پردازش سیگنال گرافی

دکتر آرش امینی

دانشگاه صنعتی شریف

مهندسی برق

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تمرین شماره 4

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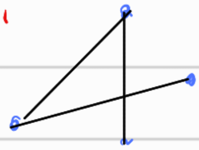
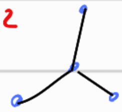
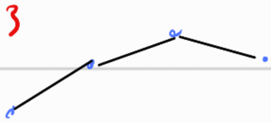
G_1  G_2  G_3 

(1

$|G_i| = N$

$L_i = V_i \Lambda_i V_i^T$

$x_i = V_i h_i \;,\; h_i \sim \mathcal{N}(0, \Lambda_i^{+})$

this produces smooth signals since $(\Lambda_i)_{kk} = 0 \iff (\Lambda_i^{+})_{kk} = 0$

and removing the 0 values, $(\Lambda_i^{+})_{kk} = \dfrac{1}{(\Lambda_i)_{kk}} \Rightarrow$ small Λ_i^{+} for large Λ_i

$\mathbb{E}(hh^T) = \Lambda_i^{+} \Rightarrow E(h_k^2) = (\Lambda_i^{+})_{kk} = \dfrac{1}{(\Lambda_i)_{kk}} \Rightarrow$ small $(h)_{(i)}$ for large $(\Lambda)_i$

large " " small "

$\mathbb{E}_{i \neq j}(h_i h_j) = 0$ 0 " " 0 "

thus $x = V_i h_i = \sum (h_i)_{(k)} V_{ki} \Rightarrow$ high amplitude for small λ → smooth

small amplitude for large λ → non smooth

0 amplitude for $0 = \lambda$ → unimportant

0 - dc value

$x = \sum_{i=1}^{n} x_i + n \;,\; n \sim \mathcal{N}(0, \sigma^2 I)$

MAP : $\max_{h_1, \ldots, h_N} \mathbb{P}(h_1, \ldots, h_K | x)$, $x_i = V_i h_i \Rightarrow h_i = V_i^T x_i$

$$\max_{h_1 \ldots h_K} P(h_1, \ldots, h_K | x) = \min_{h_1, \ldots, h_K} \frac{P(x|h_1, \ldots, h_K) P(h_1, \ldots, h_K)}{\int_{h_1, \ldots, h_K} P(x|h_1, \ldots, h_K) P(h_1, \ldots, h_K)\, dh} = \min_{h_1 \ldots h_K} P(x|h_1, \ldots, h_K) P(h_1, \ldots, h_K)$$

unimportant for MAP

$$P(h_1, \ldots, h_K) = \prod_{i=1}^{K} P(h_i) = \prod_{i=1}^{K} \frac{1}{(2\pi)^{N/2} \sqrt{\det(\Lambda^{+})}} \exp\left(-\frac{1}{2} h_i^T \Lambda_i h_i\right) \propto \exp\left(-\frac{1}{2} \sum_{i=1}^{K} h_i^T \Lambda_i h_i\right)$$

$x_i^T V_i \Lambda_i V_i^T x_i$

$= x_i^T L_i x_i$

$P(x|h_1, \ldots, h_K) = \mathbb{P}(x|x_1, \ldots, x_K) : x|x_1, \ldots, x_K \sim \mathcal{N}\left(\sum_{i=1}^{K} x_i, \sigma^2 I\right)$

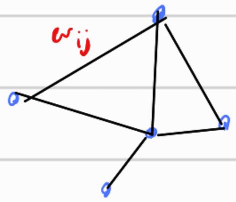
unimportant

$$\mathbb{P}(x|x_1, \ldots, x_K) = \frac{1}{(2\pi)^{N/2} \sigma^N} \exp\left(-\frac{1}{2}\left(x - \sum_{i=1}^{K} x_i\right)^T (\sigma^2 I)^{-1} \left(x - \sum_{i=1}^{K} x_i\right)\right)$$

$$\Rightarrow P(x|h_1, \ldots, h_K) \propto \exp\left(-\frac{1}{2}\frac{1}{\sigma^2} \left\| x - \sum_{i=1}^{K} x_i \right\|^2\right)$$

$$P(x|h_1, \ldots, h_K) P(h_1, \ldots, h_K) \propto \exp\left(-\frac{1}{2\sigma^2}\Big[\underbrace{\|x - \sum_{i=1}^{K} x_i\|^2 + \sigma^2 \sum_{i=1}^{K} x_i^T L_i x_i}_{\text{minimize this}}\Big]\right)$$

$$\Rightarrow \boxed{\max P(x|h_1, \ldots, h_K) P(h_1, \ldots, h_K) \sim \min_{x_1, \ldots, x_K} \|x - \sum_{i=1}^{K} x_i\|^2 + \sigma^2 \sum_{i=1}^{K} x_i^T L_i x_i}$$



