Introduction

There are 15 items of 4 different product family and the production process consists of two stages: production and packaging of items. For production process, each of the 15 items of 4 different product family have its own processing time, minor setup time, major setup time when changing between different item families and weight; for the packaging process, only items of product families 1, 2 and 3 will be packaged with their own processing time, minor and major setup times. Minor setup times and major setup times of packaging for packaging are half of those times of production process.

The goal is to schedule the production and the packaging of these items in the next two days (six shifts) while minimizing some objective function. There are 3 production and 2 packaging lines with relative capacities and for each item. There is a labor requirement for both processes; for production lines labor requirements are as shown in then table, for packaging lines, labor requirements are 12 and 15 workers respectively.

There are seven given objective functions: make span, flowtime, weighted flowtime, tardiness, weighted tardiness, number of tardy jobs, weighted number of tardy jobs. However, minimizing one objective does not necessarily mean that the other six would be minimized to the optimum as well. So, it is imperative to formulate a model that can reasonably minimize all six objectives in a reasonable time frame (CPU time).

Formulation

Sets and Indices:

- $f \in F = \{1, 2, 3, 4\}$: Set of product types f
- $j^f \in J^f$: Set of items j in type f
- $l \in L = \{1, 2, 3\}$: Set of production lines l
- $p \in P = \{4, 5\}$: Set of packing lines p
- $m \in M = L \cup P = \{1, 2, 3, 4, 5\}$: Set of lines m
- t_m: Position t of a production job for an item on machine m
    - For each machine m, there will be 7 possible slots, because even if the fastest jobs are put on the fastest line (which is line 1), and even if the major set up times are

excluded, then no more than 7 jobs will be able to be placed on one production line. The limit of 7 is in order to decrease the number of variables and constraints and make the model more efficient.

Parameters:

- P_{jf}: Minor setup time of production of item j of type f on line m
- P_{fg}: Major setup time when switching from production of an item in type f to one in type g. Note that $P_{ffm} = 0 \; \forall \; f \in F; \; \forall \; m \in M$
- S_m^C: Setup coefficient for a line m. This parameter is equal to 1 for $m \in L$ and $1/2$ for $m \in P$
- C_{jf}^p: Capacity of line 1 for production of item j of type f
- D_{jf}: Demand of item j of type f
- L_{jf_m}: Labor required for processing of item j of family f on line m
- α_m: Relative capacity of line m
- w_{jf}: Weight of job j

Decision variables:

- $x_{jf_{t_m}}: \begin{cases} 1 & if\ the\ job\ j\ of\ type\ f\ is\ placed\ in\ position\ t\ on\ machine\ m \\ 0 & otherwise \end{cases}$
- $y_{gt_m}: \begin{cases} 1 & if\ the\ job\ before\ the\ job\ in\ position\ t_m\ is\ of\ family\ g \\ 0 & otherwise \end{cases}$
    - Note: $P_{fg} y_{gt_m}$ will determine the major set up necessary for the job in position t_m. If the job before it is of the same type, then that value will be 0, since $P_{ff} = 0$
- $C_{jf_{tm}}$: Completion time of job j of family f in position t on line m
- C_{jf}: Completion time of job j of family f
- $S_{jf_{t_m}}$: Start of processing of job j of family f in position t on line m. This start time is only the processing time and does not include the minor and major – if any – setup times. It could be detached from the processing time. It is, essentially, the completion time of the setup times

Constraints:

$$\sum_{t_m \in T_m, m \in L} x_{j^f t_m} = 1 \qquad \forall\, j^f \in J^f\ \forall\, f \in F$$

- All jobs cannot be split, preempted, and must be placed in one position on the production machines

$$\sum_{t_p \in T_p, p \in P} x_{j^f t_p} = 1 \qquad \forall\, j^f \in J^f\ \forall\, f \in \{1, 2, 3\}$$

- Jobs of families 1, 2, and 3 cannot be split, preempted, and must be placed in one position on the packing machines

$$\sum_{t_p \in T_p, p \in P} x_{j^4 t_p} = 0 \qquad \forall\, j^4 \in J^4$$

- Jobs of family 4 cannot be assigned to the packing lines

$$\sum_{j \in J^f, f \in F} x_{j^f t_m} \le 1 \qquad \forall\, t_m \in T_m\ \forall\, m \in M$$

