EEN020: Home assignment 3

Exercise 1: The Fundamental Matrix

The fundamental matrix is

$$F = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ -2 & -2 & 0 \end{bmatrix}$$

The epipolar line in the second image generated by x is

$$l = \begin{bmatrix} 2 & 0 & -4 \end{bmatrix}$$

The points $(2, 0)$ and $(2, 1)$ could be projections of the same point $\mathbf{X}$ into P_2.

Exercise 2: The Fundamental Matrix

The camera centers are the null space of the camera matrices.

$$C_1^T = \text{Null}(P_1) \Longrightarrow P_1 C_1^T = 0$$

$$\begin{bmatrix} \mathbf{I} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}^T = 0 \Longrightarrow C_1 = (0, 0, 0, 1)$$

$$C_2^T = \text{Null}(P_2) \Longrightarrow P_2 C_2^T = 0$$

$$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0 \Longrightarrow \begin{cases} x_1 + x_2 + x_3 + 2x_4 = 0 \\ 2x_2 + 2x_4 = 0 \\ x_3 = 0 \end{cases}$$

$$\Longleftrightarrow \begin{cases} x_1 + -x_2 = 0 \\ x_2 + x_4 = 0 \\ x_3 = 0 \end{cases} \Longrightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \end{bmatrix} \Longleftrightarrow C_2 = \frac{1}{\sqrt{3}}(-1, -1, 0, 1)$$

The fundamental matrix F is computed as

$$F = [e_2]_x A$$

where e_2 is the epipole for camera 2 with camera matrix $P_2 = \begin{bmatrix} A & t \end{bmatrix}$.

$$e_2 = P_2 C_1 \Longleftrightarrow e_2 = \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \Longleftrightarrow e_2 = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$$

$$F = [e_2]_x A \Longleftrightarrow F = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & -2 \\ -2 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Longrightarrow F = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & -2 \\ -2 & 2 & -2 \end{bmatrix}$$

The determinant of the fundamental matrix is easy to calculate as the top left corner is a 2×2 zero matrix

$$\det(F) = 0$$

$$e_2^T F = \begin{bmatrix} 2 & 2 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & -2 \\ -2 & 2 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 4 + (-4) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$$

$$e_1 = P_1 C_2 \Longleftrightarrow e_1 = \frac{1}{\sqrt{3}} \begin{bmatrix} \mathbf{I} & 0 \end{bmatrix} \begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \end{bmatrix} \Longrightarrow e_1 = \frac{1}{\sqrt{3}} \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix}$$

$$Fe_1 = \frac{1}{\sqrt{3}} \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & -2 \\ -2 & 2 & -2 \end{bmatrix} \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{3}} \begin{bmatrix} 0 \\ 0 \\ 2+(-2) \end{bmatrix} \Longrightarrow Fe_1 = \mathbf{0}$$

The camera center of $P_1 = \begin{bmatrix} \mathbf{I} & 0 \end{bmatrix}$ and $P_2 = \begin{bmatrix} A & t \end{bmatrix}$ are the null space of the two camera matrices.

$$P_1 \begin{bmatrix} C_1 \\ 1 \end{bmatrix} = \mathbf{0} \Longleftrightarrow \begin{bmatrix} \mathbf{I} & 0 \end{bmatrix} \begin{bmatrix} C_1 \\ 1 \end{bmatrix} = \mathbf{0} \Longleftrightarrow \mathbf{I}C_1 + 0 \cdot 1 = \mathbf{0} \Longrightarrow C_1 = \mathbf{0}$$

$$P_2 \begin{bmatrix} C_2 \\ 1 \end{bmatrix} = \mathbf{0} \Longleftrightarrow \begin{bmatrix} A & t \end{bmatrix} \begin{bmatrix} C_2 \\ 1 \end{bmatrix} = \mathbf{0} \Longleftrightarrow AC_2 + t \cdot 1 = \mathbf{0} \Longrightarrow C_2 = -A^{-1}t$$

