پردازش سیگنال گرافی

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دانشگاه صنعتی شریف

مهندسی برق

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تمرین شماره 1

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$$\mu_k = \max_{\substack{S \subseteq \mathbb{R}^n \\ \dim(S)=k}} \min_{\substack{x \in S \\ x \neq 0}} \frac{x^T M x}{x^T x} = \min_{\substack{T \subseteq \mathbb{R}^n \\ \dim(T)=n-k+1}} \max_{\substack{x \in T \\ x \neq 0}} \frac{x^T M x}{x^T x}, \quad \mu_1 \geq \dots \geq \mu_n : \text{eigenvalue of } M$$ (1

الف)

$$\text{let } S = \operatorname{Span}(\{\psi_1, \psi_2, \dots, \psi_k\}) \Rightarrow \forall x \in S : \exists \alpha_i : x = \sum_{i=1}^{k} \alpha_i \psi_i$$

$$\Rightarrow \forall x \in S : \frac{x^T M x}{x^T x} = \frac{(\sum \alpha_i \psi_i)^T M (\sum \alpha_i \psi_i)}{(\sum \alpha_i \psi_i)^T (\sum \alpha_i \psi_i)} = \frac{\sum_{i=1}^{k} \mu_i \alpha_i^2}{\sum_{i=1}^{k} \alpha_i^2}$$

$$\Rightarrow \min_{x \in S} \frac{\sum \mu_i \alpha_i^2}{\sum \alpha_i^2} = \min_{\{\alpha_i\} \in \mathbb{R}^k} \frac{\sum \mu_i \alpha_i^2}{\sum \alpha_i^2} = \min_{i \in \{1..k\}} \mu_i = \mu_k = \min_{x \in S} \frac{x^T M x}{x^T x}$$

$$\Rightarrow \boxed{\text{if } (S = \operatorname{Span}(\{\psi_1, \psi_2, \dots, \psi_k\})) \Rightarrow \mu_k = \min_{x \in S} \frac{x^T M x}{x^T x}}$$

ب)

$$\forall x \in T : \exists \{\alpha_i\}_{i \in \{k..n\}} : x = \sum_{i=k}^{n} \alpha_i \psi_i \Rightarrow \frac{x^T M x}{x^T x} = \frac{\sum \alpha_i \mu_i}{\sum \alpha_i^2}$$

$$\Rightarrow \max_{x \in T} \frac{\sum \alpha_i^2 \mu_i}{\sum \alpha_i^2} = \max_{\{\alpha_i\} \in \mathbb{R}^{n-k+1}} \frac{\sum \alpha_i^2 \mu_i}{\sum \alpha_i^2} = \max_{i \in \{k..n\}} \mu_i = \mu_k = \max_{x \in T} \frac{x^T M x}{x^T x}$$

$$\text{if } \dim(S) = k \Rightarrow \exists v_i : v_i = \psi_{a_i} : S = \operatorname{Span}(\{v_1, v_2, \dots, v_n\}), \ a_i : \{1..k\} \to k \text{ from } \{1..n\}$$

$$\min_{x \in S} \frac{x^T M x}{x^T x} \overset{\text{طبق}}{=} \min_{i \in \{1..k\}} \mu_{a_i}, \quad S = \operatorname{Span}(\{\psi_{a_1}, \psi_{a_2}, \dots, \psi_{a_k}\}), \quad T = \operatorname{Span}(\{\psi_k, \dots, \psi_n\})$$

$$S \cap T = \operatorname{Span}(\{\psi_i : \psi_i \subseteq S, T\}) \neq \phi$$

$$\min_{i \in \{1..k\}} \mu_{a_i} = \mu_{\max_{i \in \{1..k\}}(a_i)}, \quad \max_{i \in \{1..k\}}(a_i) \geq k \Rightarrow \max_{i \in \{1..k\}} \mu_{a_i} \leq \mu_k \Rightarrow \boxed{\min_{x \in S} \frac{x^T M x}{x^T x} \leq \max_{x \in T} \frac{x^T M x}{x^T x}}$$

