Martingales

Martingales are a key concept in probability and in mathematical finance. The term 'martingale' may refer to very different ideas e.g. a stochastic process that has no drift. Essentially, this is the idea of a fair (random) game. We encounter Martingales through three distinct, but closely connected ideas:

1. *Martingales* as a class of stochastic process; coin toss sample space

2. *Exponential martingales*, which are a specific and extremely useful example of a martingale; $\longrightarrow$ used in option pricing

3. *Equivalent martingale measures*, where we look for a probability measure $\mathbb{Q}$ such that a given stochastic process $S(t)$ is a martingale under $\mathbb{Q}$ regardless of its nature under $\mathbb{P}$. The correspondence between the measures $\mathbb{P}$ and $\mathbb{Q}$ is done through a change of measure.

change of measure

$\tilde{P} = \{p, 1-p\}$

$Q = \{q, 1-q\}$

$\mathbb{P}$ physical planet

$\mathbb{Q}$ risk-neutral

blackboard bold

general prob. measure

We look at the triple $(\Omega, \mathcal{F}, \mathbb{P})$, called a *probability space.*

always assume its there

It forms the foundation of the *probabilistic universe*. This probability space comprises of

1. the sample space Ω

2. the filtration $\mathcal{F}$

3. the probability measure $\mathbb{P}$

Example:

The daily closing price of a risky asset, e.g. share price on the FTSE100. Over the course of a year (252 business days)

$$\Omega = \{S_1, S_2, S_3, \ldots, S_{252}\}$$

We could define an event e.g. $\Psi = \{S_i : S_i \geq 110\}$

A *random variable* (RV) Y is a function which maps from the sample space Ω to the set of real numbers

filtration

$$Y : \omega \in \Omega \longrightarrow \mathbb{R}, \qquad \mathbb{E}\left[S_6 \,\middle|\, \underbrace{R_1, \dots, R_5}\right] = S_5$$

i.e. it associates a number $Y(\omega)$ with each outcome ω.

While recording information obtained from a coin tossing game is manageable, imagine storing 100 years of S&P500 data. How can we keep track of an ever expanding sample space (in a simple yet elegant manner)?

We do this by introducing the *filtration*.

The filtration is the mathematical object that keeps track of how (the increasing flow of) information evolves; the fact that there is a binomial tree, from time 1 to time 2 there are 2 possibilities from time 2 to time 3 there are 8 possibilities in total, etc. After just 10 periods there are $2^{10} = 1024$ outcomes!

$\Omega \quad \mathcal{F}$

The filtration, $\mathcal{F}$, is an indication of how an increasing family of events builds up over time as more results become available, it is much more than just a family of events. The filtration, $\mathcal{F}$ is a set formed of all possible combinations of events $A \subset \Omega$, their unions and complements.

$\int f(x)\, dx$

always in the literature

Adapted (Measurable) Process (always in the background)

A stochastic process S_t is said to be adapted to the filtration $\mathcal{F}_t$ (or measurable with respect to $\mathcal{F}_t$, or $\mathcal{F}_t$ -adapted) if the value of S_t at time t is known given the information set $\mathcal{F}_t$.

Place prob. measure $\mathbb{P}$ on $(\Omega, \mathcal{F})$.

→ i) $\mathbb{P} : \mathcal{F} \to [0,1]$; $\mathbb{P}(\Omega) = 1$

→ ii) $A_1, A_2, \ldots$ seq of disjoint sets in $\mathcal{F}$, then

→ $\mathbb{P}\left(\bigcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} \mathbb{P}(A_k)$. For a set $A \in \mathcal{F}$, there is a prob in $[0,1]$ that outcome of random expt. will lie in set A. set them

Conditional Expectations

What makes a conditional expectation different (from an unconditional one) is information (just as in the case of conditional probability). In our probability space, $(\Omega, \mathcal{F}, \mathbb{P})$ information is represented by the filtration $\mathcal{F}$; hence a conditional expectation with respect to the (usual information) filtration seems a natural choice.

$$Y = \mathbb{E}\left[X \mid \mathcal{F}\right]$$

set of information

is the expected value of the random variable conditional upon the filtration set $\mathcal{F}$. In general

- In general Y will be a random variable

- Y will be adapted to the filtration $\mathcal{F}$.

Conditional expectations have the following useful properties: If X, Y are integrable random variables and a, b are constants then

1. **Linearity**:

$$\mathbb{E}\left[aX + bY \mid \mathcal{F}\right] = a\mathbb{E}\left[X \mid \mathcal{F}\right] + b\mathbb{E}\left[Y \mid \mathcal{F}\right]$$

2. **Tower Property (i.e. Iterated Expectations)**: if $\mathcal{F} \subset \mathcal{G}$ $\subset \mathcal{H} \subset \mathcal{Z}$

$$\mathbb{E}\left[\mathbb{E}\left[X \mid \mathcal{G}\right] \mid \mathcal{F}\right] = \mathbb{E}\left[X \mid \mathcal{F}\right]$$

This property states that if taking iterated expectations with respect to several levels of information, we may as well take a single expectation subject to the smallest set of available information.

