INSTITUT POLYTECHNIQUE DE PARIS

ENSTA IP PARIS

SOD314. Cooperative Optimization for data science

Andrea Simonetto

Academic year 2022/2023

General info

- 5 classes (2h class, 1h30 project)
- All the info are on moodle
- 2h written Exam: 28/03, with material
- Grade: a function of Exam, TP, and optional homework along the way (it's the average between exam and TP plus extra points due to homework, but the result is cropped so that it cannot be more or less than 3 points of the exam grade)
- Extra points due to homework: max 3 points in total
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Content

1. Introduction
2. Distributed Optimization
3. Intermezzo: Stochastic gradient
4. Federated Learning
5. Differential Privacy

Material: Slides, and suggested references to books/articles as we go along.

Part I
Introduction

Some recap of useful notions

- We look at continuous optimization problems (like in OPT201, OPT202), i.e.,
$$\min_{x \in X \subseteq \mathrm{R}^n} f(x), \tag{1}$$
where the set X is closed and convex, $f : \mathbf{R}^n \to \mathbf{R}$ is a convex function and the minimum is attained for a $x \in X$.

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- What you need to remember from past courses will be reviewed as we go along, but mainly:
  - Optimality conditions (KKT), which are here necessary and sufficient (under a Slater's constraint qualification assumption)
  - How to derive dual problems (Lagrangian function, dual function, problems)
  - Subgradients
  - Algorithms: How to derive first-order algorithms (e.g., gradient descent) from the optimality conditions, and how to prove their convergence and convergence rate
  - Some basic linear algebra: eigenvalues, singular value decomposition, solution of linear systems, etc..

The optimization problem of interest

- Said so, the problem we will look at in this course has the form,

$$\text{(P)} \qquad \min_{x \in X \subseteq \mathbb{R}^n} \sum_{i=1}^{N} f_i(x), \tag{2}$$

where the set X is closed and convex, $f_i : \mathbf{R}^n \to \mathbf{R}$ is a convex function for all i's and the minimum is attained for a $x \in X$.
Here N are the number of agents/players/nodes/sub-systems/etc.. that cooperate to solve the optimization problem.

Examples and applications

- Cooperative least-squares. Imagine N users collect noisy measurements $\boldsymbol{y}_i \in \mathbf{R}^n$ about a quantity $\boldsymbol{x} \in X$. A way to estimate the true value for $\boldsymbol{x}$ is to set up a least-squares problem as

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- Cooperative linear model training. Imagine N users collect input-output pairs $\boldsymbol{w}_i \in \mathbf{R}^{n-1}, \boldsymbol{y}_i \in \mathbf{R}$ (e.g., features and labels), and they want to train a global model with weights $\boldsymbol{x} \in X \subseteq \mathbf{R}^n, \boldsymbol{x} = [\theta \in \mathbf{R}^{n-1}, c \in \mathbf{R}]$. Let the local input-output mapping being affine,

$$\boldsymbol{y}_i = \theta^\top \boldsymbol{w}_i + c, \qquad \forall i,$$

  then the training problem can be written as

$$\min_{\boldsymbol{x} \in X} \sum_{i=1}^{N} \|\boldsymbol{y}_i - (\theta^\top \boldsymbol{w}_i + c)\|^2 = \sum_{i=1}^{N} f_i(\boldsymbol{x})$$

Examples and applications

- Cooperative network problems. Take a sensor network of N sensors. you want to compute the localization of the whole network based on pair-wise distance measurements (e.g., sensors can be cars, or people with their phones). Then each measurement is

$$m_{ij} = \|x_i - x_j\| + \text{noise}, \qquad x_i \in \mathbf{R}^2 \text{ is the position of node } i.$$

  You have E pair-wise measurements. Then you can write the problem as

$$\min_{x \in X \subset \mathrm{R}^{2E}} \sum_{i,j}^{E} (m_{ij} - \|x_i - x_j\|)^2 = \sum_{k=1}^{E} f_k(x)$$

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- Cooperative consensus. You have N robots at different locations $\boldsymbol{s}_i$ and moving at different speeds v_i, and you want to find the best position in space for the fastest rendez-vous:

$$\min_{x \in X \subset \mathrm{R}^3} \sum_{i}^{N} \frac{\|\boldsymbol{s}_i - x\|}{v_i} = \sum_{i=1}^{N} f_i(x)$$

Main challenges and plan for the course

- Large-scale problems: N is big or n is big and the problem cannot be solved (not stored) locally. You need to solve it on separate machines, or iteratively.
  - If n is big, usually we talk about parallel methods;
  - If N is big, usually we talk about distributed methods.

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- Distributed localization: the data that builds f_i is stored in separate machines (think: mobile phones). Then, you have communication issues and latencies..
  - Are we communicating to a server? (Cloud-based)
  - Are we communicating to each other (Peer-to-peer)
  - Common issues: asynchronicity, bi-directionality, packet-losses, communication overhead,..
  - Here: graph theory to the rescue!

The SneakerNet paradox. What is the fastest way to send large data sets?

