Static Traffic Assignment with Capacity Constraints

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Motivation

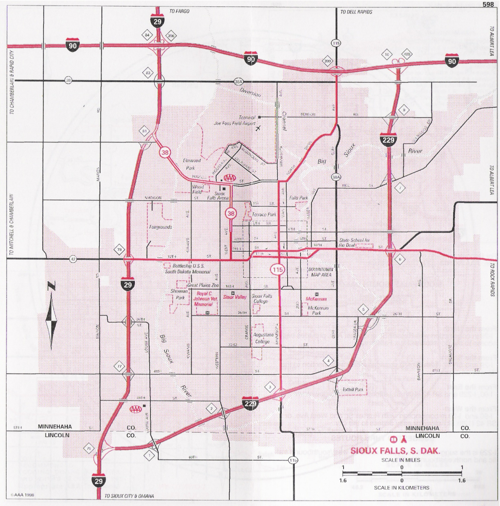
Goal: Understand how a traffic incident affects traffic patterns

Why?

- Better predict travel times with traffic incidents
- Better allocate crisis resources
- Assure the resilience of the road network

How?

Traffic incident = change in capacity constraint



Sioux Falls in 1998

Image source: *github.com/bstabler/TransportationNetworks/tree/master/SiouxFalls*

Case Study: Sioux Falls, ND

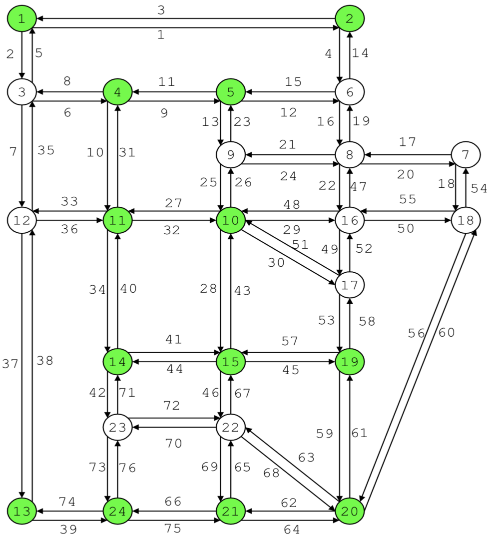


Image source: *github.com/bstabler/TransportationNetworks/tree/master/SiouxFalls*

The Static Traffic Assignment Problem [TAP]

Objective: Minimize the travel time of each driver

Travel time functions are continuous and monotonically increasing with respect to link flows $\rightarrow$ convex optimization program

Fixed demand (static problem)

Inputs	**Outputs**
Network layout (nodes, links)	Equilibrium flow on each link
Demand for each origin-destination pair	Travel time for each link

Solving the TAP $\Longleftrightarrow$ Determining the unique traffic flow equilibrium

The Wardrop Equilibrium Conditions

"User equilibrium principle": all users minimize their own travel times

- Economic theory (assumption of perfect information)

Equilibrium property: No individual is able to decrease his/her travel time by changing the route

- Equilibrium travel times for all used routes are identical and less than the travel times for unused routes

$$
\begin{aligned}
h_r > 0 \Longrightarrow t_r = t_k^* \qquad & \forall r \in \mathcal{R}_k, \forall k \in \mathcal{K} \\
h_r = 0 \Longrightarrow t_r \geq t_k^* \qquad & \forall r \in \mathcal{R}_k, \forall k \in \mathcal{K}
\end{aligned}
$$

Source: J.G. Wardrop of the Road Research Laboratory (1952)

Primal Formulation [TAP-C]