3. As a special case of the Tower property, we have

$$\mathbb{E}\left[\mathbb{E}\left[X \mid \mathcal{F}\right]\right] = \mathbb{E}\left[X\right]$$

Moral : since "no filtration" is always a smaller information set than any filtration.

4. "Taking out what is known": If X is $\mathcal{F}-$measurable, then the value of X is known once we know $\mathcal{F}$. Therefore,

$$\mathbb{E}\left[X\middle|\mathcal{F}\right] = X$$

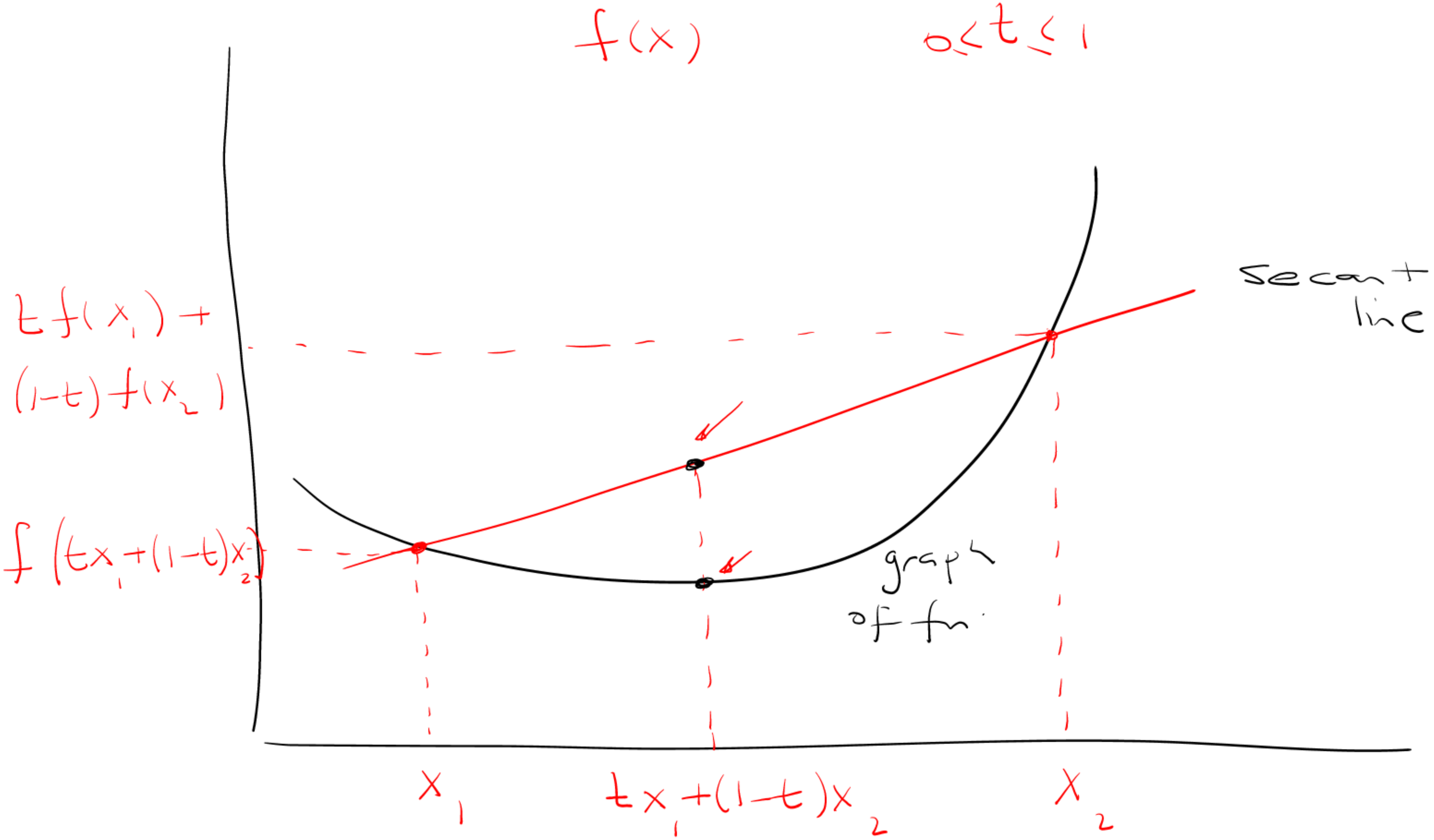
5. **Independence:** If X is independent from $\mathcal{F}$, then knowing $\mathcal{F}$ does not assist in predicting the value of X. Hence

$$\mathbb{E}\left[X\middle|\mathcal{F}\right] = \mathbb{E}\left[X\right]$$

6. **Positivity:** If $X \geq 0$ then $\mathbb{E}\left[X\middle|\mathcal{F}\right] \geq 0$.

