MVA: Detection theory and its industrial applications (2022/2023)	Homework 1

Detection theory and its industrial applications

Lecturers: *R. GROMPONE, J.M. MOREL* *(January, 2023)*

Solution by DI PIAZZA Théo

Instructions

- This file is the answer of the **Birthday Problem** AND **Gestalt commentary**.
- Part 1 : Birthday in a Class
- Part 2 : Gestalt commentary

1 Birthday in a Class

Notations

- Class of 30 students, birthdays are independant and uniformly distributed.
- For $1 \leq n \leq 30$, C_n the number of n-tuples of students of the class having the same birthday.
- $\mathbb{P}_n = \mathbb{P}(C_n \geq 1)$ the probability that there is at least one n-tuple with the same birthday.
- p_n the probability that there is at least one n-tuple and no (n+1)-tuple.

Question 1

Let's show that $\mathbb{P}_n = 1 - \sum_{i=1}^{n-1} p_i$

$$\mathbb{P}_n = \mathbb{P}(C_n \geq 1) = 1 - \mathbb{P}(C_n = 0)$$

with :

$$\mathbb{P}(C_n = 0) = \mathbb{P}(C_n = 0, C_{n-1} \geq 1) + \mathbb{P}(C_n = 0, C_{n-1} = 0)$$
$$= p_{n-1} + \mathbb{P}(C_n = 0, C_{n-1} = 0, C_{n-2} \geq 1) + \mathbb{P}(C_n = 0, C_{n-1} = 0, C_{n-2} = 0)$$
$$= ... = p_{n-1} + p_{n-2} + ... + p_2 + p_1 + \mathbb{P}(C_n = 0, C_{n-1} = 0, ..., C_1 = 0)$$
$$= p_{n-1} + p_{n-2} + ... + p_2 + p_1 = \sum_{i=1}^{n-1} p_i$$

Hence $\boxed{\mathbb{P}_n = 1 - \sum_{i=1}^{n-1} p_i}$

Let's show that $\mathbb{P}_n = \mathbb{P}_{n-1} - p_{n-1}$

$$\mathbb{P}_n = 1 - \sum_{i=1}^{n-1} p_i = 1 - \sum_{i=1}^{n-2} p_i - p_{n-1} = \mathbb{P}_{n-1} - p_{n-1}$$

Hence $\boxed{\mathbb{P}_n = \mathbb{P}_{n-1} - p_{n-1}}$

Question 2

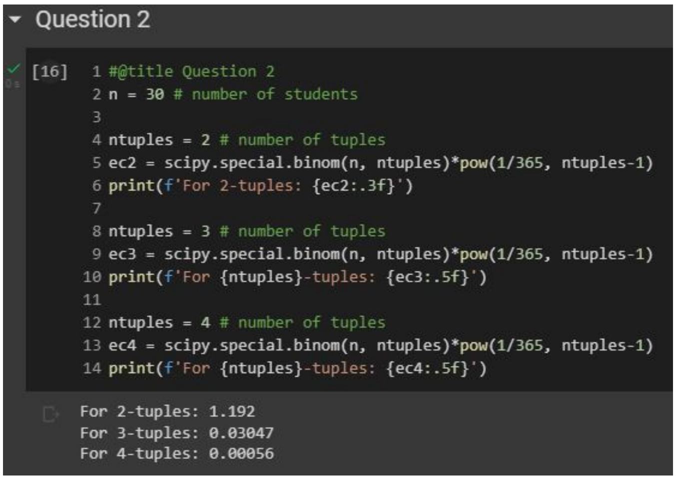
Let's show that $\mathbb{E}C_n = \binom{30}{n}\frac{1}{365^{n-1}}$

To compute it, let's define the event S_n such that : S_n = Students denoted by $s_1, ..., s_n$ have the same birthday.
Then it comes that:

$$\mathbb{E}C_n = \mathbb{E}\left[\sum_{1\leq s_1\leq ...\leq s_n\leq 30} \mathbb{1}S_n\right] = \sum_{1\leq s_1\leq ...\leq s_n\leq 30} \mathbb{E}\left[\mathbb{1}S_n\right] = C_{30}^n\mathbb{P}(S_n) = \binom{30}{n}\frac{1}{365^{n-1}}$$