- One job at most in each position

$$x_{j^f t_m} \le \sum_{i^g \in J^g\ \forall\ g \in F} x_{i^g t_m - 1} \qquad \forall\, j^f \in J^f\ \forall\, f \in F; \forall\, t_m \in (T_m - \{1_m\})\ \forall\, m \in M$$

- Do not allow a job in a position t unless there is a job in the previous position

$$C_{j^f} \ge C_{j t_m} \qquad \forall\, j^f \in J^f\ \forall\, f \in F; \forall\, t_m \in T_m\ \forall\, m \in M$$

- Getting **final** completion time of a job j of family f

$$C_{j^f t_m} \ge \sum_{i^g \in J^g, g \in F} S_{j^f t_m} + \frac{D_{j^f}}{\alpha_m \times C^p_{j^f}} - M\left(1 - x_{j^f t_m}\right)$$

$$\forall\, j \in J^f\ \forall\, f \in F; \forall\, t_m \in T_m\ \forall\, m \in M$$

- Completion time of the production part of job j is the start time of that job, plus the processing time. If the job isn't at the specified position, the constraint is nonbinding

$$S_{j^f t_m} \geq \sum_{i^g \in J^g, g \in F} C_{i^g, t_m - 1} + S_m^C P_{j^f} + \sum_g S_m^C P_{fg} y_{g t_m} - M\left(1 - x_{j^f t_m}\right)$$

$$\forall j \in J^f \ \forall f \in F; \forall t_m \in (T_m - \{1_m\}) \ \forall m \in M$$

- Start time of a job j is the completion time of the previous job on the same line plus the minor start up time, plus the major start up time if needed. If the job isn't at the specified position, then the constraint is nonbinding
- Note that $C_{j^f 0_m} = 0 \ \forall j^f \in J^f \ \forall f \in F; \forall m \in M$

$$S_{j^f t_p} \geq \sum_{\tau_l \in T_l \ \forall l \in L} S_{j^f \tau_l} + S_p^C P_{j^f} + \sum_g S_p^C P_{fg} y_{g t_p} - M\left(1 - x_{j^f t_p}\right)$$

$$\forall j \in J^f \ \forall f \in \{1, 2, 3\}; \forall t_p \in T_p \ \forall p \in P$$

- The starting time of a job j of family f on the package lines can start after that job's setup time is completed plus the setup time of that job on the package line. If the job isn't at the specified position, then the constraint is nonbinding

$$y_{g t_m} \geq \sum_{i^g \in J^g} x_{i^g, t_m - 1} \qquad \forall g \in G \ \forall t_m \in (T_m - \{1_m\}) \ \forall m \in M$$

- Major setup time is required if the job in the position before is of a different family

Boundaries:

- $x_{j^f t_m} \in \{0,1\}$ $\qquad \forall j^f \in J^f \ \forall f \in F; \ \forall t_m \in T_m \ \forall m \in M$
- $y_{g t_m} \in \{0,1\}$ $\qquad \forall g \in F; \ \forall t_m \in T_m \ \forall m \in M$
- $C_{j^f tm} \geq 0$ $\qquad \forall j^f \in J^f \ \forall f \in F; \ \forall t_m \in T_m \ \forall m \in M$
- $C_{j^f} \geq 0$ $\qquad \forall j^f \in J^f \ \forall f \in F$
- $S_{j^f t_m} \geq 0$ $\qquad \forall j^f \in J^f \ \forall f \in F; \forall t_m \in T_m \ \forall m \in M$

Extra Variables and Objective Function Formulation

For the solver to be able to calculate the objective functions accurately, extra variables may be needed:

- C_{max}: Maximum completion time
- T_{jf}: Tardiness of job j of type f
- $U_{jf} = \begin{cases} 1 & if\ job\ j\ of\ type\ f\ is\ tardy \\ 0 & otherwise \end{cases}$

Relevant Constraints:

- $C_{max} \geq C_{jf}\ \forall j^f \in J^f\ \forall\ f \in F$
- $T_{jf} \geq C_{jf} - d\ \forall\ j^f \in J^f\ \forall\ f \in F$, where $d = 48$ hours
- $M \times U_{jf} \geq T_{jf}\ \forall\ j^f \in J^f\ \forall\ f \in F$

Boundaries:

- $C_{max} \geq 0$
- $T_{jf} \geq 0\ \forall\ j^f \in J^f\ \forall\ f \in F$
- $U_{jf} \in \{0,1\}\ \forall\ j^f \in J^f\ \forall\ f \in F$