7. **Jensen's Inequality:** Let f be a convex function, then

$$f\left(\mathbb{E}\left[X\middle|\mathcal{F}\right]\right) \leq \mathbb{E}\left[f\left(X\right)\middle|\mathcal{F}\right]$$



$f(x)$
$0 \le t \le 1$
secant line
$tf(x_1) +$
$(1-t)f(x_2)$
$f(tx_1+(1-t)x_2)$
graph of fn.
x_1
$tx_1+(1-t)x_2$
x_2

Discrete Time Martingales

discrete time

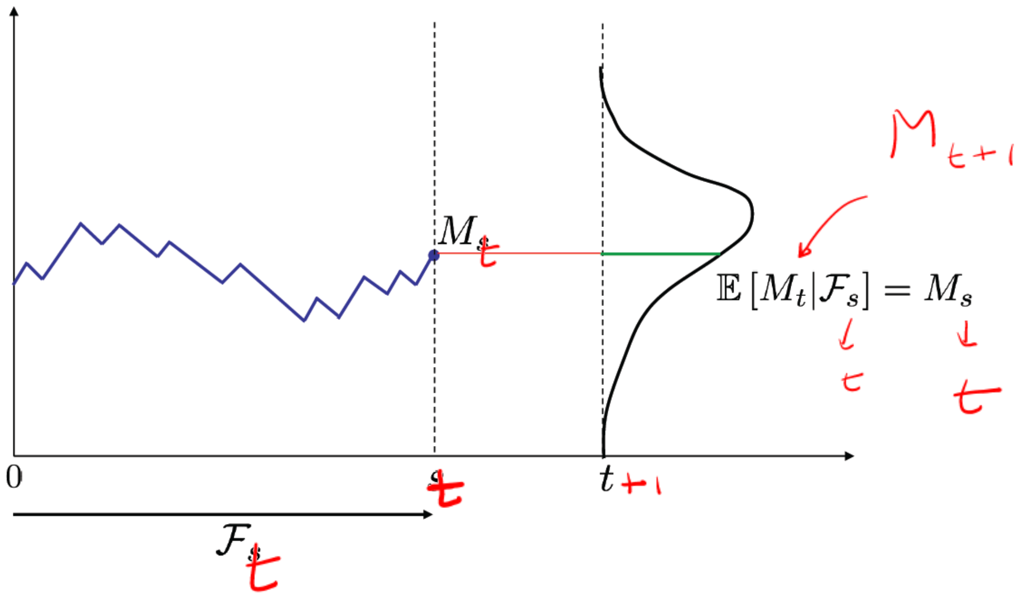
A discrete time stochastic process $\{M_t : t = 0, \ldots, T\}$ such that M_t is $\mathcal{F}_t$-measurable for $\mathbb{T} = \{0, \ldots, T\}$ is a **martingale** if $\mathbb{E}\,|M_t| < \infty$ and

finite expectation exist

$$\boxed{\mathbb{E}_t[M_{t+1}|\mathcal{F}_t] = M_t} \tag{1}$$

M_t adapted to $\mathcal{F}_t$

$s<t$



M_{t+1}

Constant mean stochastic process

The first equation represents a standard integrability condition.

Importance of finite mean existing.

The second equation tells you that the expected value of M at time $t+1$ conditional on all the information available up to time t is the value of M at time t. In short, a Martingale is a **driftless process**.

only has randomness

If we take expectation on both sides of eqn. 1, then

$$\mathbb{E}[M_{t+1}] = \mathbb{E}[M_t] = M_t$$

This is due to the **Tower Property** of conditional expectations.

Martingales are a very nice mathematical object. They $\boxed{\text{"get rid of the drift"}}$ and enable us to focus on what probabilists consider is the most interesting part: the statistical properties of purely random processes.

In addition, Doob and Meyer have developed a powerful theory centred around martingales.

Continuous Time Martingales

Next, we generalize our definitions to continuous time: A continuous time stochastic process

family of RV.s indexed with time

$$\{M_t : t \in \mathbb{R}^+\}$$

cts time

such that $\underbrace{M_t \text{ is } \mathcal{F}_t\text{-measurable}}$ for $t \in \mathbb{R}^+$ is a **martingale** if

$$\mathbb{E}\,|M_t| < \infty$$

finite mean

and

$$\boxed{\mathbb{E}_s[M_t|\mathcal{F}_s] = M_s, \quad 0 \leq s \leq t.}$$

Lévy's Martingale Characterisation: Let $X_t,\ t > 0$ be a stochastic process and let $\mathcal{F}_t$ be the filtration generated by it. X_t is a Brownian motion iff the following conditions are satisfied:

$s<t \quad \mathbb{E}[X_t | \mathcal{F}_s] = X_s$

1. $X_0 = 0$ a.s.; ✓

$$\mathbb{E}[X_t - X_s + X_s | \mathcal{F}_s] = \underbrace{\mathbb{E}(X_t - X_s | \mathcal{F}_s)}_{=0} + \underbrace{\mathbb{E}[X_s | \mathcal{F}_s]}_{=X_s}$$

2. the sample paths $t \mapsto X_t$ are continuous a.s.; ✓

new condition

3. X_t is a martingale with respect to the filtration $\mathcal{F}_t$;

4. $|X_t|^2 - t$ is a martingale with respect to the filtration $\mathcal{F}_t$.

