MVA: Detection theory and its industrial applications (2022/2023) **Homework 2**

Detection theory and its industrial applications

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Instructions

- This file is the answer of the **Hoeffding's Inequality for a Sum of Random Variables**.

Hoeffding's Inequality for a Sum of Random Variables

Notations

- $\forall\ i\ \{1, .., l\} : 0 \leq X_i \leq 1$
- $S_t = \sum_{i=1}^{l} X_i$
- $p = \mathbb{E}[\frac{S_t}{l}]$
- $p_i = \mathbb{E}[X_i]$

Question 1

Below is the plot of the function $h(p)$ on the interval $[0, 1]$.

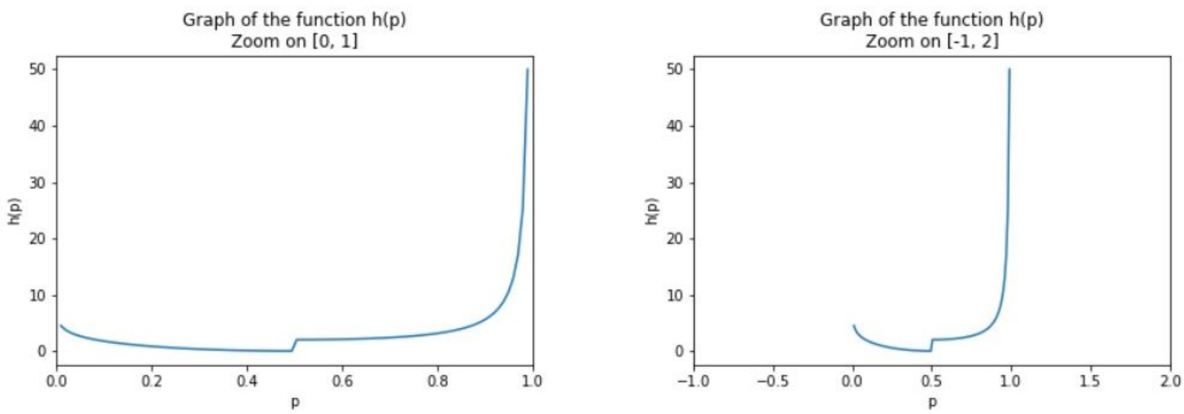

Figure 1: Question 1 with Python.

Question 2

Let X be a random variable such that $a \leq X \leq b$.
By convexity, it's known that for $x, y \in dom(f)$, and $\lambda_1, \lambda_2 \in [0, 1]$ that $f(\lambda_1 x + \lambda_2 y) \leq \lambda_1 f(x) + \lambda_2 f(y)$.

Hence as $\frac{b-x}{b-a}, \frac{x-a}{b-a} \geq 0$, it comes that :

$$e^{\lambda x} = e^{\lambda(\frac{b-x}{b-a}a + \frac{x-a}{b-a}b)} \leq \frac{b-x}{b-a}e^{\lambda a} + \frac{x-a}{b-a}e^{\lambda b}$$

$$\implies \boxed{\mathrm{E}[\mathrm{e}^{\lambda X}] \leq \frac{b-\mathbb{E}[X]}{b-a}e^{\lambda a} + \frac{\mathbb{E}[X]-a}{b-a}e^{\lambda b}}$$

Question 3

First part of the question

Let's show that $\mathbb{1}_{x \geq 0} \leq e^{\lambda x}$ for $\lambda > 0$.

- If $x < 0 \implies \mathbb{1}_{x \geq 0} = 0$, and $e^{\lambda x} \geq 0 \ \forall x < 0 \implies \mathbb{1}_{x \geq 0} \leq e^{\lambda x}$ for $x < 0$.

- Else, if $x \geq 0 \implies \mathbb{1}_{x \geq 0} = 1$, and $\min_{x \geq 0} e^{\lambda x} = e^0 = 1 \implies \mathbb{1}_{x \geq 0} \leq e^{\lambda x}$ for $x \geq 0$.

