M A T H S
I N T E R N A L   A S S E S S M E N T

Modelling a building against an earthquake through optimising building mensurations for extremities of wealth.

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Introduction:

Natural disasters pose a significant challenge to societies worldwide, with earthquakes being one of the deadliest and most destructive natural forces. Over 100,000 earthquakes, which are felt by people, happen per year and over 60,000 lives are taken. These unforeseeable events can cause catastrophic damage to infrastructure, increasing the chance of people being harmed or killed. These factors make it essential that engineers and architects can design structures that can withstand their extensive power.

As an aspiring architect, I wanted to create a mathematics internal assessment which would link with either engineering or architecture and would allow me to address a global issue prevalent in our world. I had the idea to implement earthquakes into my internal assessment after reading about the recent Syrian and Turkish earthquakes in the news, which had a 7.8 magnitude on the Richter scale, and killed a tragic 60,000 people. In this maths internal assessment I will model two buildings against an earthquake through optimising building mensurations for extremities of wealth. This IA is based on a theoretical situation where a rich country (based on Turkey) and a poor country (based on Syria) would ask for an engineer/architect to optimise their chosen building (where both buildings/cuboids have the same dimensions) based on their extremities of wealth.

Aim & Methodology:

In this IA I aim to optimise the dimensions of a building by fixing the volume and minimising the surface area for the rich country, and fixing the surface area and maximising the volume for the poor country. This was defined due to the fact that the surface area can be directly correlated with the materials needed to make the building, thus an increased surface area equals an increase in price. Therefore, the poor country would want to fix the cost (Surface area), though still want the maximum volume possible.

On the other hand, since the rich country has no limitations towards cost, the volume is fixed (so no volume is lost in the optimization of cuboid to pyramid and cone) and the surface area is minimised. The Surface area is minimised due to the fact that even though the rich country has no limitations for cost, it does not hurt to find the cheapest cost while keeping the volume constant.

The expectation for these optimizations is that for the rich country, the dimensions would increase from the cuboid to the pyramid and cone, since a cuboid, in its geometric nature, has more volume than a pyramid/cone. So for the pyramid/cone to match the cuboid's volume, it would need to increase its base area, its height or both. This is due to the fact that a (pyramid/cone)’s volume is more concentrated towards the base, whereas a cuboid’s volume is evenly distributed throughout the entire structure. As for the poor country, the opposite may be expected, since the surface area is fixed, the volume can be expected to decrease due to the same reason explained.

Symbol/Variable	Quantity represented
l	Length (m)
w	Width (m)
h	Height (m)
r	Radius (m)
π	3.1415926535…
x	x = Length = Width =
V	Volume (m^3)
A	Surface Area (m^2)
a	Base area (m^2)

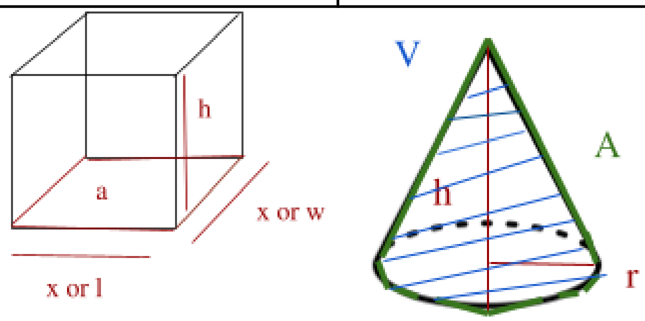


Figure 1: table and shapes defining symbols used

Rationale:

The most used geometric shapes for buildings are cuboids. However in practice, there are two much safer options against earthquakes. The pyramidal shaped building is much safer and more structurally sound against an earthquake than the cuboidal building due to three main reasons: The distribution of mass in a cuboid is evenly distributed throughout the whole geometric shape, meaning that under pressure, cuboids have the tendency to deform into a parallelogram,changing the angles of the corners, thus making it less stable. On the other hand,the pyramid has a broad base which converges to a single point, meaning that most of the mass is distributed at the bottom, thus making the structure better at resisting toppling during an earthquake. In addition, the centre of gravity of a pyramid is lower than that of a cuboid, thus it being even more stable.