$X_i \in \{0,1\}^N$, $(X_i)_{(k)} \sim \text{Bernouli}(p)$ (2

we constructed: $X_1, \dots, X_M$, $W_{ij} = \begin{cases} \dfrac{1}{1+\sum_{k=1}((x_k)_i-(x_k)_j)^2} & i \neq j \\ 0 & i=j \end{cases}$

$y \sim N(0, \sigma^2 I)$, $\mathrm{Cov}(y, X_k) = 0$ (a

$$y^T L y = \sum_{i<k} w_{ik}|y_i - y_k|^2 = \frac{1}{2}\sum_{i,k} w_{ik}|y_i - y_k|^2 \quad , \quad |y_i-y_k|^2 = y_i^2 + y_k^2 - 2y_i y_k$$

$$E(y^T L y | L) = E(y^T L y | w_{ij}) = \frac{1}{2}\sum_{i,k} w_{ik} E(|y_i - y_k|^2) \quad , \quad E(|y_i-y_k|^2) = \cancel{E}(y_i^2)^{\sigma^2} + E(y_k^2)^{\sigma^2} - 2E(y_i \cancel{y_k})$$

$$\Rightarrow E(y^T L y | L) = \sigma^2 \sum_{i,k} w_{ij} \Rightarrow E(y^T L y) = \sigma^2 \sum E(w_{ij})$$

$x_{k(j)} \backslash x_{k(i)}$	0 ($1-p$)	1 (p)
0 ($1-p$)	0	1
1 (p)	1	0

$$\Rightarrow |(x_k)_i - (x_k)_j|^2 \sim \text{Bernauli}(\underbrace{2p(1-p)}_{\tilde{p}})$$

$$\Rightarrow \sum_{k=1}^{M} ((x_k)_i - (x_k)_j)^2 \sim \text{Binomial}(M, \tilde{p})$$

$$E(w_{ij}) = E_{Z \sim Bin(M,\tilde{p})}\left[\frac{1}{1+Z}\right] = \sum_{k=0}^{M} \frac{1}{1+k}\binom{M}{k}\tilde{p}^k \tilde{q}^{M-k} = \lambda(\tilde{p}, \tilde{q})$$

$$\frac{\partial \lambda}{\partial \tilde{p}} = \sum_{k=0}^{M} \frac{k}{1+k}\binom{M}{k}\tilde{p}^{k-1}\tilde{q}^{M-k} \Rightarrow \left(1 + \tilde{p}\frac{\partial}{\partial \tilde{p}}\right)\lambda(\tilde{p},\tilde{q}) = \sum_{k=0}\binom{M}{k}\tilde{p}^k\tilde{q}^{M-k} = (\tilde{p}+\tilde{q})^M$$

$$\lambda + \tilde{p}\frac{\partial \lambda}{\partial \tilde{p}} = \frac{\partial}{\partial \tilde{p}}(\tilde{p}\lambda) = (\tilde{p}+\tilde{q})^M \Rightarrow \tilde{p}\lambda = \frac{1}{M+1}(\tilde{p}+\tilde{q})^{M+1} + h(\tilde{q})$$

$$\lambda(0,\tilde{q}) = \tilde{q}^M < \infty \Rightarrow \tilde{p}\lambda(\tilde{p},\tilde{q})\Big|_0 = 0 \Rightarrow h(\tilde{q}) = -\frac{1}{M+1}\tilde{q}^{M+1} \Rightarrow \lambda = \frac{(\tilde{p}+\tilde{q})^{M+1} - \tilde{q}^{M+1}}{(M+1)\tilde{p}}$$

$$E(w_{ij}) \xrightarrow{\tilde{q}=1-\tilde{p}} \frac{1-\tilde{q}^{M+1}}{(M+1)\tilde{p}} = E(w_{ij}) \quad , \quad \tilde{q} = 1-\tilde{p} \, , \, \tilde{p} = 2p(1-p)$$

$$E(y^T L y) = N\sigma^2 E(w_{ij}) = \boxed{N\sigma^2 \frac{1-(1-\tilde{p})^{M+1}}{(M+1)\tilde{p}} = E(y^T L y) \, , \, \tilde{p} = 2p(1-p)}$$

(b

$E\left(\sum_{1 \leq i \neq j \leq N} \lambda_i \lambda_j\right) = \,?$, we know: $\displaystyle\sum_{\substack{S \subseteq [n] \\ |S|=k}} \prod_{i \in S} \lambda_i = \sum_{\substack{S \subseteq [n] \\ |S|=k}} \det(A(S,S))$

$$\Rightarrow \sum_{1\le i\neq j\le N} \lambda_i\lambda_j = \sum_{\substack{S\subseteq[N]\\|S|=2}} \det(W(S,S)) \quad : \quad W(S,S) = \begin{vmatrix} w_{ii} & w_{ij} \\ w_{ji} & w_{jj} \end{vmatrix} = -w_{ij}^2$$