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  - Are we communicating to a server? (Cloud-based)
  - Are we communicating to each other (Peer-to-peer)
  - Common issues: asynchronicity, bi-directionality, packet-losses, communication overhead,..
  - Here: graph theory to the rescue!
- Data and functions are private: you don't want to disclose f_i or the local decision to the other players. How do you do that? (Example: training language models based on private text messages on your phone)

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- The algorithms will be of the first-order kind (more later)
- We divide the course in three parts (distributed, federated, private)

Part II
Basics

Some definitions

Definition 1 (Convex functions)

A function $f : X \subseteq \mathbf{R}^n \to \mathbf{R}$ is convex iff X is convex and

$$\text{(C1)} \quad \forall \boldsymbol{x}, \boldsymbol{y} \in X, \lambda \in [0, 1] : \quad f(\lambda \boldsymbol{x} + (1-\lambda)\boldsymbol{y}) \leq \lambda f(\boldsymbol{x}) + (1-\lambda) f(\boldsymbol{y}).$$

Multiple definitions exists, for example:

$$\text{(C1)} + f \in \mathcal{C}^1(X) \quad \Longleftrightarrow \quad \forall \boldsymbol{x}, \boldsymbol{y} \in X,\ f(\boldsymbol{x}) \geq f(\boldsymbol{y}) + \langle \nabla f(\boldsymbol{y}), \boldsymbol{x} - \boldsymbol{y} \rangle \tag{3}$$

$$\text{(C1)} + f \in \mathcal{C}^2(X) \quad \Longleftrightarrow \quad \forall \boldsymbol{x}, \boldsymbol{y} \in X,\ \nabla^2 f(\boldsymbol{x}) \succeq 0 \tag{4}$$

Proof. Homework.

Some definitions

Definition 1 (Strongly convex functions)

A convex function $f : X \subseteq \mathbf{R}^n \to \mathbf{R}$ is *m*-strongly convex iff

$$(\mathrm{SC}) \quad f(\mathbf{x}) - \frac{m}{2}\|\mathbf{x}\|^2 \text{ is convex.}$$

f doesn't need to be differentiable!
Multiple definitions exists, for example:

$$(\mathrm{SC}) + f \in \mathcal{C}^1(X) \iff \forall \mathbf{x}, \mathbf{y} \in X,\ \langle \nabla f(\mathbf{x}) - \nabla f(\mathbf{y}), \mathbf{x} - \mathbf{y} \rangle \geq m\|\mathbf{x} - \mathbf{y}\|^2 \quad (3)$$

$$(\mathrm{SC}) + f \in \mathcal{C}^2(X) \iff \forall \mathbf{x}, \mathbf{y} \in X,\ \nabla^2 f(\mathbf{x}) \succeq m I_n \quad (4)$$

Proof. Homework.

Some definitions

Definition 1 (Smooth functions)

A convex function $f : X \subseteq \mathbf{R}^n \to \mathbf{R}$ is L-smooth iff

$$\text{(LC)} \quad \frac{L}{2}\|x\|^2 - f(x) \text{ is convex.}$$

Important (LC) $\implies f \in \mathcal{C}^1(X)$!
Multiple definitions exists, for example:

$$\text{(LC)} \iff \forall x, y \in X,\ \|\nabla f(x) - \nabla f(y)\| \leq L\|x - y\| \tag{3}$$

$$\text{(LC)} \iff \forall x, y \in X,\ \frac{1}{L}\|\nabla f(x) - \nabla f(y)\|^2 \leq \langle \nabla f(x) - \nabla f(y), x - y\rangle \tag{4}$$

$$\text{(LC)} + f \in \mathcal{C}^2(X) \iff \forall x, y \in X,\ 0 \preceq \nabla^2 f(x) \preceq L I_n \tag{5}$$

Proof. Homework.

Gradient algorithm

Consider solving $\min_{x \in \mathbb{R}^n} f(x)$.
Rem: the gradient is an ascent direction, and a first-order method

Gradient algorithm

Consider solving $\min_{x \in \mathbb{R}^n} f(x)$.
Rem: the gradient is an ascent direction, and a first-order method
The simplest scheme, let $\alpha_k > 0$:

- Start with $x_0 \in \mathbf{R}^n$
- Iterate $x_{k+1} = x_k - \alpha_k \nabla f(x_k), \quad k = 0, 1, \ldots.$

There are different methods to choose α_k (either a priori or online):

- Constant: $\alpha_k = \alpha$
- Vanishing: $\alpha_k = \frac{\alpha}{\sqrt{k+1}}$
- ...

Why the difference? Convergence

Gradient algorithm: convergence recap

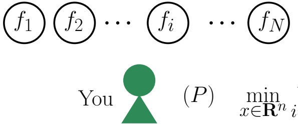
- Recap from OPT201-202: gradient converges in a well-defined sense provided the step size α_k is chosen appropriately.
- Depending on the functional class, the gradient algorithm behaves differently with the number of iterations t.
- The table below recaps the main results that one can expect:

Type	just convex	smooth	smooth+strongly convex
	First-order		
Convergence guarantee	$\|\nabla f\|, f - f^*$	$f - f^*$	$\|x - x^*\|$
Convex	$O(1/\sqrt{t})$, $O(1/t)$	$O(1/t)$, $O(1/t^2)$	$O(\rho^t)$

Homework: Revise your theory

Let's finally start!

- We want to solve problem (P) but N is too big. How do we do?



