CQF Module 1 Lecture 5

Stochastic Differential Equations – Maths and Computation

Simulating and Manipulating Stochastic Differential Equations

In this lecture. . .

- Using Itô's lemma to manipulate stochastic differential equations
- Continuous-time stochastic differential equations as discrete-time processes
- Further Kolmogorov eqⁿ's
- Simple ways of generating random numbers in Excel

  ↳ we did this in the last class
- Correlated random walks

By the end of this lecture you will be able to

- manipulate stochastic differential equations more maths
- find transition probability density functions for arbitrary stochastic differential equations further FKE BKE asset prices
- simulate stochastic differential equations

fun bit in Excel.

Introduction

In order to become comfortable with the kind of models commonly used in quantitative finance you must be able to manipulate stochastic differential equations and generate random walks numerically.

general SDEs

mathematically

numerically

$\{G_t : t \in \mathbb{R}^+\}$ $\quad t \to t+dt$ $\quad dG_t^2 = B^2(G_{t'}, t)\,dt$

Manipulating stochastic differential equations

An equation of the form $\int_0^t dG_s = \int_0^t a(G_s, s)\,ds + \int_0^t b(G_s, s)\,dX_s$

$G_t - G_0 = \dots$ $\qquad G_t = x + \int_0^t a\,ds + \int_0^t b\,dX_s \quad (**)$

SDE

$$(*) \quad dG_t = a(G_t, t)\,dt + b(G_t, t)\,dX_t \qquad G_0 = x$$

$(*) \equiv (**)$

is called a Stochastic Differential Equation (SDE) for G (or random walk for dG) and consists of two components:

1. $a(G, t)\,dt$ is deterministic – coefficient of dt is known as the **drift** or **growth**

$\mathbb{E}[dG] = \mathbb{E}[a\,dt] + \mathbb{E}[b\,dX]$
$= a\,dt + b\,\mathbb{E}[dX] = a\,dt$

2. $b(G, t)\,dX$ is random – coefficient of dX is known as the **diffusion** or **volatility**

$\mathbb{V}[dG] = \overbrace{\cancel{\mathbb{V}[a\,dt]}}^{=0} + \mathbb{V}[b\,dX]$

and we say G evolves according to (or follows) this process.

$= b^2\,\mathbb{V}[dX] = b^2\,dt$

So if for example we have a random walk

$$dS = \mu S dt + \sigma S dX \tag{1}$$

then the drift is $a\,(S,t) = \mu S$ and the diffusion is $b\,(S,t) = \sigma S$.

The process (1) is also called **Geometric Brownian Motion** (GMB) or ~~**Exponential Brownian motion**~~ (EMB) and is a popular model for a wide class of asset prices.

A quick recap

We have previously considered Itô's lemma to obtain the change in a function $f(X)$ when $X \to X + dX$, where X is a standard Brownian motion.

This jump $df = f(X + dX) - f(X)$ is given by

$G_{t+dt} - G_t = a(G_t, t)\,dt + b(G_t, t)\underbrace{\left(X_{t+dt} - X_t\right)}$

at time t

G_t is known

$$df = \frac{df}{dX}dX + \frac{1}{2}\frac{d^2 f}{dX^2}dt \qquad (2)$$

using the result

$$\boxed{\lim_{dt \to 0} dX^2 = dt.}$$

financial contract

Suppose we now wish to extend the result (2) to consider the change in an option price $V(S)$ where the underlying variable S follows a geometric Brownian motion.

(Of course, you are not supposed to know anything about options yet. Just think of manipulating functions.)

$S \to S + dS \quad \therefore \quad V(S + dS) =$

If we rewrite (1) as

$\sqrt{dt}$

$$\frac{dS}{S} = \mu \, dt + \sigma \, dX$$

then dS represents the change in asset price S in a small time interval dt.

This expression is the return on the asset.

μ is the average growth rate of the asset and σ the associated volatility (standard deviation) of the returns.

dX is an increment of a Brownian Motion, known as a Wiener process and is a Normally distributed random variable such that $dX \sim N(0, dt)$.

An obvious question we may ask is, what is the jump in $V(S + dS)$ when $S \to S + dS$?

1D TSE

$V(S+dS) = V(S) + \frac{dV}{dS} dS + \frac{1}{2} \frac{d^2V}{dS^2} dS^2$

We begin (again) by using a Taylor series as in (2), but for $V(S + dS)$ to get

$dV = V(S+dS) - V(S)$

$$V(S+dS) - V(S) = dV = \frac{dV}{dS} \underbrace{dS}_{\mu S\, dt + \sigma S\, dX} + \frac{1}{2} \frac{d^2V}{dS^2} dS^2.$$

$dS^2 = (\mu S\, dt + \sigma S\, dX)^2 = \mu^2 S^2 \underbrace{\cancel{dt^2}}_{=0} + 2\mu\sigma S^2 \underbrace{\cancel{dt\, dX}}_{=0} + \sigma^2 S^2 \underbrace{\cancel{dX^2}}_{=dt}$

Now use Itô mult. table

$\therefore dS^2 = \sigma^2 S^2 dt$

We can proceed further now as we have an expression for dS (and hence dS^2). As dt is very small, any terms in $dt^{\frac{3}{2}}$ or dt^2 are insignificant in comparison and can be ignored. So working to $O(dt)$

$$\boxed{dS^2 = \sigma^2 S^2 dt.}$$

If we substitute this into the previous expression for dV we get Itô's lemma as applied to $V(S)$:

drift diffusion

Itô III

$$dV = \underbrace{\left(\mu S \frac{dV}{dS} + \frac{1}{2}\sigma^2 S^2 \frac{d^2V}{dS^2}\right) dt}_{} + \left(\sigma S \frac{dV}{dS}\right) dX. \qquad (3)$$

Note that this is another stochastic differential equation!

It contains a predictable part and a random part.

