

Forme Algébrique
Corrigé

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Exercices.

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Exercice 6.1 [◆◆◆]

Résoudre $4z^2 + 8|z|^2 - 3 = 0$.

Soit $z \in \mathbb{C}$ et $(a, b) \in \mathbb{R}^2$ tels que $z = a + ib$. On a :

$$\begin{aligned}
 &4z^2 + 8|z|^2 - 3 = 0 \\
 \iff &4(a + ib)^2 + 8(a^2 + b^2) - 3 = 0 \\
 \iff &4a^2 + 8aib - 4b^2 + 8a^2 + 8b^2 - 3 = 0 \\
 \iff &(12a^2 + 4b^2 - 3) + i(8ab) = 0 \\
 \iff &\begin{cases} 12a^2 + 4b^2 - 3 = 0 \\ 8ab = 0 \end{cases} \\
 \iff &\begin{cases} 12a^2 + 4b^2 - 3 = 0 \\ a = 0 \end{cases} \quad \text{ou} \quad \begin{cases} 12a^2 + 4b^2 - 3 = 0 \\ b = 0 \end{cases} \\
 \iff &4b^2 - 3 = 0 \text{ ou } 12a^2 - 3 = 0 \\
 \iff &b^2 = \frac{3}{4} \text{ ou } a^2 = \frac{1}{4} \\
 \iff &b = \pm \frac{\sqrt{3}}{2} \text{ ou } a = \pm \frac{1}{2}
 \end{aligned}$$

Les solutions sont donc :

$$\left\{ -\frac{1}{2}, \frac{1}{2}, -i\frac{\sqrt{3}}{2}, i\frac{\sqrt{3}}{2} \right\}$$

□

Exercice 6.2 [◆◆◆]

Soient a et b deux nombres complexes non nuls. Montrer que :

$$\left| \frac{a}{|a|^2} - \frac{b}{|b|^2} \right| = \frac{|a - b|}{|a||b|}.$$

On a :

$$\begin{aligned}
 \left| \frac{a}{|a|^2} - \frac{b}{|b|^2} \right| &= \left| \frac{a|b|^2 - b|a|^2}{|a|^2|b|^2} \right| = \frac{|ab\bar{b} - ba\bar{a}|}{||ab|^2|} \\
 &= \frac{|ab(\bar{b} - \bar{a})|}{||ab|^2|} = \frac{|ab||\bar{a} - \bar{b}|}{|ab|^2} \\
 &= \frac{|a - b|}{|ab|} = \frac{|a - b|}{|a||b|}
 \end{aligned}$$

□

Exercice 6.3 [◆◆◆]

Soit $z \in \mathbb{C} \setminus \{1\}$, montrer que :

$$\frac{1+z}{1-z} \in i\mathbb{R} \iff |z| = 1.$$

Supposons $\frac{1+z}{1-z} \in i\mathbb{R}$. Montrons $|z| = 1$.

Soit $b \in \mathbb{R}$, on a :

$$\frac{1+z}{1-z} = ib \iff 1+z = ib - zib \iff z(1+ib) = ib - 1 \iff z = \frac{ib-1}{1+ib}$$

Ainsi, $|z| = \left| \frac{ib-1}{1+ib} \right| = \frac{\sqrt{1+b^2}}{\sqrt{1+b^2}} = 1$.

Supposons $|z| = 1$, montrons $\frac{1+z}{1-z} \in i\mathbb{R}$.

Soient $(a, b) \in \mathbb{R}$ tels que $z = a + ib$. Par supposition, $a^2 + b^2 = 1$. On a :

$$\begin{aligned} \frac{1+z}{1-z} &= \frac{1+a+ib}{1-a-ib} = \frac{(1+a+ib)(1-a+ib)}{(1-a-ib)(1-a+ib)} = \frac{1+2ib-a^2-b^2}{1-2a+a^2+b^2} \\ &= \frac{2ib}{2-2a} = \frac{ib}{1-a} = i \frac{b}{1-a} \end{aligned}$$

□

Exercice 6.4 [◆◆◆]

Soient z_1, z_2, \dots, z_n des nombres complexes non nuls de mêmes module. Démontrer que

$$\frac{(z_1 + z_2)(z_2 + z_3) \dots (z_{n-1} + z_n)(z_n + z_1)}{z_1 z_2 \dots z_n} \in \mathbb{R}. \quad (1)$$

Commençons par énoncer que :

$$\forall (i, j) \in \llbracket 1, n \rrbracket^2, \quad \frac{\overline{z_i}}{z_j} = \frac{z_j}{z_i}.$$

En effet,

$$\frac{z_i}{z_j} \cdot \frac{\overline{z_i}}{\overline{z_j}} = \left| \frac{z_i}{z_j} \right|^2 = 1 \iff \frac{\overline{z_i}}{\overline{z_j}} = \frac{z_j}{z_i}.$$

Le conjugué de (1) est :

$$\frac{(\overline{z_1} + \overline{z_2})(\overline{z_2} + \overline{z_3}) \dots (\overline{z_{n-1}} + \overline{z_n})(\overline{z_n} + \overline{z_1})}{\overline{z_1} \overline{z_2} \dots \overline{z_n}} = (1 + \frac{\overline{z_2}}{z_1})(1 + \frac{\overline{z_3}}{z_2}) \dots (1 + \frac{\overline{z_n}}{z_{n-1}})(1 + \frac{\overline{z_1}}{z_n})$$

Ainsi :

$$\begin{aligned} \frac{(\overline{z_1} + \overline{z_2})(\overline{z_2} + \overline{z_3}) \dots (\overline{z_{n-1}} + \overline{z_n})(\overline{z_n} + \overline{z_1})}{\overline{z_1} \overline{z_2} \dots \overline{z_n}} &= (1 + \frac{z_1}{z_2}) \dots (1 + \frac{z_n}{z_1}) \\ &= \frac{z_1 + z_2}{z_2} \dots \frac{z_n + z_1}{z_1} = \frac{(z_1 + z_2)(z_2 + z_3) \dots (z_{n-1} + z_n)(z_n + z_1)}{z_1 z_2 \dots z_n} \end{aligned}$$

Puisque (1) est égal à son conjugué, (1) $\in \mathbb{R}$.

