Contents

1 Entangled Relativity

1.1 Path integral formulation

Entangled relativity can be defined through a path integral:

$$Z = \int \mathcal{D}g \prod_i \mathcal{D}f_i \exp\left(-\frac{i}{2\epsilon^2}\int \mathrm{d}^4x\sqrt{-g}\frac{\mathcal{L}_m^2}{R}\right), \tag{1}$$

where R is the usual scalar curvature constructed upon the metric $g_{\mu\nu}$ and its derivatives, $\mathcal{L}_m$ is the Lagrangian density of matter fields f_i. g is the metric determinant, ϵ is a constant that has the dimension of an energy. $\int\mathcal{D}$ denotes the integration over all possible and non-redundant field configurations.

It is easy to see that this theory verifies Mach's principle only by the way it is written. Indeed, if $\mathcal{L}_m = \emptyset$, then the phase within the integral vanishes and thus, the theory cannot be defined without matter fields.

The only fundamental constants of this theory are c and ϵ. From these, it is impossible to construct an equivalent to the Planck length. A priori, this length scale does not play any role in the structure of space-time.

1.2 Scalar-tensor equivalent form and field equations

Classically, the field equations are given by the condition $\delta\Theta = 0$ where Θ is the phase in the path integral:

$$\Theta = -\frac{1}{2\epsilon^2}\int \mathrm{d}^4x\sqrt{-g}\frac{\mathcal{L}_m^2}{R}. \tag{2}$$

One can check that ϵ does not influence the classical phenomenology of the theory.

The equation of motion reads:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = -\frac{R}{\mathcal{L}_m}T_{\mu\nu} + \frac{R^2}{\mathcal{L}_m^2}\left(\nabla_\mu\nabla_\nu - g_{\mu\nu}\Box\right)\frac{\mathcal{L}_m^2}{R^2}, \tag{3}$$

where $R_{\mu\nu}$ is the usual Ricci tensor and $T_{\mu\nu}$ is the stress-energy tensor, defined by:

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta\left(\sqrt{-g}\mathcal{L}_m\right)}{\delta g^{\mu\nu}}. \tag{4}$$

The equation of motion can also be derived from the following phase:

$$\begin{aligned}\Theta' &= \frac{1}{\epsilon^2} \int \mathrm{d}^4x \sqrt{-g} \frac{1}{\kappa} \left(\frac{R}{2\kappa} + \mathcal{L}_m \right) \\ &= \frac{1}{\epsilon^2 \tilde{\kappa}} \int \mathrm{d}^4x \sqrt{-g} \left(\frac{\phi^2 R}{2\tilde{\kappa}} + \phi \mathcal{L}_m \right),\end{aligned} \tag{5}$$

provided $\mathcal{L}_m \neq \emptyset$. $\tilde{\kappa}$ is a constant. κ and ϕ are scalar fields related by $\frac{\tilde{\kappa}}{\kappa} = \phi$.

The two formulations are equivalent on-shell. Indeed, the equation of motion for κ is:

$$\kappa = -\frac{R}{\mathcal{L}_m}. \tag{6}$$

By plugging this back into (5), one gets (2).

The fields equations are:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -\frac{\tilde{\kappa}}{\phi} T_{\mu\nu} + \frac{1}{\phi^2} \left(\nabla_\mu \nabla_\nu - g_{\mu\nu} \Box \right) \phi^2, \tag{7}$$

$$\phi = -\tilde{\kappa} \frac{\mathcal{L}_m}{R}. \tag{8}$$

Let us note that the stress-energy tensor is not conserved in general as one has:

$$\nabla_\nu T^{\mu\nu} = \left(g^{\mu\nu} \mathcal{L}_m - T^{\mu\nu} \right) \frac{\nabla_\nu \phi}{\phi}. \tag{9}$$

1.3 Motion of test particles in Entangled Relativity

1.3.1 Massive particles

The stress-energy tensor for non-interactive massive particles is given by $T^{\mu\nu} = \rho U^\alpha U^\beta$, with $U^\alpha = \frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau}$ and τ is the proper time. Noting that $\mathcal{L}_m = -\rho$ and that the matter fluid current is conserved: $\nabla_\mu \left(\rho U^\mu \right) = 0$, one gets from (9):

$$U^\mu \nabla_\mu U^\nu = -\left(g^{\nu\sigma} + U^\nu U^\sigma \right) \frac{\partial_\sigma \phi}{\phi}. \tag{10}$$

As $U^\mu \nabla_\mu U^\nu = \frac{\mathrm{d}U^\nu}{\mathrm{d}\tau} + \Gamma^\nu_{\alpha\beta} U^\alpha U^\beta$, the equation of motion is different than the geodesic equation. Nonetheless, one can prove two things about solutions of this equation.

