Transition Probability Density Functions for Stochastic Differential Equations

To match the mean and standard deviation of the trinomial model with the continuous-time random walk we choose the following definitions for the probabilities

$$\begin{aligned}\phi^{+}(y,t) &= \frac{1}{2}\frac{\delta t}{\delta y^{2}}\left(B^{2}(y,t)+A(y,t)\,\delta y\right),\\ \phi^{-}(y,t) &= \frac{1}{2}\frac{\delta t}{\delta y^{2}}\left(B^{2}(y,t)-A(y,t)\,\delta y\right)\end{aligned}$$

We first note that the expected value is

$$\begin{aligned}&\phi^{+}(\delta y)+\phi^{-}(-\delta y)+\left(1-\phi^{+}-\phi^{-}\right)(0)\\ =\;&\left(\phi^{+}-\phi^{-}\right)\delta y\end{aligned}$$

We already know that the mean and variance of the continuous time random walk given by

$$dy = A(y,t)\,dt + b(y,t)\,dW$$

is, in turn,

$$\begin{aligned}\mathbb{E}[dy] &= Adt\\ \mathbb{V}[dy] &= B^{2}dt.\end{aligned}$$

So to match the mean requires

$$\left(\phi^{+}-\phi^{-}\right)\delta y = A\delta t$$

The variance of the trinomial model is $\mathbb{E}\left[u^{2}\right]-\mathbb{E}^{2}[u]$ and hence becomes

$$\begin{aligned}&(\delta y)^{2}\left(\phi^{+}+\phi^{-}\right)-\left(\phi^{+}-\phi^{-}\right)^{2}(\delta y)^{2}\\ =\;&(\delta y)^{2}\left(\phi^{+}+\phi^{-}-\left(\phi^{+}-\phi^{-}\right)^{2}\right).\end{aligned}$$

We now match the variances to get

$$(\delta y)^{2}\left(\phi^{+}+\phi^{-}-\left(\phi^{+}-\phi^{-}\right)^{2}\right) = B^{2}\delta t$$

First equation gives

$$\phi^{+} = \phi^{-} + A\tfrac{\delta t}{\delta y}$$

which upon substituting into the second equation gives

$$(\delta y)^{2}\left(\phi^{-}+\alpha+\phi^{-}-\left(\phi^{-}+\alpha-\phi^{-}\right)^{2}\right) = B^{2}\delta t$$

where $\alpha = A\frac{\delta t}{\delta y}$. This simplifies to

$$2\phi^{-}+\alpha-\alpha^{2} = B^{2}\tfrac{\delta t}{(\delta y)^{2}}$$

which rearranges to give

$$\begin{aligned}
\phi^{-} &= \frac{1}{2}\left(B^2 \tfrac{\delta t}{(\delta y)^2} + \alpha^2 - \alpha\right) \\
&= \frac{1}{2}\left(B^2 \tfrac{\delta t}{(\delta y)^2} + \left(A\tfrac{\delta t}{\delta y}\right)^2 - A\tfrac{\delta t}{\delta y}\right) \\
&= \frac{1}{2}\tfrac{\delta t}{(\delta y)^2}\left(B^2 + A^2\delta t - A\delta y\right)
\end{aligned}$$

δt is small compared with δy and so

$$\phi^{-} = \frac{1}{2}\tfrac{\delta t}{(\delta y)^2}\left(B^2 - A\delta y\right).$$

Then

$$\phi^{+} = \phi^{-} + A\tfrac{\delta t}{\delta y} = \frac{1}{2}\tfrac{\delta t}{(\delta y)^2}\left(B^2 + A\delta y\right).$$

Note

$$\left(\phi^{+} + \phi^{-}\right)(\delta y)^2 = B^2\delta t$$

Derivation of the Fokker-Planck/Forward Kolmogorov Equation

Recall that $y',\ t'$ are futures states.
We have $p(y,t;y',t') =$

$$\begin{aligned}
&\phi^{-}(y'+\delta y, t'-\delta t)\, p(y,t;y'+\delta y, t'-\delta t) \\
&+\left(1-\phi^{-}(y', t'-\delta t) - \phi^{+}(y', t'-\delta t)\right) p(y,t;y', t'-\delta t) \\
&+\phi^{+}(y'-\delta y, t'-\delta t)\, p(y,t;y'-\delta y, t'-\delta t)
\end{aligned}$$