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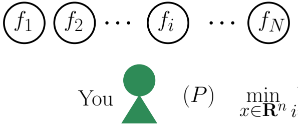

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- First possibility: you ask one device to apply a local gradient, then you get back the result and you ask a second device..

Incremental/ Gauss-Seidel Gradient

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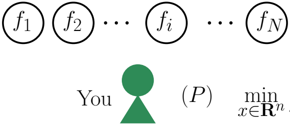

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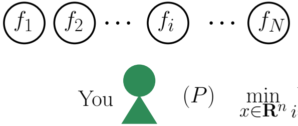

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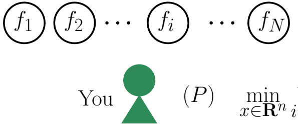

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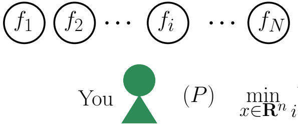

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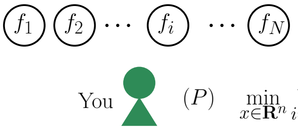

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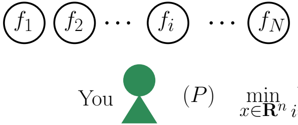

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Properties and convergence of incremental gradient

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- The data that generates f_i stays private, and if you have enough time you don't care about latencies (etc..): the scheme is quite robust
- Also known in the ML community as: online back-propagation, finite sum stochastic gradient descent (careful: step size is called the learning rate)

Theorem 2

Consider problem (P) for a m-strongly convex and L-smooth function f. Consider the incremental gradient method with step size α_k. Assume that $\|\nabla f_i(x) - \sum_i \nabla f_i(x)\| \leq G$ for all $x \in \mathbf{R}^n$, choose $\alpha_k < 2/L$, and define the quantity, $\rho_k = \max\{|1-\alpha_k m|, |1-\alpha_k L|\} < 1$.
Then convergence of the incremental gradient goes as

$$\|x_{k+1} - x^*\| \leq \rho_k \|x_k - x^*\| + \alpha_k G.$$

Corollary 3

If $\alpha_k = 1/k^s$, $0 < s < 1$ then, $\|x_k - x^*\| \leq O(1/k^s)$.

Proof

- Define $f(\mathbf{x}) = \sum_i f_i(\mathbf{x})$. Write

$$\|\mathbf{x}_{k+1} - \mathbf{x}^*\| \leq \|\mathbf{x}_k - \alpha_k \nabla f(\mathbf{x}_k) - \mathbf{x}^* + \alpha_k \nabla f(\mathbf{x}^*)\| + \alpha_k \|\nabla f_i(\mathbf{x}_k) - \nabla f(\mathbf{x}_k)\|$$

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- Use strongly convex and smoothness property to say

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- A lot of research is devoted in lifting the assumptions (strong convexity, smoothness), and relaxing the assumption on G, but the basic ideas are still valid: you have a trade-off between speed and error.

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- The scheme is less robust to asynchronicity, package drops, etc..
- The scheme is the starting point of distributed optimization (next!)

Part III
Distributed optimization

Let's go Graph theory!

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- The Laplacian matrix of a undirected graph $\mathcal{G}$ is an $|\mathcal{V}| \times |\mathcal{V}|$ symmetric matrix defined by

  $$\mathcal{L}_{\mathcal{G}} = \mathcal{D} - \mathcal{A},$$

  where $\mathcal{D}$ =diag$(d_1, \dots, d_n)$ is the degree matrix, which is the diagonal matrix formed from the vertex degrees.
  Note that $\mathcal{L}_{\mathcal{G}}\mathbf{1} = \mathbf{0}$.

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where $\mathcal{D}$ =diag$(d_1, \dots, d_n)$ is the degree matrix, which is the diagonal matrix formed from the vertex degrees.
Note that $\mathcal{L}_{\mathcal{G}}\mathbf{1} = \mathbf{0}$.
- A doubly stochastic matrix W is a matrix for which, $W\mathbf{1} = \mathbf{1}$, and $\mathbf{1}^\top W = \mathbf{1}^\top$.

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$$\mathbf{y}_{k+1} = W\mathbf{y}_k - \alpha_k \nabla_{\mathbf{y}} F(\mathbf{y}_k), \qquad F(\mathbf{y}) := \sum_{i=1}^N f_i(\mathbf{x}^i),$$

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where W is assumed doubly stochastic and it is the mixing matrix.
- DGD is peer-to-peer:
  - Each device gets and sends x_k^j / x_k^i from/to the neighbors
  - Each device computes: $x_{k+1}^i = \sum_{j=1}^N w_{ij} x_k^j - \alpha_k \nabla f_i(x_k^i), \quad \forall i$

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- What can we say about convergence? For a constant step size α, it turns out that it is not so difficult, since

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- For a general diminishing α_k sequence, it is a bit more complex but doable

DGD convergence

- We assume that the mixing matrix $W = [w_{ij}]$ is symmetric and doubly stochastic. The eigenvalues of W are real and sorted in a nonincreasing order
$$1 = \lambda_1(W) \geq \lambda_2(W) \geq \ldots \geq \lambda_N(W) \geq -1.$$
Let the second largest magnitude of the eigenvalues of W be denoted as
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$$\gamma = \max\{|\lambda_2(W)|, |\lambda_N(W)|\}$$
- Some basic questions in the decentralized/distributed setting arise: (1) When does x_k^i converge? (2) Does it converge to x^* (3) If not does consensus (i.e., $x_k^i = x_k^j$) hold asymptotically? (4) How do the properties of f_i and the network affect convergence?

