Solutions to CQF Module 1 Assignment January 2025

1. a. Itô's lemma can be used to deduce the following formula for stochastic differential equations and stochastic integrals

$$\int_0^t \frac{\partial F}{\partial X_s}\, dX_s = F\left(X_t, t\right) - F\left(X_0, 0\right) - \int_0^t \left(\frac{\partial F}{\partial s} + \frac{1}{2}\frac{\partial^2 F}{\partial X_s^2}\right) ds$$

for a function $F(X_t, t)$ where dX_t is an increment of a Brownian motion. If $X_0 = 0$ evaluate

$$\int_0^t s^2 \sin X_s \, dX_s.$$

$$\begin{array}{ll} \downarrow \dfrac{\partial F}{\partial X_t} = t^2 \sin X_t \longrightarrow & F = -t^2 \cos X_t \downarrow \\ \dfrac{\partial^2 F}{\partial X^2} = t^2 \cos X_t & \dfrac{\partial F}{\partial t} = -2t \cos X_t \end{array}$$

and substitute into the integral formula

$$\int_0^t s^2 \sin X_s \, dX_s = -t^2 \cos X_t - \int_0^t \left(-2s \cos X_s + \frac{1}{2} s^2 \cos X_s\right) ds$$

b. Suppose the stochastic process $S\left(t\right)$ evolves according to Geometric Brownian Motion (GBM), where

$$dS_t = \mu S_t dt + \sigma S_t dX_t.$$

Obtain a SDE $df\left(S, t\right)$ for each of the following functions. Here we use Itô IV

$$df = \left(\frac{\partial f}{\partial t} + \mu S_t \frac{\partial f}{\partial S_t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 f}{\partial S_t^2}\right) dt + \sigma S_t \frac{\partial f}{\partial S_t} dX_t.$$

i $f\left(S, t\right) = \alpha^t + \beta t S^n$ $\quad$ α, β are constants

$$\begin{aligned} \frac{\partial f}{\partial t} &= \alpha^t \log \alpha + \beta S^n; \ \frac{\partial f}{\partial S} = n\beta t S^{n-1}; \ \frac{\partial^2 f}{\partial S^2} = n\left(n-1\right)\beta t S^{n-2} \\ df &= \left(\alpha^t \log \alpha + \beta S^n + n\mu\beta t S^n + \frac{1}{2} n\left(n-1\right)\beta t \sigma^2 S^n\right) dt + \sigma n \beta t S^n dX_t \end{aligned}$$

ii $f\left(S, t\right) = \log tS + \cos tS$

$$\begin{aligned} \frac{\partial f}{\partial t} &= \frac{1}{t} - S \sin tS; \ \frac{\partial f}{\partial S} = \frac{1}{S} - t \sin tS; \ \frac{\partial^2 f}{\partial S^2} = -\frac{1}{S^2} - t^2 \cos tS \\ df &= \left(\frac{1}{t} - S \sin tS + \mu S\left(\frac{1}{S} - t \sin tS\right) + \frac{1}{2}\sigma^2 S^2\left(-\frac{1}{S^2} - t^2 \cos tS\right)\right) dt + \\ & \quad \sigma S\left(\frac{1}{S} - t \sin tS\right) dX_t \end{aligned}$$

You can simplify, but it just creates more terms.

2. Consider a function $V(t, S_t, r_t)$ where the two stochastic processes S_t and r_t evolve according to a two factor model given by

$$\begin{aligned} dS_t &= \mu S_t dt + \sigma S_t dX_t^{(1)} \\ dr_t &= \gamma (m - r_t) dt + c dX_t^{(2)}, \end{aligned}$$

in turn. and where $\mathbb{E}\left[dX_t^{(1)} dX_t^{(2)}\right] = \rho dt$. The parameters μ, σ, γ, m and c are constant. Let $V(t, S_t, r_t)$ be a function on $[0, T]$ with $V(0, S_0, r_0) = v$. Using Itô, deduce the integral form for $V(T, S_T, r_T)$. Begin by writing a 3D Taylor expansion for $V(t, S_t, r_t)$

$$\begin{aligned} & V(t + dt, S_t + dS, r_t + dv) - V(t, S_t, r_t) \\ = \;& \frac{\partial V}{\partial t} dt + \frac{\partial V}{\partial S} dS + \frac{\partial V}{\partial r} dr + \frac{1}{2} \frac{\partial^2 V}{\partial S^2} dS^2 + \frac{1}{2} \frac{\partial^2 V}{\partial r^2} dr^2 + \frac{\partial^2 V}{\partial r \partial S} dr dS \end{aligned}$$

