Neutron Stars Theory Computing Project

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Abstract.
This undergraduate project investigates the variation of pressure and mass of both white dwarf stars and neutron stars as a function of radius. Through the use of computational methods such as the Runge-Kutta and Newton-Raphson method, this enabled us to solve the coupled ordinary differential equations (ODEs) relating to the pressure and mass of these compact stars. We used relativistic and non-relativistic approximations which required complex adaptations to the ODEs and the associated variables, ensuring these overall calculations were increasingly accurate and reliable throughout this report. Using the boundary conditions of the mass and pressure of the compact stars, we thereby determined mass-radius relations of each type of star in turn utilizing our choice of root finding methodology.

1 Introduction

Neutron and white dwarf stars are both stellar remnants, produced towards the end of a star's lifetime. They arise when the original star burns all of its hydrogen and helium layers (including its reserves) and eventually, if the temperature and pressure are high enough, fuses the carbon and oxygen at its core. Stars that go on to become white dwarfs cannot sustain these high temperatures and so consist of a core of carbon and oxygen, however stars that go on to become neutron stars consist of a core of tightly bound neutrons due to the depletion of these carbon and oxygen levels in the latter stages of the stars' life.

Whilst a star such as our sun is stable due to hydrostatic equilibrium between thermal pressure and gravitational force, for so called *compact stars* (i.e. white dwarfs and neutron stars), the thermal pressure is insufficient to counterbalance the enormous gravitational forces from their large mass [3]. As a result, the radius of the star contracts due to the dominance of the force of gravity until another force arises that leads to a new equilibrium - the *degeneracy pressure*[6]. This degeneracy pressure does not originate due to thermal effects as there is no fusion within these compact cores, so where does it come from?

The Pauli exclusion principle states that two or more fermions cannot simultaneously occupy the same quantum state[5]. In the context of white dwarfs, the result is that some of the electrons in the plasma of the star are forced out of the lowest energy state, since it is already being occupied. As the volume of the star is compressed, electrons are further forced to higher energy levels, and the rise in kinetic energy of the electrons corresponds to the increase of the electron degeneracy pressure. We shall later show that this equilibrium between electron degeneracy pressure and gravitational force does not continue indefinitely, and in fact reaches a limit - the famous *Chandresekhar limit*.

In a similar fashion, a neutron star radiates its pressure due to the neutron decay being disallowed by this same principle. During neutron decay, a proton, electron and an anti-electron neutrino are created, however this would violate the Pauli exclusion principle if the decay took place, since all the allowed quantum mechanical lowest energy states are occupied by electrons and protons.

We can treat these two compact stars as Fermi gases due to their extremely high densities which then allows us to use the approximation of non-interacting constituent particles within the two stars in our calculations below. Various other approximations can be made for simplification, however as the report continues we reduce the severity of approximations, for example abandoning the polytropic approximation and considering the ultra-relativistic case, to improve accuracy.

2 White Dwarfs

2.1 Non-relativistic White Dwarfs

To begin with, the non-relativistic white dwarf star was initially modelled by considering the equilibrium between degeneracy pressure and Newtonian gravitational force. Here, we will broadly follow the derivation of relevant equations as done in Sagert et al.[4], although these equations for hydrostatic equilibrium are commonplace in astrophysics.
It can be stated that

$$\mathrm{d}p = \frac{\mathrm{d}F}{4\pi r^2} \quad , \quad \mathrm{d}F = -\frac{G\rho(r)4\pi r^2 \mathrm{d}r \cdot m(r)}{r^2}, \tag{1}$$

where the first expression follows from the definition of pressure and the surface area of a sphere with a radius of r, and the second from Newton's law of gravitation with the mass element $\mathrm{d}m$ expressed in terms of the density, $\rho(r)$. We define the mass density $\rho(r)$ in terms of the energy density $\epsilon(r)$ as