The norm of the velocity is conserved: Let us write $K = U^\mu U_\mu$, then:

$$\begin{aligned}\frac{\mathrm{d}K}{\mathrm{d}\tau} &= U^\mu U^\nu U^\alpha g_{\mu\nu,\alpha} + 2 g_{\mu\nu} U^\mu \frac{\mathrm{d}U^\nu}{\mathrm{d}\tau} \\ &= U^\mu U^\nu U^\alpha g_{\mu\nu,\alpha} - 2 g_{\mu\nu} U^\mu \left(\Gamma^\nu_{\alpha\beta} U^\alpha U^\beta + \left(g^{\nu\sigma} + U^\nu U^\sigma \right) \frac{\partial_\sigma \phi}{\phi} \right) \\ &= U^\mu U^\nu U^\alpha \left(g_{\mu\nu,\alpha} - 2 g_{\mu\beta} \Gamma^\beta_{\alpha\nu} \right) - 2 \left(K + 1 \right) U^\sigma \frac{\partial_\sigma \phi}{\phi} \\ &= -2 \left(K + 1 \right) \frac{\mathrm{d} \log \phi}{\mathrm{d}\tau},\end{aligned}$$

which can be solved as $K = -1 + \frac{A}{\phi^2}$ with A a constant. If at initial time, we choose $K = -1$ then it remains equal to -1 for all time.

Quantity conserved: If both the metric and the scalar field do not depend on a coordinate x^μ, then ϕU_μ is conserved:

$$\begin{aligned}\frac{\mathrm{d}\phi U_\mu}{\mathrm{d}\tau} &= \phi U^\nu U^\alpha g_{\mu\nu,\alpha} + \phi g_{\mu\nu}\frac{\mathrm{d}U^\nu}{\mathrm{d}\tau} + U_\mu U^\nu \partial_\nu \phi \\ &= \phi U^\nu U^\alpha g_{\mu\nu,\alpha} - \phi g_{\mu\nu}\left(\Gamma^\nu_{\alpha\beta}U^\alpha U^\beta + \left(g^{\nu\sigma} + U^\nu U^\sigma\right)\frac{\partial_\sigma \phi}{\phi}\right) + U_\mu U^\nu \partial_\nu \phi \\ &= \phi U^\nu U^\alpha \left(g_{\mu\nu,\alpha} - g_{\mu\beta}\Gamma^\beta_{\nu\alpha}\right) - \partial_\mu \phi \\ &= \frac{1}{2}\phi U^\nu U^\alpha g_{\nu\alpha,\mu} - \partial_\mu \phi.\end{aligned}$$

1.3.2 Massless particles

Introducing the electromagnetic Lagrangian in the phase (5) and varying it with respect to the 4-potential A_μ leads to the modified Maxwell equation in the vacuum:

$$\nabla_\nu\left(\phi F^{\mu\nu}\right) = 0 \Leftrightarrow \partial_\mu\left(\phi\sqrt{-g}F^{\mu\nu}\right) = 0. \tag{11}$$

Let us write the 4-potential as $A^\mu = a^\mu \mathrm{e}^{i\theta}$. Then $F^{\mu\nu} = i\mathrm{e}^{i\theta}\left(a^\mu k^\nu - a^\nu k^\mu\right)$, where $k_\mu = \partial_\mu \theta$. Let us work in normal coordinates. The Lorenz gauge reads $0 = \partial_\mu A^\mu = i a^\mu k_\mu \mathrm{e}^{i\theta}$.