Hence $\boxed{\mathbb{E}C_n = \binom{30}{n}\frac{1}{365^{n-1}}}$

We can check that $\mathbb{E}C_2 \approx 1.192$, $\mathbb{E}C_3 \approx 0.03047$ and $\mathbb{E}C_4 \approx 0.00056$.



```
Question 2
[16]
#@title Question 2
n = 30 # number of students

ntuples = 2 # number of tuples
ec2 = scipy.special.binom(n, ntuples)*pow(1/365, ntuples-1)
print(f'For 2-tuples: {ec2:.3f}')

ntuples = 3 # number of tuples
ec3 = scipy.special.binom(n, ntuples)*pow(1/365, ntuples-1)
print(f'For {ntuples}-tuples: {ec3:.5f}')

ntuples = 4 # number of tuples
ec4 = scipy.special.binom(n, ntuples)*pow(1/365, ntuples-1)
print(f'For {ntuples}-tuples: {ec4:.5f}')

For 2-tuples: 1.192
For 3-tuples: 0.03047
For 4-tuples: 0.00056
```

Figure 1: Question 2 with Python.

Question 3

Let's show that $\mathbb{P}(C_2 = 0) = \frac{365*...*336}{365^{30}}$

First, let's determine how many possibilities there are that no two pairs of students have the same birthday.

For the first treated student, it can be born among the 365 days. For the second treated student, it can be born among 364 days (365-the day of the treated student 1). For the third student, he can be born among 363 days (365-the days of treated students 1 and 2). And so on...

Thus the possible number of day combinations to have no pair of students born on the same day is $654 * 364 * 363 * ... * 336$.
It comes that:

$$\mathbb{P}(C_2 = 0) = \frac{365 * ... * 336}{365^{30}} = \frac{1}{365^{30}}\frac{365!}{335!} \approx 0.294$$

Finally, we can deduce that $P_2 = \mathbb{P}(C_2 \geq 1) = 1 - \mathbb{P}(C_2 = 0) = 1 - 0.294 = 0.706$

Question 4

Let's show that $p_2 = \frac{1}{365^{30}}\sum_{i=1}^{15}\frac{\prod_{j=1}^{i}\binom{32-2j}{2}}{i!}\prod_{k=0}^{29-i}(365-k)$

p_2 is the probability of having at least 1 pair of students with the same birthday and 0 triples with the same birthday. For every 30 students, there can be a maximum of 15 separate pairs ($15 * 2 = 30$).

Hence $p_2 = \sum_{i=1}^{15} \mathbb{P}(C_2 = i, C_3 = 0)$. Now, let's compute $\mathbb{P}(C_2 = i, C_3 = 0)$ for a fixed value of i (exactly i pair of students with the same birthday and 0 triples with the same birthday).

Possibilities of 2-tuple anniversary days
For the first pair treated, the number of days for their birthday available is 365. Once this day is taken by the first treated pair, there are 364 days available for the second pair. Once pairs 1 and 2 are treated, there are 363 days available. And so on... Once the i pairs are treated, $2i$ students are treated. This leaves $30-2i$ students to be processed, all with different birthdays. In total, this corresponds to $30-2i+i$ days.

The number of associated possibilities is therefore $\prod_{k=0}^{29-i}(365-k)$ for a given i.

Possibilities to form pairs between students
Once the management of the days is associated, we are interested in all possible combinations of pairs of students. Intuitively, we understand that the possibility of forming i pairs among 30 students is equal to :

$$\binom{30}{2} . \binom{28}{2} . \binom{26}{2} \ ... \ (i \ times) = \prod_{j=1}^{i} \binom{32-2j}{2}$$

By removing the redundancy of certain pair configurations, it comes to

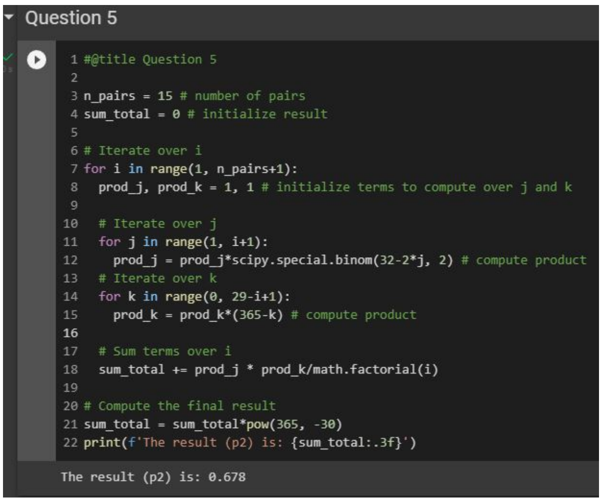
$$\prod_{j=1}^{i} \frac{\binom{32-2j}{2}}{i!}$$

