پردازش سیگنال گرافی

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مهندسی برق

برنا خدابنده ۴۰۰۱۰۹۸۹۸

تمرین شماره 2

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$\forall i=1,\dots,n : u(i)\geq 0 \Rightarrow \exists i,j : u(i)=0,\ u(j)\neq 0$ (سوال 1

(الف) $\mu u(i) = (Mu)(i) = \sum M_{ik}u(k) \geq M_{ij}u(j) > 0$

$\Rightarrow \mu u(i) > 0 \wedge u(i)=0 \Rightarrow \times \Rightarrow \boxed{\forall i\in 1,\dots,n : u(i)>0}$

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$\mu_1 = \max_{y\in\mathbb{R}^n} \frac{y^TMy}{y^Ty} = \frac{u_1^TMu_1}{u_1^Tu_1}$, let: $x(i)=|u_1(i)|$, x: a signal on graph

$x^Tx = \sum x(i)^2 = \sum u_1^2 = 1 = u_1^Tu_1$

$x^TMx = \sum M_{ij}x(i)x(j) = \sum M_{ij}|u_1(i)||u_1(j)| \overset{M_{ij}\geq 0}{=} \sum |M_{ij}u_1(i)u_1(j)|$

$\Rightarrow x^TMx = \sum|M_{ij}u_1(i)u_1(j)| \geq |\overbrace{\sum M_{ij}u_1(i)u_1(j)}^{u_1^TMu_1}| = |u_1^TMu_1| \geq u_1^TMu_1$

$\xrightarrow[x^Tx=u_1^Tu_1]{x^TMx\geq u_1^TMu_1} \frac{x^TMx}{x^Tx} \geq \frac{u_1^TMu_1}{u_1^Tu_1} = \max_{y\in\mathbb{R}^n}\frac{y^TMy}{y^Ty} \Rightarrow \boxed{x=u_1}$

Since there can't be an x making $\frac{y^TMy}{y^Ty}$ higher,

$\Rightarrow x=u_1 \Rightarrow x(i)=u_1(i) \Rightarrow \boxed{u_1(i)\geq 0}$

$u_1(i)\geq 0 \Rightarrow \boxed{u_1(i)>0 \Rightarrow \text{all are positive} \Rightarrow \text{proving (1)}}$

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let: $y(i)=|u_n(i)| \Rightarrow$ same process: $\frac{y^TMy}{y^Ty}\geq\frac{|u_n^TMu_n|}{u_n^Tu_n} = |\mu_n| \geq -\mu_n$

and: $\frac{y^TMy}{y^Ty} \leq \max_{x\in\mathbb{R}^n}\frac{x^TMx}{x^Tx} = \mu_1 \Rightarrow \boxed{\mu_1\geq-\mu_n} \Rightarrow$ proving (2)

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$\forall i\in[n] : u_1(i)>0$, $u_2^Tu_1 = 0 = \sum u_2(i)u_1(i) = 0$

$\Rightarrow \exists i,j : u_1(i)u_2(j)<0,\ u_1(j)u_2(j)>0 \Rightarrow \boxed{\exists i,j : u_2(i)<0,\ u_2(j)>0}$

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$$\left.\begin{array}{l} \text{let: } y(i)=|u_2(i)| \Rightarrow y^TMy \geq u_2^TMu_2 = \sum M_{ij}u_2(i)u_2(j) \\ \exists i,j : u_2(i)>0,\ u_2(j)<0 \end{array}\right\} \Rightarrow y^TMy > u_2^TMu_2$$

$$y^Ty = u_2^Tu_2 \Rightarrow \underbrace{\frac{u_2^TMu_2}{u_2^Tu_2}}_{\mu_2} < \frac{y^TMy}{y^Ty} \le \mu_1 \Rightarrow \boxed{\begin{array}{l}\mu_2 < \mu_1 \\ \Rightarrow \mu_2 \neq \mu_1\end{array}} \quad (3)$$

