Combinatorics 2017 HW 1009

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1. how many integers from 1 to 10000 are not squares of integers or cubes of integers?

First, let us define some sets:

- Let $U = \{1, 2, \dots, 10000\}$ be the universal set.
- Let $A = \{x \mid x \in U,\ \exists\, y \in \mathbb{Z}: y^2 = x\}$ the set of squares within U.
- Let $B = \{x \mid x \in U,\ \exists\, y \in \mathbb{Z}: y^3 = x\}$ the set of cubes within U.

Our goal is to find $\left|\overline{A} \cap \overline{B}\right|$. Using the *inclusion-exclusion principle*:

$$\left|\overline{A} \cap \overline{B}\right| = |U| - |A \cup B| = |U| - |A| - |B| + |A \cap B|$$

The cardinality of these sets:

$$|U| = 10000 \quad |A| = \left\lfloor\sqrt{10000}\right\rfloor = 100 \quad |B| = \left\lfloor\sqrt[3]{10000}\right\rfloor = 21 \quad |A \cap B| = \left\lfloor\sqrt[6]{10000}\right\rfloor = 4$$

(For x^n to be both a square and a cube, n has to be divisible by both 2 and 3, i.e. $x^n = x^{6k}$.)

Substituting these values, we get

$$\left|\overline{A} \cap \overline{B}\right| = 10000 - 100 - 21 + 4 = 9883$$

There are 9883 integers from 1 to 10000 that are neither squares nor cubes of integers.

2. How many permutations of 1, 2, 3，……，9 have at least one odd number in its natural position?

Let the property P_i denote that the number i is in its natural position. Let A_i denote the set of permutations satisfying P_i. Our goal is to find the number of permutations satisfying at least one of the properties $\{P_i\}_{i=1,3,5,7,9}$

$$x = |A_1 \cup A_3 \cup A_5 \cup A_7 \cup A_9|$$

To satisfy P_i, the number i has to be in its natural position, and the rest of the numbers can be arranged in any way:

$$|A_i| = 1 \cdot (9-1)! = 8! \qquad \sum_{i \in \{1,3,5,7,9\}} |A_i| = C(5,1) \cdot 8!$$

For satisfying two properties P_i and P_j, the numbers i and j have to be in their natural positions, while the rest can be arranged in any way:

$$\left|A_i \cap A_j\right| = 1 \cdot (9-2)! = 7! \qquad \sum_{\substack{i,j \in \{1,3,5,7,9\} \\ i \neq j}} \left|A_i \cap A_j\right| = C(5,2) \cdot 7!$$

The same method can be extended to more constraints. For all $\{P_i\}_{i=1,3,5,7,9}$ properties, we get:

$$\left|A_i \cap A_j \cap A_k \cap A_l \cap A_m\right| = 1 \cdot (9-5)! = 4! \qquad \sum_{i,j,k,l,m} \left|A_i \cap A_j \cap A_k \cap A_l \cap A_m\right| = C(5,5) \cdot 4!$$

Having defined the cardinality of these sets, we can now calculate our final value using *IEP*:

$$x = \sum_i |A_i| - \sum_{i,j} \left|A_i \cap A_j\right| + \cdots + \sum_{i,j,k,l,m} \left|A_i \cap A_j \cap A_k \cap A_l \cap A_m\right| =$$
$$= C(5,1) \cdot 8! - C(5,2) \cdot 7! + C(5,3) \cdot 6! - C(5,4) \cdot 5! + C(5,5) \cdot 4! = 157{,}824$$

Thus, there are 157,824 permutations of the numbers 1, 2, …, 9 that have at least one odd number in its natural position.

3. $x_1 + x_2 + x_3 + x_4 = 20, where\ 1 \le x_1 \le 6, 0 \le x_2 \le 7, 4 \le x_3 \le 8, 2 \le x_4 \le 6$ please calculate the number of integral solutions.