پ) برای تست کل ها، مشابهی انجام شد

$$\min_{x \in S} \frac{x^T M x}{x^T x} \leq \max_{x \in T} \frac{x^T M x}{x^T x} = \mu_k$$

$$\max_{\substack{S \subseteq \mathbb{R}^n \\ \dim(S)=k}} \min_{\substack{x \in S \\ x \neq 0}} \frac{x^T M x}{x^T x} \leq \mu_k \xrightarrow[S = \operatorname{Span}(\{\psi_1, \dots, \psi_k\})]{\text{تست است}} \boxed{\max_{\substack{S \subseteq \mathbb{R}^n \\ \dim(S)=k}} \min_{\substack{x \in S \\ x \neq 0}} \frac{x^T M x}{x^T x} = \mu_k}$$

(2

$$\gamma_k = \max_{\substack{S \subseteq \mathbb{R}^{n-1} \\ \dim(S)=k}} \ \min_{\substack{x \in S \\ x \neq 0}} \frac{x^T B x}{x^T x}$$

تعریف می‌کنیم که $\tilde{B}$ یک ماتریس $n \times n$ است به طوری که سطر و ستون
حذف شده A که به B رسیده اند، در $\tilde{B}$ برابر با صفر است.
فرض می‌کنیم که سطر و ستون l ام حذف است.

$$\tilde{B}_{ij} = \begin{cases} A_{ij} & : i,j \neq l \\ 0 & : i,j = l \end{cases} \quad , \text{let } \tilde{x}_i = \begin{cases} x_i & : i \neq l \\ \alpha & : i = l \end{cases}$$

$$\Rightarrow \tilde{x}^T \tilde{B} \tilde{x} = [x_1^T \vdots \alpha \vdots x_2^T] \left[\begin{array}{c:c:c} A_1 & \vdots & A_2 \\ \hdashline \cdots & \vdots & \cdots \\ \hdashline A_3 & \vdots & A_4 \end{array}\right] \begin{bmatrix} x_1 \\ \alpha \\ x_2 \end{bmatrix} = [x_1^T , \alpha \vdots x_2] \begin{bmatrix} A_1 x_1 + A_2 x_2 \\ 0 \\ A_3 x_1 + A_4 x_2 \end{bmatrix}$$

(3

$$\Rightarrow \tilde{x}^T \tilde{B} \tilde{x} = x_1^T (A_1 x_1 + A_2 x_2) + x_2^T (A_3 x_1 + A_4 x_2) = x^T B x$$

$$\text{if } \alpha = 0 : \tilde{x}^T A \tilde{x} = [x_1^T \vdots 0 \vdots x_2^T] \left[\begin{array}{c:c:c} A_1 & \vdots & A_2 \\ \hdashline \cdots & \cdots & \cdots \\ \hdashline A_3 & \vdots & A_4 \end{array}\right] \begin{bmatrix} x_1 \\ 0 \\ x_2 \end{bmatrix} \xRightarrow{\text{دو گزاره}} \text{if } \alpha = 0 : \tilde{x}^T \tilde{B} \tilde{x} = \tilde{x}^T A \tilde{x}$$

$$\lambda_k = \max_{\substack{S \subseteq \mathbb{R}^n \\ \dim(S)=k}} \ \min_{\substack{x \in S \\ x \neq 0}} \frac{x^T A x}{x^T x} \geq \max_{\substack{S \subseteq \mathbb{R}^n \\ \dim(S)=k}} \ \min_{\substack{x \in S \\ x \neq 0 \\ x_l = 0}} \frac{x^T A x}{x^T x} = \max_{\substack{S \subseteq \mathbb{R}^n \\ \dim(S)=k}} \ \min_{\substack{x \in S \\ x \neq 0 \\ x_l = 0}} \frac{x^T \tilde{B} x}{x^T x}$$

$$\lambda_k \geq \max_{\substack{S \subseteq \mathbb{R}^n \\ \dim(S)=k}} \ \min_{\substack{x \in S \\ x \neq 0}} \frac{x^T \tilde{B} x}{x^T x} = \max_{\substack{S \subseteq \mathbb{R}^{n-1} \\ \dim(S)=k}} \ \min_{\substack{x \in S \\ x \neq 0}} \frac{x^T B x}{x^T x} = \gamma_k \Rightarrow \boxed{\lambda_k \geq \gamma_k}$$

