# Mean Square Convergence

Consider a function $F(X)$. If

$$\mathbb{E}\left[(F(X) - l)^2\right] \longrightarrow 0$$

then we say that $F(X) = l$ in the *mean square limit*, also called *mean square convergence*. We present a full derivation of the mean square limit. Starting with the quantity:

$$\mathbb{E}\left[\left(\sum_{j=1}^{n}\Big(X(t_j) - X(t_{j-1})\Big)^2 - t\right)^2\right]$$

where $t_j = \dfrac{jt}{n} = j\Delta t$.

Hence we are saying that *up to mean square convergence,*

$$dX^2 = dt.$$

This is the symbolic way of writing this property of a Wiener process, as the partitions $\Delta t$ become smaller and smaller.

## Developing the terms inside the expectation

First, we will simplify the notation in order to deal more easily with the outer (rightmost) squaring. Let $Y(t_j) = \left(X(t_j) - X(t_{j-1})\right)^2$, then we can rewrite the expectation as:

$$\mathbb{E}\left[\left(\sum_{j=1}^{n} Y(t_j) - t\right)^2\right]$$

Expanding we have:

$$\mathbb{E}\left[(Y(t_1) + Y(t_2) + \ldots + Y(t_n) - t) \times (Y(t_1) + Y(t_2) + \ldots + Y(t_n) - t)\right]$$

The term inside the Expectation is equal to

$$
\begin{aligned}
&Y(t_1)^2 + Y(t_1)Y(t_2) + \ldots + Y(t_1)Y(t_n) - Y(t_1)t \\
&+Y(t_2)^2 + Y(t_2)Y(t_1) + \ldots + Y(t_2)Y(t_n) - Y(t_2)t \\
&\vdots \\
&+Y(t_n)^2 + Y(t_n)Y(t_1) + \ldots + Y(t_n)Y(t_{n-1}) - Y(t_n)t \\
&-tY(t_1) - tY(t_2) - \ldots - tY(t_n) + t^2
\end{aligned}
$$

Rearranging

$$
\begin{aligned}
&Y(t_1)^2 + Y(t_2)^2 + \ldots + Y(t_n)^2 \\
&2Y(t_1)Y(t_2) + 2Y(t_1)Y(t_3) + \ldots + 2Y(t_{n-1})Y(t_n) \\
&-2Y(t_1)t - 2Y(t_2)t - \ldots - 2Y(t_n)t \\
&+t^2
\end{aligned}
$$

We can now factorize to get

$$
\sum_{j=1}^{n} Y(t_j)^2 + 2\sum_{i=1}^{n}\sum_{j<i} Y(t_i)Y(t_j) - 2t\sum_{j=1}^{n} Y(t_j) + t^2
$$

Substituting back $Y(t_j) = \Big(X(t_j) - X(t_{j-1})\Big)^2$ and taking the expectation, we arrive at:

$$\begin{aligned}
&\mathbb{E}\,[\,\sum_{j=1}^{n} \Big(X(t_j) - X(t_{j-1})\Big)^4 \\
&+2\sum_{i=1}^{n}\sum_{j<i} (X(t_i) - X(t_{i-1}))^2 \Big(X(t_j) - X(t_{j-1})\Big)^2 \\
&-2t\sum_{j=1}^{n} \Big(X(t_j) - X(t_{j-1})\Big)^2 \\
&+t^2\,]
\end{aligned}$$

# Computing the expectation

By linearity of the expectation operator, we can write the previous expression as:

$$
\begin{aligned}
&\sum_{j=1}^{n} \mathbb{E}\left[\left(X(t_j) - X(t_{j-1})\right)^4\right] \\
&+2 \sum_{i=1}^{n} \sum_{j<i} \mathbb{E}\left[(X(t_i) - X(t_{i-1}))^2 \left(X(t_j) - X(t_{j-1})\right)^2\right] \\
&-2t \sum_{j=1}^{n} \mathbb{E}\left[\left(X(t_j) - X(t_{j-1})\right)^2\right] \\
&+t^2
\end{aligned}
$$

Now, since $Z(t_j) = X(t_j) - X(t_{j-1})$ follows a Normal distribution with mean 0 and variance $\dfrac{t}{n}$ $(= dt)$, it follows (standard result) that its fourth moment is equal to $3\frac{t^2}{n^2}$. We will show this shortly.