Important Example: Solve $dS = \mu S\, dt + \sigma S\, dX$

Suppose that we had a formula for $V(S)$. Let's take a very special case, let's consider

$\sigma, \mu \in \mathbb{R}$

$\exists$ closed form sol^n for a special case of a GBM

Do Itô III on $V = \log S$

$$\boxed{V(S) = \log S.}$$

Differentiating this once gives

$$\frac{dV}{dS} = \frac{1}{S}.$$

Differentiating this again gives

$$\frac{d^2V}{dS^2} = -\frac{1}{S^2}.$$

Subst in (3) $d(\log S) = \left(\mu S\left(\frac{1}{S}\right) + \frac{1}{2}\sigma^2 S^2\left(-\frac{1}{S^2}\right)\right)dt + \sigma S\left(\frac{1}{S}\right)dX$

Now from (3) we have

we are after this

$$d\left(\log S\right)=\left(\mu-\tfrac{1}{2}\sigma^{2}\right)dt+\sigma dX.$$

now

some time in future

Integrating both sides between 0 and t

$$\int_0^t d\left(\log S\right)=\int_0^t\left(\mu-\frac{1}{2}\sigma^{2}\right)d\tau+\int_0^t\sigma\,dX\qquad(t>0)$$

$$\log\frac{S_t}{S_0}=\left(\mu-\tfrac{1}{2}\sigma^{2}\right)t+\sigma\underbrace{\left(X\left(t\right)-\overset{=0}{\cancel{X}\left(0\right)}\right)}.$$

known

$$\sigma X(t)$$

$$\sigma\phi\sqrt{t}$$

$$\int_0^T : \quad S_T = S_0 \exp\left\{\left(\mu - \tfrac{1}{2}\sigma^2\right)T + \sigma\phi\sqrt{T}\right\}$$

Therefore

stock at expiry.

$$\log\left(\frac{S(t)}{S(0)}\right) = \left(\mu - \tfrac{1}{2}\sigma^2\right)t + \sigma\left(X(t) - X(0)\right)$$

I.C

Assuming $X(0) = 0$ and $S(0) = S_0$, the exact solution becomes

$$S(t) = S_0 \exp\left(\left(\mu - \tfrac{1}{2}\sigma^2\right)t + \sigma X(t)\right). \tag{4}$$

$$\int_0^t \quad S_t = S_0 e^{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma\phi\sqrt{t}} \qquad \phi \sim N(0,1)$$

$$\int_t^T : \quad S_T = S_t \exp\left[\left(\mu - \tfrac{1}{2}\sigma^2\right)(T-t) + \sigma\phi\sqrt{T-t}\right]$$

Important European option

$$\int_t^{t+dt} : \quad S_{t+dt} = S_t \exp\left\{\left(\mu - \tfrac{1}{2}\sigma^2\right)dt + \sigma\phi\sqrt{dt}\right\}$$

Ito IV

In the last example we had $V = V(S_t)$.

Now consider $V = V(t, S_t)$ $\therefore$ $t \to t + dt$; $S_t \to S_t + dS_t$

2D T.S.E : $\underbrace{V(t+dt, S_t + dS_t) - V(t, S_t)}_{dV} + \ldots$

$$V(t+dt, S_t+dS_t) = V(t, S_t) + \frac{\partial V}{\partial t} dt + \frac{\partial V}{\partial S_t} \underbrace{dS_t}_{\mu S_t dt + \sigma S_t dX_t} + \frac{1}{2}\frac{\partial^2 V}{\partial S_t^2} \underbrace{dS_t^2}_{= \sigma^2 S^2 dt}$$

$$dV = \underbrace{\left(\frac{\partial V}{\partial t} + \mu S_t \frac{\partial V}{\partial S_t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V}{\partial S_t^2} \right)}_{\text{drift}} dt + \underbrace{\sigma S_t \frac{\partial V}{\partial S_t} dX_t}_{\text{diffusion}}$$

Another example: Interest rate model

Let's take a look at the Vasicek interest rate model for short-term interest rates, and try manipulating that.

Spot rate $dr_t = (\eta - \gamma r_t)\,dt + \sigma\, dX_t = \gamma\left(\eta/\gamma - r_t\right)dt + \sigma\, dX_t$

$$dr = \gamma\left(\overline{r} - r\right)dt + \sigma dX.$$

define $\bar{r} = \dfrac{\eta}{\gamma}$

$dr = -\gamma(r_t - \bar{r})\,dt + \sigma\, dX_t$

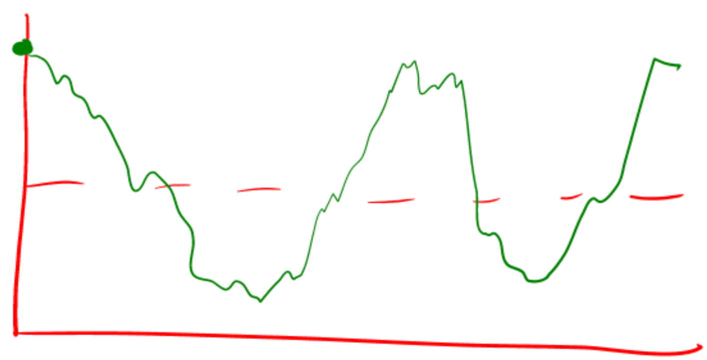
γ refers to the **reversion rate** and $\overline{r}$ denotes the $\boxed{\textbf{mean rate}}$.

Speed of reversion (speed)

long term equilibrium rate

$\eta = \gamma \bar{r}$

$\bar{r}$



mean reversion

drift mean reversion

$-\gamma(r_t - \bar{r})$

$r_t < \bar{r}$ +ve trend

$r_t > \bar{r}$ -ve trend

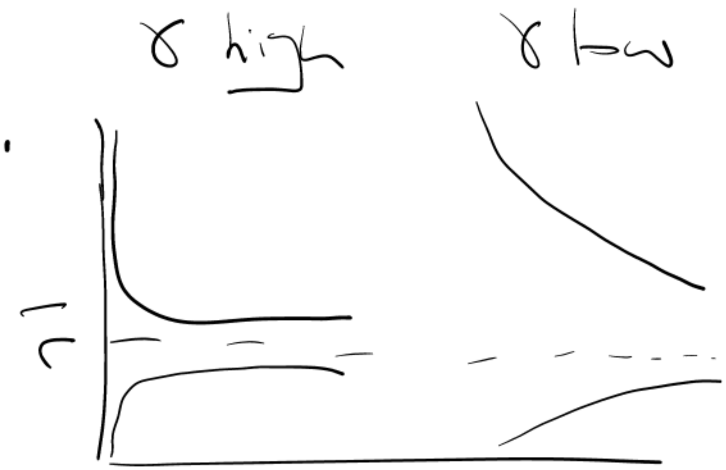
Let's look at Vasicek in the absence of randomness, i.e. set $\sigma = 0$

$$\therefore \quad dr = -\gamma(r-\bar{r})\,dt \qquad \text{ODE}$$

$$\int \frac{dr}{r-\bar{r}} = -\gamma \int dt$$

$$\log(r-\bar{r}) = -\gamma t + \text{Const.}$$

$$\therefore \quad r = \bar{r} + A e^{-\gamma t}$$



exponential decay $\qquad \dfrac{1}{\gamma}$ units of time transf$^{\underline{n}}$ / change of var. $du = dr$

