Forme Algébrique
Corrigé

DARVOUX Théo

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Exercices.

Exercice 6.1 [♦◊◊]

Résoudre $4z^2 + 8|z|^2 - 3 = 0$.
Soit $z \in \mathbb{C}$ et $(a,b) \in \mathbb{R}^2$ tels que $z = a + ib$. On a :

$$\begin{aligned}
& 4z^2 + 8|z|^2 - 3 = 0 \\
\iff & 4(a+ib)^2 + 8(a^2+b^2) - 3 = 0 \\
\iff & 4a^2 + 8aib - 4b^2 + 8a^2 + 8b^2 - 3 = 0 \\
\iff & (12a^2 + 4b^2 - 3) + i(8ab) = 0 \\
\iff & \begin{cases} 12a^2 + 4b^2 - 3 = 0 \\ 8ab = 0 \end{cases} \\
\iff & \begin{cases} 12a^2 + 4b^2 - 3 = 0 \\ a = 0 \end{cases} \quad \text{ou} \quad \begin{cases} 12a^2 + 4b^2 - 3 = 0 \\ b = 0 \end{cases} \\
\iff & 4b^2 - 3 = 0 \text{ ou } 12a^2 - 3 = 0 \\
\iff & b^2 = \frac{3}{4} \text{ ou } a^2 = \frac{1}{4} \\
\iff & b = \pm\frac{\sqrt{3}}{2} \text{ ou } a = \pm\frac{1}{2}
\end{aligned}$$

Les solutions sont donc :

$$\left\{-\frac{1}{2}, \frac{1}{2}, -i\frac{\sqrt{3}}{2}, i\frac{\sqrt{3}}{2}\right\}$$

□

Exercice 6.3 [♦♦◊]

Soit $z \in \mathbb{C} \setminus \{1\}$, montrer que :

$$\frac{1+z}{1-z} \in i\mathbb{R} \iff |z| = 1.$$

Supposons $\frac{1+z}{1-z} \in i\mathbb{R}$. Montrons $|z| = 1$.
Soit $b \in \mathbb{R}$, on a :

$$\frac{1+z}{1-z} = ib \iff 1 + z = ib - zib \iff z(1+ib) = ib - 1 \iff z = \frac{ib-1}{1+ib}$$

Ainsi, $|z| = |\frac{ib-1}{1+ib}| = \frac{\sqrt{1+b^2}}{\sqrt{1+b^2}} = 1$.
Supposons $|z| = 1$, montrons $\frac{1+z}{1-z} \in i\mathbb{R}$.
Soient $(a,b) \in \mathbb{R}$ tels que $z = a + ib$. Par supposition, $a^2 + b^2 = 1$. On a :

$$\begin{aligned}
\frac{1+z}{1-z} &= \frac{1+a+ib}{1-a-ib} = \frac{(1+a+ib)(1-a+ib)}{(1-a-ib)(1-a+ib)} = \frac{1+2ib-a^2-b^2}{1-2a+a^2+b^2} \\
&= \frac{2ib}{2-2a} = \frac{ib}{1-a} = i\frac{b}{1-a}
\end{aligned}$$

□