The Lévy characterization can be contrasted with the classical definition of a Brownian motion as a stochastic process X_t satisfying:

Standard B.M

1. $X_0 = 0$ a.s.;

2. the sample paths $t \mapsto X(t)$ are continuous a.s.;

3. **independent increments**: for $t_1 < t_2 < t_3 < t_4$ the increments $X_{t_4} - X_{t_3}$, $X_{t_2} - X_{t_1}$ are independent;

4. **normally distributed increments**: $X_t - X_s \sim N(0, |t-s|)$.

Lévy's characterization neither mentions independent increments nor normally distributed increments.

Instead, Lévy introduces two easily verifiable martingale conditions.

Itô Integrals and Martingales

Next, we explore the link between Itô integration and martingales.

Itô I

$dF = \frac{dF}{dX_t} dX_t + \frac{1}{2}\frac{d^2F}{dX_t^2} dt$

Consider the stochastic process $Y(t) = X^2(t)$. By Itô, we have

followed $\int_0^T$

$$X^2(T) = T + \int_0^T 2X(t)dX(t)$$

Taking the expectation, we get

set in stone

$$\mathbb{E}[X^2(T)] = T + \mathbb{E}\left[\int_0^T 2X(t)dX(t)\right]$$

Now, the quadratic variation property of Brownian motions implies that

$$\mathbb{E}[X^2(T)] = T$$

and hence

$$\mathbb{E}\left[\int_0^T 2X(t)dX(t)\right] = 0.$$

Therefore, the Itô integral

$$\int_0^T 2X(t)dX(t)$$

is a martingale.

In fact, this property is shared by all Itô integrals.

The Itô integral is a martingale

Let $g(t, X_t)$ be a function on $[0, T]$ and satisfying the technical condition. Then the Itô integral

$$\int_0^T g(t, X_t)dX_t$$

is a martingale.

Conclusion: So, Itô integrals are martingales.

But does the converse hold? Can we represent any martingale as an Itô integral?

The answer is yes!

In Module III

Martingale Representation Theorem: If M_t is a martingale, then there exists a function $g(t, X_t)$ satisfying the technical condition such that

$$\boxed{M_T = M_0 + \int_0^T g(t, X_t) dX_t}$$

later used in pricing

Example Using only Itô and the fact that Itô integrals are martingales, we will show that

$$\boxed{\mathbb{E}\left[X^2(T)\right] = T.}$$

Consider the function $F(t, X_t) = X_T^2$, then by Itô's lemma,

(and integration

$$
\begin{aligned}
X_T^2 &= X_0^2 + \frac{1}{2}\int_0^T 2dt + \int_0^T 2X_t dX_t \\
X_T^2 &= \int_0^T dt + 2\int_0^T X_t dX_t
\end{aligned}
$$

since $X_0 = 0$

Taking the expectation,

$$
\mathbb{E}\left[X_T^2\right] = \mathbb{E}\left[\int_0^T dt\right] + 2\mathbb{E}\left[\int_0^T X_t dX_t\right]
$$

zero

Now,

$$
\int_0^T X_t dX_t
$$

is an Itô integral and as a result $\mathbb{E}\left[\int_0^T X_t dX_t\right] = 0$

Moreover,

$$\mathbb{E}\left[\int_0^T dt\right] = \mathbb{E}\left[T\right] = T$$

We can conclude that

$$\boxed{\mathbb{E}\left[X^2(T)\right] = T}$$

As an aside, we can usually exchange the order of integration between the time integral and the expectation so that

$$\mathbb{E}\left[\int_0^T f(X_t)dt\right] = \int_0^T \mathbb{E}\left[f(X_t)\right]dt$$

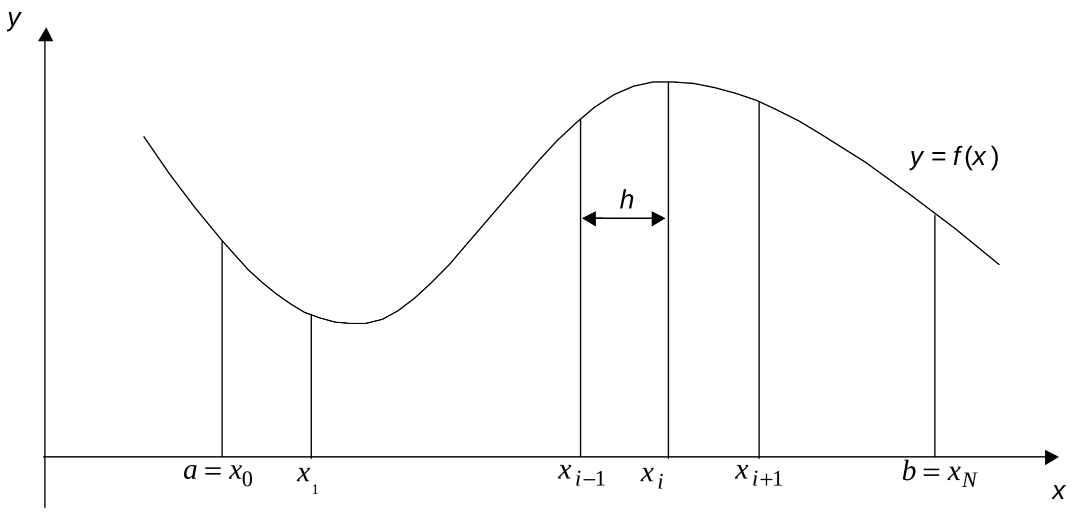
This is due to an analysis result known as **Fubini's Theorem**.