By setting $u = r - \overline{r}$, u is a solution of

$$u_0 = \alpha \qquad du = -\gamma u\, dt + \sigma dX. \quad \text{Ornstein – Uhlenbeck process}$$

$$du + \gamma u\, dt = \sigma\, dX$$

Mult. both sides by an I.F $\quad e^{\gamma t}\left(du + \gamma u\, dt\right) = \sigma e^{\gamma t} dX_t$

An analytic solution for this equation exists. To see, this write the equation as

$$d\left(ue^{\gamma t}\right) = \sigma e^{\gamma t} dX. \qquad \int_0^t d(u_s e^{\gamma s}) = \sigma \int_0^t e^{\gamma s} dX_s$$

now $\leftarrow$ $\quad$ $\rightarrow$ later

$$u_t e^{\gamma t} - u_0 e^{\gamma \times 0}$$

$$u_t e^{\gamma t} - u_0 =$$

Integrating over from zero to t gives

$$\longrightarrow u(t) = u(0)e^{-\gamma t} + \sigma \int_0^t e^{\gamma(s-t)} dX_s.$$

$\therefore$ $u = r - \overline{r} \Rightarrow r = u + \overline{r}$

This can be **integrated by parts** to give

$$u(t) = u(0)e^{-\gamma t} + \sigma\left(X(t) - \gamma\int_0^t X(s)e^{\gamma(s-t)}\, ds\right).$$

Cox-Ingersoll-Ross

$r \ll 1$

$$dr = -\gamma(r-\bar{r})\,dt + \sigma\sqrt{r}\,dX$$

sqrt process in diffusion

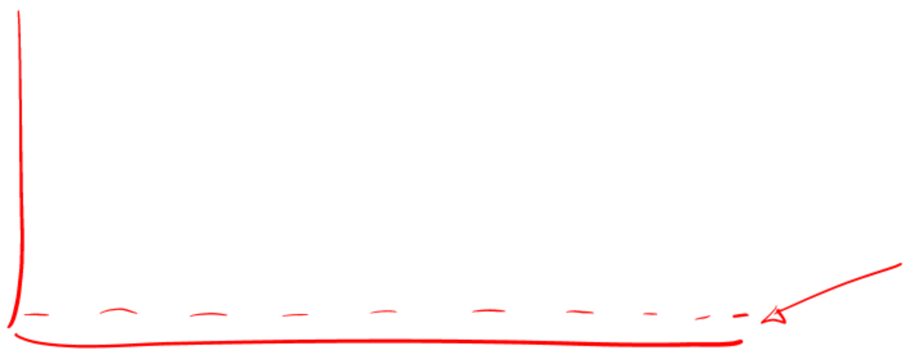
As $r \longrightarrow 0$ randomness switches off

we are left with $-\gamma(r-\bar{r})$ $-\gamma \times (-\bar{r})$

$r \to 0$

$\boxed{\eta, \gamma, \sigma}$

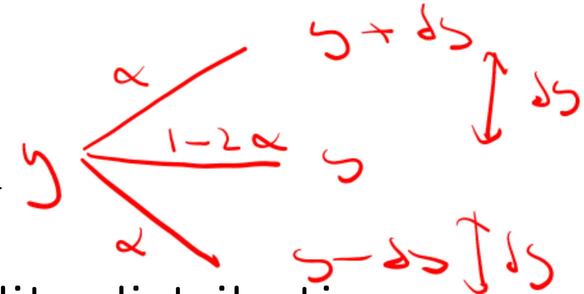
+ve



$e^{-\gamma t} \longrightarrow$ inside the integral

Revisit transition densities,

Transition probability density functions again

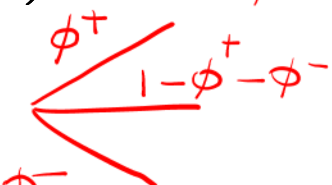


$\delta y \sim O(\sqrt{\delta t})$

Let's look at the equations governing the probability distribution for an arbitrary random walk:

$$dy = \underbrace{A(y,t)\,dt}_{\text{drift}} + \overbrace{B(y,t)\,dX}^{\text{diffusion}}$$

for the variable y.



Calc. ϕ^+, ϕ^- s.t. mean of discrete matches cts

Variance of discrete matches cts

Remember the **transition probability density function** $p(y,t;y',t')$ defined by

cts. time.

$$\text{Prob}(a < y' < b \text{ at time } t' | y \text{ at time } t) = \int_a^b p(y,t;y',t')\,dy'.$$

In words this is 'the probability that the random variable y lies between a and b at time t' in the future, given that it started out with value y at time t.'

Think of y and t as being current values with y' and t' being future values.

The transition probability density function can be used to answer questions such as

"What is the probability of the variable y being in a certain range at time t' given that it started out with value y at time t?"

The transition probability density function $p(y, t; y', t')$ satisfies two equations.

One involves derivatives with respect to the future state and time (y' and t') and is called the **forward equation.**

The other involves derivatives with respect to the current state and time (y and t) and is called the **backward equation.**

These can be derived by the same trinomial idea we used before (but the details are a lot messier for the general stochastic differential equation).

F.K.E depends on (y', t') with (y, t) fixed

B.K.E depends on (y, t) with (y', t') fixed

$$\frac{\partial p}{\partial t'} = c^2 \frac{\partial^2 p}{\partial y'^2}$$ Zero drift

The forward equation

Cutting to the chase, the transition probability density function satisfies the partial differential equation

$$dy = A(y,t)\,dt + B(y,t)\,dX_t$$

Complicated

$$\frac{\partial p}{\partial t'} = \tfrac{1}{2}\frac{\partial^2}{\partial y'^2}\left(B(y',t')^2 p\right) - \frac{\partial}{\partial y'}\left(A(y',t')p\right)$$

\$

This is the **Fokker–Planck** or **forward Kolmogorov equation**.

$$dy = A(y,t)\,dt + B(y,t)\,dX_t$$

Example: The most important example to us is that of the distribution of equity prices in the future. If we have the random walk

stock

$$dS = \overbrace{\mu S}^{A}\,dt + \overbrace{\sigma S}^{B}\,dX$$

subst A, B in S

then the forward equation becomes

F.K.E for a stock.

$$\frac{\partial p}{\partial t'} = \frac{1}{2}\frac{\partial^2}{\partial S'^2}\left(\sigma^2 S'^2 p\right) - \frac{\partial}{\partial S'}\left(\mu S' p\right).$$

transf / subst, I.C.