DGD convergence: statement

Theorem 4 (DGD convergence)

Consider Problem (P) *and its solution via a decentralized gradient descent algorithm with constant step size* α, *and doubly stochastic communication matrix* W *with* $\gamma < 1$.
Let convex functions f_i *be* L_i*-smooth and let* $L = \max_i\{L_i\}$. *Let the mean value be*

$$\bar{x}_k = \frac{1}{N}\sum_{i=1}^{N} x_k^i.$$

If α *is chosen small enough and in particular* $\leq O(1/L)$, *then*

1. *Consensus:*

$$\|x_k^i - \bar{x}_k\| \to O(\frac{\alpha}{1-\gamma});$$

2. *Convergence*

$$f(\bar{x}_k) - f^* \to O(\frac{\alpha}{1-\gamma}).$$

DGD convergence: statement, cont'ed

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- We can see how for a constant step size we obtain a constant error bound (not so surprising, since we are changing the cost function!)
- When α is diminishing, you can obtain a zero error bound, but the analysis is more complicated, and you require typically extra assumptions.

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- Package drops, asynchronous communication. This is often less hard to deal with, if the instantaneous W stay symmetric and doubly stochastic. **Typically**, the idea is to define a finite set of communication of matrices $\{W^r\}_r$ and at each time k we randomly pick from them. If each W^r has the same structural properties of W and the eigenvalues of a restricted sequence of W^r have the same properties of the eigenvalues of W, then we can extend convergence results.

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- Gossip protocols: asynchronous and directed. A node wakes up and transmits to some of its neighbors, it also analyses what it has received, then it sleeps again. This is **very realistic** but **quite hard to do**. Here latencies..

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- We have now a term that tracks the gradient values and "integrates" the errors

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Theorem 5 (Gradient Tracking convergence)

Consider Problem (P) and its solution via a gradient tracking algorithm with constant step size α, and doubly stochastic communication matrix W with $w_{ij} \geq 0$. Let convex functions f_i be L-smooth and strongly convex. If α is chosen small enough and in particular $\leq O(1/L)$, then

1. *Consensus:*

$$\|x_k^i - \bar{x}_k\| \to 0;$$

2. *Convergence*

$$\|\bar{x}_k - x^*\| \to 0.$$

And convergence is linear.

Proof: arXiv:1906.10760

Gradient tracking explanation

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- On the good side, we have a first-order primal method that is fully distributed and allows us to reach zero consensus and residual error.
- On the minus side, we communicate quite a lot in terms of gradients, so we lose in privacy

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- Why would I care about communication or distributed when we can all upload on the cloud? (Think BIG: large-data files, server farms, or many sensors, etc..)

Sample references

1. The standard book: *Dimitri P. Bertsekas and John N. Tsitsiklis*, **Parallel and Distributed Computation: Numerical Methods**, 1997, `https://web.mit.edu/dimitrib/www/pdc.html`
2. Two recent articles:
   Kun Yuan, Qing Ling, Wotao Yin, **On the Convergence of Decentralized Gradient Descent**, arXiv:1310.7063 and SIAM Journal on Optimization, 2016
   Giuseppe Notarstefano, Ivano Notarnicola, Andrea Camisa, **Distributed Optimization for Smart Cyber-Physical Networks**, arXiv:1906.10760 and Foundations and Trends in Systems and Control, 2019
3. Many variants out there.

Class 2

Let's revisit our problem

- We remind that the problem at hand is

$$(P) \qquad \min_{x \in \mathrm{R}^n} \sum_{i=1}^{N} f_i(x).$$

and we proceed as in the first class by endowing each device with a copy of x, so that the problem becomes,

$$(P') \qquad \min_{x^i \in \mathrm{R}^n, i=1,\dots,N} \sum_{i=1}^{N} f_i(x^i) \quad \text{subject to } x^i = x^j, \forall i \sim j$$

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- Can we now look at its dual problem?

Let's revisit duality

- Consider the problem

$$\min_{x \in \mathbf{R}^n} f(x) \quad \text{subject to } Ax = b,$$

for a convex function $f : \mathbf{R}^n \to \mathbf{R}$, and matrix $A \in \mathbf{R}^{p \times n}$, vector $b \in \mathbf{R}^p$ (with b in the image of A).

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- The Lagrangian dual function is then,

$$q(\lambda) = \inf_{x \in \mathbb{R}^n} L(x, \lambda) = -(f^\star(-A^\top \lambda) + \lambda^\top b),$$

where $f^\star$ is the conjugate function of f, i.e., $f^\star(y) = \sup_x \{y^\top x - f(x)\}$.
Homework. Prove the last equality.

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- For this course strong duality always holds because I don't look at dualizing inequality constraints.