Since $dX_t^2 \to dt$ in the mean square limit for each $i = 1, 2$, we see that

$$dS_t^2 \to \sigma^2 S_t^2 dt,$$

$$dr_t^2 \to c^2 dt,$$

Also, since $dX_t^{(1)} dX_t^{(2)} = \rho dt$, we see that

$$dS_t dr_t \to \rho c \sigma S_t dt$$

This gives us a *bivariate* version of Itô's Lemma, the SDE for V is given by

$$\begin{aligned} dV = \;& \left(\frac{\partial V}{\partial t} + \mu S_t \frac{\partial V}{\partial S} + \gamma (m - r_t) \frac{\partial V}{\partial r} \right. \\ & \left. + \frac{1}{2} \sigma^2 S_t^2 \frac{\partial^2 V}{\partial S^2} + \frac{1}{2} c^2 \frac{\partial^2 V}{\partial r^2} + \rho c \sigma S_t \frac{\partial^2 V}{\partial r \partial S} \right) dt \\ & + \sigma S_t \frac{\partial V}{\partial S} dX_t^{(1)} + c \frac{\partial V}{\partial r} dX_t^{(2)} \end{aligned}$$

Integrating over $[0, t]$, we get

$$\begin{aligned} V(t, S_t, r_t) = \;& \underbrace{V(0, S_0, r_0)}_{=v} + \int_0^t \left(\frac{\partial V}{\partial \tau} + \mu S_\tau \frac{\partial V}{\partial S} + \gamma (m - r_\tau) \frac{\partial V}{\partial r} \right. \\ & \left. + \frac{1}{2} \sigma^2 S_\tau^2 \frac{\partial^2 V}{\partial S^2} + \frac{1}{2} c^2 \frac{\partial^2 V}{\partial r^2} + \rho c \sigma S_\tau \frac{\partial^2 V}{\partial r \partial S} \right) d\tau \\ & + \int_0^t \sigma S_\tau \frac{\partial V}{\partial S} dX_\tau^{(1)} + \int_0^t c \frac{\partial V}{\partial r_\tau} dX_\tau^{(2)} \end{aligned}$$

3. An equity price S evolves according to Geometric Brownian Motion

$$dS_t = \mu S_t dt + \sigma S_t dX_t,$$

where μ and σ are constants. We know that an explicit solution is

$$S_t = S_0 e^{\left(\mu - \sigma^2/2\right)t + \sigma X_t}$$

where S_0 is S_t at time $t = 0$.
By working through all the integration steps, deduce that the expected value of S_t at time $t > 0$, given S_0, is

$$\mathbb{E}\left[\left. S_t \right| S_0\right] = S_0 e^{\mu t}.$$

You are required to present all your integration steps to obtain the expectation.
We wish to calculate the expected terminal value of this risky asset. So

$$\begin{aligned} \mathbb{E}\left[S_t\right] &= S_0 \mathbb{E}\left[\exp\left\{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma X_t\right\}\right] \\ &= S_0 e^{\left(\mu - \frac{1}{2}\sigma^2\right)t} \mathbb{E}\left[e^{\sigma X_t}\right] \end{aligned}$$

At this point we have removed from the expectation operator all terms that are not random. This leaves the B.M term. We can write the expectation without the condition S_0 and arrive at the same solution but mathematically it is sensible to say that this value is used to get values at future times.

$$\mathbb{E}\left[\left. S(t) \right| S_0\right] = \mathbb{E}\left[S_0 e^{\left(\mu - \sigma^2/2\right)t + \sigma X_t}\right] = S_0 e^{\left(\mu - \sigma^2/2\right)t} \mathbb{E}\left[e^{\sigma X_t}\right]$$

Now let's recall $X_t \sim N(0, t)$. So we can write down the transition pdf of X_t as we did in class which is

$$\frac{1}{\sqrt{2\pi t}} e^{-x^2/2t}$$

Consider a random variable X with pdf $p(x)$. If we have to calculate the expected value of $h(X)$ it is written

$$\mathbb{E}\left[h(X)\right] = \int_{-\infty}^{\infty} h(x) p(x) dx$$

Now to calculate the shortened expectation $\mathbb{E}\left[e^{\sigma X_t}\right] = \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} e^{\sigma x} e^{-x^2/2t} dx = \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} e^{\sigma x - x^2/2t} dx$