$$\begin{aligned}0 &= \partial_\mu\left(\phi \mathrm{e}^{i\theta}\left(a^\mu k^\nu - a^\nu k^\mu\right)\right) \\ &= \partial_\mu \phi \mathrm{e}^{i\theta}\left(a^\mu k^\nu - a^\nu k^\mu\right) + i\phi k_\mu \mathrm{e}^{i\theta}\left(a^\mu k^\nu - a^\nu k^\mu\right) + \phi \mathrm{e}^{i\theta}\left(a^\mu \partial_\mu k^\nu - a^\nu \partial_\mu k^\mu\right) \\ &= \mathrm{e}^{i\theta}\left[\partial_\mu \phi\left(a^\mu k^\nu - a^\nu k^\mu\right) - i\phi a^\nu k^\mu k_\mu + \phi\left(a^\mu \partial_\mu k^\nu - a^\nu \partial_\mu k^\mu\right)\right] \\ &= \mathrm{e}^{i\theta} a^\mu\left[\partial_\mu \phi k^\nu - k^\alpha \partial_\alpha \phi \delta^\nu_\mu - i\phi \delta^\nu_\mu k^\alpha k_\alpha + \phi \partial_\mu k^\nu - \phi \delta^\nu_\mu \partial_\alpha k^\alpha\right].\end{aligned}$$

After simplifying by $\mathrm{e}^{i\theta} a^\mu$, taking the real part and the imaginary part and recovering general coordinates, the previous equation yields:

$$k_\alpha k^\alpha = 0, \tag{12}$$

$$k^\nu \partial_\mu \phi - \delta^\nu_\mu k^\alpha \partial_\alpha \phi + \phi \nabla_\mu k^\nu - \phi \delta^\nu_\mu \nabla_\alpha k^\alpha = 0. \tag{13}$$

Contracting by either k_ν or k^μ yields:

$$\begin{aligned}\phi k_\mu \nabla_\alpha k^\alpha &= -k_\mu k^\alpha \partial_\alpha \phi, \\ k^\nu \nabla_\alpha k^\alpha &= k^\mu \nabla_\mu k^\nu.\end{aligned}$$

Finally this yields the equation of motion:

$$k^\mu \nabla_\mu k^\nu = g^{\mu\nu} k_\mu \nabla_\alpha k^\alpha = -g^{\mu\nu} k_\mu k^\alpha \frac{\partial_\alpha \phi}{\phi} = -k^\nu k^\alpha \frac{\partial_\alpha \phi}{\phi}. \tag{14}$$

Similar properties as in the massive case can be shown for the massless case:

The norm of the velocity is conserved: Let us write $K = k^\mu k_\mu$ and define the affine parameter λ such that $\frac{\mathrm{d}}{\mathrm{d}\lambda} = k^\mu \partial_\mu$ then:

$$\begin{aligned}\frac{\mathrm{d}K}{\mathrm{d}\lambda} &= k^\mu k^\nu k^\alpha g_{\mu\nu,\alpha} + 2g_{\mu\nu} k^\mu \frac{\mathrm{d}k^\nu}{\mathrm{d}\lambda} \\ &= k^\mu k^\nu k^\alpha g_{\mu\nu,\alpha} - 2g_{\mu\nu} k^\mu\left(\Gamma^\nu_{\alpha\beta} k^\alpha k^\beta + k^\nu k^\sigma \frac{\partial_\sigma \phi}{\phi}\right) \\ &= k^\mu k^\nu k^\alpha\left(g_{\mu\nu,\alpha} - 2g_{\mu\beta}\Gamma^\beta_{\alpha\nu}\right) - 2K k^\sigma \frac{\partial_\sigma \phi}{\phi} \\ &= -2K \frac{\mathrm{d}\log \phi}{\mathrm{d}\lambda},\end{aligned}$$

which can be solved as $K = \frac{A}{\phi^2}$ with A a constant. If at initial time, we choose $K = 0$ then it remains equal to 0 for all time.