↳ 1D F.K.E.

The solution of this representing a stock price starting at $S' = S$ at $t' = t$ is

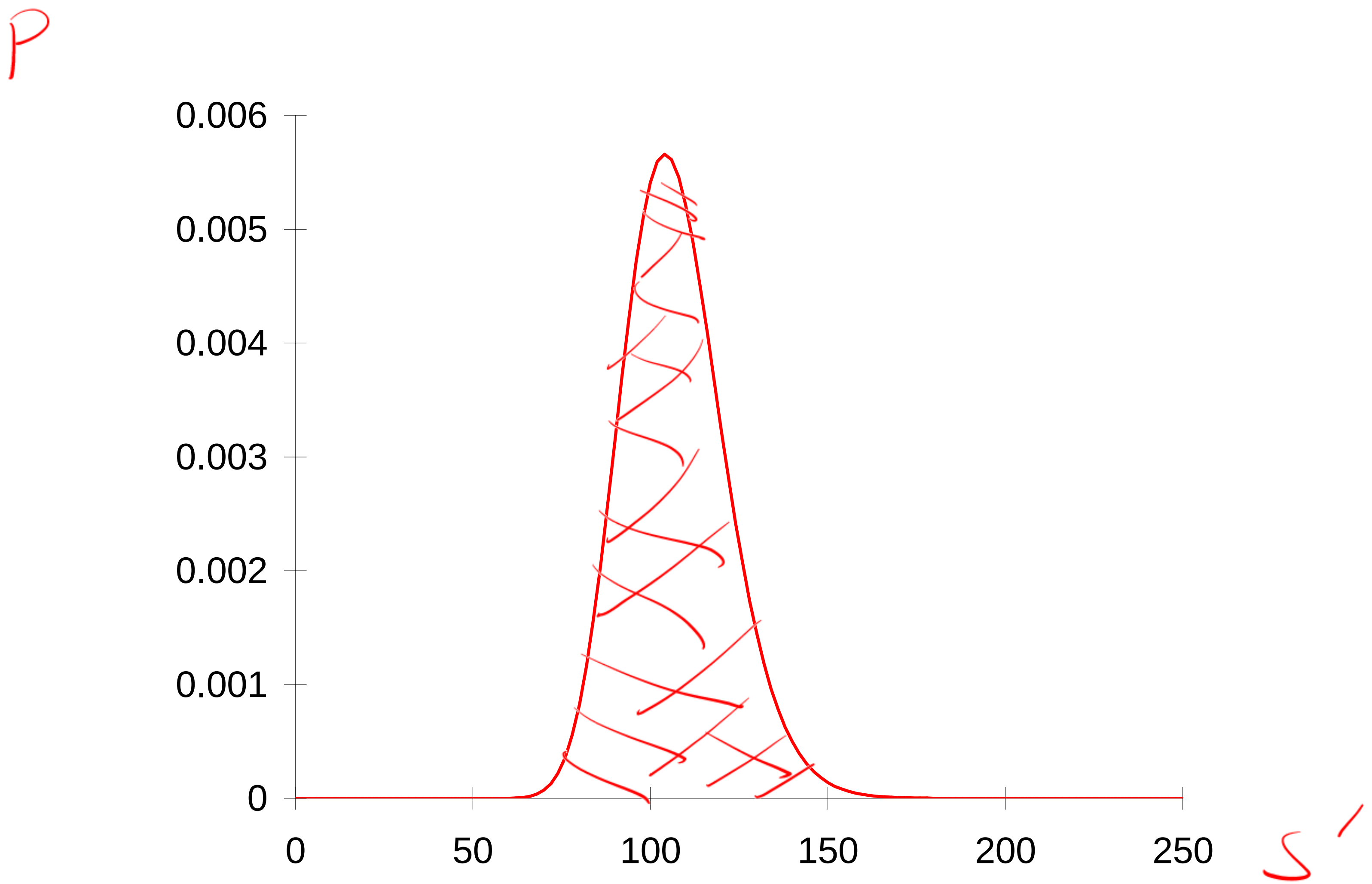
stock is (S, t)

later time stock = S'

$$p(\underbrace{S,t}_{\text{I.C.}};\underbrace{S',t'}_{\text{var.s.}}) = \frac{1}{\sigma S'\sqrt{2\pi(t'-t)}}e^{-\left(\log(S/S')+(\mu-\frac{1}{2}\sigma^2)(t'-t)\right)^2/2\sigma^2(t'-t)}$$

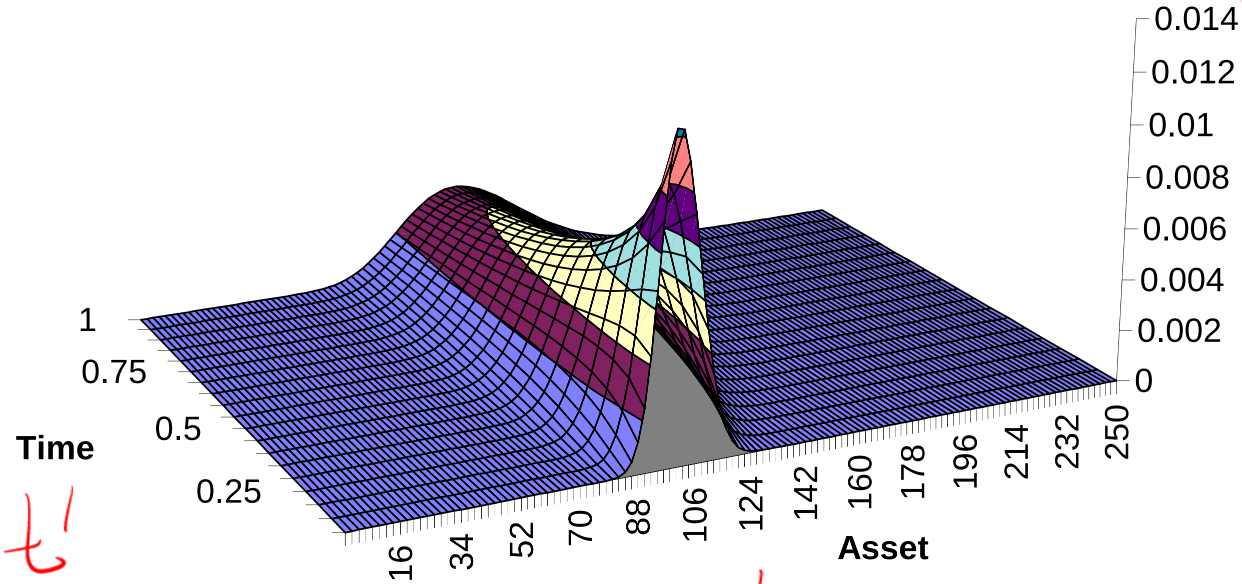
In mod. 3 (B.S.E)

look at MIL3



The probability density function for the lognormal random walk, after a certain time.

