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Research project

Screening techniques for the LASSO in large dimension

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Abstract

As part of the 4th grade in the INSA's Mathematics Department, it is proposed to follow a "research project". This project allows engineering students to discover a research field and research methods by being supervised by a researcher. Several subjects and supervisors were proposed. The one I chose is about Sparse representation and was supervised by Cedric Herzet with the help of Clement Elvira.

Over the past decade, many articles focused on the development of screening techniques. The aim of these techniques is to identify null-coefficients of the optimal solution of the LASSO problem without needing to solve it. This is useful to reduce the problem dimension when it is an issue for classical iterative algorithms. LASSO has many applications : image processing, signal processing, statistics, ... In this report, we will explain and review most of the screening techniques. We will see first how to construct test allowing to identify null-coefficients of the solution and then and how to use these tests in practice. Finally, we will see how to generalize such tests to other problems.

Keywords : Screening, Dynamic screening, SAFE region, LASSO, High dimensional optimization, ISTA, LARS

Acknowledgements

First, I would like to thank the Mathematics Department of INSA Rennes for providing such good study program. The teaching and opportunities given to students are of high quality. Thanks for allowing me to follow this research project. I would also like to thank Cédric Herzet and Clément Elvira for having supervised me for six months. I learned a lot from them and this project will help me a lot, I am positive, for the rest of my professional career. Finally, I would like to thank the three researchers who took time to answer my questions about their work.

Contents

1 Introduction

"Modelling data is the way we, scientists, believe that information should be explained and handled"

– Michael Elad

1.1 Sparse representation

Sparse representation is the subject of lots of studies since few decades. The principle of this particular representation of data is to find a model depending only on few parameters. Usually, models representing data are used as "black boxes" and combines lots of parameters. In contradiction, sparse models represents data with as few parameters as possible. Given a set of data $y = (y_1, \dots, y_m)$ and a matrix of parameters $A = [a_{ij}]_{(1 \leq i \leq n; 1 \leq j \leq m)}$ (called a *dictionary*), the aim is to find a *sparse vector* x (*i.e* a vector with lot of null-coefficients) satisfying $y \simeq Ax$. Knowing a solution for this problem gives a way to resume y by a linear combination of only few parameters.

There are lots of applications for this particular manner of representing data. First, it can be useful to approximate a long vector into a longer but sparse vector with the help of parameters. In practice, to process a sparse vector in memory, only the coordinates and the values of the non-null coefficients are stored. A sparse vector can be larger than a full (non-sparse) vector but easier to store in memory. Sparse representation is a way to compress data. An other use of sparse representation is to construct a model to resume data which is understandable by humans. Indeed, if we can resume data with a small number of parameters, we can hope to find an real explanation of the influence of those parameter on the data. The whole challenge of sparse representation is to find a solution as sparse as possible but which can resume the data with accurately.

Here is a simple example of sparse representation application. Consider y as the average lifetime of different countries and A some parameters (GPD, health services quality, percentage of illiterates, average happiness, etc...) for each country. The problem is to resume y with the parameters of A. Solving this problem with sparse techniques can lead to a model which gives average lifetime in function of GPD and average happiness only. In reality, average lifetime doesn't rely only on those two parameters but this model gives a convenient way to understand lifetime. With this model, it is simple to extrapolate some results or to predict lifetime of a country knowing only on the GDP and the average happiness of this country. Other models can give a solution depending on all the parameters but it is more difficult to really understand how each parameter acts on data. Sparse representation has lots of application in image processing, media compression, medical imaging, signal processing ... [Pap+18][Rud+17]

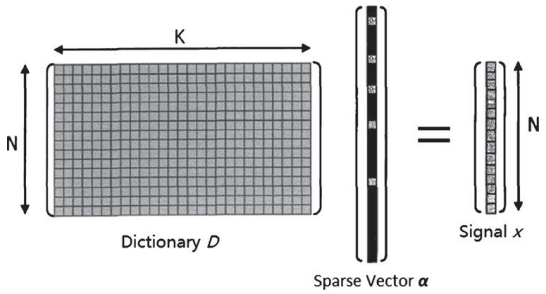

Figure 1: A sparse representation α of the vector x using the dictionary D

1.2 From sparse representation to LASSO

A way to express the sparse representation problem is under the following form :

$$\mathcal{P}_{ideal}(t) = \min_x \left\{ \|y - Ax\|_2, \|x\|_0 \leq t, t \in \mathbb{N} \right\} \tag{1}$$

$\mathcal{P}_{ideal}$ is called the ℓ_0-minimization problem. Here, $y \in \mathbb{R}^m$ is a non-sparse vector, $x \in \mathbb{R}^n$ is the sparse representation of y through the dictionary A. $\|x\|_0$ denotes the number of non-null coefficients of x. t is a fixed integer as small as possible. Indeed, the aim of the sparse representation is to resume the information contained in y into another sparse vector x so that's why we want to minimize the number of non-null coefficients of x. We also want to have

a good approximation Ax of y and that is why we want to minimize the gap between y and Ax. To have a good approximation of y by Ax, the dimension of x must be larger than the dimension of y. Unfortunately, this problem is NP-hard [FR13]. To solve this problem, studies mainly focuses on its relaxation which is

$$\mathcal{P}_{relaxed}(\lambda) : \min_x \left\{ \|y - Ax\|_2 + \lambda \|x\|_0, \lambda \geq 0 \right\} \tag{2}$$

Where λ is called a *regularization parameter*. The second term of the objective function penalizes non-sparse vectors as they increase this function. That is how we can hope to have a sparse solution. The difference between the solution of (1) and its relaxation depends on λ [Ela10]. The 0-norm is not convenient to solve $\mathcal{P}_{relaxed}(\lambda)$ because it is hard to deal with its mathematical expression. That's why we focus on an other problem, the LASSO (or ℓ_1-minimization) problem :

$$\boxed{\mathcal{P}(\lambda) : \min_x \left\{ \frac{1}{2} \|y - Ax\|_2^2 + \lambda \|x\|_1, \lambda \geq 0 \right\}} \tag{3}$$

The 1-norm is more convenient to work with and we will see the LASSO problem has some very good properties, especially its dual problem. The 0-norm is replaced by the 1-norm but the solution x^* of (3) is still sparse. In fact, we can prove that $\|x^*\|_0 \leq m$ so x^* will be more sparse than y [FR13]. We can hope that solving the LASSO leads to a solution, worst than the solution of (1), but which can still resume sparingly and accurately y. As $\mathcal{P}(\lambda)$ is convex, the optimal solution x^* is unique.

