ICFP Master - Theoretical Physics

Motion of test-particles in Entangled Relativity

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Introduction

According to Einstein, a relativistic theory of gravitation must satisfy three principles: the principles of relativity, equivalence and the relativity of inertia [1].

The *principle of relativity*, or *principle of generalised covariance*, asserts that the laws of physics must be the same in all reference frames, inertial or not. This principle is verified whenever the laws of physics are stated in a tensorial way.

The *principle of equivalence* states that the effects of gravitation are the same as the effects of acceleration in the observer's reference frame. It was this principle that led Einstein to consider a curved manifold to describe space-time, where the metric of this manifold alone contains all the information about the gravitational field.

The *principle of relativity of inertia*, or *principle of Mach* asserts that it is impossible to define motion, and even space-time itself, without matter. Thus, there can be no vacuum solution.

This last principle is not verified by General Relativity. In fact, many solutions of the vacuum are known today (eg. Minkowski solution). Einstein added the cosmological constant to General Relativity equations to prevent these solutions from existing [2]. However, vacuum solutions with a non-zero cosmological constant are also known today (eg. de Sitter solution).

In addition, when General Relativity and the Standard Model are written in path integral form, three constants appear: $\hbar$ Planck's constant, c the causal structure constant of space-time and $\tilde{\kappa}$ Einstein's constant, linked to Newton's constant G.

These constants are used to form Planck units such as energy, length and Planck time. Because of Heisenberg's uncertainty principle, if we wanted to look at what happens at the Planck length, we would have to mobilise an energy equal to the Planck energy. GR then predicts the creation of a black hole: a singularity hidden behind a horizon.

So the notion of space-time becomes ambiguous, and we can no longer define the path integral. We then introduce a threshold energy as the upper limit of the path integral.

Entangled Relativity is a theory which, because of the way it is written, verifies Mach's principle. Moreover, we cannot define a Planck length, so the structure of the universe could be conserved whatever the scale.

Contents

1 Entangled Relativity

1.1 Path integral formulation

Entangled relativity can be defined through a path integral [3]:

$$Z_{ER} = \int \mathcal{D}g \prod_i \mathcal{D}f_i \exp\left(-\frac{i}{2\epsilon^2}\int \mathrm{d}^4x\sqrt{-g}\frac{\mathcal{L}_m^2}{R}\right), \tag{1}$$

where R is the usual scalar curvature constructed upon the metric $g_{\mu\nu}$ and its derivatives, $\mathcal{L}_m$ is the Lagrangian density of matter fields f_i. g is the metric determinant, ϵ is a constant that has the dimension of an energy. $\int \mathcal{D}$ denotes the integration over all possible and non-redundant field configurations.

It is easy to see that this theory verifies Mach's principle only by the way it is written. Indeed, if $\mathcal{L}_m = \emptyset$, then the phase within the integral vanishes and thus, the theory cannot be defined without matter fields.

The only fundamental constants of this theory are c and ϵ. From these, it is impossible to construct an equivalent to the Planck length. A priori, this length scale does not play any role in the structure of space-time.

1.2 Scalar-tensor equivalent form and field equations

Classically, the field equations are given by the condition $\delta\Theta = 0$ where Θ is the phase in the path integral [3, 4, 5]:

$$\Theta = -\frac{1}{2\epsilon^2}\int \mathrm{d}^4x\sqrt{-g}\frac{\mathcal{L}_m^2}{R}. \tag{2}$$

One can check that ϵ does not influence the classical phenomenology of the theory.

The equation of motion reads:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = -\frac{R}{\mathcal{L}_m}T_{\mu\nu} + \frac{R^2}{\mathcal{L}_m^2}\left(\nabla_\mu\nabla_\nu - g_{\mu\nu}\Box\right)\frac{\mathcal{L}_m^2}{R^2}, \tag{3}$$

where $R_{\mu\nu}$ is the usual Ricci tensor and $T_{\mu\nu}$ is the stress-energy tensor, defined by:

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta\left(\sqrt{-g}\mathcal{L}_m\right)}{\delta g^{\mu\nu}}. \tag{4}$$

The equation of motion can also be derived from the following phase [6]:

$$\begin{aligned}\Theta' &= \frac{1}{\epsilon^2}\int \mathrm{d}^4x\sqrt{-g}\frac{1}{\kappa}\left(\frac{R}{2\kappa} + \mathcal{L}_m\right) \\ &= \frac{1}{\epsilon^2\tilde{\kappa}}\int \mathrm{d}^4x\sqrt{-g}\left(\frac{\phi^2 R}{2\tilde{\kappa}} + \phi\mathcal{L}_m\right),\end{aligned} \tag{5}$$

provided $\mathcal{L}_m \neq \emptyset$. $\tilde{\kappa}$ is a constant. κ and ϕ are scalar fields related by $\frac{\tilde{\kappa}}{\kappa} = \phi$.

The two formulations are equivalent on-shell. Indeed, the equation of motion for κ is [4, 6]:

$$\kappa = -\frac{R}{\mathcal{L}_m}. \tag{6}$$

By plugging this back into (5), one gets (2).

The fields equations are:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{\tilde{\kappa}}{\phi}T_{\mu\nu} + \frac{1}{\phi^2}\left(\nabla_\mu\nabla_\nu - g_{\mu\nu}\Box\right)\phi^2, \tag{7}$$

$$\phi = -\tilde{\kappa}\frac{\mathcal{L}_m}{R}. \tag{8}$$

Let us note that the stress-energy tensor is not conserved in general as one has:

$$\nabla_\nu T^{\mu\nu} = \left(g^{\mu\nu}\mathcal{L}_m - T^{\mu\nu}\right)\frac{\nabla_\nu \phi}{\phi}. \tag{9}$$

1.3 General Relativity as a limit

In tensor-scalar theories, an evolution equation for the scalar field ϕ can be obtained by taking the trace of the Einstein equation. Here, the trace of (7) gives :

$$\begin{aligned} \frac{3}{\phi^2}\Box\phi^2 &= \frac{\tilde{\kappa}T}{\phi} + R \\ &= \frac{\tilde{\kappa}}{\phi}\left(T - \mathcal{L}_m\right). \end{aligned} \tag{10}$$

For matter in the form of dust without pressure or electromagnetic radiation, the right hand side of equation (10) cancels out.

Indeed, for dust $\mathcal{L}_m = -\rho$ where ρ is the mass density and $T = -\rho + 3p = -\rho$ with p the pressure, zero here. For electromagnetic radiation, we have $T = 0$ and $\mathcal{L}_m \propto B^2 - E^2 = 0$.

Therefore, in most cases encountered in the universe, the scalar field ϕ is not sourced and can be taken constant equal to 1, with the normalisation such that $\tilde{\kappa}$ corresponds to the measured Einstein constant. One can note that in this case, the solutions of General Relativity are also solutions of Entangled Relativity.