However, after further research, I came to the conclusion that a conical shaped building is the most structurally sound geometric shape against an earthquake. This is due to the fact that the cone is even better than the pyramid in distributing mass. A cone is uniform along the vertical axis making it even more resistant to toppling. Additionally, the centre of gravity of a cone is lower than that of both a pyramid and a cuboid. Finally, the lack of edges and corners in a cone (which are the weakness of a structure) makes it that the stress (energy) in the building is distributed evenly throughout, providing an even greater stability.

Furthermore, a cone's circular base is stronger than a triangular or rectangular base against lateral forces, due to its lack of vertices and corners.

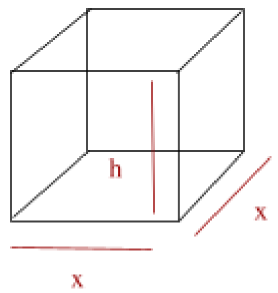
In summary a conical building is the optimal geometric structure against an earthquake, due to its better distribution of mass, lowered centre of gravity and increased structural integrity. However a pyramidal structure is still better than a cuboidal.

Fixing the mensurations:

The dimensions of the fixed cuboid were set as 10 metres for the height ($h = 10$), 20 metres for the length ($l = 20$) and 20 metres for the width ($w = 20$). These dimensions were inspired by the Washington DC, Brazilian embassy, which by an approximation, the conclusion of these measurements was made.

-Cuboidal building

Firstly, a cube was drawn for reference:



Then, the formula for volume and surface area of a cube were used:

$$V = lwh \text{ OR } V = hx^2$$
$$10 \cdot 20^2 = 4000m^3$$

For Surface Area:

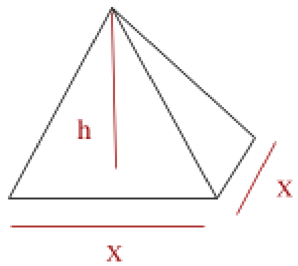
$$A = 2(lw + lh + hw) \text{ OR } A = 2(x^2 + 2hx)$$
$$A = 2(20^2 + 2(10 \cdot 20)) = 1600m^2$$

Rich country optimization (Fixing volume, minimising surface area):

The reason behind fixing the volume and minimising surface area for the rich country is that, since the rich country has no limitations for cost, the rich country would want to keep their building's volume while only changing the shape. However, since we can, it does not hurt to minimise the surface area to make it as little as possible while maintaining the same volume through the optimization.

-Optimising Pyramidical building

Firstly, a pyramid was drawn for reference:



Then, the fixed volume of the cube, was set equal to the pyramid volume formula, which is:

$$V = \frac{lwh}{3} \text{ OR } V = \frac{hx^2}{3}$$

$$\frac{hx^2}{3} = 4000$$

Then, the equation was solved for h:

$$hx^2 = 4000 \cdot 3$$

$$h = \frac{12000}{x^2}$$

This was done so the height could be optimised where the volume is kept fixed, and the surface area minimised (which will be the next step).

Now, the value of h will be substituted into the square base pyramid surface area formula, which is:

$$A = 2l\sqrt{(\frac{l}{2})^2 + h^2} + l\hat{}2 \quad \text{OR} \quad A = 2x\sqrt{(\frac{x}{2})^2 + h^2} + x^2$$

This formula represents the surface area of a square pyramid. The $2l$ equals the total area of all the 4 triangle faces of the pyramid, which is $2l$ or wl. The $\sqrt{(\frac{l}{2})^2 + h^2}$ equals the slant height of the pyramid, It is calculated through the Pythagorean theorem as the square root of the sum of the square of half the base length (l/2) and the square of the height (h). The final part of the formula, $+ l\hat{}2$ equals the area of the square base of the pyramid.

$$A = 2x\sqrt{(\frac{x}{2})^2 + (\frac{12000}{x^2})^2} + x^2$$

[Substitution of h]

Now, this equation will be plotted in terms of x, where the minimum point will give the optimised x value with the minimal surface area.

