DI PIAZZA
Théo

HW1 - Convex Optimization

Exercice 1

1) $C = \{ x \in \mathbb{R}^n \mid \alpha_i \leq x_i \leq \beta_i, \ i = 1, \dots, n \}$

A rectangle is a finite intersection of half-spaces

A half-space is a convex set

An intersection of convex sets is convex

$\Rightarrow$ C is a convex set

2) $C = \{ x \in \mathbb{R}^2_+ \mid x_1 x_2 \geq 1 \}$

Let $x = (x_1, x_2) \in C$ and $y = (y_1, y_2) \in C$ and $\lambda \in [0, 1]$

$z = \lambda x + (1-\lambda) y \in C$ if $z_1 z_2 \geq 1$, where $z = (z_1, z_2)$

$$z_1 z_2 = (\lambda x_1 + (1-\lambda) y_1)(\lambda x_2 + (1-\lambda) y_2) = \lambda^2 + (1-\lambda)^2 + \lambda(1-\lambda)(x_1 y_2 + y_1 x_2)$$
$$= 2\lambda^2 + 1 - 2\lambda + \lambda(1-\lambda)(x_1 y_2 + y_1 x_2) = 1 + \underbrace{\lambda(1-\lambda)}_{\geq 0}(x_1 y_2 + y_1 x_2 - 2)$$

Then $z_1 z_2 \geq 1$ if $x_1 y_2 + y_1 x_2 - 2 \geq 0$

$x, y \in C$ then $x_1 y_2 + y_1 x_2 \geq \frac{x_1}{y_1} + \frac{y_1}{x_1} = \frac{x_1^2 + y_1^2}{x_1 y_1} = \frac{(x_1 - y_1)^2 + 2 x_1 y_1}{x_1 y_1}$

$$= 2 + \underbrace{\frac{(x_1 - y_1)^2}{x_1 y_1}}_{\geq 0}$$

$\Rightarrow x_1 y_2 + y_1 x_2 - 2 \geq 0 \Rightarrow z_1 z_2 \geq 1 \Rightarrow z = \lambda x + (1-\lambda) y \in C$

$\Rightarrow$ C is a convex set

3) $C = \{x \mid \|x - x_0\|_2 \leq \|x - y\|_2 \ \forall y \in S\}$ where $S \subset \mathbb{R}^n$

Let $y \in S$, we define $C_y = \{x \mid \|x - x_0\|_2 \leq \|x - y\|_2 , y \in S\}$

Then $x \in C_y \iff \|x - x_0\|_2 \leq \|x - y\|_2$

$$\iff \|x - x_0\|_2^2 \leq \|x - y\|_2^2$$

$$\iff \|x\|_2^2 + \|x_0\|_2^2 - 2x^T x_0 \leq \|x\|_2^2 + \|y\|_2^2 - 2x^T y$$

$$\iff x^T (y - x_0) \leq \frac{\|y\|_2^2 - \|x_0\|_2^2}{2}$$

$$\iff (y - x_0)^T x \leq \frac{\|y\|_2^2 - \|x_0\|_2^2}{2}$$

$$\iff x \in \left\{ z \mid (y - x_0)^T z \leq \frac{\|y\|_2^2 - \|x_0\|_2^2}{2} , y \in S \right\}$$

which is the definition of a half-space $\Rightarrow C_y$ is convex set

Finally, C is the intersection of half-spaces $\Rightarrow C$ is a convex set

4) $C = \{x \mid \operatorname{dist}(x, S) \leq \operatorname{dist}(x, T)\}$ where $S, T \subseteq \mathbb{R}^n$

and $\operatorname{dist}(x, S) = \inf\{\|x - y\|_2 \mid y \in S\}$

We define a simple example such that $S = \{-1, 1\}$ and $T = \{0\}$

Then $C =]-\infty, -\frac{1}{2}] \cup [\frac{1}{2}, +\infty[$ which is not a convex set.

5) $C = \{ x \mid x + S_2 \subseteq S_1 \}$ where $S_1, S_2 \subseteq \mathbb{R}^n$
and S_1 convex

We define $x \in C$ and $y \in C$ and $\lambda \in [0,1]$
and $z = \lambda x + (1-\lambda) y$

Let $k \in S_2$, then $z + k = \lambda x + (1-\lambda) y + k + \lambda k - \lambda k$
$= \lambda \underbrace{(x+k)}_{\in x+S_2 \subseteq S_1} + (1-\lambda) \underbrace{(y+k)}_{\in x+S_2 \subseteq S_1}$

Then $\lambda (x+k) + (1-\lambda)(y+k) = z + k \in S_1$
since S_1 is a convex set.

