin{bmatrix} I_{N/M} & 0 & \cdots & 0 \\ 0 & & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & & 0 \end{bmatrix} = \begin{bmatrix} I_{N/M} & 0 \\ 0 & 0 \end{bmatrix}$$

$$\bar{A} = DB^M D^T = [I_{N/M}\ \ 0] \begin{bmatrix} (A^M)_{1,1} & 0 \\ (A^M)_{2,1} & (A^M)_{2,2} \end{bmatrix} \begin{bmatrix} I_{N/M} \\ 0 \end{bmatrix} = (A^M)_{1,1}$$

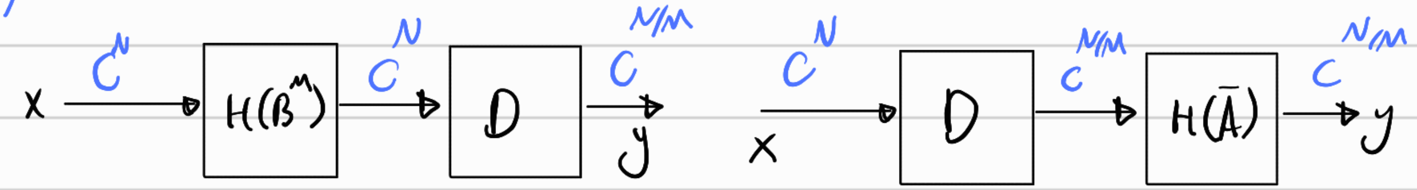
$$DB^M = [I_{N/M}\ \ 0] \begin{bmatrix} (A^M)_{1,1} & 0 \\ (A^M)_{2,1} & (A^M)_{2,2} \end{bmatrix} = [\underbrace{\cancel{(A^M)_{1,1}}}_{\bar{A}}\ \ 0] = \bar{A} D$$

$$\Rightarrow DB^M = \bar{A} D \Rightarrow D(B^M)^k = (DB^M)(B^M)^{k-1} = \bar{A} D (B^M)^{k-1} \underset{\text{same pattern}}{=} (\bar{A})^k D$$

$$\Rightarrow D(B^M)^k = (\bar{A})^k D \Rightarrow D H(B^M) = D\left(\sum_{k=0}^{L} h_k (B^M)^k\right) = \sum_{k=0}^{L} h_k D (B^M)^k$$

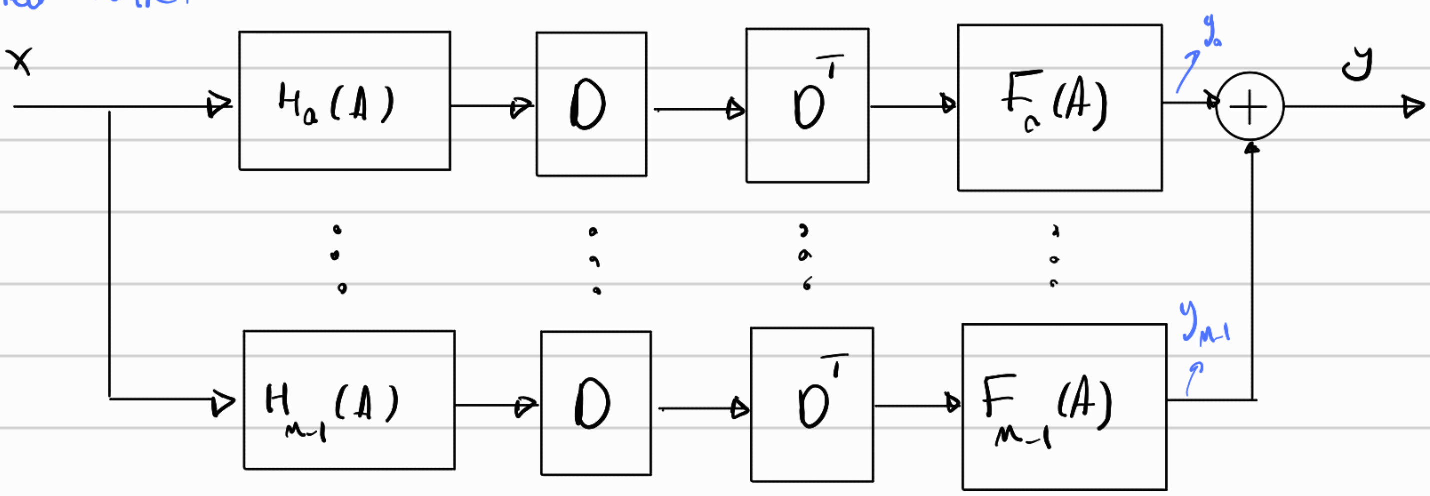
$$= \sum_{k=0}^{L} h_k \bar{A}^k D = \left(\sum_{k=0}^{L} h_k \bar{A}^k\right) D = \boxed{H(\bar{A}) D = D H(B^M)}$$