$$A(x) = 2x\sqrt{(\frac{x}{2})^2 + (\frac{12000}{x^2})^2} + x^2$$

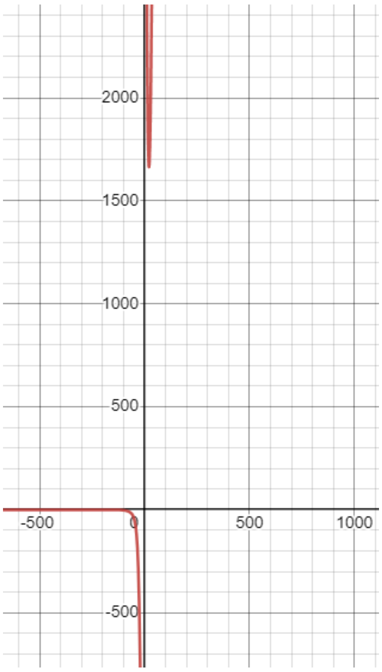


Figure 2: full plotted graph (desmos)

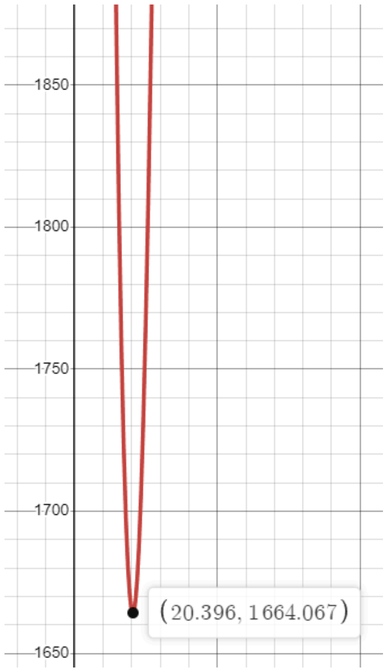


Figure 3: Zoomed in plotted graph (desmos)

After the graph is plotted, it can be seen that there are two curves, an asymptotic curve on the negative x and y axis, and a "quadratic" curve on the positive x and y-axis, since there is no such thing as negative surface area or negative length/width the asymptotic curve is ignored. The minimum point of the "quadratic" curve shows the minimal surface area (y-axis) which is 1664.06, and the minimum length/width (x-axis), 20.39. Now the optimal height can be calculated by substituting the x-axis value, 20.4, into the height equation.

$$h = \frac{12000}{(20.39)^2}$$

$$h = 28.83m$$

Finally, we can correct our answer by inputting the x and h into the volume equation where we would get the fixed volume, of 4000m^3.

$$\frac{20.4^2 \cdot 28.835}{3} = 3999.99 \approx 4000$$

Concluding the final mensurations of the optimised pyramid:

$$x = 20.39m$$

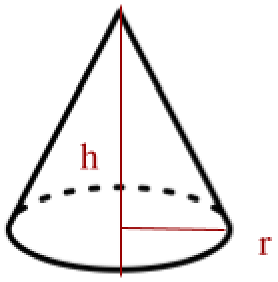
$$h = 28.83m$$

$$V = 4000m^3$$

$$A = 1664.06m^2$$

-Optimising Conical building

Firstly, a cone was drawn for reference:



(shape from https://www.pngwing.com/en/free-png-aherb)

Then, the fixed volume of the cube, was set equal to the cone volume formula, which is:

$$V = \frac{h\pi r^2}{3}$$

$$\frac{h\pi r^2}{3} = 4000$$

Then, the equation was solved for h:

$$h\pi r^2 = 4000 \cdot 3$$

$$h = \frac{12000}{\pi r^2}$$

The reason for solving for h and for the substitution can be found in the appendix.

Now, the value of h will be substituted into the cone surface area formula, which is:

$$A = \pi r (r + \sqrt{h^2 + r^2})$$

This formula represents the surface area of a cone. The $\pi r\,(r) = \pi r^2$ is the area of the circular base, which when added to $\sqrt{h^2 + r^2}$, the slant height of the cone, the total surface area is calculated. (To see how slant height was calculated go to appendix)

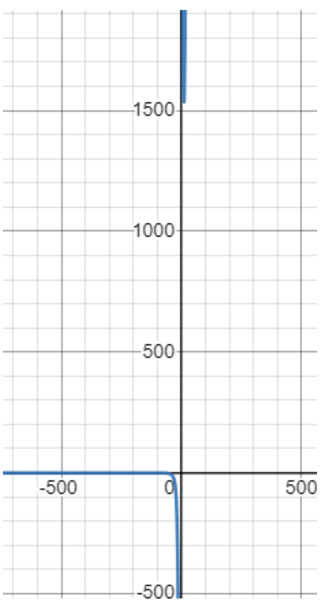
$$A = \pi r (r + \sqrt{(\frac{12000}{\pi r^2})^2 + r^2})$$