$E(w_{ij}^2)$: independent from i,j

$$\Rightarrow \mathbb{E}\Big(\sum_{1\le i\neq j\le N} \lambda_i\lambda_j\Big) = \mathbb{E}\Big(-\sum_{1\le i\neq j\le N} w_{ij}^2\Big) = -\binom{N}{2}\mathbb{E}(w_{ij}^2)$$

$$E(w_{ij}^2) = \mathbb{E}_{Z\sim \text{Binomial}(m,\tilde{p})}\Big(\big(\frac{1}{1+Z}\big)^2\Big) = \sum_{k=0}^{m} \frac{1}{(1+k)^2}\binom{m}{k}\tilde{p}^{k}\tilde{q}^{m-k} = \theta(\tilde{p},\tilde{q})$$

similiarly: $(1+\tilde{p}\frac{\partial}{\partial\tilde{p}})^2\theta(\tilde{p},\tilde{q}) = (\tilde{p}+\tilde{q})^m \rightsquigarrow (1+\tilde{p}\frac{\partial}{\partial\tilde{p}})\theta(\tilde{p},\tilde{q}) = \lambda(\tilde{p},\tilde{q})$

$1+\tilde{p}\frac{\partial}{\partial\tilde{p}} = \frac{\partial}{\partial\tilde{p}}\tilde{p} \rightsquigarrow \frac{\partial}{\partial\tilde{p}}(\tilde{p}\theta) = \lambda = \frac{(\tilde{p}+\tilde{q})^{m+1}-\tilde{q}^{m+1}}{(m+1)\tilde{p}}$ $\rightsquigarrow$ very hard problem

we may approximate via Chao and Strawderman, Negative moment.

$$E\Big(\frac{1}{(1+X)^2}\Big) = \text{Var}\Big(\frac{1}{1+X}\Big) + \mathbb{E}\Big(\frac{1}{1+X}\Big)^2 = \text{Var}\Big(\frac{1}{1+X}\Big) + \Big[\frac{1-\tilde{q}^{m+1}}{(m+1)\tilde{p}}\Big]^2$$

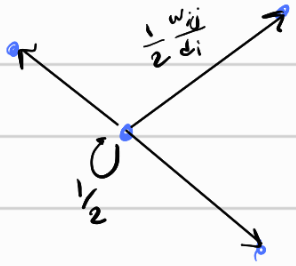
$$\text{Var}\Big(\frac{1}{1+X}\Big) \xrightarrow{\text{expand near mean}} \approx \text{Var}\Big(\frac{1}{1+E(X)+\delta}\Big) = \text{Var}\Big(-\frac{\delta}{(1+E(X))^2}\Big)$$

$$\approx \frac{\text{Var}(\delta)}{(1+E(X))^4} = \frac{\text{Var}(X)}{(1+E(X))^4} = \frac{m\tilde{p}\tilde{q}}{(1+m\tilde{p})^4}$$

$$\Rightarrow \mathbb{E}\Big(\frac{1}{(1+X)^2}\Big) \approx \frac{m\tilde{p}\tilde{q}}{(1+m\tilde{p})^4} + \Big[\frac{1-\tilde{q}^{m+1}}{(m+1)\tilde{p}}\Big]^2$$

$$\Rightarrow \boxed{\mathbb{E}\Big(\sum\lambda_i\lambda_j\Big) \approx -\binom{N}{2}\Big\{\frac{m\tilde{p}\tilde{q}}{(1+m\tilde{p})^4} + \Big[\frac{1-\tilde{q}^{m+1}}{(m+1)\tilde{p}}\Big]\Big\} \quad \tilde{q}=1-\tilde{p},\ \tilde{p}=2p(1-p)}$$

approximates



(3

p: vector length N, denoting $(p)_{(k)} = \mathbb{P}(X=k)$

starting from a: $p_0 = \delta_a$

$\tilde{W}$: transition matrix, $p_{t+1} = \tilde{W} p_t$

$\omega_1 \geq \omega_2 \geq \dots \geq \omega_n$: eigenvalues, $\psi_1, \psi_2, \dots, \psi_n$: eigenvectors.

(g

$$\mathbb{P}_{t+1}(X=k) = \sum_{\ell \in E(k,:)} P_t(X=\ell)\, P_{\ell \to k} = \frac{1}{2} \times P_t(X=k) + \sum_{\ell=1}^{n} \frac{1}{2} \frac{W_{k\ell}}{D_\ell} P_t(X=\ell)$$

$$= \frac{1}{2} P_t(X=k) + \frac{1}{2} \sum (D^{-1}W)_{k\ell}\, P(X=\ell) \Rightarrow P_{t+1} = \left(\frac{1}{2} I + \frac{1}{2} (D^{-1}W)^T\right) P_t$$

$$\Rightarrow \tilde{W} = \frac{1}{2}\left(I + (D^{-1}W)^T\right) \underset{\hookrightarrow W = W^T}{=} \boxed{\frac{1}{2} D^{1/2} (I + A) D^{-1/2} = \tilde{W}}, \quad A = D^{-1/2} W D^{-1/2}$$

$$\tilde{W} \psi_i = \omega_i \psi_i = \frac{1}{2} D^{1/2} (I + A) D^{-1/2} \psi_i = \omega_i \psi_i \Rightarrow (I + A)(D^{-1/2} \psi_i) = 2\omega_i D^{-1/2} \psi_i$$

$$\Rightarrow A(D^{-1/2} \psi_i) = \boxed{(\underbrace{2\omega_i - 1}_{\lambda_i})(\underbrace{D^{-1/2} \psi_i}_{v_i})} \Rightarrow 2\omega_i - 1 \in \Lambda_A \Rightarrow 2\omega_i - 1 = \mu_i$$