Objective Functions:

- $z_{makespan} = C_{max}$
- $$z_{flowtime} = \sum_{j^f \in J^f\ \forall\ f \in F} C_{jf}$$
- $$z_{weighted\ flowtime} = \sum_{j^f \in J^f\ \forall\ f \in F} w_{jf} C_{jf}$$
- $$z_{Tardiness} = \sum_{j^f \in J^f\ \forall\ f \in F} T_{jf}$$
- $$z_{Weighted\ Tardiness} = \sum_{j^f \in J^f\ \forall\ f \in F} w_{jf} T_{jf}$$
- $$z_{Tardy\ Jobs} = \sum_{j^f \in J^f\ \forall\ f \in F} U_{jf}$$

- $$z_{Weighted\ Tardy\ Jobs} = \sum_{j^f \in J^f\ \forall\ f \in F} w_{jf} U_{jf}$$

Results

After running the solver for a maximum of 10 minutes per objective function, the results are summarized in table 1. All of the models reached a solution within the first minute of the solve start but didn't find a better one for the remainder of the 10-minute timespan.

Objective Function	Max Completion Time	Flow Time	Weighted Flow Time	Total Tardiness	Total Weighted Tardiness	Number of Tardy Jobs	Weighted Number of Tardy Jobs	Time Taken (Secs)	Node Count
Max Completion Time	**112.18**	926.77	4558.77	349.48	1688.63	10	44	601	286562
Flow Time	135.57	**841.50**	4374.66	295.98	1404.61	7	38	1202	397748
Weighted Flow Time	138.43	921.79	**4063.13**	336.21	1106.97	9	40	601	405258
Total Tardiness	135.47	933.67	4854.90	**295.29**	1411.75	9	46	1804	287333
Total Weighted Tardiness	141.14	914.43	4063.20	327.86	**1062.17**	8	36	2406	324439
Number of Tardy Jobs	142.75	899.93	4292.44	307.84	1161.64	**6**	28	3008	253954
Weighted Number of Tardy Jobs	135.90	884.56	4344.93	302.20	1344.35	6	**27**	3610	339838

Table 1: Objective function values, time taken, and node count of each model.

Makespan Model

The Gantt Chart is given in Figure 2:

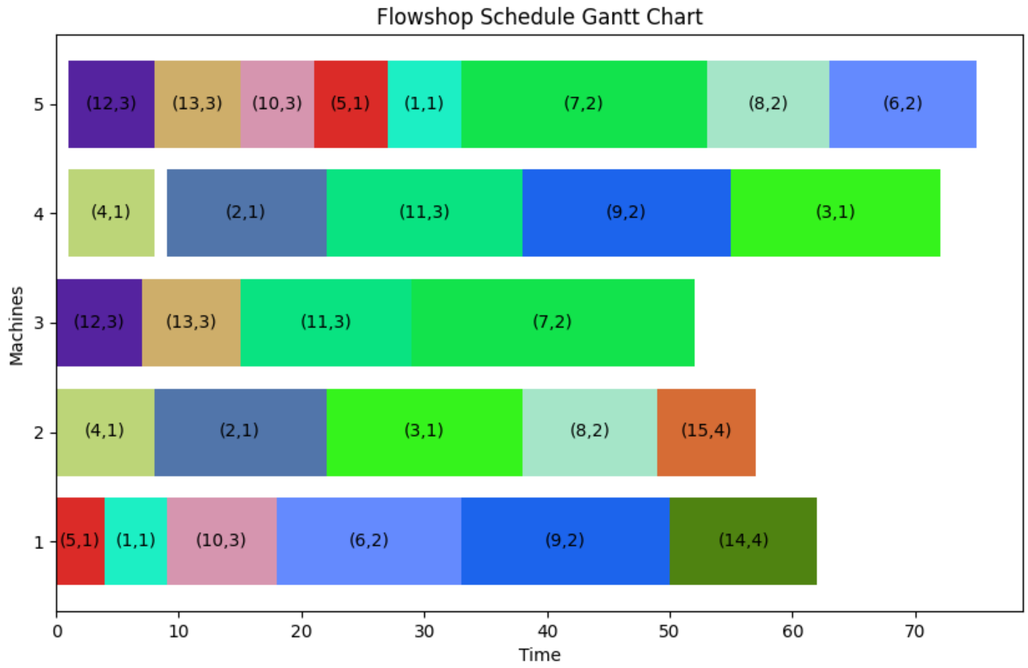

Figure 2: Gantt Chart of the makespan model. Note: the bars are labeled as (job, family). The bars do not include the setup times.