Variables: Let f_a be the traffic flow on link a

Let h_r be the traffic flow on route r

$$
\begin{aligned}
\text{[TAP-C]} \quad \min_f \sum_{a \in \mathcal{A}} T_a(f_a) &= \sum_{a \in \mathcal{A}} \int_0^{f_a} t_a(s)ds \\
\text{s.t.} \quad \sum_{k \in \mathcal{K}} \sum_{r \in R_k} \delta_{ra} h_r &= f_a, \forall a \in \mathcal{A} && (1) \\
\sum_{r \in \mathcal{R}_k} h_r &= d_k, \forall k \in \mathcal{K} && (2) \\
h_r &\geq 0, \forall r \in \mathcal{R}_k, \forall k \in \mathcal{K} && (3) \\
f_a &\leq u_a, \forall a \in \mathcal{A} && (4)
\end{aligned}
$$

Remarks: strictly convex objective, linear constraints; number of routes for each O-D pair could be very large; computationally expensive [1]

KKT Conditions and Dual Variables

$$
\begin{aligned}
\sum_{k\in\mathcal{K}}\sum_{r\in R_k}\delta_{ra}h_r &= f_a, \quad \forall a\in\mathcal{A}\\
\sum_{r\in\mathcal{R}_k} h_r &= d_k, \quad \forall k\in\mathcal{K}\\
h_r &\geq 0, \forall r\in\mathcal{R}_k, \quad \forall k\in\mathcal{K}\\
f_a &\leq u_a, \quad \forall a\in\mathcal{A}\\
\nu_a &\geq 0, \quad \forall a\in\mathcal{A}\\
\nu_a\left(f_a-u_a\right) &= 0, \quad \forall a\in\mathcal{A}\\
h_r(\pi_k-\bar{c}_r(f)) &= 0, \quad \forall r\in\mathcal{R}_k, \forall k\in\mathcal{K}\\
\pi_k &\leq \bar{c}_r(f), \quad \forall r\in\mathcal{R}_k, \forall k\in\mathcal{K}
\end{aligned}
$$

where the 'generalized route travel cost,' $\bar{c}_r(f)=\sum_{a\in\mathcal{A}}\left(t_a(f_a)+\nu_a\right)\delta_{ra}$
Interpretation of dual variables: π_k - min generalized travel cost
ν_a - queuing delay of saturated links

Solving for Dual from Primal Solution

Given the optimal primal variables f_a^*'s and h_r^*'s, we can solve the KKT conditions to get optimal dual variables, which can be simplified as a linear system:

$$\begin{aligned} \nu_a &\geq 0, \quad \forall a \in \mathcal{A} \\ \nu_a &= 0, \quad \forall a \in \tilde{\mathcal{A}} \\ \pi_k - \sum_{a \in \mathcal{A}} \left(t_a(f_a^*) + \nu_a \right) \delta_{ra} &= 0, \quad \forall r \in \tilde{\mathcal{R}}_k, \forall k \in \mathcal{K} \end{aligned}$$

where $\tilde{\mathcal{A}} = \{a \in \mathcal{A} : f_a \neq u_a\}$, $\tilde{\mathcal{R}}_k = \{r \in \mathcal{R}_k : h_r \neq 0\}$

It's really easy to find the optimal dual variables given the optimal primal variables.

Sensitivity Analysis and an upper bound

- What does constraint perturbation mean?

$$
\begin{aligned}
f \leq u &\rightarrow& f \leq u - \delta u \\
T^\star(u - \delta u) &\geq& T^\star(u) - {\nu^\star}^T(-\delta u) \\
T^\star(u - \delta u) &\geq& T^\star(u) + {\nu^\star}^T \delta u
\end{aligned}
$$

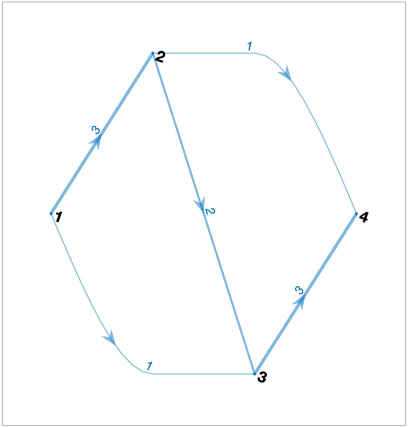
- A lower bound on the cost and not flows!
- For an upper bound on the cost – Feasibility analysis