Nodis

Itô Integral

Recall the usual Riemann definition of a definite integral

$$\int_a^b f(x)\, dx$$



$h = \frac{b-a}{N}$

which represents the area under the curve between $x = a$ and $x = b$, where the curve is the graph of $f(x)$ plotted against x.

Assuming f is a "well behaved" function on $[a, b]$, there are many different ways (which all lead to the same value for the definite integral).

Start by partitioning $[a, b]$ into N intervals with end points $x_0 = a < x_1 < x_2 < \quad \ldots\ldots \quad < x_{N-1} < x_N = b$, where the length of an interval $dx = x_i - x_{i+1}$ tends to zero as $N \to \infty$. So there are N intervals and $N + 1$ points x_i.

Discretising x gives

$$x_i = a + idx$$

Now consider the definite integral

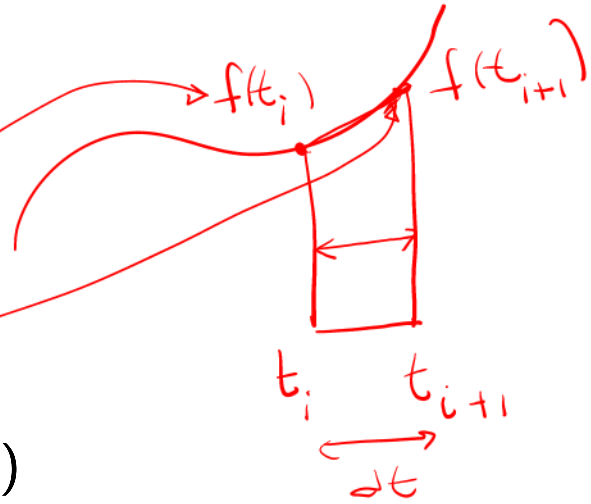
$\delta x = \frac{T}{N} = h$

$$\int_0^T f(t)\, dt.$$

With Riemann integration there are a number of ways we can approximate this:

1. left hand rectangle rule;

$$\int_0^T f(t)\,dt = \lim_{N\to\infty} \sum_{i=0}^{N-1} f(t_i)\,\overbrace{(t_{i+1} - t_i)}^{dt}$$

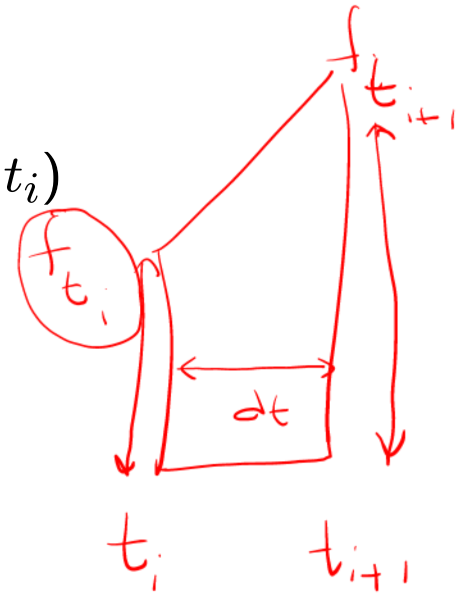


2. right hand rectangle rule;

$$\int_0^T f(t)\,dt = \lim_{N\to\infty} \sum_{i=0}^{N-1} f(t_{i+1})\,(t_{i+1} - t_i)$$

3. trapezium rule;

$$\int_0^T f(t)\,dt = \lim_{N\to\infty} \sum_{i=0}^{N-1} \underbrace{\frac{1}{2}\left(f(t_i) + f(t_{i+1})\right)}\,(t_{i+1} - t_i)$$



4. midpoint rule

$$\int_0^T f(t)\,dt = \lim_{N\to\infty} \sum_{i=0}^{N-1} f\left(\tfrac{1}{2}(t_i + t_{i+1})\right)(t_{i+1} - t_i)$$

In the limit $N \to \infty$, $f(t)$ we get the same value for each definition of the definite integral, provided the function is integrable.

Now consider the stochastic integral of the form

general formula stoch. integral

$$\int_0^T f(t, X)\,dX = \int_0^T f(t, X(t))\,dX(t)$$

where $X(t)$ is a Brownian motion. We can define this integral as

$$\lim_{N\to\infty} \sum_{i=0}^{N-1} f(t_i, X_i)(X_{i+1} - X_i),$$

($\equiv$ left hand)

where $X_i = X(t_i)$, or as

$$\lim_{N\to\infty} \sum_{i=0}^{N-1} f(t_{i+1}, X_{i+1})(X_{i+1} - X_i),$$

(right hand)

or as

$$\lim_{N\to\infty} \sum_{i=0}^{N-1} f\left(t_{i+\frac{1}{2}}, X_{i+\frac{1}{2}}\right)(X_{i+1} - X_i),$$

where $t_{i+\frac{1}{2}} = \underbrace{\frac{1}{2}(t_i + t_{i+1})}$ and $X_{i+\frac{1}{2}} = X\left(t_{i+\frac{1}{2}}\right)$ or in many other ways. So clearly drawing parallels with the above Riemann form.

