Combinatorics 2018 HW 5.1

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1. A basket of fruit is being arranged out of apples, bananas, and oranges. What is the smallest number of pieces of fruits that should be put in the basket in order to guarantee that either there are at least 8 apples or at least 6 bananas or at least 9 oranges?

Let us denote the number of apples with n_a, the number of bananas with n_b, and the number of oranges with n_o. Our goal is to find the smallest n

$$n = n_a + n_b + n_o$$

such that

$$n_a \geq 8 \quad \vee \quad n_b \geq 6 \quad \vee \quad n_o \geq 9$$

The solution is to simply satisfy the smallest constraint, i.e.

$$n_a = 0 \quad n_b = 6 \quad n_o = 0$$

Thus the smallest number of pieces of fruit is 6, for which we need to put 0 apples, 6 bananas, and 0 oranges into the basket.

If we assume that while picking fruits, we cannot know what fruit we pick next, then we have to use the *pigeonhole principle*. The extreme case that does not satisfy our constraints is

$$n_a = 7 \quad n_b = 5 \quad n_o = 8 \quad n = 20$$

If we pick one more fruit, we are guaranteed to have chosen either 8 apples or 6 bananas or 9 oranges (or more). **Thus, in this case, the answer is 21.**

2. Show that for any given 52 integers there exists two of them whose sum, or else whose difference, is divisible by 100.

$\forall \{a_1, \dots, a_{52} \mid a_i \in \mathbb{Z}\}\colon\ \exists\ a_i, a_j\ \ (i \neq j)\colon\ a_i + a_j = 100k\ \lor\ a_i - a_j = 100k,\ k \in \mathbb{Z}.$

Let us regard the possible remainders w.r.t. 100 in pairs:

$$(0,100)$$

$$(1,99)$$

$$(2,98)$$

$$\dots$$

$$(r, 100 - r)$$

$$\dots$$

$$(50,50)$$

We have 51 such pairs, from which we want to choose 52 elements. As the *pigeonhole principle* states, there is going to be at least one pair that we chose multiple times. Let this pair be denoted by $(r, 100 - r)$. There are two possibilities:

a) We chose r (or $100 - r$) twice. That means that there are (at least) two numbers that have the same remainder w.r.t 100. The difference of these numbers is divisible by 100.

$$a_i = 100k_i + r \quad a_j = 100k_j + r \quad a_i - a_j = 100(k_i - k_j)$$

b) We chose both r and $(100 - r)$. The sum of these numbers if divisible by 100.

$$a_i = 100k_i + r \quad a_j = 100k_j + 100 - r \quad a_i + a_j = 100(k_i + k_j + 1)$$

Thus, regardless of how we choose 52 elements, there is guaranteed to be two of them whose difference or sum is divisible by 100.