In this report, we will *not* focus on finding a good λ, which is a whole subject of research. We will assume that we want to solve the LASSO for a *given* λ. In the following, we will consider that data and parameters are normalized (*i.e* $\|y\|_2 = 1$ and $\|a_i\|_2 = 1, \forall i = 1, \ldots, n$ where $A = [a_1, \ldots, a_n]$).

1.3 Dimension problem

The LASSO problem is well known and there are many algorithms which can solve problems of this type. One of the most famous is the ISTA algorithm [Ami09], which is a gradient-descent based algorithm. This algorithm used to solve (3) requires the computation of eigenvalues of A^TA which in a square matrix of the dimension of x and other manipulations of the matrix A. In some cases, like image processing, y can have a very large dimension (high-resolution picture for example). We said that x have to be even larger than y. The computation of the eigenvalues of A^TA can be very tricky because of its dimension. Even the load in memory of A^TA can be very challenging for a standard computer. This problem is not only specific to the ISTA algorithm but it is a common problem we have to face while solving LASSO in large dimension cases. For the sequel, we consider that

- A computation needing $o(m)$ operations is well processable
- A computation needing $o(nm)$ operations is processable
- A computation needing $o(n^2)$, $o(nm^2)$ or more operations is hard to process

Since a decade, techniques called *screening* techniques [GVR10] are developed to face this dimensionality problem. The aim is to identify some of the null coefficients of the solution with tests *before* or *during* the solving process. In fact, finding out some null-coefficients of the solution allows to reduce the dimension of the problem. If we know that $x_i^* = 0$, it is possible to exclude this coefficient and the i-th column of A during the resolution process so as to reduce the dimension of the problem. Those techniques have to satisfy two conditions to be useful when facing the dimension problem. We say that a screening test is a "good" test if :

- The test can identify lots of null-coefficients of x^* so as to reduce at most the dimension of the problem.
- The test is cheap to compute so it is worth it to be done.

2 LASSO and screening procedures

To face the dimensionality issue of the LASSO when dealing with large scaled data, tests able to identify null-coefficients of the solutions before and during the solving process can be proceeded. In this section, we will see how to construct such tests which are mostly based on geometrical intuitions.

2.1 Primal and Dual LASSO

Classical algorithms used to solve the LASSO are based on its primal form. In contrast, screening techniques are based on its dual form. In fact, the dual LASSO problem has some very interesting properties to find out null-coefficients of the solution. Dual LASSO problem can be parametrized as :

$$\boxed{\mathcal{D}(\lambda) : \max_{\theta \in \mathbb{R}^m} \left\{ \frac{1}{2}\|y\|_2^2 - \frac{\lambda^2}{2}\|\theta - \frac{y}{\lambda}\|_2^2, \quad |\theta^T a_i| \leq 1, \ i = 1, \dots, n \right\}} \tag{4}$$

For the dual LASSO, we assume that the point y/λ is outside the feasible region ($i.e$ $\lambda \leq 1$). If y/λ is inside the feasible region $\mathcal{F} = \cap_{i=1}^n \{\theta, |\theta^T a_i| \leq 1\}$, we can easily find the optimal solution which is $\theta^* = y/\lambda$. In the opposite case, the dual solution θ^* is the projection of y/λ onto the feasible region. As we supposed the normalization of y and the columns of A, we have that y and $a_i, i \in 1, \dots, n$ are in the unit sphere $\mathcal{S}$. Applying KKT conditions on the dual problem leads to the two following links between dual and primal solutions x^* and θ^* :

$$\begin{aligned} & y = Ax^* + \theta^* \\ & \theta^{*T} a_i \in \begin{cases} sign\{x_i^*\} & \text{if } x_i^* \neq 0 \\ [-1; 1] & \text{if } x_i^* = 0 \end{cases} \quad \forall i \in \{1, \dots, n\} \end{aligned} \tag{5}$$

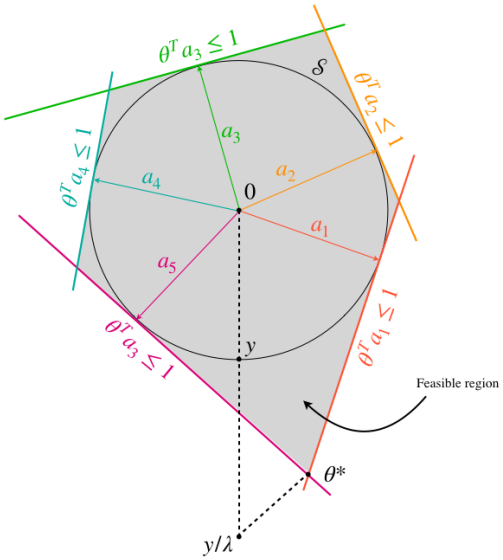

Figure 2: Geometry of the dual LASSO problem

2.2 Ideal screening test

We can observe that (5) are very interesting to *screen* ($i.e$ identify null-coefficient) the solution of the LASSO. Suppose that we know the dual solution θ^*, we can compute $\theta^{*T} a_i$ for each $i = 1, \dots, n$ and if $|\theta^{*T} a_i| < 1$, then we can conclude that $x_i^* = 0$ without solving (3). We can write this ideal screening test as

$$|\theta^{*T} a_i| < 1 \Longrightarrow x_i^* = 0 \tag{6}$$

It is important to notice that if $|\theta^{*T} a_i| \geq 1$, we can *not* conclude on the nullity of x_i^*. Here, the complexity of the test is in $o(m)$ and to test all the coefficient of x^*, we need $o(nm)$ operations. So this ideal screening test is cheap to compute. Unfortunately, to proceed this test, we need to know the optimal solution of the dual LASSO. This solution is as difficult to obtain as the primal solution so that is why we call this test the "ideal" test. Though, we will see in the sequel that all screening techniques relies on the fact that we can deduce if a coefficient of x^* is null based on the KKT conditions. The aim of all these techniques is to prove for a given a_i that $|\theta^{*T} a_i| < 1$ without knowing θ^*.

2.3 SAFE test

The basic idea of screening tests is to remark that if we can find a region (a *test region*) Θ which contains θ^*, we can derive that

$$\boxed{\mu(a_i) := \max_{\theta \in \Theta} |\theta^T a_i| < 1 \Longrightarrow |\theta^{*T} a_i| < 1 \Longrightarrow x_i^* = 0} \tag{7}$$

This test is called a "SAFE" test [GVR10] because it allows to identify null-coefficients without mistakes. However, if we find that $|\theta^{*T} a_i| \geq 1$, x_i^* can be either null or not. The relevance of the test remains of finding a "good" region Θ. A geometrical intuition is to say that the smaller Θ is, the better it is. Indeed, if Θ is small, we can hope that $\mu(a_i)$ is also small. We can prove that if $\Theta_1 \subset \Theta_2$, then Θ_1 will screen at least as many coefficients as Θ_2 [XWR14]. An other criterion to find a good region Θ is that Θ doesn't make $\mu(a_i)$ hard to compute. We have to compute that value for each $a_1, \ldots, a_n$. If it is too hard to compute, we will also face a dimensionality problem as $n = dim(x)$ (which is supposed very large).