$$\lambda_k = \min_{\substack{T \in \mathbb{R}^n \\ \dim(T) = n-k+1}} \ \max_{\substack{x \in T \\ x \neq 0}} \frac{x^T A x}{x^T x} \leq \min_{\substack{T \in \mathbb{R}^n \\ \dim(T) = n-k+1}} \ \max_{\substack{x \in T \\ x \neq 0 \\ x_l = 0}} \frac{x^T A x}{x^T x} = \min_{\substack{T \in \mathbb{R}^n \\ \dim(T) = n-k+1}} \ \max_{\substack{x \in T \\ x \neq 0}} \frac{x^T \tilde{B} x}{x^T x}$$

$$= \min_{\substack{T \subseteq \mathbb{R}^{n-1} \\ \dim(T) = n-k}} \ \max_{\substack{x \in T \\ x = 0}} \frac{x^T B x}{x^T x} = \min_{\substack{T \subseteq \mathbb{R}^{n-1} \\ \dim(T) = n-(k-1)+1}} \ \max_{\substack{x \in T \\ x = 0}} \frac{x^T B x}{x^T x} = \boxed{\gamma_{k-1} \geq \lambda_k}$$

$$\Rightarrow \forall k \in [n] : \gamma_k \leq \lambda_k \leq \gamma_{k-1}$$

$$\Rightarrow \boxed{\lambda_1 \geq \gamma_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n-1} \geq \gamma_{n-1} \geq \lambda_n}$$

$A = AB - BA$, $A, B \in \mathbb{C}^{2\times 2}$ $\rightsquigarrow$ prove: $A^2 = 0$ (3

$P(x) = (x-\lambda_1)(x-\lambda_2) \Rightarrow P(A) = (A-\lambda_1 I)(A-\lambda_2 I) = 0$ $\quad : A \begin{cases} \lambda_1, v_1 \\ \lambda_2, v_2 \end{cases}$

$Av_1 = (AB-BA)v_1 = ABv_1 - \lambda_1 Bv_1 = (A-\lambda_1 I)Bv_1$

$(A-\lambda_2 I)Av_1 = \lambda_1(A-\lambda_2 I)v_1 = \overbrace{(A-\lambda_2 I)(A-\lambda_1 I)}^{0}Bv_1 = 0 = \lambda_1(\lambda_1-\lambda_2)v_1$

$\Rightarrow \begin{cases} \lambda_1 = 0 \\ \lambda_1 = \lambda_2 = \lambda \end{cases}$: we can write the same for v_2 $\Rightarrow \begin{cases} \lambda_1 = \lambda_2 \\ \lambda_1 = \lambda_2 = 0 \end{cases} \Rightarrow \boxed{\lambda_1 = \lambda_2 = \lambda}$

$\rightsquigarrow P(A) = (A-\lambda I)^2 = 0$

$\mathrm{Tr}(A) = \mathrm{Tr}(AB-BA) = \mathrm{Tr}(AB) - \mathrm{Tr}(BA) = 0 = \mathrm{Tr}(A)$

$\Rightarrow \lambda_1 + \lambda_2 = 2\lambda = 0 \Rightarrow \boxed{\lambda = 0} \rightsquigarrow \boxed{P(A) = A^2 = 0}$

$$A=\begin{bmatrix} B & C \\ C^T & D \end{bmatrix},\ C\in\mathbb{R}^{n\times m},\ D\in\mathbb{R}^{m\times m},\ B\in\mathbb{R}^{n\times n}$$ (14

(الف

$$\lambda_{min}(A)\leq\frac{x^TAx}{x^Tx}\leq\lambda_{max}(A)\ ,\quad \text{let}\quad x=[x_1^T,x_2^T]^T,\ \|x\|=1$$

$$\Rightarrow\lambda_{min}(A)\leq x^TAx\leq\lambda_{max}(A)\ \Rightarrow\ \lambda_{max}(A)=\max_{\substack{x\in\mathbb{R}^{n+m}\\ \|x\|=1}}x^TAx$$

$$x^TAx=[x_1^T,x_2^T]\begin{bmatrix} B & C \\ C^T & D \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=[x_1^T,x_2^T]\begin{bmatrix} Bx_1+Cx_2 \\ C^Tx_1+Dx_2 \end{bmatrix}=\begin{matrix}x_1^TBx_1+x_1^TCx_2\\+x_2^TC^Tx_1+x_2^TDx_2\end{matrix}$$