$p(S', t')$



t'

S'

pick a time t' and look at a cross-section

The probability density function for the lognormal random walk evolving through time.

A special case

The steady-state distribution

something which settles in the 'long run'

Some random walks have a steady-state distribution.

That is, in the long run as $t' \to \infty$ the distribution $p(y, t; y', t')$ as a function of y' settles down to be independent of the starting state y and time t. Possible examples are stochastic differential equation models for interest rates, inflation, volatility.

Some random walks have no such steady state even though they have a time-independent equation. For example the lognormal random walk either grows without bound or decays to zero.

p behaves indep. of t'

$$\Rightarrow \quad p = p(y') \quad \text{and} \quad \frac{\partial p}{\partial t'} = 0$$

If there is a steady-state distribution $p_\infty(y')$ then it satisfies the *ordinary* differential equation

$t' \to \infty$ and a steady state soln exists

F.K.E : for steady state

$$\frac{1}{2}\frac{d^2}{dy'^2}\left(B^2 p_\infty\right) - \frac{d}{dy'}\left(A p_\infty\right) = 0.$$

$p_\infty \equiv p$ in the steady state case

Example: The Vasicek model

$$dr = \overbrace{\gamma\left(\bar{r} - r\right)}^{A} dt + \overbrace{\sigma}^{B} dX.$$

The steady-state distribution $p_\infty(r')$ satisfies

2nd order ODE.

$$\frac{1}{2}\sigma^2\frac{d^2 p_\infty}{dr'^2} + \gamma\frac{d}{dr'}\left(\overset{r' - \bar{r}}{(\bar{r} - r')}\, p_\infty\right) = 0.$$

$\frac{1}{2}\sigma^2 p_\infty''(r') = -\gamma \frac{d}{dr'}\left((r' - \bar{r})p_\infty\right)$

The solution is

$$\frac{1}{2}\sigma^2 p_\infty''(r') = -\gamma \frac{d}{dr'}\left((r'-\bar{r})p_\infty\right)$$

Now integrate both sides

$$\frac{1}{2}\sigma^2 \frac{dp}{dr} = -\gamma(r-\bar{r})p + \text{const}.$$

Const $= 0$ $\because$ of r' being a RV and p_∞ a pdf

$$\frac{1}{2}\sigma^2 \frac{dp}{dr} = -\gamma(r-\bar{r})p$$

$$p_\infty = \frac{1}{\sigma}\sqrt{\frac{\gamma}{\pi}}\; e^{-\frac{\gamma(\bar{r}-r')^2}{\sigma^2}}.$$

$$-\frac{\gamma(r'-\bar{r})^2}{\sigma^2} \qquad -\frac{1}{2}\,\frac{2\gamma(r'-\bar{r})^2}{\sigma^2}$$

mean

$$\frac{(r'-\bar{r})^2}{\sigma^2/2\gamma}$$

In other words, the interest rate r is Normally distributed with mean $\bar{r}$ and standard deviation $\sigma/\sqrt{2\gamma}$.

$$\int\frac{dp}{p} = -\frac{2\gamma}{\sigma^2}\int(r-\bar{r})\,dr = -\frac{\cancel{2}\gamma}{\sigma^2}\int\frac{1}{\cancel{2}}\frac{d}{dr}(r-\bar{r})^2\,dr \qquad \frac{\sigma^2}{2\gamma}$$

$$\log p = -\frac{\gamma}{\sigma^2}(r-\bar{r})^2 + \text{Const}.$$

$$p = A e^{-\frac{\gamma}{\sigma^2}(r-\bar{r})^2}$$

A normalising const.

$$\int_{\mathbb{R}} p_\infty(r')\,dr' = 1 \qquad A\int_{-\infty}^{\infty} e^{-\frac{\gamma}{\sigma^2}(r-\bar{r})^2}\,dr = 1$$

Let $x = \frac{\sqrt{\gamma}}{\sigma}(r-\bar{r})$

$$\frac{\sigma}{\sqrt{\gamma}}\,dx = dr$$

$$\frac{\sigma}{\sqrt{\gamma}}A\int_{-\infty}^{\infty} e^{-x^2}\,dx = 1$$

$$\frac{\sigma}{\sqrt{\gamma}}\sqrt{\pi}\,A = 1 \qquad A = \sqrt{\frac{\gamma}{\pi}}\,\frac{1}{\sigma}$$

The backward equation

Now we come to the backward equation. This will be useful if we want to calculate probabilities of reaching a specified final state from various initial states.

The transition probability density function satisfies the **backward Kolmogorov equation**

$$\frac{\partial p}{\partial t} + \frac{1}{2}B(y,t)^2\frac{\partial^2 p}{\partial y^2} + A(y,t)\frac{\partial p}{\partial y} = 0.$$

-rV

looks like a B.S.E

reminds us

Full derivation is the extra notes to follow at the end of this session

Simulating the lognormal random walk

The lognormal random walk model for assets can be written in continuous time as

cts time
$$dS = \mu S\, dt + \sigma S\, dX.$$

discrete time $\delta S = \mu S \delta t + \sigma S \phi \sqrt{\delta t}$

In discrete time this is

S_i is S at t_i $S(t_i)$

$$S_{i+1} - S_i = S_i \left(\mu\, \delta t + \sigma \phi\, \delta t^{1/2} \right).$$

index notation

To generate representative simulations of possible asset paths we must obviously work in discrete time.

The random walk on a spreadsheet

The random walk can be written as a 'recipe' for generating S_{i+1} from S_i:

now

$$S_{i+1} = S_i \left(1 + \mu\, \delta t + \sigma \phi\, \delta t^{1/2}\right).$$

stock at next time step

We can easily simulate the model using a spreadsheet.