(h

Perron-Frobenius: $\mu_1 : \forall i : v(i) > 0$, $\mu_1 \geq -\mu_n$, $\mu_1 > \mu_2$: if $M_{ij} \geq 0$

$A v_i = \mu_i v_i$, $A = D^{-1/2} W D^{-1/2}$, let: x : $(x)_{(i)} = \sqrt{d_i}$ $\Rightarrow x = [\sqrt{d_1} \dots \sqrt{d_N}]^T$

$\therefore Ax = D^{-1/2} W D^{-1/2} x = D^{-1/2} W D^{-1/2} [\sqrt{d_1} \dots \sqrt{d_N}]^T = D^{-1/2} W \mathbb{1} = D^{-1/2} [d_1, \dots, d_N]^T$

$\therefore Ax = [\sqrt{d_1}, \dots, \sqrt{d_N}]^T = x = Ax \Rightarrow x$ is an eigenvector of A with $\mu = 1$

$\forall i : (x)_{(i)} > 0 \Rightarrow x = u_1$, $1 = \mu_1$, $\mu_1 = 2\omega_1 - 1 = 1 \Rightarrow \boxed{\omega_1 = 1}$

$v_1 = D^{-1/2} \psi_1 \Rightarrow \psi_1 = D^{1/2} u_1 = D^{1/2} [\sqrt{d_1}, \dots, \sqrt{d_N}]^T = \boxed{[d_1, \dots, d_N]^T = v_1 = d}$

$\mu_1 = 1$, $\mu_2 < \mu_1 = 1 \Rightarrow \mu_i < 1$ ($\hookrightarrow \omega_i < 1$), $\mu_n \geq -\mu_1 = -1 \Rightarrow \underbrace{2\omega_n - 1 \geq -1}_{\omega_n \geq 0}$ $\Big\} \Rightarrow \boxed{0 \leq \omega_i < 1}$

(c

$$P_{t+1} = \tilde{W} P_t \Rightarrow P_t = \tilde{W}^t P_0 \,,\quad 0 \le \omega_i \le 1 \Rightarrow \tilde{W}^t : \text{converges for large } t$$

$$\tilde{W} = \psi \omega \psi^{-1} \Rightarrow \tilde{W}^t = \psi \omega^t \psi^{-1} \Rightarrow \lim_{t\to\infty} \tilde{W}^t = \psi \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 0 \end{pmatrix} \psi^{-1}$$

next step is the same

$$\tilde{W}^t : \text{converges} \Rightarrow \exists \pi : |\pi| < \infty : \pi = \lim_{t\to\infty} \tilde{W}^t P_0 \Rightarrow \tilde{W}\pi = \pi$$

$$\tilde{W}\pi = \pi \Rightarrow \pi = \psi_1 = d \leadsto \text{note that } |\pi|_1 = 1 \Rightarrow \boxed{\pi = \frac{d}{|d|_1} = \frac{[d_1, \cdots, d_N]^T}{\sum_{i=1}^{\tilde{N}} d_i}}$$

(d

$$P_t = \tilde{W}^t P_0 = \tilde{W}^t \delta_a = D^{1/2}\left(\frac{I+A}{2}\right)^t D^{-1/2}\delta_a \Rightarrow D^{-1/2}P_t = \left(\frac{I+A}{2}\right)^t (D^{-1/2}\delta_a)$$

$$D^{-1/2}\pi = \lim_{t\to\infty}\left(\frac{I+A}{2}\right)^t (D^{-1/2}\delta_a) \Rightarrow D^{-1/2}(P_t - \pi) = \left[\left(\frac{I+A}{2}\right)^t - \lim_{t\to\infty}\left(\frac{I+A}{2}\right)^t\right](D^{-1/2}\delta_a)$$

$$\left|\left(\frac{I+A}{2}\right)^t - \lim_{t\to\infty}\left(\frac{I+A}{2}\right)^t\right|_2 = \sigma_{max}\left(\left(\frac{I+A}{2}\right)^t - \lim_{t\to\infty}\left(\frac{I+A}{2}\right)^t\right) \underset{n\times n}{\overset{\text{symmetric}}{\leadsto}} \lambda_{max}(\cdots)$$