[Substitution of h]

Now, this equation will be plotted in terms of r, where the minimum point will give the optimised r value with the minimal surface area.

$$A(r) = \pi r (r + \sqrt{(\frac{12000}{\pi r^2})^2 + r^2})$$

Reasoning behind ignoring negative axis can be found in the appendix.



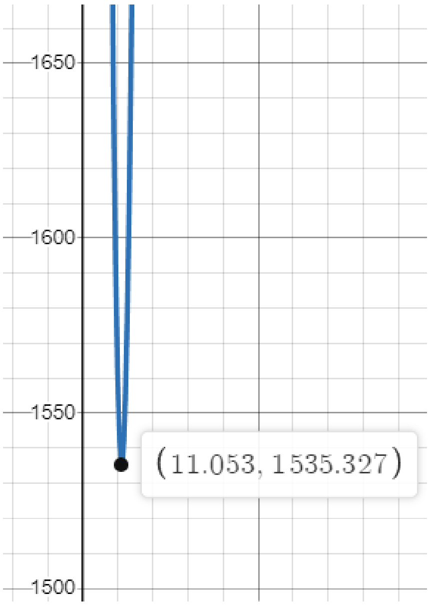


Figure 4: full plotted graph (desmos) *Figure 5: Zoomed in plotted graph (desmos)*

The minimum point of the "quadratic" curve shows the minimal surface area (y-axis) which is 1535.33, and the minimum length/width (x-axis), 11.05. Now the optimal height can be calculated by substituting the x-axis value, 11.05, into the height equation:

$$h = \frac{12000}{\pi(11.05)^2}$$

$$h = 31.28m$$

Finally, we can correct our answer by inputting the x and h into the volume equation where we would get the fixed volume, of 4000m^3.

$$\frac{\pi 11.05^2 \cdot 31.283}{3} = 4000.01 \approx 4000$$

Concluding the final mensurations of the optimised cone:

$$x = 11.05m$$
$$h = 31.28m$$
$$V = 4000m^3$$
$$A = 1535.33m^2$$

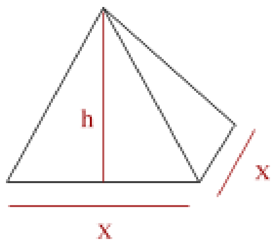
Summary:

With these both optimizations we can see that the cost (surface area) slightly increased when optimising the pyramid, (1600m^2→1664.06m^2) and the height increased, (10m→28.8m) however the length/width stayed around the same (20m→20.83m). On the other hand, the cone's cost slightly decreased (1600m^2→1535.33m^2) and the height increased (10m→31.28m), while the length/width decreased almost by half (20m→11.05m), this shows that the conical building is a much better option than the pyramid, since it is cheaper to produce while fixing the volume of the original cuboidal building, and is safer than the pyramidal.

Poor country optimization (Fixing surface area, maximising volume):

The reason behind fixing the surface area and maximising volume for the poor country is that, since the rich country has a limited budget, they would want to fix their cost (surface area), while still getting the maximum amount of volume possible.

-Optimising Pyramidical building



Firstly, the fixed surface area of the cube, will be set equal to the pyramid surface area formula:

$$2x\sqrt{(\frac{x}{2})^2 + h^2} + x^2 = 1600$$

Then, the equation was solved for h:

$$2x\sqrt{\frac{x^2}{4} + h^2} + x^2 = 1600$$

$$2x\sqrt{\frac{x^2+4h^2}{4}} + x^2 = 1600$$

$$2x\,\frac{\sqrt{x^2+4h^2}}{2} + x^2 = 1600$$

$$x\sqrt{x^2 + 4h^2} + x^2 = 1600$$

$$x\sqrt{x^2 + 4h^2} = 1600 - x^2$$

$$\sqrt{x^2 + 4h^2} = \frac{1600-x^2}{x}$$

$$x^2 + 4h^2 = \frac{(1600-x^2)^2}{x^2}$$

$$x^4 + 4x^2h^2 = (1600 - x^2)^2$$

$$4x^2h^2 = (1600 - x^2)^2 - x^4$$

$$4x^2h^2 = 1600^2 - 3200x^2 + x^4 - x^4$$

$$4x^2h^2 = 1600^2 - 3200x^2$$

$$h^2 = \frac{1600^2-3200x^2}{4x^2}$$

$$h = \frac{\sqrt{1600^2-3200x^2}}{2x}$$

The reason for solving for h and for the substitution can be found in the appendix.