The phase (5) is written in the same way as the phase of the General Relativity path integral:

$$\Theta_{RG} = \frac{1}{\hbar c}\int \mathrm{d}^4x\sqrt{-g}\left(\frac{R}{2\tilde{\kappa}} + \mathcal{L}_m\right). \tag{11}$$

To restore General Relativity, the value of ϵ needs to be fixed at Planck energy:

$$\epsilon = \sqrt{\frac{\hbar c}{\tilde{\kappa}}}. \tag{12}$$

1.4 The Standard Model as a limit

In this section, we consider $\mathcal{L}_m$ as the Lagrangian of the Standard Model.

If $R = cst$, the action integral $\int \sqrt{-g} \mathrm{d}^4 x \frac{\mathcal{L}_m^2}{cst}$ is certainly not the standard model, even less so if $cst = 0$, a priori.

This problem is easily solved by understanding the central role played by the field ϕ (or κ).

In General Relativity, the trace of Einstein's equation gives:

$$\tilde{\kappa} = -\frac{R}{T}. \tag{13}$$

This way of writing the equation suggests that $\tilde{\kappa}$ diverges when $T = 0$. This is not the case, because we also have $R = 0$ and $\tilde{\kappa}$ keeps its value.

The source term of the equation (10) is negligible in front of the source term of the first-order metric fluctuation equation :

$$\left| \frac{\kappa}{3} \left(T - \mathcal{L}_m \right) \right| < \left| \frac{\kappa \rho}{2} \right|. \tag{14}$$

The scalar field is at best of the same order of magnitude as the terms of the post-Newtonian expansion of the metric in General Relativity. So if we neglect gravity, i.e. variations of the metric, we also neglect variations of $\kappa = -\frac{R}{\mathcal{L}_m}$ (or φ). In this case, $\kappa = \tilde{\kappa}$ (or $\phi = 1$). Thus, the path integral of Entangled Relativity can be written, in an approximate form :

$$Z_{ER} \approx \int \prod_i \mathcal{D} f_i \exp \left(\frac{i}{\epsilon^2 \tilde{\kappa}} \int \mathrm{d}^4 x \mathcal{L}_m \right), \tag{15}$$

which is exactly the path integral of the Standard Model with gravity neglected.

1.5 Motion of test particles in Entangled Relativity

In this part, the equation of motion for test particles are derived from the field equations seen above. c is taken equal to 1.

1.5.1 Massive particles

The stress-energy tensor for non-interactive massive particles is given by $T^{\mu\nu} = \rho U^\alpha U^\beta$, with $U^\alpha = \frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau}$ is the 4-velocity and τ is the proper time. Noting that $\mathcal{L}_m = -\rho$ and that the matter fluid current is conserved: $\nabla_\mu \left(\rho U^\mu \right) =$

0, one gets from (9) [7]:

$$U^\mu \nabla_\mu U^\nu = -\left(g^{\nu\sigma} + U^\nu U^\sigma\right) \frac{\partial_\sigma \phi}{\phi}. \tag{16}$$

As $U^\mu \nabla_\mu U^\nu = \frac{\mathrm{d}U^\nu}{\mathrm{d}\tau} + \Gamma^\nu_{\alpha\beta} U^\alpha U^\beta$, the equation of motion is different than the geodesic equation. Nonetheless, one can prove two things about solutions of this equation.

The norm of the velocity is conserved: Let us write $K = U^\mu U_\mu$, then:

$$\begin{aligned}
\frac{\mathrm{d}K}{\mathrm{d}\tau} &= U^\mu U^\nu U^\alpha g_{\mu\nu,\alpha} + 2 g_{\mu\nu} U^\mu \frac{\mathrm{d}U^\nu}{\mathrm{d}\tau} \\
&= U^\mu U^\nu U^\alpha g_{\mu\nu,\alpha} - 2 g_{\mu\nu} U^\mu \left(\Gamma^\nu_{\alpha\beta} U^\alpha U^\beta + \left(g^{\nu\sigma} + U^\nu U^\sigma\right) \frac{\partial_\sigma \phi}{\phi} \right) \\
&= U^\mu U^\nu U^\alpha \left(g_{\mu\nu,\alpha} - 2 g_{\mu\beta} \Gamma^\beta_{\alpha\nu} \right) - 2 \left(K + 1\right) U^\sigma \frac{\partial_\sigma \phi}{\phi} \\
&= -2 \left(K + 1\right) \frac{\mathrm{d} \log \phi}{\mathrm{d}\tau},
\end{aligned}$$

which can be solved as $K = -1 + \frac{A}{\phi^2}$ with A a constant. If at initial time, we choose $K = -1$ then it remains equal to -1 for all times.

Quantity conserved: If both the metric and the scalar field do not depend on a coordinate x^μ, then ϕU_μ is conserved:

$$\begin{aligned}
\frac{\mathrm{d}\phi U_\mu}{\mathrm{d}\tau} &= \phi U^\nu U^\alpha g_{\mu\nu,\alpha} + \phi g_{\mu\nu} \frac{\mathrm{d}U^\nu}{\mathrm{d}\tau} + U_\mu U^\nu \partial_\nu \phi \\
&= \phi U^\nu U^\alpha g_{\mu\nu,\alpha} - \phi g_{\mu\nu} \left(\Gamma^\nu_{\alpha\beta} U^\alpha U^\beta + \left(g^{\nu\sigma} + U^\nu U^\sigma\right) \frac{\partial_\sigma \phi}{\phi} \right) + U_\mu U^\nu \partial_\nu \phi \\
&= \phi U^\nu U^\alpha \left(g_{\mu\nu,\alpha} - g_{\mu\beta} \Gamma^\beta_{\nu\alpha} \right) - \partial_\mu \phi \\
&= \frac{1}{2} \phi U^\nu U^\alpha g_{\nu\alpha,\mu} - \partial_\mu \phi.
\end{aligned}$$

1.5.2 Massless particles

Introducing the electromagnetic Lagrangian ($\mathcal{L}_m = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu}$) in the phase (5) and varying it with respect to the 4-potential A_μ leads to the modified Maxwell equation in the vacuum:

$$\nabla_\nu \left(\phi F^{\mu\nu}\right) = 0 \Leftrightarrow \partial_\mu \left(\phi \sqrt{-g} F^{\mu\nu}\right) = 0. \tag{17}$$

Following the steps of [8] and [9], let us write the 4-potential as $A^\mu = (a^\mu + \epsilon b^\mu)\,\mathrm{e}^{i\theta/\epsilon}$. Using the Lorenz gauge ($\nabla_\mu A^\mu = 0$) and defining $k_\mu = \partial_\mu \theta$, the first two leading orders ($1/\epsilon^2$ and $1/\epsilon$) of equation (17) give :

$$k_\mu k^\nu = 0$$

$$a^\mu \nabla_\nu k^\nu + 2k^\nu \nabla_\nu a^\mu = \left(k^\mu a^\nu - k^\nu a^\mu\right) \frac{\partial_\nu \phi}{\phi}.$$

Contracting with k_μ gives :

$$k^\mu \nabla_\mu k^\nu = 0 \tag{18}$$

because at leading order, the Lorenz gauge is $k_\mu a^\mu = 0$.