The method is called the **Euler method**.

Euler–Maruyama method $\qquad \phi \sim N(0,1)$

Start with an initial stock price, say, 100. S_0

And a couple of parameters, $\mu = 0.1$ and $\sigma = 0.2$, say, that best represent the asset in question.

Decide on a (small) time step, $\delta t = 0.01$, say. $\frac{T}{100}$

Now start picking random numbers!

$S_0 = 100$

First time step: The random number is. . . 0.12. So

$$S_1 = S_{i+1} = \underbrace{100}_{S_0}\,(1 + 0.1 \times 0.01 + 0.2 \times 0.1 \times 0.12) = 100.34.$$

Second time step: The random number is. . . -0.25. So

$$S_2 \quad S_{i+1} = \underbrace{100.34}_{S_1}\,(1 + 0.1 \times 0.01 + 0.2 \times 0.1 \times (-0.25)) = 99.94.$$

And so on.

In this simulation there are several input parameters, which remain constant:

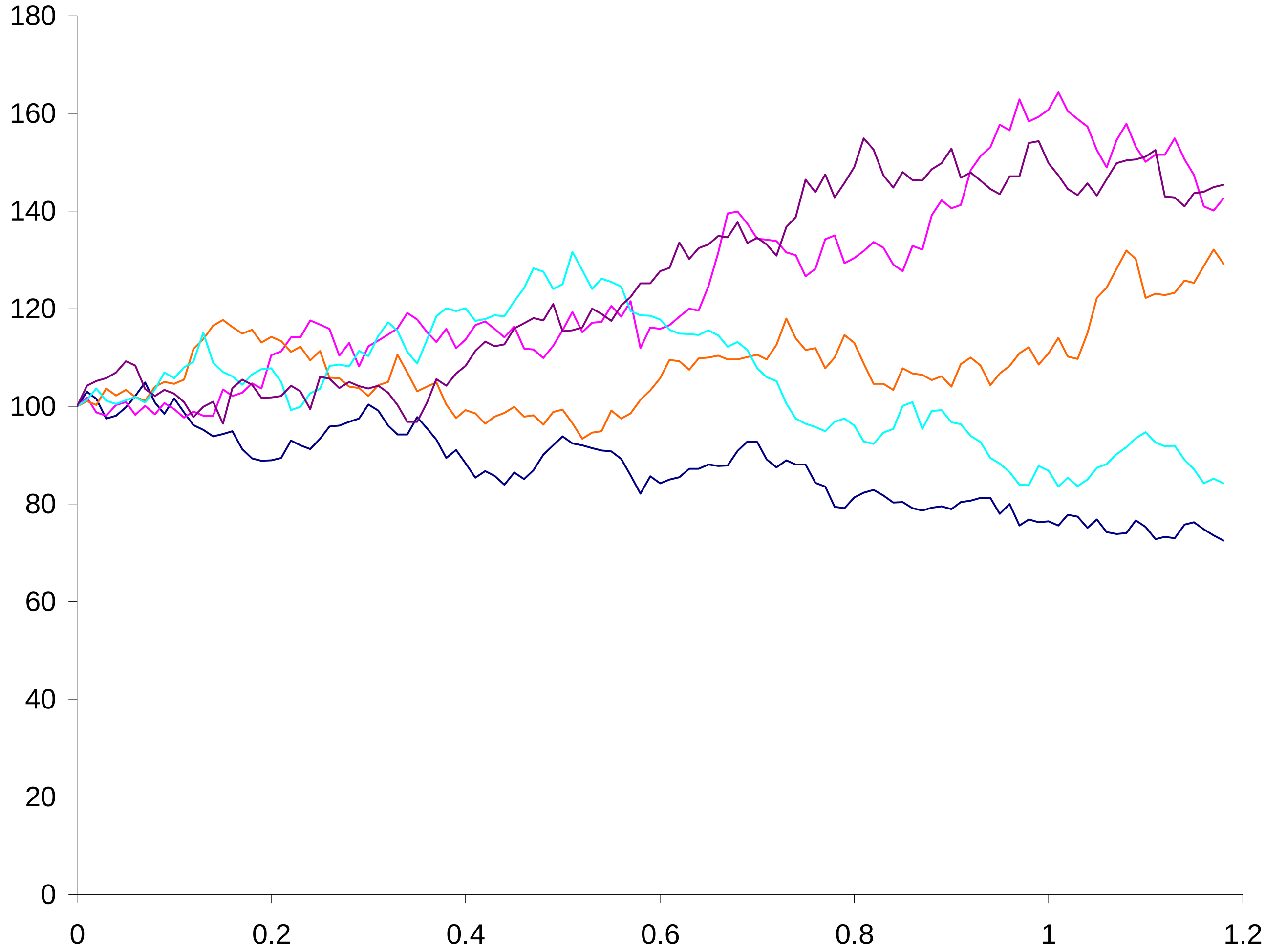
- a starting value for the asset
- a time step δt
- the drift rate μ
- the volatility σ
- the total number of time steps

Then, at each time step, we must choose a random number ϕ from a Normal distribution.

	A	B	C	D	E	F	G
1	**Asset**	**100**		**Time**	**Asset**		
2	**Drift**	**0.15**		0	100		
3	**Volatility**	**0.25**		0.01	96.10692		
4	**Timestep**	**0.01**		0.02	96.99647		
5				0.03	94.76352		
6				0.04	91.46698		
7				0.05	88.83325		
8				0.06	88.42727		
9				0.07	90.62882		
10				0.08	88.80545		
11							
12							
13				0.11	84.93865		

=D4+$B$4

=E7*(1+$B$2*$B$4+$B$3*SQRT($B$4)*(RAND()+RAND()+RAND()+RAND()
+RAND()+RAND()+RAND()+RAND()+RAND()+RAND()+RAND()+RAND()-6))


180
160
140
120
100
80
60
40
20
0
0
0.2
0.4
0.6
0.8
1
1.2

Simulating other random walks

This method is not restricted to the lognormal random walk.

Later in the course we will be modeling interest rates as stochastic differential equations.