$$\lambda_i\left(\frac{I+A}{2}\right) = \omega_i \Rightarrow \lambda_i\left(\left(\frac{I+A}{2}\right)^t\right) = \omega_i^t \,,\quad \lambda_i\left(\lim_{t\to\infty}\left(\frac{I+A}{2}\right)^t\right) = \begin{cases} 1 & i=1 \\ 0 & \text{o.w.} \end{cases}$$

$$\Rightarrow \lambda_i\left(\left(\frac{I+A}{2}\right)^t - \lim_{t\to\infty}\left(\frac{I+A}{2}\right)^t\right) = \left\{\begin{matrix} 0 & i=1 \\ \omega_i^t & \text{o.w.} \end{matrix}\right\} \Rightarrow \lambda_{max} = \omega_2^t$$

$$|D^{-1/2}(P_t - \pi)|_2 = |[\cdots](D^{-1/2}\delta_a)|_2 \le |\cdots|_2 |D^{-1/2}\delta_a| = \omega_2^t |D^{-1/2}\delta_a|$$

$$D^{-1/2}\delta_a = \frac{1}{\sqrt{d(a)}}\delta_a \Rightarrow |D^{-1/2}\delta_a|_2 = \frac{1}{\sqrt{d(a)}}$$

$$\underset{\forall b}{|D^{-1/2}v|_2 \ge |(D^{-1/2}v)(b)|} = \frac{1}{\sqrt{d(b)}}|v(b)| \Rightarrow |D^{-1/2}(P_t - \pi)|_2 \ge \frac{1}{\sqrt{d(b)}}|P_t(b) - \pi(b)|$$

$$\Rightarrow \frac{1}{\sqrt{d(b)}}|P_t(b) - \pi(b)| \le \frac{1}{\sqrt{d(a)}}\omega_2^t \Rightarrow \boxed{|P_t(b) - \pi(b)| \le \sqrt{\frac{d(b)}{d(a)}}\,\omega_2^t}$$

$X = [x_1, x_2, \dots, x_n]^T \in \mathbb{R}^{n\times m}$, $Z \in \mathbb{R}^{n\times n}$: $Z_{ij} = |x_i - x_j|^2$ (4

(a

$$\min_{W\in\mathcal{W}_n} \|Z\circ W\|_{1,1} + 2\sigma^2 \sum_{i,j} W_{ij}(\log(W_{ij})-1)$$, $\mathcal{W}_n$: set of all valid weight matrices

$$\|Z\circ W\|_{1,1} = \sum_{i,j} |Z_{ij}W_{ij}| = \sum_{i,j} Z_{ij}W_{ij}$$

$$W^* = \arg\min_{W\in\mathcal{W}_n} \sum W_{ij}Z_{ij} + 2\sigma^2\sum W_{ij}(\log(W_{ij})-1)$$

$$= \arg\min_{W\in\mathcal{W}_n}\sum_{i,j} W_{ij}(Z_{ij}+2\sigma^2(\log(W_{ij})-1))$$ ⇝ seperable cost

$$\Rightarrow W^*_{ij} = \arg\min_{w\geq 0} w(Z_{ij}+2\sigma^2(\log(w)-1)) \Rightarrow \frac{\partial L}{\partial w}=0 \Rightarrow Z_{ij}+2\sigma^2(\log(w^*)-1)+w^*\times\frac{2\sigma^2}{w^*}$$

$$\Rightarrow Z_{ij}+2\sigma^2\log(w^*)=0 \Rightarrow \log(w^*) = -\frac{Z_{ij}}{2\sigma^2} \Rightarrow \boxed{W^*_{ij} = \exp\left(-\frac{Z_{ij}}{2\sigma^2}\right)}$$

(b

$$W^* = \arg\min_{W\in\mathcal{W}_n} \|Z\circ W\|_{1,1} - \alpha \mathbb{1}^T\log(W\mathbb{1})+\beta\|W\|_F^2 = F(Z,\alpha,\beta)$$

$$F(Z,\alpha,\beta) = \gamma \arg\min_{W\in\mathcal{W}_n} L(\gamma W,\alpha,\beta) = \gamma\arg\min_{W\in\mathcal{W}_n} |Z\circ(\gamma W)|_{1,1} - \alpha\mathbb{1}^T\log(\gamma W\mathbb{1}) + \beta\|\gamma W\|_F$$

Constant

$$F(Z,\alpha,\beta) = \gamma\arg\min_{W\in\mathcal{W}_n} \gamma|Z\circ W| - \alpha\mathbb{1}^T\log(W\mathbb{1}) - \cancel{c} + \beta\gamma^2\|W\|_F$$

$$= \gamma \arg\min_{W\in\mathcal{W}_n} \gamma|Z\circ W| - \alpha \mathbb{1}^T \log(W\mathbb{1})+\beta\gamma^2\|W\|_F$$, $\arg\min \frac{L(x)}{\alpha} = \arg\min L(x)$

$$= \gamma\arg\min_{W\in\mathcal{W}_n} |Z\circ W|_{1,1} - \frac{\alpha}{\gamma}\mathbb{1}^T\log(W\mathbb{1})+\beta\gamma\|W\|_F = \gamma F\left(Z,\frac{\alpha}{\gamma},\gamma\beta\right)$$