(18) is the usual null-geodesic equation. It will be integrated once using the usual conserved quantities (ie. the norm of k^μ is zero and k_μ is conserved if the metric does not depend on x^μ.)

2 Motion around spherical and non-rotating star

In this part, we will use the equation of motions derived in the previous part to derive the precession of orbits and the deviation of light around a central, spherical, non-rotating and charged object. First, these quantities will be derived in General Relativity, then in Entangled Relativity.

2.1 General Relativity: Reissner-Nordström black hole

In General Relativity, the exterior solution of a spherical, non-rotating star of mass M and charge Q is given by the Reissner-Nordström metric [10]:

$$\mathrm{d}s^2 = -f(r)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{f(r)} + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\varphi^2\right), \tag{19}$$

with $f(r) = 1 - \frac{r_s}{r} + \frac{r_Q^2}{r^2}$, $r_s = \frac{2GM}{c^2}$ and $r_Q = \frac{GQ^2}{4\pi\varepsilon_0 c^4}$.

Both massive and massless test particles follow geodesic. The equations of motion can be integrated once as:

$$\begin{aligned}
\theta &= \frac{\pi}{2}\\
E &= c^2\left(1 - \frac{r_S}{r} + \frac{r_Q^2}{r^2}\right)\dot{t}\\
L &= r^2\dot{\varphi}\\
\dot{r}^2 &= \frac{E^2}{c^2} - c^2\left(1 - \frac{r_S}{r} + \frac{r_Q^2}{r^2}\right)\left(\varepsilon + \frac{L^2}{r^2c^2}\right)
\end{aligned} \tag{20}$$

where E and L are the energy and the angular momentum per unit of mass.

To study the precession of orbits or the deviation of light, the function $r(\varphi)$ is needed. $\dot{r} = r'\dot{\varphi} = L\frac{r'}{r^2}$ where a prime corresponds to $\frac{\mathrm{d}}{\mathrm{d}\varphi}$. To do so, L is assumed to be non-zero. Then, we are going to use the dimensionless variable $x = \frac{r_S}{r}$. The equation of motion for x is:

$$\begin{aligned} L^2 x'^2 &= E^2 - \left(1 - x + \alpha^2 x^2\right)\left(\varepsilon + L^2 x^2\right) \\ &= E^2 - \varepsilon + \varepsilon x - x^2\left(L^2 + \varepsilon\alpha^2\right) + L^2 x^3 - \alpha^2 L^2 x^4, \end{aligned} \tag{21}$$

after rewriting L as $\frac{L}{r_S c}$ and E as $\frac{E}{c^2}$ and with $\alpha = \frac{r_Q}{r_s}$. Differentiating again:

$$2\left(L^2 x'' + \left(L^2 + \varepsilon\alpha^2\right)x\right) = \varepsilon + 3L^2x^2 - 4\alpha^2 L^2 x^3. \tag{22}$$

2.1.1 Precession of orbits

Let us suppose that $\varepsilon = 1$. (22) becomes:

$$x'' + \left(1 + \frac{\alpha^2}{L^2}\right)x = \frac{1}{2L^2} + \frac{3}{2}x^2.$$

At first order, it can be solved as: $x_0 = \frac{1}{p}\left(1 + e\cos(\omega\phi)\right)$ with $p = 2\left(L^2 + \alpha^2\right)$, $\omega^2 = 1 + \frac{\alpha^2}{L^2}$ and e depends on α, L and E.

At second order, the solution is $x = x_0 + x_1$. x_1 is solution of $x_1'' + \omega^2 x_1 = \frac{3}{2}x_0^2$. The solution is given by:

$$x_1 = \frac{3}{2\omega^2 p^2}\left(1 + \frac{e^2}{2} + e\omega\varphi\sin(\omega\varphi) - \frac{e^2}{6}\cos(2\omega\varphi)\right).$$

Let us find the maxima of x. At first order in $1/p$, they are such that $\omega\varphi_n = 2\pi n$. At second order in $1/p$:

$$x' = -\frac{e\omega}{p}\sin\omega\varphi + \frac{3e}{2p^2\omega}\sin\omega\varphi + \frac{3e}{2p^2}\varphi\cos\omega\varphi + \frac{e^2}{2p^2\omega}\sin 2\omega\varphi.$$

$\varphi_0 = 0$ is still solution. At this order $\omega\varphi_1 = 2\pi + \delta$ is solution if: $\delta = \frac{3\pi}{p\omega}$. Then, $\varphi_1 = \frac{2\pi}{\omega} + \frac{6\pi}{2p\omega^2} = 2\pi + \frac{3\pi}{p} - \frac{\pi\alpha^2}{L^2}$. The precession of the orbit is:

$$\Delta\varphi = \frac{3\pi}{p} - \frac{\pi\alpha^2}{L^2}$$

2.1.2 Deviation of light

Let us suppose that $\varepsilon = 0$. (22) becomes:

$$x'' + x = \frac{3}{2}x^2 - 2\alpha^2 x^3.$$

At first and second orders in x, the deviation of light does not depend on the charge of the central object. It is the same as in Schwarzschild's metric. The solution at first and second order is:

$$x_0 = \frac{\sin\varphi}{b},$$

$$x_1 = \frac{3}{4b^2}\left(1 + \frac{1}{3}\cos 2\varphi\right).$$

At first order in $1/b$, $x = 0$ when $\varphi = \varphi_+ = \pi$ and $\varphi = \varphi_- = 0$. At second order in $1/b$, the zeros of x are $\varphi_+ = \pi + \delta_+$ and $\varphi_- = \delta_-$ if $\delta_\pm = \pm\frac{1}{b}$ so the total deviation is $\chi = \pi - \varphi_+ + \varphi_- = -\frac{2}{b}$.

Let us go to the next order. $x = x_0 + x_1 + x_2$ with x_0 and x_1 defined as above. x_2 is solution of $x'' + x = 3x_0x_1 - 2\alpha^2 x_0^3$ and it yields $x_2 = \frac{1}{b^3}\left(\frac{3}{4}\alpha^2 - \frac{15}{16}\right)\varphi\cos\varphi - \frac{1}{b^3}\left(\frac{\alpha^2}{8} + \frac{3}{32}\right)\sin\varphi\cos 2\varphi$.