The following is a stochastic differential equation model for an interest rate, that goes by the name of an **Ornstein-Uhlenbeck process** (an example of a mean-reverting random walk), or when used in an interest rate context the **Vasicek model**:

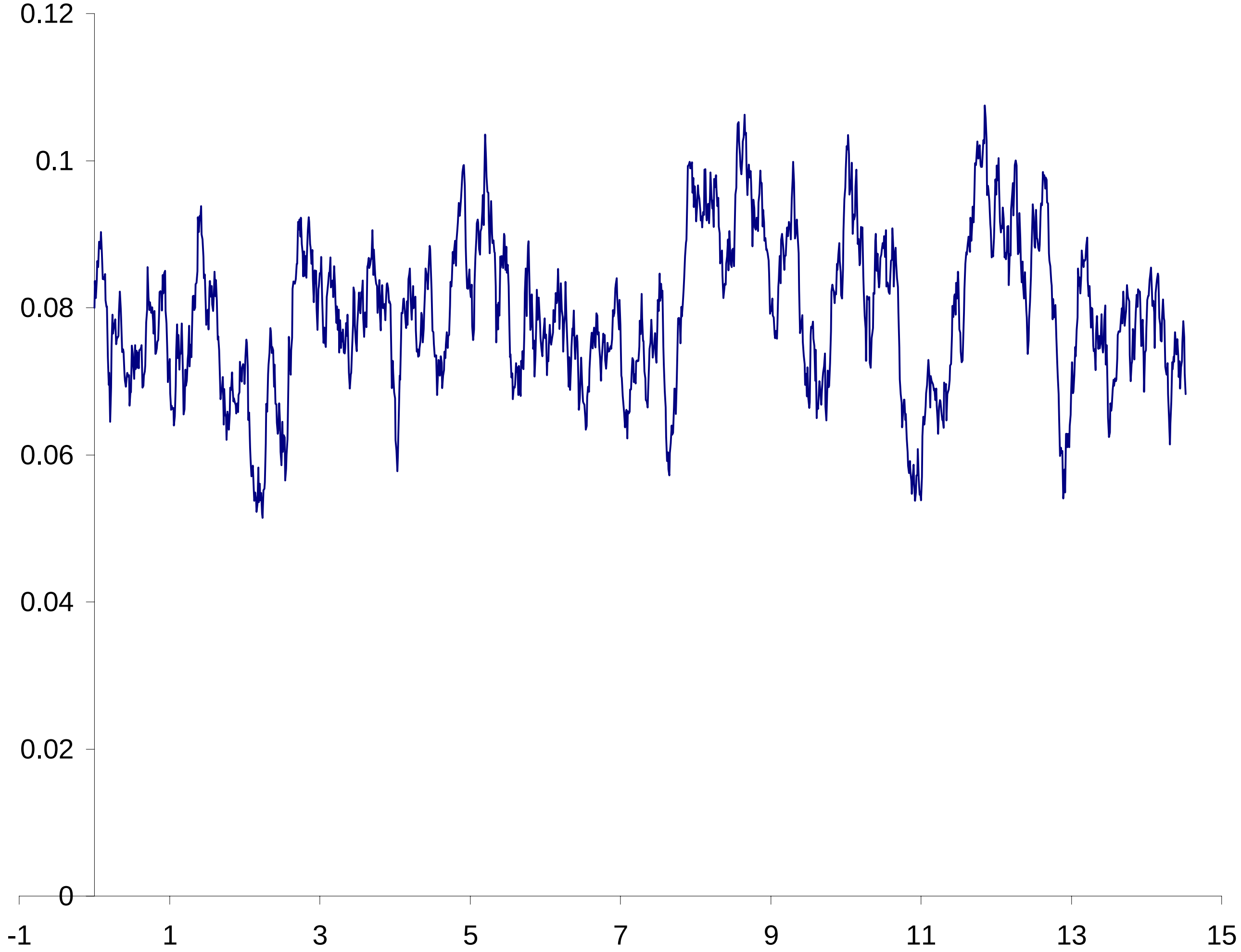
$$dr = \gamma\,(\overline{r} - r)\,dt + \sigma dX.$$

$$dr = (\eta - \gamma r)\,dt + \sigma dX$$

In discrete time we can approximate this by

$$r_{i+1} = r_i + (\eta - \gamma r_i)\,\delta t + \sigma \phi \sqrt{\delta t}$$

$$r_{i+1} = r_i + \gamma\,(\overline{r} - r_i)\,dt + \sigma\phi\;\delta t^{1/2}.$$


0.12
0.1
0.08
0.06
0.04
0.02
0
-1
1
3
5
7
9
11
13
15

Producing correlated random numbers

We will often want to simulate paths of correlated random walks.

We may want to examine the statistical properties of a portfolio of stocks, or value a convertible bond under the assumption of random asset price and random interest rates.

Multi-factor model $\Rightarrow$ at least 2 sources of randomness

Suppose e.g. 2 stocks $V = V(S_1, S_2, t)$

$$\frac{dS_i}{S_i} = \mu_i\,dt + \sigma_i\,dX_t^{(i)} \qquad i = 1, 2$$

$$\mathbb{E}[\phi_1 \phi_2] = \rho\,dt \qquad \mathbb{E}[dX^{(1)} dX^{(2)}] = \rho\,dt$$

Example:

Assets S_1 and S_2 both follow lognormal random walks with correlation ρ.

In continuous time we write

$\phi_1\sqrt{dt}$

$$dS_1 = \mu_1 S_1\,dt + \sigma_1 S_1\,dX_1,$$

$$dS_2 = \mu_2 S_2\,dt + \sigma_2 S_2\,dX_2, \longrightarrow \phi_2\sqrt{dt}$$

with

$\mathbb{E}(\phi_1\phi_2)$

$$\longrightarrow E[dX_1\,dX_2] = \rho\,dt.$$

In discrete time these become

Stock 1

$$S_{1_{i+1}} - S_{1_i} = S_{1_i}\left(\mu_1\,\delta t + \sigma_1 \phi_1\,\delta t^{1/2}\right)$$

$\phi_1 \to \varepsilon_1$

and

Stock 2

$$S_{2_{i+1}} - S_{2_i} = S_{2_i}\left(\mu_2\,\delta t + \sigma_2 \phi_2\,\delta t^{1/2}\right)$$

$\phi_2 \to \rho\,\varepsilon_1 + \sqrt{1-\rho^2}\,\varepsilon_2$

with

$$E[\phi_1\,\phi_2] = \rho.$$

Q: How can we choose a ϕ_1 and a ϕ_2 which are both Normally distributed, both have mean zero and standard deviation of one, and with a correlation of ρ between them?

$\phi_i \sim N(0,1)$

$\mathbb{E}[\phi_1 \phi_2] = \rho$

A: This can be done in two steps, first pick two *uncorrelated* Normally distributed random variables, and then combine them.