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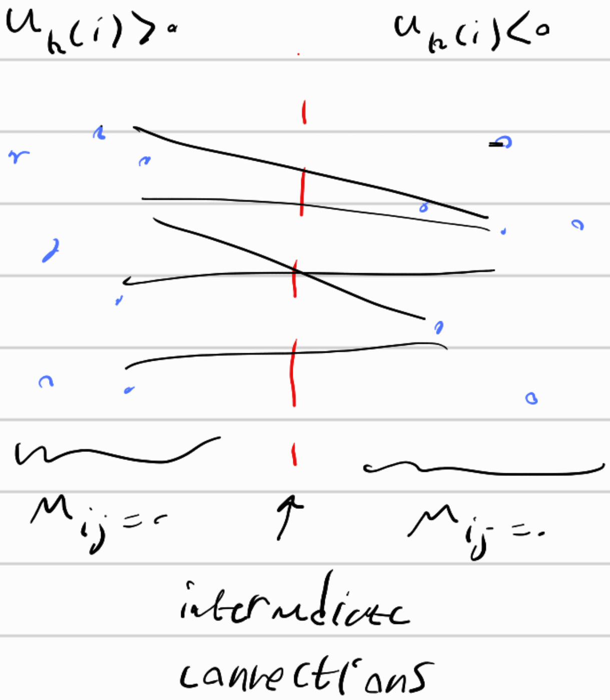
if $\mu_1 = -\mu_n \Rightarrow$ all "$\geq$" should become "$=$" $\Rightarrow y = u_1 \Rightarrow u_1(i) = |u_n(i)|$

$$, \sum |M_{ij} u_n(i) u_n(j)| = -\sum M_{ij} u_n(i) u_n(j) = y^TMy = u_1^TMu = \sum M_{ij} u_1(i) u_1(j)$$

$\rightarrow$ negating all terms is the only way.

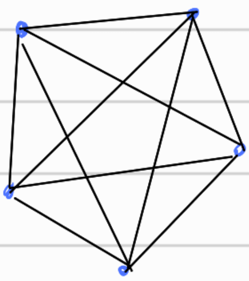
$$\Rightarrow \forall i,j : M_{ij} = 1 \Rightarrow u_1(i) u_1(j) = -u_n(i) u_n(j) = |u_n(i) u_n(j)|$$

$$\Rightarrow \forall i,j : M_{ij} = 1 : u_n(i) u_n(j) \leq 0 \quad \begin{cases} \text{if } u_n(i) \geq 0 \Rightarrow u_n(j) \leq 0 \\ \Rightarrow u_n(i) = u_1(i),\ u_n(j) = -u_1(j) \\ \text{same for } u_n(j) \geq 0 \end{cases}$$



$\Rightarrow$ $\boxed{\text{2 part graph}}$

$$\boxed{\begin{array}{c}\mu_1 = -\mu_n \\ u_1(i) = |u_n(i)| \\ M_{ij} = 1 \Leftrightarrow u_n(i) u_n(j) \leq 0\end{array}}$$



$: A_{ij}: \begin{cases}1 & i\neq j \\ 0 & i=j\end{cases} \Rightarrow L_{ij} = D_{ij} - W_{ij}: \begin{cases}-1 & i\neq j \\ n-1 & i=j\end{cases}$ سوال ۲)

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$\Lambda_L = \{\text{set of eigenvalues of } L\}$, $\underset{v_1}{V_L} = \{\text{set of eigenvectors of } L\}$

$$L = \begin{bmatrix} n-1 & -1 & -1 & \cdots & -1 \\ -1 & n-1 & -1 & \cdots & -1 \\ -1 & -1 & n-1 & \cdots & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & -1 & \cdots & n-1 \end{bmatrix} = nI + \overbrace{\begin{bmatrix} -1 & \cdots & -1 \\ \vdots & \ddots & \vdots \\ -1 & \cdots & -1 \end{bmatrix}}^{O_{-1}}$$

$$\Lambda_L = \{\lambda : \det(\lambda I - L) = 0\} = \{\lambda : \det((\lambda - n)I - O_{-1}) = 0\} = n + \Lambda_{O_{-1}}$$

$$O_{-1} v = -\begin{bmatrix} \sum v(j) \\ \vdots \\ \sum v(j) \end{bmatrix} \Rightarrow \text{if } O_{-1} v = \lambda v \Rightarrow \lambda v(j) = \sum_{k=0}^{n-1} v(k) \begin{cases} v: \text{dc signal} \\ \lambda = 0, v \in R^n_{/dc} \end{cases}$$

$$O_{-1}\begin{bmatrix}1\\ \vdots \\ 1\end{bmatrix} = \begin{bmatrix}-n \\ \vdots \\ -n\end{bmatrix} = -nv \Rightarrow \underbrace{\lambda = -n}_{1}, \underbrace{\lambda = 0}_{\dim(R^n_{/dc}) = n-1}$$

$$\Lambda_{O_{-1}} = \{-n, \overbrace{0, \cdots, 0}^{n-1}\} \Rightarrow \boxed{\Lambda_L = \{0, n, \cdots, n\} \Rightarrow \lambda_1 = 0, \lambda_2 = \cdots = \lambda_n = n}$$