Hence for $\lambda > 0$,

$$\boxed{1_{x \geq 0} \leq e^{\lambda x}}$$

Second part of the question

Let's show that $\mathbb{P}[S_t \geq l(p+t)] \leq e^{-\lambda(p+t)l} \prod_{i=1}^{l} \mathbb{E}[e^{\lambda X_i}]$

Starting with the first part of the question applied to $1_{S_t - \mathbb{E}[S_t] - lt \geq 0}$, it comes that :

$$1_{S_t - \mathbb{E}[S_t] - lt \geq 0} \leq e^{\lambda 1_{S_t - \mathbb{E}[S_t] - lt}} \implies \mathbb{E}[1_{S_t - \mathbb{E}[S_t] - lt \geq 0}] \leq \mathbb{E}[e^{\lambda(S_t - \mathbb{E}[S_t] - lt)}]$$

$$\implies \mathbb{P}[S_t - \mathbb{E}[S_t] - lt \geq 0] = \mathbb{P}[S_t - pl - lt \geq 0] = \mathbb{P}[S_t \geq l(p+t)] \leq e^{-\lambda(\mathbb{E}[S_t] + lt)} \mathbb{E}[e^{\lambda S_t}]$$

$$\implies \boxed{\mathrm{P}[\mathrm{S}_t \geq l(p+t)] \leq e^{-\lambda l(p+t)} \prod_{i=1}^{l} \mathbb{E}[e^{\lambda X_i}]} \text{ since } X_1, ... X_l \perp\!\!\!\perp$$

Question 4

With the result of Question 2, it comes that :

$$\mathbb{E}[e^{\lambda X_i}] \leq \frac{1 - \mathbb{E}[X_i]}{1} e^0 + \frac{\mathbb{E}[X_i]}{1} e^{\lambda} = 1 - \mathbb{E}[X_i] + e^{\lambda} \mathbb{E}[X_i] = 1 - p_i + e^{\lambda} p_i$$

Hence,

$$\boxed{\prod_{i=1}^{l} \mathbb{E}[X_i] \leq \prod_{i=1}^{l} (1 - p_i + e^{\lambda} p_i)}$$

Finally, if the X_i's are Bernouilli random variables, their values are in the range $[0, 1]$ since the support of a Bernouilli random variable is $\{0, 1\}$. So the above inequality holds.

Question 5

Let's show that $(\prod_{i=1}^{l} a_i)^{\frac{1}{l}} \leq \frac{1}{l} \sum_{i=1}^{l} a_i$

Since the log function is concave, the Jensen's inequality can be used :

$$log(\frac{\sum_{i=1}^{l} a_i}{l}) \geq \sum_{i=1}^{l} \frac{log(a_i)}{l} = \frac{1}{l} \sum_{i=1}^{l} log(a_i) = \frac{1}{l} log(\prod_{i=1}^{l} a_i) = log((\prod_{i=1}^{l} a_i)^{\frac{1}{l}})$$

By taking the antilogs on the left and right sides, it comes that :

$$\boxed{(\prod_{i=1}^{l} a_i)^{\frac{1}{l}} \leq \frac{1}{l} \sum_{i=1}^{l} a_i}$$

Question 6

Let's show that $\prod_{i=1}^{l} \mathbb{E}[e^{\lambda X_i}] \leq (1-p-pe^{\lambda})^l$
Starting with Question 4, it comes that :

$$\prod_{i=1}^{l} \mathbb{E}[X_i] \leq \prod_{i=1}^{l}(1-p_i+e^{\lambda}p_i) = \prod_{i=1}^{l}(1-p_i+e^{\lambda}p_i) = ((\prod_{i=1}^{l}(1-p_i+e^{\lambda}p_i))^{\frac{1}{l}})^l$$

Then with Question 5, it comes that :

$$\prod_{i=1}^{l} \mathbb{E}[X_i] \leq (\frac{1}{l}\sum_{i=1}^{l}(1-p_i+e^{\lambda}p_i))^l = (1-\frac{\sum_{i=1}^{l}\mathbb{E}[X_i]}{l} - \frac{\sum_{i=1}^{l}\mathbb{E}[X_i]}{l}e^{\lambda})^l = (1-p-pe^{\lambda})^l$$

So finally,

$$\boxed{\prod_{i=1}^{l} \mathbb{E}[e^{\lambda X_i}] \leq (1-p-pe^{\lambda})^l}$$

Question 7

From Question 3 and Question 6, it comes that:

$$\mathbb{P}[S_t \geq (p+t)l] \leq e^{-\lambda(p+t)l}\prod_{i=1}^{l}\mathbb{E}[e^{\lambda X_i}] \leq e^{-\lambda(p+t)l}(1-p+pe^{\lambda})^l$$

To find the minimum in λ, let's define : $f(\lambda) = log(e^{-\lambda(p+t)l}(1-p+pe^{\lambda})^l)$ since the log function is positive and monotone on its domain definition.