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- We remind that the problem at hand is

$$(P'') \qquad \min_{\boldsymbol{y} \in \mathrm{R}^{Nn}} F(\boldsymbol{y}) \quad \text{subject to } A\boldsymbol{y} = 0$$

whose Lagrangian is

$$L(\boldsymbol{y}, \lambda) = F(\boldsymbol{y}) + \lambda^\top A\boldsymbol{y}$$

and dual problem,

$$\max_{\lambda} q(\lambda) := \inf_{\boldsymbol{y}} \{F(\boldsymbol{y}) + \lambda^\top A\boldsymbol{y}\} = -F^\star(-A^\top \lambda).$$

Let's revisit our problem

- We remind that the problem at hand is

$$(P'') \qquad \min_{\boldsymbol{y}\in\mathbb{R}^{Nn}} F(\boldsymbol{y}) \quad \text{subject to } A\boldsymbol{y} = 0$$

whose Lagrangian is

$$L(\boldsymbol{y}, \lambda) = F(\boldsymbol{y}) + \lambda^\top A\boldsymbol{y}$$

and dual problem,

$$\max_{\lambda} q(\lambda) := \inf_{\boldsymbol{y}} \{F(\boldsymbol{y}) + \lambda^\top A\boldsymbol{y}\} = -F^\star(-A^\top\lambda).$$

- Can we set up a dual ascent algorithm?

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- What's $\partial q(\lambda_k)$?
- Start with

$$\partial_\lambda q(\lambda_k) \quad = \quad -A\partial_\lambda[-F^\star(-A^\top \lambda_k)]$$

Then, we know that for convex functions $(\partial_v F)^{-1}(v) = \partial_u F^\star(u)$, and in addition let,

$$\boldsymbol{y}^*(\lambda) := \arg\min_{\boldsymbol{y}} L(\boldsymbol{y}, \lambda) \iff \boldsymbol{y}^*(\lambda) = (\partial F)^{-1}(-A^\top \lambda)$$

Therefore,

$$\partial q(\lambda_k) = A\boldsymbol{y}^*(\lambda_k) \qquad \text{residual map!}$$

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- Solve locally

$$\mathbf{y}^*(\lambda) := \arg\min_{\mathbf{y}} L(\mathbf{y}, \lambda_k) = \arg\min_{\mathbf{x}^1,\ldots,\mathbf{x}^N} \sum_{i=1}^{n} f_i(\mathbf{x}^i) + \lambda_k^\top A[\mathbf{x}^1, \ldots, \mathbf{x}^N]^\top$$

Upon dividing $A = [A_1|\ldots|A_N]$,

$$\iff \mathbf{x}^{i,*}(\lambda_k) := \arg\min_{\mathbf{x}_i} f_i(\mathbf{x}^i) + \lambda_k^\top A_i \mathbf{x}^i \qquad \forall i,$$

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- Send and update $\boldsymbol{x}^{i,*}(\lambda_k)$ to all, and gather,

$$\partial q(\lambda_k) = A\boldsymbol{y}^*(\lambda_k) = \sum_i A_i \boldsymbol{x}^{i,*}(\lambda_k)$$

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- Update then, $\lambda_{k+1} = \lambda_k + \alpha_k \sum_i A_i \mathbf{x}^{i,*}(\lambda_k)$, for all $k = 0, 1, \ldots$.

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  Let A be the collection of $x^i - x^j$ for the edge set. Then the associated dual variable is λ^{ij} and that can live on the edge i, j.

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  Let A be the collection of $x^i - x^j$ for the edge set. Then the associated dual variable is λ^{ij} and that can live on the edge i, j.
- The last means that also λ^{ij} can also be done locally!
  $\lambda_{k+1}^{ij} = \lambda_k^{ij} + \alpha_k[x^{i,*}(\lambda_k) - x^{j,*}(\lambda_k)]$, for all $k = 0, 1, \ldots$

Dual decomposition: a picture

- Consider a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, undirected. We indicate $i \sim j$ if there is an edge between i and j. We also define A as

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- This has two local updates and one communication round (synchronous, undirected)

Dual decomposition: convergence I

- We look now at the convergence of dual decomposition. We need to recall some of the properties of the dual function. In particular, let $\sigma(A)$ be the singular values of A.
  If f is m-strongly convex then $-q$ is $\sigma^2_{\max}(A)/m$ smooth;
  If f is L-smooth then $-q$ is $\sigma^2_{\min}(A)/L$ strongly convex;
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Theorem 6 (Dual decomposition convergence)

Consider problem (P) and the dual decomposition approach for a certain connection matrix A. If f_i's are m-strongly convex and L-smooth, then we can select $\alpha < 2m/\sigma^2_{\max}(A)$ and obtaining a linear rate of convergence as,

$$\|\lambda_k - \lambda^*\| \leq \left[\max\{|1 - \alpha\frac{\sigma^2_{\max}(A)}{m}|, |1 - \alpha\frac{\sigma^2_{\min}(A)}{L}|\}\right]^k \|\lambda_0 - \lambda^*\|$$

and

$$\|x^i_k - x^*\| \leq \frac{\sigma_{\max}(A)}{m}\|\lambda_k - \lambda^*\|.$$

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- First $\lambda^* = \lambda_0^* + \lambda_1^*$, with $\lambda_0^* \in \mathrm{im}(A)$ and $\lambda_1^* \in \mathrm{null}(A^\top)$. One can show that λ_0^* is unique and we concentrate on that one, since λ_1^* is redundant $A^\top \lambda_0^* = 0$.