For x to vanish, φ_+ has to be $\pi + \frac{1}{b} - \frac{\pi}{b^2}\left(\frac{3}{4}\alpha^2 - \frac{15}{16}\right)$ while φ_- does not change. The deviation is now:

$$\chi = -\frac{2}{b} + \frac{\pi}{b^2}\left(\frac{3}{4}\alpha^2 - \frac{15}{16}\right).$$

2.2 Entangled Relativity : The Jordan frame

In the Jordan frame (ie. as defined in (5), (6) and (7)), the solution of the field equation is given by [11, 12, 13]:

$$\mathrm{d}s^2 = -\lambda_0^2\mathrm{d}t^2 + \lambda_r^{-2}\mathrm{d}r^2 + \rho^2\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2\right), \tag{23}$$

$$\phi = \frac{1}{\left(1 - \frac{r_-}{r}\right)^{\frac{2}{13}}}, \tag{24}$$

$$A = -\frac{Q}{r}\mathrm{d}t, \tag{25}$$

with:

$$\begin{aligned}
\lambda_0^2 &= \left(1-\frac{r_+}{r}\right)\left(1-\frac{r_-}{r}\right)^{\frac{15}{13}},\\
\lambda_r^2 &= \left(1-\frac{r_+}{r}\right)\left(1-\frac{r_-}{r}\right)^{\frac{7}{13}},\\
\rho^2 &= r^2\left(1-\frac{r_-}{r}\right)^{\frac{6}{13}}.
\end{aligned} \tag{26}$$

r_+ and r_- are parameters related to the mass M and the charge Q of the star:

$$r_S = \frac{2GM}{c^2} = r_+ + \frac{11}{13}r_-, \qquad r_Q^2 = \frac{GQ^2}{4\pi\varepsilon_0 c^4} = \frac{12}{13}r_+r_-. \tag{27}$$

Even though it is not possible to take $r_- = 0$ because it would yield $Q = 0$ and $\mathcal{L}_m$ would vanish, if r_- tends to zero, the solution is well approximated by Schwarschild's solution.

Taking all the results shown previously into account, the equations of motion can be integrated once:

$$\begin{aligned}
\theta &= \frac{\pi}{2}\\
E &= \phi^\varepsilon \lambda_0^2 \dot{t}\\
L &= \phi^\varepsilon \rho^2 \dot{\varphi}\\
-\varepsilon &= -\frac{E^2}{\phi^{2\varepsilon}\lambda_0^2} + \frac{\dot{r}^2}{\lambda_r^2} + \frac{L^2}{\phi^{2\varepsilon}\rho^2}
\end{aligned} \tag{28}$$

with $\varepsilon = 0$ for massless particles and 1 for massive particles. E and L can be interpreted as the energy per unit of mass and the angular momentum per unit of mass. The last equation can be written as:

$$\dot{r}^2 = \frac{\lambda_r^2}{\phi^{2\varepsilon}\lambda_0^2}E^2 - \lambda_r^2\left(\varepsilon + \frac{L^2}{\phi^{2\varepsilon}\rho^2}\right). \tag{29}$$

Expanding the functions in the equations yields:

$$L = r^2\left(1-\frac{r_-}{r}\right)^{\frac{6-2\varepsilon}{13}}\dot{\varphi},$$

$$\dot{r}^2 = \frac{E^2}{c^2}\left(1-\frac{r_-}{r}\right)^{\frac{4\varepsilon-8}{13}} - c^2\left(1-\frac{r_+}{r}\right)\left(1-\frac{r_-}{r}\right)^{\frac{7}{13}}\left(\varepsilon + \frac{L^2}{c^2r^2\left(1-\frac{r_-}{r}\right)^{\frac{6-4\varepsilon}{13}}}\right).$$

After going from $r(\lambda)$ to $r(\varphi)$ (assuming $L \neq 0$), the equation of motion becomes:

$$L^2\left(1-\frac{r_-}{r}\right)^{-\frac{4}{13}}\left(\frac{r'}{r^2}\right)^2=\frac{E^2}{c^2}-c^2\left(1-\frac{r_+}{r}\right)\left(1-\frac{r_-}{r}\right)^{\frac{15-4\varepsilon}{13}}\left(\varepsilon+\frac{L^2}{c^2r^2\left(1-\frac{r_-}{r}\right)^{\frac{6-4\varepsilon}{13}}}\right). \tag{30}$$

As done in General Relativity, we will denote $x = \frac{r_S}{r}$. We also denote $u = \frac{r_+}{r_S}$ and $v = \frac{r_-}{r_S}$.

2.2.1 Precession of orbits

Let us suppose that $\varepsilon = 1$. The equation of motion becomes:

$$L^2x'^2\left(1-vx\right)^{-\frac{4}{13}}=E^2-\left(1-ux\right)\left(1-vx\right)^{\frac{11}{13}}\left(\varepsilon+L^2x^2\left(1-vx\right)^{-\frac{2}{13}}\right).$$

Differentiating and expanding this equation to the x^3 order:

$$2L^2\left(x''+\Omega^2x\right)=\varepsilon-\frac{\alpha^2L^2}{3}x'^2+L^2x^2\left(3+\frac{1}{6}\alpha^2\right)-4L^2\alpha^2x^3, \tag{31}$$

with $\Omega^2 = 1 + \frac{13\alpha^2}{12L^2}$ and we have expanded u and v in terms of α^2, meaning $u = 1 - \frac{11}{12}\alpha^2$ and $v = \frac{13}{12}\alpha^2$ and keep in this equation only terms in α^2.

At first order in x, it can be solved as $x_0 = \frac{1}{p}\left(1 + e\cos\Omega\varphi\right)$ with $p = 2L^2 + \frac{13}{6}\alpha^2$.

At second order in x, the solution is $x = x_0 + x_1$, with x_1 solution of:

$$x''+\Omega^2x=\left(\frac{3}{2}+\frac{\alpha^2}{12}\right)x_0^2-\frac{\alpha^2}{6}(x_0')^2.$$

Solving this yields:

$$x_1=C_1+C_2\varphi\sin\Omega\varphi+C_3\cos 2\Omega\varphi,$$

with

$$C_1=\frac{1}{p^2\Omega^2}\left(\left(\frac{3}{2}+\frac{\alpha^2}{12}\right)\left(\frac{e^2}{2}+1\right)-\frac{\alpha^2}{12}e^2\Omega^2\right),$$

$$C_2=\frac{e}{2p^2\Omega}\left(3+\frac{\alpha^2}{6}\right),$$

$$C_3=-\frac{1}{3p^2\Omega^2}\left(\frac{e^2}{2}\left(\frac{3}{2}+\frac{\alpha^2}{12}\right)+\frac{e^2}{12}\alpha^2\Omega^2\right).$$

At this order, $\Omega\varphi_1 = 2\pi + \delta$ is solution of the equation $x_0' + x_1' = 0$ if $\delta = \frac{3\pi}{p\Omega} + \frac{\pi\alpha^2}{6p\Omega}$ meaning that $\varphi_1 = \frac{2\pi}{\Omega} + \frac{3\pi}{p\Omega^2} + \frac{\pi\alpha^2}{6p\Omega^2} = 2\pi + \frac{3\pi}{2L^2} - \frac{\pi\alpha^2}{L^2}$. At this order of approximation (ie. first non zero order for the charge and at order $1/p$ for the solution x), the precession is the same as in General Relativity.

