**HW W6-2:**

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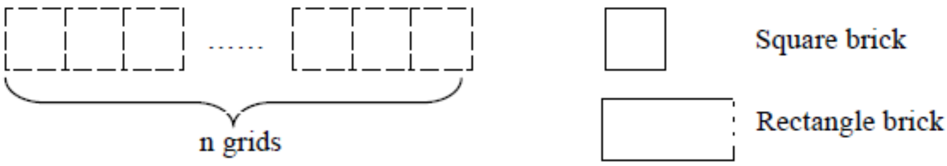
1. A worker is tiling a road with n square grids, there are two different kinds of bricks available, one is the square brick and the other is the rectangle brick which covers 2 grids. Please count the total different ways the road can be tiled.




Let $T(n)$ denote the solution for $n$, i.e. the number of tilings for $n$ square grids. We can directly write a recursive formula for $T(n)$ as:

$$T(n) = T(n-1) + T(n-2) \qquad T(0) = 1 \quad T(1) = 1$$

The intuition behind this is:

a) If we already have a valid tiling of $n-1$ grids, there is only one way to finish it: use a single square brick.
b) If we already have a valid tiling of $n-2$ grids, there are two ways to finish it: we either use two square bricks, or a single rectangle brick. The latter case is included in a) so we do not have to count it here.

Let us first rearrange the formula:

$$T(n) - T(n-1) - T(n-2) = 0$$

This is a linear homogeneous recurrence relation. The characteristic equation is

$$q^2 - q - 1 = 0 \quad \rightarrow \; q_1 = \frac{1-\sqrt{5}}{2} \;\; q_2 = \frac{1+\sqrt{5}}{2}$$

Our candidate closed-form formula for $T(n)$ becomes:

$$T(n) = c_1 \cdot \left(\frac{1-\sqrt{5}}{2}\right)^n + c_2 \cdot \left(\frac{1+\sqrt{5}}{2}\right)^n$$

Let us use the initial conditions to derive the values of these constants:

$$\left.\begin{aligned} 1 &= c_1 + c_2 \\ 1 &= \left(\frac{1-\sqrt{5}}{2}\right) \cdot c_1 + \left(\frac{1+\sqrt{5}}{2}\right) \cdot c_2 \end{aligned}\right\} \rightarrow \quad \begin{aligned} c_1 &= \frac{5-\sqrt{5}}{10} \\ c_2 &= \frac{5+\sqrt{5}}{10} \end{aligned}$$

Plugging this back into our candidate formula, we get:

$$T(n) = \left(\frac{5-\sqrt{5}}{10}\right) \cdot \left(\frac{1-\sqrt{5}}{2}\right)^n + \left(\frac{5+\sqrt{5}}{10}\right) \cdot \left(\frac{1+\sqrt{5}}{2}\right)^n$$

**The formula above gives us the number of ways a grid of size $n$ can be tiled.**

**2. How many different ways to color n grids in a line with red, white or blue colors but no two adjacent grids are colored with red?**

Let $X(n)$ denote the solution for $n$, i.e. the number of ways to color $n$ grids in a line with the colors R, B, W s.t. no two adjacent grids are colored R.

Let $R(n)$ denote the number of valid colorings for $n$ grids ending with R.
Let $BW(n)$ denote the number of valid colorings for $n$ grids ending with B or W.

We know that

$$X(n) = R(n) + BW(n)$$

Given a coloring for a grid of size $n-1$, we can construct a coloring for a grid of size $n$: if the last square is R, we have two options (B, W); if the last square is B or W, we have three options (R, B, W).

Using these insights, we can construct $X(n)$ by partitioning $X(n-1)$ into solutions ending with R and ones ending with B or W:

$$X(n) = 2 \cdot R(n-1) + 3 \cdot BW(n-1)$$

A solution for n grids ending in R is constructed by getting the solutions for $n-1$ ending in B or W and appending an R:

$$R(n) = 1 \cdot BW(n-1)$$

A solution for n grids ending in B/W is constructed by getting the solutions for $n-1$ and appending a B or W:

$$BW(n) = 2 \cdot X(n-1)$$

Combining the 3 equations above, we get the linear homogeneous recurrence relation

$$X(n) - 6 \cdot X(n-2) - 4 \cdot X(n-3) = 0 \quad \begin{cases} X(0) = 1 \\ X(1) = 3 \\ X(2) = 8 \end{cases}$$

… whose characteristic equation is

$$q^3 - 6q - 4 = 0 \quad \begin{cases} q_1 = -2 \\ q_2 = 1 - \sqrt{3} \\ q_3 = 1 + \sqrt{3} \end{cases}$$

Now, we have our new candidate

$$X(n) = c_1(-2)^n + c_2\left(1-\sqrt{3}\right)^n + c_3\left(1+\sqrt{3}\right)^n$$

Using the initial conditions, we can derive the values of the coefficients:

$$\left.\begin{aligned} 1 &= c_1 + c_2 + c_3 \\ 3 &= -2\cdot c_1 + \left(1-\sqrt{3}\right)\cdot c_2 + \left(1+\sqrt{3}\right)\cdot c_3 \\ 8 &= 4\cdot c_1 + \left(1-\sqrt{3}\right)^2\cdot c_2 + \left(1+\sqrt{3}\right)^2\cdot c_3 \end{aligned}\right\} \rightarrow \begin{cases} c_1 = 0 \\ c_2 = \dfrac{\sqrt{3}-2}{2\sqrt{3}} \\ c_3 = \dfrac{\sqrt{3}+2}{2\sqrt{3}} \end{cases}$$

Thus, the closed-form solution for $X(n)$ is:

$$X(n) = \frac{\sqrt{3}-2}{2\sqrt{3}}\left(1-\sqrt{3}\right)^n + \frac{\sqrt{3}+2}{2\sqrt{3}}\left(1+\sqrt{3}\right)^n$$

**Using this formula, we can directly calculate the number of valid colorings for a grid of size n.**