It comes that :

$$f(\lambda) = l(-\lambda(p+t) + log(1-p+pe))$$

$$\implies \frac{\partial f(\lambda)}{\partial \lambda} = l(-\lambda(p+t) + \frac{pe^{\lambda}}{1-p+pe^{\lambda}}) = 0$$

$$\iff \frac{pe^{\lambda}}{1-p+pe^{\lambda}} = p+t$$

$$\iff e^{\lambda} = \frac{(p+t)(1-p)}{p(1-p-t)} \iff \boxed{\lambda = log(\tfrac{(p+t)(1-p)}{p(1-p-t)})}$$

To show that λ is positive when $0 < t < 1-p$, it's needed to show that $\frac{(1-p)(p+t)}{p(1-p-t)} > 1$. Hence :

$$\frac{(1-p)(p+t)}{p(1-p-t)} > 1 \iff p+t-p^2-pt > p-p^2-pt \iff t > 0$$

.
Hence,

$$\boxed{\lambda = log(\tfrac{(p+t)(1-p)}{p(1+p+t)}) > 0 \text{ when } 0 < t < 1-p}$$

Finally, let's obtain the first Hoeffding inequality.

$$\mathbb{P}[S_t \geq (p+t)l] \leq e^{-\lambda(p+t)l}(1-p+pe^{\lambda})^l = (e^{-\lambda(p+t)}(1-p+pe^{\lambda}))^l$$

Moreover, it can be shown that :

$$e^{-\lambda(p+t)}(1-p+pe^{\lambda}) = \left(\frac{(p+t)(1-p)}{p(1-p-t)}\right)^{-p-t}\left(1-p+p\frac{(p+t)(1-p)}{p(1-p-t)}\right)^1$$

$$= \left(\frac{p}{p+t}\right)^{p+t}\left(\frac{1-p}{1-p-t}\right)^{-p-t}\left(\frac{(1-p)(1-p-t)+(p+t)(1-p)}{1-p-t}\right)^{1}$$
$$= \left(\frac{p}{p+t}\right)^{p+t}\left(\frac{1-p}{1-p-t}\right)^{-p-t}\left(\frac{(1-p)(1-p-t+p+t)}{1-p-t}\right)^{1}$$
$$= \left(\frac{p}{p+t}\right)^{p+t}\left(\frac{1-p}{1-p-t}\right)^{-p-t}\left(\frac{1-p}{1-p-t}\right)^{1}$$
$$= \left(\frac{p}{p+t}\right)^{p+t}\left(\frac{1-p}{1-p-t}\right)^{1-p-t}$$

So finally,

$$\boxed{\mathrm{P}[S_t \geq (p+t)l] \leq \left(\frac{p}{p+t}\right)^{l(p+t)}\left(\frac{1-p}{1-p-t}\right)^{l(1-p-t)}}$$

Question 8

Let's define $G(t,p) = \frac{p+t}{t^2}log(\frac{p+t}{p}) + \frac{1-p-t^2}{t^2}log(\frac{1-p-t}{1-p})$, it comes that :

$$\frac{\partial G(t,p)}{\partial t} = \frac{t^2-2t(p+t)}{t^4}log(\frac{p+t}{p}) + \frac{\frac{1}{p}}{\frac{p+t}{p}}\frac{p+t}{t^2} + \frac{-t^2-2t(1-p-t)}{t^4}log(\frac{1-p-t}{1-p}) + \frac{1-p-t}{t^2}\frac{\frac{-1}{1-p}}{\frac{1-p-t}{1-p}}$$
$$= \frac{t^2-2t(p+t)}{t^4}log(\frac{p+t}{p}) + \frac{1}{t}^2 + \frac{-t^2-2t(1-p-t)}{t^4}log(\frac{1-p-t}{1-p}) - \frac{1}{t^2}$$
$$\Longrightarrow t^2\frac{\partial G(t,p)}{\partial t} = (1-2\frac{p+t}{t})log(\frac{p+t}{p}) - (1+2\frac{1-p-t}{t})log(\frac{1-p-t}{1-p})$$
$$= -(1-2\frac{p+t}{t})log(1-\frac{t}{t+p}) - (-1+2\frac{1-p}{t})log(1-\frac{t}{1-p})$$
$$= -(1-2\frac{p+t}{t})log(1-\frac{t}{t+p}) + (1-2\frac{1-p}{t})log(1-\frac{t}{1-p})$$
$$= (1-2\frac{1-p}{t})log(1-\frac{t}{1-p}) - (1-2\frac{p+t}{t})log(1-\frac{t}{t+p})$$
$$\Longrightarrow \boxed{t^2\frac{\partial G(t,p)}{\partial t} = H(\frac{t}{1-p}) - H(\frac{t}{t+p})}$$