Finally, by combining all the previous explanations, we find that :

$$\boxed{p_2 = \frac{1}{365^{30}} \sum_{i=1}^{15} \frac{\prod_{j=1}^{i} \binom{32-2j}{2}}{i!} \prod_{k=0}^{29-i}(365-k)}$$

Question 5

Below is the computer program that computes p_2. Please see Figure 2.


Question 5

```python
#@title Question 5

n_pairs = 15 # number of pairs
sum_total = 0 # initialize result

# Iterate over i
for i in range(1, n_pairs+1):
  prod_j, prod_k = 1, 1 # initialize terms to compute over j and k

  # Iterate over j
  for j in range(1, i+1):
    prod_j = prod_j*scipy.special.binom(32-2*j, 2) # compute product
  # Iterate over k
  for k in range(0, 29-i+1):
    prod_k = prod_k*(365-k) # compute product

  # Sum terms over i
  sum_total += prod_j * prod_k/math.factorial(i)

# Compute the final result
sum_total = sum_total*pow(365, -30)
print(f'The result (p2) is: {sum_total:.3f}')
```

```
The result (p2) is: 0.678
```

Figure 2: Question 5 with Python.

Question 6

$$\mathbb{P}_3 = \mathbb{P}_2 - p_2 = 0.706 - 0.678 \approx 0.028$$

Question 7

Let's show that $p_3 = \frac{1}{365^{30}} \sum_{i=1}^{10} \frac{\prod_{j=1}^{i} \binom{33-3j}{3}}{i!} \left[\prod_{k=0}^{29-2i} (365-k) + \sum_{l=1}^{[\frac{30-3i}{2}]} \frac{\prod_{m=1}^{l} \binom{30-3i+2-2m}{2}}{l!} \prod_{n=0}^{29-2i-l} (365-n) \right]$

To start, $p_3 = \mathbb{P}(C_3 \geq 1, C_4 = 0) = \sum_{i=1}^{10} \mathbb{P}(C_3 = i, C_4 = 0)$

Where $\mathbb{P}(C_3 = i, C_4 = 0) = \frac{\text{Number of possibilities to choose from the 3-tuples of the event}}{365^{30}}$

Possibilities to form 3-tuples between students

Once the management of the days is associated, we are interested in all possible combinations of 3-tuples of students. Intuitively, we understand that the possibility of forming i 3-tuples among 30 students is equal to :

$$\binom{30}{3} \cdot \binom{27}{3} \cdot \binom{24}{3} \ ... \ (i \ times) = \prod_{j=1}^{i} \binom{33-3j}{3}$$

By removing the redundancy of certain 3-tuple configurations, it comes to

$$\prod_{j=1}^{i} \frac{\binom{33-3j}{3}}{i!}$$

Possibilities of 3-tuple anniversary days - unique birthday outside 3-tuples

For the first 3-tuple treated, the number of days for their birthday available is 365. Once this day is taken by the first treated 3-tuple, there are 364 days available for the second 3-tuple. Once 3-tuples 1 and 2 are treated, there are 363 days available. And so on... Once the i 3-tuples are treated, $3i$ students are treated. This leaves $30 - 3i$ students to be processed, all with different birthdays. In total, this corresponds to $30 - 3i + i$ days.

The number of associated possibilities is therefore $\prod_{k=0}^{29-2i} (365-k)$ for a given i.

Possibilities of 3-tuple anniversary days - pairs of birthday outside 3-tuples

This time we consider l the number of 2-tuples of students who have the same birthday, apart from the triplets. Since there are already i 3-tuples, there can be at most $\left[\frac{30-3i}{2}\right]$ outside the 3-tuples.

With reasoning similar to that just shown in Question 4, there are $\frac{\prod_{m=1}^{l} \binom{30-3i+2-2m}{2}}{l!}$ possibilities of forming these 2-tuples outside the 3-tuples.

