Proof for maximal correlation (1/)

Define the operator-valued kernel $k(x,y) = \left(\sqrt{k_i(x,y)k_j(x,y)}\right)_{i,j=1}^n$. Take $x_1, .., x_N$ and $Y_1, .., Y_N \in \mathbb{R}^n$. Consider

$$A = \sum_{k,l=1}^{N} Y_k k(x_k, x_l) Y_l = \sum_{k,l,i,j=1}^{N} (Y_k)_i k_{ij}(x_k, x_l)(Y_l)_j \overset{?}{\geq} 0 \tag{18}$$

This is equivalent to proving that the following matrix is positive definite (up to reoderdering, similar matrices have same spectrum):

$$M := \begin{pmatrix} K_1 & \sqrt{K_1} \circ \sqrt{K_2} & \dots & \sqrt{K_1} \circ \sqrt{K_n} \\ \sqrt{K_1} \circ \sqrt{K_2} & K_2 & \dots & \sqrt{K_2} \circ \sqrt{K_n} \\ \vdots & \vdots & \ddots & \vdots \\ \sqrt{K_1} \circ \sqrt{K_n} & \sqrt{K_2} \circ \sqrt{K_n} & \dots & K_n \end{pmatrix} \tag{19}$$

Where $(K_i)_{kl} = k_i(x_k, x_l)$ and $\sqrt{(K_i)}_{kl} = \sqrt{k_i(x_k, x_l)}$ and $\circ$ is the elementwise product

Proof for maximal correlation (2/)

Observe that

$$M = \left[\begin{pmatrix} \sqrt{K_1} \\ \vdots \\ \sqrt{K_n} \end{pmatrix} \begin{pmatrix} I & \dots & I \end{pmatrix} \right] \circ \left[\begin{pmatrix} I \\ \vdots \\ I \end{pmatrix} \begin{pmatrix} \sqrt{K_1} & \dots & \sqrt{K_n} \end{pmatrix} \right] \quad (20)$$

The elementwise product ($\circ$, called Hadamard product), is the principal submatrix of the Kronnecker product, ie for $A, B \in \mathbb{R}^{p \times p}$, J a "selection matrix", we have

$$A \circ B = J^T (A \otimes B) J \quad (21)$$

Proof for maximal correlation (3/)

So

$$M = J^T \left(\left[\begin{pmatrix} \sqrt{K_1} \\ \vdots \\ \sqrt{K_n} \end{pmatrix} \begin{pmatrix} I & \dots & I \end{pmatrix} \right] \otimes \left[\begin{pmatrix} I \\ \vdots \\ I \end{pmatrix} \begin{pmatrix} \sqrt{K_1} & \dots & \sqrt{K_n} \end{pmatrix} \right] \right) J \tag{22}$$

We have the following property

$$(AC) \otimes (BD) = (A \otimes B)(C \otimes D) \tag{23}$$

$$M = J^T \left(\left[\begin{pmatrix} \sqrt{K_1} \\ \vdots \\ \sqrt{K_n} \end{pmatrix} \otimes \begin{pmatrix} I \\ \vdots \\ I \end{pmatrix} \right] \left[\begin{pmatrix} I & \dots & I \end{pmatrix} \otimes \begin{pmatrix} \sqrt{K_1} & \dots & \sqrt{K_n} \end{pmatrix} \right] \right) J \tag{24}$$

Proof for maximal correlation (3/)

Since $(A \otimes B)^T = (A^T) \otimes (B^T)$

$$M = J^T \left(\left[\begin{pmatrix} \sqrt{K_1} \\ \vdots \\ \sqrt{K_n} \end{pmatrix} \otimes \begin{pmatrix} I \\ \vdots \\ I \end{pmatrix} \right] \left[\begin{pmatrix} I \\ \vdots \\ I \end{pmatrix} \otimes \begin{pmatrix} \sqrt{K_1} \\ \vdots \\ \sqrt{K_n} \end{pmatrix} \right]^T \right) J \quad (25)$$

Taking the transpose, since $M = M^T$,

$$M = J^T \left(\left[\begin{pmatrix} I \\ \vdots \\ I \end{pmatrix} \otimes \begin{pmatrix} \sqrt{K_1} \\ \vdots \\ \sqrt{K_n} \end{pmatrix} \right] \left[\begin{pmatrix} \sqrt{K_1} \\ \vdots \\ \sqrt{K_n} \end{pmatrix} \otimes \begin{pmatrix} I \\ \vdots \\ I \end{pmatrix} \right]^T \right) J \quad (26)$$