数值分析第三次大作业

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题目 1

关于 x, y, t, u, v, w 的下列方程组:

$$\begin{cases} 0.5\cos t + u + v + w - x = 2.67 \\ t + 0.5\sin u + v + w - y = 1.07 \\ 0.5t + u + \cos v + w - x = 3.74 \\ t + 0.5u + v + \sin w - y = 0.79 \end{cases}$$

以及关于 z, t, u 的下列二维数表确定了一个二元函数 $z = f(x, y)$。

表 1.1: 二维数表

z \ y / t	0	0.4	0.8	1.2	1.6	2
0	-0.5	-0.34	0.14	0.94	2.06	3.5
0.2	-0.42	-0.5	-0.26	0.3	1.18	2.38
0.4	-0.18	-0.5	-0.5	-0.18	0.46	1.42
0.6	0.22	-0.34	-0.58	-0.5	-0.1	0.62
0.8	0.78	-0.02	-0.5	-0.66	-0.5	-0.02
1	1.5	0.46	-0.26	-0.66	-0.74	-0.5

1. 试用数值方法求出 $f(x, y)$ 在区域 $D = \{(x, y) | 0 \le x \le 0.8, 0.5 \le y \le 1.5\}$ 上的一个近似表达式：

$$p(x, y) = \sum_{r=0}^{k} \sum_{s=0}^{k} c_{rs} x^r y^s$$

要求 $p(x, y)$ 最小的 k 值达到以下的精度：

$$\sigma = \sum_{i=0}^{10} \sum_{j=0}^{20} [f(x_i, y_j) - p(x_i, y_j)]^2 \le 10^{-7}$$

其中，$x_i = 0.08i, y_j = 0.5 + 0.05j$

2. 计算 $f(x_i^*, y_j^*), p(x_i^*, y_j^*) \qquad (i = 1, 2, \cdots, 8; j = 1, 2, \cdots, 5)$ 的值，以观察 $p(x, y)$ 逼近 $f(x, y)$ 的效果，其中,$x_i^* = 0.1i, y_j^* = 0.5 + 0.2j$

说明:

1. 用迭代方法求解非线性方程组时，要求近似解向量 $\boldsymbol{x}^{(k)}$ 满足以下精度

$$\frac{\|\boldsymbol{x}^{(k)} - \boldsymbol{x}^{(k-1)}\|_\infty}{\|\boldsymbol{x}^{(k)}\|_\infty} \le 10^{-12}$$

2. 作二元插值时，要使用分片二次代数插值。

3. 要由程序自动确定最小的 k 值。

4. 打印以下内容：

   (a) 全部源程序;

   (b) 数表:$\{x_i, y_j, f(x_i, y_j)\} \qquad (i = 0, 1, 2, \cdots, 10; j = 0, 1, 2, \cdots, 20)$;

   (c) 选择过程的 k, σ 值;

   (d) 达到精度要求时的 k 和 σ 值以及 $p(x, y)$ 中的系数 $c_{rs}(r = 0, 1, \cdots, k; s = 0, 1, \cdots, k)$;

   (e) 数表:$\{x_i^*, y_j^*, f(x_i^*, y_j^*), p(x_i^*, y_j^*)\} \qquad (i = 1, 2, \cdots, 8; j = 1, 2, \cdots, 5)$。

5. 采用 f 型输出 x_i, y_j, x_i^*, y_j^* 的准确值，其余实型数采用 e 型输出并且至少显示 12 位有效数字。

算法设计方案 2

2.1 方案综述

1. 将 $x_i = 0.08i, y_j = 0.5 + 0.05j \quad (i = 0, 1, \cdots, 10; j = 0, 1, \cdots, 20)$ 代入非线性方程组 (2.1) 中，用Newton 迭代法解出 t_{ij} 和 u_{ij};
2. 对数表 $z(t, u)$ 进行分片双二次插值, 求得 $z_{ij} = \hat{z}(t_{ij}, u_{ij})$
3. 根据 z_{ij} 的值进行曲面拟合，要求精度 $\sigma \leq 10^{-7}$, 得拟合函数

$$p(x, y) = \sum_{r=0}^{k} \sum_{s=0}^{k} c_{rs} x^r y^s$$

4. 创建新的数据点集 $x_i^* = 0.1i, y_j^* = 0.5{+}0.2j \quad (i = 1, 2, \cdots, 8; j = 1, 2, \cdots, 5)$, 并代入非线性方程组 (2.1) 中，用Newton 迭代法解出 t^* 和 u^*, 再用分片双二次插值计算出 $f(x^*, y^*)$, 将其与 $p(x^*, y^*)$ 输出并观察比较。[1]

[1]算法流程图见第9页。

2.2 Newton 迭代法

$$\begin{cases} 0.5\cos t + u + v + w - x = 2.67 \\ t + 0.5\sin u + v + w - y = 1.07 \\ 0.5t + u + \cos v + w - x = 3.74 \\ t + 0.5u + v + \sin w - y = 0.79 \end{cases} \tag{2.1}$$

对于该非线性方程组方程组来说，x, y 为已知量，需解出 t, u, v, w。设 $\boldsymbol{x} = (t, u, v, w)^T$, 并设定精度水平 $\varepsilon = 10^{-12}$ 和最大迭代次数 M 先在 $\boldsymbol{x}^*$ 附近选取 $\boldsymbol{x}^{(0)} = (t^{(0)}, u^{(0)}, v^{(0)}, w^{(0)})^T$ 然后迭代[2]:

$$\boldsymbol{x}^{(k+1)} = \boldsymbol{x}^{(k)} - [\boldsymbol{F}'(\boldsymbol{x}^{(k)})]^{-1}\boldsymbol{F}(\boldsymbol{x}^{(k)})$$

具体算法如下：

Algorithm 1 Newton's method

```
Set x^(0) ∈ D and k = 0
while k<M do
    Compute F(x^(k)) and F'(x^(k))
    Compute Δx^(k) = −[F'(x^(k))]^(−1) F(x^(k))
    if ‖Δx^(k)‖/‖x^(k+1)‖ ≤ ε then
        x* = x^(k)
        Break
    end if
    x^(k+1) = x^(k) + Δx^(k)
    k=k+1
end while
```

其中，基于方程组 (2.1) 的 $\boldsymbol{F}(\boldsymbol{x})$及其雅可比矩阵$\boldsymbol{F}'(\boldsymbol{x})$ 分别为:

$$F(\boldsymbol{x}) = \begin{bmatrix} 0.5 * cos(t) + u + v + w - x - 2.67 \\ t + 0.5 * sin(u) + v + w - y - 1.07 \\ 0.5 * t + u + cos(v) + w - x - 3.74 \\ t + 0.5 * u + v + sin(w) - y - 0.79 \end{bmatrix}$$

$$F'(\boldsymbol{x}) = \begin{bmatrix} -0.5 * \sin(t) & 1 & 1 & 1 \\ 1 & 0.5 * \cos(u) & 1 & 1 \\ 0.5 & 1 & -\sin(v) & 1 \\ 1 & 0.5 & 1 & \cos(w) \end{bmatrix}$$

[2]迭代的终止条件为 $\|\Delta\boldsymbol{x}^{(k)}\|/\|\boldsymbol{x}^{(k+1)}\| \le \varepsilon$, 若 $k > M$ 时仍未达到迭代精度，则迭代失败。

2.3 分片双二次插值

前面我们将 $\{(x_i, y_j)\}$ 代入非线性方程组 (2.1) 中，然后用Newton 迭代法解出了 t_{ij} 和 u_{ij}.

在这一节中我们需要根据表1.1对 $z(t,u)$ 进行分片双二次插值，求得 $z_{ij} = \hat{z}(t_{ij}, u_{ij}) \qquad (i = 0, 1, 2, \cdots, 10; j = 0, 1, 2, \cdots, 20)$.