Concerning the choice of birthdays, there are i 3-tuples which corresponds to $3i$ pupils, l 2-tuples which corresponds to $2i$ pupils, so that leaves 30-3i-2l pupils whose birthdays are different from each other. This gives a total of possibilities of birthdays of : $\sum_{k=0}^{29-2i-l} (365-k)$.

Finally, putting all these terms together, it is clear that :

$$\boxed{p_3 = \frac{1}{365^{30}} \sum_{i=1}^{10} \frac{\prod_{j=1}^{i} \binom{33-3j}{3}}{i!} \left[\prod_{k=0}^{29-2i} (365-k) + \sum_{l=1}^{[\frac{30-3i}{2}]} \frac{\prod_{m=1}^{l} \binom{30-3i+2-2m}{2}}{l!} \prod_{n=0}^{29-2i-l} (365-n) \right]}$$

Question 8

```python
#@title Question 8

n_triplet = 10 # number of 3-tuplet
sum_total = 0 # intialize result

# Iterate over i
for i in range(1, n_triplet+1):
  prod_j, prod_k, sum_l, prod_m, prod_n = 1, 1, 0, 1, 1

  # Iterate over j
  for j in range(1, i+1):
    prod_j = prod_j*scipy.special.binom(33-3*j, 3) # compute product over j
  prod_j = prod_j/math.factorial(i)

  # Iterate over k
  for k in range(0, 29-2*i+1):
    prod_k = prod_k*(365-k) # compute product over k

  # Iterate over l
  for l in range(1, math.floor((30-3*i)/2)+1):
    # Iterate over m
    for m in range(1, l+1):
      prod_m = prod_m*scipy.special.binom(30-3*i+2-2*m, 2) # compute product over m
    prod_m = prod_m/math.factorial(l)
    # Iterate over n
    for n in range(0, 29-2*i-l+1):
      prod_n = prod_n*(365-n) # compute product over n

    print(prod_n)
    sum_l = prod_m*prod_n # compute sum over l

  # Sum terms over i
  sum_total += prod_j*(prod_k+sum_l)

# Compute the final result
sum_total = sum_total*pow(365, -30)
print(f'The result (p3) is: {sum_total:.6f}')
```

```
The result (p3) is: 0.027998
```

Figure 3: Question 8 with Python.

Question 8

Above is the computer program that computes p_3:
Hence, $\mathbb{P}_4 = \mathbb{P}_3 - p_3 \approx 0.0285 - 0.0280 \approx 5.4 * 10^{-4}$

Question 9

From questions 4 and 7, it follows that :

$$p_n = \frac{1}{365^{30}} \sum_{i=1}^{\left[\frac{30}{n}\right]} \frac{\prod_{j=1}^{i} \binom{30+n-nj}{n}}{i!} \left[\prod_{k=0}^{29-ni+i} (365-k) + \sum_{p=2}^{n-1} \sum_{l=1}^{\left[\frac{30-ni}{p}\right]} \prod_{m=1}^{l} \frac{\binom{30+ni+p-pm}{p}}{l!} \prod_{k=0}^{29-pi-pl+l} (365-k) \right]$$

Question 10

- Let's show that $\mathbb{E}C_{30} = \mathbb{P}_{30} = \frac{1}{365^{29}}$

  According to the definition:

$$\mathbb{P}_{30} = \mathbb{P}(C_{30} \geq 1) = \frac{365}{365^{30}} = \frac{1}{365^{29}}$$

Moreover with Question 2:

$$\mathbb{E}C_{30} = \binom{30}{30}\frac{1}{365^{29}} = \frac{1}{365^{29}}$$

Hence, $\boxed{\mathbb{E}C_{30} = \mathbb{P}_{30} = \frac{1}{365^{29}}}$

- Let's show that $\mathbb{E}C_{29} = \frac{30}{365^{28}}$ and $\mathbb{P}_{29} = \frac{30*364+1}{365^{29}}$

With Question 2:

$$\mathbb{E}C_{29} = \binom{30}{29}\frac{1}{365^{29}} = \frac{30!}{29!}\frac{1}{365^{28}} = \frac{30}{365^{28}}$$