The epipoles e_1 and e_2 are calculated by projecting the camera centers through the camera matrices as

$$e_1 = P_1 C_2 \Longleftrightarrow e_1 = \begin{bmatrix} \mathbf{I} & 0 \end{bmatrix} \begin{bmatrix} -A^{-1}t \\ 1 \end{bmatrix} \Longleftrightarrow e_1 = -A^{-1}t$$

$$e_2 = P_2 C_1 \Longleftrightarrow e_2 = \begin{bmatrix} A & t \end{bmatrix} \begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix} \Longleftrightarrow e_2 = t$$

For $F = [t]_x A$ the epipoles should fulfill $e_2^T F = 0$ and $Fe_1 = 0$

$$e_2^T F = t^T [t]_x A \Longleftrightarrow ([t]_x^T t)^T A \Longleftrightarrow e_2^T F = (-[t]_x t)A \Longleftrightarrow e_2^T F = -(t \times t)A \Longleftrightarrow e_2^T F = 0$$

$$Fe_1 = [t]_x A(-A^{-1}t) \Longleftrightarrow Fe_1 = -[t]_x \mathbf{I}t \Longleftrightarrow Fe_1 = -(t \times t) \Longleftrightarrow Fe_1 = 0$$

Exercise 3: The Fundamental Matrix

$$F = N_2^T \tilde{F} N_1$$

Computer Exercise 1: The Fundamental Matrix

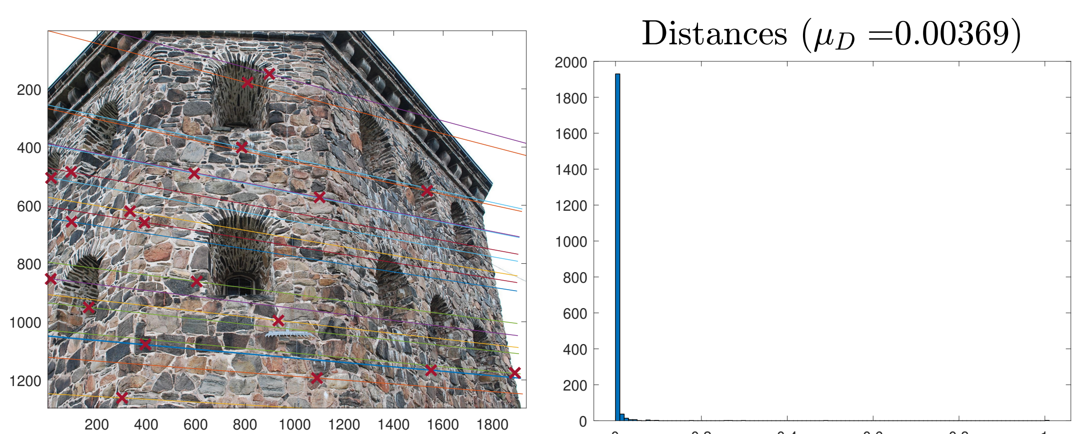

Figure 1: Points normalized before solving the DLT yields good results as seen by the epipolar lines being close to the image points. A histogram of the distance between epipolar line and corresponding image point can be seen to the right with the mean distance μ_d. Below is the fundamental matrix for the unnormalized points.

$$F_{UN,1} = \begin{bmatrix} 0 & 0 & -0.0058 \\ 0 & 0 & 0.00269 \\ -0.0073 & 0.0265 & 1 \end{bmatrix}$$