因为表1.1为 6×6 的数表，故可设:

$$t_i = ih \qquad (i = 0, 1, \cdots, 5)$$

$$u_i = j\tau \qquad (j = 0, 1, \cdots, 5)^{3}$$

对于给定的 (t,u), 如果 (t,u) 满足：

$$t_i - \frac{h}{2} < t \leq t_i + \frac{h}{2},^{4} \qquad 2 \leq i \leq 3$$

$$u_j - \frac{\tau}{2} < u \leq u_j + \frac{\tau}{2},^{5} \qquad 2 \leq j \leq 3$$

那么应选择 $(t_k, u_r) \quad (k = i-1, i, i+1; r = j-1, j, j+1)$ 为插值节点。

若 t 满足：

$$t \leq t_1 + \frac{h}{2}$$

或

$$t > t_3 + \frac{h}{2}$$

则相应地选取 $i = 1$ 或 $i = 4$

同样的，若 u 满足：

$$u \leq u_1 + \frac{\tau}{2}$$

或

$$u > u_3 + \frac{\tau}{2}$$

则相应地选取 $j = 1$ 或 $j = 4$

最后得到插值多项式为

$$\boxed{\hat{z}(t,u) = \sum_{k=i-1}^{i+1} \sum_{r=j-1}^{j+1} l_k(t)\tilde{l}_r(u)z(t_k, u_r)} \tag{2.2}$$

其中，

$$l_k(t) = \prod_{\substack{m=i-1 \\ m \neq k}}^{i+1} \frac{t - t_m}{t_k - t_m} \qquad (k = i-1, i, i+1)$$

$$\tilde{l}_r(u) = \prod_{\substack{n=j-1 \\ n \neq r}}^{j+1} \frac{u - u_n}{u_r - u_n} \qquad (r = j-1, j, j+1)$$

---

[3] 其中: $\quad h = 0.2, \tau = 0.4$

[4] 计算 i,j 时有个小技巧: 可取 $i = \lfloor \frac{t}{h} + 0.5 \rfloor$

[5] 同样的，$j = \lfloor \frac{u}{\tau} + 0.5 \rfloor$

2.4 曲面拟合

设在三维坐标系 $Oxyu$ 中给定 $(m+1)\times(n+1)$ 个点:

$$\mathfrak{D}=\{(x_i,y_j),z_{ij}\}\qquad(i=0,1,\cdots,m;j=0,1,\cdots,n)\tag{2.3}$$

选定 $M+1$ 个 x 的函数 $\{\varphi_r(x)\}_{r=0}^{M}$ 和 $N+1$ 个 y 的函数 $\{\psi_s(y)\}_{s=0}^{N}$

以函数组 $\{\varphi_r(x)\psi_s(y)\}\quad(r=0,1,\cdots,M;s=0,1,\cdots,N)$ 为基函数，构成以 $\{c_{rs}\}$ 为参数的曲面族

$$p(x,y)=\sum_{r=0}^{M}\sum_{s=0}^{N}c_{rs}\varphi_r(x)\psi_s(y)\tag{2.4}$$

若参数 $\{c_{rs}^*\}$ 使得

$$L(\boldsymbol{C})=\sum_{i=0}^{m}\sum_{j=0}^{n}\left[\sum_{r=0}^{M}\sum_{s=0}^{N}c_{rs}\varphi_r(x_i)\psi_s(y_j)-u_{ij}\right]^2\tag{2.5}$$

在 $\boldsymbol{C}=\boldsymbol{C}^*$ 处取到最小值 $L(\boldsymbol{C}^*)$，则称相应曲面 $p^*(x,y)$ 为在曲面族 (2.4) 中按最小二乘原则确定的对于数据 (2.3) 的拟合曲面。

设：

$$\boldsymbol{B}=\left[\varphi_r(x_i)\right]_{(m+1)\times(M+1)}$$
$$\boldsymbol{G}=\left[\psi_s(y_j)\right]_{(n+1)\times(N+1)}$$
$$\boldsymbol{U}=\left[u_{ij}\right]_{(m+1)\times(n+1)}$$
$$\boldsymbol{C}=\left[c_{rs}\right]_{(M+1)\times(N+1)}$$

可证得[6]，拟合曲面的系数矩阵为

$$\boldsymbol{C}=(\boldsymbol{B}^T\boldsymbol{B})^{-1}\boldsymbol{B}^T\boldsymbol{U}\boldsymbol{G}(\boldsymbol{G}^T\boldsymbol{G})^{-1}\tag{2.6}$$

在本实验中:

$$\boldsymbol{B}=\begin{bmatrix}1 & x_0 & {x_0}^2 & \cdots & {x_0}^k\\ 1 & x_1 & {x_1}^2 & \cdots & {x_1}^k\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 1 & {x_{10}}^2 & {x_{10}}^2 & \cdots & {x_{10}}^k\end{bmatrix}_{11\times(k+1)}$$

$$\boldsymbol{G}=\begin{bmatrix}1 & y_0 & {y_0}^2 & \cdots & {y_0}^k\\ 1 & y_1 & {y_1}^2 & \cdots & {y_1}^k\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 1 & {y_{20}}^2 & {y_{20}}^2 & \cdots & {y_{20}}^k\end{bmatrix}_{21\times(k+1)}$$

$$\boldsymbol{U}=\left[u_{ij}\right]_{(m+1)\times(M+1)}=\left[z_{ij}\right]_{11\times21}=\left[f(x_i,y_j)\right]_{11\times21}$$

[6]在第五章的讨论中，本文将给出一种与教材不同的证明方法

在计算 (2.6) 式时，需要求矩阵的逆，此处可采用 Gauss 消元法。
解出 c_{rs}^k 后，可得：

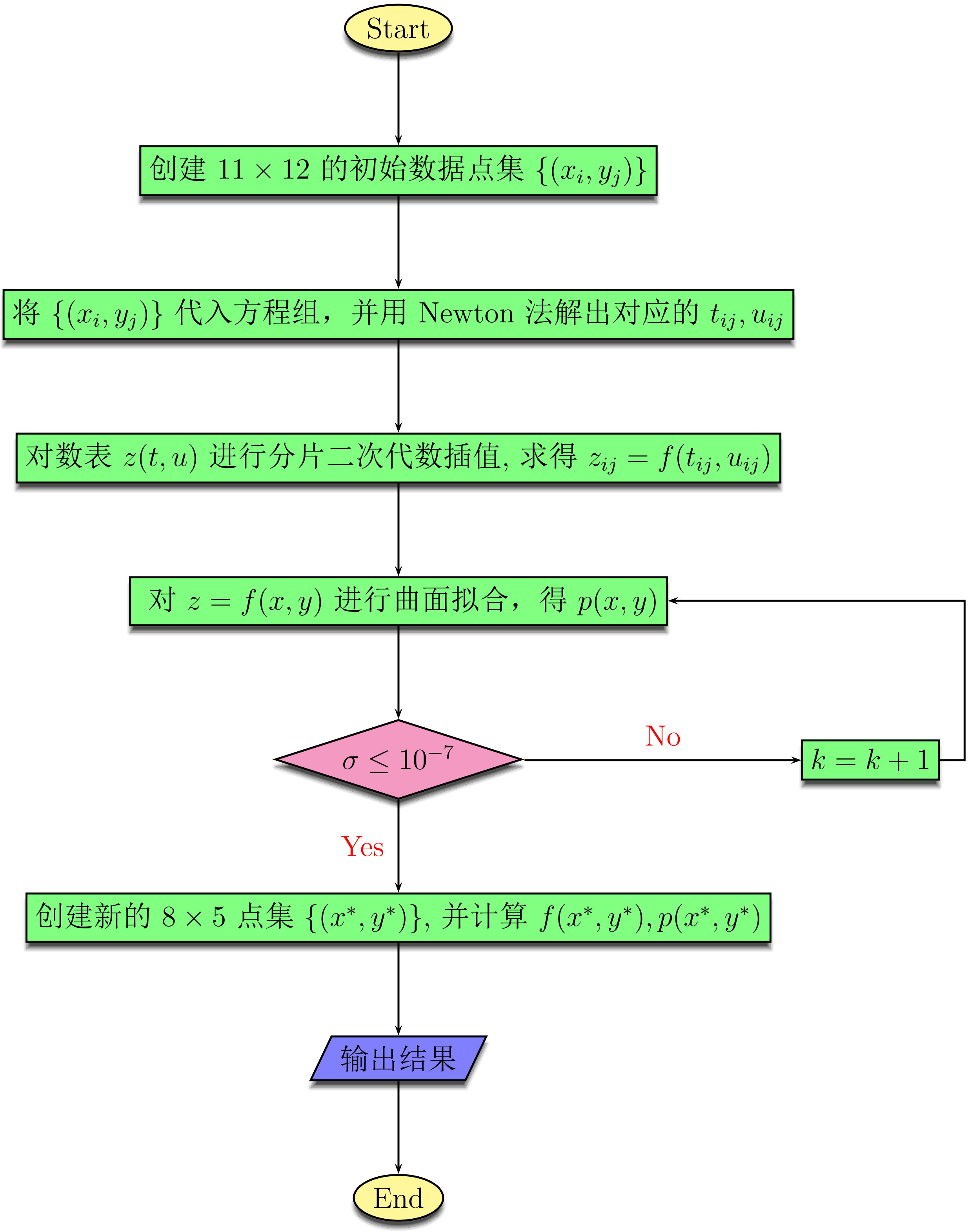
$$p^{(k)}(x, y) = \sum_{r,s=0}^{k} c_{rs}^k x^r y^s$$

其算法如下：[7]

Algorithm 2 Surface Fitting

```
Set k = 0, σ = 1
while σ > ε do
    Compute C = (B^T B)^{-1} B^T U G (G^T G)^{-1}
    Compute P = B C G^T
    Compute σ = ||P − U||_E^2
    k=k+1
    if k > N then
        Break
    end if
end while
```

[7]写成这样的矩阵形式能有更直观的理解，详细请参看第五章


Start
创建 11×12 的初始数据点集 $\{(x_i, y_j)\}$
将 $\{(x_i, y_j)\}$ 代入方程组，并用 Newton 法解出对应的 t_{ij}, u_{ij}
对数表 $z(t,u)$ 进行分片二次代数插值, 求得 $z_{ij} = f(t_{ij}, u_{ij})$
对 $z = f(x,y)$ 进行曲面拟合，得 $p(x,y)$
$\sigma \leq 10^{-7}$
No
$k = k+1$
Yes
创建新的 8×5 点集 $\{(x^*, y^*)\}$, 并计算 $f(x^*, y^*), p(x^*, y^*)$
输出结果
End