We will focus on two types of regions : spherical regions and "dome" regions (the intersection between a sphere and a half-space). As these regions are simple to describe with constraints, we can hope that $\mu(a_i)$ will be easy to compute. We will try to find an expression of $\mu(a_i)$ regardless of the problem so as to directly construct a test which allows to screen x^* for any problem. In other word, for a given region Θ, we want to compute $\mu(a_i)$ and test if $\mu(a_i) < 1$ to hopefully screen x_i^* by solving the following problem :

$$\mu(a_i) = \left\{ \begin{array}{ll} \max\limits_{\theta} & |\theta^T a_i| \\ \text{s.t.} & \theta \in \Theta \end{array} \right. , \quad \forall i \in \{1, \ldots, n\} \tag{8}$$

2.4 Spherical regions

Spherical regions are the simplest regions to work with to construct screening tests. They are building blocks for more complicated tests. A spherical region can be expressed as $\Theta := \{\theta, \|\theta - q\|_2 \leq r\}$ (where q and r are parameters). In this case, the problem to solve to find $\mu(a_i)$ is

$$\mu(a_i) = \left\{ \begin{array}{ll} \max\limits_{\theta} & |\theta^T a_i| \\ \text{s.t.} & \|\theta - q\|_2 \leq r \end{array} \right. \tag{9}$$

With a simple drawing, we can deduce that the optimal solution is $q + r a_i$ so the optimal value of this problem is $\mu(a_i) = |q^T a_i| + r$. We can then deduce a spherical screening test as follows :

$$\boxed{|q^T a_i| < 1 - r \Longrightarrow x_i^* = 0} \tag{10}$$

Such test is really simple to proceed. As $dim(a_i) = dim(y) = m$, the inner product $|q^T a_i|$ is cheap to compute. The complexity to test all the coefficients of x^* is $o(nm)$ so we can say that this test is cheap to proceed. It remains to find r and q to construct a region Θ containing θ^* which allows to screen as many coefficients of x^* as possible. This is called the *parameter selection problem* [XWR14]. We will see in the next section different ways to find the values of q and r.

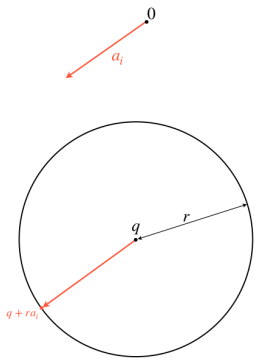

Figure 3: Finding the solution of (9) graphically

2.5 Dome regions

Dome tests are very close to sphere tests presented above. The idea is to construct a region Θ containing θ^* which is the intersection of a sphere and a half-space. A dome can be described as $\mathcal{D}_{(q,r;n,c)} = \{\theta, n^T\theta \leq c, \|\theta - q\|_2 \leq r\}$. To ensure that the half-space crosses the sphere and effectively forms a dome, we can introduce $\psi_d := (n^Tq - c)/r$. To form a dome, we must have $q - \psi_d rn$ inside the sphere which is equivalent to $|\psi_d| \leq 1$. The larger ψ_d is, the smaller and better the dome region is to screen x^*.

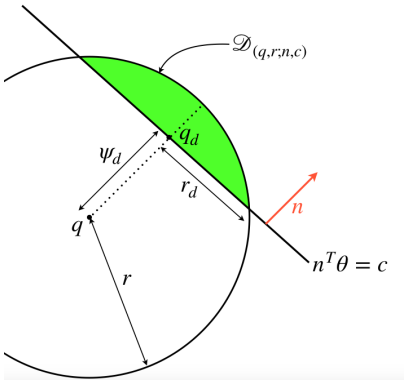

Figure 4: A dome formed with a sphere of parameters q, r and a half-space of parameters n, c.

For the dome test, we want to find $\mu(a_i) = \max_{\theta\in\Theta} |\theta^T a_i|$ so as to hopefully screen x_i^*. This can be done by solving the following problem

$$\mu(a_i) = \begin{cases} \max\limits_{\theta} & \theta^T a_i \\ \text{s.t.} & \|\theta - q)\|_2 \leq r \\ & n^T\theta \leq c \end{cases} \tag{11}$$

For a dome $\mathcal{D}_{(q,r;n,c)}$, assuming that $|\psi_d| \leq 1$, the solution is $\mu(a_i) = q^Ta_i + M(n^Ta_i, 1)$, where

$$M(t_1, t_2) = \begin{cases} rt_2 & \text{if } t_1 < -\psi_r t_2 \\ -\psi_r rt_1 + r\sqrt{t_2^2 - t_1^2}\sqrt{1 - \psi_r^2} & \text{if } t_1 \geq -\psi_r t_2 \end{cases} \tag{12}$$

So as $\mu(a_i) < 1 \iff \max\{q^Ta_i + M(n^Ta_i, 1); q^Tb + M(-n^Ta_i, 1)\} < 1$, the resulting dome test is

$$\boxed{M(-n^Ta_i, 1) - 1 < q^Ta_i < 1 - M(n^Ta_i, 1) \Longrightarrow x_i^* = 0} \tag{13}$$

As in spherical regions, it remains to find the values of the parameters q, r, n and c that allow to construct a region containing θ^* which can well screen x^*. We will see in the next section how to solve the parameter selection problem for a dome test.