Moreover:

$$\mathbb{P}_{29} = \mathbb{P}(C_{29} \geq 1) = \mathbb{P}(C_{29} \geq 1, C_{30} = 0) + \mathbb{P}(C_{29} \geq 1, C_{30} \geq 1)$$
$$= \mathbb{P}(C_{29} = 1, C_{30} = 0) + \mathbb{P}(C_{30} = 1) = \frac{30 * 364}{365^{29}} + \frac{1}{365^{29}} = \frac{30 * 364 + 1}{365^{29}}$$

Since the left term means *"Only one student is born on a different day than others (so 30 students can be chosen within the 364 available birthdays)."*

Hence, $\boxed{\mathbb{E}C_{29} = \frac{30}{365^{28}} \text{ and } \mathbb{P}_{29} = \frac{30 * 364 + 1}{365^{29}}}$

Question 11

From what has been shown Question 1, 2 and 10. It comes that :

$$\frac{\mathbb{E}C_2 - \mathbb{P}_2}{\mathbb{P}_2} = \frac{1.192 - 0.706}{0.706} = 0.6884 = 68.84\%$$
$$\frac{\mathbb{E}C_3 - \mathbb{P}_3}{\mathbb{P}_3} = \frac{0.034 - .0285}{.0285} = 0.2175 = 21.75\%$$
$$\frac{\mathbb{E}C_4 - \mathbb{P}_4}{\mathbb{P}_4} = \frac{5.6 * 10^{-4} - 5.3 * 10^{-4}}{5.3 * 10^{-4}} = 0.0566 = 5.66\%$$
$$\frac{\mathbb{E}C_{29} - \mathbb{P}_29}{\mathbb{P}_29} = \frac{\frac{30}{365^{28}} - \frac{30*364+1}{365^{29}}}{\frac{30*364+1}{365^{29}}} = 0.0027 = 0.27\%$$
$$\frac{\mathbb{E}C_{30} - \mathbb{P}_30}{\mathbb{P}_30} = \frac{\frac{1}{365^{29}} - \frac{1}{365^{29}}}{\frac{1}{365^{29}}} = 0 = 0\%$$

Question 12

Thanks to Markov inequalities, we have that $\forall n,\ \mathbb{P}_n \leq \mathbb{E}C_n$.

It means that, the smaller the value of $\mathbb{E}C_n$, the closer $\mathbb{P}_n$ tends to be to $\mathbb{E}C_n$ because it is bounded by $\mathbb{E}C_n$.

It is remarkable that from $n \geq 3$, the value $\mathbb{E}C_3$, this is why $\mathbb{E}C_n$ and $\mathbb{P}_n$ are so close for $n \geq 3$.

2 Gestalt commentary

The image

Figure 4: Image for the commentary (Source: https://www.flickr.com/photos/illektronik/4447759537).

The commentary

A symmetry on the vertical axis is observed.

On the sides, the columns are identical. On each column one can observe a reflection of light towards the interior of the image. They are evenly spaced, which gives an effect of depth. The columns form vertical lines parallel to each other. The vertical lines become shorter with depth.

On each side, the 3 columns support an other column fixed to the ceiling. On this column a continuity and vanishing perspective can be observed.

In the centre a car can be seen. For the commentary, this car can be interpreted as an object that hides the centre of the room. Around this object positioned in the centre we notice a multi-directional light (reflected on the floor) so the intensity decreases when it moves away from the centre. On this car, a rectangle is observed on a black background, on which white shapes of a consistent colour (the letters of the plate) can be observed. This creates a break with the rest of the object, which is also black (car boot).

On ce center, 2 bright red circles are present. They mask 2 white circles (phars masking by lighting). This is amodal completition.

On the ceiling we observe 2 tubes of white light and a vanishing effect.
On the ceiling are observed lines that are parallel to each other, evenly spaced. These lines seem to become shorter and shorter as they go along, hence the effect of depth. They appear to be perpendicular to the lines delineating the columns that support the you.

The ground is uniformly grey in colour with discontinuities in the position and intensity of darker patches. On this floor, apart from these stains, no distinct shape is recognisable. The ground is separated by a horizontal black line (at the level of the car's wheels).

In the background is a barrier with vertical and evenly spaced bars parallel to each other. They are masked by some objects (columns, car) but they remain recognisable: amodal completition.