源程序 3

```
#include<stdio.h>
#include<math.h>
#include<string.h>
#define N 100
double X[N];//t,u,v,w
void F(double x,double y,double Fx[N]){
    Fx[1]=0.5*cos(X[1])+X[2]+X[3]+X[4]-x-2.67;
    Fx[2]=X[1]+0.5*sin(X[2])+X[3]+X[4]-y-1.07;
    Fx[3]=0.5*(X[1])+X[2]+cos(X[3])+X[4]-x-3.74;
    Fx[4]=X[1]+0.5*X[2]+X[3]+sin(X[4])-y-0.79;
}

void JF(double x,double y,double A[N][N]){
    A[1][1]=-0.5*sin(X[1]); A[1][2]=1; A[1][3]=1; A[1][4]=1;
    A[2][1]=1; A[2][2]=0.5*cos(X[2]); A[2][3]=1; A[2][4]=1;
    A[3][1]=0.5;A[3][2]=1; A[3][3]=-sin(X[3]);  A[3][4]=1;
    A[4][1]=1; A[4][2]=0.5; A[4][3]=1; A[4][4]=cos(X[4]);
}

void clear(double a[N][N]){
    for(int i=0;i<N;i++)
        for(int j=0;j<N;j++)
            a[i][j]=0;
}

void put(double a[N][N],int n,int m){
    for(int i=1;i<=n;i++){
        for(int j=1;j<=m;j++)
            printf("C[%d][%d]=%.12e \n",i-1,j-1,a[i][j]);
        printf("\n");
    }
    printf("\n");
}

```

```
double Maxtrix_x(double a[N][N],double b[N][N],double c[N][N],int
     m,int p,int n){
    double s=0;
    clear(c);
    for(int i=1;i<=m;i++)
        for(int j=1;j<=n;j++)
            for(int k=1;k<=p;k++)
                c[i][j]+=a[i][k]*b[k][j];
}

void Inverse(double C[N][N],double B[N][N],double n){
    double m,A[N][N];
    clear(B);
    clear(A);
    for(int i=1;i<=n;i++)
        for(int j=1;j<=n;j++)
            A[i][j]=C[i][j];
    for(int k=1;k<=n;k++)B[k][k]=1;
    for(int k=1;k<n;k++){
        for(int i=k+1;i<=n;i++){
            m=A[i][k]/A[k][k];
            for(int j=1;j<=n;j++){
                A[i][j]-=m*A[k][j];
                B[i][j]-=m*B[k][j];
            }
        }
    }
    for(int k=n;k;k--){
        for(int i=n;i>k;i--){
            m=A[k][i];
            for(int j=1;j<=n;j++){
                A[k][j]-=m*A[i][j];
                B[k][j]-=m*B[i][j];
            }
        }
        m=A[k][k];
        for(int j=1;j<=n;j++){
            A[k][j]/=m;
            B[k][j]/=m;
        }
    }
}

void Transpose(double A[N][N],double B[N][N],int n,int m){
    clear(B);
    for(int i=1;i<=n;i++)
        for(int j=1;j<=m;j++)
            B[j][i]=A[i][j];
}
```

```

double MaxX(double x[N],int n){
    double max=0;
    for(int i=1;i<=n;i++)
        if(fabs(x[i])>max)max=fabs(x[i]);
}

void Newton(double x,double y){
    int n=4,k;
    double eps=1e-12,max,A[N][N],Fx[N],B[N][N],dX[N];
    for(k=1;k<=1000;k++){
        memset(Fx,0,sizeof(Fx));
        memset(B,0,sizeof(B));
        memset(dX,0,sizeof(dX));
        F(x, y, Fx);
        JF(x, y, A);
        Inverse(A,B,n);
        for(int i=1;i<=n;i++)
            for(int j=1;j<=n;j++)
                dX[i]+=Fx[j]*B[i][j];
        max=0;
        for(int i=1;i<=n;i++)
            if(fabs(X[i])>max)max=fabs(X[i]);
        if((MaxX(dX,n)/max)<eps)
            return ;
        for(int i=1;i<=n;i++)X[i]-=dX[i];
    }
//  printf("wrong!\n");
}

double Interpolation(double t,double u){
    double h=0.2,tau=0.4;
    double z[N][N],sum=0,p;
    int i,j;
z[0][0]=-0.5 ; z[0][1]=-0.34; z[0][2]= 0.14; z[0][3]= 0.94; z[0][4]= 2.06; z[0][5]= 3.5;
z[1][0]=-0.42; z[1][1]=-0.5 ; z[1][2]=-0.26; z[1][3]= 0.3 ; z[1][4]= 1.18; z[1][5]= 2.38;
z[2][0]=-0.18; z[2][1]=-0.5 ; z[2][2]=-0.5 ; z[2][3]=-0.18; z[2][4]= 0.46; z[2][5]= 1.42;
z[3][0]= 0.22; z[3][1]=-0.34; z[3][2]=-0.58; z[3][3]=-0.5 ; z[3][4]=-0.1 ; z[3][5]= 0.62;
z[4][0]= 0.78; z[4][1]=-0.02; z[4][2]=-0.5 ; z[4][3]=-0.66; z[4][4]=-0.5 ; z[4][5]=-0.02;
z[5][0]= 1.5 ; z[5][1]= 0.46; z[5][2]=-0.26; z[5][3]=-0.66; z[5][4]=-0.74; z[5][5]=-0.5;
    if(t<0.3)i=1;
    else if(t>0.7)i=4;
    else if(0.3<=t && t<=0.7)i=int(t/h+0.5);
```

```
    if(u<0.6)j=1;
    else if(u>1.4)j=4;
    else if(0.6<=u && u<=1.4)j=int(u/tau+0.5);

    for(int k=i-1;k<=i+1;k++)
        for(int r=j-1;r<=j+1;r++){
            p=1;
            for(int m=i-1;m<=i+1;m++){
                if(m==k)continue;
                p*=(t-h*m)/(h*k-h*m);
            }
            for(int n=j-1;n<=j+1;n++){
                if(n==r)continue;
                p*=(u-tau*n)/(tau*r-tau*n);
            }
            p*=z[k][r];
            sum+=p;
        }
    return sum;
}

double Surface_Fitting(double x[N],double y[N],double z[N][N],
    double C[N][N],int k){
    double B[N][N],G[N][N],I[N][N],J[N][N],sigma=0;
    for(int i=1;i<=11;i++)
        for(int j=1;j<=k+1;j++)
            B[i][j]=pow(x[i],j-1);
    for(int i=1;i<=21;i++)
        for(int j=1;j<=k+1;j++)
            G[i][j]=pow(y[i],j-1);

    Transpose(B,I,11,k+1);
    Maxtrix_x(I,B,J,k+1,11,k+1);
    Inverse(J,C,k+1);
    Maxtrix_x(C,I,J,k+1,k+1,11);
    Maxtrix_x(J,z,I,k+1,11,21);
    Maxtrix_x(I,G,J,k+1,21,k+1);
    Transpose(G,I,21,k+1);
    Maxtrix_x(I,G,C,k+1,21,k+1);
    Inverse(C,I,k+1);
    Maxtrix_x(J,I,C,k+1,k+1,k+1);

    Maxtrix_x(B,C,I,11,k+1,k+1);
    Transpose(G,J,21,k+1);
    Maxtrix_x(I,J,G,11,k+1,21);
    for(int i=1;i<=11;i++)
        for(int j=1;j<=21;j++)
           sigma+=(G[i][j]-z[i][j])*(G[i][j]-z[i][j]);
    return sigma;
```

```
}

int main(){
    freopen("Work.out","w",stdout);
    double x[N],y[N],Z[N][N],C[N][N],sigma,eps=1e-7;
    int k;
    for(int i=0;i<=10;i++)
        for(int j=0;j<=20;j++){
            x[i+1]=0.08*i;
            y[j+1]=0.5+0.05*j;
            for(int k=1;k<=4;k++)X[k]=1;
            Newton(x[i+1],y[j+1]);
            Z[i+1][j+1]=Interpolation(X[1],X[2]);
            printf("x[%d]=%lf y[%d]=%lf f(x,y)=%.12e \n",i,x[i+1],j,y
                [j+1],Z[i+1][j+1]);
        }
    for(k=0;k<=10;k++){
        sigma=Surface_Fitting(x,y,Z,C,k);
        printf("K=%d,sigma=%.12e\n",k,sigma);
        if(fabs(sigma)<eps)break;
    }
    put(C,k+1,k+1);

    sigma=0;
    for(int i=1;i<=8;i++)
    for(int j=1;j<=5;j++){
        x[i]=0.1*i;
        y[j]=0.5+0.2*j;
        for(k=1;k<=4;k++)X[k]=1;
        Newton(x[i],y[j]);
        sigma=Interpolation(X[1],X[2]);
        printf("x[%d]=%lf y[%d]=%lf f(x,y)=%.12e ",i,x[i],j,y[j],
            sigma);
        sigma=0;
        for(int r=0;r<=k;r++)
        for(int s=0;s<=k;s++){
            sigma+=C[r+1][s+1]*pow(x[i],r)*pow(y[j],s);
        }
        printf("f(x,y)=%.12e\n",i,j,sigma);


    }

    return 0;
}
```