Very Important: In the case of a stochastic variable $dX(t)$ the value of the stochastic integral **does** depend on which definition we choose.

In the case of a stochastic integral, the definition

$$\longrightarrow I = \lim_{N\to\infty} \sum_{i=0}^{N-1} f(t_i, X_i)\underbrace{(X_{i+1} - X_i)},$$

(left hand rule)

$dG = A(G,t)\,dt + \underbrace{B(G,t)}\,dX \leftarrow X_{t+dt} - X_t$

is special. This definition results in the **Itô Integral**.

↳ definition of Itô integral

It is special because it is **non-anticipatory** given that we are at time t_i we know $X_i = X(t_i)$ and therefore we know $f(t_i, X_i)$. The only uncertainty is in the $X_{i+1} - X_i$ term.

Compare this to a definition such as

t_i $\quad X_{t_i}$ $\quad X_{t_{i+1}}$

$$\lim_{N\to\infty} \sum_{i=0}^{N-1} \underbrace{f(t_{i+1}, X_{i+1})}\,\underbrace{(X_{i+1} - X_i)},$$

which is **anticipatory**; given that at time t_i we know X_i but are uncertain about the future value of X_{i+1}. Thus we are uncertain about *both* the value of

$$f(t_{i+1}, X_{i+1})$$

and the value of $(X_{i+1} - X_i)$ — there exists uncertainty in both of these quantities. That is, evaluation of this integral requires us to anticipate the future value of X_{i+1} so that we may evaluate $f(t_{i+1}, X_{i+1})$.

We can use Itô $\int_0^T X^2 \, dX$ $\qquad \int_0^T \frac{dF}{dX_t} dX_t = F(X_t) - F(X_0) - \frac{1}{2} \int_0^T \frac{d^2F}{dX_t^2} dt$

The main thing to note about Itô integrals is that I is a random variable (unlike the deterministic case). Additionally, since I is essentially the limit of a sum of normal random variables, then by the CLT I is also normally distributed, and can be characterized by its mean and variance.

Example: Show that Itô's lemma implies that

$$3\int_0^T X^2 dX = X(T)^3 - X(0)^3 - 3\int_0^T X(t)\, dt.$$

Show that the result also can be found by writing the integral

X_{i+1}

$$3\int_0^T X^2 dX = \lim_{N\to\infty} 3 \sum_{i=0}^{N-1} X_i{}^2 (X_{i+1} - X_i)$$

definition

Hint: use $3b^2(a-b) = \underbrace{a^3 - b^3 - 3b(a-b)^2 - (a-b)^3}$.

X_i

The Itô integral here is defined as

$$\int_0^T 3X^2(t)\, dX(t) = \lim_{N\to\infty} \sum_{i=0}^{N-1} 3\overset{b}{X_i^2}(\overset{a}{X_{i+1}} - \overset{b}{X_i})$$

Now note the hint:

$$3b^2(a-b) = a^3 - b^3 - 3b(a-b)^2 - (a-b)^3$$

hence

$$\begin{aligned} &\equiv 3X_i^2(X_{i+1} - X_i) \\ &= \underbrace{X_{i+1}^3}_{a^3} - \underbrace{X_i^3}_{b^3} - \underbrace{3X_i(X_{i+1} - X_i)^2 - (X_{i+1} - X_i)^3}_{3b(a-b)^2 - (a-b)^3}, \end{aligned}$$

so that

$$\sum_{i=0}^{N-1} 3X_i^2(X_{i+1} - X_i) =$$

$$\boxed{\sum_{i=0}^{N-1} X_{i+1}^3} - \boxed{\sum_{i=0}^{N-1} X_i^3} - \boxed{\sum_{i=0}^{N-1} 3X_i(X_{i+1} - X_i)^2}$$
$$-\sum_{i=0}^{N-1} (X_{i+1} - X_i)^3$$

$$\left(X_1^3 + X_2^3 + \cdots + X_N^3\right) - \left(X_0^3 + \cdots + X_{N-1}^3\right)$$

Now the first two expressions above give

$$\sum_{i=0}^{N-1} X_{i+1}^3 - \sum_{i=0}^{N-1} X_i^3 = X_N^3 - X_0^3$$
$$= \boxed{X(T)^3} - X(0)^3 .$$

In the limit $N \to \infty$, i.e. $dt \to 0$, $(X_{i+1} - X_i)^2 \to dt$, so

$$\lim_{N\to\infty} \sum_{i=0}^{N-1} 3X_i (X_{i+1} - X_i)^2 = \int_0^T 3X(t)\, dt$$

Finally $(X_{i+1} - X_i)^3 = \overbrace{(X_{i+1} - X_i)^2}^{dt} \cdot \overbrace{(X_{i+1} - X_i)}^{dt^{1/2}}$ which when $N \to \infty$ behaves like $dX^2 dX \sim O\left(dt^{3/2}\right) \longrightarrow 0$.