Let us reformulate the equation:

$$y_1 + y_2 + y_3 + y_4 = 13 \quad \text{where} \quad \begin{cases} 0 \le y_1 \le 5 \\ 0 \le y_2 \le 7 \\ 0 \le y_3 \le 4 \\ 0 \le y_4 \le 4 \end{cases}$$

Let P_i denote the property that the constraint for y_i is violated. Let A_i denote the set of integral solutions with property P_i. Our target is

$$x = \left|\overline{A_1} \cap \overline{A_2} \cap \overline{A_3} \cap \overline{A_4}\right| = |S| - |A_1 \cup A_2 \cup A_3 \cup A_4|$$

When counting the number of solutions violating P_i, we can replace y_i with $(z_i + c_i + 1)$ where c_i is the upper constraint associated with y_i and $z_i \ge 0$. The solutions of the resulting equations (without taking any upper bound into account) are exactly the solutions of the original equation violating P_i.

$$\begin{array}{rlrl}
- & y_1 + y_2 + y_3 + y_4 = 13 & |S| & = C(16,3) \\
P_1 & z_1 + y_2 + y_3 + y_4 = 13 - 6 = 7 & |A_1| & = C(10,3) \\
P_2 & y_1 + z_2 + y_3 + y_4 = 13 - 8 = 5 & |A_2| & = C(8,3) \\
P_3 & y_1 + y_2 + z_3 + y_4 = 13 - 5 = 8 & |A_3| & = C(11,3) \\
P_4 & y_1 + y_2 + y_3 + z_4 = 13 - 5 = 8 & |A_4| & = C(11,3) \\
P_1 \land P_2 & z_1 + z_2 + y_3 + y_4 = 13 - 6 - 8 = -1 & |A_1 \cap A_2| & = 0 \\
P_1 \land P_3 & z_1 + y_2 + z_3 + y_4 = 13 - 6 - 5 = 2 & |A_1 \cap A_3| & = C(5,3) \\
P_1 \land P_4 & z_1 + y_2 + y_3 + z_4 = 13 - 6 - 5 = 2 & |A_1 \cap A_4| & = C(5,3) \\
P_2 \land P_3 & y_1 + z_2 + z_3 + y_4 = 13 - 8 - 5 = 0 & |A_2 \cap A_3| & = C(3,3) \\
P_2 \land P_4 & y_1 + z_2 + y_3 + z_4 = 13 - 8 - 5 = 0 & |A_2 \cap A_4| & = C(3,3) \\
P_3 \land P_4 & y_1 + y_2 + z_3 + z_4 = 13 - 5 - 5 = 3 & |A_3 \cap A_4| & = C(6,3) \\
P_1 \land P_2 \land P_3 & z_1 + z_2 + z_3 + y_4 = 13 - 6 - 8 - 5 = -6 & |A_1 \cap A_2 \cap A_3| & = 0 \\
P_1 \land P_2 \land P_4 & z_1 + z_2 + y_3 + z_4 = 13 - 6 - 8 - 5 = -6 & |A_1 \cap A_2 \cap A_4| & = 0 \\
P_1 \land P_3 \land P_4 & z_1 + y_2 + z_3 + z_4 = 13 - 6 - 5 - 5 = -3 & |A_1 \cap A_3 \cap A_4| & = 0 \\
P_2 \land P_3 \land P_4 & y_1 + z_2 + z_3 + z_4 = 13 - 8 - 5 - 5 = -5 & |A_2 \cap A_3 \cap A_4| & = 0 \\
P_1 \land P_2 \land P_3 \land P_4 & z_1 + z_2 + z_3 + z_4 = 13 - 8 - 6 - 5 - 5 = -10 & |A_1 \cap A_2 \cap A_3 \cap A_4| & = 0
\end{array}$$

From this, we can get the final result using IEP:

$$\begin{array}{lllllllllll}
C(16,3) \\
- & C(10,3) & - & C(8,3) & - & C(11,3) & - & C(11,3) \\
+ & 0 & + & C(5,3) & + & C(5,3) & + & C(3,3) & + & C(3,3) & + \; C(6,3) \\
- & 0 & - & 0 & - & 0 & - & 0 \\
+ & 0 \\
= 96
\end{array}$$

There are 96 integral solutions to the given equation with the given constraints.

