Know your Itô - When and Where?

The wonderful world of Itô calculus

In this document W_t represents a Brownian Motion. This summarises all the results developed in class for obtaining SDEs/differentials

Itô multiplication table

$\times$	dt	dW_t
dt	$dt^2 = 0$	$dtdW_t = 0$
dW_t	$dW_t dt = 0$	$dW_t^2 = dt$

Itô I

$F = F(W_t)$.

For example functions including W_t^n; $\exp(W_t)$; $\log W_t$; $W_t \cos W_t$

$$dF = \frac{dF}{dW_t} dW_t + \frac{1}{2} \frac{d^2 F}{dW_t^2} dt.$$

Itô II

$F = F(t, W_t)$.

For example functions including $W_t^n + t^2; t\exp(W_t)$; $\log W_t + \cos tW_t$

$$dF = \left(\frac{\partial F}{\partial t} + \frac{1}{2}\frac{\partial^2 F}{\partial W_t^2}\right)dt + \frac{\partial F}{\partial W_t}dW_t.$$

Itô III

$F = F(S_t)$ where the process S_t satisfies the SDE

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

For example functions including S_t^n; $\exp(S_t)$; $\log S_t$; $S_t \sin S_t$

$$dF = \left(\mu S_t \frac{dF}{dS_t} + \frac{1}{2}\sigma^2 S_t^2 \frac{d^2F}{dS^2} \right) dt + \sigma S_t \frac{dF}{dS_t} dW_t.$$

Itô IV

$F = F\left(t, S_t\right)$ where the process S_t satisfies the SDE

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

For example functions including $t^3 + S_t^n$; $\exp\left(tS_t\right)$; $\exp\left(S_t\right) + \log tS_t$; $t^2 S_t \sin S_t$

$$dF = \left. \quad \frac{\partial F}{\partial t} + \mu S_t \frac{\partial F}{\partial S_t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 F}{\partial S_t^2}\right) dt + \sigma S_t \frac{\partial F}{\partial S_t} dW_t.$$

Itô V

This is also a basic version of higher dimensional Itô for functions of the form $F = F\left(t, S_t^{(1)}, S_t^{(2)}\right)$ where the process $S_t^{(i)}$ for $i = 1, 2$ satisfies the two-factor model

$$dS_t^{(i)} = \mu_i S_t^{(i)} dt + \sigma_i S_t^{(i)} dW_t^{(i)};\ \ i = 1, 2;\ \ dW_1 dW_2 = \rho dt$$

for $|\rho| < 1$. It can be used for a function like $F\left(t, S_t^{(1)}, S_t^{(2)}\right) = t^3 + \left(S_t^{(1)}\right)^2 + S_t^{(1)} \cos S_t^{(2)}$. Hence $dF =$

$$\left(\frac{\partial F}{\partial t} + \mu_1 S_t^{(1)} \frac{\partial F}{\partial S_t^{(1)}} + \mu_2 S_t^{(2)} \frac{\partial F}{\partial S_t^{(2)}} + \frac{1}{2}\sigma_1^2 S_t^{(1)^2} \frac{\partial^2 F}{\partial S_t^{(1)^2}} + \frac{1}{2}\sigma_2^2 S_t^{(2)^2} \frac{\partial^2 F}{\partial S_t^{(2)^2}} + \right.$$

$$\left. \rho\sigma_1\sigma_2 S_t^{(1)} S_t^{(2)} \frac{\partial^2 F}{\partial S_t^{(1)} \partial S_t^{(2)}}\right) dt$$

$$+\sigma_1 S_t^{(1)} \frac{\partial F}{\partial S_t^{(1)}} dW_t^{(1)} + \sigma_2 S_t^{(2)} \frac{\partial F}{\partial S_t^{(2)}} dW_t^{(2)}$$