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- Start with $\lambda_0 \in \mathrm{im}(A)$; then, it is direct that:

$$\lambda_{k+1} = \lambda_k + \alpha A \mathbf{x} \in \mathrm{im}(A)$$

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- Then, can we say that (restricted strong convexity)

$$(\partial q(\lambda) - \partial q(\lambda'))^\top(\lambda' - \lambda) \geq \sigma_{\underline{\min}}^2/L\|\lambda' - \lambda\|^2, \quad \forall \lambda, \lambda' \in \mathrm{im}(A)?$$

with $\sigma_{\underline{\min}}$ the minimum non-zero singular value of A?

Dual decomposition: convergence II

- Yes, substitute $\partial q(\lambda) = A \partial F^\star(-A^\top \lambda)$, then,

$$\begin{aligned}(\partial F^\star(-A^\top\lambda) - \partial F^\star(-A^\top\lambda'))^\top A^\top(\lambda' - \lambda) = \\ (\partial F^\star(-A^\top\lambda) - \partial F^\star(-A^\top\lambda'))^\top(-A^\top\lambda + A^\top\lambda') \geq \\ 1/L\|A^\top(\lambda' - \lambda)\|^2 \geq \sigma^2_{\min}/L\|\lambda' - \lambda)\|^2.\end{aligned}$$

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- So the Theorem becomes: we can select $\alpha < 2m/\sigma^2_{\mathrm{max}}(A)$ and $\lambda_0 \in \mathrm{im}(A)$, and obtaining a linear rate of convergence as,

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- Note: $\lambda_0 = \mathbf{0} \in \mathrm{im}(A)$

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- Convergence may be complicated when communication is directed, or asynchronous, or you have latencies, as in the first class

The Alternating Direction Method of Multipliers (ADMM)

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- Consider the problem:

$$\begin{array}{ll} \min\limits_{x \in \mathbb{R}^n, y \in \mathbb{R}^p} & f(x) + g(y) \\ \text{subject to} & Ax + By = c. \end{array}$$

For well-defined matrices and vectors, as well as convex functions f, g.

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  For well-defined matrices and vectors, as well as convex functions f, g.
- Define the augmented Lagrangian,

$$L(\mathbf{x}, \mathbf{y}, \lambda) := f(\mathbf{x}) + g(\mathbf{y}) + \lambda^\top(A\mathbf{x} + B\mathbf{y} - c) + \frac{\beta}{2}\|A\mathbf{x} + B\mathbf{y} - c\|^2,$$

  for any scalar $\beta > 0$.

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- ADMM is similar to an incremental approach, but on two variables. It is also similar to dual decomposition, but on two variables
- It looks more complicated than dual ascent, is it better in some sense?
- Assume strong duality holds and primal and dual solutions exist, then convergence is ensure for any $\beta > 0$ in a weak sense.

ADMM: convergence in some special cases

- Assume f to be m-strongly convex and L-smooth and that a primal-dual solution exists (g can be a generic convex function). This is a special case and ADMM generally converges with minimal assumptions, but in a much weaker sense, so we keep here the stronger assumptions.

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Theorem 7 (ADMM convergence)

With the assumptions in place,

$$\|\lambda_k - \lambda^*\| \leq \varrho^k \|\lambda_0 - \lambda^*\| \qquad \varrho = \max\left\{\left|\frac{1-\beta\frac{\sigma^2_{\max}(A)}{m}}{1+\beta\frac{\sigma^2_{\max}(A)}{m}}\right|, \left|\frac{1-\beta\frac{\sigma^2_{\min}(A)}{L}}{1+\beta\frac{\sigma^2_{\min}(A)}{L}}\right|\right\}$$

and,

$$\|x_k - x^*\| \leq \frac{\|A\|}{m}\|\lambda_k - \lambda^*\|.$$

Proof?

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- The proof goes as follows: ADMM is an application of a special algorithm (the Douglas-Rachford splitting) applied to the dual of our initial problem. The Douglas-Rachford splitting converges in a certain way given functional properties. We derive the dual of those functional properties and (as in the dual decomposition case) the convergence of the dual algorithm (ADMM).

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- This is why the result looks very similar to the result of the dual decomposition. ADMM is a dual algorithm.

Proof: step I, Douglas-Rachford splitting

- Consider the problem,

$$\min_{x \in \mathrm{R}^n} f(x) + g(x),$$

with f and g convex closed and proper (CCP). Consider now the following method to find a solution x^*.

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- Start at a certain z_0 and iterate for all $k \in \mathbb{N}$:

$$x_k = \mathrm{prox}_{\beta f}(z_k) \tag{6a}$$

$$z_{k+1} = z_k + \mathrm{prox}_{\beta g}(2x_k - z_k) - x_k, \tag{6b}$$

where $\mathrm{prox}_{\beta\phi}$ is the usual prox operator:

$$\mathrm{prox}_{\beta\phi}(u) = \arg\min_v \{\phi(v) + \frac{1}{2\beta}\|v - u\|^2\}$$

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- In particular if f is m-strongly convex and L-smooth, then, for all $\beta > 0$

$$\|z_{k+1} - z^*\| \leq \varrho^k \|z_k - z^*\|, \qquad \varrho = \max\left\{\left|\frac{1-\beta L}{1+\beta L}\right|, \left|\frac{1-\beta m}{1+\beta m}\right|\right\}$$

