Sheet 1 Solutions

1. Consider the Forward Kolmogorov equation (FKE), given by

$$\frac{\partial p}{\partial t'} = c^2 \frac{\partial^2 p}{\partial y'^2} \tag{1.1}$$

for the transition density function $p(y, t\ ; y', t')$; $c^2 \in \mathbb{R}^+$. The states (y, t) are past and are **fixed** while (y', t') refers to future ones and are variables. By simple substitution show that

$$p(y, t; y', t') = \frac{1}{2c\sqrt{\pi(t'-t)}} \exp\left(-\frac{(y'-y)^2}{4c^2(t'-t)}\right), \tag{1.2}$$

<u>satisfies</u> the FKE. **You may drop the (y, t) from your working as they won't change.**

$$\begin{aligned}
\frac{\partial p}{\partial t'} &= -\frac{1}{2c\sqrt{\pi}}\left(\frac{1}{2(t'-t)^{3/2}}\right)\exp(\cdots) + \frac{1}{2c\sqrt{\pi}}\left(\frac{1}{(t'-t)^{1/2}}\right)\left(\frac{(y'-y)^2}{4c^2(t'-t)^2}\right)\exp(\cdots). \\
&= -\frac{1}{4c\sqrt{\pi}(t'-t)^{3/2}}\exp(\cdots) + \frac{1}{8c^3\sqrt{\pi}}\frac{(y'-y)^2}{(t'-t)^{5/2}}\exp(\cdots) \qquad (1)
\end{aligned}$$

$$\frac{\partial p}{\partial y'} = \frac{1}{2c\sqrt{\pi}}\left(\frac{1}{(t'-t)^{1/2}}\right)\left(\frac{(y'-y)}{2c^2(t'-t)}\right)\exp(\cdots) = \frac{1}{4c^3\sqrt{\pi}}\frac{(y'-y)}{(t'-t)^{3/2}}\exp(\cdots)$$

and

$$\begin{aligned}
\frac{\partial^2 p}{\partial y'^2} &= \frac{-1}{4c^3\sqrt{\pi}}\left(\frac{1}{(t'-t)^{3/2}}\right)\exp(\cdots) + \frac{1}{4c^3\sqrt{\pi}}\left(\frac{1}{(t'-t)^{3/2}}\right)(y'-y)\left(\frac{(y'-y)}{2c^2(t'-t)}\right)\exp(\cdots) \\
&= \frac{-1}{4c^3\sqrt{\pi}(t'-t)^{3/2}}\exp(\cdots) + \frac{1}{8c^5\sqrt{\pi}}\frac{(y'-y)^2}{(t'-t)^{5/2}}\exp(\cdots) \qquad (2)
\end{aligned}$$

Substituting (1) and (2) in to (1.1) gives the result.

Show that (1.2) satisfies

$$\int_{\mathbb{R}} p(y, t; y', t')dy' = 1.$$

This requires integration by substitution

$$\frac{1}{2c\sqrt{\pi(t'-t)}}\int_{\mathbb{R}} \exp\left(-\frac{(y'-y)^2}{4c^2(t'-t)}\right) dy'$$

Put $u = \frac{y'-y}{2c\sqrt{t'-t}} \to 2c\sqrt{t'-t}du = dy'$

$$\begin{aligned}
&= \frac{1}{2c\sqrt{\pi(t'-t)}} 2c\sqrt{t'-t}\int_{\mathbb{R}} e^{-u^2} du \\
&= \frac{1}{\sqrt{\pi}}\sqrt{\pi} = 1
\end{aligned}$$

2. Consider a **symmetric** random walk which starts with a marker placed at a point x at time s; written (x,s). Suppose at a later time $t>s$ the marker is at y; the future state denoted (y,t). The marker can move in step sizes of δy in a time step of δt. At the previous step the marker must have been at one of $(y-\delta y, t-\delta t)$ or $(y+\delta y, t-\delta t)$. The transition probability density function of the position y of the diffusion at a later time t, is written $p(x,s;y,t)$. Derive the Forward Equation

$$\frac{\partial p}{\partial t}=\tfrac{1}{2}\frac{\partial^2 p}{\partial y^2}.$$

You may omit the dependence on (x,s) in your working as they will not change.

$$p(y',t')=\tfrac{1}{2}p(y'+\delta y,t'-\delta t)+\tfrac{1}{2}p(y'-\delta y,t'-\delta t)$$

Taylor series expansion gives

$$\begin{aligned}
p(y'+\delta y,t'-\delta t) &= p(y',t')-\frac{\partial p}{\partial t'}\delta t+\frac{\partial p}{\partial y'}\delta y+\tfrac{1}{2}\frac{\partial^2 p}{\partial y'^2}\delta y^2+\ldots\\
p(y'-\delta y,t'-\delta t) &= p(y',t')-\frac{\partial p}{\partial t'}\delta t-\frac{\partial p}{\partial y'}\delta y+\tfrac{1}{2}\frac{\partial^2 p}{\partial y'^2}\delta y^2+\ldots
\end{aligned}$$