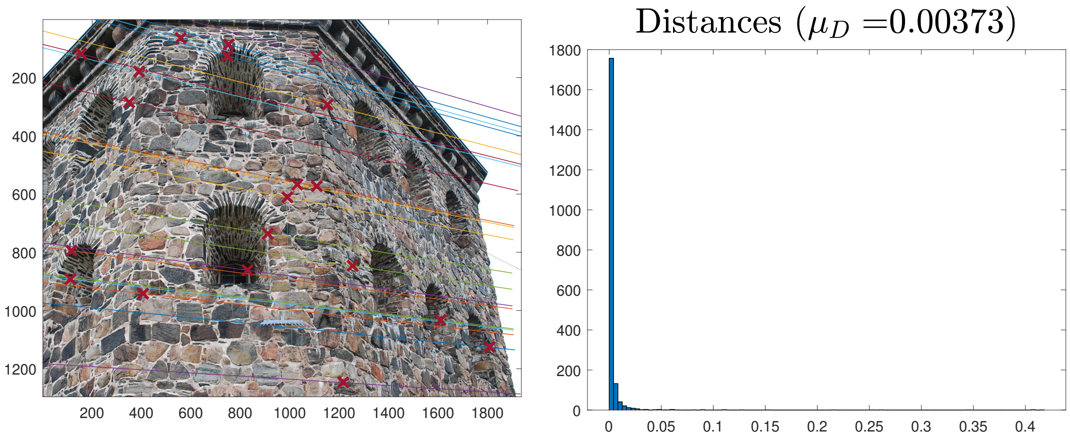

Figure 2: Points are not normalized before solving the DLT. This yields visibly good results but lacks in numerical accuracy compared to when we normalize the points first. Below is the fundamental matrix for the unnormalized points.

$$F_{UN,2} = \begin{bmatrix} 0 & 0 & 0.0058 \\ 0 & 0 & -0.0266 \\ -0.0072 & 0.0262 & 1 \end{bmatrix}$$

Exercise 4: The Fundamental Matrix

We have

$$X_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \Longrightarrow \mathbf{X}_1 = \begin{bmatrix} X_1 \\ 1 \end{bmatrix} \Longrightarrow \mathbf{X}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix}$$

and

$$X_2 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \Longrightarrow \mathbf{X}_2 = \begin{bmatrix} X_2 \\ 1 \end{bmatrix} \Longrightarrow \mathbf{X}_2 = \begin{bmatrix} 3 \\ 2 \\ 1 \\ 1 \end{bmatrix}$$

We have the camera matrices $P_1 = \begin{bmatrix} \mathbf{I} & 0 \end{bmatrix}$ and $P_2 = \begin{bmatrix} [e_2]_x F & e_2 \end{bmatrix}$ Instead of proving this for the two different points $\mathbf{X}_1$ and $\mathbf{X}_2$ we could prove it for a general point

$$\mathbf{X} = \begin{bmatrix} X \\ \lambda \end{bmatrix}$$

The projection of the point $\mathbf{X}$ in camera 1 is

$$x_1 = P_1 \mathbf{X} \longrightarrow x_1 = \mathbf{I} X + 0 \lambda \longrightarrow x_1 = X$$

and in camera 2

$$x_2 = P_2 \mathbf{X} \Longrightarrow [e_2]_x F X + e_2 \lambda \Longrightarrow x_2 = e_2 \times (FX) + e_2 \lambda$$

$$\begin{aligned} x_2^T F x_1 &= \left(e_2 \times (FX) + e_2 \lambda\right)^T FX \\ \Longleftrightarrow & \left(e_2 \times (FX)\right)^T FX + e_2^T \Lambda F X \end{aligned}$$