Proof: step II, The dual problem and its properties

- Start from our problem:

$$\begin{array}{ll} \min_{\mathbf{x}\in\mathbb{R}^n,\mathbf{y}\in\mathbb{R}^p} & f(\mathbf{x}) + g(\mathbf{y}) \\ \text{subject to} & A\mathbf{x} + B\mathbf{y} = c, \end{array}$$

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- Dual problem is,

$$\max_{\lambda} \inf_{\mathbf{x},\mathbf{y}} \{f(\mathbf{x}) + g(\mathbf{y}) + \lambda^\top(A\mathbf{x} + B\mathbf{y} - c)\}$$

and so,

$$\min_{\lambda} f^\star(-A^\top\lambda) + g^\star(-A^\top\lambda) + c^\top\lambda$$

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- Consider "f"$= f^\star(-A^\top\lambda) + c^\top\lambda$ and "g"$= g^\star(-B^\top\lambda)$
- As before we know that "f" is $\sigma^2_{\min}/L$-strongly convex and $\sigma^2_{\max}/m$-smooth

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- Apply DRS "f" $= f^\star(-A^\top\lambda) + c^\top\lambda$ and "g" $= g^\star(-B^\top\lambda)$ and use its convergence properties.

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- Apply DRS "f"$= f^\star(-A^\top \lambda) + c^\top \lambda$ and "g"$= g^\star(-B^\top \lambda)$ and use its convergence properties.
- **Do some computational gymnastic: check out the references**
- Arrive at the ADMM iterations

ADMM: distributed optimization

- Let's go back to:

$$\begin{aligned} \min_{\mathbf{x}\in\mathbb{R}^n, \mathbf{y}\in\mathbb{R}^p} \quad & f(\mathbf{x}) + g(\mathbf{y}) \\ \text{subject to} \quad & A\mathbf{x} + B\mathbf{y} = c, \end{aligned}$$

and our problem,

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- Later, we will see a different cloud-based splitting instead.
- Compactify $\mathbf{x}^i = \mathbf{y}^{ij}$ as, $A\mathbf{x} + B\mathbf{y} = 0$. Careful here that A is not full row rank, so linear convergence requires more work, but possible.

ADMM: distributed algorithm

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- For all nodes update:

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which is equivalent to,

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- Communicate $\mathbf{x}^i_{k+1}$ to your neighbors
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$$\mathbf{y}^{ij}_{k+1} = \arg\min_{\mathbf{y}^{ij}}\{(\lambda_k)^\top(A\mathbf{x}_{k+1} + B\mathbf{y}) + \frac{\beta}{2}\|A\mathbf{x}_{k+1} + B\mathbf{y}\|^2\}$$

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1. Start with $\lambda^{ij}_0 = 0$, and $\boldsymbol{y}^{ij}_0$ then,
2. Each node updates, $\boldsymbol{x}^i_{k+1} = \arg\min_{\boldsymbol{x}^i}\{f_i(\boldsymbol{x}^i) + \sum_{j\in\mathcal{N}^i}\frac{\beta}{2}\|\boldsymbol{x}^i - \boldsymbol{y}^{ij}_k\|^2\}$
3. Each node communicates $\boldsymbol{x}^i_{k+1}$ to its neighbors
4. Each node updates, $\boldsymbol{y}^{ij}_{k+1} = \frac{\boldsymbol{x}^i_{k+1}+\boldsymbol{x}^j_{k+1}}{2}$

ADMM: Example in model fitting

- Consider the task of training a model via the convex problem,

$$\min_{x \in \mathbb{R}^n} \ell(Ax - b) + r(x),$$

where $\ell : \mathbf{R}^p \to \mathbf{R}$ is a convex loss function, $A \in \mathbf{R}^{p \times n}$ is the feature matrix, $b \in \mathbf{R}^p$ is the output vector, x are the parameters of the model, and $r : \mathbf{R}^n \to \mathbf{R}$ is a convex regularization function.

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- r is often separable too. For instance if r is the Tikhonov regularization (aka ridge penalty): $r(x) = \nu\|x\|_2^2$, or if r is the lasso penalty: $r(x) = \nu\|x\|_1$
- Typically you have a modest number of features but a very large number of training examples (i.e., $m \gg n$).

ADMM: Example in model fitting II

- The goal is to solve the problem in a distributed way, with each processor handling a subset of the training data. This is useful either when there are so many training examples that it is inconvenient or impossible to process them on a single machine or when the data is naturally collected or stored in a distributed fashion. This includes, for example, online social network data, webserver access logs, wireless sensor networks, and many cloud computing applications more generally.