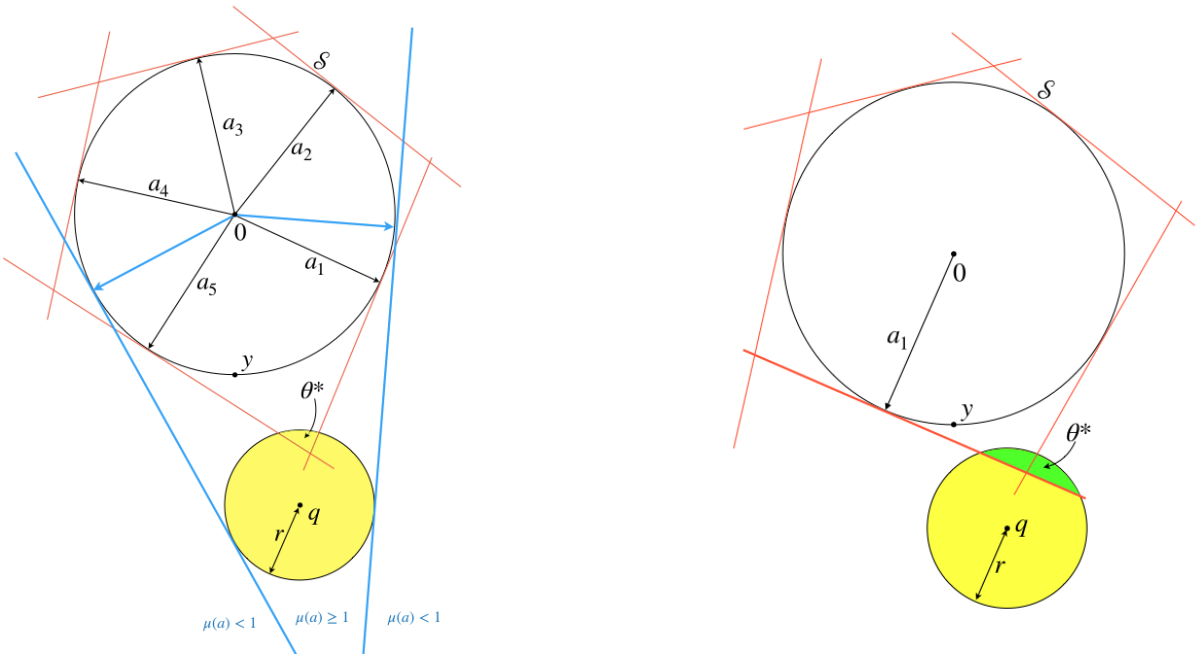

Figure 5: Left : A spherical region (in yellow) containing θ^*. In this case, we can affirm that $x_2^* = x_3^* = x_4^* = 0$ as we have $\mu(a_2) < 1$, $\mu(a_3) < 1$ and $\mu(a_4) < 1$. Right : Geometrical construction of a dome region (in green) containing θ^* using a sphere of center q and radius r and the half space defined with the constraint $|\theta^T a_1| \leq 1$.

3 Parameter selection problem

In the previous section, we have seen two types of regions which leads to screening tests cheap to compute. Those regions are constructed thanks to some parameters influencing on the region's capacity to well screen x^*. We also want to construct regions containing θ^*. In this section, we will see, for spherical and dome regions, different ways to find the parameters used to construct those regions.

3.1 Simple sphere tests

We recall that a sphere defined as $\Theta = \{\theta, \|\theta - q\|_2 \leq r\}$ containing θ^* leads to the screening test $|q^T a_i| < 1 - r \Rightarrow x_i^* = 0$. The center of the sphere is q and its radius is r. The main idea to find q and r is to assume that we know a *dual-feasible point* θ_f. In the dual LASSO, θ^* is the projection of y/λ onto the feasible region $\mathcal{F}$. In other word, θ^* is the closest dual-feasible point to y/λ. We can easily deduce that

$$\|\theta^* - y/\lambda\|_2 \leq \|\theta_f - y/\lambda\|_2 \tag{14}$$

for any dual-feasible point θ_f. If we know such dual-feasible point, we can set $q = y/\lambda$ and $r = \|\theta_f - y/\lambda\|_2$ to construct a region containing θ^*. The region then constructed is

$$\boxed{\Theta = \left\{\theta, \|\theta - y/\lambda\|_2 \leq \|\theta_f - y/\lambda\|_2\right\}} \tag{15}$$

Of course, we have that $\theta^* \in \Theta$. The ability to well screen x^* will directly depends of the dual point used to construct the region. In fact, the closer θ_f is from θ^*, the smaller the radius of the sphere is and the more coefficients the test will screen. As we know how to construct a region for a given dual-feasible point θ_f, we now have to find a way to get dual-feasible points which leads to good screening tests.

3.1.1 Finding a dual-feasible point

A way to construct a feasible point is to compute $\lambda_{max} = \max_i |y^T a_i|$ and the corresponding a^* satisfying $\lambda_{max} = y^T a^*$ ($a^* \in \{\pm a_i, i = 1, \ldots, n\}$). We have that $\lambda_{max} \geq \lambda$. Knowing that, we can construct the dual-feasible point $\theta_f := y/\lambda_{max}$ ($|\theta_f a_i| \leq 1 \ \forall i$ as $\|y\|_2 = 1$). Geometrically, this operation consists of sliding toward the origin the point y/λ to the edge of the feasible region $\mathcal{F}$. The first point which meets the feasible region is y/λ_{max}. By doing this, we try to find a dual-feasible point close to y/λ. As the radius of the sphere is $\|\theta_f - y/\lambda\|_2$, we hope to find a short radius and construct a screening test which will well screen x^*.

We can generalize this technique to find a dual-feasible point starting with any dual point. It is called *dual-scaling* [GVR10]. Taking any dual point θ, we can construct a dual-feasible point θ_f as follows :

$$\theta_f = \begin{cases} \theta & \text{if} \quad \|A^T\theta\|_\infty \leq 1 \\ \theta/\|A^T\theta\|_\infty & \text{otherwise} \end{cases} \tag{16}$$

By doing that, we slide the point θ toward the origin until it reaches the feasible region $\mathcal{F}$. As for y/λ_{max}, we try to find a small radius for the sphere we are building so as to construct a good screening test. One thing to notice is that computing $\|A^T\theta\|_\infty$ costs $o(nm^2)$ operations. This is an issue as we try to solve the LASSO in large dimension. We will see further how to fix this problem.

3.1.2 ST1, ST2 and ST3

Sphere test 1 (ST1) is the simplest test we can construct for spherical regions. In this test, we choose the dual-feasible point y/λ_{max} to construct the test region. Obviously, we assume that λ_{max} has already been computed. We can then set q and r to construct the test-sphere :

$$ST1 \quad : \quad q = y/\lambda \quad r = 1/\lambda - 1/\lambda_{max} \tag{17}$$

This leads to the following screening test :

$$|y^T a_i| < \lambda - 1 + \lambda/\lambda_{max} \Rightarrow x_i^* = 0 \tag{18}$$

ST2 and ST3 are a little different from ST1. We won't find other dual point to set q and r but we will use the fact that $\|\theta - y/\lambda\|_2 \leq 1/\lambda - 1/\lambda_{max}$ (θ^* is in ST1 sphere) and that ${\theta^*}^T a_i \leq 1, \forall i$ (θ^* is feasible) to find other values of q and r such that the region created contains θ^*. Developing these inequalities leads to the following parameter for ST2 and ST3 regions :

$$\begin{array}{lll} ST2 & : & q = y/\lambda_{max} \qquad\qquad r = 2\sqrt{1/\lambda_{max}^2 - 1}(\lambda_{max}/\lambda - 1) \\ ST3 & : & q = y/\lambda - (\lambda_{max}/\lambda - 1)a^* \quad r = \sqrt{1/\lambda_{max}^2 - 1}(\lambda_{max}/\lambda - 1) \end{array} \tag{19}$$

The ST3 region always outperforms the ST2 test as ST3 $\subset$ ST2 (ST3 will screen more coefficients than ST2). For problems where $\lambda_{max} \geq \sqrt{3}/2$, ST2 outperforms ST1 test [XXR11].