Quantity conserved: If the metric does not depend on a coordinate x^μ, then ϕk_μ is conserved:

$$\begin{aligned}
\frac{\mathrm{d}\phi k_\mu}{\mathrm{d}\lambda} &= \phi k^\nu k^\alpha g_{\mu\nu,\alpha} + \phi g_{\mu\nu}\frac{\mathrm{d}k^\nu}{\mathrm{d}\lambda} + k_\mu k^\nu \partial_\nu \phi \\
&= \phi k^\nu k^\alpha g_{\mu\nu,\alpha} - \phi g_{\mu\nu}\left(\Gamma^\nu_{\alpha\beta}k^\alpha k^\beta + k^\nu k^\sigma \frac{\partial_\sigma \phi}{\phi}\right) + k_\mu k^\nu \partial_\nu \phi \\
&= \phi k^\nu k^\alpha \left(g_{\mu\nu,\alpha} - g_{\mu\beta}\Gamma^\beta_{\nu\alpha}\right) \\
&= \frac{1}{2}\phi k^\nu k^\alpha g_{\nu\alpha,\mu}.
\end{aligned}$$

2 Motion around spherical and non-rotating star

2.1 The metric

To ensure that $\mathcal{L}_m \neq \emptyset$, let us suppose that the star is electrically charged, inducing an electromagnetic field outside of it.

$$\mathcal{L}_m = -\frac{1}{2}F_{\mu\nu}F^{\mu\nu}, \tag{15}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \tag{16}$$

with A_μ the 4-potential. Another field equation must be added, it is obtained by varying (5) with respect to A_μ:

$$\nabla_\nu \left(\phi F^{\mu\nu}\right) = 0. \tag{17}$$

The solution of the field equations is given by:

$$\mathrm{d}s^2 = -\lambda_0^2 \mathrm{d}t^2 + \lambda_r^{-2}\mathrm{d}r^2 + \rho^2\left(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\varphi^2\right), \tag{18}$$

$$\phi = \frac{1}{\left(1 - \frac{r_-}{r}\right)^{\frac{2}{13}}}, \tag{19}$$

$$A = -\frac{Q}{r}\mathrm{d}t, \tag{20}$$

with:

$$\begin{aligned}
\lambda_0^2 &= \left(1 - \frac{r_+}{r}\right)\left(1 - \frac{r_-}{r}\right)^{\frac{15}{13}}, \\
\lambda_r^2 &= \left(1 - \frac{r_+}{r}\right)\left(1 - \frac{r_-}{r}\right)^{\frac{7}{13}}, \\
\rho^2 &= r^2\left(1 - \frac{r_-}{r}\right)^{\frac{6}{13}}.
\end{aligned} \tag{21}$$

r_+ and r_- are parameters related to the mass M and the charge Q of the star:

$$r_S = \frac{2GM}{c^2} = r_+ + \frac{11}{13}r_-, \qquad r_Q^2 = \frac{GQ^2}{4\pi\varepsilon_0 c^4} = \frac{12}{13}r_+ r_-. \tag{22}$$

Even though it is not possible to take $r_- = 0$ because it would yield $Q = 0$ and $\mathcal{L}_m$ would vanish, if r_- tends to zero, the solution is well approximated by Schwarschild's solution.

2.2 Solving the equations of motion

Taking all the results shown previously into account, the equations of motion can be integrated once:

$$\begin{aligned} \theta &= \frac{\pi}{2} \\ E &= \phi\lambda_0^2\dot{t} \\ L &= \phi\rho^2\dot{\varphi} \\ -\varepsilon &= -\frac{E^2}{\phi^2\lambda_0^2} + \frac{\dot{r}^2}{\lambda_r^2} + \frac{L^2}{\phi^2\rho^2} \end{aligned} \tag{23}$$

with $\varepsilon = 0$ for massless particles and 1 for massive particles. The last equation can be written as:

$$\dot{r}^2 = \frac{\lambda_r^2}{\phi^2\lambda_0^2}E^2 - \lambda_r^2\left(\varepsilon + \frac{L^2}{\phi^2\rho^2}\right). \tag{24}$$

2.3 General Relativity: Reissner-Nordström black hole

The equations of motion for a test particle around a Reissner-Nordström black hole are:

$$\begin{aligned} \theta &= \frac{\pi}{2} \\ E &= c^2\left(1 - \frac{r_S}{r} + \frac{r_Q^2}{r^2}\right)\dot{t} \\ L &= r^2\dot{\varphi} \\ \dot{r}^2 &= \frac{E^2}{c^2} - c^2\left(1 - \frac{r_S}{r} + \frac{r_Q^2}{r^2}\right)\left(\varepsilon + \frac{L^2}{r^2c^2}\right) \end{aligned} \tag{25}$$

with E and L are the energy and the angular momentum per unit of mass. $r_s = \frac{2GM}{c^2}$ is the Schwarzschild radius and $r_Q^2 = \frac{GQ^2}{4\pi\varepsilon_0 c^4}$ is related to the charge of the black hole.