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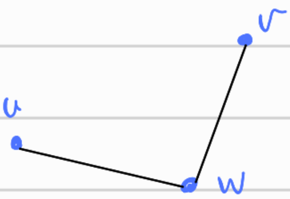

$$A = \begin{array}{c|c} & \begin{matrix} u & w & v \end{matrix} \\ \begin{matrix} u \\ w \\ v \end{matrix} & \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \end{array} \Rightarrow L = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} = [l_1, l_1 + l_2, l_2]$$

$Lv = \mu v$: if $\mu = 1 \Rightarrow Lv = v \Rightarrow (I - A)v = 0 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} v = 0$

$\Rightarrow \boxed{v = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}}$: $Lv = v$, $\text{trace}(L) = \text{trace}(D) = 2 \times |E| = 4$

$= 0 + 1 + \lambda_3 = 4 \Rightarrow \boxed{\lambda_3 = 3} \Rightarrow \lambda_1 = 0, \lambda_2 = 1, \lambda_3 = 3$

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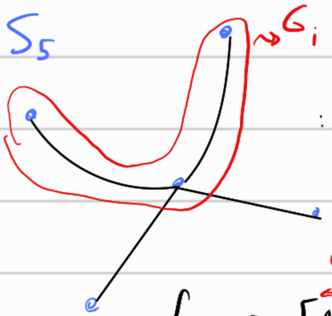
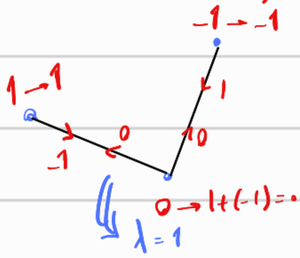
S_5



$$: L_n = \begin{bmatrix} I_{n-1} & \begin{matrix} -1 \\ \vdots \\ -1 \end{matrix} \\ \begin{matrix} -1 & -1 & \cdots & -1 \end{matrix} & n-1 \end{bmatrix}$$

n inter connections, center, G_1, center →, G_1

, pattern from S_3: $v_3 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ G_1



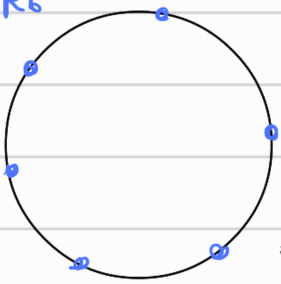
if $v = [1\ -1\ 0\ \cdots\ 0]^T \Rightarrow L_n v = v$ ✓ now, all such choices also apply

$\Rightarrow \forall v \in \{v_{i_0 i}: \begin{cases} 1 & : i = i_0 \\ -1 & : i = i_1 \neq i_0 \\ 0 & o.w. \end{cases}\}: Lv = v$: N(choices): $n - 1 - 1 = n - 2$

$\Rightarrow \boxed{(n-2) \times (\lambda = 1)}$, $\boxed{\lambda_1 = 0}$, $tr(L) = 2|E| = 2(n-1) = n - 2 + \lambda_n = 2n - 2$

$$\Rightarrow \lambda_n = n \Rightarrow \boxed{\lambda_1 = 0,\ \lambda_2 = \lambda_3 = \dots = \lambda_{n-1} = 1,\ \lambda_n = n}$$

R_6



$$L = \begin{bmatrix} 2 & -1 & \cdots & -1 \\ -1 & 2 & -1 & \vdots \\ \vdots & & \ddots & -1 \\ -1 & \cdots & -1 & 2 \end{bmatrix} = \begin{bmatrix} l_1^T \\ 1 \text{ shift } l_1^T \\ 2 \text{ shift } l_1^T \\ \vdots \\ n\text{-}1 \text{ shift } l_1^T \end{bmatrix} \rightsquigarrow \text{circulant matrix}$$

$$\Rightarrow PL = LP : P \Rightarrow \text{shift matrix} \Rightarrow V_P = V_L$$

$$V_P : Pv = \lambda v \Rightarrow \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \lambda \begin{bmatrix} v_n \\ v_1 \\ v_2 \\ \vdots \\ v_{n-1} \end{bmatrix} \Rightarrow v_j(k) = \lambda_j^k,\ \lambda_j^n = 1 \Rightarrow \lambda_j = \omega^j : \omega = e^{\frac{2\pi i}{n}}$$

$$\Rightarrow v_j(k) = \omega^{jk} = e^{\frac{2\pi i}{n} jk} : \text{columns of dft matrix},\ \lambda_j = e^{\frac{2\pi i}{n} j} \text{ (for } P\text{)}$$