$$\Rightarrow x^TAx=x_1^TBx_1+x_2^TDx_2+2x_1^TCx_2=\lambda_{max}(A):\ x=eig_{max}(A)$$

$$\text{if }(x_1=0)\Rightarrow x^TAx=x_2^TDx_2\Rightarrow\lambda_{max}(A)=\max_{x_2\in\mathbb{R}^m}\frac{x_2^TDx_2}{x_2^Tx_2}=\lambda_{max}(D)$$

$$\text{if }(x_2=0)\xrightarrow{\text{به طور مشابه}}\lambda_{max}(A)=\lambda_{max}(B)$$

$$\Rightarrow\lambda_{max}(B)+\lambda_{max}(D)=2\lambda_{max}(A)\geq\lambda_{max}(A)+\lambda_{min}(A)$$

$$\Rightarrow\boxed{\lambda_{max}(A)+\lambda_{min}(A)\leq\lambda_{max}(B)+\lambda_{max}(D)}$$ بازای $x_1=0\vee x_2=0$

$$\text{general}:\ \lambda_{max}(A)=x_1^TBx_1+x_2^TDx_2+2x_1^TCx_2$$ کلی‌تر:

$$\lambda_{min}(A)\leq y^TAy=\left[-\frac{\|x_2\|}{\|x_1\|}x_1^T,\ \frac{\|x_1\|}{\|x_2\|}x_2^T\right]\begin{bmatrix} B & C \\ C^T & D \end{bmatrix}\begin{bmatrix} -\frac{\|x_2\|}{\|x_1\|}x_1 \\ \frac{\|x_1\|}{\|x_2\|}x_2 \end{bmatrix}$$

$$=\left[-\frac{\|x_2\|}{\|x_1\|}x_1^T,\ \frac{\|x_1\|}{\|x_2\|}x_2^T\right]\begin{bmatrix} -\frac{\|x_2\|}{\|x_1\|}Bx_1+\frac{\|x_1\|}{\|x_2\|}Cx_2 \\ -\frac{\|x_2\|}{\|x_1\|}C^Tx_1+\frac{\|x_1\|}{\|x_2\|}Dx_2 \end{bmatrix}=\frac{\|x_2\|^2}{\|x_1\|^2}x_1^TBx_1+\frac{\|x_1\|^2}{\|x_2\|^2}x_2^TDx_2-2x_1^TCx_2$$

$$\Rightarrow\lambda_{min}(A)+\lambda_{max}(A)\leq\frac{\|x_2\|^2}{\|x_1\|^2}x_1^TBx_1+\frac{\|x_1\|^2}{\|x_2\|^2}x_2^TDx_2-\cancel{2x_1^TCx_2}+x_1^TBx_1+x_2^TDx_2+\cancel{2x_1^TCx_2}$$

$$\Rightarrow\lambda_{min}(A)+\lambda_{max}(A)\leq\frac{\overbrace{\|x_1\|^2+\|x_2\|^2}^{1}}{\|x_1\|^2}x_1^TBx_1+\frac{\overbrace{\|x_1\|^2+\|x_2\|^2}^{1}}{\|x_2\|^2}x_2^TDx_2=\overbrace{\frac{x_1^TBx_1}{x_1^Tx_1}}^{\leq\lambda_{max}(B)}+\overbrace{\frac{x_2^TDx_2}{x_2^Tx_2}}^{\leq\lambda_{max}(D)}$$

$$\Rightarrow\boxed{\lambda_{min}(A)+\lambda_{max}(A)\leq\lambda_{max}(B)+\lambda_{max}(D)}$$ $\|x_1\|\neq0\wedge\|x_2\|\neq0$

که قضیه را به طور کلی اثبات می‌کند.

ب) طبق استقرا، اگر فرض کنیم برای k برقرار است:

$$A = \begin{bmatrix} M_{1,1} & M_{1,2} & \cdots & M_{1,k} \\ M_{1,2}^T & M_{2,2} & \cdots & M_{2,k} \\ \vdots & \vdots & \ddots & \vdots \\ M_{1,k}^T & M_{2,k}^T & \cdots & M_{k,k} \end{bmatrix}$$