Expand each of the terms in Taylor series about the point y', t' to find

$$p(y,t;y'+\delta y, t'-\delta t) = p(y,t;y',t') + \delta y\frac{\partial p}{\partial y'} + \frac{1}{2}\delta y^2\frac{\partial^2 p}{\partial y'^2} - \delta t\frac{\partial p}{\partial t'} + ...,$$

$$p(y,t;y', t'-\delta t) = p(y,t;y',t') - \delta t\frac{\partial p}{\partial t'} + ...,$$

$$p(y,t;y'-\delta y, t'-\delta t) = p(y,t;y',t') - \delta y\frac{\partial p}{\partial y'} + \frac{1}{2}\delta y^2\frac{\partial^2 p}{\partial y'^2} - \delta t\frac{\partial p}{\partial t'} + ...,$$

$$\phi^{+}(y'-\delta y, t'-\delta t) = \phi^{+}(y',t') - \delta y\frac{\partial \phi^{+}}{\partial y'} + \frac{1}{2}\delta y^2\frac{\partial^2 \phi^{+}}{\partial y'^2} - \delta t\frac{\partial \phi^{+}}{\partial t'} + ...,$$

$$\phi^{+}(y', t'-\delta t) = \phi^{+}(y',t') - \delta t\frac{\partial \phi^{+}}{\partial t'} + ...,$$

$$\phi^{-}(y'+\delta y, t'-\delta t) = \phi^{-}(y',t') + \delta y\frac{\partial \phi^{-}}{\partial y'} + \frac{1}{2}\delta y^2\frac{\partial^2 \phi^{-}}{\partial y'^2} - \delta t\frac{\partial \phi^{-}}{\partial t'} + ...,$$

$$\phi^{-}(y', t'-\delta t) = \phi^{-}(y',t') - \delta t\frac{\partial \phi^{-}}{\partial t'} + ...,$$

Substituting in our equation for $p(y,t;y',t')$, ignoring terms smaller than δt, noting that $\delta y \sim O\left(\sqrt{\delta t}\right)$, gives

$$\frac{\partial p}{\partial t'} = -\frac{\partial}{\partial y'}\left(\frac{1}{\delta y}\left(\phi^{+}-\phi^{-}\right)p\right)+\frac{1}{2}\frac{\partial^{2}}{\partial y'^{2}}\left(\left(\phi^{+}-\phi^{-}\right)p\right).$$

Noting the earlier results

$$\begin{aligned} A &= \frac{(\delta y)^{2}}{\delta t}\left(\frac{1}{\delta y}\left(\phi^{+}-\phi^{-}\right)\right), \\ B^{2} &= \frac{(\delta y)^{2}}{\delta t}\left(\phi^{+}+\phi^{-}\right) \end{aligned}$$

gives the *forward equation*

$$\frac{\partial p}{\partial t'} = \frac{1}{2}\frac{\partial^{2}}{\partial y'^{2}}\left(B^{2}\left(y',t'\right)p\right)-\frac{\partial}{\partial y'}\left(A\left(y',t'\right)p\right)$$

The initial condition used is

$$p\left(y,t;y',t'\right)=\delta\left(y'-y\right)$$

As an example consider the important case of the distribution of stock prices. Given the random walk for equities, i.e. Geometric Brownian Motion

$$\frac{dS}{S}=\mu dt+\sigma dW.$$

So $A(S',t')=\mu S'$ and $B(S',t')=\sigma S'$. Hence the forward equation becomes

$$\frac{\partial p}{\partial t'} = \frac{1}{2}\frac{\partial^{2}}{\partial S'^{2}}\left(\sigma^{2}S'^{2}p\right)-\frac{\partial}{\partial S'}\left(\mu S'p\right).$$

More on this and solution technique later, but note that a transformation reduces this to the one dimensional heat equation and the *similarity reduction method* which follows is used.