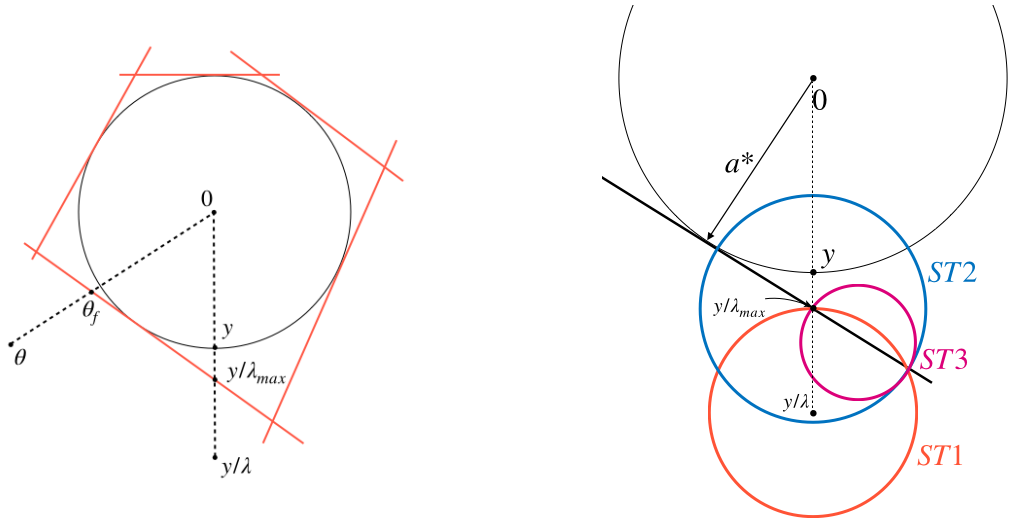

Figure 6: Left : Dual points y/λ and θ and dual-scaled points y/λ_{max} and θ_f. ST1, ST2 and ST3 test regions.

3.2 Projection-based tests

To construct a spherical region, we only have to know a dual feasible point. We already used dual-scaling to construct ST1, ST2 and ST3 but it was costly in operations. Now, we will see how to find a dual feasible point using the property of the projection operator [Wan+13] [MH16].

3.2.1 Using the non-expensiveness property

We recall that the projection on a closed convex is *non-expensive* : given a closed convex $\mathcal{C}$, then for any x and y, we have

$$\|P_{\mathcal{C}}(x) - P_{\mathcal{C}}(y)\| \leq \|x - y\| \tag{20}$$

We have seen that θ^* is the projection of y/λ onto the feasible region which is closed and convex, so $\theta^* = P_{\mathcal{F}}(y/\lambda)$. In the following, we will denote $\theta^* \equiv \theta^*(\lambda)$ to emphasize the fact that the solution of the dual LASSO depends on the regularization parameter λ.

Suppose that we know a dual solution $\theta^*(\lambda')$ of the LASSO problem for a particular regularization parameter λ'. Of course, $\theta^*(\lambda')$ is a feasible point for this problem. As the constraints of the LASSO do not depend on the regularization parameter, $\theta^*(\lambda')$ is also feasible for the LASSO for any regularization parameter and in particular for the regularisation parameter λ of (3). To construct a sphere test, we only need a dual feasible point. If we can find easily a solution of the LASSO for a particular λ', we can set $\theta_f = \theta^*(\lambda')$ in (15) so as to construct a spherical test region. An other way to create a test region is to use the non-expensiveness of the projection. As $P_{\mathcal{C}}(y/\lambda') = \theta^*(\lambda')$ and $P_{\mathcal{C}}(y/\lambda) = \theta^*(\lambda)$, we have

$$\boxed{\|\theta^*(\lambda) - \theta^*(\lambda')\|_2 \leq |1/\lambda' - 1/\lambda|} \tag{21}$$

It is then possible to construct a spherical test with $q = \theta^*(\lambda')$ and $r = |1/\lambda' - 1/\lambda|$. We see that the closer λ' is from λ, the smaller the radius of the sphere is so the better the test is.

Finding $\theta^*(\lambda')$ is not easy because we have to solve the dual LASSO and we will still face the dimension problem. But there is a favourable case. We said that the point y/λ_{max} is feasible. So the solution of the LASSO for the regularization parameter λ_{max} is the point y/λ_{max} itself as it is equal to its projection onto $\mathcal{F}$. This allows to create a spherical region test by setting $q = y/\lambda_{max}$ and $r = 1/\lambda - 1/\lambda_{max}$ and to deduce the following screening test :

$$|y^T a_i| < \lambda_{max} + 1 - \lambda_{max}/\lambda \Rightarrow x_i^* = 0 \tag{22}$$

Finding a solution $\theta^*(\lambda')$ for any λ' is heavy to compute but in the sequel, we will see how to find such solution where λ' is closed to λ with few computations so as to construct sharper tests based on projection.

3.2.2 Using the firmly non-expensiveness property

It is possible to go further to define new screening tests using other projection properties. In fact, the projection operator over a closed-convex set is *firmly non-expensive* :

$$\|P_{\mathcal{C}}(x) - P_{\mathcal{C}}(y)\|^2 \leq \langle x - y; P_{\mathcal{C}}(x) - P_{\mathcal{C}}(y)\rangle \tag{23}$$

Knowing this property leads to the following inequality [HM15] :

$$\boxed{\|\theta^*(\lambda) - (\theta + \frac{1}{2}(\frac{y}{\lambda} - \frac{y'}{\lambda'}))\|_2 \leq \frac{1}{2}\|\frac{y}{\lambda} - \frac{y'}{\lambda'}\|_2} \tag{24}$$

Where $\lambda \neq \lambda'$, θ is a dual-feasible point and $y' \in \mathcal{Y}(\theta)$ with $\mathcal{Y}(\theta) := \{y', y' = \lambda'\theta + \sum_{\{i, a_i^T\theta = 1\}} x(i)a_i, x \geq 0\}$.