Hence putting together gives

$$X(T)^3 - X(0)^3 - \int_0^T 3X(t)\, dt$$

which is consistent with Itô's lemma.

The other important property that the Itô integral has is that it is a martingale. We know that

$$X_{i+1} - X_i$$

is a martingale; i.e. in the context

$$\mathbb{E}\left[X_{i+1} - X_i\right] = 0.$$

t_{i+1} or $t_{i+\frac{1}{2}}$

Since

$$\mathbb{E}\left[\sum_{i=0}^{N-1} f\left(t_i, X_i\right)\left(X_{i+1} - X_i\right)\right] = \sum_{i=0}^{N-1} f\left(t_i, X_i\right)\mathbb{E}\left[X_{i+1} - X_i\right] = 0$$

Thus

$$\mathbb{E}\left[\int_0^T f\left(t, X\left(t\right)\right) dX\left(t\right)\right] = 0.$$

This is, essentially a consequence of the Itô integral being non-anticipatory, as discussed earlier. No other stochastic integral has this property.

Exercise We know from Itô's lemma that

$$4\int_0^T X^3(t)\,dX(t) = X^4(T) - X^4(0) - 6\int_0^T X^2(t)\,dt$$

Show from the definition of the Itô integral that the result can also be found by initially writing the integral

$$4\int_0^T X^3 dX = \lim_{N\to\infty} 4\sum_{i=0}^{N-1} X_i^3\left(X_{i+1} - X_i\right)$$

Hint: use $4b^3(a-b) = a^4 - b^4 - 4b(a-b)^3 - 6b^2(a-b)^2 - (a-b)^4$.

Proving that a Continuous Time Stochastic Process is a Martingale

Consider a stochastic process $Y(t)$ solving the following SDE:

$$(*) \qquad dY(t) = f(Y_t, t)dt + g(Y_t, t)dX(t), \qquad Y(0) = Y_0$$

How can we tell whether $Y(t)$ is a martingale?

The answer has to do with the fact that Itô integrals are martingales.

$Y(t)$ is a martingale if and only if it satisfies the martingale condition

M. P.

$$\mathbb{E}_s[Y_t|\mathcal{F}_s] = Y_s, \quad 0 \leq s \leq t$$

Let's start by integrating the SDE $(*)$ between s and t to get an exact form for $Y(t)$:

$$Y(t) = Y(s) + \int_s^t f(Y_u, u)du + \int_s^t g(Y_u, u)dX(u)$$

Taking the expectation conditional on the filtration at time s, we get

$$
\begin{aligned}
\mathbb{E}\left[Y_t|\mathcal{F}_s\right] &= \mathbb{E}\left[Y(s) + \int_s^t f(Y_u,u)du + \int_s^t g(Y_u,u)dX(u)|\mathcal{F}_s\right] \\
\mathbb{E}_s\left[Y_t|\mathcal{F}_s\right] &= Y(s) + \underbrace{\mathbb{E}\left[\int_s^t f(Y_u,u)du|\mathcal{F}_s\right]}_{\text{drift}}
\end{aligned}
$$

where the last line follows from the fact that an Itô integral is a martingale, $\therefore$

$$
\mathbb{E}\left[\int_s^t g(Y_u,u)dX(u)|\mathcal{F}_s\right] = \int_s^s g(Y_u,u)dX(u) = 0.
$$

So, $Y(t)$ is a martingale iff

$$
\mathbb{E}\left[\int_s^t f(u)du|\mathcal{F}_s\right] = 0
$$

This condition is satisfied only if $f(Y_t,t) = 0$ for all t. Returning to our SDE, we conclude that $Y(t)$ is a martingale iff it is of the form

$$
dY(t) = g(Y_t,t)dX(t), \qquad Y(0) = Y_0
$$

pure randomness

Test: Given G_t obtain its S.D.E. If there is no drift $\Rightarrow$ G_t is a martingale

Exponential Martingales

Let's start with a motivating example.

Consider the stochastic process $Y(t)$ satisfying the SDE

Special condition

$$dY(t) = f(t)dt + g(t)dX(t), \qquad Y(0) = Y_0 \tag{2}$$

where $f(t)$ and $g(t)$ are two time-dependent functions and $X(t)$ is a standard Brownian motion.

Define a new process $Z(t) = e^{Y(t)}$.

How should we choose $f(t)$ if we want the process $Z(t)$ to be a martingale?

i.e dZ_t to have zero drift.

