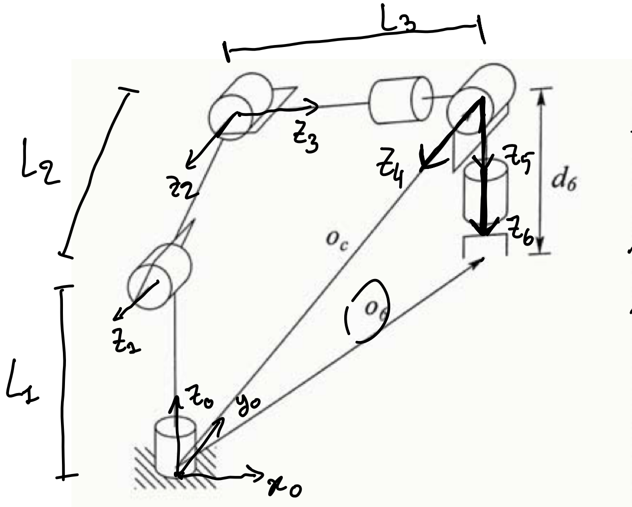
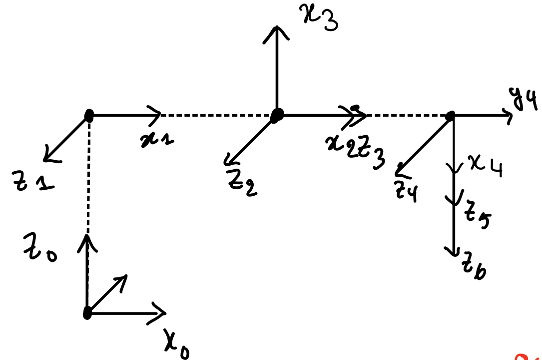
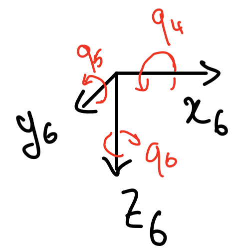


$\tilde{\theta}_i$	d_i	a_i	α_i
0	L_1	0	$\pi/2$
0	0	L_2	0
$\pi/2$	0	0	$\pi/2$
π	L_5	0	$\pi/2$
$\frac{\pi}{2}$	0	0	$\pi/2$
0	d_6	0	0





final XYZ rotation

Once we found q_1, q_2, q_3 for wrist position, we can find the orientation of end-effector at that point (before actuating q_4, q_5, q_6). Then we match a Euler from this state $(q_1, q_2, q_3, q_4=0, q_5=0, q_6=0)$ to the desired orientation.

Given ${}^0R_6(q_1,q_2,q_3,q_4=0, q_5=0, q_6=0)$ and ${}^0R_6(\alpha, \beta, \gamma)$, find q_4, q_5, q_6.

$$ {}^3T_{6,\text{Desired}} = \left[{}^0T_6 (q_1,q_2,q_3,q_4=0, q_5=0, q_6=0) \right]^{-1} \cdot T_{\text{Desired}} $$

$\rightarrow ({}^0T_{6,\text{Desired}})$

$\hookrightarrow$ extract ${}^3R_{6,\text{Desired}} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \longrightarrow$ set equal to $R_{\text{Euler},XYZ}$

Algorithm 2RIK

1. $c_2 \leftarrow \frac{\|\mathbf{x}_D\|^2 - L_1^2 - L_2^2}{2L_1 L_2}$.
2. **if** $|c_2| > 1$ **then return** $\emptyset$.
3. **if** $c_2 = 1$ **then return** $\{(atan2(x_y, x_x), 0)\}$.
4. **if** $c_2 = -1$ and $\mathbf{x}_D \neq 0$ **then return** $\{(atan2(x_y, x_x), \pi)\}$.
5. **if** $c_2 = -1$ and $\mathbf{x}_D = 0$ **then return** $\{(q_1, \pi) \quad | \quad q_1 \in [0, 2\pi)\}$
6. Otherwise, let $q_2^{(1)} \leftarrow \cos^{-1} c_2$ and $q_2^{(2)} \leftarrow -\cos^{-1} c_2$.
7. Compute $\theta = atan2(x_y, x_x)$.
8. **for** $k \in \{1, 2\}$ **do**
9. $\quad q_1^{(k)} = \theta - atan2(L_2 \sin q_2^{(k)}, L_1 + L_2 \cos q_2^{(k)})$.
10. **return** $\{(q_1^{(1)}, q_2^{(1)}), (q_1^{(2)}, q_2^{(2)})\}$.