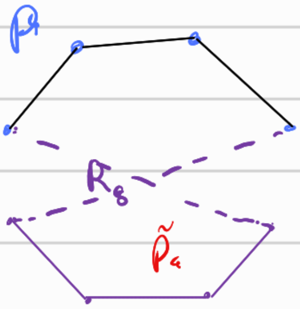
$$L v_j = \begin{bmatrix} 2 & -1 & \cdots & -1 \\ -1 & 2 & -1 & \vdots \\ \vdots & & \ddots & -1 \\ -1 & \cdots & -1 & 2 \end{bmatrix} \begin{bmatrix} \omega^0 \\ \omega^j \\ \vdots \\ \omega^{j(n-1)} \end{bmatrix} = \begin{bmatrix} (2 - \omega^j - \omega^{j*}) \\ (2 - \omega^j - \omega^{j*})\omega \\ \vdots \\ (2 - \omega^j - \omega^{j*})\omega^{n-1} \end{bmatrix} = \underbrace{(2 - \omega^j - \omega^{j*})}_{\lambda_j} v_j = 2\left(1 - \cos\left(\tfrac{2\pi}{n} j\right)\right) v_j$$

$$2 - e^{\frac{2\pi i j}{n}} - e^{-\frac{2\pi i j}{n}}$$

$$\Rightarrow \boxed{\lambda_k = 2\left(1 - \cos\left(\tfrac{2\pi k}{n}\right)\right)},\ v_j(k) = \omega^{jk} = \cos\left(\tfrac{2\pi k j}{n}\right) + i \sin\left(\tfrac{2\pi k j}{n}\right),\ \lambda_j \in \mathbb{R}$$

$$\Rightarrow \text{if } v \in V_L \Rightarrow Re(v), Imag(v) \in V_L \Rightarrow \left\{ \boxed{\begin{aligned} x_k(i) &= \cos\left(\tfrac{2\pi k i}{n}\right) \\ y_k(i) &= \sin\left(\tfrac{2\pi k i}{n}\right) \end{aligned}} \right. \text{ for the same } \lambda$$

P_n



$$L(R_{2n}) = \begin{bmatrix} 2 & -1 & \cdots & -1 & 0 \\ -1 & 2 & & & \vdots \\ \vdots & & \ddots & & -1 \\ -1 & & & & \vdots \\ \vdots & \cdots & -1 & -1 & 2 \end{bmatrix} = \underbrace{\left[\begin{array}{cc|cc} 1 & & -1 & \\ & 1 & & -1 \\ \hline -1 & & 1 & \\ & -1 & & 1 \end{array}\right]}_{P_n, \tilde{P}_n} + \left[\begin{array}{c|c} \overbrace{\begin{matrix} 1 & -1 & \\ -1 & 2 & -1 \\ & -1 & \ddots \end{matrix}}^{P_n} & 0 \\ \hline 0 & \underbrace{\begin{matrix} \ddots & -1 \\ -1 & \end{matrix}}_{\tilde{P}} \end{array}\right]$$

we want to reduce to P_n

$$L(R_{2n}) \begin{bmatrix} I \\ I \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} P_n \\ \tilde{P}_n \end{bmatrix} \text{ (stacked)} \Rightarrow \text{reduce stage needed}$$

$$\hookrightarrow \boxed{[I \ \ I]\, L(R_{2n}) \begin{bmatrix} I \\ I \end{bmatrix} = 2L(P_n)}$$

$$L(P_n)\phi = \frac{1}{2}[I \ \ I]\, L(R_{2n}) \begin{bmatrix} I \\ I \end{bmatrix} \phi$$

$$= \frac{1}{2}[I \ \ I]\, L(R_{2n}) \underbrace{\begin{bmatrix} \phi \\ \phi \end{bmatrix}}_{\psi} = \frac{\lambda}{2}[I \ \ I] \begin{bmatrix} \phi \\ \phi \end{bmatrix} = \boxed{\lambda\phi = L(P_n)\phi}$$

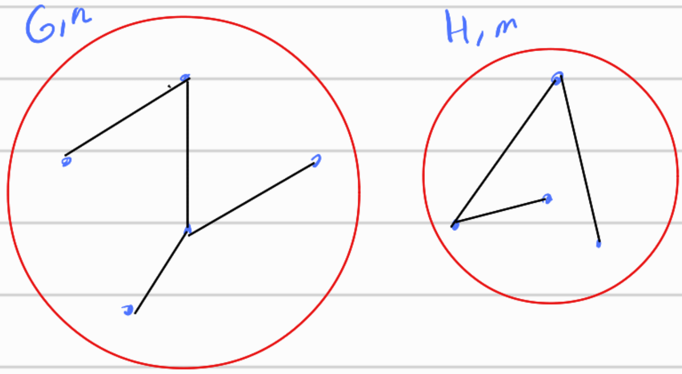
$$V_j(R_{2n}) = \begin{bmatrix} \omega^0 \\ \omega^j \\ \omega^{2j} \\ \vdots \\ \omega^{nj} \\ \vdots \\ \omega^{(2n-1)j} \end{bmatrix} = \begin{bmatrix} \Phi_j \\ \omega^{nj}\Phi_j \end{bmatrix}, \text{ if } \omega^{nj} = 1 \Rightarrow \checkmark, \quad \omega = e^{\frac{\pi i}{n}} \Rightarrow \text{if: } e^{j\pi i} = 1 \Rightarrow j: \text{even} \Rightarrow \psi = \begin{bmatrix} \phi \\ \phi \end{bmatrix}$$