Shifts	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**
Labor Required	59	60	68	68	68	57	45	15

Table 3: Labor requirements for the makespan model.

Comparison

To evaluate between these models, a score system will be established; for each model, each objective function value it generated is compared with the other models, and if the objective function value is better than another model, it gets one point. Another measure is to calculate the relative closeness for each of the models' objective functions compared with the best performance for that objective function, meaning for a model m: $\frac{\min_{n \in models} z_{obj,n}}{z_{obj,m}}$. For both measures, the higher the better. The results are in figure 2:

Objective Function	Total Score	Total Relative Closeness
Max Completion Time	16	5.49
Flow Time	33	6.08

Weighted Flow Time	27	5.9
Total Tardiness	21	5.57
Total Weighted Tardiness	31	6.12
Number of Tardy Jobs	33	6.51
Weighted Number of Tardy Jobs	37	6.48

Table 3: Performance of each model according to total score and total gap.

The Weighted Number of Tardy Jobs model is the one that should be chosen. Its Gantt Chart is in figure 4:

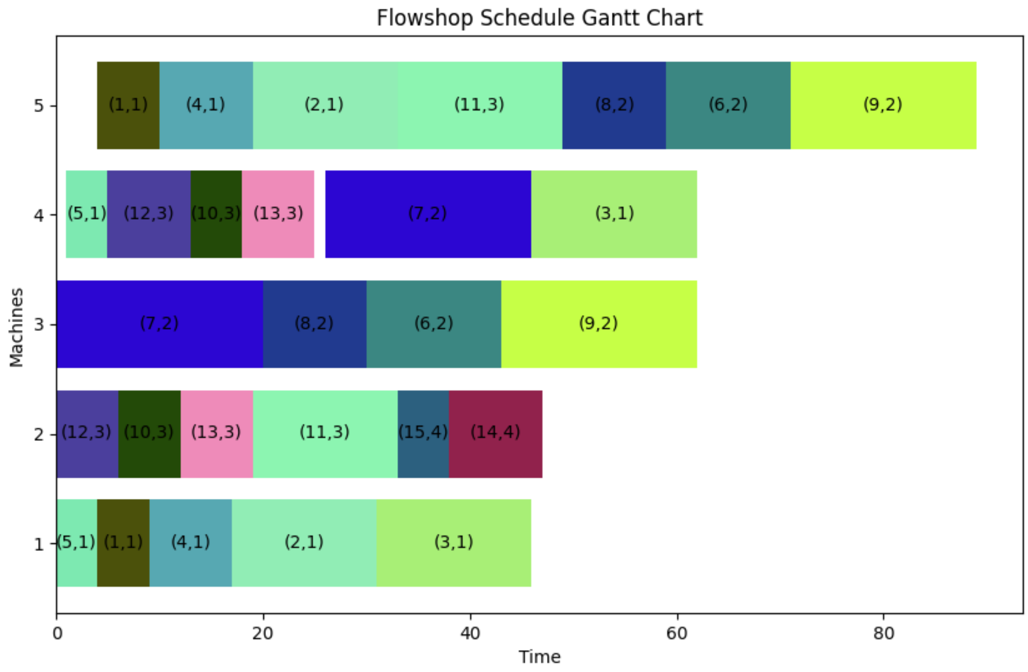

Figure 4: Gantt Chart of the Weighted Number of Tardy Jobs model.

In this scenario, compared to the single-stage case, there are idle times, visible in the packing lines, but not too much.

In both cases, family 1 jobs were assigned to machine 1, due to its higher opportunity cost for having major setup times. However, in this scenario, machine 3 saw no major setup either.

In the previous scenario, there were way less tardy jobs, but this scenario suffers from way more tardiness in general, presumably because of the extra step.

Policies/Suggestions

Since there are not many idle times, it alludes to the fact that the bottleneck is not the packing machines themselves, but the time to produce/pack. This also indicates that batching the products may not help too much.

Reducing major setup times can greatly decrease the level of tardiness.

Furthermore, postponing the deadline would decrease the level of tardiness as well.