计算结果 4

表 4.1: 数表 $x_i, y_j, f(x_i, y_j)$

x_i	y_j	$f(x_i, y_j)$
x_0 =0.000000	y_0 =0.500000	$f(x_0, y_0)$ =4.465040184807e-001
x_0 =0.000000	y_1 =0.550000	$f(x_0, y_1)$ =3.246832629277e-001
x_0 =0.000000	y_2 =0.600000	$f(x_0, y_2)$ =2.101596866827e-001
x_0 =0.000000	y_3 =0.650000	$f(x_0, y_3)$ =1.030436083160e-001
x_0 =0.000000	y_4 =0.700000	$f(x_0, y_4)$ =3.401895562676e-003
x_0 =0.000000	y_5 =0.750000	$f(x_0, y_5)$ =-8.873581363800e-002
x_0 =0.000000	y_6 =0.800000	$f(x_0, y_6)$ =-1.733716327497e-001
x_0 =0.000000	y_7 =0.850000	$f(x_0, y_7)$ =-2.505346114666e-001
x_0 =0.000000	y_8 =0.900000	$f(x_0, y_8)$ =-3.202765063876e-001
x_0 =0.000000	y_9 =0.950000	$f(x_0, y_9)$ =-3.826680661097e-001
x_0 =0.000000	y_{10} =1.000000	$f(x_0, y_{10})$ =-4.377957667384e-001
x_0 =0.000000	y_{11} =1.050000	$f(x_0, y_{11})$ =-4.857589414438e-001
x_0 =0.000000	y_{12} =1.100000	$f(x_0, y_{12})$ =-5.266672548835e-001
x_0 =0.000000	y_{13} =1.150000	$f(x_0, y_{13})$ =-5.606384797965e-001
x_0 =0.000000	y_{14} =1.200000	$f(x_0, y_{14})$ =-5.877965387677e-001
x_0 =0.000000	y_{15} =1.250000	$f(x_0, y_{15})$ =-6.082697790899e-001
x_0 =0.000000	y_{16} =1.300000	$f(x_0, y_{16})$ =-6.221894528764e-001
x_0 =0.000000	y_{17} =1.350000	$f(x_0, y_{17})$ =-6.296883781856e-001
x_0 =0.000000	y_{18} =1.400000	$f(x_0, y_{18})$ =-6.308997600028e-001
x_0 =0.000000	y_{19} =1.450000	$f(x_0, y_{19})$ =-6.259561525454e-001

x_i	y_j	$f(x_i, y_j)$
x_0 =0.000000	y_{20} =1.500000	$f(x_0, y_{20})$ =-6.149885466094e-001
x_1 =0.080000	y_0 =0.500000	$f(x_1, y_0)$ =6.380152265113e-001
x_1 =0.080000	y_1 =0.550000	$f(x_1, y_1)$ =5.066117551467e-001
x_1 =0.080000	y_2 =0.600000	$f(x_1, y_2)$ =3.821763692774e-001
x_1 =0.080000	y_3 =0.650000	$f(x_1, y_3)$ =2.648634911537e-001
x_1 =0.080000	y_4 =0.700000	$f(x_1, y_4)$ =1.547802002848e-001
x_1 =0.080000	y_5 =0.750000	$f(x_1, y_5)$ =5.199268349094e-002
x_1 =0.080000	y_6 =0.800000	$f(x_1, y_6)$ =-4.346804020491e-002
x_1 =0.080000	y_7 =0.850000	$f(x_1, y_7)$ =-1.316010567885e-001
x_1 =0.080000	y_8 =0.900000	$f(x_1, y_8)$ =-2.124310883088e-001
x_1 =0.080000	y_9 =0.950000	$f(x_1, y_9)$ =-2.860045510580e-001
x_1 =0.080000	y_{10} =1.000000	$f(x_1, y_{10})$ =-3.523860789794e-001
x_1 =0.080000	y_{11} =1.050000	$f(x_1, y_{11})$ =-4.116554565222e-001
x_1 =0.080000	y_{12} =1.100000	$f(x_1, y_{12})$ =-4.639049115188e-001
x_1 =0.080000	y_{13} =1.150000	$f(x_1, y_{13})$ =-5.092367247005e-001
x_1 =0.080000	y_{14} =1.200000	$f(x_1, y_{14})$ =-5.477611179623e-001
x_1 =0.080000	y_{15} =1.250000	$f(x_1, y_{15})$ =-5.795943883391e-001
x_1 =0.080000	y_{16} =1.300000	$f(x_1, y_{16})$ =-6.048572588895e-001
x_1 =0.080000	y_{17} =1.350000	$f(x_1, y_{17})$ =-6.236734213318e-001
x_1 =0.080000	y_{18} =1.400000	$f(x_1, y_{18})$ =-6.361682484133e-001
x_1 =0.080000	y_{19} =1.450000	$f(x_1, y_{19})$ =-6.424676566901e-001
x_1 =0.080000	y_{20} =1.500000	$f(x_1, y_{20})$ =-6.426971026996e-001
x_2 =0.160000	y_0 =0.500000	$f(x_2, y_0)$ =8.400813957666e-001
x_2 =0.160000	y_1 =0.550000	$f(x_2, y_1)$ =6.997641656732e-001
x_2 =0.160000	y_2 =0.600000	$f(x_2, y_2)$ =5.660614423517e-001
x_2 =0.160000	y_3 =0.650000	$f(x_2, y_3)$ =4.391716081176e-001
x_2 =0.160000	y_4 =0.700000	$f(x_2, y_4)$ =3.192421380408e-001
x_2 =0.160000	y_5 =0.750000	$f(x_2, y_5)$ =2.063761923874e-001
x_2 =0.160000	y_6 =0.800000	$f(x_2, y_6)$ =1.006385238914e-001
x_2 =0.160000	y_7 =0.850000	$f(x_2, y_7)$ =2.060740067837e-003
x_2 =0.160000	y_8 =0.900000	$f(x_2, y_8)$ =-8.935402476698e-002
x_2 =0.160000	y_9 =0.950000	$f(x_2, y_9)$ =-1.736269688648e-001

x_i	y_j	$f(x_i, y_j)$
x_2 =0.160000	y_{10} =1.000000	$f(x_2, y_{10})$ =-2.507999561599e-001
x_2 =0.160000	y_{11} =1.050000	$f(x_2, y_{11})$ =-3.209322694446e-001
x_2 =0.160000	y_{12} =1.100000	$f(x_2, y_{12})$ =-3.840977350046e-001
x_2 =0.160000	y_{13} =1.150000	$f(x_2, y_{13})$ =-4.403821754175e-001
x_2 =0.160000	y_{14} =1.200000	$f(x_2, y_{14})$ =-4.898811523126e-001
x_2 =0.160000	y_{15} =1.250000	$f(x_2, y_{15})$ =-5.326979655338e-001
x_2 =0.160000	y_{16} =1.300000	$f(x_2, y_{16})$ =-5.689418792921e-001
x_2 =0.160000	y_{17} =1.350000	$f(x_2, y_{17})$ =-5.987265495151e-001
x_2 =0.160000	y_{18} =1.400000	$f(x_2, y_{18})$ =-6.221686297503e-001
x_2 =0.160000	y_{19} =1.450000	$f(x_2, y_{19})$ =-6.393865356972e-001
x_2 =0.160000	y_{20} =1.500000	$f(x_2, y_{20})$ =-6.504993507878e-001
x_3 =0.240000	y_0 =0.500000	$f(x_3, y_0)$ =1.051515091803e+000
x_3 =0.240000	y_1 =0.550000	$f(x_3, y_1)$ =9.029274308310e-001
x_3 =0.240000	y_2 =0.600000	$f(x_3, y_2)$ =7.605802668596e-001
x_3 =0.240000	y_3 =0.650000	$f(x_3, y_3)$ =6.247151981456e-001
x_3 =0.240000	y_4 =0.700000	$f(x_3, y_4)$ =4.955197560009e-001
x_3 =0.240000	y_5 =0.750000	$f(x_3, y_5)$ =3.731340427746e-001
x_3 =0.240000	y_6 =0.800000	$f(x_3, y_6)$ =2.576567488723e-001
x_3 =0.240000	y_7 =0.850000	$f(x_3, y_7)$ =1.491505594102e-001
x_3 =0.240000	y_8 =0.900000	$f(x_3, y_8)$ =4.764698677337e-002
x_3 =0.240000	y_9 =0.950000	$f(x_3, y_9)$ =-4.684932320146e-002
x_3 =0.240000	y_{10} =1.000000	$f(x_3, y_{10})$ =-1.343567603849e-001
x_3 =0.240000	y_{11} =1.050000	$f(x_3, y_{11})$ =-2.149133449274e-001
x_3 =0.240000	y_{12} =1.100000	$f(x_3, y_{12})$ =-2.885737006348e-001
x_3 =0.240000	y_{13} =1.150000	$f(x_3, y_{13})$ =-3.554063647857e-001
x_3 =0.240000	y_{14} =1.200000	$f(x_3, y_{14})$ =-4.154913964886e-001
x_3 =0.240000	y_{15} =1.250000	$f(x_3, y_{15})$ =-4.689182499695e-001
x_3 =0.240000	y_{16} =1.300000	$f(x_3, y_{16})$ =-5.157838831247e-001
x_3 =0.240000	y_{17} =1.350000	$f(x_3, y_{17})$ =-5.561910752001e-001
x_3 =0.240000	y_{18} =1.400000	$f(x_3, y_{18})$ =-5.902469305629e-001
x_3 =0.240000	y_{19} =1.450000	$f(x_3, y_{19})$ =-6.180615482412e-001
x_3 =0.240000	y_{20} =1.500000	$f(x_3, y_{20})$ =-6.397468392579e-001