Substituting into the above

$$\begin{aligned}
p(y',t') &= \tfrac{1}{2}\left(p(y',t')-\frac{\partial p}{\partial t'}\delta t+\frac{\partial p}{\partial y'}\delta y+\tfrac{1}{2}\frac{\partial^2 p}{\partial y'^2}\delta y^2\right)\\
&\quad+\tfrac{1}{2}\left(p(y',t')-\frac{\partial p}{\partial t'}\delta t-\frac{\partial p}{\partial y'}\delta y+\tfrac{1}{2}\frac{\partial^2 p}{\partial y'^2}\delta y^2\right)
\end{aligned}$$

$$0=-\frac{\partial p}{\partial t'}\delta t+\tfrac{1}{2}\frac{\partial^2 p}{\partial y'^2}\delta y^2$$

$$\frac{\partial p}{\partial t'}=\tfrac{1}{2}\frac{\delta y^2}{\delta t}\frac{\partial^2 p}{\partial y'^2}$$

Now take limits. This only makes sense if $\frac{\delta y^2}{\delta t}$ is $O(1)$, i.e. $\delta y^2\sim O(\delta t)$ and letting $\delta y,\ \delta t\longrightarrow 0$ gives the equation

$$\frac{\partial p}{\partial t'}=\tfrac{1}{2}\frac{\partial^2 p}{\partial y'^2}$$

3. A FKE of the following form is given

$$\frac{\partial p}{\partial t}=\tfrac{1}{2}\frac{\partial^2 p}{\partial y^2},\tag{3.1}$$

for the transition probability density function $p(y,t)$. At time t, the diffusion has position y. Assume a solution of (3.1) exists and takes the following form

$$p(y,t)=t^{-1/2}f(\eta);\ \eta=\frac{y}{t^{1/2}}.$$

Solve (3.1) to show that a particular solution of this is

$$p(y,t)=\frac{1}{\sqrt{2\pi t}}\exp\left(-\frac{y^2}{2t}\right).$$

We will need the appropriate derivatives

$$\frac{\partial \eta}{\partial y}=t^{-1/2};\ \frac{\partial \eta}{\partial t}=-\frac{1}{2}yt^{-3/2}$$

write

$$p(y,t) = t^{-1/2} f(\eta)$$

therefore

$$\frac{\partial p}{\partial y} = \frac{\partial p}{\partial \eta}\frac{\partial \eta}{\partial y} = t^{-1/2} f'(\eta) \times t^{-1/2} = t^{-1} f'(\eta)$$

$$\frac{\partial^2 p}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial p}{\partial y}\right) = \frac{\partial}{\partial y}\left(t^{-1} f'(\eta)\right) = t^{-3/2} f''(\eta)$$

$$\begin{aligned}
\frac{\partial p}{\partial t} &= t^{-1/2}\frac{\partial}{\partial t} f(\eta) - \frac{1}{2} t^{-3/2} f(\eta) \\
&= t^{-1/2}\left(-\frac{1}{2} y t^{-3/2}\right) f'(\eta) - \frac{1}{2} t^{-3/2} f(\eta) \\
&= -\frac{1}{2}\eta t^{-3/2} f'(\eta) - \frac{1}{2} t^{-3/2} f(\eta)
\end{aligned}$$

and then substituting

$$\begin{aligned}
\frac{\partial p}{\partial t} &= -\frac{1}{2} t^{-3/2}\left(\eta f'(\eta) + f(\eta)\right) \\
\frac{\partial^2 p}{\partial y^2} &= t^{-3/2} f''(\eta)
\end{aligned}$$

gives

$$-\frac{1}{2} t^{-3/2}\left(\eta f'(\eta) + f(\eta)\right) = \frac{1}{2} t^{-3/2} f''(\eta)$$

simplifying to the ODE

$$-\left(f + \eta f'\right) = f''.$$

We have an exact derivative on the lhs, i.e. $\frac{d}{d\eta}(\eta f) = f + \eta f'$, hence

$$-\frac{d}{d\eta}(\eta f) = f''$$

and we can integrate once to get

$$-\eta f = f' + K.$$

We set $K = 0$ in order to get the correct solution, i.e.

$$-\eta f = f'$$

which can be solved as a simple first order variable separable equation:

$$f(\eta) = A\exp\left(-\tfrac{1}{2}\eta^2\right)$$

A is a normalizing constant, so write

$$\underbrace{A\int_{\mathbb{R}} \exp\left(-\tfrac{1}{2}\eta^2\right) d\eta}_{=\sqrt{2\pi}} = 1 \rightarrow A = \frac{1}{\sqrt{2\pi}}$$

$$p(y,t) = t^{-1/2} f(\eta) \text{ becomes } p(y,t) = \frac{1}{\sqrt{2\pi t}}\exp\left(-\frac{y^2}{2t}\right).$$