Firstly we know that $Z(t_j) \sim N\left(0, \frac{t}{n}\right)$, i.e.

$$\mathbb{E}\left[Z(t_j)\right] = 0, \quad \mathbb{V}\left[Z(t_j)\right] = \frac{t}{n}$$

therefore we can construct its PDF. For any random variable $\psi \sim N\left(\mu, \sigma^2\right)$ its probability density is given by

$$p(\psi) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2}\frac{(\psi-\mu)^2}{\sigma^2}\right)$$

hence for $Z(t_j)$ the PDF is

$$p(z) = \frac{1}{\sqrt{t/n}\sqrt{2\pi}} \exp\left(-\frac{1}{2}\frac{z^2}{t/n}\right)$$

$$\begin{aligned}
\mathbb{E}\left[\left(X(t_j) - X(t_{j-1})\right)^4\right] &= \mathbb{E}\left[Z^4\right] \\
&= 3\frac{t^2}{n^2} \qquad \text{for } j = 1, \ldots, n
\end{aligned}$$

So

$$\begin{aligned}\mathbb{E}\left[Z^4\right] &= \int_{\mathbb{R}} Z^4 p(z)\,dz \\ &= \sqrt{\frac{n}{2t\pi}}\int_{\mathbb{R}} Z^4 \exp\left(-\frac{1}{2}\frac{z^2}{t/n}\right) dz\end{aligned}$$

now put

$$u = \frac{z}{\sqrt{t/n}} \longrightarrow du = \sqrt{n/t}dz$$

Our integral becomes

$$\sqrt{\frac{n}{2t\pi}}\int_{\mathbb{R}} \left(\sqrt{\frac{t}{n}}u\right)^4 \exp\left(-\frac{1}{2}u^2\right)\sqrt{\frac{t}{n}}du$$

$$
\begin{aligned}
&= \sqrt{\frac{1}{2\pi}}\frac{t^2}{n^2}\int_{\mathbb{R}} u^4 \exp\left(-\tfrac{1}{2}u^2\right) du \\
&= \frac{t^2}{n^2}.\sqrt{\frac{1}{2\pi}}\int_{\mathbb{R}} u^4 \exp\left(-\tfrac{1}{2}u^2\right) du \\
&= \frac{t^2}{n^2}.\mathbb{E}\left[u^4\right].
\end{aligned}
$$

So the problem reduces to finding the fourth moment of a standard normal random variable. Here we do not have to explicitly calculate any integral. Two ways to do this.

Either use the MGF and obtain the fourth moment to be three.

Or the other method is to make use of the fact that the kurtosis of the standardised normal distribution is 3.

That is

$$
\mathbb{E}\left[\frac{(\phi-\mu)^4}{\sigma^4}\right] = \mathbb{E}\left[\frac{(\phi-0)^4}{1^4}\right] = 3.
$$

Hence $\mathbb{E}\left[u^4\right] = 3$ and we can finally write $3\dfrac{t^2}{n^2}$.

and

$$\mathbb{E}\left[\Big(X(t_j) - X(t_{j-1})\Big)^2\right] = \frac{t}{n} \qquad \text{for } j = 1, \ldots, n$$

Because of the single summation, the fourth moment and the variance multiplied by $t$ actually recur $n$ times. Because of the double summation, the product of variances occurs $\frac{n(n-1)}{2}$ times.

We can now conclude that the expectation is equal to:

$$\begin{aligned} & 3n\frac{t^2}{n^2} + n(n-1)\frac{t^2}{n^2} - 2tn\frac{t}{n} + t^2 \\ = \; & 3\frac{t^2}{n} + t^2 - \frac{t^2}{n} - 2t^2 + t^2 = 2\frac{t^2}{n} \\ = \; & O\left(\frac{1}{n}\right) \end{aligned}$$

So, as our partition becomes finer and finer and $n$ tends to infinity, the quadratic variation will tend to $t$ *in the mean square limit*.