x_i	y_j	$f(x_i, y_j)$
x_4 =0.320000	y_0 =0.500000	$f(x_4, y_0)$ =1.271246751483e+000
x_4 =0.320000	y_1 =0.550000	$f(x_4, y_1)$ =1.115002018147e+000
x_4 =0.320000	y_2 =0.600000	$f(x_4, y_2)$ =9.646077272157e-001
x_4 =0.320000	y_3 =0.650000	$f(x_4, y_3)$ =8.203473694751e-001
x_4 =0.320000	y_4 =0.700000	$f(x_4, y_4)$ =6.824476781795e-001
x_4 =0.320000	y_5 =0.750000	$f(x_4, y_5)$ =5.510852085975e-001
x_4 =0.320000	y_6 =0.800000	$f(x_4, y_6)$ =4.263923859018e-001
x_4 =0.320000	y_7 =0.850000	$f(x_4, y_7)$ =3.084629956332e-001
x_4 =0.320000	y_8 =0.900000	$f(x_4, y_8)$ =1.973571296919e-001
x_4 =0.320000	y_9 =0.950000	$f(x_4, y_9)$ =9.310562085940e-002
x_4 =0.320000	y_{10} =1.000000	$f(x_4, y_{10})$ =-4.285992234034e-003
x_4 =0.320000	y_{11} =1.050000	$f(x_4, y_{11})$ =-9.483392529689e-002
x_4 =0.320000	y_{12} =1.100000	$f(x_4, y_{12})$ =-1.785729903640e-001
x_4 =0.320000	y_{13} =1.150000	$f(x_4, y_{13})$ =-2.555537790546e-001
x_4 =0.320000	y_{14} =1.200000	$f(x_4, y_{14})$ =-3.258401501575e-001
x_4 =0.320000	y_{15} =1.250000	$f(x_4, y_{15})$ =-3.895069883634e-001
x_4 =0.320000	y_{16} =1.300000	$f(x_4, y_{16})$ =-4.466382045995e-001
x_4 =0.320000	y_{17} =1.350000	$f(x_4, y_{17})$ =-4.973249517677e-001
x_4 =0.320000	y_{18} =1.400000	$f(x_4, y_{18})$ =-5.416640326994e-001
x_4 =0.320000	y_{19} =1.450000	$f(x_4, y_{19})$ =-5.797564797951e-001
x_4 =0.320000	y_{20} =1.500000	$f(x_4, y_{20})$ =-6.117062881476e-001
x_5 =0.400000	y_0 =0.500000	$f(x_5, y_0)$ =1.498321052482e+000
x_5 =0.400000	y_1 =0.550000	$f(x_5, y_1)$ =1.334998632066e+000
x_5 =0.400000	y_2 =0.600000	$f(x_5, y_2)$ =1.177125123739e+000
x_5 =0.400000	y_3 =0.650000	$f(x_5, y_3)$ =1.025024055020e+000
x_5 =0.400000	y_4 =0.700000	$f(x_5, y_4)$ =8.789600231744e-001
x_5 =0.400000	y_5 =0.750000	$f(x_5, y_5)$ =7.391451087035e-001
x_5 =0.400000	y_6 =0.800000	$f(x_5, y_6)$ =6.057448714871e-001
x_5 =0.400000	y_7 =0.850000	$f(x_5, y_7)$ =4.788838610666e-001
x_5 =0.400000	y_8 =0.900000	$f(x_5, y_8)$ =3.586506258818e-001
x_5 =0.400000	y_9 =0.950000	$f(x_5, y_9)$ =2.451022361964e-001
x_5 =0.400000	y_{10} =1.000000	$f(x_5, y_{10})$ =1.382683509285e-001

x_i	y_j	$f(x_i, y_j)$
x_5 =0.400000	y_{11} =1.050000	$f(x_5, y_{11})$ =3.815486540699e-002
x_5 =0.400000	y_{12} =1.100000	$f(x_5, y_{12})$ =-5.525282116814e-002
x_5 =0.400000	y_{13} =1.150000	$f(x_5, y_{13})$ =-1.419868808137e-001
x_5 =0.400000	y_{14} =1.200000	$f(x_5, y_{14})$ =-2.220944390959e-001
x_5 =0.400000	y_{15} =1.250000	$f(x_5, y_{15})$ =-2.956352324598e-001
x_5 =0.400000	y_{16} =1.300000	$f(x_5, y_{16})$ =-3.626795115028e-001
x_5 =0.400000	y_{17} =1.350000	$f(x_5, y_{17})$ =-4.233061642240e-001
x_5 =0.400000	y_{18} =1.400000	$f(x_5, y_{18})$ =-4.776010361325e-001
x_5 =0.400000	y_{19} =1.450000	$f(x_5, y_{19})$ =-5.256554266672e-001
x_5 =0.400000	y_{20} =1.500000	$f(x_5, y_{20})$ =-5.675647436551e-001
x_6 =0.480000	y_0 =0.500000	$f(x_6, y_0)$ =1.731892740383e+000
x_6 =0.480000	y_1 =0.550000	$f(x_6, y_1)$ =1.562034577209e+000
x_6 =0.480000	y_2 =0.600000	$f(x_6, y_2)$ =1.397216918208e+000
x_6 =0.480000	y_3 =0.650000	$f(x_6, y_3)$ =1.237801006739e+000
x_6 =0.480000	y_4 =0.700000	$f(x_6, y_4)$ =1.084087532678e+000
x_6 =0.480000	y_5 =0.750000	$f(x_6, y_5)$ =9.363227723149e-001
x_6 =0.480000	y_6 =0.800000	$f(x_6, y_6)$ =7.947044490537e-001
x_6 =0.480000	y_7 =0.850000	$f(x_6, y_7)$ =6.593871980282e-001
x_6 =0.480000	y_8 =0.900000	$f(x_6, y_8)$ =5.304875868400e-001
x_6 =0.480000	y_9 =0.950000	$f(x_6, y_9)$ =4.080886854542e-001
x_6 =0.480000	y_{10} =1.000000	$f(x_6, y_{10})$ =2.922442012295e-001
x_6 =0.480000	y_{11} =1.050000	$f(x_6, y_{11})$ =1.829822068536e-001
x_6 =0.480000	y_{12} =1.100000	$f(x_6, y_{12})$ =8.030849403543e-002
x_6 =0.480000	y_{13} =1.150000	$f(x_6, y_{13})$ =-1.579041305164e-002
x_6 =0.480000	y_{14} =1.200000	$f(x_6, y_{14})$ =-1.053445516210e-001
x_6 =0.480000	y_{15} =1.250000	$f(x_6, y_{15})$ =-1.883980906096e-001
x_6 =0.480000	y_{16} =1.300000	$f(x_6, y_{16})$ =-2.650071493189e-001
x_6 =0.480000	y_{17} =1.350000	$f(x_6, y_{17})$ =-3.352378389040e-001
x_6 =0.480000	y_{18} =1.400000	$f(x_6, y_{18})$ =-3.991645038868e-001
x_6 =0.480000	y_{19} =1.450000	$f(x_6, y_{19})$ =-4.568681433016e-001
x_6 =0.480000	y_{20} =1.500000	$f(x_6, y_{20})$ =-5.084349932782e-001
x_7 =0.560000	y_0 =0.500000	$f(x_7, y_0)$ =1.971221786400e+000