In [ref], a code can be found to integrate the equations of motion in both theories : General Relativity and Entangled Relativity. The code is implemented to integrate the equations in both cases, massive and massless particles.

In the massive case, an orbit with the same parameters in both theories was integrated. As the computational time was very long, only one orbit was tested. The parameters of the orbits (namely, its starting point, its starting velocity and L) were chosen to fit into the approximations made in the calculations above and to ensure a suitable computational time.

The aim of the simulation is to plot the precession of the orbit as a function of the charge of the central object (more precisely, the dimensionless parameter α). α was taken in the interval $[0, 1/2]$ as it is the only values it can take in General Relativity. It is the $|Q| < M$ limit in Reissner-Nordström metric to ensure that the singularity is hidden behind an horizon.

Once the equation of motion is integrated, a function is used to calculate the precession. It finds the moments φ and $\varphi + h$ where $x'(\varphi) > 0$ and $x'(\varphi + h) < 0$ where h is the integration step. The root is in the interval $[\varphi, \varphi + h]$ and Brent's method is used to find the root of x' in this interval. It is the periastron of the orbit. Then we have to subtract the initial value of φ, which was taken to be zero and subtract 2π.

As seen in Figure 1, the results given by the simulation are really close to the expected values. A fit was made to quantify how close the calculated values are from the expected values. The relative variation is always below one percent. The code can be found in the *Equations of Motion* notebook at [14].

2.2.2 Deviation of light

Let us suppose that $\varepsilon = 0$. The equations of motion becomes:

$$L^2 x'^2 (1 - vx)^{-\frac{4}{13}} = E^2 - L^2 x^2 (1 - ux)(1 - vx)^{\frac{9}{13}} .$$

Differentiating and expanding this equation to the x^3 order and to the α^2 order:

$$x'' + x = -\frac{\alpha^2}{6} x'^2 + x^2 \left(\frac{3}{2} + \frac{1}{12}\alpha^2 \right) - 2\alpha^2 x^3, \tag{32}$$

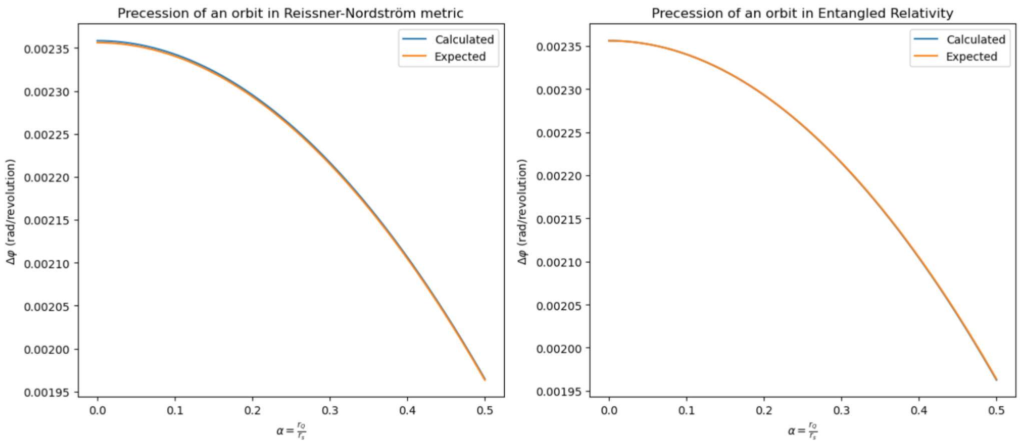

Figure 1: Precession of an orbit in General Relativity (left) and in Entangled Relativity (right) in terms of α. The blue curves are calculated thanks to the simulation, the orange curves are the expected values. The orbit is such that $x(0) = 0.003$ and $L = \sqrt{2000} \approx 44.72$.

At first and second order in x, this solves as:

$$x_0 = \frac{\sin\varphi}{b},$$

$$x_1 = \frac{1}{4b^2}\left(3 - \frac{\alpha^2}{6} + \left(1 + \frac{\alpha^2}{6}\right)\cos 2\varphi\right).$$

At first order in $1/b$, $x = 0$ when $\varphi = \varphi_+ = \pi$ and $\varphi = \varphi_- = 0$. At second order in $1/b$, the zeros of x are $\varphi_+ = \pi + \delta_+$ and $\varphi_- = \delta_-$ if $\delta_\pm = \pm\frac{1}{b}$ so the total deviation is $\chi = -\frac{2}{b}$.

Let us go to the next order in x. At this order, $x = x_0 + x_1 + x_2$ with x_0 and x_1 defined as above. x_2 is solution of $x'' + x = \left(3 + \frac{\alpha^2}{6}\right) x_0 x_1 - \frac{\alpha^2}{3} x_0' x_1' - 2\alpha^2 x_0^3$ and it yields:

$$x_2 = \frac{A}{b^3}\varphi\cos\varphi - \frac{B}{b^3}\sin\varphi\cos 2\varphi,$$

with

$$A = -\frac{\alpha^2}{24}\left(1 + \frac{\alpha^2}{6}\right) + \frac{3}{4}\alpha^2 - \frac{1}{8}\left(3 + \frac{\alpha^2}{6}\right)\left(3 - \frac{\alpha^2}{6}\right) + \frac{1}{16}\left(3 + \frac{\alpha^2}{6}\right)\left(1 + \frac{\alpha^2}{6}\right),$$

$$B = -\frac{\alpha^2}{48}\left(1 + \frac{\alpha^2}{6}\right) + \frac{1}{8}\alpha^2 + \frac{1}{32}\left(3 + \frac{\alpha^2}{6}\right)\left(1 + \frac{\alpha^2}{6}\right).$$

For x to vanish at this order, φ_+ has to be $\pi + \frac{1}{b} - \frac{\pi}{b^2}A$ while φ_- does not change. The deviation is now, at α^2 order:

$$\chi = -\frac{2}{b} + \frac{\pi}{b^2}\left(\frac{3}{4}\alpha^2 - \frac{15}{16}\right).$$

At this order of approximation (ie. in $1/b^2$ and α^2 orders), the deviation of light is also the same as in General Relativity.