Next we complete the square on the exponent $\sigma x - x^2/2t$ by adding and subtracting the same term

$$\begin{aligned} -\frac{1}{2t}\left(-2t\sigma x - x^2\right) &= -\frac{1}{2t}\left(-2t\sigma x + x^2 + \sigma^2 t/2 - \sigma^2 t/2\right) \\ &= -\left(x - \sigma t\right)^2/2t + \sigma^2 t/2 \end{aligned}$$

$$\begin{aligned} &= \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} e^{-(x-\sigma t)^2/2t} e^{\sigma^2 t/2} dx \\ &= e^{\sigma^2 t/2} \underbrace{\frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} e^{-(x-\sigma t)^2/2t} dx}_{=1} \end{aligned}$$

We have done the integration by substitution, let $u = \dfrac{x - \sigma t}{\sqrt{2t}}$ hence $\sqrt{2t}du = dx$. The limits remain the same.

$$\frac{\sqrt{2t}}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} e^{-u^2} du = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-u^2} du = \frac{1}{\sqrt{\pi}} \times \sqrt{\pi} = 1$$

So $\mathbb{E}\left[\left. S(t) \right| S_0\right] = S_0 e^{\left(\mu - \sigma^2/2\right)t} e^{\sigma^2 t/2} = S_0 e^{\mu t}$

4. In the steady state case, there is no time dependency, hence the Fokker Planck PDE becomes an ODE with

$$\frac{1}{2}\sigma^2\frac{d^2p_\infty}{du^2}+\theta\frac{d}{du}\left(up_\infty\right)=0$$

$p_\infty = p_\infty\left(u\right)$. The prime notation and subscript have been dropped simply for convenience at this stage. To solve the steady-state equation: Integrate wrt u

$$\frac{1}{2}\sigma^2\frac{dp}{du}+\theta\left(up\right)=k$$

where k is a constant of integration and can be calculated from the condtions, that as

$$u\rightarrow\infty:\left\{\begin{array}{l}\dfrac{dp}{du}\rightarrow 0\\ p\rightarrow 0\end{array}\right.\quad\Rightarrow k=0$$

which gives

$$\frac{1}{2}\sigma^2\frac{dp}{du}=-\theta\left(up\right),$$

a first order variable separable equation. So

$$\begin{aligned}\frac{1}{2}\sigma^2\int\frac{dp}{p} &= -\theta\int u du\rightarrow\\ \frac{1}{2}\sigma^2\ln p &= -\theta\left(\frac{u^2}{2}\right)+C\,,\qquad C\ \text{ is arbitrary.}\\ \ln p &= -\frac{\theta}{\sigma^2}u^2+c\end{aligned}$$

Rearranging and taking exponentials of both sides to give

$$p=\exp\left(-\frac{\theta}{\sigma^2}u^2+c\right)=A\exp\left(-\frac{\theta}{\sigma^2}u^2\right)$$

and we know as p_∞ is a PDF

$$\int_{-\infty}^{\infty}p_\infty\ du'=1\rightarrow A\int_{-\infty}^{\infty}e^{-\left(\frac{\theta}{\sigma^2}u^2\right)}du=1$$

A few (related) ways to calculate A. Now use the error function, i.e. $\int_{-\infty}^{\infty}e^{-x^2}dx=\sqrt{\pi}$. So put

$$x=\sqrt{\frac{\theta}{\sigma^2}}u\rightarrow dx=\sqrt{\frac{\theta}{\sigma^2}}du$$

which transforms the integral above

$$\frac{A\sigma}{\sqrt{\theta}}\int_{-\infty}^{\infty}e^{-x^2}\ dx=1\rightarrow A\sigma\sqrt{\frac{\pi}{\theta}}=1\therefore A=\frac{1}{\sigma}\sqrt{\frac{\theta}{\pi}}.$$

This allows us to finally write the steady-state transition PDF as

$$p_\infty=\frac{1}{\sigma}\sqrt{\frac{\theta}{\pi}}\exp\left(-\frac{\theta}{\sigma^2}u'^{\,2}\right).$$