x_i	y_j	$f(x_i, y_j)$
x_7 =0.560000	y_1 =0.550000	$f(x_7, y_1)$ =1.795329599501e+000
x_7 =0.560000	y_2 =0.600000	$f(x_7, y_2)$ =1.624067113228e+000
x_7 =0.560000	y_3 =0.650000	$f(x_7, y_3)$ =1.457830582708e+000
x_7 =0.560000	y_4 =0.700000	$f(x_7, y_4)$ =1.296954649752e+000
x_7 =0.560000	y_5 =0.750000	$f(x_7, y_5)$ =1.141718105447e+000
x_7 =0.560000	y_6 =0.800000	$f(x_7, y_6)$ =9.923495333243e-001
x_7 =0.560000	y_7 =0.850000	$f(x_7, y_7)$ =8.490326633294e-001
x_7 =0.560000	y_8 =0.900000	$f(x_7, y_8)$ =7.119113522641e-001
x_7 =0.560000	y_9 =0.950000	$f(x_7, y_9)$ =5.810941589219e-001
x_7 =0.560000	y_{10} =1.000000	$f(x_7, y_{10})$ =4.566585132334e-001
x_7 =0.560000	y_{11} =1.050000	$f(x_7, y_{11})$ =3.386544961394e-001
x_7 =0.560000	y_{12} =1.100000	$f(x_7, y_{12})$ =2.271082557696e-001
x_7 =0.560000	y_{13} =1.150000	$f(x_7, y_{13})$ =1.220250891932e-001
x_7 =0.560000	y_{14} =1.200000	$f(x_7, y_{14})$ =2.339221963760e-002
x_7 =0.560000	y_{15} =1.250000	$f(x_7, y_{15})$ =-6.881870197104e-002
x_7 =0.560000	y_{16} =1.300000	$f(x_7, y_{16})$ =-1.546493442129e-001
x_7 =0.560000	y_{17} =1.350000	$f(x_7, y_{17})$ =-2.341526664587e-001
x_7 =0.560000	y_{18} =1.400000	$f(x_7, y_{18})$ =-3.073910919133e-001
x_7 =0.560000	y_{19} =1.450000	$f(x_7, y_{19})$ =-3.744348623481e-001
x_7 =0.560000	y_{20} =1.500000	$f(x_7, y_{20})$ =-4.353605565359e-001
x_8 =0.640000	y_0 =0.500000	$f(x_8, y_0)$ =2.215667863688e+000
x_8 =0.640000	y_1 =0.550000	$f(x_8, y_1)$ =2.034201133607e+000
x_8 =0.640000	y_2 =0.600000	$f(x_8, y_2)$ =1.856955143619e+000
x_8 =0.640000	y_3 =0.650000	$f(x_8, y_3)$ =1.684358164161e+000
x_8 =0.640000	y_4 =0.700000	$f(x_8, y_4)$ =1.516776352400e+000
x_8 =0.640000	y_5 =0.750000	$f(x_8, y_5)$ =1.354519041151e+000
x_8 =0.640000	y_6 =0.800000	$f(x_8, y_6)$ =1.197844086673e+000
x_8 =0.640000	y_7 =0.850000	$f(x_8, y_7)$ =1.046963049419e+000
x_8 =0.640000	y_8 =0.900000	$f(x_8, y_8)$ =9.020460838023e-001
x_8 =0.640000	y_9 =0.950000	$f(x_8, y_9)$ =7.632264776629e-001
x_8 =0.640000	y_{10} =1.000000	$f(x_8, y_{10})$ =6.306048219543e-001
x_8 =0.640000	y_{11} =1.050000	$f(x_8, y_{11})$ =5.042528145972e-001

x_i	y_j	$f(x_i, y_j)$
x_8 =0.640000	y_{12} =1.100000	$f(x_8, y_{12})$ =3.842167155457e-001
x_8 =0.640000	y_{13} =1.150000	$f(x_8, y_{13})$ =2.705204766410e-001
x_8 =0.640000	y_{14} =1.200000	$f(x_8, y_{14})$ =1.631685723996e-001
x_8 =0.640000	y_{15} =1.250000	$f(x_8, y_{15})$ =6.214855811676e-002
x_8 =0.640000	y_{16} =1.300000	$f(x_8, y_{16})$ =-3.256661939682e-002
x_8 =0.640000	y_{17} =1.350000	$f(x_8, y_{17})$ =-1.210165348444e-001
x_8 =0.640000	y_{18} =1.400000	$f(x_8, y_{18})$ =-2.032513996228e-001
x_8 =0.640000	y_{19} =1.450000	$f(x_8, y_{19})$ =-2.793303595584e-001
x_8 =0.640000	y_{20} =1.500000	$f(x_8, y_{20})$ =-3.493199575400e-001
x_9 =0.720000	y_0 =0.500000	$f(x_9, y_0)$ =2.464684222659e+000
x_9 =0.720000	y_1 =0.550000	$f(x_9, y_1)$ =2.278058979398e+000
x_9 =0.720000	y_2 =0.600000	$f(x_9, y_2)$ =2.095251250840e+000
x_9 =0.720000	y_3 =0.650000	$f(x_9, y_3)$ =1.916718127997e+000
x_9 =0.720000	y_4 =0.700000	$f(x_9, y_4)$ =1.742854628776e+000
x_9 =0.720000	y_5 =0.750000	$f(x_9, y_5)$ =1.573998427334e+000
x_9 =0.720000	y_6 =0.800000	$f(x_9, y_6)$ =1.410434835231e+000
x_9 =0.720000	y_7 =0.850000	$f(x_9, y_7)$ =1.252401750608e+000
x_9 =0.720000	y_8 =0.900000	$f(x_9, y_8)$ =1.100094409628e+000
x_9 =0.720000	y_9 =0.950000	$f(x_9, y_9)$ =9.536698512613e-001
x_9 =0.720000	y_{10} =1.000000	$f(x_9, y_{10})$ =8.132510552489e-001
x_9 =0.720000	y_{11} =1.050000	$f(x_9, y_{11})$ =6.789307429659e-001
x_9 =0.720000	y_{12} =1.100000	$f(x_9, y_{12})$ =5.507748485043e-001
x_9 =0.720000	y_{13} =1.150000	$f(x_9, y_{13})$ =4.288256769731e-001
x_9 =0.720000	y_{14} =1.200000	$f(x_9, y_{14})$ =3.131047717398e-001
x_9 =0.720000	y_{15} =1.250000	$f(x_9, y_{15})$ =2.036155140327e-001
x_9 =0.720000	y_{16} =1.300000	$f(x_9, y_{16})$ =1.003454782409e-001
x_9 =0.720000	y_{17} =1.350000	$f(x_9, y_{17})$ =3.268565186571e-003
x_9 =0.720000	y_{18} =1.400000	$f(x_9, y_{18})$ =-8.765306591329e-002
x_9 =0.720000	y_{19} =1.450000	$f(x_9, y_{19})$ =-1.724672478188e-001
x_9 =0.720000	y_{20} =1.500000	$f(x_9, y_{20})$ =-2.512302207523e-001
x_{10} =0.800000	y_0 =0.500000	$f(x_{10}, y_0)$ =2.717811109467e+000
x_{10} =0.800000	y_1 =0.550000	$f(x_{10}, y_1)$ =2.526399501255e+000

x_i	y_j	$f(x_i, y_j)$
x_{10} =0.800000	y_2 =0.600000	$f(x_{10}, y_2)$ =2.338411386860e+000
x_{10} =0.800000	y_3 =0.650000	$f(x_{10}, y_3)$ =2.154329377280e+000
x_{10} =0.800000	y_4 =0.700000	$f(x_{10}, y_4)$ =1.974574556652e+000
x_{10} =0.800000	y_5 =0.750000	$f(x_{10}, y_5)$ =1.799510579099e+000
x_{10} =0.800000	y_6 =0.800000	$f(x_{10}, y_6)$ =1.629448220554e+000
x_{10} =0.800000	y_7 =0.850000	$f(x_{10}, y_7)$ =1.464650043751e+000
x_{10} =0.800000	y_8 =0.900000	$f(x_{10}, y_8)$ =1.305334967651e+000
x_{10} =0.800000	y_9 =0.950000	$f(x_{10}, y_9)$ =1.151682621307e+000
x_{10} =0.800000	y_{10} =1.000000	$f(x_{10}, y_{10})$ =1.003837419906e+000
x_{10} =0.800000	y_{11} =1.050000	$f(x_{10}, y_{11})$ =8.619123372279e-001
x_{10} =0.800000	y_{12} =1.100000	$f(x_{10}, y_{12})$ =7.259923711112e-001
x_{10} =0.800000	y_{13} =1.150000	$f(x_{10}, y_{13})$ =5.961377115201e-001
x_{10} =0.800000	y_{14} =1.200000	$f(x_{10}, y_{14})$ =4.723866279136e-001
x_{10} =0.800000	y_{15} =1.250000	$f(x_{10}, y_{15})$ =3.547580958979e-001
x_{10} =0.800000	y_{16} =1.300000	$f(x_{10}, y_{16})$ =2.432541841813e-001
x_{10} =0.800000	y_{17} =1.350000	$f(x_{10}, y_{17})$ =1.378622225247e-001
x_{10} =0.800000	y_{18} =1.400000	$f(x_{10}, y_{18})$ =3.855677032640e-002
x_{10} =0.800000	y_{19} =1.450000	$f(x_{10}, y_{19})$ =-5.469859593446e-002
x_{10} =0.800000	y_{20} =1.500000	$f(x_{10}, y_{20})$ =-1.419496597088e-001