system sketch:




filter bank:

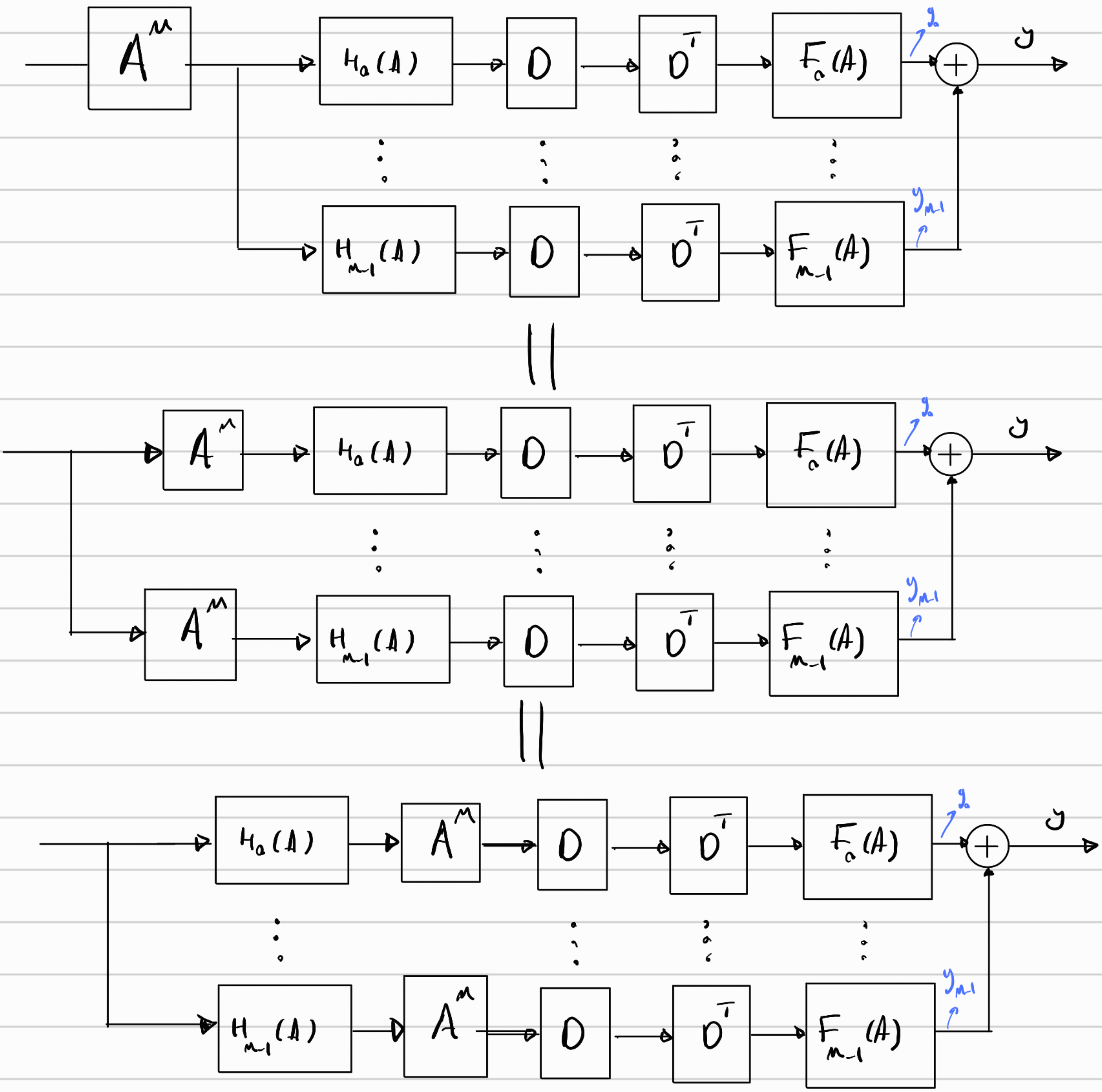
(b



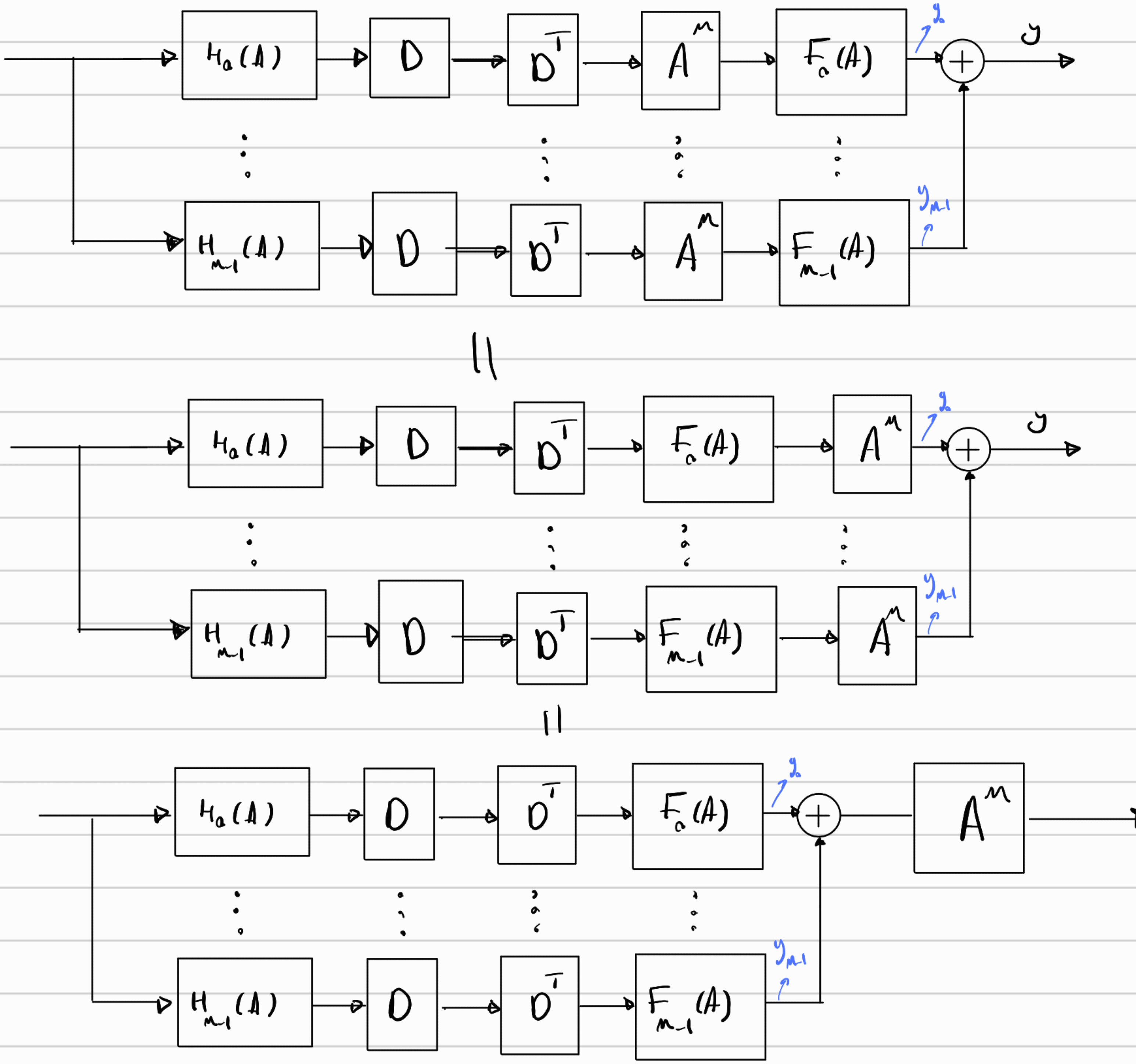
$$D^T D A^M = \begin{bmatrix} I_{N/M} \\ 0 \\ \vdots \\ 0 \end{bmatrix} [I_{N/M}\ 0 \cdots 0] \begin{bmatrix} (A^M)_{1,1} & (A^M)_{1,2} \\ (A^M)_{2,1} & (A^M)_{2,2} \end{bmatrix} = \begin{bmatrix} (A^M)_{1,1} & (A^M)_{1,2} \\ 0 & 0 \end{bmatrix}$$