Now, the value of h will be substituted into the square base pyramid surface area formula:

$$V = \frac{x^2 \cdot \frac{\sqrt{1600^2 - 3200x^2}}{2x}}{3}$$

Now, this equation will be plotted in terms of x, where the maximum point will give the optimised x value with the maximum volume.

$$V(x) = \frac{x^2 \cdot \frac{\sqrt{1600^2 - 3200x^2}}{2x}}{3}$$

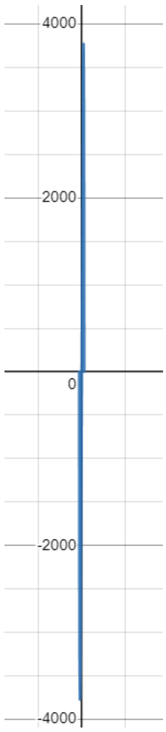


Figure 5: full plotted graph (desmos)

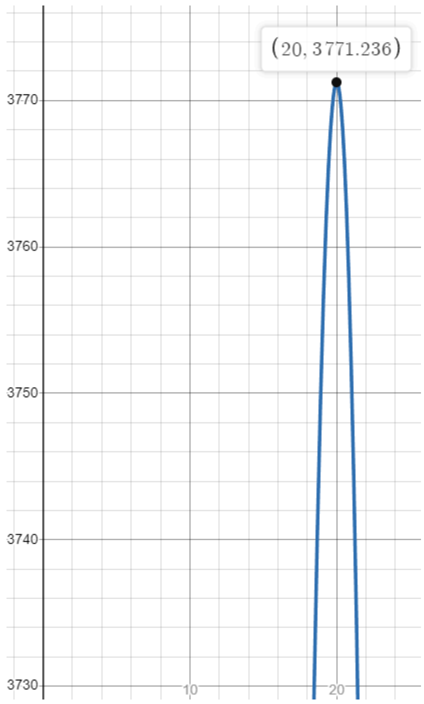


Figure 6: Zoomed in plotted graph (desmos)

Reasoning behind ignoring negative axis can be found in the appendix.

The maximum point of the positive-axis “quadratic” curve shows the maximum volume(y-axis) which is 3771.24, and the length/width (x-axis), 20. Since the maximum volume is 3771.24, there is no reason for plotting a line at y = 4000, which would find two optimal x values. Now the optimal height can be calculated by substituting the x-axis value, 20, into the height equation:

$$h = \frac{\sqrt{1600^2 - 3200(20)^2}}{2(20)}$$

$$h = 28.28m$$

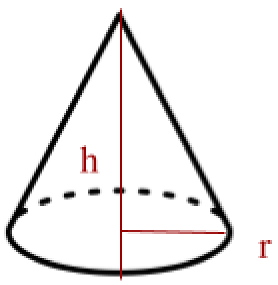
Finally, we can correct our answer by inputting the x and h into the pyramid surface area equation where we would get the fixed surface area of 1600m^2.

$$2(20)\sqrt{(\frac{20}{2})^2 + 28.284^2} + 20^2 = 1599.99 \approx 1600$$

Concluding the final mensurations of the optimised pyramid:

$$x = 20m$$
$$h = 28.28m$$
$$V = 3771.24m^3$$
$$A = 1600m^2$$

-Optimising Conical building



(shape from https://www.pngwing.com/en/free-png-aherb)