As $e_2 \times (FX)$ is perpendicular to FX we have

$$\left(e_2 \times (FX)\right)^T FX = 0$$

We know that e_2 is in the null space of F^T which we can utilize as

$$\lambda e_2^T F X = \lambda (F^T e_2)^T F X = 0$$

This shows that the projections of both the points X_1 and X_2 will fulfill the epipolar constraint as it's fulfilled for any point $\mathbf{X} = \begin{bmatrix} X & 1 \end{bmatrix}^T$.
The camera center of camera 2 can be calculated by utilizing the knowledge that the epipole for camera 1 is in the nullspace of the fundamental matrix F

$$e_1 = P_1 C_2 \iff F e_1 = F P_1 C_2 \iff F P_1 C_2 = 0$$

By having $C_2^T = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix}$ and $FP_1 = \begin{bmatrix} F & 0 \end{bmatrix}$ we have

$$\begin{bmatrix} F & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0 \implies \begin{cases} x_2 + x_3 = 0 \\ x_1 = 0 \\ x_2 + x_3 = 0 \end{cases} \implies \begin{cases} x_1 = 0 \\ x_2 = -x_3 \end{cases}$$

This leads to the camera center of camera 2 being equal to

$$C_2 = \frac{1}{\sqrt{3}} \begin{bmatrix} 0 \\ -1 \\ 1 \\ 1 \end{bmatrix}$$

Computer Exercise 2: The Fundamental Matrix

Figure 3: Shows the resulting projection of the 3D points and the corresponding image points. As we can see the result looks good and the error is low. The resulting 3D points however doesn't look like a model of the object.

$$P_1 = \begin{bmatrix} 420.3 & 0 & 851.3 & 0 \\ 0 & 333.1 & 667.7 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$P_2 = \begin{bmatrix} -0.7 & 2.6 & 79.0 & -443.2 \\ 2.4 & -8.8 & -355.6 & -76.1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Exercise 5(Optional): The Essential Matrix

Given $[t]_x = USV^T$ show that the eigenvalues of $[t]_x^T[t]_x$ are the squared singular values

$$[t]_x^T [t]_x = (USV^T)^T (USV^T)$$

$$\iff VS^T U^T U S V^T$$

$$\iff VS^TSV^T$$

as $U^TU = \mathbf{I}$. Given that $[t]_x \in \mathbf{R}^{m\times n}$, due to $[t]_x$ being skew symmetric we have $m = n$ which will be used for the following dimensions. Given this we have the diagonal matrix

$$S \in \mathbf{R}^{n\times n} \qquad S = diag(\sigma_1, \sigma_2, ..., \sigma_n)$$

according to the SVD of $[t]_x$. This leads to S^TS being

$$S^TS \in \mathbf{R}^{n\times n}, \qquad S^TS = diag(\sigma_1^2, \sigma_2^2, ..., \sigma_n^2)$$

As $V^TV = VV^T = \mathbf{I}$ we get

$$[t]_x^T[t]_x = VS^TSV^T \iff [t]_x^T[t]_xV = VS^TS$$

for

$$V \in \mathbf{R}^{n\times n}, \qquad V = (w_1, w_2, ..., w_n) \text{ with } w_n \in \mathbf{R}^{nx1}$$

If we take the i:th element of the diagonal matrix S^TS and corresponding i:th column of V we get

$$[t]_x^T[t]_xw_i = \sigma_i^2 w_i$$

This is the eigenvalue equation for $[t]_x^T[t]_x$ with eigenvector w_i and eigenvalue $\lambda_i = \sigma_i^2$ thus showing that the eigenvalues of $[t]_x^T[t]_x$ are the squared singular values.
I'll use the property that $A^T = -A$ for a skew symmetric matrix A along with $[t]_xA = t \times A$ to show that the eigenvalues of $[t]_x^T[t]_x$ fulfill

$$-t \times (t \times w) = \lambda w$$

Proof:

$$[t]_x^T[t]_xw_i = \lambda_i w_i \iff -[t]_x(t \times w_i) = \lambda_i w_i \iff -t \times (t \times w_i) = \lambda_i w_i$$

