Forme Algébrique
Corrigé

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Exercices.

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Exercice 6.1 [⧫◊◊]

Résoudre $4z^2 + 8|z|^2 - 3 = 0$.
Soit $z \in \mathbb{C}$ et $(a, b) \in \mathbb{R}^2$ tels que $z = a + ib$. On a :

$$\begin{aligned}
& 4z^2 + 8|z|^2 - 3 = 0 \\
\iff & 4(a+ib)^2 + 8(a^2+b^2) - 3 = 0 \\
\iff & 4a^2 + 8aib - 4b^2 + 8a^2 + 8b^2 - 3 = 0 \\
\iff & (12a^2 + 4b^2 - 3) + i(8ab) = 0 \\
\iff & \begin{cases} 12a^2 + 4b^2 - 3 = 0 \\ 8ab = 0 \end{cases} \\
\iff & \begin{cases} 12a^2 + 4b^2 - 3 = 0 \\ a = 0 \end{cases} \quad \text{ou} \quad \begin{cases} 12a^2 + 4b^2 - 3 = 0 \\ b = 0 \end{cases} \\
\iff & 4b^2 - 3 = 0 \text{ ou } 12a^2 - 3 = 0 \\
\iff & b^2 = \frac{3}{4} \text{ ou } a^2 = \frac{1}{4} \\
\iff & b = \pm\frac{\sqrt{3}}{2} \text{ ou } a = \pm\frac{1}{2}
\end{aligned}$$

Les solutions sont donc :

$$\left\{-\frac{1}{2}, \frac{1}{2}, -i\frac{\sqrt{3}}{2}, i\frac{\sqrt{3}}{2}\right\}$$

□

Exercice 6.2 [⧫◊◊]

Soient a et b deux nombres complexes non nuls. Montrer que :

$$\left|\frac{a}{|a|^2} - \frac{b}{|b|^2}\right| = \frac{|a-b|}{|a||b|}.$$

On a :

$$\begin{aligned}
\left|\frac{a}{|a|^2} - \frac{b}{|b|^2}\right| &= \left|\frac{a|b|^2 - b|a|^2}{|a|^2|b|^2}\right| = \frac{|ab\bar{b} - ba\bar{a}|}{||ab|^2|} \\
&= \frac{\left|ab(\bar{b} - \bar{a})\right|}{||ab|^2|} = \frac{|ab||\bar{a} - \bar{b}|}{|ab|^2} \\
&= \frac{|a-b|}{|ab|} = \frac{|a-b|}{|a||b|}
\end{aligned}$$

□

Exercice 6.3 [♦♦◊]

Soit $z \in \mathbb{C} \setminus \{1\}$, montrer que :

$$\frac{1+z}{1-z} \in i\mathbb{R} \iff |z| = 1.$$

Supposons $\frac{1+z}{1-z} \in i\mathbb{R}$. Montrons $|z| = 1$.
Soit $b \in \mathbb{R}$, on a :

$$\frac{1+z}{1-z} = ib \iff 1 + z = ib - zib \iff z(1+ib) = ib - 1 \iff z = \frac{ib-1}{1+ib}$$

Ainsi, $|z| = |\frac{ib-1}{1+ib}| = \frac{\sqrt{1+b^2}}{\sqrt{1+b^2}} = 1$.
Supposons $|z| = 1$, montrons $\frac{1+z}{1-z} \in i\mathbb{R}$.
Soient $(a, b) \in \mathbb{R}$ tels que $z = a + ib$. Par supposition, $a^2 + b^2 = 1$. On a :

$$\begin{aligned}\frac{1+z}{1-z} &= \frac{1+a+ib}{1-a-ib} = \frac{(1+a+ib)(1-a+ib)}{(1-a-ib)(1-a+ib)} = \frac{1+2ib-a^2-b^2}{1-2a+a^2+b^2} \\ &= \frac{2ib}{2-2a} = \frac{ib}{1-a} = i\frac{b}{1-a}\end{aligned}$$

□

Exercice 6.4 [♦◊◊]

Soient $z_1, z_2, \dots, z_n$ des nombres complexes non nuls de mêmes module. Démontrer que

$$\frac{(z_1+z_2)(z_2+z_3)\dots(z_{n-1}+z_n)(z_n+z_1)}{z_1z_2\dots z_n} \in \mathbb{R}. \tag{1}$$

Commençons par énoncer que :

$$\forall (i,j) \in [\![1,n]\!]^2, \qquad \frac{\overline{z_i}}{\overline{z_j}} = \frac{z_j}{z_i}.$$

En effet,

$$\frac{z_i}{z_j} \cdot \frac{\overline{z_i}}{\overline{z_j}} = \left|\frac{z_i}{z_j}\right|^2 = 1 \iff \frac{\overline{z_i}}{\overline{z_j}} = \frac{z_j}{z_i}.$$