$\gamma=\sqrt{\frac{\alpha}{\beta}}$ ⇒ $\gamma\arg\min |Z\circ W| - \frac{1}{\theta}\mathbb{1}^T\log(W\mathbb{1})+\frac{1}{\theta}\|W\|_F$ ⇝ $\sqrt{\frac{\alpha}{\beta}}F\left(\frac{Z}{\sqrt{\alpha\beta}},1,1\right)$

$\theta = \frac{1}{\sqrt{\alpha\beta}}$

$$\boxed{F(Z,\alpha,\beta) = \gamma F\left(Z,\frac{\alpha}{\gamma},\gamma\beta\right) \overset{\gamma=\alpha}{=} \alpha F(Z,1,\alpha\beta) \overset{\gamma=\sqrt{\alpha/\beta}}{=} \sqrt{\frac{\alpha}{\beta}}F\left(\frac{Z}{\sqrt{\alpha\beta}},1,1\right)}$$

$x \sim \mathcal{N}(0, L^{+})$: $x_1, x_2, \dots, x_m \overset{iid}{\sim} \mathcal{N}(0, L^{+})$, $X = [x_1, x_2, \dots, x_m] \in \mathbb{R}^{n \times m}$ (5

here we will find the maximum likelihood estimator for L

$$f(X|L) = \prod_{i=1}^{m} f(x_i|L) = \mathcal{L}(L;X)$$

$$S = XX^T$$

we will instead minimize the negative log likelihood

$$\ell(L;X) = -\log(\mathcal{L}(L;X)) = -\sum_{i=1}^{m} \log(f(x_i|L))$$

since L has a zero eigenvalue, and $\mu = 0$ respectively, analysing the signal from the respective eigenvector is pointless, for our statistics

we do not consider zero eigenvalues, $\tilde{L}$: L without zero eigenvals

unimportant for minimization

$$f(x_i|L) = \frac{\sqrt{|\det(\tilde{L})|}}{(2\pi)^{n/2}} \exp\left(-\frac{1}{2} x_i^T L x_i\right) \Rightarrow -\log(f(x_i|L)) = \frac{1}{2} x_i^T L x_i - \frac{1}{2} \log(|\det(\tilde{L})|) + \frac{n}{2} \log(2\pi)$$

unimportant

ℓ'

$$\Rightarrow \ell'(L;X) = \sum_{i=1}^{m} x_i^T L x_i - \log(\det(\tilde{L}))$$

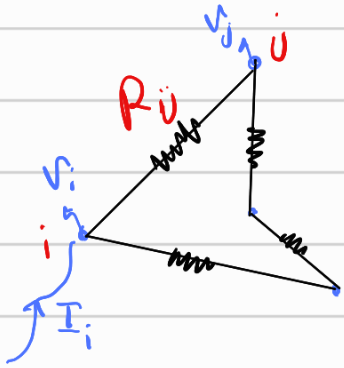
$$Tr(LS) = Tr(LXX^T) = Tr(X^T L X) = \sum_{i=1}^{m} (X^T L X)_{ii} = \sum_{i=1}^{n} x_i^T L x_i$$

$$\Rightarrow \ell'(L;X) = Tr(LS) - m\log(\det(\tilde{L}))$$

and the best L is chosen from the loss function.

$$\boxed{\min_{L \in \mathcal{L}_n^T} Tr(LS) - m\log(\det(\tilde{L}))}$$

$R_{eff}(u,v)$: effective resistance between nodes u, v (6



$R_{ij} = \frac{1}{\omega_{ij}}$: resistance between nodes i,j (a

V_i : voltage of node i, I_i input current of node i

if $I_i = -I_j = I \rightsquigarrow V_i - V_j = R_{eff}(i,j)\, I$

$$I_i = \sum_{j\in N} \frac{V_i - V_j}{R_{ij}} = \sum_j (V_i - V_j) w_{ij} = \overbrace{V_i \sum_j w_{ij}}^{(Dv)(i)} - \overbrace{\sum_j w_{ij} V_j}^{(Wv)(i)} = (Lv)(i) = I_i \Rightarrow I = Lv$$

assume : $I = \delta_u - \delta_v$ (in from node u out from node v)