$\Rightarrow$ C is a convex set.

DI PIAZZA

Théo

HW1 - Convex optimization

Exercice 2

1) $f(x_1, x_2) = x_1 x_2$ on $\mathbb{R}^2_{++}$, f is twice differentiable on $\mathbb{R}^2_{++}$

With $\nabla f(x) = (x_2 \quad x_1)$ and $\nabla^2 f(x) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

And $|\nabla^2 f - \lambda I_2| = \lambda^2 - 1$ with $\lambda \in \mathbb{R}$

Then 1 and -1 are eigen values of $\nabla^2 f(x)$

$\Rightarrow$ The function is neither convex nor concave on $\mathbb{R}^2_{++}$

2) $f(x_1, x_2) = \dfrac{1}{x_1 x_2}$ on $\mathbb{R}^2_{++}$, f is twice differentiable on $\mathbb{R}^2_{++}$

With $\nabla f(x) = \left(\dfrac{-1}{x_1^2 x_2} \quad \dfrac{-1}{x_1 x_2^2}\right)$ and $\nabla^2 f(x) = \dfrac{1}{x_1 x_2}\begin{pmatrix} \frac{2}{x_1^2} & \frac{1}{x_1 x_2} \\ \frac{1}{x_1 x_2} & \frac{2}{x_2^2} \end{pmatrix}$

Then, for $u = (a, b) \in \mathbb{R}^2$: $u^T \nabla^2 f(x) u = (a \ b) \begin{pmatrix} \frac{2}{x_1^3 x_2} & \frac{1}{x_1^2 x_2^2} \\ \frac{1}{x_1^2 x_2^2} & \frac{2}{x_1 x_2^3} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$

$$\Rightarrow 2g(a, b) = u^T \nabla^2 f(x) u = 2\left(\frac{a^2}{x_1^3 x_2} + \frac{b^2}{x_1 x_2^3} + \frac{ab}{x_1^2 x_2^2}\right)$$

So we have $g(a, b) = v^2 + w^2 + vw$ with $v = \dfrac{a}{x_1^{3/2} x_2^{1/2}}$ and $w = \dfrac{b}{x_1^{1/2} x_2^{3/2}}$

* If we assume that $v < 0 < w$ and $-v \leq w$

Then $w^2 + vw \geq 0 \Rightarrow g(a,b) \geq 0$

* If we assume that $v < 0 < w$ and $-v > w$

Then $v^2 + vw > 0 \Rightarrow g(a,b) > 0$

$\Rightarrow$ So the Hessian of $f(x_1, x_2)$ is positive semi definite $\Rightarrow f(x_1, x_2)$ is convex on $\mathbb{R}^2_{++}$.

3) $f(x_1, x_2) = \frac{x_1}{x_2}$ on $\mathbb{R}^2_{++}$, f is twice differentiable on $\mathbb{R}^2_{++}$

With $\nabla f(x_1, x_2) = \begin{pmatrix} \frac{1}{x_2} & -\frac{x_1}{x_2^2} \end{pmatrix}$ and $\nabla^2 f(x_1, x_2) = \begin{pmatrix} 0 & -\frac{1}{x_2^2} \\ -\frac{1}{x_2^2} & \frac{2x_1}{x_2^3} \end{pmatrix}$

We define $g(\lambda) = \det(\nabla^2 f(x) - \lambda I_2) = -\lambda\left(\frac{2x_1}{x_2^3} - \lambda\right) - \frac{1}{x_2^4} = \lambda^2 - 2\lambda \frac{x_1}{x_2^3} - \frac{1}{x_2^4}$

with $\nabla g = \frac{4(x_1^2 + x_2^2)}{x_2^6} > 0 \Rightarrow g(\lambda) = 0$ has 2 real solution λ_1 and λ_2.

where $\lambda_1 = \frac{x_1 - \sqrt{x_1^2 + x_2^2}}{x_2^3} < 0$ and $\lambda_2 = \frac{x_1}{x_2^3} + \frac{\sqrt{x_1^2 + x_2^2}}{x_2^3} > 0$

Then the Hessian of f is not positive or negative semidefinite.