Clearly, we can deduce a spherical test based on this inequality with $q = \theta + (y/\lambda - y'/\lambda')/2$ and $r = \|y/\lambda - y'/\lambda'\|_2/2$. We can see that the ability to well screen x^* depends on the choice of y' and θ. When θ is close to θ^*, y' is such that r is small (*i.e* y' close to y). This leads to a good screening test. Since $y' \in \mathcal{Y}(\theta)$ and $y \in \mathcal{Y}(\theta^*(\lambda))$, one should therefore choose θ such that $\mathcal{Y}(\theta)$ is close to $\mathcal{Y}(\theta^*(\lambda))$. With $\lambda' \simeq \lambda$, we can choose $\theta = \theta^*(\lambda')$.

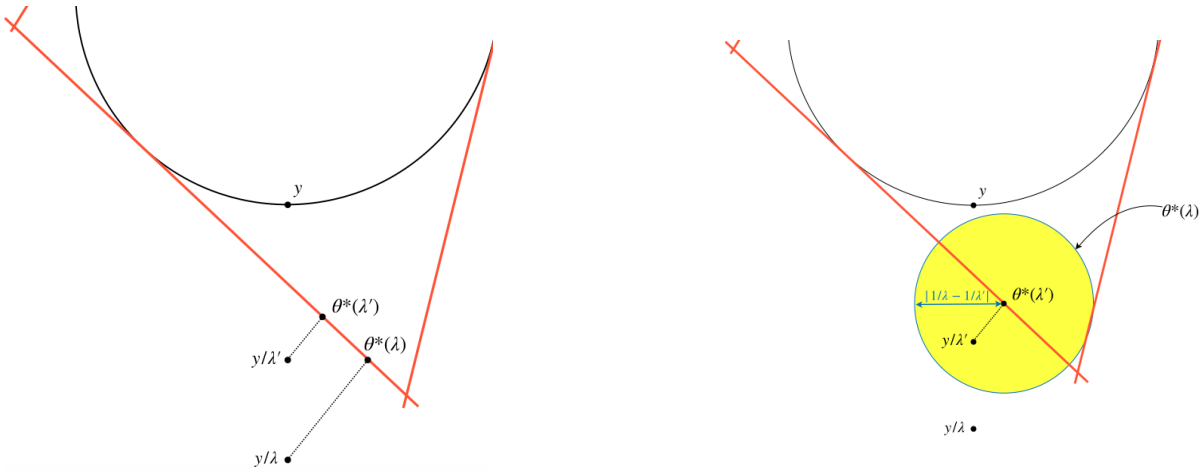

Figure 7: Left : Solutions of the LASSO for two different regularization parameter. Right : Projection-based test construction using the solution $\theta^*(\lambda')$

3.3 GAP sphere test

The GAP sphere test [Ndi+15] is based on the duality gap between primal and dual solutions of the LASSO. If $P_\lambda(x)$ and $D_\lambda(\theta)$ denotes the objective value of the primal and dual at points x and θ, we know that as *strong duality holds* for the LASSO [FR13], $P_\lambda(x^*) - D_\lambda(\theta^*) = 0$. In the following, we will denote

$$GAP(x, \theta) = P_\lambda(x) - D_\lambda(\theta) \tag{25}$$

In the GAP test, we construct a sphere with a dual-feasible point as center. This point can be obtained with dual-scaling for example. To find the radius of the sphere, we use the fact that $\|y - Ax\|_2^2/2$ is 1-Lipschitz to derive that for any $x \in \mathbb{R}^n$ and any dual-feasible point θ_f,

$$\|\theta^* - \theta_f\|_2 \leq \sqrt{\frac{2GAP(x, \theta_f)}{\lambda^2}} \tag{26}$$

If we know any primal point $x \in \mathbb{R}^n$, it is possible to compute its associated dual point and then dual-scale this point to obtain θ_f so as to create a GAP sphere with $q = \theta_f$ and $r = \sqrt{2GAP(x, \theta_f)/\lambda^2}$ and apply the standard sphere test (10). But those operations are heavy in calculations. We will see further how to deal with this problem. One thing to notice is that the radius of the sphere is small if the point x chosen is close to the optimum point x^*. We know that smaller the test region is, better the test is. We will see further how to use the output of some classical algorithms to construct sphere with a radius going to 0. In theory, a sphere with zero-radius (*i.e* the point θ^*) leads to a test which allows to identify all the null-coefficients of x^*, which is very interesting to reduce at most the dimension of the problem.

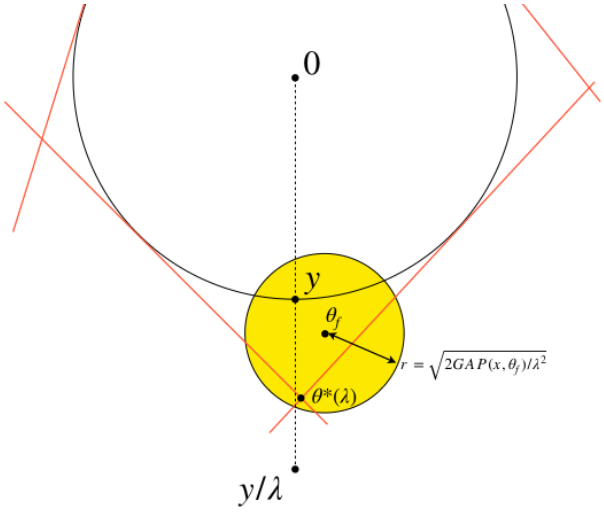

Figure 8: GAP sphere constructed with a dual-feasible point θ_f

3.4 Dome test

We said that to create a dome region, we have to find parameters q, r, n and c allowing to construct a dome containing θ^* as small a possible to well screen x^*. As the dome is the intersection of a sphere and a half space, we can use any sphere described in the previous sphere tests as we know that those sphere contains θ^*. We also know that θ^* is a feasible point so it is contained in any of the half spaces $|\theta^T a_i| \leq 1$ formed by the constraints. Taking the dome as the intersection of any sphere constructed before and one of those half space ensure to create a non-empty dome as at least θ^* is inside. A half space formed with the constraint $|\theta^T a_i| \leq 1$ has for parameters $n = a_i$ and $c = 1$.