$\Rightarrow$ if $I^* = \delta_u - \delta_v \Rightarrow (v)(u) - (v)(v) = R_{eff}(u,v) = (\delta_u - \delta_v)^T v^*$

$$Lv^* = I^* = \delta_u - \delta_v \xrightarrow[\text{we project along } R^n/\text{dc}\ \ R^n/N(L)]{L \text{ is irreversible}} v^* = L^+(\delta_u - \delta_v) \Rightarrow \boxed{R_{eff}(u,v) = (\delta_u - \delta_v)^T L^+ (\delta_u - \delta_v)}$$

we saw that : $Lv = I$, in standalone : $(I)(u) = 0 \Rightarrow (Lv)(u) = 0$ (b

$$(Lv)(u) = 0 = d(u)v(u) - (Wv)(u) = d(u)v(u) - \sum_s w_{su} v(s)$$

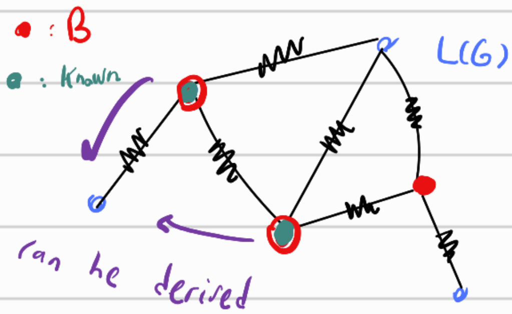
$$\Rightarrow \boxed{v(u) = \frac{1}{d(u)} \sum_{(s,u)\in E(G)} w_{su} v(s)}$$, on another note:

$$v^T L v = \sum_{i,j} w_{ij} \|v_i - v_j\|^2 \rightsquigarrow \nabla_{v_{B^c}} = 0 \rightsquigarrow \frac{\partial}{\partial v_u}\Big|_{\forall u \in V\setminus B} = 0 = \sum_{s:(s,u)\in E(G)} 2w_{us}(v(u) - v(s)) = 0$$

$\Rightarrow v(u)\sum w_{su} - \sum w_{su} v(s) = 0 \Rightarrow$ same equation as before! $\Rightarrow \boxed{\nabla_u (v^T L v) = 0}$

$u = V/B$

(c



we need to find L_B : $v_{(B)}^T L_B v_{(B)} = v^T L_G v$

prove that L_B is a valid laplacian.

$\forall u \in V/B : (Lv)(u) = 0$ (harmonic)

$$v^T L v = \sum_u v(u)(Lv)(u) = \sum_{u \in B} v(u)(Lv)(u) + \sum_{u \in V/B} v(u)\overset{0}{\cancel{(Lv)(u)}}$$

$$\Rightarrow v^T L v = \sum_{u \in B} v(u)(Lv)(u) = \sum_{u \in B}\sum_{s} v(u) L_{us} v(s)$$

consider only removing node $k \rightsquigarrow B = V/\{k\}$

$$\Rightarrow v^T L v = \sum_{\substack{u \in V \\ u \neq k}} \sum_{s} v(u) L_{us} v(s) = \sum_{\substack{u \in V \\ u \neq k}} v(u) \sum_{s} L_{us} v(s)$$

$$= \sum_{\substack{u \in V \\ u \neq k}} v(u) \left[L_{uk} v(k) + \sum_{\substack{s \in V \\ s \neq k}} L_{us} v(s) \right], \quad (Lv)(k) = 0 \rightsquigarrow v(k) = \frac{1}{d(k)} \sum_{\substack{s \in V \\ s \neq k}} w_{sk} v(s)$$

$$\rightsquigarrow v^T L v = \sum_{\substack{u \in V \\ u \neq k}} v(u) \sum_{\substack{s \in V \\ s \neq k}} \left(L_{su} + \frac{L_{ku}}{d(k)} w_{sk} \right) v(s)$$

→ if L' is valid we have an algorithm to remove nodes

$$= \sum_{u \in B} \sum_{s \in B} v(u) \underbrace{\left(L_{su} + \frac{w_{sk}}{d(k)} L_{ku} \right)}_{L'} v(s) = v(B)^T L' v(B)$$

$L' \rightsquigarrow$ spreading L_{ku} evenly, except the diagonal

$$L = \underbrace{\begin{pmatrix} & \vdots_{k} & \\ \cdots_{k} & & \\ & & \end{pmatrix}}_{n} \rightsquigarrow L' = \underbrace{\begin{pmatrix} L_{ii} & & \\ & & \\ & & \end{pmatrix}}_{n-1} \qquad L_{ij} + \frac{L_{kj}}{d(k)} w_{ik} \ (s \neq k)$$

L' is a valid laplacian

$$\sum_j^{n-1} L'_{ij} = L_{ii} + \sum_{i \neq k} L_{ij} + \frac{L_{kj}}{d(k)} \overset{L_{ik}}{w_{ik}} = L_{ii} + \underbrace{\sum_{i \neq k} L_{ij} + L_{kj}}_{\sum L_{ij}} \sum_j \cancel{\frac{w_{ik}}{d(k)}} = 0 \ \checkmark$$

for $V \to B = V/\{k\}$, we found L', continue this to find $L'', L''', \ldots$

thus in general, for B $\boxed{\text{we can find } L_B : v(B)^T L_B v(B) = v^T L v}$

$s,u \in B$

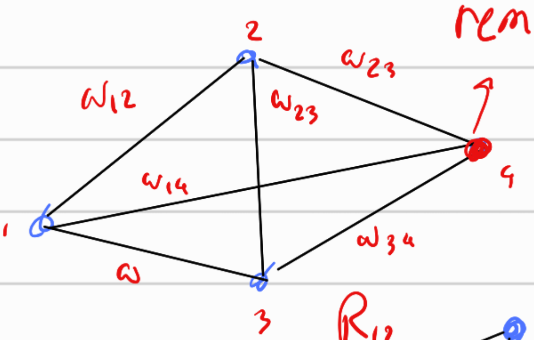
$L^{n\times n} \longrightarrow L'^{(n-1)\times(n-1)}$ (removing node k) : $L'_{su} = L_{su} + \frac{w_{sk}}{d(k)} L_{ku}$

node removal algorithm.