پس بردار ویژه‌های زوج R_{2n} این‌ها را رعایت می‌کنند.

$$\lambda_j(R_{2n}) = 2\left(1 - \cos\left(\frac{\pi j}{n}\right)\right) \Rightarrow j = 2k \Rightarrow \boxed{\lambda_k = 2\left(1 - \cos\left(\frac{2\pi k}{n}\right)\right)}$$

$$V_k(P_n) = \phi_k = \begin{bmatrix} \omega'^0 \\ \omega'^k \\ \vdots \\ \omega'^{(n-1)k} \end{bmatrix}, \quad \omega' = e^{\frac{2\pi i}{n}} \xrightarrow[\text{same}]{} \begin{cases} \boxed{x_k(i) = \cos\left(\frac{2\pi k i}{n}\right)} \\ \boxed{y_k(i) = \sin\left(\frac{2\pi k i}{n}\right)} \end{cases}$$

G, n $\quad$ H, m



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$$\forall i \le n: \ L_G \Psi_i = \lambda_i \Psi_i$$
$$, \ \forall k \le m: \ L_H \Phi_k = \gamma_k \Phi_k$$

let: $Q = G \times H \Rightarrow V_Q = V_G \times V_H$, $A_Q = A_G \otimes I_m + I_n \otimes A_H$

$$L_Q = D_Q - A_Q = D_G \otimes I_m + I_n \otimes D_H - A_Q = L_G \otimes I_m + I_n \otimes L_H$$

$\Rightarrow$ let: $\beta_{ij}(a,b) = \Psi_i(a)\Phi_j(b) \Rightarrow \beta_{ij} = \Psi_i \otimes \Phi_j$

$$L_Q \beta_{ij} = (L_G \otimes I_m + I_n \otimes L_H)\Psi_i \otimes \Phi_j = (\overset{\lambda_i \Psi_i}{\cancel{L_G \Psi_i}}) \otimes \Phi_j + \Psi_i \otimes (\overset{\gamma_j \Phi_j}{\cancel{L_H \Phi_j}})$$

$= (\lambda_i + \gamma_j)\Psi_i \otimes \Phi_j = \boxed{(\lambda_i + \gamma_j)\beta_{ij} = L_Q \beta_{ij}}$, we have $n \times m$ β_{ij} values, and $L_{Q_{mn \times mn}}$ so these are all the eigenvalues.

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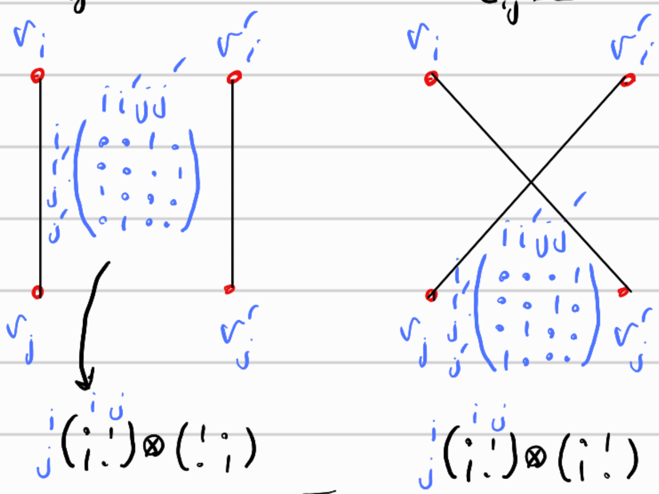
$e_{ij} = +1$ $\qquad$ $e_{ij} = -1$



$$\begin{cases} A_G \\ E_G : E_{G(ij)} = e_{ij} \\ A_H((i,i'),(j,j')) = A_{G(i,j)} \otimes \begin{cases} \overset{T_1}{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}} : e_{ij} = -1 \\ \underset{T_{-1}}{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}} : e_{ij} = 1 \end{cases} \end{cases}$$

$$A_1 = A_G \odot \underbrace{\left(\frac{\mathbb{1}\mathbb{1}^T + E_G}{2}\right)}_{e_{ij}=1}, \quad A_2 = A_G \odot \underbrace{\left(\frac{\mathbb{1}\mathbb{1}^T - E_G}{2}\right)}_{e_{ij}=-1}$$