We introduced the quantity $\psi_d = (a_i^T q - c)/r$ and said that the larger ψ_d, the smaller the dome. To construct a good dome region, we have to find $a_d \in \{a_i, i = 1, \ldots, n\}$ which leads to the smallest dome for a chosen sphere of parameters q and r. We have to solve

$$a_d \in \begin{cases} \underset{a_i, i \in 1,\ldots,n}{\text{argmax}} & \frac{a_i^T q - c}{r} = \underset{a_i, i \in 1,\ldots,n}{\text{argmax}} \, a_i^T q \\ \text{s.t.} & |\frac{a_i^T q - c}{r}| \leq 1 \end{cases} \tag{27}$$

The solution depends on the sphere chosen to construct the dome test. But the problem leads to the a_i the most correlated with q. If we choose $q = y/\lambda$ as in ST1 test for example, then $a_d = a^*$ (where a^* is the same as the one

defined in the ST1 test) and the resulting dome is $\mathcal{D}_{(y/\lambda,1/\lambda-1/\lambda_{max};a^*,1)}$.

An other way to choose the parameters of the half space is to rely of the fact that θ^* is the projection of y/λ onto the closed convex region $\mathcal{F}$. Then, for any $\theta_f \in \mathcal{F}$, we have

$$(y/\lambda - \theta^*)^T(\theta_f - \theta^*) \leq 0 \tag{28}$$

Suppose that we know a dual solution $\theta^*(\lambda_0)$ of the LASSO for a particular λ_0. We know that $\theta^*(\lambda_0)$ lies on the boundaries of $\mathcal{F}$ so for any $\theta_f \in \mathcal{F}$, $(y_0/\lambda_0 - \theta^*(\lambda_0))^T\theta_f \leq (y_0/\lambda_0 - \theta^*(\lambda_0))^T\theta^*(\lambda_0)$. We can conclude that $\mathcal{F}$ is bounded by the half space $n_0^T\theta_f \leq c_0$ with

$$r_0 = \|y_0/\lambda_0 - \theta^*(\lambda_0)\|_2 \qquad n_0 = (y_0/\lambda_0 - \theta^*(\lambda_0))/r_0 \qquad c_0 = n_0^T\theta^*(\lambda_0)$$

So knowing a dual-feasible point and a solution $\theta(\lambda_0)$ can also lead to a dome test.

As the dome is included in the sphere used to construct it, domes are always better than the spheres used to screen x^*. For any sphere test, we can sharpen the test by constructing a dome test based on the sphere but this costs the computation of a_d. In the appendix, we explain an algorithm allowing to construct a smaller sphere based on the geometrical properties of the dome. This allows to construct sequentially smaller and smaller domes and spheres to have better and better tests.

4 Screening in practice

In the two last sections, we presented a way to construct screening regions and a way to choose the parameters of these regions. The computations of such parameters is not easy because we need to dual-scale a point or to find the solution of the LASSO for a given regularization parameter, different from λ. In this section, we will see how to take advantage of classical algorithms usually used to solve the LASSO so as to compute such points at virtually no cost.

4.1 Static vs Dynamic screening

The scheme of a screening test using a dual-feasible point presented before is simple. We first compute a dual-feasible point, then we construct a test region and finally, we apply the screening test to all the coefficients of x^*. This allows to reduce once the dimension of the problem but we have to pay a heavy cost to compute the dual-feasible point ($o(nm^2)$ operations). In these tests, we only reduce once the dimension, we call such tests *static screening tests*. We now introduce a new approach : *dynamic screening* [Bon+14a] [FG17]. The principle is to take advantage of known iterative algorithms used to solve the LASSO. Those algorithms output a sequence of points converging to x^*. Using these points, we can construct dual points (and dual-feasible points) closer and closer to θ^* to construct smaller and smaller regions. The regions constructed successively will screen better and better x^*.

We can ask ourselves a question : why doing more screening operations (which requires each the computation of a dual feasible point) if we can only screen x^* once ? The answer is that some iterative algorithms used to solve the LASSO already compute dual feasible points during their process. The aim of dynamic screening is to re-use the values computed by theses algorithms to construct a screening test without needing to compute a dual-feasible point. In addition, as the dual feasible points computed by those iterative algorithms are closer and closer to x^*, the tests constructed iteratively will be better and better to screen x^*.

To illustrate these two approaches, we can simply describe the scheme of a classical iterative algorithm used to solve the LASSO as a loop computing a sequence of points $x_k \to x^*$ ending with a stopping criterion (based on $\|x_{k+1} - x_k\|$ for example). We call "doAlgorithmIteration()" the procedure which computes the successive terms x_k. This procedure is also useful as it computes a dual-feasible point θ_f. We call "doScreeningTest(θ_f, A, x)" the dimensionality reduction operation of x and A proceeded with a screening test based on the dual-feasible point θ_f. Finally, we call "computeDualFeasiblePoint()" the procedure which computes a dual-feasible point (using (16) for example).

Algorithm 1: Static screening

Data: y, A, λ, x_0
Result: x^*, the optimal solution of (3)

$\theta_f \leftarrow$ computeDualFeasiblePoint()
$A,\ x_0 \leftarrow$ doScreeningTest(θ_f, A, x_0)
repeat
  $x_{k+1} \leftarrow$ doAlgorithmIteration(x_k)
until *A stopping criterion*
return x_{k+1}

Algorithm 2: Dynamic screening

Data: y, A, λ, x_0
Result: x^*, the optimal solution of (3)

repeat
  $\tilde{x_k}, \theta_f \leftarrow$ doAlgorithmIteration(x_k)
  $A_{k+1},\ x_{k+1} \leftarrow$ doScreeningTest($\theta_f, A_k, \tilde{x_k}$)
until *A stopping criterion*
return x_{k+1}

4.2 Constructing a test with dynamic screening

Dynamic screening relies on the fact that classical algorithms designed to solve LASSO compute sequence of points which converges to x^*. We will use dual-feasible points already computed by these algorithms to construct screening tests with almost no calculations.

4.2.1 Dual-scaling computations using ISTA

ISTA algorithm [Ami09] [Ela10] computes iteratively points which converges to x^*. One step of an ISTA iteration is to compute the value $A^T(Ax_k - y)$. With (5), we can notice that

$$A^T(Ax_k - y) = A^T\theta_k \tag{29}$$

To dual-scale θ_k, we need to compute $\|A^T\theta_k\|_\infty$ which only requires $o(m)$ operations if $A^T\theta_k$ is already computed. Thus, during an ISTA iteration, we can first store the value of $Ax_k - y = \theta_k$, then the value of $A^T\theta_k$ so as to compute the dual feasible point $\tilde{\theta_k}$ by dual-scaling with only $o(m)$ operations. We can then perform a cheap ST1 test which allows to reduce the dimension of the problem. The test iteratively constructed will be better and better to screen coefficients as $x_k \to x^*$ (and $\theta_k \to \theta^*$). Therefore, adding a screening procedure at each ISTA iteration only requires $o(m)$ operations to obtain a dual-feasible point and $o(nm)$ operations to proceed the screening test.