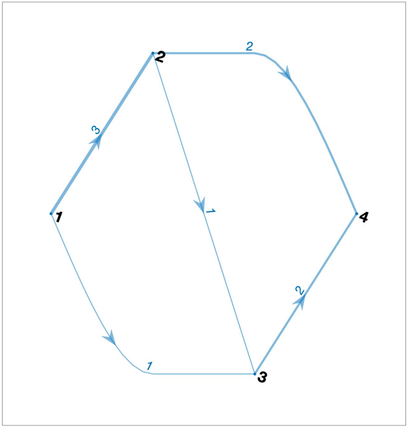
$$\min_{f \in \mathcal{F}} T(f) \leq T(\bar{f}) \text{ for some } \bar{f} \in \mathcal{F}$$

- Given solutions to TAP, reroute them to make them feasible!

An example – The Braess network

$$t(f) = \begin{bmatrix} 5 & f & 0 & f & 5 \end{bmatrix}^T$$
$$u = \begin{bmatrix} 5 & 5 & 2 & 5 & 5 \end{bmatrix}^T \to T = 12$$
$$u - \delta u = \begin{bmatrix} 5 & 5 & 1 & 5 & 5 \end{bmatrix}^T \to T = 13$$





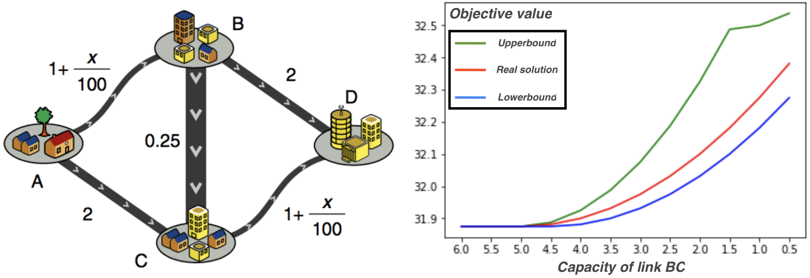


Figure 1: The Braess network and the results of the capacity constraints

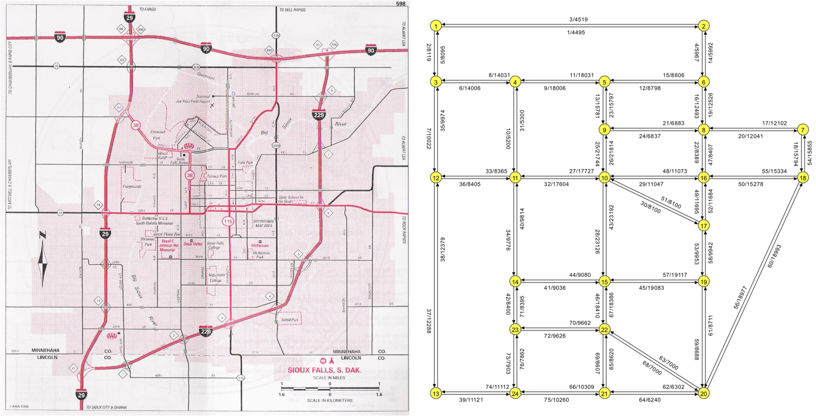


Figure 2: The Sioux Falls network and the results of the unconstrained problem

Results 3

- Computing the path is NP-hard
- CVX does not work
- Need Frank-Wolfe (gradient descent)
- Add the capacity constraints (projected gradient descent)
- Plot the results

Conclusion

- What did we do?
  - Traffic Assignment Problem with Capacity constraints
  - Primal and Dual problems
  - Sensitivity analysis and upper bounds
  - Implementation on real networks
- Adding constraints to the primal convex program does not translate to a direct addition to the KKT conditions
- Meaning of dual parameters

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Thank You!

Questions?