The Steady-State Distribution

As the name suggests 'steady state' refers to time independent. Random walks for interest rates and volatility can be modelled with stochastic differential equations which have steady-state distributions. So in the long run, i.e. as $t' \longrightarrow \infty$ the distribution $p(y,t;y',t')$ settles down and becomes independent of the starting state y and t. The partial derivatives in the forward equation now become ordinary ones and the unsteady term $\frac{\partial p}{\partial t'}$ vanishes.

The resulting forward equation for the steady-state distribution $p_{\infty}(y')$ is governed by the ordinary differential equation

$$\frac{1}{2}\frac{d^{2}}{dy'^{2}}\left(B^{2}p_{\infty}\right)-\frac{d}{dy'}\left(Ap_{\infty}\right)=0.$$

Example: The Vasicek model for the spot rate r evolves according to the stochastic differential equation

$$dr = \gamma\left(\overline{r} - r\right)dt + \sigma dW$$

Write down the Fokker-Planck equation for the transition probability density function for the interest rate r in this model.

Now using the steady-state version for the forward equation, solve this to find the **steady state** probability distribution $p_\infty\left(r'\right)$, given by

$$p_\infty = \tfrac{1}{\sigma}\sqrt{\frac{\gamma}{\pi}}\exp\left(-\tfrac{\gamma}{\sigma^2}\left(r' - \overline{r}\right)^2\right).$$

Solution:

For the SDE $dr = \gamma\left(\overline{r} - r\right)dt + \sigma dW$ where drift $= \gamma\left(\overline{r} - r\right)$ and diffusion is σ the Fokker Planck equation becomes

$$\frac{\partial p}{\partial t'} = \frac{1}{2}\sigma^2\frac{\partial^2 p}{\partial r'^{\,2}} - \gamma\frac{\partial}{\partial r'}\left(\left(\overline{r} - r'\right)p\right)$$

where $p = p\left(r',\ t'\right)$ is the transition PDF and the variables refer to future states. In the steady state case, there is no time dependency, hence the Fokker Planck PDE becomes an ODE with

$$\frac{1}{2}\sigma^2\frac{d^2 p_\infty}{dr^2} - \gamma\frac{d}{dr}\left(\left(\overline{r} - r\right)p_\infty\right) = 0$$

$p_\infty = p_\infty\left(r\right)$. The prime notation and subscript have been dropped simply for convenience at this stage. To solve the steady-state equation:

Integrate wrt r

$$\frac{1}{2}\sigma^2\frac{dp}{dr} - \gamma\left(\left(\overline{r} - r\right)p\right) = k$$

where k is a constant of integration and can be calculated from the conditions, that as $r \to \infty$

$$\left\{\begin{array}{c} \dfrac{dp}{dr} \to 0 \\ p \to 0 \end{array}\right. \Rightarrow k = 0$$

which gives

$$\frac{1}{2}\sigma^2\frac{dp}{dr} = -\gamma\left(\left(r - \overline{r}\right)p\right),$$

a first order variable separable equation. So

$$\begin{aligned} \frac{1}{2}\sigma^2\int\frac{dp}{p} &= -\gamma\int\left(\left(r - \overline{r}\right)\right)dr \to \\ \frac{1}{2}\sigma^2\ln p &= -\gamma\left(\frac{r^2}{2} - \overline{r}r\right) + C, \qquad C \text{ is arbitrary.} \end{aligned}$$

Rearranging and taking exponentials of both sides to give

$$p = \exp\left(-\tfrac{2\gamma}{\sigma^2}\left(\frac{r^2}{2} - \overline{r}r\right) + D\right) = E\exp\left(-\tfrac{2\gamma}{\sigma^2}\left(\frac{r^2}{2} - \overline{r}r\right)\right)$$