It's easy to show that $w = t$ is an eigenvector with the eigenvalue $\lambda = 0$ as

$$-t \times (t \times w_i) = -t \times (t \times t) = 0$$

as $t \times t = 0$ for all matrices t. This shows that λ has to be 0 given that $t \neq 0$ as

$$-t \times (t \times t) = \lambda t \iff 0 = \lambda t \implies \lambda = 0$$

Given that w is perpendicular we can calculate the eigenvalue through the formula given in the assignment. If $t \perp w$ then $t \cdot w = 0$ and $t \cdot t = \|t\|^2$

$$-(t \times (t \times w)) = -((t\cdot w)t - (t\cdot t)w)$$

$$\iff -(t \times (t \times w)) = -(-\|t\|^2 w)$$

$$\iff [t]_x^T[t]_xw = \|t\|^2 w$$

This shows that the eigenvalue of the eigenvector w that is perpendicular to t is equal to $\|t\|^2$. We also have the eigenvector $w' = -w$ which is also perpendicular to t that satisfies the the eigenvector equation with the eigenvalue $\lambda = \|t\|^2$. The eigenvector $w = -t$ is also valid as it satisfies the eigenvector equation with the eigenvalue $\lambda = 0$.
As the eigenvalues of $[t]_x^T[t]_x$ are the square of the singular values of $[t]_x$, shown above, it's easy to show that the singular values are $\sigma_1 = 0$, $\sigma_2 = \|t\|$ and $\sigma_3 = -\|t\|$. The singular values are the square root of the eigenvalues

$$\sigma_1 = \sqrt{\lambda_1} \implies \sigma_1 = \sqrt{0} = 0$$

$$\sigma = \sqrt{\lambda_2} \implies \sigma = \sqrt{\|t\|^2} = \pm\|t\| \implies \begin{cases} \sigma_2 = \|t\| \\ \sigma_3 = -\|t\| \end{cases}$$

A singular value decomposition of $E = [t]_xR$ is

$$E = US'R$$

with

$$S' = SV^T$$

Computer Exercise 3: The Essential Matrix

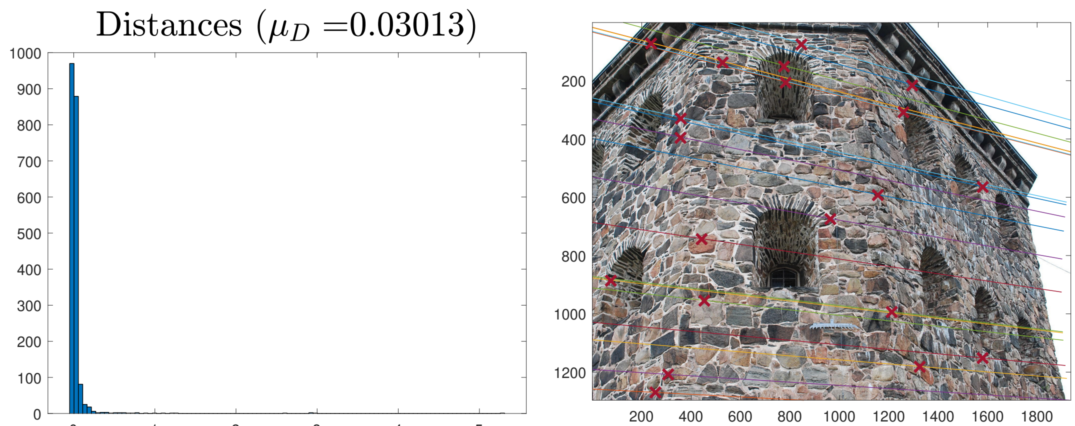

Figure 4: To the left is a histogram over the distances between the epipolar lines and the image points with the mean distance μ_d. To the right is the epipolar lines and the image points.

$$E = \begin{bmatrix} -8.9 & -1\,005.8 & 377.1 \\ 1\,252.5 & 78.4 & -2\,448.2 \\ -472.8 & 2\,550.2 & 1 \end{bmatrix}$$

