Module 1 Assignment January 2025

Instructions: This is a <u>non-assessed</u> assignment; there are a number of exercises to help consolidate some of the key material covered in module 1. The aim should be to work through the numerous problems in this sheet to help measure understanding and performance. Complete solutions will follow. Throughout this sheet X_t or X both refer to a standard Brownian motion.

1. a. Calculate the Itô integral given by

$$\int_0^t s^2 \sin X_s \, dX_s.$$

   b. Suppose the stochastic process $S(t)$ evolves according to Geometric Brownian Motion (GBM), where

$$dS = \mu S dt + \sigma S dX_t.$$

   Obtain a SDE $df(S,t)$ for each of the following functions

   i $f(S,t) = \alpha^t + \beta t S^n \qquad \alpha, \beta$ are constants

   ii $f(S,t) = \log tS + \cos tS$

2. Consider a function $V(t, S_t, r_t)$ where the two stochastic processes S_t and r_t evolve according to a two factor model given by

$$\begin{aligned} dS_t &= \mu S_t dt + \sigma S_t dX_t^{(1)} \\ dr_t &= \gamma (m - r_t) dt + c dX_t^{(2)}, \end{aligned}$$

in turn. and where $\mathbb{E}\left[dX_t^{(1)} dX_t^{(2)}\right] = \rho dt$. The parameters μ, σ, γ, m and c are constant. Let $V(t, S_t, r_t)$ be a function on $[0, T]$ with $V(0, S_0, r_0) = v$. Using Itô, deduce the integral form for $V(T, S_T, r_T)$.

3. An equity price S evolves according to a Geometric Brownian Motion

$$dS_t = \mu S_t dt + \sigma S_t dX_t,$$

where μ and σ are constants. We know that an explicit solution is

$$S_t = S_0 e^{\left(\mu - \sigma^2/2\right)t + \sigma X_t}$$

where S_0 is S_t at time $t = 0$.
By working through all the integration steps, deduce that the expected value of S_t at time $t > 0$, given S_0, is

$$\mathbb{E}\left[S_t \middle| S_0 \right] = S_0 e^{\mu t}.$$

You are required to present all your integration steps to obtain the expectation.

4. Consider the diffusion process for the state variable u which evolves according to the Ornstein-Uhlenbeck process

$$du = -\theta u dt + \sigma dX_t.$$

Both θ and σ are constants. Obtain the <u>**steady state**</u> probability distribution $p_\infty(u)$, which is given by

$$\sqrt{\frac{\theta}{\sigma^2 \pi}} \exp\left(-\frac{\theta}{\sigma^2} u^2\right).$$

By looking at p_∞, write down the mean and standard deviation for this distribution.

5. Consider the spot rate r, which evolves according to the popular form

$$dr = u(r)\,dt + \nu r^{\beta} dX_t \ , \tag{1}$$

where ν and β are constants.
Suppose such a model has a **steady state transition probability density function** $p_{\infty}(r)$ that satisfies the forward Kolmogorov Equation.
Show that this implies that the drift structure of (1) is given by

$$u(r) = \nu^2 \beta r^{2\beta-1} + \frac{1}{2}\nu^2 r^{2\beta} \frac{d}{dr}(\log p_{\infty}).$$

6. In this question $t \geq 0$.

   a. $Y_t = X_t^2 + k$, where k is a constant. Is Y_t a martingale?

   b. For which values of k is the process

   $$Y_t = X_t^4 - 6tX_t^2 + kt^2,$$

   a martingale?

   c. Define $Y_t = t^2 X_t - 2\int_0^t sX_s ds$. Is Y_t a martingale?

   d. Show that $X_t = \cosh(\theta X_t)\, e^{-\theta^2 t/2}$; $\theta \in \mathbb{R}$, is a martingale.

7. Consider the following model, where the risk-free interest rate $r = 0$:

ω	$S(0)$	$S(1)$	$S(2)$
ω_1	S	aS	a^2S
ω_2	S	aS	S
ω_3	S	$a^{-1}S$	S
ω_4	S	$a^{-1}S$	$a^{-2}S$

S is the initial asset value at $t = 0$ and $a > 1$ is a constant.

   a. Find all the one period risk-neutral probabilities and the corresponding probability measure $\mathbb{Q}$ on $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$. Confirm that $\mathbb{E}^{\mathbb{Q}}[X]$ is the fair price, where X is the payoff function.

   b. Now consider a model where in each period the asset can either double or half. Show that the value of an option struck at the initial asset value S is $S/3$.