表 4.2: 选择过程

迭代次数	计算精度
K=0	$\sigma = 1.442880771836e + 002$
K=1	$\sigma = 3.220908973638e + 000$
K=2	$\sigma = 4.659960033271e - 003$
K=3	$\sigma = 1.721175379142e - 004$
K=4	$\sigma = 3.309534300188e - 006$
K=5	$\sigma = 2.541973048217e - 008$

表 4.3: 系数矩阵 C

$C[0][0] = 2.021187388920e + 000$	$C[0][1] = -3.668183485046e + 000$	$C[0][2] = 7.087201308459e - 001$
$C[0][3] = 8.491593282670e - 001$	$C[0][4] = -4.161784546450e - 001$	$C[0][5] = 6.748737231828e - 002$
$C[1][0] = 3.192051731843e + 000$	$C[1][1] = -7.419162022998e - 001$	$C[1][2] = -2.695372858783e + 000$
$C[1][3] = 1.629345907830e + 000$	$C[1][4] = -4.837870921474e - 001$	$C[1][5] = 6.043032779417e - 002$
$C[2][0] = 2.570841806009e - 001$	$C[2][1] = 1.578821773641e + 000$	$C[2][2] = -4.610247723758e - 001$
$C[2][3] = -8.384333364666e - 002$	$C[2][4] = 1.033623972908e - 001$	$C[2][5] = -2.126520453021e - 002$
$C[3][0] = -2.693288041046e - 001$	$C[3][1] = -7.298612487502e - 001$	$C[3][2] = 1.075305416249e + 000$
$C[3][3] = -8.061323901638e - 001$	$C[3][4] = 3.024259777740e - 001$	$C[3][5] = -4.588455485646e - 002$
$C[4][0] = 2.174746167584e - 001$	$C[4][1] = -1.784562694957e - 001$	$C[4][2] = -7.222350058146e - 002$
$C[4][3] = 2.431393063162e - 001$	$C[4][4] = -1.412377151428e - 001$	$C[4][5] = 2.649111476785e - 002$
$C[5][0] = -5.590456307618e - 002$	$C[5][1] = 1.432065665213e - 001$	$C[5][2] = -1.362862847891e - 001$
$C[5][3] = 4.073630926723e - 002$	$C[5][4] = 3.766563044337e - 003$	$C[5][5] = -2.666031705303e - 003$

表 4.4: 数表 $f(x_i^*, y_j^*), p(x_i^*, y_j^*)$

x_i^*	y_j^*	$f(x_i^*, y_j^*)$	$p(x_i^*, y_j^*)$
x_1^* =0.100000	y_1^* =0.700000	$f(x_1^*, y_1^*)$ =1.947204079177e-001	$p(x_1^*, y_1^*)$ =1.947303951907e-001
x_1^* =0.100000	y_2^* =0.900000	$f(x_1^*, y_2^*)$ =-1.830370791887e-001	$p(x_1^*, y_2^*)$ =-1.830418956336e-001
x_1^* =0.100000	y_3^* =1.100000	$f(x_1^*, y_3^*)$ =-4.454976469148e-001	$p(x_1^*, y_3^*)$ =-4.455000035598e-001
x_1^* =0.100000	y_4^* =1.300000	$f(x_1^*, y_4^*)$ =-5.975667076413e-001	$p(x_1^*, y_4^*)$ =-5.975588581971e-001
x_1^* =0.100000	y_5^* =1.500000	$f(x_1^*, y_5^*)$ =-6.464595939011e-001	$p(x_1^*, y_5^*)$ =-6.464460483472e-001
x_2^* =0.200000	y_1^* =0.700000	$f(x_2^*, y_1^*)$ =4.059791892882e-001	$p(x_2^*, y_1^*)$ =4.059895504399e-001
x_2^* =0.200000	y_2^* =0.900000	$f(x_2^*, y_2^*)$ =-2.251595837462e-002	$p(x_2^*, y_2^*)$ =-2.252113238658e-002
x_2^* =0.200000	y_3^* =1.100000	$f(x_2^*, y_3^*)$ =-3.382208160396e-001	$p(x_2^*, y_3^*)$ =-3.382240139857e-001
x_2^* =0.200000	y_4^* =1.300000	$f(x_2^*, y_4^*)$ =-5.444378315219e-001	$p(x_2^*, y_4^*)$ =-5.444304518316e-001
x_2^* =0.200000	y_5^* =1.500000	$f(x_2^*, y_5^*)$ =-6.473613385679e-001	$p(x_2^*, y_5^*)$ =-6.473479955995e-001
x_3^* =0.300000	y_1^* =0.700000	$f(x_3^*, y_1^*)$ =6.347771951510e-001	$p(x_3^*, y_1^*)$ =6.347874320293e-001
x_3^* =0.300000	y_2^* =0.900000	$f(x_3^*, y_2^*)$ =1.588011688394e-001	$p(x_3^*, y_2^*)$ =1.587963265625e-001
x_3^* =0.300000	y_3^* =1.100000	$f(x_3^*, y_3^*)$ =-2.073656941709e-001	$p(x_3^*, y_3^*)$ =-2.073686066688e-001
x_3^* =0.300000	y_4^* =1.300000	$f(x_3^*, y_4^*)$ =-4.653579068978e-001	$p(x_3^*, y_4^*)$ =-4.653499246673e-001
x_3^* =0.300000	y_5^* =1.500000	$f(x_3^*, y_5^*)$ =-6.202709530749e-001	$p(x_3^*, y_5^*)$ =-6.202571796714e-001
x_4^* =0.400000	y_1^* =0.700000	$f(x_4^*, y_1^*)$ =8.789600231744e-001	$p(x_4^*, y_1^*)$ =8.789698088632e-001
x_4^* =0.400000	y_2^* =0.900000	$f(x_4^*, y_2^*)$ =3.586506258818e-001	$p(x_4^*, y_2^*)$ =3.586461262622e-001
x_4^* =0.400000	y_3^* =1.100000	$f(x_4^*, y_3^*)$ =-5.525282116814e-002	$p(x_4^*, y_3^*)$ =-5.525550275112e-002
x_4^* =0.400000	y_4^* =1.300000	$f(x_4^*, y_4^*)$ =-3.626795115028e-001	$p(x_4^*, y_4^*)$ =-3.626710656197e-001
x_4^* =0.400000	y_5^* =1.500000	$f(x_4^*, y_5^*)$ =-5.675647436551e-001	$p(x_4^*, y_5^*)$ =-5.675506875505e-001
x_5^* =0.500000	y_1^* =0.700000	$f(x_5^*, y_1^*)$ =1.136610910158e+000	$p(x_5^*, y_1^*)$ =1.136620257569e+000
x_5^* =0.500000	y_2^* =0.900000	$f(x_5^*, y_2^*)$ =5.749803409475e-001	$p(x_5^*, y_2^*)$ =5.749759828899e-001
x_5^* =0.500000	y_3^* =1.100000	$f(x_5^*, y_3^*)$ =1.159923767920e-001	$p(x_5^*, y_3^*)$ =1.159892117227e-001
x_5^* =0.500000	y_4^* =1.300000	$f(x_5^*, y_4^*)$ =-2.385683040123e-001	$p(x_5^*, y_4^*)$ =-2.385604240142e-001
x_5^* =0.500000	y_5^* =1.500000	$f(x_5^*, y_5^*)$ =-4.914343936557e-001	$p(x_5^*, y_5^*)$ =-4.914210766559e-001
x_6^* =0.600000	y_1^* =0.700000	$f(x_6^*, y_1^*)$ =1.406041798905e+000	$p(x_6^*, y_1^*)$ =1.406050548790e+000
x_6^* =0.600000	y_2^* =0.900000	$f(x_6^*, y_2^*)$ =8.059414940631e-001	$p(x_6^*, y_2^*)$ =8.059375035947e-001
x_6^* =0.600000	y_3^* =1.100000	$f(x_6^*, y_3^*)$ =3.044292210453e-001	$p(x_6^*, y_3^*)$ =3.044256744904e-001
x_6^* =0.600000	y_4^* =1.300000	$f(x_6^*, y_4^*)$ =-9.501613009962e-002	$p(x_6^*, y_4^*)$ =-9.500895352837e-002
x_6^* =0.600000	y_5^* =1.500000	$f(x_6^*, y_5^*)$ =-3.939023077456e-001	$p(x_6^*, y_5^*)$ =-3.938900917542e-001
x_7^* =0.700000	y_1^* =0.700000	$f(x_7^*, y_1^*)$ =1.685783515309e+000	$p(x_7^*, y_1^*)$ =1.685791033487e+000
x_7^* =0.700000	y_2^* =0.900000	$f(x_7^*, y_2^*)$ =1.049881153064e+000	$p(x_7^*, y_2^*)$ =1.049878006340e+000