Complete the square to get

$$\begin{aligned} p &= E\exp\left(-\tfrac{\gamma}{\sigma^2}\left[(r-\overline{r})^2-\overline{r}^2\right]\right) \\ p_\infty &= A\exp\left(-\tfrac{\gamma}{\sigma^2}(r'-\overline{r})^2\right). \end{aligned}$$

There is another way of performing the integration on the rhs. If we go back to $-\gamma\int(r-\overline{r})\,dr$ and write as

$$-\gamma\int\tfrac{1}{2}\tfrac{d}{dr}(r-\overline{r})^2\,dr=\frac{-\gamma}{2}(r-\overline{r})^2$$

to give

$$\frac{1}{2}\sigma^2\ln p=\frac{-\gamma}{2}(r-\overline{r})^2+C.$$

Now we know as p_∞ is a PDF

$$\begin{aligned} \int_{-\infty}^{\infty}p_\infty\;dr' &= 1\rightarrow \\ A\int_{-\infty}^{\infty}\exp-\left(\tfrac{\gamma}{\sigma^2}(r'-\overline{r})^2\right)\;dr' &= 1 \end{aligned}$$

A few (related) ways to calculate A. Now use the error function, i.e.

$$\int_{-\infty}^{\infty}e^{-x^2}dx=\sqrt{\pi}$$

So put

$$x=\sqrt{\tfrac{\gamma}{\sigma^2}}(r'-\overline{r})\rightarrow dx=\sqrt{\tfrac{\gamma}{\sigma^2}}dr'$$

which transforms the integral above

$$\frac{A\sigma}{\sqrt{\gamma}}\int_{-\infty}^{\infty}e^{-x^2}\;dx=1\rightarrow A\sigma\sqrt{\frac{\pi}{\gamma}}=1$$

therefore

$$A=\tfrac{1}{\sigma}\sqrt{\frac{\gamma}{\pi}}.$$

This allows us to finally write the steady-state transition PDF as

$$p_\infty=\tfrac{1}{\sigma}\sqrt{\frac{\gamma}{\pi}}\exp\left(-\tfrac{\gamma}{\sigma^2}(r'-\overline{r})^2\right).$$

The *backward equation* is obtained in a similar way to the forward
$p(y,t;y',t')=$

$$\begin{aligned} &\phi^+(y,t)\,p(y+\delta y,t+\delta t;y',t') \\ &+\left(1-\phi^-(y,t)-\phi^+(y,t)\right)p(y,t+\delta t;y',t') \\ &+\phi^-(y,t)\,p(y-\delta y,t+\delta t;y',t') \end{aligned}$$

and expand using Taylor. The resulting PDE is

$$\frac{\partial p}{\partial t} + \frac{1}{2} B^2 (y,t) \frac{\partial^2 p}{\partial y^2} + A(y,t) \frac{\partial p}{\partial y} = 0.$$

So the forward equation can be obtained from the backward equation using the transformation $t' = T - t$,

$$\frac{\partial p}{\partial t} = \frac{1}{2} \sigma^2 S^{^2} \frac{\partial^2 p}{\partial S^2} + \mu S \frac{\partial p}{\partial S}.$$

Write $p = p(S', t)$ as $p = p(\xi, t)$ where $\xi = \log S'$

$$\frac{\partial p}{\partial S'} = \frac{1}{S'} \frac{\partial p}{\partial \xi}; \;\; \frac{\partial^2 p}{\partial S'^2} = \frac{1}{S'^2} \left(\frac{\partial^2 p}{\partial \xi^2} - \frac{\partial p}{\partial \xi} \right).$$

To solve, reduce to a 1D heat equation initially.

This can be solved with a starting condition of $S' = S$ at $t' = t$ to give the transition pdf

$$p(S, t; S', T) = \frac{1}{\sigma S' \sqrt{2\pi (t'-t)}} e^{-\left(\log(S/S') + \left(\mu - \frac{1}{2}\sigma^2\right)(t'-t)\right)^2 / 2\sigma^2 (t'-t)}.$$