□

Exercice 6.5 [◆◆◆]

Soient a, b deux nombres complexes tels que $\bar{a}b \neq 1$ et $c = \frac{a-b}{1-\bar{a}b}$. Montrer que

$$(|c| = 1) \iff (|a| = 1 \text{ ou } |b| = 1).$$

Supposons $|c| = 1$. Montrons que $|a| = 1$ ou $|b| = 1$.

On a :

$$\begin{aligned} |c| &= 1 \\ \iff |c|^2 &= \frac{(a-b)(\bar{a}-\bar{b})}{(1-\bar{a}b)(1-\bar{a}b)} = \frac{|a|^2 - a\bar{b} - \bar{a}b + |b|^2}{1 - a\bar{b} - \bar{a}b + |a|^2|b|^2} = 1 \\ \iff |a|^2 - a\bar{b} - \bar{a}b + |b|^2 &= 1 - a\bar{b} - \bar{a}b + |a|^2|b|^2 \\ \iff |a|^2 + |b|^2 - |a|^2|b|^2 &= 1 \\ \iff |a|^2(1 - |b|^2) &= 1 - |b|^2 \end{aligned}$$

Si on suppose $|b| \neq 1$, on obtient : $|c| = 1 \iff |a|^2 = \frac{1-|b|^2}{1-|b|^2} = 1$ donc $|a| = 1$.

Si on suppose $|a| \neq 1$, on obtient : $|c| = 1 \iff |b|^2 = \frac{1-|a|^2}{1-|a|^2} = 1$ donc $|b| = 1$.

Supposons $|a| = 1$. On a :

$$|c| = \left| \frac{a-b}{1-\bar{a}b} \right| = \left| \frac{a-b}{\bar{a}a - \bar{a}b} \right| = \left| \frac{1}{\bar{a}} \right| \left| \frac{a-b}{a-b} \right| = |a| = 1$$

Supposons $|b| = 1$. On a :

$$|c| = \left| \frac{a-b}{1-\bar{a}b} \right| = \left| \frac{a-b}{\bar{b}b - \bar{a}b} \right| = \left| \frac{1}{\bar{b}} \right| \left| \frac{a-b}{\bar{b}-\bar{a}} \right| = |b| \left| \frac{a-b}{a-b} \right| = |b| = 1$$

□

Exercice 6.6 [◆◆◆]

Pour $n \in \mathbb{N}^*$, calculer $R^2 + S^2$ où

$$R = \sum_{0 \leq 2k \leq n} (-1)^k \binom{n}{2k} \quad \text{et} \quad S = \sum_{0 \leq 2k+1 \leq n} (-1)^k \binom{n}{2k+1}.$$

On a :

$$(1+i)^n = \sum_{k=0}^n \binom{n}{k} i^k = \sum_{0 \leq 2k \leq n} \binom{n}{2k} i^{2k} + \sum_{0 \leq 2k+1 \leq n} \binom{n}{2k+1} i^{2k+1} = R + iS$$

Ainsi :

$$\begin{cases} R = \operatorname{Re}((1+i)^n) = 2^{\frac{n}{2}} \cos\left(\frac{n\pi}{4}\right) \\ S = \operatorname{Im}((1+i)^n) = 2^{\frac{n}{2}} \sin\left(\frac{n\pi}{4}\right) \end{cases}$$

Finalement, $R^2 + S^2 = 2^n (\cos^2(\frac{n\pi}{4}) + \sin^2(\frac{n\pi}{4})) = 2^n$.

□

Exercice 6.7 [◆◆◆]

Soit $ABCD$ un parallélogramme.

Montrer que $AC^2 + BD^2 = AB^2 + BC^2 + CD^2 + DA^2$

Soient $(z, z') \in \mathbb{R}$. Les points A, B, C, D d'affixes $0, z, z + z', z'$ forment un parallélogramme.

Alors :

$$\begin{cases} AC^2 = |z + z'|^2 \\ BD^2 = |z - z'|^2 \\ AB^2 = CD^2 = |z|^2 \\ BC^2 = DA^2 = |z'|^2 \end{cases}$$

On a :

$$\begin{aligned} AC^2 + BD^2 &= |z + z'|^2 + |z - z'|^2 = (z + z')(\bar{z} + \bar{z}') + (z - z')(\bar{z} - \bar{z}') \\ &= z\bar{z} + z\bar{z}' + z'\bar{z} + z'\bar{z}' + z\bar{z} - z\bar{z}' - z'\bar{z} + z'\bar{z}' \\ &= |z|^2 + |z'|^2 + |z|^2 + |z'|^2 \\ &= AB^2 + BC^2 + CD^2 + DA^2 \end{aligned}$$

□