$$A_k = \begin{bmatrix} M_{1,1} & M_{1,2} & \cdots & M_{1,k} \\ M_{1,2}^T & M_{2,2} & \cdots & M_{2,k} \\ \vdots & \vdots & \ddots & \vdots \\ M_{1,k}^T & M_{2,k}^T & \cdots & M_{k,k} \end{bmatrix}, \quad (k-1)\lambda_{min}(A_k) + \lambda_{max}(A_k) \leq \sum_{i=1}^{k} \lambda_{max}(M_{i,i})$$

$$\Rightarrow \lambda_{max}(A_k) \leq \sum_{i=1}^{k} \lambda_{max}(M_{i,i}) - (k-1)\lambda_{min}(A_k)$$

$$A_{k+1} = \begin{bmatrix} M_{1,1} & M_{1,2} & \cdots & M_{1,k+1} \\ M_{1,2}^T & M_{2,2} & \cdots & M_{2,k+1} \\ \vdots & \vdots & \ddots & \vdots \\ M_{1,k+1}^T & M_{2,k+1}^T & \cdots & M_{k+1,k+1} \end{bmatrix} = \begin{bmatrix} A_k & M_{1,k+1} \\ M_{1,k+1}^T & M_{k+1,k+1} \end{bmatrix} \xrightarrow{\text{از الف}} \lambda_{min}(A_{k+1}) + \lambda_{max}(A_{k+1}) \leq \lambda_{max}(A_k) + \lambda_{max}(M_{k+1,k+1})$$

$$\Rightarrow \lambda_{min}(A_{k+1}) + \lambda_{max}(A_{k+1}) \leq \underbrace{\lambda_{max}(M_{k+1,k+1}) + \sum_{i=1}^{k} \lambda_{max}(M_{i,i})}_{\sum_{i=1}^{k+1} \lambda_{max}(M_{i,i})} - (k-1)\lambda_{min}(A_k)$$

طبق سوال ۲:

$$\lambda_{min}(A_k) \geq \lambda_{min}(A_{k+1})$$

$$\Rightarrow \lambda_{min}(A_{k+1}) + \lambda_{max}(A_{k+1}) \leq \sum_{i=1}^{k+1} \lambda_{max}(M_{i,i}) - (k-1)\lambda_{min}(A_{k+1})$$

قضیه برای $k+1$:

$$\Rightarrow \boxed{k\lambda_{min}(A_{k+1}) + \lambda_{max}(A_{k+1}) \leq \sum_{i=1}^{k+1} \lambda_{max}(M_{i,i})}$$

پس طبق استقرا، قضیه برای $k=2$ که درست است، برای $k+1$ درست است پس قضیه درست است.

(5

$$\det(A)=\sum_{\pi\in S_n}\left(\operatorname{sgn}(\pi)\prod_{i=1}^{n}A(i,\pi(i))\right),\quad A\in\mathbb{R}^n \ : \ \pi:\{1,\dots,n\}$$ جایگشت های با

$$\chi_A(x)=\det(xI-A)=\sum_{k=0}^{n}(-1)^k\sigma_k(A)x^{n-k},\quad \operatorname{sgn}(\pi)=\begin{cases}+1 & \text{even permutation}\\ -1 & \text{odd permutation}\end{cases}$$

prove: $\sigma_k(A)=\sum_{\substack{S\subseteq[n]\\|S|=k}}\prod_{i\in S}\lambda_i=\sum_{\substack{S\subseteq[n]\\|S|=k}}\det(A(S,S))$, $A(S,S)$: choose columns and rows in S

تمام انتخاب های k تایی

ضرب کلی

$$\det(xI-A)=\prod_{i=1}^{n}(x-\lambda_i)=\sum_{k=0}^{n}x^{n-k}\left(\text{جمع تمام ضرب } k \text{ تایی } (-\lambda) \text{ ها}\right)=\sum_{k=0}^{n}x^{n-k}\left\{\sum_{\substack{S\subseteq[n]\\|S|=k}}\prod_{i\in S}(-\lambda)_i\right\}$$