Le conjugué de (1) est :

$$\frac{(\overline{z_1}+\overline{z_2})(\overline{z_2}+\overline{z_3})\dots(\overline{z_{n-1}}+\overline{z_n})(\overline{z_n}+\overline{z_1})}{\overline{z_1z_2}\dots\overline{z_n}} = (1+\frac{\overline{z_2}}{\overline{z_1}})(1+\frac{\overline{z_3}}{\overline{z_2}})\dots(1+\frac{\overline{z_n}}{\overline{z_{n-1}}})(1+\frac{\overline{z_1}}{\overline{z_n}})$$

Ainsi :

$$\begin{aligned}\frac{(\overline{z_1}+\overline{z_2})(\overline{z_2}+\overline{z_3})\dots(\overline{z_{n-1}}+\overline{z_n})(\overline{z_n}+\overline{z_1})}{\overline{z_1z_2}\dots\overline{z_n}} &= (1+\frac{z_1}{z_2})\dots(1+\frac{z_n}{z_1}) \\ =&\frac{z_1+z_2}{z_2}\dots\frac{z_n+z_1}{z_1} = \frac{(z_1+z_2)(z_2+z_3)\dots(z_{n-1}+z_n)(z_n+z_1)}{z_1z_2\dots z_n}\end{aligned}$$

Puisque (1) est égal à son conjugué, (1) $\in \mathbb{R}$. □

Exercice 6.5 [♦♦◊]

Soient a, b deux nombres complexes tels que $\overline{a}b \neq 1$ et $c = \frac{a-b}{1-\overline{a}b}$. Montrer que

$$(|c| = 1) \iff (|a| = 1 \text{ ou } |b| = 1).$$

Supposons $|c| = 1$. Montrons que $|a| = 1$ ou $|b| = 1$.
On a :

$$\begin{aligned} & |c| = 1 \\ \iff & |c|^2 = \frac{(a-b)(\overline{a}-\overline{b})}{(1-\overline{a}b)(1-a\overline{b})} = \frac{|a|^2 - a\overline{b} - b\overline{a} + |b|^2}{1 - a\overline{b} - \overline{a}b + |a|^2|b|^2} = 1 \\ \iff & |a|^2 - a\overline{b} - \overline{a}b + |b|^2 = 1 - a\overline{b} - \overline{a}b + |a|^2|b|^2 \\ \iff & |a|^2 + |b|^2 - |a|^2|b|^2 = 1 \\ \iff & |a|^2(1 - |b|^2) = 1 - |b|^2 \end{aligned}$$

Si on suppose $|b| \neq 1$, on obtient : $|c| = 1 \iff |a|^2 = \frac{1-|b|^2}{1-|b|^2} = 1$ donc $|a| = 1$.
Si on suppose $|a| \neq 1$, on obtient : $|c| = 1 \iff |b|^2 = \frac{1-|a|^2}{1-|a|^2} = 1$ donc $|b| = 1$.
Supposons $|a| = 1$. On a :

$$|c| = \left|\frac{a-b}{1-\overline{a}b}\right| = \left|\frac{a-b}{\overline{a}a - \overline{a}b}\right| = \left|\frac{1}{\overline{a}}\right| \left|\frac{a-b}{a-b}\right| = |a| = 1$$

Supposons $|b| = 1$. On a :

$$|c| = \left|\frac{a-b}{1-\overline{a}b}\right| = \left|\frac{a-b}{\overline{b}b - \overline{a}b}\right| = \left|\frac{1}{b}\right| \left|\frac{a-b}{\overline{b}-\overline{a}}\right| = |b|\frac{|a-b|}{|a-b|} = |b| = 1$$

□

Exercice 6.6 [♦♦♦]

Pour $n \in \mathbb{N}^*$, calculer $R^2 + S^2$ où

$$R = \sum_{0 \leq 2k \leq n} (-1)^k \binom{n}{2k} \qquad \text{et} \qquad S = \sum_{0 \leq 2k+1 \leq n} (-1)^k \binom{n}{2k+1}.$$

On a :

$$(1+i)^n = \sum_{k=0}^{n} \binom{n}{k} i^k = \sum_{0 \leq 2k \leq n} \binom{n}{2k} i^{2k} + \sum_{0 \leq 2k+1 \leq n} \binom{n}{2k+1} i^{2k} \cdot i = R + iS$$

Ainsi :

$$\begin{cases} R = \operatorname{Re}\left((1+i)^n\right) = 2^{\frac{n}{2}} \cos(\frac{n\pi}{4}) \\ S = \operatorname{Im}\left((1+i)^n\right) = 2^{\frac{n}{2}} \sin(\frac{n\pi}{4}) \end{cases}$$

Finalement, $R^2 + S^2 = 2^n(\cos^2(\frac{n\pi}{4}) + \sin^2(\frac{n\pi}{4})) = 2^n$.

□