If we compare this to

$$p\left(x\right)=\frac{1}{\sigma\sqrt{2\pi}}\exp\left(-\frac{1}{2\sigma^2}\left(x-\mu\right)^2\right)$$

we find u' is normally distributed with mean 0. For the variance write $-\frac{\theta}{\sigma^2}u'^{\,2}$ as

$$-\frac{1}{2}.2.\frac{\theta}{\sigma^2}u'^{\,2}\equiv-\frac{1}{2}\frac{1}{\sigma^2/2\theta}u'^{\,2}$$

to give the variance as $\sigma^2/2\theta$ and standard deviation is $\sqrt{\sigma/2\theta}$

5. The steady state equation for our model becomes

$$\frac{1}{2}\nu^2\frac{d^2}{dr^2}\left(r^{2\beta}p_\infty(r)\right)-\frac{d}{dr}\left(u(r)\,p_\infty(r)\right)=0$$

This can be simply integrated once to give

$$\begin{aligned}\frac{1}{2}\nu^2\frac{d}{dr}\left(r^{2\beta}p_\infty(r)\right)-\left(u(r)\,p_\infty(r)\right) &= \text{const}\\ \frac{1}{2}\nu^2\left(r^{2\beta}\frac{dp_\infty}{dr}\right)+\nu^2\beta r^{2\beta-1}p_\infty(r)-\left(u(r)\,p_\infty(r)\right) &= \text{const}\end{aligned}$$

The constant of integration is zero because as r becomes large

$$\left.\begin{array}{c}p_\infty(r)\\ \frac{dp_\infty}{dr}\end{array}\right\}\longrightarrow 0$$

$$\begin{aligned}u(r)\,p_\infty(r) &= \frac{1}{2}\nu^2 r^{2\beta}\frac{dp_\infty}{dr}+\nu^2\beta r^{2\beta-1}p_\infty(r)\\ u(r) &= \frac{1}{2}\nu^2 r^{2\beta}\frac{1}{p_\infty(r)}\frac{dp_\infty}{dr}+\nu^2\beta r^{2\beta-1}\end{aligned}$$

We can write $\dfrac{1}{p_\infty}\dfrac{dp_\infty}{dr}$ as $\dfrac{d}{dr}(\log p_\infty)$

$$u(r)=\frac{1}{2}\nu^2 r^{2\beta}\frac{d}{dr}(\log p_\infty)+\nu^2\beta r^{2\beta-1}$$

6. In this question $t\geq 0$.

   a. There are two ways to argue that this is not a martingale. Either say that X_t^2-t is a martingale hence Y_t cannot be. Or do Itô to get a SDE For which values of k is the process

   $$dY_t=\frac{dY_t}{dX_t}dX_t+\frac{1}{2}\frac{d^2Y_t}{dX_t^2}dt=dt+2X_t dX_t$$

   which is not driftless.

   b.

   $$Y_t=X_t^4-6tX_t^2+kt^2,\ t\geq 0,$$

   a martingale? The problem is asking you to calculate the value of k such that Y_t has zero drift. Using Itô

   $$dY_t=\left(\frac{\partial Y_t}{\partial t}+\frac{1}{2}\frac{\partial^2 Y_t}{\partial X_t^2}\right)dt+\frac{\partial Y_t}{\partial X_t}dX_t$$

   $$\frac{\partial Y_t}{\partial t}=-6X_t^2+2kt;\ \frac{\partial Y_t}{\partial X_t}=4X_t^3-12tX_t;\ \frac{\partial^2 Y_t}{\partial X_t^2}=12X_t^2-12t$$

   $$\begin{aligned}\frac{\partial Y_t}{\partial t}+\frac{1}{2}\frac{\partial^2 Y_t}{\partial X_t^2} &= 0\to -6X_t^2+2kt+6X_t^2-6t=0\\ k &= 3.\end{aligned}$$

   c. Define $Y_t=t^2X_t-2\displaystyle\int_0^t sX_s ds$. Is Y_t a martingale? Easiest way is to define a new process $Z_t=t^2X_t$ with $Z_0=0$. To check that Y_t is a martingale it is enough to see if the drift of Z_t is $2\displaystyle\int_0^t sX_s ds$. Itô on $Z_t=t^2X_t$ gives $dZ_t=2tX_tdt+t^2dX_t$. Now integrate over $[0,t]$

   $$\begin{aligned}\int_0^t dZ_s &= \int_0^t 2sX_s ds+\int_0^t s^2 dX_s\\ Z_t &= t^2X_t+2\int_0^t sX_s ds\end{aligned}$$

hence

$$Y_t = 2\int_0^t s^2 dX_s$$

which is an Itô integral and hance a martingale.