$$\rho(r) = \frac{\epsilon(r)}{c^2}, \tag{2}$$

where c is the speed of light, in accordance with special relativity. Following some rearranging, and considering the required equality between the degeneracy pressure and gravitational force for an equilibrium state, one should find that two coupled ODEs can be written as

$$\frac{\mathrm{d}m}{\mathrm{d}r} = \frac{4\pi r^2 \epsilon(r)}{c^2} \tag{3}$$

$$\frac{\mathrm{d}p}{\mathrm{d}r} = -\frac{G\epsilon(r)m(r)}{c^2 r^2}. \tag{4}$$

These are the *structure equations* of a star.

We now make use of the polytropic equation of state (EoS)

$$p(\epsilon) = K_{nrel/rel}\epsilon^{\gamma}, \tag{5}$$

whereby $K_{nrel/rel}$ and γ has varying values depending on relativistic and non-relativistic calculations. Utilising the polytropic EoS above, our ODEs take the form

$$\frac{\mathrm{d}\bar{m}}{\mathrm{d}r} = \frac{4\pi r^2}{M_\odot c^2}\left(\frac{p(r)}{K}\right)^{1/\gamma} \tag{6}$$

$$\frac{\mathrm{d}p}{\mathrm{d}r} = -\frac{R_0 p(r)^{1/\gamma}\bar{m}(r)}{r^2 K^{1/\gamma}}, \tag{7}$$

where $\bar{m}(r)$ is the dimensionless mass, $\bar{m}(r) = \frac{m(r)}{M_\odot}$ and R_0 is half the Schwarzchild radius, $R_0 = \frac{GM_\odot}{c^2}$. We will use the units of Pascals (Pa) for the pressure within the compact star due to convenience and convention. We shall not give a full derivation of the polytropic EoS here, however it should be noted that this arises by considering the matter of a white dwarf to consist of an degenerate ideal Fermi gas.

These can now be solved for a given K and γ. For the non-relativistic case,

$$K_{nrel} = \frac{\hbar^2}{15\pi^2 m_e}\left(\frac{3\pi^2}{m_N \eta c^2}\right)^{\gamma}, \tag{8}$$

where the adiabatic index $\gamma = 5/3$ and the ratio of mass to atomic number $\eta = A/Z = 2$, since a white dwarf can be assumed to be predominantly made up of the isotopes ^{16}O and ^{12}C.

Before continuing with our calculations, we need to first establish boundary conditions to in order to fully solve the ODEs. The first boundary condition we set is the initial central pressure, p_0 at the core of the star (i.e. when $r = 0$). Our second boundary condition is that the mass enclosed at the centre of the star will be 0, which intuitively makes sense since if we draw a sphere with $r = 0$, we enclose no matter. Using these boundary conditions will now allow us to decouple the ODEs in Equation 6 and Equation 7.

Choosing $p_0 = 2.33002 \cdot 10^{22}$Pa, we get a graph shown in Figure 1 using our classical Newtonian code, representing the mass-radius and pressure-radius relations within a white dwarf. The calculated mass and radius are $M = 0.44543M_\odot$ and $R = 11,383$km respectively.

This graph fits fairly well with the data given in [4] where Sagert and others obtained values of $M = 0.40362M_\odot$ and $R = 10,919$km.

We then modelled how the radii and mass varied for the non-relativistic white dwarf as a function of the initial central pressures. The results are shown in Figure 2. Similar to the case for a single central pressure, values of R and M were calculated by considering when the calculated pressure reached zero