x_i^*	y_j^*	$f(x_i^*, y_j^*)$	$p(x_i^*, y_j^*)$
x_7^* =0.700000	y_3^* =1.100000	$f(x_7^*, y_3^*)$ =5.082937839397e-001	$p(x_7^*, y_3^*)$ =5.082908357562e-001
x_7^* =0.700000	y_4^* =1.300000	$f(x_7^*, y_4^*)$ =6.614879670648e-002	$p(x_7^*, y_4^*)$ =6.615634386620e-002
x_7^* =0.700000	y_5^* =1.500000	$f(x_7^*, y_5^*)$ =-2.768343417776e-001	$p(x_7^*, y_5^*)$ =-2.768223818751e-001
x_8^* =0.800000	y_1^* =0.700000	$f(x_8^*, y_1^*)$ =1.974574556652e+000	$p(x_8^*, y_1^*)$ =1.974581029227e+000
x_8^* =0.800000	y_2^* =0.900000	$f(x_8^*, y_2^*)$ =1.305334967651e+000	$p(x_8^*, y_2^*)$ =1.305332339748e+000
x_8^* =0.800000	y_3^* =1.100000	$f(x_8^*, y_3^*)$ =7.259923711112e-001	$p(x_8^*, y_3^*)$ =7.259890468767e-001
x_8^* =0.800000	y_4^* =1.300000	$f(x_8^*, y_4^*)$ =2.432541841813e-001	$p(x_8^*, y_4^*)$ =2.432607739624e-001
x_8^* =0.800000	y_5^* =1.500000	$f(x_8^*, y_5^*)$ =-1.419496597088e-001	$p(x_8^*, y_5^*)$ =-1.419392172271e-001

讨论 5

5.1 向量求导在曲线拟合中的应用

在曲线拟合中，我发现如果应用向量的求导法则，那么能大大简化证明的过程.
首先给出条件:

$$\boldsymbol{c} = (c_0, c_1, \cdots, c_n)^T$$

$$\boldsymbol{y} = (y_0, y_1, \cdots, y_m)^T$$

$$\boldsymbol{A} = \begin{bmatrix} \varphi_0(x_0) & \varphi_1(x_0) & \cdots & \varphi_k(x_0) \\ \varphi_0(x_1) & \varphi_1(x_1) & \cdots & \varphi_k(x_1) \\ \vdots & \vdots & \ddots & \vdots \\ \varphi_0(x_m) & \varphi_1(x_m) & \cdots & \varphi_k(x_m) \end{bmatrix}$$

然后我们将误差函数写成向量相乘的形式:

$$\begin{aligned} \boldsymbol{F}(\boldsymbol{c}) &= \sum_{i=0}^{m} \left[f(x_i) - \sum_{k=0}^{n} c_k \varphi_k(x_i) \right]^2 && (5.1) \\ &= (\boldsymbol{y} - \boldsymbol{A}\boldsymbol{c})^T (\boldsymbol{y} - \boldsymbol{A}\boldsymbol{c}) && (5.2) \\ &= \boldsymbol{y}^T \boldsymbol{y} - 2\boldsymbol{c}^T \boldsymbol{A}^T \boldsymbol{y} + C^T \boldsymbol{A}^T \boldsymbol{A}\boldsymbol{c} && (5.3) \end{aligned}$$

然后应用向量求导法则，令其关于 $\boldsymbol{c}$ 的偏导为 0

$$\frac{\partial \boldsymbol{F}}{\partial \boldsymbol{c}} = -2\boldsymbol{A}^T Y + 2\boldsymbol{A}^T \boldsymbol{A}\boldsymbol{c} = 0 \tag{5.4}$$

即可解得所求的 $\boldsymbol{c}$:

$$\boxed{\boldsymbol{c} = (\boldsymbol{A}^T \boldsymbol{A})^{-1} \boldsymbol{A}^T \boldsymbol{y}} \tag{5.5}$$

5.2 矩阵求导在曲面拟合中的应用

收到之前的启发，我开始思考，在曲面拟合中，也能用类似的方法简化流程吗？

在翻阅了一些关于矩阵求导的资料后，我开始了尝试，最后成功得出了相同的结果，这是将曲线拟合从一维推广到二维的情况，只要我们注意观察比较，就能发现其中的相似之处并得到启发，而且如果关于矩阵的掌握熟练的话，那么计算的过程将会更简单自然。

首先受到式 (5.1) 的启发，我们第一步应该尝试着将误差函数转化为矩阵乘积的形式:

$$\boldsymbol{B} = \left[\varphi_r(x_i)\right]_{(m+1)\times(M+1)} = \begin{bmatrix} \varphi_0(x_0) & \varphi_1(x_0) & \cdots & \varphi_M(x_0) \\ \varphi_0(x_1) & \varphi_1(x_1) & \cdots & \varphi_M(x_1) \\ \vdots & \vdots & \ddots & \vdots \\ \varphi_0(x_m) & \varphi_1(x_m) & \cdots & \varphi_M(x_m) \end{bmatrix}$$

$$\boldsymbol{G} = \left[\psi_s(y_j)\right]_{(n+1)\times(N+1)} = \begin{bmatrix} \psi_0(y_0) & \psi_1(y_0) & \cdots & \psi_N(y_0) \\ \psi_0(y_1) & \psi_1(y_1) & \cdots & \psi_N(y_1) \\ \vdots & \vdots & \ddots & \vdots \\ \psi_0(y_n) & \psi_1(y_n) & \cdots & \psi_N(y_n) \end{bmatrix}$$

$$\boldsymbol{C} = \left[c_{rs}\right]_{(M+1)\times(N+1)}$$

我们仔细观察上面的三个矩阵，可以发现: 如果将 $c_{rs}\varphi_r(x_i)\psi_s(y_j)$ 排成一个 $(m+1)\times(n+1)$ 的矩阵 $\boldsymbol{P}$，那么这个矩阵可以拆分成矩阵 $\boldsymbol{B}, \boldsymbol{C}, \boldsymbol{G}^T$ 的乘积。[1]

$$\left[\varphi_r(x_i)c_{rs}\psi_s(y_j)\right]_{(m+1)\times(n+1)} = \boldsymbol{P} = \boldsymbol{BCG}^T \tag{5.6}$$

而式 (2.5) 则刚好为矩阵 $\boldsymbol{BCG}^T - \boldsymbol{U}$ 的 Euclid—范数的平方, 即：

$$L(\boldsymbol{C}) = \|\boldsymbol{BCG}^T - \boldsymbol{U}\|_E^2 \tag{5.7}$$

我们想让 $L(\boldsymbol{C})$ 取最小值，这时应该使 $L(\boldsymbol{C})$ 在 $\boldsymbol{C}^*$ 处的导数 $L(\boldsymbol{C}^*) = 0$，于是这时我们用矩阵的求导法则，对 $L(\boldsymbol{C})$ 求关于 $\boldsymbol{C}$ 的偏导，得:

$$\boxed{\frac{\partial L}{\partial \boldsymbol{C}} = 2\boldsymbol{B}^T(\boldsymbol{BCG}^T - \boldsymbol{U})\boldsymbol{G}} \tag{5.8}$$

关于式 (5.8) 的证明如下:

$$L(\boldsymbol{C}) = \|\boldsymbol{BCG}^T - \boldsymbol{U}\|_E^2 \tag{5.9}$$

$$= \sum_{i=0}^{m}\sum_{j=0}^{n}\left[\sum_{r=0}^{M}\sum_{s=0}^{N}\varphi_r(x_i)c_{rs}\psi_s(y_j) - u_{ij}\right]^2 \tag{5.10}$$

[1] 在 $k < N$ 的情况下，它都成立

$$\frac{\partial L}{\partial c_{rs}} = 2\sum_{i=0}^{m}\sum_{j=0}^{n}\Big[\psi_s(y_j)\varphi_r(x_i)c_{rs}\psi_s(y_j)\varphi_r(x_i) - \psi_s(y_j)u_{ij}\varphi_r(x_i)\Big] \tag{5.11}$$

$$= 2\sum_{i=0}^{m}\sum_{j=0}^{n}\psi_s(y_j)[\varphi_r(x_i)c_{rs}\psi_s(y_j)]\varphi_r(x_i) - 2\sum_{i=0}^{m}\sum_{j=0}^{n}\psi_s(y_j)u_{ij}\varphi_r(x_i) \tag{5.12}$$

$$\left[\frac{\partial L}{\partial c_{rs}}\right]_{(M+1)\times(N+1)} = 2\boldsymbol{B}^T(\boldsymbol{B}\boldsymbol{C}\boldsymbol{G}^T)\boldsymbol{G} - 2\boldsymbol{B}^T\boldsymbol{U}\boldsymbol{G} \tag{5.13}$$

我们可以发现式 (5.12) 到式 (5.13) 的转化和式 (5.6) 是类似的。

当式 (5.8) 为 0 时，解得：

$$\boldsymbol{C} = (\boldsymbol{B}^T\boldsymbol{B})^{-1}\boldsymbol{B}^T\boldsymbol{U}\boldsymbol{G}(\boldsymbol{G}^T\boldsymbol{G})^{-1} \tag{5.14}$$

这和式 (2.6) 是相同的，于是我们找到了一种更简洁的方法证明了这个结论。

可见，如果我们能掌握一些关于矩阵求导的知识，在数值分析中对简化运算是有非常大帮助的。