Computer Exercise 4: The Essential Matrix

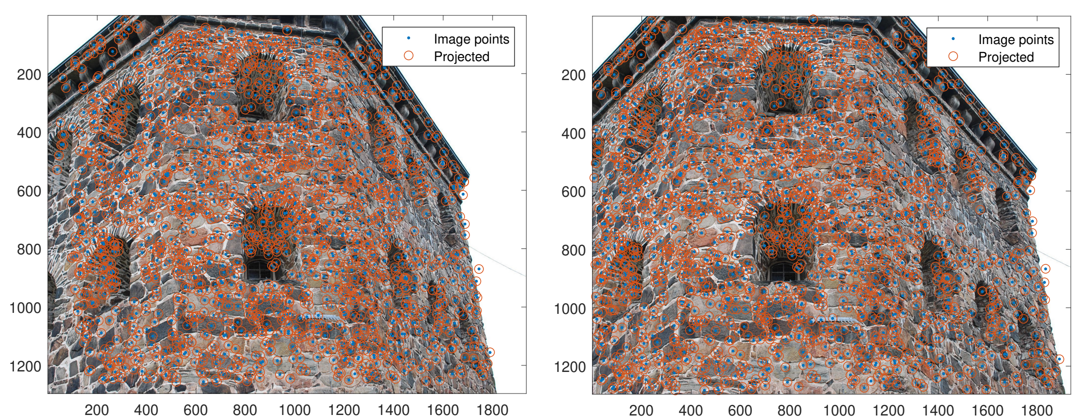

Figure 5: The image points and the projected points match well for the two cameras.

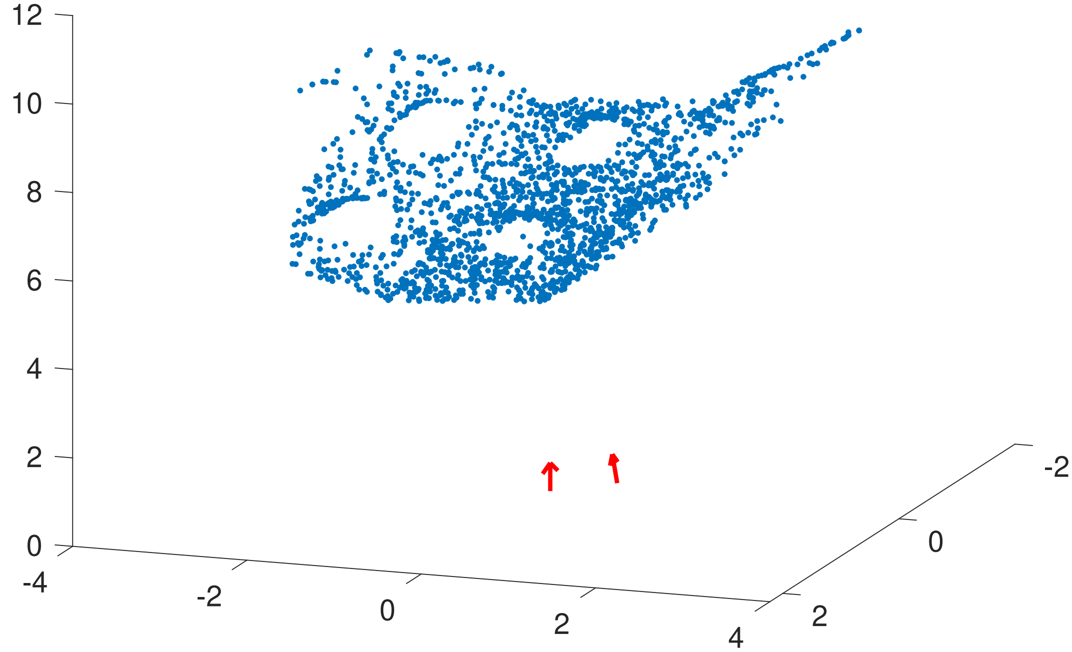

Figure 6: 3D points and the two cameras. Looks like a good model of the two images which is what was expected as we had calibrated cameras.