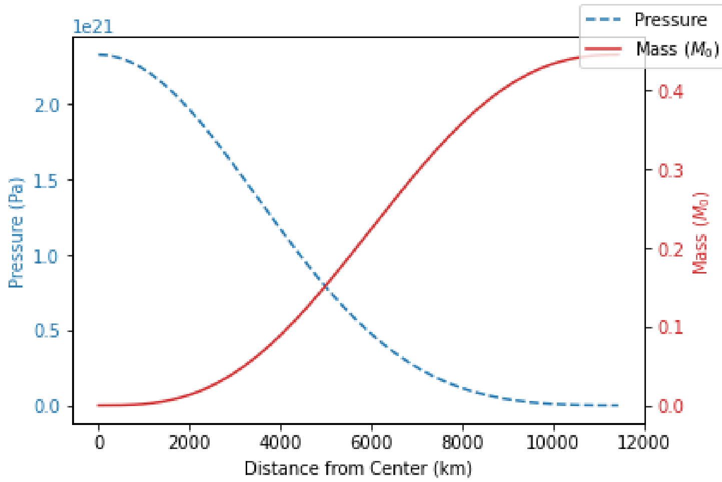

Fig. 1. The mass and pressure distribution for a white dwarf star as a function of radius with p_0=2.33002 $\cdot$ 10^{22} Pa.

(i.e. we had reached the edge of the star). From here, however, we stored these values then repeated the process for a different central pressure and added the new values to our existing array. This required functionalisation of our code, in addition to the solving of differential equations already implemented previously, for which SciPy was used [7].

2.2 Relativistic Corrections

Whilst our previous approach works sufficiently well for certain central pressures, it begins to breakdown as they increase towards some threshold. In fact, we were previously implicitly satisfying the condition that $k_F \ll m_e c$, where k_F is the Fermi momentum, such that we were in the non-relativistic regime. Now we shall extend our analysis to cover the relativistic case, where $k_F \gg m_e c$.

For a white dwarf, the energy density is dominated by the nucleons whilst the pressure is mostly from the electrons. Forgoing derivations, we shall simply state that

$$\epsilon_e(x) = \frac{\epsilon_0}{8}\left[(2x^3 + x)(1 + x^2)^{1/2} - \sinh^{-1}(x)\right] \tag{9}$$

$$p_e(x) = \frac{\epsilon_0}{24}\left[(2x^3 - 3x)(1 + x^2)^{1/2} + 3\sinh^{-1}(x)\right], \tag{10}$$

where

$$\epsilon_0 = \frac{m_e^4 c^5}{\pi^2 \hbar^3}, \quad x = \frac{k_F}{m_e c}. \tag{11}$$

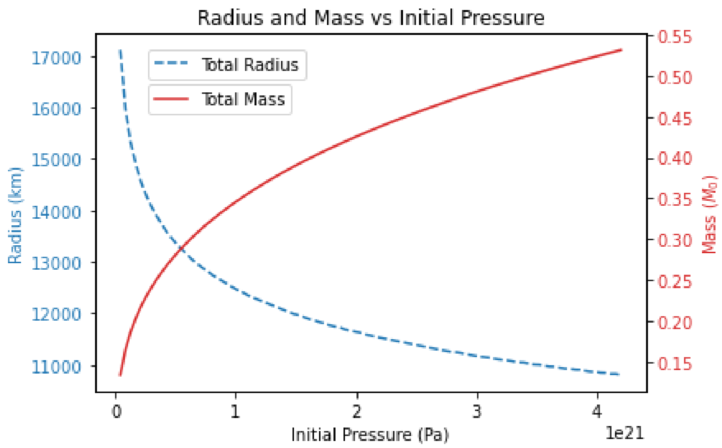

Fig. 2. The mass and radius of a white dwarf as a function of the initial pressures.

`We may extend this section, although we have mainly applied relativistic corrections to the neutron star.`

3 Neutron Stars

3.1 Non-relativistic

The non-relativistic regime for neutron stars is much like that for white dwarfs. Before discussion of results, we shall briefly outline the main differences between the two, however.