$\Rightarrow f$ is not convex or concave on $\mathbb{R}^2_{++}$

4) $f(x_1, x_2) = x_1^{\alpha} x_2^{1-\alpha}$ where $0 \leq \alpha \leq 1$ on $\mathbb{R}^2_{++}$, f is twice differentiable on $\mathbb{R}^2_{++}$

With $\nabla f(x_1, x_2) = \begin{pmatrix} \alpha x_1^{\alpha-1} x_2^{1-\alpha} & (1-\alpha) x_1^{\alpha} x_2^{-\alpha} \end{pmatrix}$

and $\nabla^2 f(x_1, x_2) = \underbrace{\alpha(1-\alpha) x_1^{\alpha} x_2^{-\alpha}}_{\geq 0} \begin{pmatrix} -\frac{x_2}{x_1^2} & \frac{1}{x_1} \\ \frac{1}{x_1} & -\frac{1}{x_2} \end{pmatrix} = \underbrace{\alpha(1-\alpha) x_1^{\alpha} x_2^{1-\alpha}}_{\geq 0} \begin{pmatrix} \frac{1}{x_1^2} & -\frac{1}{x_1 x_2} \\ -\frac{1}{x_1 x_2} & \frac{1}{x_2^2} \end{pmatrix}$

We define $g(\lambda) = \begin{vmatrix} -\frac{x_2}{x_1^2} - \lambda & \frac{1}{x_1} \\ \frac{1}{x_1} & -\frac{1}{x_2^2} - \lambda \end{vmatrix} = \begin{vmatrix} \frac{1}{x_1^2} - \lambda & -\frac{1}{x_1 x_2} \\ -\frac{1}{x_1 x_2^2} & \frac{1}{x_2^2} - \lambda \end{vmatrix} = \lambda^2 - \lambda\left(\frac{1}{x_1^2} + \frac{1}{x_2^2}\right)$

when $\Delta g = \left(\frac{1}{x_1^2} + \frac{1}{x_2^2}\right)^2 > 0$ then $g(\lambda) = 0$ has 2 real solution λ_1 and λ_2.

$\lambda_1 = 0$ and $\lambda_2 = -\left(\frac{1}{x_1^2} + \frac{1}{x_2^2}\right) < 0 \Rightarrow$ The Hessian of f is negative semi definite.

$\Rightarrow f$ is concave on $\mathbb{R}^2_{++}$

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HW1 - Convex Optimization

Exercice 3

1) $f(X) = Tr(X^{-1})$ on $dom f = S_{++}^n$

Let's define $h(t) = f(A + tB)$ with $A \succ 0$ and $B \subset S^n$

$\Rightarrow h(t) = f(A+tB) = Tr((A+tB)^{-1}) = Tr(A^{-1}(I + tA^{-1/2}BA^{-1/2})^{-1})$

$= Tr(A^{-1}(I + tQ\Lambda Q^T)^{-1})$ where $A^{-1/2}BA^{-1/2} = Q\Lambda Q^T$ (eigenvalue decomposition)

$= Tr(A^{-1}Q(I + t\Lambda)^{-1}Q^T)$

$= Tr(Q^T A^{-1} Q (I + t\Lambda)^{-1})$

$= \sum_{i=1}^{n} \underbrace{(Q^T A^{-1} Q)_{ii}}_{\geq 0} \times \underbrace{\frac{1}{1 + t\lambda_i}}_{\text{convex}}$ where λ_i are eigenvalues of $A^{-1/2}BA^{-1/2}$

$\Rightarrow h(t)$ is a positive weighted sum of convex functions

$\Rightarrow$ The function f is convex on $dom f$.

2) $f(X, y) = y^T X^{-1} y$ on $dom f = S_{++}^n \times \mathbb{R}^n$

Let's show that $epi f$ is convex.

$epi f = \{(X, y, t) \in S_{++}^n \times \mathbb{R}^n \times \mathbb{R} \mid f(X, y) \leq t\}$

$\Rightarrow epi f = \{(X, y, t) \mid X \succ 0, y^T X^{-1} y \leq t\}$

With $\begin{bmatrix} X & y \\ y^T & t \end{bmatrix}$ a block matrix with $X \in S_{++}^n$ and $\det(X) \neq 0$

Therefore using the Schur complement condition, we can write that:

$$\operatorname{epi} f = \{(X, y, t) \mid \begin{bmatrix} X & y \\ y^T & t \end{bmatrix} \succcurlyeq 0,\ X \succ 0\}$$

where conditions are linear matrix inequality in (X, y, t)

$\Rightarrow \operatorname{epi} f$ is convex $\Leftrightarrow$ f is convex on $\operatorname{dom} f$

3) $f(X) = \sum_{i=1}^{m} \sigma_i(X)$ on $\operatorname{dom} f = S^n$

where $\sigma_1(X), \ldots, \sigma_n(X)$ are singular values of X.