Firstly, the fixed surface area of the cube, will be set equal to the cone surface area formula:

$$\pi r(r + \sqrt{h^2 + r^2}) = 1600$$

Then, the equation was solved for h:

$$\pi r(r + \sqrt{h^2 + r^2}) = 1600$$
$$\pi r^2 + \pi r\sqrt{h^2 + r^2} = 1600$$
$$\pi r\sqrt{h^2 + r^2} = 1600 - \pi r^2$$
$$\sqrt{h^2 + r^2} = \frac{1600-\pi r^2}{\pi r}$$
$$h^2 + r^2 = \frac{(1600-\pi r^2)^2}{\pi^2 r^2}$$
$$\pi^2 r^2 h^2 + \pi^2 r^4 = (1600 - \pi r^2)^2$$
$$\pi^2 r^2 h^2 = (1600 - \pi r^2)^2 - \pi^2 r^4$$
$$\pi^2 r^2 h^2 = 1600^2 - 3200\pi r^2 + \pi^2 r^4 - \pi^2 r^4$$
$$\pi^2 r^2 h^2 = 1600^2 - 3200\pi r^2$$
$$h^2 = \frac{1600^2-3200\pi r^2}{\pi^2 r^2}$$
$$h = \frac{\sqrt{1600^2-3200\pi r^2}}{\pi r}$$

The reason for solving for h and for the substitution can be found in the appendix.

Now, the value of h will be substituted into the cone surface area formula:

$$V = \frac{\pi r^2 \cdot \frac{\sqrt{1600^2-3200\pi r^2}}{\pi r}}{3}$$

Now, this equation will be plotted in terms of r, where the maximum point will give the optimised x value with the maximum volume.

$$V(r) = \frac{\pi r^2 \cdot \frac{\sqrt{1600^2 - 3200\pi r^2}}{\pi r}}{3}$$

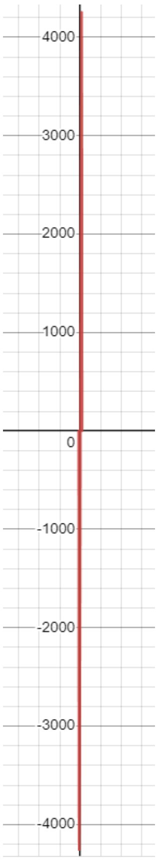


Figure 6: full plotted graph (desmos)

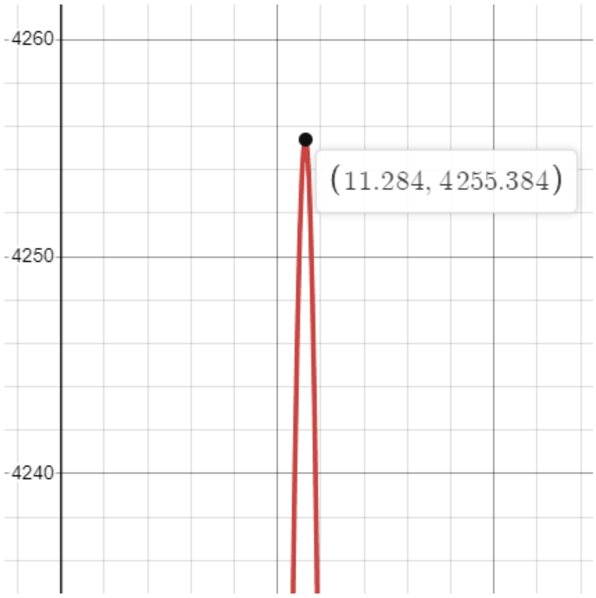


Figure 7: Zoomed in plotted graph (desmos)

Reasoning behind ignoring negative axis can be found in the appendix.

The maximum point of the positive-axis "quadratic" curve shows the maximum volume (y-axis) which is 4225.38, and the length/width (x-axis), 11.28. Now the optimal height can be calculated by substituting the x-axis value, 11.28, into the height equation:

$$h = \frac{\sqrt{1600^2 - 3200\pi(11.28)^2}}{\pi 11.28}$$

$$h = 31.94$$

Finally, we can correct our answer by inputting the x and h into the cone surface area equation where we would get the fixed surface area of 1600m^2.

$$\pi 11.28(11.28 + \sqrt{31.936^2 + 11.28^2}) = 1599.97 \approx 1600$$

Concluding the final mensurations of the optimised pyramid:

$$x = 11.28m$$

$$h = 31.94m$$

$$V = 4225.38m^3$$

$$A = 1600m^2$$

Poor country	
Pyramid	
x	20m
y	28.8m
V	3771.24m^3
A	1600m^2
Cone	
x	11.28m
y	31.94m
V	4225.38m^3
A	1600m^2