$$A_H = A_1 \otimes T_1 + A_2 \otimes T_{-1} = \frac{1}{2}\Big[\overbrace{(A_G \odot \mathbb{1}\mathbb{1}^T)}^{A_G} \otimes T_1 + \overbrace{(A_G \odot \mathbb{1}\mathbb{1}^T)}^{A_G} \otimes T_2 + (A_G \odot E_G) \otimes T_1 - (A_G \odot E_G) \otimes T_2\Big]$$

$$\Rightarrow A_H = A_G \otimes \overbrace{\begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}}^{M_1} - (\overbrace{A_G \odot \tilde{E}_G}^{A_S}) \otimes \underbrace{\begin{bmatrix} -1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}}_{M_2}$$

$$M_1, M_2: \begin{cases} M_1\begin{pmatrix}1\\-1\end{pmatrix}=0 & M_1\begin{pmatrix}1\\1\end{pmatrix}=\begin{pmatrix}1\\1\end{pmatrix} \\ M_2\underbrace{\begin{pmatrix}1\\-1\end{pmatrix}}_{m_1}=-\begin{pmatrix}1\\-1\end{pmatrix} & M_2\underbrace{\begin{pmatrix}1\\1\end{pmatrix}}_{m_2}=0 \end{cases}$$

let: $\Psi_{i,j}=\psi_i\otimes m_j$

$$A_H\Psi_{i,1}=(A_G\otimes M_1+A_S\otimes M_2)(\psi_i\otimes m_1)=$$

$$(A_G\psi_i)\otimes(\cancelto{0}{M_1m_1})-(A_S\psi_i)\otimes(\cancelto{-m_1}{M_2m_1})$$

$$=-(A_S\psi_i)\otimes m_1 \Rightarrow \forall\psi_i\in\Lambda_{A_S}: A_H\Psi_{i,1}=\lambda_i\Psi_{i,1}$$

$$A_H\Psi_{i,2}=(A_G\otimes M_1+A_S\otimes M_2)(\psi_i\otimes m_2)=$$

$$(A_G\psi_i)\otimes(\cancelto{m_2}{M_1m_2})+(A_S\psi_i)\otimes(\cancelto{0}{M_2m_2})$$

$$=(A_G\psi_i)\otimes m_2 \Rightarrow \forall\psi_i\in\Lambda_{A_G}: A_H\Psi_{i,2}=\lambda_i\Psi_{i,2}$$

formally:

$$\forall\psi_i\in U_G,\ v=\psi_i\otimes\begin{bmatrix}1\\1\end{bmatrix}: A_Hv=\lambda_iv$$
$$A_G\psi_i=\lambda_i\psi_i$$

$$\forall\bar{\phi}_i\in U_G,\ v=\bar{\phi}_i\otimes\begin{bmatrix}1\\-1\end{bmatrix}: A_Hv=\sigma_iv$$
$$A_G\bar{\phi}_i=\sigma_i\bar{\phi}_i$$

$$\Rightarrow \boxed{\Lambda_H=\Lambda_G\cup\Lambda_S}\ ,\ \boxed{U_H=\left(U_G\otimes\begin{bmatrix}1\\1\end{bmatrix}\right)\cup\left(U_S\otimes\begin{bmatrix}1\\-1\end{bmatrix}\right)}$$

$$A_S=A_G\odot\bar{E}_0$$

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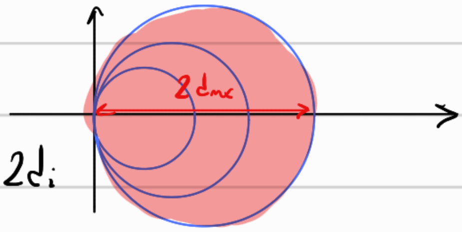
$C_i = d_i,\ R_i = \sum_{i\neq j} n_{ij} = -d_i \Rightarrow$

$\hookrightarrow$ circles touching $(0,0)$ with $D_i = 2d_i$



$\Rightarrow \boxed{\lambda_{mx} \leq 2d_{mx}}$

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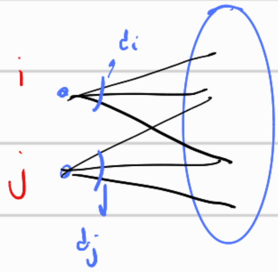

$$\left.\begin{array}{l}\lambda_1 = 0,\ V_1: \text{dc signal} \\ R^n \perp dc = \{v \in R^n \mid \sum_i v(i) = 0\}\end{array}\right\} \Rightarrow \lambda_2 = \min_{\substack{x \in R^n \perp dc \\ x \neq 0}} \frac{x^T L x}{x^T x} \Rightarrow \forall x \in R^n \perp dc: \frac{x^T L x}{x^T x} \geq \lambda_2$$

let: $X = [0 \dots 0\ \overset{i}{1}\ 0 \dots \overset{j}{-1}\ 0 \dots 0]^T \Rightarrow LX = [\longleftrightarrow d_i \longleftrightarrow -d_j \longleftrightarrow]$