$\varepsilon_i \sim N(0,1)$

uncorrelated $\mathbb{E}[\varepsilon_1 \varepsilon_2] = 0$

$\mathbb{E}[\varepsilon_1] = 0 = \mathbb{E}[\varepsilon_2] \quad \mathbb{E}[\varepsilon_1^2] = 1 = \mathbb{E}[\varepsilon_2^2] \quad \mathbb{E}[\varepsilon_1 \varepsilon_2] = 0$

Step 1: Choose uncorrelated ϵ_1 and ϵ_2, both Normally distributed with zero means and standard deviations of one.

$\mathbb{E}[\phi_1] = 0 = \mathbb{E}[\phi_2] \quad \mathbb{E}[\phi_i^2] = 1 \quad \mathbb{E}[\phi_1 \phi_2] = \rho$

Step 2: Convert these independent Normal numbers into correlated Normals by taking a linear combination.

① Set $\phi_1 = \varepsilon_1$

form a linear combination

② $\phi_2 = \alpha \varepsilon_1 + \beta \varepsilon_2 \quad \alpha, \beta \in \mathbb{R}$

$$\phi_1 = \epsilon_1$$

$\phi_1 = \varepsilon_1$

$\phi_2 = \rho \varepsilon_1 + \sqrt{1-\rho^2}\, \varepsilon_2$

$\mathbb{E}[\phi_1 \phi_2] = \mathbb{E}[\varepsilon_1(\alpha \varepsilon_1 + \beta \varepsilon_2)]$

$$\phi_2 = \rho\, \epsilon_1 + \sqrt{1-\rho^2}\, \epsilon_2.$$

$= \alpha \mathbb{E}[\varepsilon_1^2] + \beta \mathbb{E}[\varepsilon_1 \varepsilon_2] = \alpha \times 1 + \beta \times 0 = \rho \Rightarrow \boxed{\alpha = \rho}$

$\mathbb{E}[\phi_2^2] = \mathbb{E}[(\alpha \varepsilon_1 + \beta \varepsilon_2)^2] = \mathbb{E}[\alpha^2 \varepsilon_1^2 + 2\alpha\beta \varepsilon_1 \varepsilon_2 + \beta^2 \varepsilon_2^2] = 1$

$\alpha^2 \mathbb{E}[\underbrace{\varepsilon_1^2}_{=1}] + 2\alpha\beta \mathbb{E}[\underbrace{\varepsilon_1 \varepsilon_2}_{=0}] + \beta^2 \mathbb{E}[\underbrace{\varepsilon_2^2}_{=1}] = 1 \quad \therefore \quad \alpha^2 + \beta^2 = 1$

$\beta = \sqrt{1-\rho^2}$

Check:

$$E[\phi_1^2] = 1,$$

$$E\left[\phi_2^2\right] = E\left[\rho^2\epsilon_1^2 + 2\rho\sqrt{1-\rho^2}\epsilon_1\epsilon_2 + (1-\rho^2)\epsilon_2^2\right]$$

$$= \rho^2 + 0 + (1-\rho^2) = 1,$$

and

$$E\left[\phi_1\phi_2\right] = E\left[\rho\epsilon_1^2 + \sqrt{1-\rho^2}\epsilon_1\epsilon_2\right] = \rho.$$

And Normality?

Weighted sums of Normally distributed numbers are themselves Normally distributed!

If $X_i \sim N(\mu_i, \sigma_i^2)$ for $i = 1, \ldots, n$ then

$$\sum_{i=1}^{n} w_i X_i \sim N\left(\sum_{i=1}^{n} w_i \mu_i, \sum_{i=1}^{n} w_i^2 \sigma_i^2\right).$$

Higher Dimensional Itô i.e $V = V(S_1, S_2, t)$

Using the set-up on p.36 for 2 stocks and their correlation..

$t \to t + dt$, $S_i \to S_i + dS_i$ $\quad i = 1, 2$

2D T.S.E

$$V(t+dt, S_1 + dS_1, S_2 + dS_2) = V(t, S_1, S_2) + \frac{\partial V}{\partial t} dt + \frac{\partial V}{\partial S_1} dS_1$$

$$+ \frac{\partial V}{\partial S_2} dS_2 + \frac{1}{2} \frac{\partial^2 V}{\partial S_1^2} dS_1^2 + \frac{1}{2} \frac{\partial^2 V}{\partial S_2^2} dS_2^2$$

$$+ \frac{\partial^2 V}{\partial S_1 \partial S_2} dS_1 dS_2$$

Use $dS_i^2 = \sigma_i^2 S_i^2 dt$ $\qquad dS_1 dS_2 = \rho \sigma_1 \sigma_2 S_1 S_2 dt$

$$dV = \left(\frac{\partial V}{\partial t} + \mu_1 S_1 \frac{\partial V}{\partial S_1} + \mu_2 S_2 \frac{\partial V}{\partial S_2} + \frac{1}{2} \frac{\partial^2 V}{\partial S_1^2} + \frac{1}{2} \frac{\partial^2 V}{\partial S_2^2} + \rho \sigma_1 \sigma_2 S_1 S_2 \frac{\partial^2 V}{\partial S_1 \partial S_2} \right) dt$$

$$+ \sigma_1 S_1 \frac{\partial V}{\partial S_1} dX_t^{(1)} + \sigma_2 S_2 \frac{\partial V}{\partial S_2} dX_t^{(2)}$$

Itô V

Itô VI G_t satisfies $dG_t = A(G_t, t)dt + B(G_t, t)dX$

Let $F = F(t, G_t)$

Then SDE for F is

$$dF = \left(\frac{\partial F}{\partial t} + A \frac{\partial F}{\partial G} + \frac{1}{2} B^2 \frac{\partial^2 F}{\partial G^2} \right) dt + B \frac{\partial F}{\partial G} dX$$

[General Itô]

Close-form soln

$$S_{i+1} = S_i \exp\left\{\left(\mu - \frac{1}{2}\sigma^2\right)\delta t + \sigma\phi\sqrt{\delta t}\right\}$$

Summary

$$S_{i+1} = S_i\left(1 + \mu\,\delta t + \sigma\phi\sqrt{\delta t}\right) \qquad \phi \sim N(0,1)$$

Please take away the following important ideas

- With the right tool (Itô's lemma) you can examine functions of stochastic variables

- Partial differential equations can be used for finding probability density functions for arbitrary random walks

- Simulating random walks can be very easy indeed