ضرب k تا

$$=\sum(-1)^k x^{n-k}\left\{\underbrace{\sum_{\substack{S\subseteq[n]\\|S|=k}}\prod_{i\in S}\lambda_i}_{\sigma_k(A)}\right\}=\sum(-1)^k x^{n-k}\sigma_k(A)\xrightarrow{\forall x\in\mathbb{C}}\boxed{\sigma_k(A)=\sum_{\substack{S\subseteq[n]\\|S|=k}}\prod_{i\in S}\lambda_i}$$

$$\chi_A(x)=\det(xI-A)=\sum_{k=0}^{n}(-1)^k\sigma_k(A)x^{n-k}=\sum_{\pi\in S_n}\left(\operatorname{sgn}(\pi)\prod_{i=1}^{n}(xI-A)(i,\pi(i))\right)$$

$$\sum_{\pi\in S_n}\left(\operatorname{sgn}(\pi)\prod_{i=1}^{n}\left[x\delta(i,\pi(i))-A(i,\pi(i))\right]\right)$$

$$\prod_{i=1}^{n}\left[x\delta(i,\pi(i))-A(i,\pi(i))\right]=(x\delta(1,\delta(1))-A(1,\delta(1)))\dots(x\delta(n,\delta(n))-A(n,\delta(n)))$$

$$=\sum_{k=1}^{n}x^{n-k}\left[\sum_{\substack{S\subseteq[n]\\|S|=k}}\prod_{i\notin S}\delta(i,\pi(i))\prod_{i\in S}(-A(i,\pi(i)))\right]=\sum_{k=1}^{n}(-1)^k\left[\sum_{\substack{S\subseteq[n]\\|S|=k}}\prod_{i\notin S}\delta(i,\pi(i))\prod_{i\in S}A(i,\pi(i))\right]x^{n-k}$$

$$\chi_A(x)=\sum_{\pi\in S_n}\operatorname{sgn}(\pi)\sum_{k=1}^{n}(-1)^k\left[\sum_{\substack{S\subseteq[n]\\|S|=k}}\prod_{i\notin S}\delta(i,\pi(i))\prod_{i\in S}A(i,\pi(i))\right]x^{n-k}$$

$$=\sum_{k=1}^{n}(-1)^k x^{n-k}\sum_{\pi\in S_n}\operatorname{sgn}(\pi)\left[\sum_{\substack{S\subseteq[n]\\|S|=k}}\prod_{i\notin S}\delta(i,\pi(i))\prod_{i\in S}A(i,\pi(i))\right]$$

$$=\sum_{k=1}^{n}(-1)^k x^{n-k}\left[\sum_{\substack{S\subseteq[n]\\|S|=k}}\sum_{\pi\in S_n}\operatorname{sgn}(\pi)\prod_{i\notin S}\delta(i,\pi(i))\prod_{i\in S}A(i,\pi(i))\right]$$

$\pi_S=\{\pi:\forall i\notin S:\pi(i)=i\}$

جمع را روی π_S انجام میدیم.

$$\sum_{k=1}^{n}(-1)^k x^{n-k}\left[\sum_{\substack{S\subseteq[n]\\|S|=k}}\sum_{\pi\in\pi_S}\operatorname{sgn}(\pi)\prod_{i\in S}A(i,\pi(i))\right]=\sum_{k=1}^{n}(-1)^k x^{n-k}\left[\sum_{\substack{S\subseteq[n]\\|S|=k}}\underbrace{\sum_{\pi\in\pi_S}\operatorname{sgn}(\pi)\prod_{i=1}^{k}A(S(i),\pi(S(i)))}_{\det(A(S,S))}\right]$$

$$=\sum_{k=1}^{n}(-1)^k x^{n-1}\left[\underbrace{\sum_{\substack{S\subseteq n\\|S|=k}}\det(A(S,S))}_{\sigma_k(A)}\right]\Rightarrow\boxed{\sigma_k(A)=\sum_{\substack{S\subseteq[n]\\|S|=k}}\det(A(S,S))}$$

سوال 6)

$A = A^T,\ \det(B) \neq 0,\ C = BAB^T$

الف)

$\forall v \in N(C) : Cv = 0 \Rightarrow BAB^Tv = 0 \Rightarrow A(B^Tv) = 0\ ,\ \forall v \neq 0 : B^Tv \neq 0$

$\Rightarrow \forall v \in N(BAB^T) : \exists w = B^Tv \neq 0 : Aw = 0 \Rightarrow \boxed{\dim(N(BAB^T)) = \dim(N(A))}$

$\Rightarrow B^T : N(BAB^T) \to N(A) : 1 \to 1 \text{ mapping}$

ب)

$\gamma_1, \dots, \gamma_n$: positive eigenvalues of C, Γ_k : subspace of $v_1, \dots, v_k$