Firstly, as one might presuppose upon hearing the name of the star, neutron stars are indeed composed of neutrons; we shall then consider them to be characterised by an EoS of a Fermi gas with neutrons. Then, similar to the Fermi gas of electrons, in the non-relativistic case,

$$\epsilon(x) \simeq \frac{\epsilon_0}{3}x^3, \quad p(x) \simeq \frac{\epsilon_0}{15}x^5, \tag{12}$$

which allows us to describe the EoS with the polytrope

$$p(\epsilon) = K_{nrel}\epsilon^{5/3}. \tag{13}$$

Here,

$$K_{nrel} = \frac{\hbar^2}{15\pi^2 m_n}\left(\frac{3\pi^2}{m_N \eta c^2}\right)^{\gamma}, \tag{14}$$

with $\gamma = 5/3$ and $\eta = 1$.

The result of this model for a range of pressures is shown in Figure 3.

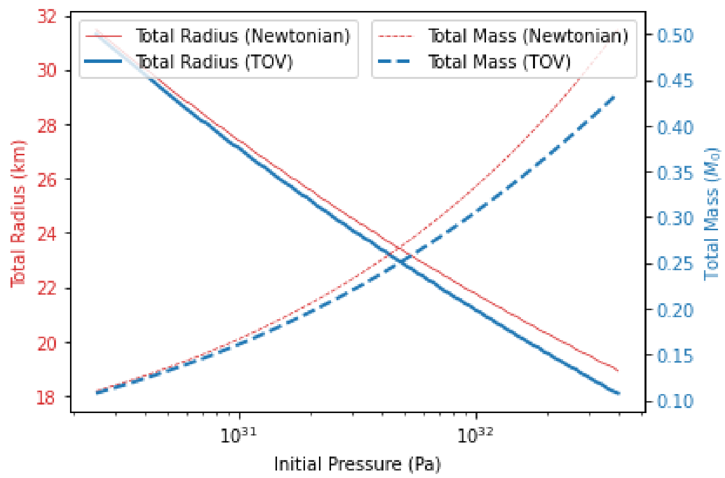

Fig. 3. The mass and radius of a neutron star as a function of the central pressure p_0. Plots using the Newtonian equations and the TOV correction are overlayed for comparison.

Note that we have in fact presented two sets of data for the radii and mass curves. The 'TOV' curve comes from the Tolman-Oppenheimer-Volkoff equation [2],

$$\frac{\mathrm{d}p}{\mathrm{d}r} = -\frac{G\epsilon(r)m(r)}{c^2r^2}\left(1+\frac{p(r)}{\epsilon(r)}\right)\left(1+\frac{4\pi r^3 p(r)}{m(r)c^2}\right)\left(1-\frac{2Gm(r)}{c^2 r}\right)^{-1}, \tag{15}$$

which contains three correction factors from general relativity. The derivation of this equation is beyond the scope of this project, however we shall point out that it assumes an isotropic, static, ideal fluid sphere in hydrostatic equilibrium. We see good agreement between the two methods for lower pressures, followed by a slight divergence as the pressures increases. This is what we might expect, since the GR corrections become more impactful as the gravity increases.

3.2 Relativistic Corrections

As described in subsection 2.2, the polytropic approximation for neutron stars will also not be sufficient in the ultra-relativistic case. Again, with Equation 9 and Equation 10 - except of course now in the case of neutrons - we have two functions of the Fermi momentum. Whilst a direct analytical solution is impossible, for a given pressure, one can employ a root finding algorithm to find the corresponding function of Fermi momentum, x, and then substitute this to find a value for $\epsilon(x)$. More explicitly,

$$p(x) - p = 0 \tag{16}$$

is used to find x for a given p. The root finding algorithm employed was the Newton-Raphson method, as described by Izaac [1].

The values produced by this method are presented in Figure 4, which is analogous to Figure 3 with the difference that this abandonment of the polytropic approximation allows us to accurately calculate radii and masses for initial pressures of far greater magnitude.