$$A^M D^T D = \begin{bmatrix} (A^M)_{1,1} & (A^M)_{1,2} \\ (A^M)_{2,1} & (A^M)_{2,2} \end{bmatrix} \begin{bmatrix} I_{N/M} \\ 0 \\ \vdots \\ 0 \end{bmatrix} [I_{N/M}\ 0 \cdots 0] = \begin{bmatrix} (A^M)_{1,1} & 0 \\ (A^M)_{2,1} & 0 \end{bmatrix}$$

$$\Rightarrow A^M D^T D = D^T D A^M \text{ iff } (A^M)_{2,1} = 0,\ (A^M)_{1,2} = 0$$



$A^m D^T D \neq D^T D A^m$

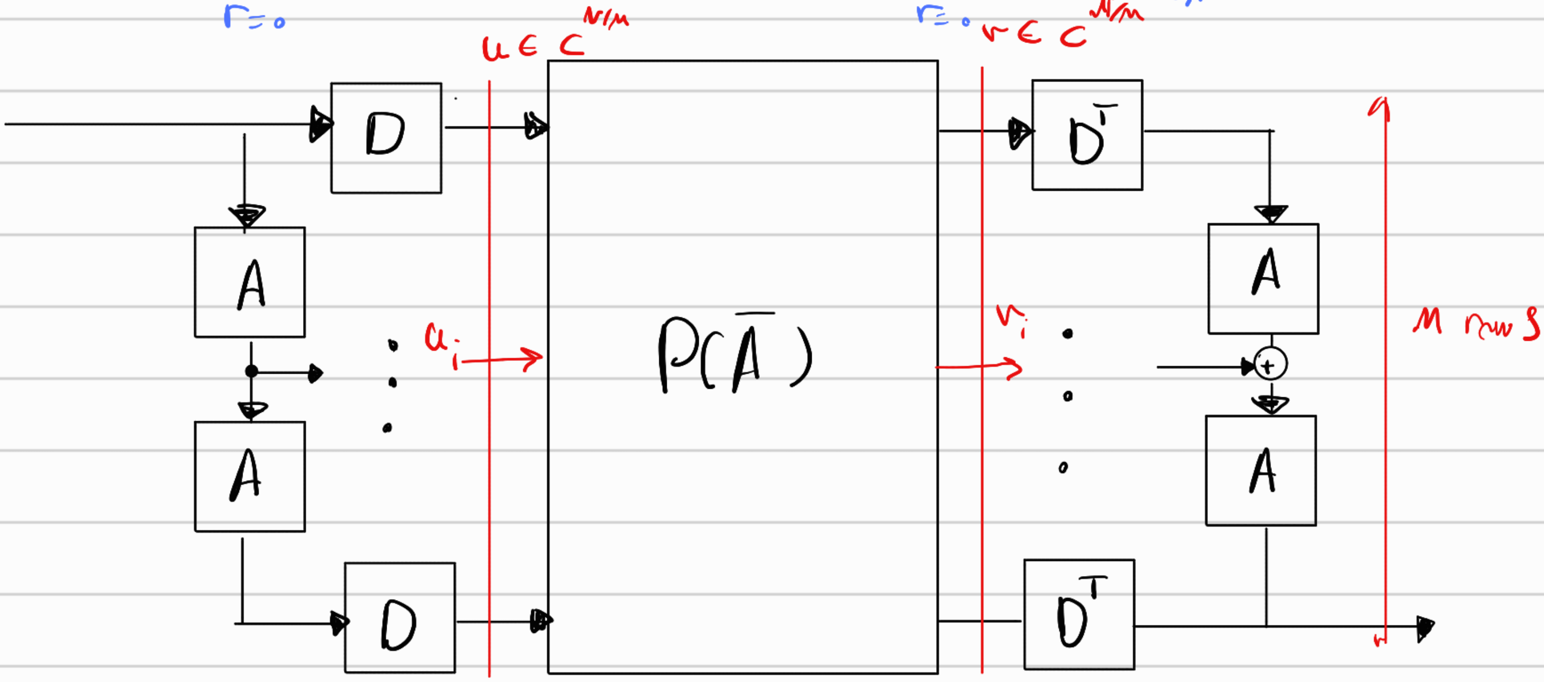



in general, the described filter-bank is not MGSI and depends on $D^T D A^m \overset{?}{=} A^m D^T D$, which would be true iff $(A^m)_{1,2} = 0$, $(A^m)_{2,1} = 0$. here we checked $SA^m = A^m S$ in the general form, for standard GSI we can simply use $m = 1$.

(b

we proved this in the last part. $(A^M)_{1,2}=0,\ (A^M)_{2,1}=0$

$H_i(A)=\sum_{r=0}^{M-1} A^r E_{i,r}(A^M)$, $F_i(A)=\sum_{r=0}^{M-1} A^{M-r-1} R_{i,r}(A^M)$ (c




$u_i = DA^i x \in C^{N/M}$ ; $u = \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{M-1} \end{bmatrix} = \begin{bmatrix} DA^0 \\ DA^1 \\ \vdots \\ DA^{M-1} \end{bmatrix} x = \mathrm{diag}_M(D) \begin{bmatrix} A^0 \\ A^1 \\ \vdots \\ A^{M-1} \end{bmatrix} x \in C^N$

$\underbrace{\begin{pmatrix} D & & & \\ & D & & \\ & & \ddots & \\ & & & D \end{pmatrix}}_{M \text{ times}}$

MIMO system:

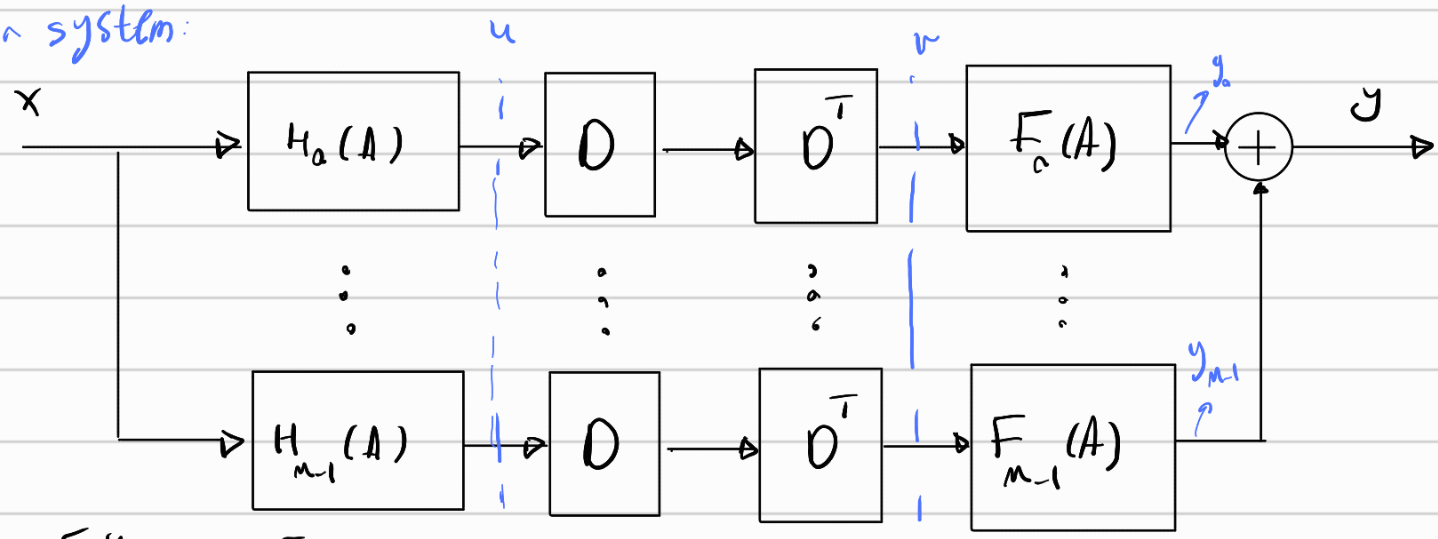
$$v_i = \sum_{j=0}^{M-1} \underbrace{P(\bar A)_{ij}}_{N/M \times N/M} u_j \Rightarrow v = \begin{bmatrix} v_0 \\ v_1 \\ \vdots \\ v_{M-1} \end{bmatrix} = \underbrace{P(\bar A) u}_{N\times N},\quad P(\bar A) = \left[P(\bar A)_{ij}^{N/M\times N/M}\right]_{N\times N}$$

$$y = \sum_{i=0}^{M-1} A^{M-i-1} D^T v_i = \left[A^{M-1}D^T \ A^{M-2}D^T \ \dots \ A^1 D^T \ A^0 D^T\right] \begin{bmatrix} v_0 \\ v_1 \\ \vdots \\ v_{M-1} \end{bmatrix}$$

$$\Rightarrow y = [A^{M-1} \dots A^1 A^0]\, \mathrm{diag}_M(D^T) v = [A^{M-1} \dots A^1 A^0]\, \mathrm{diag}_M(D)^T v$$

$$\Rightarrow y = [A^{M-1} \dots A^1 A^0]\, \mathrm{diag}(D)^T P(\bar A)\, \mathrm{diag}_M(D) \begin{bmatrix} A^0 \\ A^1 \\ \vdots \\ A^{M-1} \end{bmatrix} x$$

other system:




$$u = \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{M-1} \end{bmatrix} = \begin{bmatrix} H_0(A) \\ \vdots \\ H_{M-1}(A) \end{bmatrix} x, \quad v = \operatorname{diag}(D^T D)\, u, \quad y = [F_0(A) \cdots F_{M-1}(A)]\, v$$

$$\Rightarrow y = [F_0(A) \cdots F_{M-1}(A)] \operatorname{diag}(D^T D) \begin{bmatrix} H_0(A) \\ \vdots \\ H_{M-1}(A) \end{bmatrix} x$$

$$H_i(A) = \sum_{r=0}^{M-1} A^r E_{i,r}(A^M), \quad F_i(A) = \sum_{r=0}^{M-1} A^{M-r-1} R_{i,r}(A^M)$$

$$\Rightarrow F_i(A) = [A^{M-1} \cdots A^1 A^0] \underbrace{\begin{bmatrix} R_{i,0}(A^M) \\ R_{i,1}(A^M) \\ \vdots \\ R_{i,M-1}(A^M) \end{bmatrix}}_{R_i(A^M)} = [A^{M-1} \cdots A^0]\, R_i(A^M)$$

$$H_i(A) = \underbrace{[E_{i,0}(A^M) \; E_{i,1}(A^M) \cdots A_{i,M-1}(A^M)]}_{E_i(A^M)} \begin{bmatrix} A^0 \\ A^1 \\ \vdots \\ A^{M-1} \end{bmatrix} = E_i(A^M) \begin{bmatrix} A^0 \\ A^1 \\ \vdots \\ A^{M-1} \end{bmatrix}$$