4. For the permutation P=P1 P2 P3 P4 of {1,2,3,4}, how many feasible permutations are there if we constrain that P1≠2，P2≠2、3， P3≠3、4， P4≠4? (4 points)

Let A_i denote the set of permutations violating the constraint for P_i. Our goal is to find

$$n = |\overline{A_1} \cap \overline{A_2} \cap \overline{A_3} \cap \overline{A_4}| = |S| - |A_1 \cup A_2 \cup A_3 \cup A_4|$$

Let us count the possibilities:

$$\begin{array}{rlrl}
A_1 & \underline{2}\,\underline{a}\,\underline{b}\,\underline{c} & |A_1| & = 1 \cdot 3! = 6 \\
A_2 & \underline{a}\,\underline{2}\,\underline{b}\,\underline{c} \;\; \underline{a}\,\underline{3}\,\underline{b}\,\underline{c} & |A_2| & = 2 \cdot 3! = 12 \\
A_3 & \underline{a}\,\underline{b}\,\underline{3}\,\underline{c} \;\; \underline{a}\,\underline{b}\,\underline{4}\,\underline{c} & |A_3| & = 2 \cdot 3! = 12 \\
A_4 & \underline{a}\,\underline{b}\,\underline{c}\,\underline{4} & |A_4| & = 1 \cdot 3! = 6 \\
A_1 \cap A_2 & \underline{2}\,\underline{3}\,\underline{a}\,\underline{b} & |A_1 \cap A_2| & = 1 \cdot 2! = 2 \\
A_1 \cap A_3 & \underline{2}\,\underline{a}\,\underline{3}\,\underline{b} \;\; \underline{2}\,\underline{a}\,\underline{4}\,\underline{b} & |A_1 \cap A_3| & = 2 \cdot 2! = 4 \\
A_1 \cap A_4 & \underline{2}\,\underline{a}\,\underline{b}\,\underline{4} & |A_1 \cap A_4| & = 1 \cdot 2! = 2 \\
A_2 \cap A_3 & \underline{a}\,\underline{2}\,\underline{3}\,\underline{b} \;\; \underline{a}\,\underline{2}\,\underline{4}\,\underline{b} \;\; \underline{a}\,\underline{3}\,\underline{4}\,\underline{b} & |A_2 \cap A_3| & = 3 \cdot 2! = 6 \\
A_2 \cap A_4 & \underline{a}\,\underline{2}\,\underline{b}\,\underline{4} \;\; \underline{a}\,\underline{3}\,\underline{b}\,\underline{4} & |A_2 \cap A_4| & = 2 \cdot 2! = 4 \\
A_3 \cap A_4 & \underline{a}\,\underline{b}\,\underline{3}\,\underline{4} & |A_3 \cap A_4| & = 1 \cdot 2! = 2 \\
A_1 \cap A_2 \cap A_3 & \underline{2}\,\underline{3}\,\underline{4}\,\underline{a} & |A_1 \cap A_2 \cap A_3| & = 1 \\
A_1 \cap A_2 \cap A_4 & \underline{2}\,\underline{3}\,\underline{a}\,\underline{4} & |A_1 \cap A_2 \cap A_4| & = 1 \\
A_1 \cap A_3 \cap A_4 & \underline{2}\,\underline{a}\,\underline{3}\,\underline{4} & |A_1 \cap A_3 \cap A_4| & = 1 \\
A_2 \cap A_3 \cap A_4 & \underline{a}\,\underline{2}\,\underline{3}\,\underline{4} & |A_2 \cap A_3 \cap A_4| & = 1 \\
A_1 \cap A_2 \cap A_3 \cap A_4 & - & |A_1 \cap A_2 \cap A_3 \cap A_4| & = 0
\end{array}$$

From this, we can get the final result using *IEP*:

$$n = 4! - 3!\,(1 + 2 + 2 + 1) + 2!\,(1 + 2 + 1 + 3 + 2 + 1) - 4 + 0 = 4$$

… i.e. there are 4 permutations that satisfy the given constraints.

Using the permutation generator from project 2, we can check the result:

```
bool check_constraints(const std::vector<int>& vec) {
    return (vec[0] != 2) && (vec[1] != 2 && vec[1] != 3) &&
           (vec[2] != 3 && vec[2] != 4) && (vec[3] != 4);
}

int main() {
    Lexicographic<int> g = {1, 2, 3, 4};
    do {
      if (check_constraints(g.get())) print_vector(g.get());
    } while (g.next());
}
```

The output is the 4 permutations: `1 4 2 3`, `3 4 1 2`, `3 4 2 1`, `4 1 2 3`.