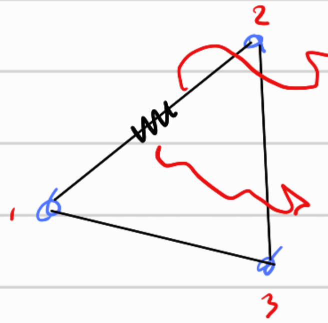
perserves degree

$$\boxed{(L_B)_{su} = L_{su} + \sum_{k\in V/B} \frac{w_{sk}}{d(k)} L_{ku}, \quad s,u \in B}$$

removing



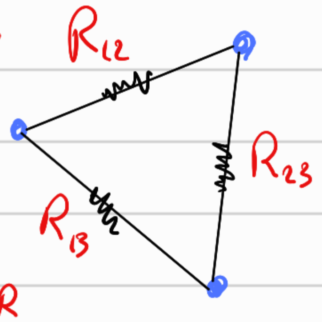
$\rightsquigarrow$



$\overset{\frac{1}{R_{12}}}{w_{12}} + \underbrace{\frac{w_{14}w_{24}}{w_{14}+w_{24}+w_{34}}}_{}$ $\quad \frac{1}{R_{24}R_{14}}(R_{14}\|R_{24}\|R_{34})$

1d

3-nodes :

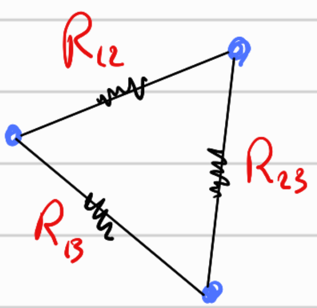


we should prove the triangle inequality for these three nodes.

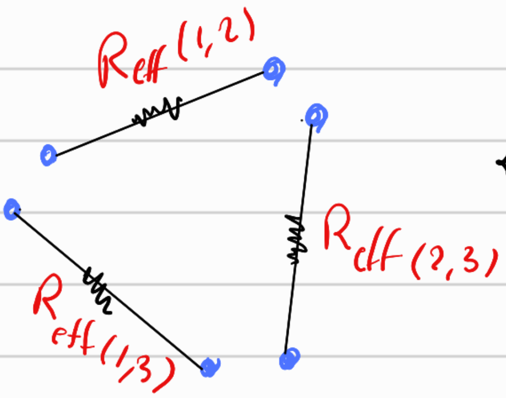
2-nodes :

$$\rightsquigarrow L = \begin{pmatrix} \frac{1}{R} & -\frac{1}{R} \\ -\frac{1}{R} & \frac{1}{R} \end{pmatrix} \rightsquigarrow L^{+} = \begin{pmatrix} R & -R \\ -R & R \end{pmatrix}\frac{1}{4}$$

$R_{eff}(1,2) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}^T \begin{pmatrix} \frac{R}{4} & -\frac{R}{4} \\ -\frac{R}{4} & \frac{R}{4} \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = R$ ✓✓✓ (meets expectations)



$\rightsquigarrow$



$$\left\{ \begin{array}{l} w'_{12} = \overset{1/R_{12}}{w_{12}} + \overset{\frac{1}{R_{23}+R_{13}}}{\dfrac{w_{13}w_{23}}{w_{13}+w_{23}}} \\ \\ \Rightarrow R_{eff(1,2)} = R_{12}\|(R_{13}+R_{23}) \end{array} \right.$$

proves our intuition

$R_{12}\to R_3,\ R_{23}\to R_1,\ R_{13}\to R_2$

$$\left.\begin{array}{l} R_{eq_1} = R_1\|(R_2+R_3) \\ R_{eq2} = R_2\|(R_1+R_3) \\ R_{eq3} = R_3\|(R_1+R_2) \end{array}\right\} \Rightarrow R_{eq1}+R_{eq2} = \frac{R_1(R_2+R_3)}{R_1+R_2+R_3} + \frac{R_2(R_1+R_3)}{R_1+R_2+R_3} = \frac{\overbrace{R_3(R_1+R_2)}^{R_{eq3}} + \overbrace{2R_1R_2}^{\geq 0}}{R_1+R_2+R_3}$$

$\Rightarrow \boxed{R_{eq1} + R_{eq2} \geq R_{eq3} \rightsquigarrow \text{valid metric}}$