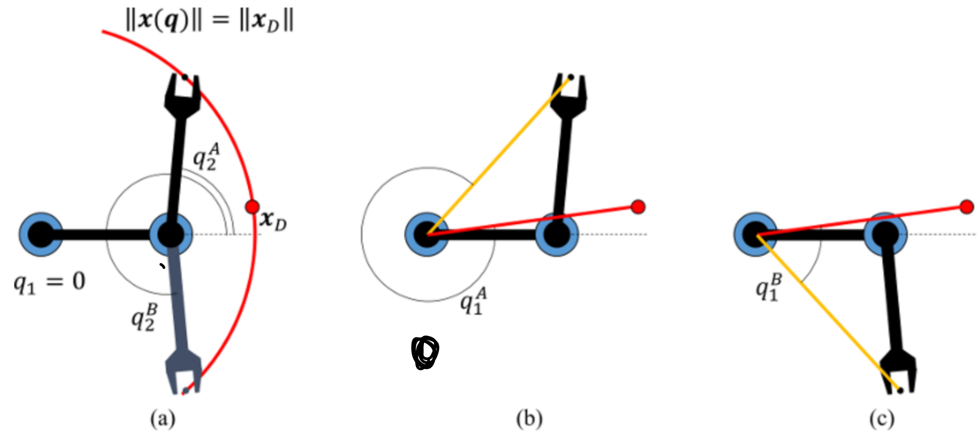


Figure 2 The key steps for solving the planar 2R IK problem. (a) Determine a value for q_2 that puts the end effector point on the circle centered on the origin with radius equal to the distance between the origin and the target point. Here there are 2 solutions. Then, determine the values of q_1 that the end effector is rotated onto the target, with q_2 fixed to its first (b) and second (c) solutions.

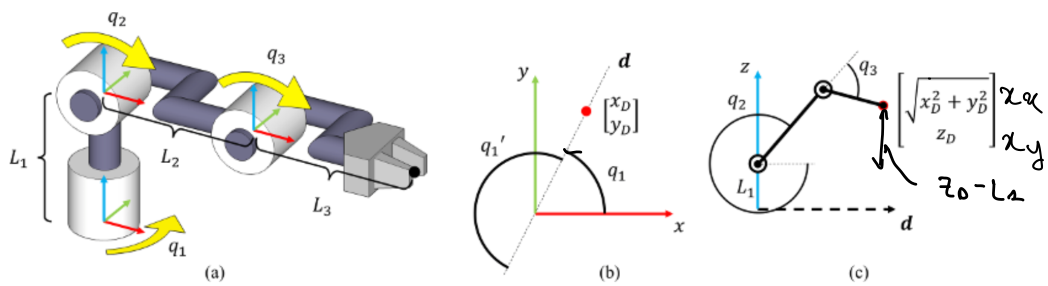


Figure 3. (a) Inverse kinematics for a 3R robot reaching a point in space. (b) The first joint angle is determined by the heading of the target point in the $x-y$ plane (up to an addition of π). (c) The second and third angles are determined by rotating the frame of reference to the plane containing the target point and the z axis, and determining solutions for the planar 2R IK problem in this plane. (Note that since q_2 and q_3 rotate about the y axis, they are measured by the clockwise angle.)

Imagine observing the robot along a plane orthogonal to the 2nd and 3rd joint axes. The rotated x coordinate is the term $\sqrt{x_D^2 + y_D^2}$, and the rotated z coordinate remains the same, z_D. In this rotated system, we can use the planar 2R solution to calculate the possible values of q_2 and q_3. However, we should accommodate the fact that joints 2 and 3 rotate about the y axis, and positive rotations about the y axis causes a negative movement in z. Namely, we should compute all values of

$$(q_2, q_3) \in 2RIK(L_2, L_3, (\sqrt{x_D^2 + y_D^2}, -z_D + L_1)). \tag{10}$$

This contributes up to 2 solutions. For each of those solutions, the value $q_1 = atan2(y_D, x_D)$ orients the first joint in the proper direction. Fig. 3 illustrates these solutions.