$\Lambda_k = \{ B^Ty : y \in \Gamma_k^c \}$

$$\lambda_k = \max_{\substack{S \subseteq \mathbb{R}^n \\ \dim(S)=k}} \min_{\substack{x \in S \\ x \neq 0}} \frac{x^TAx}{x^Tx} \geq \min_{\substack{x \in \Lambda_k \\ x \neq 0}} \frac{x^TAx}{x^Tx} = \min_{\substack{y \in \Gamma_n^c \\ y \neq 0}} \frac{(B^Ty)^TA(B^Ty)}{\|B^Ty\|^2} \leq \lambda_k$$

I

$$\min_{\substack{x \in \mathbb{R}^n \\ x \neq 0}} \frac{x^TBAB^Tx}{x^Tx} = \min_{i \in [n]} \gamma_i > 0 \Rightarrow \forall x \in \mathbb{R}^n, x \neq 0 : x^TBAB^Tx > 0$$

$$\text{I},\ \|B^Ty\|^2 \geq 0 \Rightarrow \lambda_k \geq \min_{\substack{y \in \Gamma_n^c \\ y \neq 0}} \frac{\overbrace{(B^Ty)^TA(B^Ty)}^{\geq 0}}{\underbrace{\|B^Ty\|^2}_{\geq 0}} \geq 0 \Rightarrow \boxed{\lambda_k \geq 0}$$

مقدار k امین λ مثبت

پ)

$\gamma_1, \dots, \gamma_n$: negative eigenvalues of C, Γ_k : subspace of $v_1, \dots, v_k$

$\Lambda_k = \{ B^Ty : y \in \Gamma_k^c \}$

$$\lambda_{n+1-k} = \min_{\substack{T \in \mathbb{R}^n \\ \dim(S)=k}} \max_{\substack{x \in T \\ x \neq 0}} \frac{x^TAx}{x^Tx} \leq \max_{\substack{x \in \Lambda_k \\ x \neq 0}} \frac{x^TAx}{x^Tx} = \max_{\substack{y \in \Gamma_n^c \\ y \neq 0}} \frac{(B^Ty)^TA(B^Ty)}{\|B^Ty\|^2} \geq \lambda_{n+1-k}$$

I

$$\max_{\substack{x \in \mathbb{R}^n \\ x \neq 0}} \frac{x^TBAB^Tx}{x^Tx} = \max_{i \in [n]} \gamma_i < 0 \Rightarrow \forall x \in \mathbb{R}^n, x \neq 0 : x^TBAB^Tx < 0$$

$$\text{I},\ \|B^Ty\|^2 \geq 0 \Rightarrow \lambda_{n+1-k} \geq \min_{\substack{y \in \Gamma_n^c \\ y \neq 0}} \frac{\overbrace{(B^Ty)^TA(B^Ty)}^{\leq 0}}{\underbrace{\|B^Ty\|^2}_{\geq 0}} \leq 0 \Rightarrow \boxed{\lambda_{n-k+1} \geq 0}$$

مقدار k امین λ منفی

$n_+ + n_- + n_0 = n$

$$\left.\begin{array}{l} n_+(A) \geq n_+(BAB^T) \\ n_-(A) \geq n_-(BAB^T) \\ n_0(A) = n_0(B) \end{array}\right\} \Rightarrow \boxed{\begin{array}{l} n_+(A) = n_+(BAB^T) \\ n_-(A) = n_-(BAB^T) \\ n_0(A) = n_0(BAB^T) \end{array}}$$

اینرسی یکی

سوال ۳

$A \preceq B \iff \forall x \in \mathbb{R}^n : x^T(B-A)x \geq 0 \Rightarrow \forall x \in \mathbb{R}^n : x^T B x \geq x^T A x$

$$\lambda_k(A) = \max_{\substack{S \subseteq \mathbb{R}^n \\ \dim(S)=k}} \min_{\substack{x \in S \\ x \neq 0}} \frac{x^T A x}{x^T x} \leq \max_{\substack{S \subseteq \mathbb{R}^n \\ \dim(S)=k}} \min_{\substack{x \in S \\ x \neq 0}} \frac{x^T B x}{x^T x} = \lambda_k(B) \Rightarrow \boxed{\lambda_k(A) \leq \lambda_k(B)}$$