To study the precession of orbits or the deviation of light, the function $r(\varphi)$ is needed. $\dot{r} = r'\dot{\varphi} = L\frac{r'}{r^2}$ where a prime corresponds to $\frac{\mathrm{d}}{\mathrm{d}\varphi}$. To do so, L is assumed to be non-zero. Then, we are going to use the dimensionless variable $x = \frac{r_S}{r}$. The equation of motion for x is:

$$\begin{aligned} L^2x'^2 &= E^2 - \left(1 - x + \alpha^2x^2\right)\left(\varepsilon + L^2x^2\right) \\ &= E^2 - \varepsilon + \varepsilon x - x^2\left(L^2 + \varepsilon\alpha^2\right) + L^2x^3 - \alpha^2L^2x^4, \end{aligned} \tag{26}$$

after rewriting L as $\frac{L}{r_S c}$ and E as $\frac{E}{c^2}$ and with $\alpha = \frac{r_Q}{r_s}$. Differentiating again:

$$2\left(L^2x'' + \left(L^2 + \varepsilon\alpha^2\right)x\right) = \varepsilon + 3L^2x^2 - 4\alpha^2L^2x^3. \tag{27}$$

2.3.1 Precession of orbits

Let us suppose that $\varepsilon = 1$. After noticing that the last term in (27) is negligible here, the equation becomes:

$$x'' + \left(1 + \frac{\alpha^2}{L^2}\right)x = \frac{1}{2L^2} + \frac{3}{2}x^2.$$

At first order, it can be solved as: $x_0 = \frac{1}{p}\left(1 + e\cos(\omega\phi)\right)$ with $p = 2\left(L^2 + \alpha^2\right)$, $\omega^2 = 1 + \frac{\alpha^2}{L^2}$ and e depends on α, L and E.

At second order, the solution is $x = x_0 + x_1$, with

$$x_1 = \frac{3}{2\omega^2p^2}\left(1 + \frac{e^2}{2} + e\omega\varphi\sin(\omega\varphi) - \frac{e^2}{6}\cos(2\omega\varphi)\right).$$

Let us find the maximum of x. At first order, they are such that $\omega\varphi_n = 2\pi n$. At second order:

$$x' = -\frac{e\omega}{p}\sin\omega\varphi + \frac{3e}{2p^2\omega}\sin\omega\varphi + \frac{3e}{2p^2}\varphi\cos\omega\varphi + \frac{e^2}{2p^2\omega}\sin 2\omega\varphi.$$

$\varphi_0 = 0$ is still solution. At second order $\omega\varphi_1 = 2\pi + \delta$ is solution if: $\delta = \frac{3\pi}{p\omega}$. So, $\varphi_1 = \frac{2\pi}{\omega} + \frac{6\pi}{2p\omega^2} = 2\pi + \frac{3\pi}{p} - \frac{\pi\alpha^2}{L^2}$.

2.3.2 Deviation of light

Let us suppose that $\varepsilon = 0$. (27) becomes:

$$x'' + x = \frac{3}{2}x^2 - 2\alpha^2 x^3.$$

At first order, the deviation of light does not depend on the charge of the black hole. It is the same as in Schwarzschild's metric. The solution at first and second order is:

$$x_0 = \frac{\sin\varphi}{b},$$

$$x_1 = \frac{3}{4b^2}\left(1 + \frac{1}{3}\cos 2\varphi\right).$$

At first order, $x = 0$ when $\varphi = \varphi_+ = \pi$ and $\varphi = \varphi_- = 0$. At second order, the zeroes of x are $\varphi_+ = \pi + \delta_+$ and $\varphi_- = \delta_-$ if $\delta_\pm = \pm\frac{1}{b}$ so the total deviation is $\chi = \pi - \varphi_+ + \varphi_- = -\frac{2}{b}$.