$\Rightarrow x^T L x = d_1 + d_2,\ x^T x = 2 \Rightarrow \frac{x^T L x}{x^T x} = \boxed{\frac{d_i + d_j}{2} \geq \lambda_2}$

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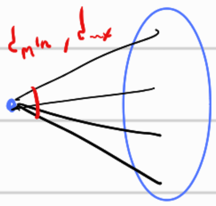

case 1: $\forall x \in R^n \perp dc: \lambda_2 \leq \frac{x^T L x}{x^T x}$

let: $X = [1\ 1 \dots 1\ \underset{\uparrow i}{-(n-1)}\ 1 \dots 1]^T \Rightarrow X(k) = \begin{cases} -(n-1) & k = i \\ 1 & \text{o.w.} \end{cases}$

$\Rightarrow X = \mathbb{1} - n[0 \dots 0\ 1\ 0 \dots 0]^T \Rightarrow LX = L\mathbb{1} - nL\begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \end{bmatrix} = -nl_i = n[\overset{\text{connections}}{\longleftrightarrow} -d_{min} \longleftrightarrow]^T$

$x^T L x = (\mathbb{1}^T - n[0 \dots 0\ 1\ 0 \dots 0])(-nl_i) = -n\cancel{\mathbb{1}^T l_i}^{\,0} + n^2 d_{min} = x^T L x = n^2 d_{min}$

$x^T x = \overbrace{1 + 1 + \dots + 1}^{n-1} + (n-1)^2 = (n-1) + (n-1)^2 = n(n-1) \Rightarrow \frac{x^T L x}{x^T x} = \boxed{\frac{n}{n-1} d_{min} \geq \lambda_2}$

same process for λ_n, using $\forall x: \lambda_n \geq \frac{x^T L x}{x^T x} \Rightarrow \boxed{\frac{n}{n-1} d_{mx} \leq \lambda_n}$

\# using the same x

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$T = \{\{v_1, v_2, \dots, v_n\}: \forall i,j,\ i \neq j: v_i^T v_j = 0\}$, $J_t = \max_{X \in T} \sum_{\substack{x_i \in X \\ |X| = t}} \frac{x_i^T M x_i}{x_i^T x_i} \Rightarrow x_i$ are orthogonal

$$\Rightarrow J_{t,max} = \max_{\substack{x_1 \in R^n \\ x_1 \neq 0}} \frac{x_1^T M x_1}{x_1^T x_1} + \max_{\substack{x_2 \in R^n \\ x_2 \neq 0 \\ x_2 \perp x_1}} \frac{x_2^T M x_2}{x_2^T x_2} + \max_{\substack{x_3 \in R^n \\ x_3 \neq 0 \\ x_3 \perp x_1, x_2}} \frac{x_3^T M x_3}{x_3^T x_3} + \dots$$

$$J_{t,mx} = \min_{\substack{T \in R^n \\ \dim(T) = n}} \max_{\substack{x \in T \\ x \neq 0}} \frac{x^T M x}{x^T x} + \min_{\substack{T \in R^n \\ \dim(T) = n-1}} \max_{\substack{x \in T \\ x \neq 0}} \frac{x^T M x}{x^T x} + \dots = \sum_{i=1}^{t} \overbrace{\min_{\substack{T \in R^n \\ \dim(T) = n-i+1}} \max_{\substack{x \in T \\ x \neq 0}} \frac{x^T M x}{x^T x}}^{\mu_i} = \sum_{i=1}^{t} \mu_i = J_{t,mx} \geq \sum_{x_i \in X} \frac{x_i^T M x_i}{x_i^T x_i}$$

let: $X = \{\underbrace{\begin{bmatrix} 1 \\ \vdots \end{bmatrix}}_{e_1}, \underbrace{\begin{bmatrix} 0 \\ 1 \\ \vdots \end{bmatrix}}_{e_2}, \dots\} \Rightarrow J_{t,x} = \sum_{i=1}^{n} \frac{e_i^T M e_i}{e_i^T e_i} = \sum \frac{m_{ii}}{1} = \boxed{\sum_{i=1}^{n} d_i \leq J_{t,mx} = \sum_{i=1}^{t} \mu_i}$