$f(X) = \sum_{i=1}^{m} \sigma_i(X) = \operatorname{tr}(X^T X)^{1/2}$ since

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HW1 Convex Optimization

Exercice 4

$$K_m^+ = \{x \in \mathbb{R}^m \mid x_1 \geq x_2 \geq \dots \geq x_m \geq 0\}$$

1) * K_{m+} is defined by m homogenous linear inequalities

$\Rightarrow K_{m+}$ is a closed cone. And it is convex.

* If $x = (m, m-1, m-2, \dots, 1) \Rightarrow x \in K_m^+ \Rightarrow K_m^+$ is nonempty

* Let $x = (x_1, \dots, x_m) \in K_m^+$ such that $x_1 \geq x_2 \geq, \dots \geq x_m$

Then $y = (y_1, \dots, y_m) = -x$ is such that $y_1 \leq y_2 \leq \dots \leq y_m$

$\Rightarrow y = -x \notin K_m^+ \Rightarrow K_m^+$ does not contain an entire line

$\Rightarrow K_{m+}$ is pointed

$\Rightarrow K_{m+}$ is a closed, solid and pointed convex cone $\Rightarrow K_{m+}$ is a proper cone

2) Dual cone of K_{m+} is $K_{m+}^* = \{y \mid y^T x \geq 0 \ \forall x \in K_{m+}\}$

Then $y^T x = \sum_{i=1}^{m} y_i x_i = y_1 x_1 + y_2 x_2 + \dots + y_m x_m$

$$= y_1 \underbrace{(x_1 - x_2)}_{\geq 0} + (y_1 + y_2)\underbrace{(x_2 - x_3)}_{\geq 0} + \dots + \sum_{i=1}^{n-1} y_i \underbrace{(x_n - x_{n+1})}_{\geq 0} + \sum_{i=1}^{n} y_i \underbrace{x_n}_{\geq 0}$$

$\Rightarrow y^T x \geq 0 \iff \forall j \in \{1, 2, \dots, m\} : \sum_{i=1}^{j} y_i \geq 0$

$$\Rightarrow K_{m+}^* = \left\{ y \mid \sum_{i=1}^{j} y_i \geq 0, \forall j \in \{1, 2, \dots, m\} \right\}$$

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Exercice 3

1) $f(x) = \max_{i=1,\dots,m} x_i$ on $\mathbb{R}^m$

The conjugate of f is $f^*(y) = \sup_{x \in \mathrm{dom} f} (y^T x - f(x))$

$\Rightarrow f^*(y) = \sup_{x \in \mathrm{dom} f} y^T x - \max_i x_i$

First, we remark that $y^T x - f(x) \leq \max_i x_i \left(\sum_{i=1}^m y_i - 1\right)$

Then to maximize $y^T x$, we can take x as a constant variable.

For this, we define $x = t\mathbb{1}$ where $\mathbb{1} = (1, 1, \dots, 1)$ and $t \in \mathbb{R}$

$\Rightarrow y^T x - f(x) = t\left(\sum_{i=1}^m y_i - 1\right)$

* If $\sum_{i=1}^m y_i > 1 \Rightarrow y^T x - f(x) \xrightarrow[t \to +\infty]{} +\infty$ Then $f^*(y) = +\infty$

* Else if $\sum_{i=1}^m y_i < 1 \Rightarrow y^T f(x) - f(x) \xrightarrow[t \to -\infty]{} +\infty$ Then $f^*(y) = +\infty$

* Else $\left(\sum_{i=1}^m y_i = 1\right)$, 2 cases are possible:

<u>1st case</u> $\rightarrow \forall i \in \{1, 2, \dots, m\}, y_i \geq 0 \Rightarrow y^T x - f(x) \leq 0$ and $y^T x - f(x) = 0$ if $x = 0$

$\Rightarrow f^*(y) = 0$

<u>2nd case</u> $\longrightarrow \exists y_j$ such that $y_j < 0$ $(j \in \{1, \dots, m\})$

We redefine x: $\begin{cases} x_j = t & \text{if } i = j \\ x_i = 0 & \text{if } i \neq j \end{cases}$ $\Rightarrow y^T x - f(x) = t y_j - \max(0, t) \xrightarrow[t \to -\infty]{} +\infty$

$\Rightarrow f^*(y) = +\infty$

Continuation of Exercice 5, Question 1

So finally: $f^*(y) = \begin{cases} 0 \text{ if } y \geqslant 0 \text{ and } \sum_{i=1}^{n} y_i = 1 \\ +\infty \text{ otherwise} \end{cases}$