However, there is a small catch: we could have also modified q_1 by a factor of π, giving a flipped coordinate system for the 2R solution where the robot "reaches backwards". As a result, there are 2 additional solutions:

$$(q_2', q_3') \in 2RIK(L_2, L_3, (-\sqrt{x_D^2 + y_D^2}, -z_D + L_1)). \tag{11}$$

each associated with the value $q_1' = atan2(y_D, x_D) + \pi$. Hence, altogether, up to 4 solutions exist.

Algorithm to choose q_1, q_2, q_3 for a given $X_\omega = \begin{bmatrix} x_\omega \\ y_\omega \end{bmatrix}$

1. Select $q_1^{(k)} = \{ q_1^{(1)} = atan2(y_\omega, x_\omega) ; q_1^{(2)} = atan2(y_\omega, x_\omega) + \pi \}$, k becomes constant
   ↳ based on minimum change from current q_1.

2. a)
$$\begin{cases} k=1: & (q_2, q_3)^{(1)} \in 2RIK(L_2, L_3, [+\sqrt{x_\omega^2 + y_\omega^2}, z_D - L_1]) \\ k=2: & (q_2, q_3)^{(2)} \in 2RIK(L_2, L_3, [\underbrace{-\sqrt{x_\omega^2 + y_\omega^2}}_{x_x}, \underbrace{z_D - L_1}_{x_y}]) \end{cases}$$

To be consistent with 2RIK

for $k=1,2$, do 2RIK with new $x_D = \begin{bmatrix} x_x = \pm\sqrt{x_\omega^2 + y_\omega^2} \\ x_y = z_D - L_1 \end{bmatrix}$

This new is position of the wrist relative to joint 2 (not origin otherwise 2RIK would not apply)

Algorithm 2RIK

1. $c_2 \leftarrow \frac{\|\mathbf{x}_D\|^2 - L_1^2 - L_2^2}{2L_1L_2}$.
2. **if** $|c_2| > 1$ **then return** $\emptyset$.
3. **if** $c_2 = 1$ **then return** $\{(atan2(x_y, x_x), 0)\}$.
4. **if** $c_2 = -1$ and $\mathbf{x}_D \neq 0$ **then return** $\{(atan2(x_y, x_x), \pi)\}$.
5. **if** $c_2 = -1$ and $\mathbf{x}_D = 0$ **then return** $\{(q_1, \pi) \mid q_1 \in [0, 2\pi)\}$
6. Otherwise, let $q_2^{(1)} \leftarrow \cos^{-1} c_2$ and $q_2^{(2)} \leftarrow -\cos^{-1} c_2$.
7. Compute $\theta = atan2(x_y, x_x)$.
8. **for** $k \in \{1, 2\}$ **do**
9. $\quad q_1^{(k)} = \theta - atan2(L_2 \sin q_2^{(k)}, L_1 + L_2 \cos q_2^{(k)})$.
10. **return** $\{(q_1^{(1)}, q_2^{(1)}), (q_1^{(2)}, q_2^{(2)})\}$.

1. $$c_3 = \frac{\|x_D\|^2 - L_1^2 - L_2^2}{2L_1L_2} = \frac{\sqrt{x_\omega^2 + y_\omega^2}^{\,2} + (z_D - L_1)^2 - L_1^2 - L_2^2}{2L_1L_2}$$

2. **if** $|c_3| > 1$ **then return** $\emptyset$.
3. **if** $c_3 = 1$ **then return** $\{(atan2(x_y, x_x), 0)\}$.
4. **if** $c_3 = -1$ and $\mathbf{x}_D \neq 0$ **then return** $\{(atan2(x_y, x_x), \pi)\}$.
5. **if** $c_3 = -1$ and $\mathbf{x}_D = 0$ **then return** $\{(q_1, \pi) \mid q_1 \in [0, 2\pi)\}$

$\left.\right\}$ w/ $x_x = \pm\sqrt{x_\omega^2 + y_\omega^2}$, $x_y = z_D - L_1$

6. Otherwise: $q_3^{(i=1)} = \cos^{-1} c_3$ and $q_3^{(i=2)} = -\cos^{-1} c_3$

7. Compute $\theta\_temp = atan2\left(z_D - L_1, \pm\sqrt{x_\omega^2 + y_\omega^2}\right)$

8. $q_2^{(i)} = \theta\_temp - atan2\left(L_3 \sin q_3^{(i)}, L_2 + L_3 \cos q_3^{(i)}\right)$

9. return $(q_2, q_3)^{(k)} \in \left\{ \left(q_2^{(1)}, q_3^{(1)}\right)^{(k)}; \left(q_2^{(2)}, q_3^{(2)}\right)^{(k)} \right\}$ or other edge case

b) 4 solutions max: $(q_2, q_3) \in \left\{ \left(q_2^{(1)}, q_3^{(1)}\right)^{(1)}; \left(q_2^{(2)}, q_3^{(2)}\right)^{(1)}; \left(q_2^{(1)}, q_3^{(1)}\right)^{(2)}; \left(q_2^{(2)}, q_3^{(2)}\right)^{(2)} \right\}$

Select (q_2, q_3) within joint limits and minimum movement.