$$\begin{bmatrix} H_0(A) \\ H_1(A) \\ \vdots \\ H_{M-1}(A) \end{bmatrix} = \underbrace{\begin{bmatrix} E_0(A^M) \\ E_1(A^M) \\ \vdots \\ E_{M-1}(A^M) \end{bmatrix}}_{E(A^M)} \begin{bmatrix} A^0 \\ A^1 \\ \vdots \\ A^{M-1} \end{bmatrix} = E(A^M) \begin{bmatrix} A^0 \\ A^1 \\ \vdots \\ A^{M-1} \end{bmatrix}, \quad E(A^M) = \begin{bmatrix} E_{0,0}(A^M) & \cdots & E_{0,M-1}(A^M) \\ \vdots & \ddots & \vdots \\ E_{M-1,0}(A^M) & \cdots & E_{M-1,M-1}(A^M) \end{bmatrix}$$

$$[F_0(A) \; F_1(A) \cdots F_{M-1}(A)] = [A^{M-1} \cdots A^1 A^0] \overbrace{[R_0(A^M) \; R_1(A^M) \cdots R_{M-1}(A^M)]}^{R^T(A^M)}$$

$$\Rightarrow [F_0(A)F_1(A)\cdots F_{m-1}(A)] = [A^{m-1}\cdots A^1A^0]\,R^T((A^m)^T)$$

$$R(A^m) = \begin{bmatrix} R_{0,0}(A^m) & R_{0,1}(A^m) & \cdots & R_{0,m-1}(A^m) \\ R_{1,0}(A^m) & & \ddots & \vdots \\ \vdots & & & \vdots \\ R_{m-1,0}(A^m) & \cdots & \cdots & R_{m-1,m-1}(A^m) \end{bmatrix}, \quad \underbrace{A = A^T \Rightarrow A^m = (A^m)^T}_{\text{needed constrains}}$$

in noble, only $\bar{A}$ matters

$$\Rightarrow y = [A^{m-1}\cdots A^1A^0]\,R^T(A^m)\,\mathrm{diag}_m(D)^T\,\mathrm{diag}_m(D)\,E(A^m)\begin{bmatrix} A^0 \\ A^1 \\ \vdots \\ A^{m-1}\end{bmatrix}x$$

$$B^m = \begin{pmatrix} (A^m)_{1,1} & 0 \\ \cancel{(A^m)_{2,1}}\ 0 & (A^m)_{2,2} \end{pmatrix} = A^m \Rightarrow DH(A^m) = H(\bar{A})D$$

$$DE_{in}(A^m) = E_{in}(\bar{A})D \Rightarrow \mathrm{diag}(D)\,E(A^m) = E(\bar{A})\,\mathrm{diag}(D)$$

$$DR_{in}(A^m) = R_{in}(\bar{A})D \Rightarrow \mathrm{diag}(D)\,R(A^m) = R(\bar{A})\,\mathrm{diag}(D)$$

$$\Rightarrow R^T(A^m)\,\mathrm{diag}(D^T) = \mathrm{diag}(D)^T R^T(\bar{A})$$

$$\Rightarrow y = [A^{m-1}\cdots A^0]\,\mathrm{diag}(D^T)\,R^T(\bar{A})\,E(\bar{A})\,\mathrm{diag}(D)\begin{bmatrix} A^0 \\ \vdots \\ A^{m-1}\end{bmatrix}x$$

equivalence $\Rightarrow$ $$y = [A^{m-1}\cdots A^0]\,\mathrm{diag}(D^T)\,P(\bar{A})\,\mathrm{diag}(D)\begin{bmatrix} A^0 \\ \vdots \\ A^{m-1}\end{bmatrix}x$$

$$\Rightarrow \boxed{\bar{A} = \bar{A}^T, \quad P(\bar{A}) = R^T(\bar{A})\,E(\bar{A})}$$

$$\Rightarrow \boxed{P_{ij}(\bar{A}) = \sum_{r=0}^{m-1} R_{ri}(\bar{A})\,E_{rj}(\bar{A})}$$