In [ref], the code to integrate ray of lights can be found. It uses the same functions as in the massive case. Note that the ε in the exponent is never taken into account in the code. If we take a closer look the equation of motion, the ε dependency in the exponents is the same. Therefore, when $\epsilon = 0$, the exponents can have the same value as when $\varepsilon = 1$ and it gives the right results. This was used to simplify the code and have the same function to integrate the equation of motion in both massive and massless cases.

For the same reasons as in the massive case, only one ray of light with given parameters was simulated in both theories. The only useful parameter is the impact parameter b. It was chosen to fit the approximations made in the calculations above and to ensure a suitable computational time.

In this part, the aim is to plot the deviation of this ray of light as a function of the charge of the central object. Once the equation of motion is integrated, a function is used to compute the deviation. It finds two consecutive zeros of the solution $x(\varphi)$. This points correspond to when the ray of light is at space infinity. Taking the difference of these roots gives the deviation.

In Figure 2, the simulated and calculated deviation are really close. A fit was made to quantify how close the calculated values are from the expected values. The relative variation is always below one percent. The code can be found in the *Equations of Motion* notebook at [14].

In conclusion, the precession of orbits and the deviation of light are the same in both theories at the first order of approximation. There is no way to distinguish the two theories by looking at orbits or deviated ray of lights. It is not surprising that when the charge tends to zero, the results are the same as the solution in Entangled Relativity gives back Schwarzschild metric. It is although surprising that the charge dependenct is the same at first order.

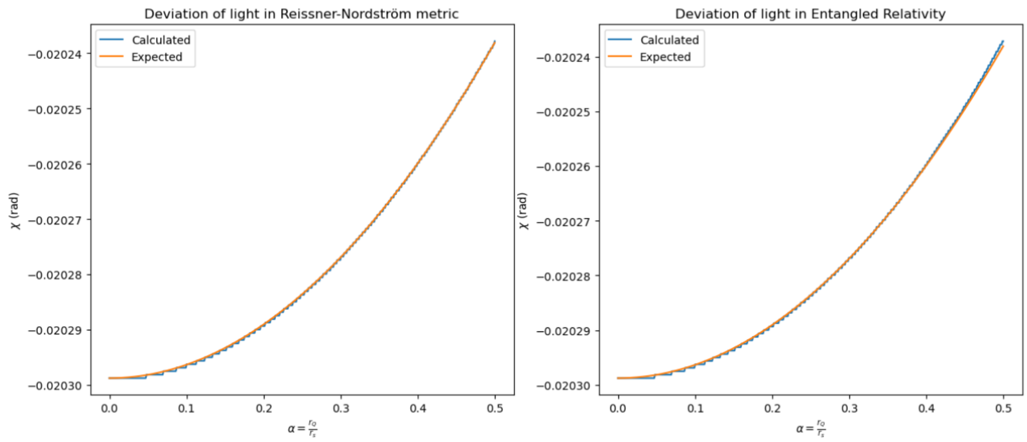

Figure 2: Deviation of a ray of light in General Relativity (left) and in Entangled Relativity (right) in terms of α. The blue curves are calculated thanks to the simulation, the orange curves are the expected values. The ray of light is such that $x(0) = 0$ and $x'(0) = 0.01$, such that $b \approx 100$.

2.3 Entangled Relativity : The Einstein frame

To confirm the previous results, one can express Entangled Relativity in the Einstein frame. In this part, we will derive the precession of orbits in the Einstein frame. Let us recall that the action of Entangled Relativity is :

$$S \propto \int \mathrm{d}^4x\sqrt{-g}\left(\frac{\phi^2 R}{2\tilde{\kappa}} + \phi\mathcal{L}_m\right).$$

After performing the conformal transformation $g_{\mu\nu} \longrightarrow g^*_{\mu\nu} = \phi^2 g_{\mu\nu}$, the action becomes [6]:

$$S = \int \mathrm{d}^4x\sqrt{-g^*}\left(\frac{R^*}{2\tilde{\kappa}} - \frac{12\beta^2}{\tilde{\kappa}}g^{*\mu\nu}\partial_\mu\xi\,\partial_\nu\xi + e^{-2\beta\xi}\mathcal{L}^*_m\right),$$

where every star denotes a quantity calculated with the star metric $g^*_{\mu\nu}$. ξ is such that $\phi = \mathrm{e}^{-2\beta\xi}$. Note that the action of electromagnetism is conformal invariant.

Taking $\beta = \frac{1}{2\sqrt{3}}$ allows an equivalence between Entangled Relativity and Einstien-Maxwell-dilaton gravitation.

In this frame, the field equations are:

$$\begin{cases} \nabla_\nu\left(\mathrm{e}^{-2\alpha\xi}F^{\mu\nu}\right) = 0 \\ \Box\xi = -\dfrac{\alpha}{2}\mathrm{e}^{-2\alpha\xi}F_{\mu\nu}F^{\mu\nu} \\ R_{\mu\nu} = 2\nabla_\mu\xi\nabla_\nu\xi + \mathrm{e}^{-2\alpha\xi}T_{\mu\nu}. \end{cases} \tag{33}$$

with $\alpha = \frac{1}{2\sqrt{3}}$.

In the Einstein frame, the solution (23) becomes [15, 12]:

$$\mathrm{d}s^2 = -f(r)\mathrm{d}t^2 + f(r)^{-1}\mathrm{d}r^2 + h(r)\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2\right), \tag{34}$$

with :

$$f(r) = \left(1 - \frac{r_+}{r}\right)\left(1 - \frac{r_-}{r}\right)^{\frac{11}{13}}$$
$$h(r) = r^2\left(1 - \frac{r_-}{r}\right)^{\frac{2}{13}}.$$

The scalar field and the 4-potential A^μ remain unchanged frome (24) and (25) under the conformal transformation.

Performing the same conformal transformation on the equation of motion for massive particles (16) gives back the usual geodesic equation:

$$U^\nu \nabla_\nu U^\mu = 0.$$

This means that in the Einstein frame, the action of a test particle is given by the usual action:

$$S \propto \int \mathrm{d}\lambda \sqrt{-g^*_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}, \tag{35}$$

where $\dot{x}^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}$. Varying this action with respect to x^μ gives the geodesic equation, provided $\lambda = a\tau + b$ with τ the proper time.

Plugging the conformal transformation into (35) yields the action of massive test particles in the Jordan frame:

$$S \propto \int \mathrm{d}\lambda \phi \sqrt{-g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}. \tag{36}$$

Varying this action with respect to x gives back the equation of motion (16).