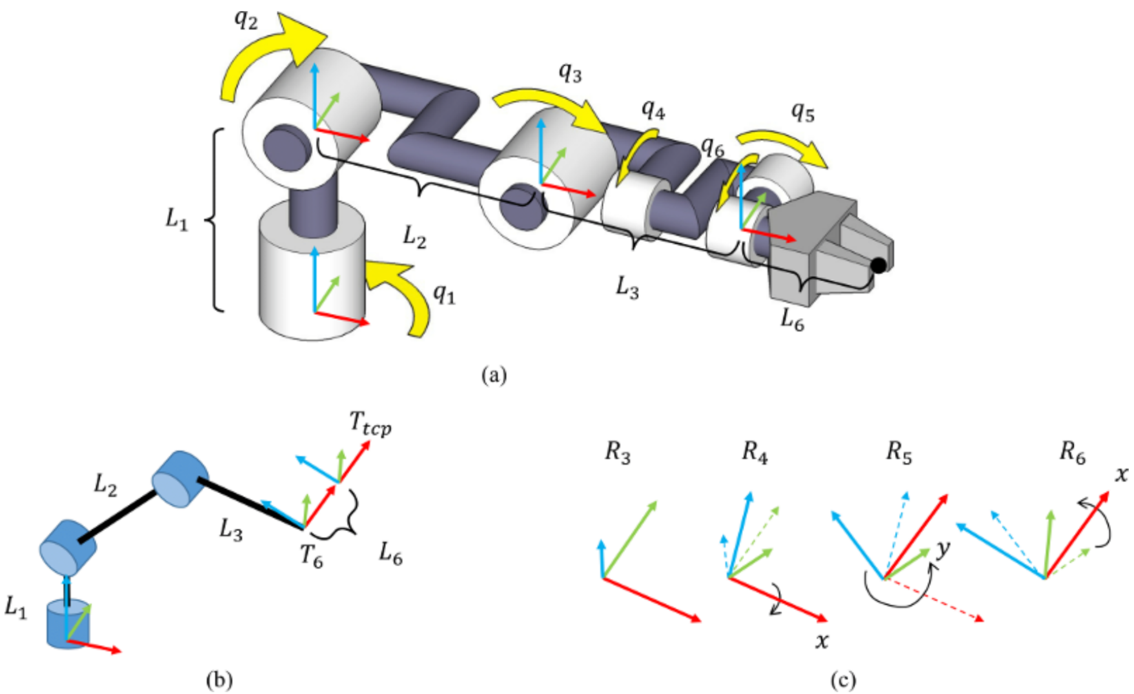
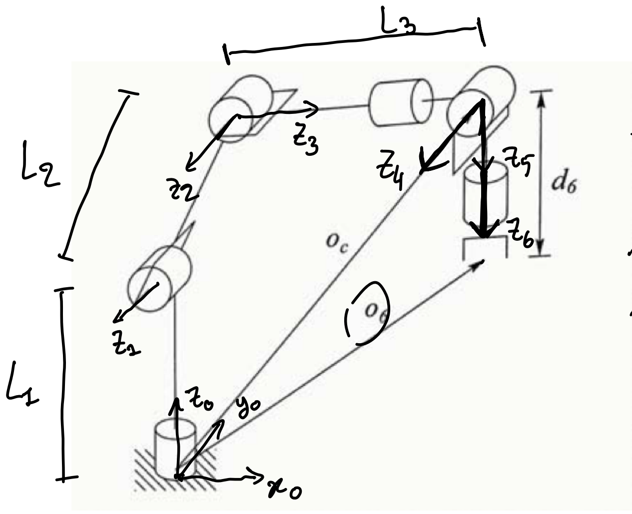
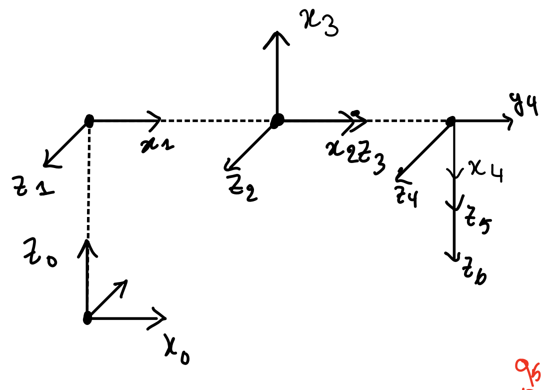
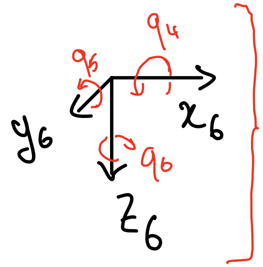