Let us go to the next order. At this order, $x = x_0 + x_1 + x_2$ with x_0 and x_1 defined as above. x_2 is solution of $x'' + x = 3x_0x_1 - 2\alpha^2 x_0^3$ and it yields $x_2 = \frac{1}{b^3}\left(\frac{3}{4}\alpha^2 - \frac{15}{16}\right)\varphi\cos\varphi - \frac{1}{b^3}\left(\frac{\alpha^2}{8} + \frac{3}{32}\right)\sin\varphi\cos 2\varphi$.

For x to vanish at this order, φ_+ has to be $\pi + \frac{1}{b} - \frac{\pi}{b^2}\left(\frac{3}{4}\alpha^2 - \frac{15}{16}\right)$ while φ_- does not change. The deviation is now:

$$\chi = -\frac{2}{b} + \frac{\pi}{b^2}\left(\frac{3}{4}\alpha^2 - \frac{15}{16}\right).$$

2.4 Entangled Relativity: the charged black hole

As done in the previous section, the precession of orbits and the deviation of light are going to be calculated around a charged black hole of entangled relativity. Recall the equations of motion (23):

$$L = r^2\left(1 - \frac{r_-}{r}\right)^{\frac{4}{13}}\dot{\varphi},$$

$$\left(\left(1 - \frac{r_-}{r}\right)^{\frac{2}{13}}\dot{r}\right)^2 = \frac{E^2}{c^2} - c^2\left(1 - \frac{r_+}{r}\right)\left(1 - \frac{r_-}{r}\right)^{\frac{11}{13}}\left(\varepsilon + \frac{L^2}{c^2r^2\left(1 - \frac{r_-}{r}\right)^{\frac{2}{13}}}\right).$$

After going from $r(\lambda)$ to $r(\varphi)$ (assuming $L \neq 0$), the equation of motion becomes:

$$L^2\left(1 - \frac{r_-}{r}\right)^{-\frac{4}{13}}\left(\frac{r'}{r^2}\right)^2 = \frac{E^2}{c^2} - c^2\left(1 - \frac{r_+}{r}\right)\left(1 - \frac{r_-}{r}\right)^{\frac{11}{13}}\left(\varepsilon + \frac{L^2}{c^2r^2\left(1 - \frac{r_-}{r}\right)^{\frac{2}{13}}}\right).$$

As done in General Relativity, let us denote $x = \frac{r_S}{r}$. We also denote $u = \frac{r_+}{r_S}$ and $v = \frac{r_-}{r_S}$:

$$L^2x'^2(1 - vx)^{-\frac{4}{13}} = E^2 - (1 - ux)(1 - vx)^{\frac{11}{13}}\left(\varepsilon + L^2x^2(1 - vx)^{-\frac{2}{13}}\right).$$

Differentiating and expanding this equation to the x^3 order:

$$2L^2\left(x''+\Omega^2 x\right)=\varepsilon-\frac{\alpha^2L^2}{3}x'^2+L^2x^2\left(3+\frac{1}{6}\alpha^2\right)-4L^2\alpha^2x^3, \tag{28}$$

with $\Omega^2=1+\varepsilon\frac{13\alpha^2}{12L^2}$ and we have expanded u and v in terms of α^2, meaning $u=1-\frac{11}{12}\alpha^2$ and $v=\frac{13}{12}\alpha^2$ and keep only terms in α^2.