Rich country	
Pyramid	
x	20.39m
y	28.83m
V	4000m^3
A	1664.06m^2
Cone	
x	11.05m
y	31.28m
V	4000m^3
A	1535.33m^2

Applying mensurations to a real world situation:

Now, we will apply these mensurations into the theoretical situation. "According to The World Steel Association, steel is the most widely used material for building earthquake resistant buildings." So steel will be the material of choice for the frame (what was calculated throughout the IA) for both rich and poor countries, since it is the most earthquake resistant material.

Cost provided item	Unit	Average cost
Steel building construction	Per m2	£200-300
Steel house (complete)	Per m2	£1250
Steel frame building + composite cladding	Per m2	£250-350

Figure 8: Steel infrastructure prices table from (https://www.checkatrade.com/blog/cost-guides/steel-building-prices/)

The cost was chosen to be the average of £250-350 which is £300 per m^2. This was done by using the table above, which shows the costs of steel infrastructures based on the UK market. Thus the steel frame building + the composite cladding will be utilised. Using this price we can calculate the total cost of all the optimizations done in the IA:

Original cuboidal Building:

$$A = 1600m^2$$
$$1600 \cdot 300 = £480,000$$

Rich country optimization (Fixing volume, minimising surface area):

Pyramidal building:

$$A = 1664.06m^2$$
$$1664.04 \cdot 300 = £499,212$$
$$V = 4000m^3$$

Conical building:

$$A = 1535.33m^2$$
$$1535.33 \cdot 300 = £460,599$$
$$V = 4000m^3$$

Poor country optimization (Fixing Surface area, maximising volume):

Pyramidal building:

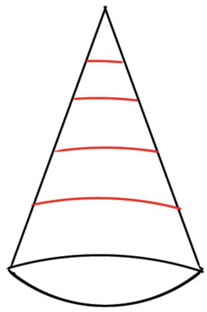
$$A = 1600m^2$$
$$1600 \cdot 300 = £480,000$$
$$V = 3771.24m^3$$

Conical building:

$$A = 1600m^2$$
$$1600 \cdot 300 = £480,000$$
$$V = 4225.38m^3$$

Evaluation & Conclusion:

To draw a final inference, in this IA, I modelled a building against an earthquake through optimising building mensurations for extremities of wealth. Through the modelling and analysis of the models, the conclusion was made that, the conical building, is indeed the best geometric shape for a building against an earthquake, due to its better distribution of mass, lowered centre of gravity and increased structural integrity. Additionally, when optimising for a fixed surface area with a maximum volume, the result was a larger surface area than the starting cuboid ($4000m^3$→$4225.38m^3$). As for optimising a minimum surface area with a fixed volume, the result was a cheaper building with the same original volume ($£480,00 \rightarrow £460,599$). "So why aren't all buildings conical?" You might ask, well, even with all of these advantages there is one major disadvantage which overcomes them all, practicality. Due to a cone's slant heights which converge to a point, all floors within a conical shaped building would decrease in volume the higher it goes, making it very inefficient and impractical, as shown in the image below:



However, if the customer's only concern is safety against earthquakes, and finds practicality negligible, a conical shaped building would be the most optimal geometric shape.

Appendix:

Explanation for solving for h-

The reason for solving for h and inputting it into the surface area (or volume formula) was so that the height could be optimised where the volume (or surface area) is kept fixed, and the surface area (or volume) is minimised (or maximised)

Explanation for ignoring negative axis-

The reason for ignoring the negative axis is that there were two curves in all graphs; an asymptotic (or another "quadratic" regarding the poor optimisation) curve on the negative x and y axis, and a "quadratic" curve on the positive x and y-axis, since there is no such thing as negative surface area/volume or negative length/width the negative axis was ignored.

How Cone slant height was calculated-

Slant height was calculated through the Pythagorean theorem $a^2 = b^2 + c^2$: the square root of the sum of the square of the radius (r) and the square of the height (h). $L = \sqrt{h^2 + r^2}$

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