Combinatorics HW 5-2

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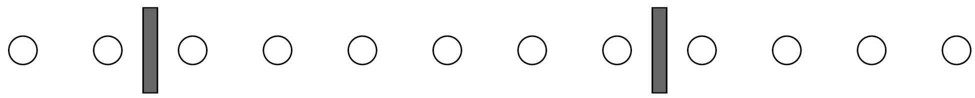
1. **Integer composition: Integer 5 is partitioned into orderly partitions which are made up by numbers 1,2,3,4. Such as (1+1+3, or 1+3+1 or 2+3, 4+1,....) How many different ways are there?**

Let us first discuss the problem of partitioning n into k partitions. An equivalent problem is: Given n identical balls, how do we partition them into k groups? We can get the answer by placing $k-1$ bars into the $n-1$ gaps between the balls:



We can easily calculate the number of ways to do this:

$$p = C(n-1, k-1)$$

Now let us go back to the original question. When partitioning the number 5, we can partition it into 2, 3, 4, or 5 groups (1 is not considered as we cannot use the element 5). These are independent cases, so we have to use the *addition principle* to get the final result:

$$p_5 = \sum_{k=2}^{5} C(5-1, k-1) = C(4,1) + C(4,2) + C(4,3) + C(4,4) = 15$$

Thus, there are 15 ways to partition 5 into orderly partitions of 1, 2, 3, 4.

2. **Integer partition: How many ways to partition n into several numbers that the order between numbers is ignored. Please write the corresponding generating function.**

Ways of choosing number 1:

$$1 + x + x^2 + x^3 + \cdots$$

Ways of choosing number 2:

$$1 + x^2 + x^4 + x^6 + \cdots$$

… etc. These can be viewed as steps, so we can use the *multiplication principle* to get the corresponding generating function:

$$G(x) = (1 + x + x^2 + \cdots)(1 + x^2 + x^4 + \cdots) \ldots = \prod_{k=1}^{\infty}\left(\sum_{i=0}^{\infty} x^{k \cdot i}\right)$$

Alternatively, we can represent these terms as fractions:

$$G(x) = \frac{1}{1-x} \cdot \frac{1}{1-x^2} \cdot \cdots = \prod_{k=1}^{\infty}\left(\frac{1}{1-x^k}\right)$$

To get the partition number of n, we need to find the coefficient of x^n.

(Note: In practice, it's enough to go until $k = n$, as for partitioning n we certainly cannot use numbers larger than it.)

3. **Provide proof that the partition number of the summation of the partitioning of integer n into odd numbers, is equaled to the partition number of n being partitioned into the self-conjugated Ferrers Diagram.**

Let A denote the set of partitions of n consisting of distinct odd numbers only.
Let B denote the set of self-conjugated Ferrers diagrams corresponding to n.

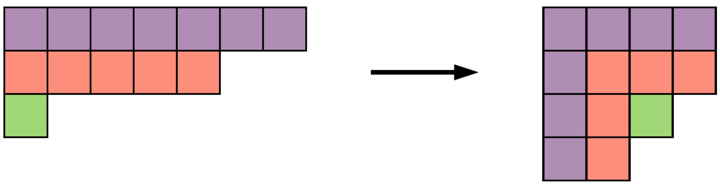
To prove that $|A| = |B|$, I will show that there is a one-to-one correspondence between elements of these two sets.

a) A → B

For any element or A (i.e. a partitioning of n using distinct odd numbers only) we can construct a self-conjugated Ferrers diagram of n. Let us consider the partitioning

$$(a_1, a_2, \dots, a_k) \in A \quad \left(\sum_i a_i = n; \quad \forall i: a_i \equiv 2k_i + 1; \quad \forall i, j: \ i < j \Rightarrow a_i > a_j \right)$$

From this, we can directly construct a Ferrers diagram F_1 such that row i of the diagram contains a_i squares. Then, we can transform this into a self-conjugated Ferrers Diagram F_2 by *folding* the rows and rearranging them like this:

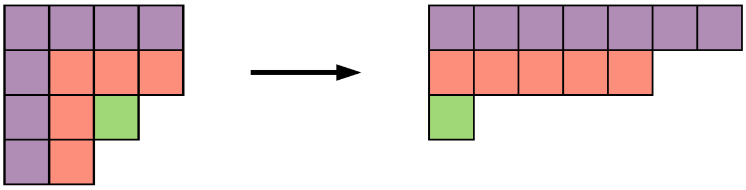


As the number of squares in each row of F_1 is odd, this kinds of symmetric folding is always possible.

As the number of squares in the rows of F_1 is guaranteed to be decreasing (F_1 is a valid Ferrer diagram and the numbers a_i are all distinct), F_2 is also guaranteed to be a valid Ferrers diagram.

b) B → A

On the other hand, given a self-conjugated Ferrers diagram F_1, we can always construct the corresponding partitioning F_2 by simply reversing this process:



As F_1 is self-conjugated (i.e. symmetric w.r.t. the diagonal), each "**Γ**" section of F_1 (corresponding to one row of F_2) is guaranteed to have an odd number of squares.

It is also easy to see that the number of squares in F_1 will be decreasing.

<u>Conclusion</u>

I have shown that for each partitioning of n consisting of distinct odd numbers only, there is a corresponding self-conjugated Ferrers diagram of n and vice-versa. This means that there is a one-to-one correspondence between the elements of these two sets, which in turn **proves that the number of elements in these two sets are equal.**