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- Suppose you associate a number of training examples to a number N of processors (in the extreme $N = m$). Then, you may solve

$$\begin{aligned} \min_{\mathbf{x}^j \in \mathbf{R}^n, \mathbf{y} \in \mathbf{R}^n} \quad & \sum_{j=1}^{N} \ell_j(A_j \mathbf{x}^j - b_j) + r(\mathbf{y}), \\ \text{subject to} \quad & \mathbf{x}^j = \mathbf{y}, \quad i = 1, \ldots, N. \end{aligned}$$

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- This can be solved via ADMM

ADMM: Example in model fitting III

- Each processor solves

$$x_{k+1}^j = \arg\min_{x^j} \{\ell_j(A_j x^j - b^j) + \lambda_k^j (x^j - y_k) + \frac{\beta}{2}\|x^j - y_k\|^2\}$$

$$= \arg\min_{x^j} \{\ell_j(A_j x^j - b^j) + \frac{\beta}{2}\|x^j - y_k + \frac{\lambda_k^j}{\beta}\|^2\}$$

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$$\mathbf{x}^j_{k+1} = \arg\min_{\mathbf{x}^j} \{\ell_j(A_j\mathbf{x}^j - b^j) + \lambda^j_k(\mathbf{x}^j - \mathbf{y}_k) + \frac{\beta}{2}\|\mathbf{x}^j - \mathbf{y}_k\|^2\}$$

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- Communication to the cloud of $\mathbf{x}^j_{k+1}$

ADMM: Example in model fitting III

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\end{aligned}
$$

- Communication to the cloud of x^j_{k+1}
- On the cloud we solve,

$$
\begin{aligned}
y_{k+1} &= \arg\min_{y}\{r(y) + \sum_{j=1}^{N}\lambda^j_k(x^j_{k+1} - y) + \frac{\beta}{2}\|x^j_{k+1} - y\|^2\} \\
&= \arg\min_{y}\{r(y) + \sum_{j=1}^{N}\frac{\beta}{2}\|x^j_{k+1} - y - \frac{\lambda^j_k}{\beta}\|^2\}
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$$\mathbf{y}_{k+1} = \arg\min_{\mathbf{y}}\{r(\mathbf{y}) + \sum_{j=1}^N \lambda^j_k(\mathbf{x}^j_{k+1} - \mathbf{y}) + \frac{\beta}{2}\|\mathbf{x}^j_{k+1} - \mathbf{y}\|^2\}$$
$$= \arg\min_{\mathbf{y}}\{r(\mathbf{y}) + \sum_{j=1}^N \frac{\beta}{2}\|\mathbf{x}^j_{k+1} - \mathbf{y} - \frac{\lambda^j_k}{\beta}\|^2\}$$

- Update $\lambda^j_{k+1} = \lambda^j_k + \beta(\mathbf{x}^j_{k+1} - \mathbf{y}_{k+1})$ and communicate back to processors $\mathbf{y}_{k+1}, \lambda^j_{k+1}$

ADMM: Example in model fitting IV

- The data never leaves the processors

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$$\boldsymbol{x}_{k+1}^j = \arg\min_{\boldsymbol{x}^j}\{\|A_j\boldsymbol{x}^j - b_j\|^2) + \frac{\beta}{2}\|\boldsymbol{x}^j - \boldsymbol{y}_k + \frac{\lambda_k^j}{\beta}\|^2\}$$
$$= (A_j^\top A_j + \beta I)^{-1}(A_j^\top b_j - \beta\boldsymbol{y}_k + \lambda_k^j).$$

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- Communication to the cloud of $\boldsymbol{x}^j_{k+1}$

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$$\begin{aligned} \boldsymbol{x}_{k+1}^j &= \arg\min_{\boldsymbol{x}^j}\{\|A_j\boldsymbol{x}^j - b_j\|^2) + \frac{\beta}{2}\|\boldsymbol{x}^j - \boldsymbol{y}_k + \frac{\lambda_k^j}{\beta}\|^2\} \\ &= (A_j^\top A_j + \beta I)^{-1}(A_j^\top b_j - \beta\boldsymbol{y}_k + \lambda_k^j). \end{aligned}$$

- Communication to the cloud of $\boldsymbol{x}_{k+1}^j$
- On the cloud we solve,

$$\boldsymbol{y}_{k+1} = \arg\min_{\boldsymbol{y}}\{\nu\|\boldsymbol{y}\|_1 + \sum_{j=1}^N \frac{\beta}{2}\|\boldsymbol{x}_{k+1}^j - \boldsymbol{y} + \frac{\lambda_k^j}{\beta}\|^2\} = S_{\nu/\beta}(\bar{\boldsymbol{x}}_{k+1} + \frac{\bar{\lambda}_k}{\beta})$$

ADMM: Example in model fitting IV

- The data never leaves the processors
- Different updates of $\boldsymbol{y}$ depending on r and on $\boldsymbol{x}$ depending on ℓ.
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- Update $\lambda^j_{k+1} = \lambda^j_k + \beta(\boldsymbol{x}^j_{k+1} - \boldsymbol{y}_{k+1})$ and communicate back to processors $\boldsymbol{y}_{k+1}, \lambda^j_{k+1}$

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- Asynchronicity, latencies, package losses, all add to the difficulty in proving convergence of the algorithm
- Research in this domain is very rich and active!

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- ADMM in particular offers multiple ways to distributed computations across multiple devices and it has received a lot of attention in recent years
- ADMM works by splitting the problem into a part that can be solved locally, and a part that we can afford to solve sharing information
- ADMM can be applied to many settings (multi-core, cloud-computing, distributed computing, etc..)

Sample references

1. *Stephen Boyd, Neal Parikh, Eric Chu, Borja Peleato, and Jonathan Eckstein*, **Distributed Optimization and Statistical Learning via the Alternating Direction Method of Multipliers**, Foundations and Trend in Machine Learning, 2010
2. *Ernest Ryu, Stephen Boyd*, **A Primer on Monotone Operator Methods**, Appl. Comput. Math., 2016
3. Many variants out there.