Use Itô III

$dY = f(t)\,dt + g(t)\,dX_t$

$dY^2 = g^2(t)\,dt$

III

Consider the process $Z(t) = e^{Y(t)}$. Applying Itô to the function we obtain:

$\frac{dZ}{dY} = e^Y = \frac{d^2Z}{dY^2}$

$$
\begin{aligned}
dZ(t) &= \frac{dZ}{dY}dY(t) + \frac{1}{2}\frac{d^2Z}{dY^2}dY^2(t) \\
&= \frac{dZ}{dY}\left(f(t)dt + g(t)dX(t)\right) + \frac{1}{2}\frac{d^2Z}{dY^2}g^2(t)dt \\
&= e^{Y(t)}\left(f(t) + \frac{1}{2}g^2(t)\right)dt + e^{Y(t)}g(t)dX(t) \\
dZ &= Z(t)\left[\underbrace{\cancel{\left(f(t) + \frac{1}{2}g^2(t)\right)}}\,dt + g(t)dX(t)\right]
\end{aligned}
$$

$Z(t)$ is a martingale if and only if it is a driftless process.

Therefore for $Z(t)$ to be a martingale we must have

$$\boxed{f(t) + \frac{1}{2}g^2(t) = 0}$$

$dY = -\frac{1}{2}g^2(t)\,dt + g(t)\,dX_t$

This is only possible if

$$f(t) = -\frac{1}{2}g^2(t)$$

Going back to the process $Y(t)$, we must have

$$dY(t) = -\frac{1}{2}g^2(t)dt + g(t)dX(t), \qquad Y(0) = Y_0$$

implying that (upon integration)

$$Y(T) = Y_0 - \frac{1}{2}\int_0^T g^2(t)dt + \int_0^T g(t)dX(t)$$

$Z_T = e^{Y_T}$

Hence, in terms of $Z(t)$:

$$dZ(t) = Z(t)g(t)dX(t).$$

Using the earlier relationship, we can write $Z(T) = e^{Y(T)}$.

$e^{Y_0} = Z_0$

Let's simplify this $Z(T) =$

$$\exp\left\{Y_0 - \frac{1}{2}\int_0^T g^2(t)dt + \int_0^T g(t)dX(t)\right\}$$

to give

$$Z(T) = Z_0 \exp\left\{-\frac{1}{2}\int_0^T g^2(t)dt + \int_0^T g(t)dX(t)\right\}$$

Because the stochastic process $Z(t)$ is the exponential of another process (namely $Y(t)$) and because it is a martingale, we call $Z(t)$ an **exponential martingale**.

We have actually just stumbled upon a much more general and very important result.

Key Condition (Novikov Condition)

A trading strategy $(\phi, \psi) = \{(\phi_t, \psi_t)\,; t \in [0, ..., T]\}$ is a previsible process in that $\phi_t \in \mathcal{F}_{t-}$.

A stochastic process Υ_t satisfies the *Novikov condition* if

$$\mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^T \gamma_s^2 ds\right)\right] < \infty$$

where is a γ_t previsible process.

will cover in detailed extra notes

$\mathbb{P}$ $\mathbb{Q}$

Changing Probability Measure

You have seen in the Binomial Model lecture that there is more than just one probability measure.

Indeed, the lecture introduced you to the distinction between the "real" or "physical" probability measure, which we encounter every day on our Bloomberg or Reuters screen, and the so-called "risk-neutral" measure, which is used for derivative pricing.

Probability measures are by no means unique. We will see in the next lecture (3.2) that the powerful arsenal of martingale techniques enables us, under certain assumptions, to change measure and transpose our problem subject to the real world measure into an equivalent problem formulated as a martingale under a different measure.

For now, we just outline the rules that allow us to define equivalent measures.

Equivalent Measure

If two measures $\mathbb{P}$ and $\mathbb{Q}$ share the same sample space Ω and if $\boxed{\mathbb{P}(A) = 0}$ implies $\boxed{\mathbb{Q}(A) = 0}$ for all subset A, we say that $\mathbb{Q}$ is **absolutely continuous** with respect to $\mathbb{P}$ and denote this by $\boxed{\mathbb{Q} << \mathbb{P}}$.

The key point is that all impossible events under $\mathbb{P}$ remain impossible under $\mathbb{Q}$. The probability mass of the possible events will be distributed differently under $\mathbb{P}$ and $\mathbb{Q}$. In short "it is alright to tinker with the probabilities as long as we do not tinker with the (im)possibilities"

If $\mathbb{Q} << \mathbb{P}$ and $\mathbb{P} << \mathbb{Q}$ then the two measures are said to be **equivalent**, denoted by $\boxed{\mathbb{P} \sim \mathbb{Q}}$.

This extremely important result is formalized in the **Radon Nikodym Theorem**.

The main interest of a change of measure is to make difficult problems easier to solve. While some problems might be extremely difficult to tackle under the real-world measure $\mathbb{P}$, it might be possible to find an equivalent measure $\mathbb{Q}$ under which they are much easier to solve.

As a result, the change of measure techniques have become a cornerstone not only of modern probability but also of mathematical finance, where they are widely used in asset pricing.

Three prob. measures $\mathbb{P}$, $\mathbb{Q}$, $\mathbb{Z}$

$\mathbb{P} \sim \mathbb{Q}$

$\mathbb{P} \not\sim \mathbb{Z}$

$\mathbb{Q} \not\sim \mathbb{Z}$