2.4.1 Precession of orbits

Let us suppose that $\varepsilon=1$. (28) becomes:

$$x''+\left(1+\frac{13\alpha^2}{12L^2}\right)x=\frac{1}{2L^2}+\left(\frac{3}{2}+\frac{\alpha^2}{12}\right)x^2-\frac{\alpha^2}{6}x'^2-2\alpha^2x^3.$$

At first order, it can be solved as $x_0=\frac{1}{p}\left(1+e\cos\Omega\varphi\right)$ with $p=2L^2+\frac{13}{6}\alpha^2$.
At second order, the solution is $x=x_0+x_1$, with x_1 solution of:

$$x''+\Omega^2x=\left(\frac{3}{2}+\frac{\alpha^2}{12}\right)x^2-\frac{\alpha^2}{6}x'^2.$$

Solving this yields:

$$x_1=C_1+C_2\varphi\sin\Omega\varphi+C_3\cos2\Omega\varphi,$$

with

$$C_1=\frac{1}{p^2\Omega^2}\left(\left(\frac{3}{2}+\frac{\alpha^2}{12}\right)\left(\frac{e^2}{2}+1\right)-\frac{\alpha^2}{12}e^2\Omega^2\right),$$

$$C_2=\frac{e}{2p^2\Omega}\left(3+\frac{\alpha^2}{6}\right),$$

$$C_3=-\frac{1}{3p^2\Omega^2}\left(\frac{e^2}{2}\left(\frac{3}{2}+\frac{\alpha^2}{12}\right)+\frac{e^2}{12}\alpha^2\Omega^2\right).$$

At this order, $\Omega\varphi_1=2\pi+\delta$ is solution of the equation $x_0'+x_1'=0$ if $\delta=\frac{3\pi}{p\Omega}+\frac{\pi\alpha^2}{6p\Omega}$ meaning that $\varphi_1=\frac{2\pi}{\Omega}+\frac{3\pi}{p\Omega^2}+\frac{\pi\alpha^2}{6p\Omega^2}=2\pi+\frac{3\pi}{2L^2}-\frac{\pi\alpha^2}{L^2}$. At this order of approximation, the precession is the same as in General Relativity.

2.4.2 Deviation of light

Let us suppose that $\varepsilon=0$, (28) becomes:

$$x''+x=\left(\frac{3}{2}+\frac{\alpha^2}{12}\right)x^2-\frac{\alpha^2}{6}x'^2-2\alpha^2x^3.$$

At first and second order, this solves as:

$$x_0=\frac{\sin\varphi}{b},$$

$$x_1=\frac{1}{4b^2}\left(3-\frac{\alpha^2}{6}+\left(1+\frac{\alpha^2}{6}\right)\cos2\varphi\right).$$

At first order, $x=0$ when $\varphi=\varphi_+=\pi$ and $\varphi=\varphi_-=0$. At second order, the zeroes of x are $\varphi_+=\pi+\delta_+$ and $\varphi_-=\delta_-$ if $\delta_\pm=\pm\frac{1}{b}$ so the total deviation is $\chi=-\frac{2}{b}$.

Let us go to the next order. At this order, $x = x_0 + x_1 + x_2$ with x_0 and x_1 defined as above. x_2 is solution of $x'' + x = \left(3 + \frac{\alpha^2}{6}\right) x_0 x_1 - \frac{\alpha^2}{3} x_0' x_1' - 2\alpha^2 x_0^3$ and it yields:

$$x_2 = \frac{A}{b^3} \varphi \cos\varphi - \frac{B}{b^3} \sin\varphi \cos 2\varphi,$$

with

$$A = -\frac{\alpha^2}{24}\left(1 + \frac{\alpha^2}{6}\right) + \frac{3}{4}\alpha^2 - \frac{1}{8}\left(3 + \frac{\alpha^2}{6}\right)\left(3 - \frac{\alpha^2}{6}\right) + \frac{1}{16}\left(3 + \frac{\alpha^2}{6}\right)\left(1 + \frac{\alpha^2}{6}\right),$$

$$B = -\frac{\alpha^2}{48}\left(1 + \frac{\alpha^2}{6}\right) + \frac{1}{8}\alpha^2 + \frac{1}{32}\left(3 + \frac{\alpha^2}{6}\right)\left(1 + \frac{\alpha^2}{6}\right).$$

For x to vanish at this order, φ_+ has to be $\pi + \frac{1}{b} - \frac{\pi}{b^2} A$ while φ_- does not change. The deviation is now, at α^2 order:

$$\chi = -\frac{2}{b} + \frac{\pi}{b^2}\left(\frac{3}{4}\alpha^2 - \frac{15}{16}\right).$$

At this order of approximation, the deviation of light is also the same as in General Relativity.