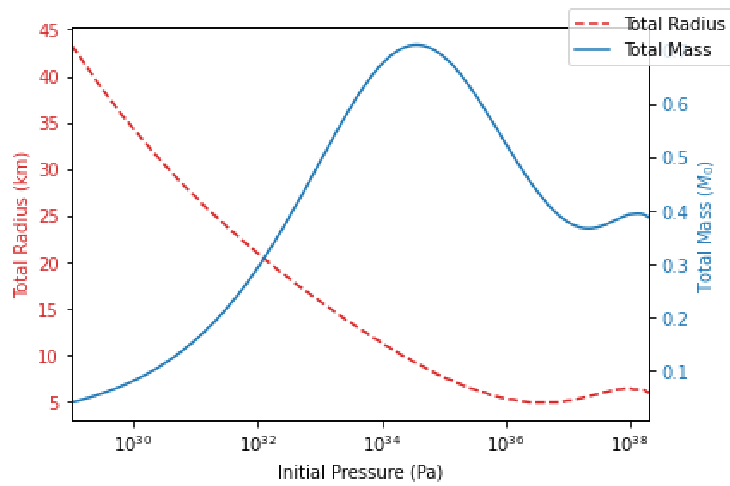

Fig. 4. The mass and radius of a neutron star as a function of the initial central pressures, p_0.

Figure 5 is a plot of the mass versus radius for the neutron star, as calculated using the previous root finding method. We calculated a maximum total mass $M = 0.710\,\mathrm{M}_\odot$ and total radius $R = 9.13\,\mathrm{km}$, which corresponded to an initial central pressure of $p_0 = 3.59 \cdot 10^{34}\,\mathrm{Pa}$. Comparison with $M = 0.71\,\mathrm{M}_\odot$ and $R = 9.5\,\mathrm{km}$ from Oppenheimer and Volkoff [2] as well as with $M = 0.712\,\mathrm{M}_\odot$ and $R = 9.14\,\mathrm{km}$ from Sagert et al. [4] shows remarkable agreement.

4 Conclusion

In this project we have derived and explored the structure equations of stars. Looking at compact stars, concepts from statistical mechanics and special relativity were used to derive equations of state, simplified initially with polytropic approximations and then extended to be more broadly applicable and accurate with numerical methods. Non-relativistic and relativistic cases were considered, as well as the difference between Newtonian and GR predictions for a neutron star.

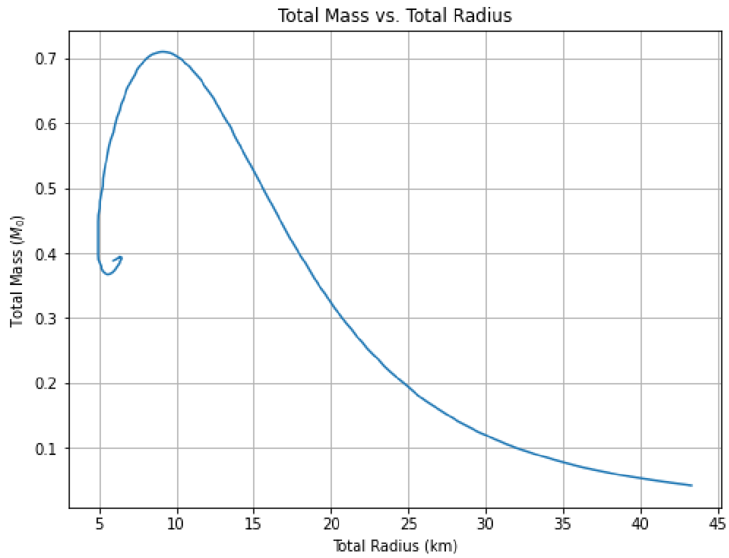

Fig. 5. Plot of the mass as a function of the radius for a relativistic neutron star. The maximum total mass is $M = 0.710\,\mathrm{M}_\odot$ and total radius $R = 9.13\,\mathrm{km}$.

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