d. Show that $X_t = \cosh(\theta X_t)\, e^{-\theta^2 t/2};\ \theta \in \mathbb{R}$, is a martingale.

$$F(X_t, t) = \cosh(\theta X_t)\, e^{-\theta^2 t/2}$$

Using Itô

$$dF = \left(\frac{\partial F}{\partial t} + \frac{1}{2}\frac{\partial^2 F}{\partial X_t^2}\right) dt + \frac{\partial F}{\partial X_t} dX_t$$

So checking that $\frac{\partial F}{\partial t} + \frac{1}{2}\frac{\partial^2 F}{\partial X_t^2} = 0$, i.e. a driftless process.

$$\begin{aligned}
\frac{\partial F}{\partial t} &= -\frac{1}{2}\theta^2 \cosh(\theta X_t)\, e^{-\theta^2 t/2} \\
\frac{\partial F}{\partial X_t} &= \theta \sinh(\theta X_t)\, e^{-\theta^2 t/2};\ \frac{\partial^2 F}{\partial X_t^2} = \theta^2 \cosh(\theta X_t)\, e^{-\theta^2 t/2}
\end{aligned}$$

$$\frac{\partial F}{\partial t} + \frac{1}{2}\frac{\partial^2 F}{\partial X_t^2} = -\frac{\theta^2}{2}\cosh(\theta X_t)\, e^{-\theta^2 t/2} + \frac{1}{2}\left(\theta^2 \cosh(\theta X_t)\, e^{-\theta^2 t/2}\right) = 0$$

Hence a Martingale.

7. Consider the following model risk-free interest rate $r = 0$:

ω	$S(0)$	$S(1)$	$S(2)$
ω_1	S	aS	a^2S
ω_2	S	aS	S
ω_3	S	$a^{-1}S$	S
ω_4	S	$a^{-1}S$	$a^{-2}S$

S is the initial asset value at $t = 0$ and $a > 1$ is a constant. Asset:

$$
\begin{tikzcd}[row sep=small]
 & & a^2S \\
 & aS \arrow[ur] \arrow[dr] & \\
S(0)=S \arrow[ur] \arrow[dr] & & S \\
 & \frac{1}{a}S \arrow[ur] \arrow[dr] & \\
 & & \frac{1}{a^2}S
\end{tikzcd}
$$

Option:

$$
\begin{tikzcd}[row sep=small]
 & & a^2S - S \\
 & V_1 \arrow[ur] \arrow[dr] & \\
V_3 \arrow[ur] \arrow[dr] & & 0 \\
 & V_2 \arrow[ur] \arrow[dr] & \\
 & & 0
\end{tikzcd}
$$

a. Find all the one period risk-neutral probabilities and the corresponding probability measure $\mathbb{Q}$ on $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$. Confirm that $\mathbb{E}^{\mathbb{Q}}[X]$ is the fair price.
These are when $r = 0$,

$$
\begin{aligned}
q(\text{up}) &= \frac{s - s_d}{s_u - s_d} \\
q(\text{down}) &= \frac{s_u - s}{s_u - s_d}
\end{aligned}
$$

For each time-step we have the probabilities:

$$
\begin{aligned}
q(\text{up}) &= \frac{S - \frac{1}{a}S}{aS - \frac{1}{a}S} = \frac{1}{a+1}, \\
q(\text{down}) &= \frac{aS - S}{aS - \frac{1}{a}S} = \frac{a}{a+1}.
\end{aligned}
$$

$\therefore$

$$
\begin{aligned}
q(\omega_1) &= \frac{1}{(a+1)^2} \\
q(\omega_2) &= \frac{a}{(a+1)^2} \\
q(\omega_3) &= \frac{a}{(a+1)^2} \\
q(\omega_4) &= \frac{a^2}{(a+1)^2}.
\end{aligned}
$$

So the expected value is:

$$\mathbb{E}^{\mathbb{Q}}[X] = \sum_{\omega} q(\omega) X(\omega) = q(\omega_1)\left(a^2 S - S\right) + 0 + 0 + 0 = \frac{a-1}{a+1} S,$$

b. Now consider a model where in each period the asset can either double or half. Show that the value of an option struck at the initial asset value S is $S/3$.
This is a special case of the above model when $a = 2$. Substituting in $a = 2$ into the option gives

$$\frac{2-1}{2+1} S = \frac{1}{3} S.$$