HW w7-1

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1. Using 5 numbers 1，2，3，4, 5 to fill in 1x n grids, each grid is filled with one digit. If there are odd number of grids that have 1 written on them, and an even number of grids with 2, please write the corresponding exponential generating function and figure out how many arrangements there for 1*6 grids?____

Basically, we need to count all n-digit numbers that have

1. an odd number of 1's,
2. an even number of 2's,
3. any number of 3-4-5's.

When talking about such numbers (grids), the order of the digits matter, i.e. we are dealing with permutations. Let us use exponential generating functions to count them.

$$G_e(x) = \left(\frac{x}{1!} + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\right)\left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right)\left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots\right)^3$$

The first term corresponds to 1., the second one to 2., the third one to 3. We know that

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \qquad e^{-x} = 1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots$$

… thus

$$\frac{x}{1!} + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots = \frac{1}{2}(e^x - e^{-x})$$

… and

$$1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots = \frac{1}{2}(e^x + e^{-x})$$

Using these

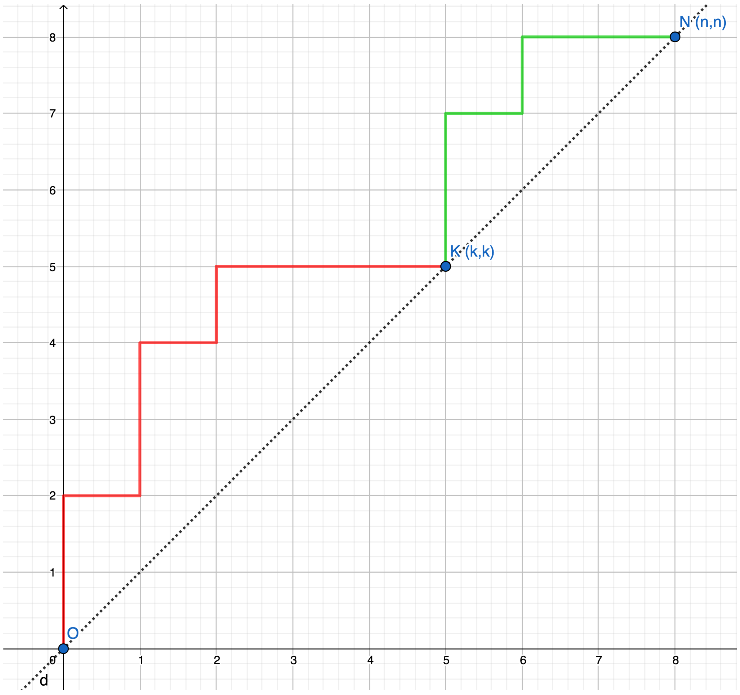
$$G_e(x) = \frac{1}{2}(e^x - e^{-x})\frac{1}{2}(e^x + e^{-x})e^{3x} = \frac{1}{4}(e^{2x} - e^{-2x})e^{3x} = \frac{1}{4}(e^{5x} - e^x) =$$

$$= \frac{1}{4}\left(\sum_{n=0}^{\infty}\frac{5^n x^n}{n!} - \sum_{n=0}^{\infty}\frac{x^n}{n!}\right) = \sum_{n=0}^{\infty}\left(\frac{5^n - 1}{4}\right)\frac{x^n}{n!} \;\rightarrow\; a_n = \frac{5^n - 1}{4}$$

For $n = 6$, there are $\frac{5^6-1}{4} = 3906$ such grids.

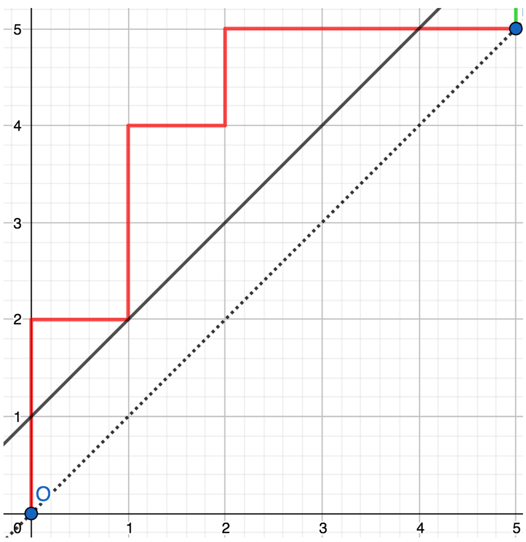
2. There are six people in a library queuing up, three of them want to return the book "Interviewing Skills", and 3 of them want to borrow the same book. If at the beginning, all the books of "Interviewing Skills" are out of stock in the library, how many ways can these people line up?____

The number of ways these people can line up properly corresponds to a Dyck path of length 6. Let us derive the number of Dyck paths for the general case.



Every Dyck-path corresponds to one lattice path from $(0{,}0)$ to (n,n) that can touch but does not go below the line $y = x$. Let $D(n)$ denote the number of such paths.

Let $K = (k,k)$ be the point where the path first touches the dotted line. Path segments from O to K that do not touch the dotted line at any other point correspond to *shifted* Dyck-paths from $(0{,}1)$ to $(k-1,k)$:



The number of such Dyck-paths is

$$1 \cdot D(k-1) \cdot 1$$

The number of line segments from K to N (potentially touching the line at other points too) is

$$D(n-k)$$

Thus, using the *multiplication principle*, we can get the number of Dyck-paths from O to N that touch the line at K for the first time

$$D(k-1) \cdot D(n-k)$$

Now let us go back to the problem of counting all Dyck-paths from O to N. We can partition these paths based on where they touch the dotted line for the first time (K). Then we can use the *addition principle* to arrive at the final count:

$$D(n) = \sum_{k=1}^{n} D(k-1) \cdot D(n-k) = D(0) \cdot D(n-1) + \cdots + D(n-1) \cdot D(0)$$

where $D(0) = 1$. This is exactly the formula for the Catalan numbers. Next, let us derive the closed-form solution for the Catalan numbers.

Let us consider the following generator function for the Catalan numbers

$$c(x) = \sum_{n=0}^{\infty} C_n x^n = C_0 + C_1 x + C_2 x^2 + \cdots$$

By taking the second power of this, we get

$$\begin{aligned} c^2(x) &= (C_0 C_0) + (C_1 C_0 + C_0 C_1)x + (C_2 C_0 + C_1 C_1 + C_0 C_2)x^2 + \cdots = \\ &= C_1 + C_2 x + C_3 x^2 + \cdots \end{aligned}$$

Let us subtract these two functions

$$c(x) - x \cdot c^2(x) = C_0 \quad \rightarrow \quad x \cdot c^2(x) - c(x) + 1 = 0$$

Solving this quadratic equation for $c(x)$, we get

$$c(x) = \frac{1 \pm \sqrt{1-4x}}{2x}$$

With Taylor-series approximation, we get

$$c(x) = \sum_{n=0}^{\infty} \binom{2n}{n} \frac{x^n}{n+1}$$

From which we can find the n'th Catalan number by finding the coefficient of x^n:

$$C_n = \binom{2n}{n}\frac{1}{n+1}$$

Getting back to the library question: If we do not differentiate between the individuals, the number of ways they can line up properly corresponds to a Dyck path of length 6, i.e. $D(3)$.

$$D(3) = C_3 = \binom{6}{3}\frac{1}{4} = 5$$

Taking their order into account two, we need to calculate the permutations of two groups of 3 people each:

$$D(3) \cdot 3! \cdot 3! = 5 \cdot 3! \cdot 3! = 180$$

Thus, those 6 people can line up 180 or 5 different ways, depending whether we take the order into account or not.