Following the usual way to integrate once the equation of geodesic, the equations of motion are:

$$\begin{aligned} \theta &= \frac{\pi}{2} \\ E &= f(r)\dot{t} \\ L &= h(r)\dot{\varphi} \\ -1 &= -\frac{E^2}{f(r)} + \frac{\dot{r}^2}{f(r)} + \frac{L^2}{h(r)} \end{aligned} \tag{37}$$

The last equation becomes then

$$\dot{r}^2 = E^2 - f(r)\left(1 + \frac{L^2}{h(r)}\right).$$

Changing the variable of integration to φ, plugging the expressions of f and h and performing the change of variable from r to x, gives the following equation

$$L^2 x'^2 (1 - vx)^{-\frac{4}{13}} = E^2 - (1 - ux)(1 - vx)^{\frac{11}{13}}\left(1 + L^2 x^2 (1 - vx)^{-\frac{2}{13}}\right).$$

This equation is exactly the same that was derived in the Jordan frame. If we expand to x^3 and α^2 order and solve for the precession, the same result will be found.

Here again, it is the same result. The only difference, in this frame, is the expression of the conserved quantities. They depend on the proper time τ or the affine parameter λ. To get rid of the proper time dependency, we will express the equation of motion in terms of the time coordinate x^0.

2.4 Equation of motion in terms of time coordinate

Instead of using the proper time (or an affine parameter) as the variable in the equation of motion, let us change it to the coordinate $x^0 = t$.

$$\frac{\mathrm{d}x^i}{\mathrm{d}\lambda} = \frac{\mathrm{d}t}{\mathrm{d}\lambda}\frac{\mathrm{d}x^i}{\mathrm{d}t}$$

$$\frac{\mathrm{d}^2x^i}{\mathrm{d}\lambda^2} = \frac{\mathrm{d}^2x^i}{\mathrm{d}t^2}\left(\frac{\mathrm{d}t}{\mathrm{d}\lambda}\right)^2 + \frac{\mathrm{d}x^i}{\mathrm{d}t}\frac{\mathrm{d}^2t}{\mathrm{d}\lambda^2}$$

where $i = 1, 2, 3$.

In this part, we will consider a generic metric:

$$\mathrm{d}s^2 = -f\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{g} + h\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2\right), \tag{38}$$

with, f, g and h are arbitrary functions of the coordinate r.

2.4.1 Massless particles : the geodesic equation

Recall the geodesic equation:

$$\frac{\mathrm{d}^2x^\mu}{\mathrm{d}\lambda^2} = -\Gamma^\mu_{\alpha\beta}\frac{\mathrm{d}x^\alpha}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\beta}{\mathrm{d}\lambda}.$$

In terms of time coordinate, it becomes:

$$\frac{\mathrm{d}^2 x^i}{\mathrm{d}t^2} = -\Gamma^i_{\alpha\beta}\frac{\mathrm{d}x^\alpha}{\mathrm{d}t}\frac{\mathrm{d}x^\beta}{\mathrm{d}t} + \frac{\mathrm{d}x^i}{\mathrm{d}t}\Gamma^t_{\alpha\beta}\frac{\mathrm{d}x^\alpha}{\mathrm{d}t}\frac{\mathrm{d}x^\beta}{\mathrm{d}t}. \tag{39}$$

Taking $i = 2$ (ie. $x^i = \theta$), the equation can still be solved as $\theta = \frac{\pi}{2}$. From now, this is the value that is going to be used.

Taking $i = 3$ (ie., $x^i = \varphi$), equation (39) becomes:

$$\frac{\mathrm{d}^2\varphi}{\mathrm{d}t^2} = \frac{\mathrm{d}\varphi}{\mathrm{d}t}\frac{\mathrm{d}r}{\mathrm{d}t}\frac{f'}{f} - \frac{\mathrm{d}\varphi}{\mathrm{d}t}\frac{\mathrm{d}\varphi}{\mathrm{d}t}\frac{h'}{h},$$

where a prime corresponds to a derivative with respect to r. This equation can be integrated once:

$$\frac{\mathrm{d}\varphi}{\mathrm{d}t} = \frac{f}{h}L, \tag{40}$$

with L a constant.

Taking $i = 1$ (ie., $x^i = r$), equation (39) becomes:

$$\frac{\mathrm{d}^2 r}{\mathrm{d}t^2} = \left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2\frac{f'}{f} - \frac{gf'}{2} + \left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2\frac{g'}{2g} + \left(\frac{\mathrm{d}\varphi}{\mathrm{d}t}\right)^2\frac{gh'}{2}.$$

Multiplying by $\frac{2}{f^2 g}\frac{\mathrm{d}r}{\mathrm{d}t}$ and replacing $\frac{\mathrm{d}\varphi}{\mathrm{d}t}$ by (40):

$$\frac{2}{f^2 g}\frac{\mathrm{d}r}{\mathrm{d}t}\frac{\mathrm{d}^2 r}{\mathrm{d}t^2} = -\frac{f'}{f^2}\frac{\mathrm{d}r}{\mathrm{d}t} + \left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^3\left(\frac{2f'}{f^3 g} + \frac{g'}{f^2 g^2}\right) + L^2\frac{h'}{h^2}\frac{\mathrm{d}r}{\mathrm{d}t}$$

which can be integrated as:

$$K = \frac{1}{f} - \frac{L^2}{h} - \frac{1}{f^2 g}\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2,$$

with K a constant. It can be rewritten as:

$$\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 = fg - f^2 g\left(K + \frac{L^2}{h}\right). \tag{41}$$

Recall that the norm of the 4-velocity has to vanish on the trajectory:

$$0 = \left(\frac{\mathrm{d}t}{\mathrm{d}\lambda}\right)^2\left(-f + \frac{1}{g}\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 + h\left(\frac{\mathrm{d}\varphi}{\mathrm{d}t}\right)^2\right) = -\left(\frac{\mathrm{d}t}{\mathrm{d}\lambda}\right)^2 f^2 K,$$

So, we have to take $K = 0$, as f and $\frac{\mathrm{d}t}{\mathrm{d}\lambda}$ do not vanish in the general case.

From (41), it is possible to have an equation on $\frac{\mathrm{d}r}{\mathrm{d}\varphi}$:

$$\frac{L^2}{h^2}\left(\frac{\mathrm{d}r}{\mathrm{d}\varphi}\right)^2 = \frac{g}{f} - g\frac{L^2}{h}$$

$$\Leftrightarrow$$

$$\frac{f}{gh^2}\left(\frac{\mathrm{d}r}{\mathrm{d}\varphi}\right)^2 = \frac{1}{L^2} - \frac{f}{h}.$$

In General Relativity, Entangled Relativity in the Einstein frame and in the Jordan frame, this last equation is the same as the previous ones. The deviation of light still does not change here.