General Itô

Consider a process G_t satisfying the SDE

$$dG_t = A\left(G_t, t\right) dt + B\left(G_t, t\right) dW_t$$

for a general drift $A\left(G_t, t\right)$ and diffusion $B\left(G_t, t\right)$ respectively, and $F = F\left(t, G_t\right)$, then

$$dF = \left. \quad \frac{\partial F}{\partial t} + A\left(G_t, t\right) \frac{\partial F}{\partial G_t} + \frac{1}{2} B^2\left(G_t, t\right) \frac{\partial^2 F}{\partial G_t^2} \right) dt + B\left(G_t, t\right) \frac{\partial F}{\partial G_t} dW_t.$$

Itô Product Rule

Let X_t, Y_t be two processes, where

$$\begin{aligned} dX_t &= a(t, X_t)\,dt + b(t, X_t)\,dW_t^{(1)}, \\ dY_t &= c(t, Y_t)\,dt + d(t, Y_t)\,dW_t^{(2)} \end{aligned}$$

Two-dimensional form of Taylor with $f(X_t, Y_t) = X_t Y_t$

$$df = \frac{\partial f}{\partial X_t} dX_t + \frac{\partial f}{\partial Y_t} dY_t + \frac{1}{2} \frac{\partial^2 f}{\partial X_t^2} dX_t^2 + \frac{1}{2} \frac{\partial^2 f}{\partial Y_t^2} dY_t^2 + \frac{\partial^2 f}{\partial X_t \partial Y_t} dX_t dY_t$$

$$\begin{array}{lll} \frac{\partial f}{\partial X_t} = Y_t & & \frac{\partial f}{\partial Y_t} = X_t \\ \frac{\partial^2 f}{\partial X_t^2} = 0 & \frac{\partial^2 f}{\partial Y_t^2} = 0 & \frac{\partial^2 f}{\partial X_t \partial Y_t} = 1 \end{array}$$

to give

$$d(X_t Y_t) = X_t dY_t + Y_t dX_t + dX_t dY_t.$$

Itô Quotient Rule

By applying the earlier two-dimensional form of Taylor used in the product rule with $f(X_t, Y_t) = \dfrac{X_t}{Y_t}$

$$\begin{array}{lll} \dfrac{\partial f}{\partial X_t} = 1/Y_t & & \dfrac{\partial f}{\partial Y_t} = -X_t/Y_t^2 \\ \dfrac{\partial^2 f}{\partial X_t^2} = 0 & \dfrac{\partial^2 f}{\partial Y_t^2} = 2X_t/Y_t^3 & \dfrac{\partial^2 f}{\partial X_t \partial Y_t} = -1/Y_t^2 \end{array}$$

which gives

$$d\left(\frac{X_t}{Y_t}\right) = \frac{X_t}{Y_t}\left(\frac{dX_t}{X_t} - \frac{dY_t}{Y_t} - \frac{dX_t dY}{XY_t} + \left(\frac{dY_t}{Y_t}\right)^2\right)$$

Stochastic Integration Formula I

$$\int_0^t \frac{dF}{dW_t} dW_t = F\left(W_t\right) - F\left(W_0\right) - \tfrac{1}{2}\int_0^t \frac{d^2F}{dW_s^2} ds$$

Example:

$$\int_0^t W_t \sin W_t dW_t$$

Stochastic Integration Formula II

$$\int_0^t \frac{\partial F}{\partial W_t} dW_t = F(t, W_t) - F(0, W_0) - \int_0^t \left(\frac{\partial F}{\partial s} + \frac{1}{2} \frac{\partial^2 F}{\partial W_s^2} \right) ds$$

Example:

$$\int_0^t \left(t W_t + e^{W_t} \right) dW_t$$

Itô Integral - definition

$$\int_0^T f(t, W_t)\, dW_t = \lim_{N \to \infty} \sum_{i=0}^{N-1} f(t_i, W_i)(W_{i+1} - W_i),$$

where $W_i \equiv W_{t_i}$