Figure 4 (a) A 6DOF industrial robot with two shoulder DOFs, an elbow, and three wrist DOFs. The order of rotational axes from base to tip are ZYYXYX. The task space consists of both position and orientation. (b) The first step solves q_1, q_2, and q_3 by determining the wrist location and solving a 3R IK problem. (c) Then, the residual relative rotation between the end effector and "forearm" is solved using the solution to XYX Euler angles.



$\tilde{\theta}_i$	d_i	a_i	α_i
0	L_1	0	$\pi/2$
0	0	L_2	0
$\pi/2$	0	0	$\pi/2$
π	L_5	0	$\pi/2$
$\frac{\pi}{2}$	0	0	$\pi/2$
0	d_6	0	0





final XYZ notation

Once we found q_1, q_2, q_3 for wrist position, we can find the orientation of end-effector at that point (before actuating q_4, q_5, q_6). Then we match a Euler from this state $(q_1, q_2, q_3, q_4=0, q_5=0, q_6=0)$ to the desired orientation.

Given ${}^0R_6(q_1, q_2, q_3, q_4=0, q_5=0, q_6=0)$ and ${}^0R_6(\alpha, \beta, \gamma)$, find q_4, q_5, q_6.

$$ {}^3T_{6,\text{Desired}} = \left[{}^0T_6(q_1, q_2, q_3, q_4=0, q_5=0, q_6=0)\right]^{-1} \cdot T_{\text{Desired}} $$

$\rightarrow ({}^0T_{6,\text{Desired}})$

$\Rightarrow$ extract ${}^3R_{6,\text{Desired}} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \longrightarrow$ set equal to $R_{\text{Euler},XYZ}$

R6_XYZ =

$$\begin{pmatrix} \cos(q_5)\cos(q_6) & -\cos(q_5)\sin(q_6) & \sin(q_5) \\ \cos(q_4)\sin(q_6)+\cos(q_6)\sin(q_4)\sin(q_5) & \cos(q_4)\cos(q_6)-\sin(q_4)\sin(q_5)\sin(q_6) & -\cos(q_5)\sin(q_4) \\ \sin(q_4)\sin(q_6)-\cos(q_4)\cos(q_6)\sin(q_5) & \cos(q_6)\sin(q_4)+\cos(q_4)\sin(q_5)\sin(q_6) & \cos(q_4)\cos(q_5) \end{pmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = {}^3R_{6,\text{desired}}$$

$$\left.\begin{aligned} q_5 &= \operatorname{atan2}\left(r_{13},\ \pm\sqrt{r_{11}^2 + r_{12}^2}\right) \\ q_4 &= \operatorname{atan2}\left(\frac{-r_{23}}{\cos(q_5)},\ \frac{r_{33}}{\cos(q_5)}\right) \\ q_6 &= \operatorname{atan2}\left(\frac{-r_{12}}{\cos(q_5)},\ \frac{r_{11}}{\cos(q_5)}\right) \end{aligned}\right\}$$

degenerates if $q5 = -\frac{\pi}{2}$ or $q5 = \frac{\pi}{2}$:

<u>If $q_5 = -\frac{\pi}{2}$</u>

$$q_5 = q_{5\_previous}$$
$$q_4 = 0$$
$$q_6 = \operatorname{atan2}(r_{21},\ r_{32})$$

<u>If $q_5 = \frac{\pi}{2}$</u>

$$q_5 = q_{5\_previous}$$
$$q_4 = 0$$
$$q_6 = \operatorname{atan2}(r_{32},\ r_{22})$$