2.4.2 Massive particles

Recall the modified geodesic equation (16):

$$\frac{\mathrm{d}^2x^\mu}{\mathrm{d}\lambda^2} = -\Gamma^\mu_{\alpha\beta}\frac{\mathrm{d}x^\alpha}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\beta}{\mathrm{d}\lambda} - g^{\mu\nu}\frac{\partial_\nu\phi}{\phi} - \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}\frac{\partial_\nu\phi}{\phi}.$$

In terms of time coordinate, it becomes:

$$\left(\frac{\mathrm{d}t}{\mathrm{d}\lambda}\right)^2\left(\frac{\mathrm{d}^2x^i}{\mathrm{d}t^2} + \Gamma^i_{\alpha\beta}\frac{\mathrm{d}x^\alpha}{\mathrm{d}t}\frac{\mathrm{d}x^\beta}{\mathrm{d}t} - \frac{\mathrm{d}x^i}{\mathrm{d}t}\Gamma^t_{\alpha\beta}\frac{\mathrm{d}x^\alpha}{\mathrm{d}t}\frac{\mathrm{d}x^\beta}{\mathrm{d}t}\right) = -g^{i\nu}\frac{\partial_\nu\phi}{\phi}. \qquad (42)$$

One can notice that the terms in $\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}\frac{\partial_\nu\phi}{\phi}$ cancel each other out from the equation for $\frac{\mathrm{d}^2t}{\mathrm{d}\lambda^2}$ and $\frac{\mathrm{d}^2x^i}{\mathrm{d}\lambda^2}$. The same metric as in the massless case is considered here. The scalar field ϕ depends only on the variable r, that is why the term in $g^{t\nu}\frac{\partial_\nu\phi}{\phi}$ vanishes.

Taking $i = 2$ (ie. $x^i = \theta$), the equation can still be solved as $\theta = \frac{\pi}{2}$. From now, this is the value that is going to be used.

Taking $i = 3$ (ie., $x^i = \varphi$), equation (42) becomes:

$$\frac{\mathrm{d}^2\varphi}{\mathrm{d}t^2} = \frac{\mathrm{d}\varphi}{\mathrm{d}t}\frac{\mathrm{d}r}{\mathrm{d}t}\frac{f'}{f} - \frac{\mathrm{d}\varphi}{\mathrm{d}t}\frac{\mathrm{d}\varphi}{\mathrm{d}t}\frac{h'}{h}.$$

This equation can be integrated once:

$$\frac{\mathrm{d}\varphi}{\mathrm{d}t} = \frac{f}{h}L, \qquad (43)$$

with L a constant.

Taking $i = 1$ (ie., $x^i = r$), equation (39) becomes:

$$\left(\frac{\mathrm{d}t}{\mathrm{d}\lambda}\right)^2 \left(\frac{\mathrm{d}^2 r}{\mathrm{d}t^2} - \left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 \frac{f'}{f} + \frac{gf'}{2} - \left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 \frac{g'}{2g} - \left(\frac{\mathrm{d}\varphi}{\mathrm{d}t}\right)^2 \frac{gh'}{2}\right) = -g\frac{\phi'}{\phi}.$$

Multiplying by $\frac{2}{f^2 g}\frac{\mathrm{d}r}{\mathrm{d}t}$ and replacing $\frac{\mathrm{d}\varphi}{\mathrm{d}t}$ by (43):

$$\left(\frac{\mathrm{d}t}{\mathrm{d}\lambda}\right)^2 \frac{\mathrm{d}K}{\mathrm{d}t} = \frac{2}{f^2}\frac{\mathrm{d}}{\mathrm{d}t}\log\phi.$$

with $K = \frac{1}{f} - \frac{L^2}{h} - \frac{1}{f^2 g}\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2$, not necessarily a constant.

If proper time is defined such that the norm of the velocity is -1, then we have the equation:

$$1 = \left(\frac{\mathrm{d}t}{\mathrm{d}\lambda}\right)^2 f^2 K$$

Then the equation of motion can be integrated as:

$$K = K_0 \phi^2$$

If we redefine affinely λ then we can set $K_0 = 1$.

The equation is then:

$$\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 = fg - f^2 g\phi^2\left(1 + \frac{L^2}{h\phi^2}\right) \tag{44}$$

From (44), it is possible to have an equation on $\frac{\mathrm{d}r}{\mathrm{d}\varphi}$:

$$\frac{L^2}{h^2}\left(\frac{\mathrm{d}r}{\mathrm{d}\varphi}\right)^2 = \frac{g}{f} - g\phi^2\left(1 + \frac{L^2}{h\phi^2}\right)$$

$$\Leftrightarrow$$

$$\frac{f}{gh^2}L\left(\frac{\mathrm{d}r}{\mathrm{d}\varphi}\right)^2 = 1 - f\phi^2\left(1 + \frac{L^2}{h\phi^2}\right). \tag{45}$$

In General Relativity, $f = g$ and $h = r^2$ as defined in (19). (45) gives back the right equation.

In Entangled relativity, in the Jordan frame, f, g and h are given by (26). If these are implemented in (45), it gives back (30).

In the Einstein frame, the term in ϕ in equation (42) is not here. So the equation on r becomes:

$$\frac{\mathrm{d}K}{\mathrm{d}t} = 0.$$

K is a constant and by a affine redefinition of λ, it can be equal to 1. The equation can then be rewritten as:

$$\frac{f}{gh^2} L \left(\frac{\mathrm{d}r}{\mathrm{d}\varphi}\right)^2 = 1 - f\left(1 + \frac{L^2}{h}\right). \tag{46}$$

and with $f = g$ and h defined in (34), it gives back the same equation of motion as in section 2.3.

In conclusion, the equations of motion are the same in all cases. Whatever way used to derive them yields the same precession of orbits and the same deviation of rays of light. In this section, as we do not take into account the proper time in the first place, we get an unambiguous definition of the conserved quantity L which can be interpreted as the angular momentum.

There is still the question of which of the Einstein frame or the Jordan frame proper times is the one the particle is experiencing. Proper time should be then derived from first principles, by writing the action of fermions [16]:

$$\Theta = \int \mathrm{d}^4x \frac{e^2}{\epsilon\kappa} \overline{\Psi} \left(i\not{D} - \frac{1}{\lambda_C}\right) \Psi,$$

and study how the scalar field κ influenced the mass of the fermions, even through the Higgs mechanism.

Conclusion

Even if Entangled Relativity is defined in a way that is very different from General Relativity, I failed at differentiating the theories in the aspects I studied during this internship. First of all, we thought about gravitational waves, but that seemed a complicated task for such a short amount of time, so we focused on the motion of test particles. I studied the precession of orbits and the deviation of light around a spherical, static and charged object. At the first non-zero order of approximation in the charge of the central object, which is that of our measurements, the two theories agree. The results have been verified in the Einstein frame and in the Jordan frame, and even by getting rid of proper time by considering the equations in terms of time coordinate. Finally, it would seem that more research is needed on the proper time of test particles, which would have to be derived from first principles to find out what proper time is experienced by the test particle.

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