Then with Taylor series, it can be shown that :

$$log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} + \mathcal{O}(x^6)$$
$$\Longrightarrow H(x) = (1-\frac{2}{x})log(1-x) = (1-\frac{2}{x})(-x-\frac{x^2}{2}-\frac{x^3}{3}-\frac{x^4}{4}-\frac{x^5}{5}+\mathcal{O}(x^6))$$
$$\Longrightarrow H(x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} + 2 + x + \frac{2}{3}x^2 + \frac{x^3}{2} + \frac{2}{5}x^5 + ...$$
$$\Longrightarrow H(x) = 2 + (\frac{2}{3}-\frac{1}{2})x^2 + (\frac{2}{4}-\frac{1}{3})x^3 + (\frac{2}{5}-\frac{1}{4})x^4 + ...$$

Then, to show that $H(x)$ is increasing for $0 < x < 1$, it can be shown that :

$$H'(x) = 2(\frac{2}{3}-\frac{1}{2})x + 3(\frac{2}{4}-\frac{1}{3})x^2 + 4(\frac{2}{5}-\frac{1}{4})x^3 > 0 \text{ for } 0 < x < 1 \Longrightarrow \boxed{H(x) \text{ is increasing for } 0 < x < 1.}$$

Then, it can be shown that :

$$\frac{\partial G(t,p)}{\partial t} > 0 \iff \frac{1}{t^2}(H(\frac{t}{1-p}) - H(\frac{t}{t+p})) > 0 \iff H(\frac{t}{1-p}) - H(\frac{t}{t+p}) > 0$$
$$\iff \frac{t}{1-p} > \frac{t}{t+p} \text{ since } H(x) \text{ is increasing with } 0 < x < 1$$
$$\iff t(p+t) > t(1-p) \iff t > 1-2p$$

Hence,

$$\boxed{\frac{\partial G(t,p)}{\partial t} > 0 \iff t > 1-2p}$$

Finally, we got that :

$$\frac{\partial G(1-2p,p)}{\partial t} = 0$$

which leads to :

$$G(1-2p,p) = \frac{p+1-2p}{(1-2p)^2}log(\frac{p+1-2p}{(1-2p)^2}) + \frac{1-p-1+2p}{(1-2p)^2}log(\frac{1-p-1+2p}{1-p})$$
$$= \frac{1-p}{(1-2p)^2}log(\frac{1-p}{p}) + \frac{p}{(1-2p)^2}log(\frac{p}{1-p}) = \frac{1-2p}{(1-2p)^2}log(\frac{1-p}{p})$$

Hence,

$$\boxed{h(p) = \frac{1}{1-2p}log(\frac{1-p}{p})}$$

Question 9

It has been shown that $G(t,p)$ attains its minimum for $t = 1-2p$. If $1-2p \leq 0$, it comes that $G(t,p)$ attains its minimum when $t \to 0$. In such case, we have that :

$$\boxed{\lim_{t\to 0} G(t,p) = \frac{1}{2p(1-p)} = h(p)}$$

Finally we got have :

$$h(\frac{1}{2}) = \frac{1}{2\frac{1}{2}(1-\frac{1}{2})} = 2$$

Moreover $h(p)$ attains its minimum when $p = \frac{1}{2}$ since :

$$h'(p) = \frac{4p-2}{(2p(1-p))^2} = 0 \iff p = \frac{1}{2}$$

So finally we have that $\boxed{h(p) \geq h(\frac{1}{2}) = 2}$ and the proof is complete :

$$P\left[S_t \geq (p+t)l\right] \leq (\frac{p}{p+t})^{l(p+t)}(\frac{1-p}{1-p-t})^{l(1-p-t)} \leq e^{-lt^2h(p)} \leq e^{-2lt^2}$$

where

$$h(p) = \frac{1}{1-2p}log(\frac{1-p}{p}) \text{ if } 0 < p < \frac{1}{2}$$
$$\text{and } h(p) = \frac{1}{2p(1-p)} \text{ if } \frac{1}{2} \leq p < 1$$

End of homework 2 - Thank you for reading.