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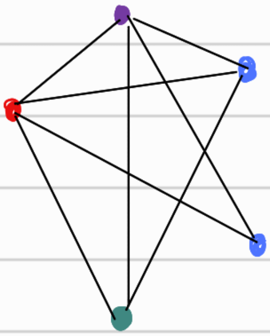
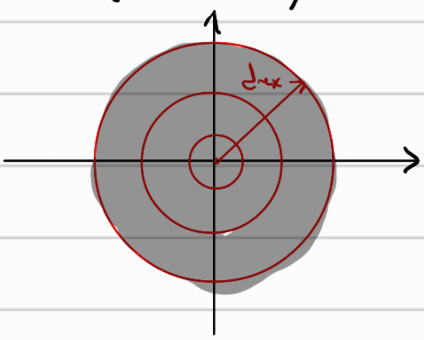


$\forall x \in R^2: \frac{x^TMx}{x^Tx} \le \lambda_{max}$, let: $x = \mathbb{1}$

$\Rightarrow x^TMx = \sum m_{ij} = 2|E| = \sum_{i=1}^{n} d_i = \mathbb{1}^TM\mathbb{1}$, $\mathbb{1}^T\mathbb{1} = n$

$\Rightarrow \lambda_{max} \ge \frac{\mathbb{1}^TM\mathbb{1}}{\mathbb{1}^T\mathbb{1}} = \frac{\sum_{i=1}^{n} d_i}{n} = d_{avg} \Rightarrow \boxed{d_{avg} \le \lambda_{max}(M)}$

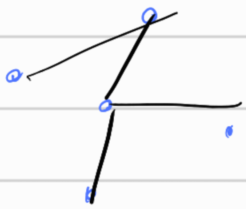
$M = \begin{pmatrix} 0 & & \\ & \ddots & \\ & & 0 \end{pmatrix}$, $R_i = \sum_{j \ne i} |m_{ij}| = d_i$, $c_i = m_{ii} = 0$ $\Rightarrow$ circles with center at (0,0) with radius d_i



$\rightsquigarrow$ all λ are inside $\Rightarrow \boxed{\lambda_{max}(M) \le d_{max}}$

$\Rightarrow \boxed{d_{avg} \le \lambda_{max}(M) \le d_{max}}$

ب)



G_1 reducing nodes $\rightsquigarrow$

G_2

, $\chi(G_2) = \chi(G_1)$

G_2: can't be reduced anymore

G_2: critical $\Rightarrow \forall i \in [n]: d_i \ge \chi(G_2) - 1 \Rightarrow d_{avg} \ge \chi(G_2) - 1$

$\rightarrow \chi(G_2) \le d_{avg} + 1 \le \lambda_{max}(G_2) + 1 \underset{\text{entanglement}}{\le} \lambda_{max}(G_1) + 1$

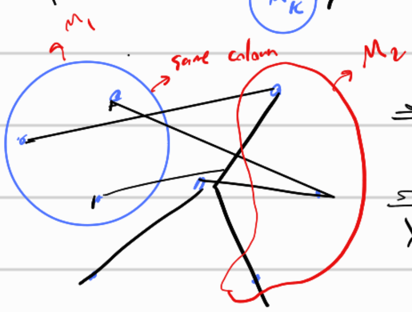
$\Rightarrow \chi(G_2) = \chi(G_1) \le \lambda_{max}(M) + 1 \le \lfloor \lambda_{max}(M) \rfloor + 1 \Rightarrow \boxed{\chi(G) \ge \lfloor \lambda_{max}(M) \rfloor + 1}$

$trace(M) = 0 \Rightarrow \lambda_{min}(G) \le 0$

ج)

$A_G = \begin{pmatrix} M_1 & & & \\ & M_2 & & \\ & & \ddots & \\ & & & M_K \end{pmatrix}$, $K = \chi(G)$

$\left.\begin{array}{l} \Rightarrow \lambda_{max}(G) + (K-1)\lambda_{min}(G) \le \sum_{i=1}^{K} \lambda(M_i) \\ M_i: \text{disconnected graphs} \Rightarrow \lambda_{max}(M_i) = 0 \end{array}\right\} \Rightarrow \lambda_{max}(G) + (k-1)\lambda_{min}(G) \le 0$



$\Rightarrow \lambda_{min}(G)(K-1) + \lambda_{max}(G) \le 0 \Rightarrow \lambda_{min}(G)\chi(G) \le \lambda_{min}(G) - \lambda_{max}(G)$

$\xrightarrow[\lambda_{min}(G) \le 0]{} \boxed{\chi(G) \ge 1 - \frac{\lambda_{max}(G)}{\lambda_{min}(G)}}$