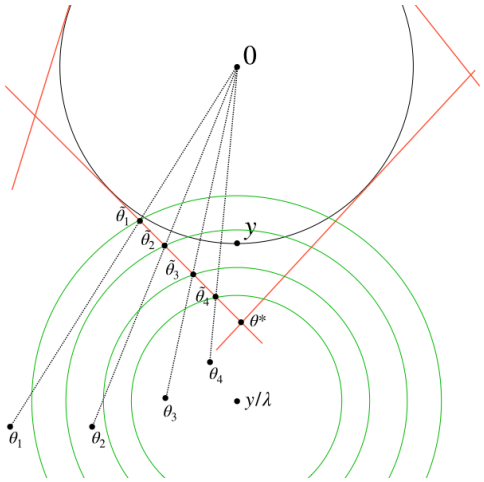

Figure 9: Points constructed by ISTA and dual-scaled points used to construct ST1 spheres iteratively smaller

Dynamic screening can be extended to other test regions which requires a feasible dual point to be constructed but as we have to determine a new region for each iteration, we have to ensure that test are not too heavy to proceed. It is also possible to use other gradient-descend algorithms such as TwiST, FISTA or SpaRSA (which doesn't compute $A^T(Ax_k - y)$ at each iteration) but it will be necessary to find a way to compute a dual-feasible point using algorithm iterations [Bon+14b].

4.2.2 Dynamic GAP screening

An other way to take advantage of the ISTA algorithm is to use GAP test. The interesting point with the GAP sphere is that if we know a sequence $(x_k)_{k\in\mathbb{N}}$ such that $x_k \to x^*$ and dual-feasible points $\theta_k \to \theta$, then $GAP(x_k, \theta_k) \to 0$. With (26), we know that as the radius of the sphere goes to 0, test constructed are better and better. We can using ISTA to extract these two sequences. At each iteration of the algorithm, we can extract a point x_k outputted by ISTA and a dual-feasible point $\tilde{\theta_k}$ with cheap calculations as explained before. Then, we can construct a GAP sphere for each ISTA iteration using :

$$\|\theta^* - \tilde{\theta_k}\|_2 \leq \sqrt{\frac{2GAP(x_k, \tilde{\theta_k})}{\lambda^2}}$$

The radius of the GAP sphere created will go to 0 and theoretically, we will be able to identify all the null-coefficients of x^* so as to reduce as much as possible the dimension of the problem. This is a very powerful way to combine screening techniques and ISTA as we only need $o(m)$ operations to construct the test and $o(nm)$ operations to screen x^* at each iteration.

4.3 Sequential screening

We introduced the ISTA algorithm so as to do dynamic screening. We now introduce a similar screening technique : *sequential screening*. Instead of using a sequence $x_k \to x^*$ for a fixed regularization parameter λ, we will use a sequence $(x^*(\lambda_k), \lambda_k) \to (x^*(\lambda), \lambda)$ of optimal points for different regularization parameters such that $\lambda_k \to \lambda$.

4.3.1 Taking advantage of the LARS

In the LASSO, the choice of the regularization parameter in very important to control whether to prioritize the sparsity or the accuracy of the approximation of y by Ax. LARS algorithm [Efr+04] [Ela10] was developed to solve this problem. This algorithm produces a sequence $(x^*(\lambda_0), \lambda_0), (x^*(\lambda_1), \lambda_1), \dots$ with $\lambda_0 > \lambda_1 > \dots$. This allows so study the sparsity of the solution in function of the regularization parameter. We will use the outputs of this algorithm to produce screening test with almost no cost.

With the relation (5), we can construct dual solution of the LASSO for different regularization parameters using the outputs of the LARS. With this sequence $\theta^*(\lambda_0), \dots$, it is possible to proceed projection-based screening tests which needs such points. The LARS produces outputs with decreasing regularization parameters. The projection based test constructed with these outputs until the regularization parameter of (3) is reached are better and better to screen $x^*(\lambda)$. It is also possible to use those points (which are actually dual-feasible for (3)) to construct tests which needs a dual-feasible point.

Actually, the LARS starts with $\lambda_0 = \lambda_{max}$ which produces a solution $x^*(\lambda_0)$ where all coefficients are equal to zero. Then, the regularization decreases at each iteration and some of the coefficients of the outputs $x^*(\lambda_k)$ becomes non-null. One this to notice is that :

$$\text{If } \lambda_i > \lambda_j, \text{ then } x^*(\lambda_i) \neq 0 \implies x^*(\lambda_j) \neq 0 \tag{30}$$

We see that it is possible to identify some of the coefficients of $x^*(\lambda)$ which can't be null (the *active set*) by looking at the coefficients of $x^*(\lambda')$ for $\lambda' > \lambda$. If we know that, we can reduce the number of coefficients to test during a screening test. We can use the LARS outputs $x^*(\lambda_0), \dots, x^*(\lambda_K)$ with $\lambda_0 = \lambda_{max}$ and $\lambda_K \geq \lambda$ by identifying non-null coefficients of $x^*(\lambda_0), \dots, x^*(\lambda_K)$ so as to deduce coefficients of the active-set of $x^*(\lambda)$. Then, we can reduce the number of screening tests to proceed as we identified some coefficients which can't be null. One thing to know is that the LARS is very good to compute the few first terms of the output so in practice, only the few first outputs can be used.

4.3.2 Sequential projection-based tests

Projection-based tests need a dual optimal point of the dual LASSO for a particular λ' in order to be constructed. We said that if λ' is close to λ, the region created can screen lots of the coefficients of x^*. We can use the outputs $(x^*(\lambda_0), \lambda_0), \dots, (x^*(\lambda_K), \lambda_K)$ of the LARS with λ_K close to λ. Each output will allows to construct a dual optimal point $\theta^*(\lambda_k)$ using the relation (5) with $o(nm + m)$ operations and a projection-based test. As the dual-feasible points iteratively constructed as closer and closer to $\theta^*(\lambda)$, the projection-based test with be better and better.

One thing very important to notice is that LARS outputs *approximations*. Projection-based tests rely on inequalities using the *exact* knowledge of the points used. Using LARS can lead to mistakes while screening coefficients. Tests constructed are not SAFE. To face this issue, some *relaxed* tests can be constructed while larger inequalities taking into account the approximation of the points used [HM15].

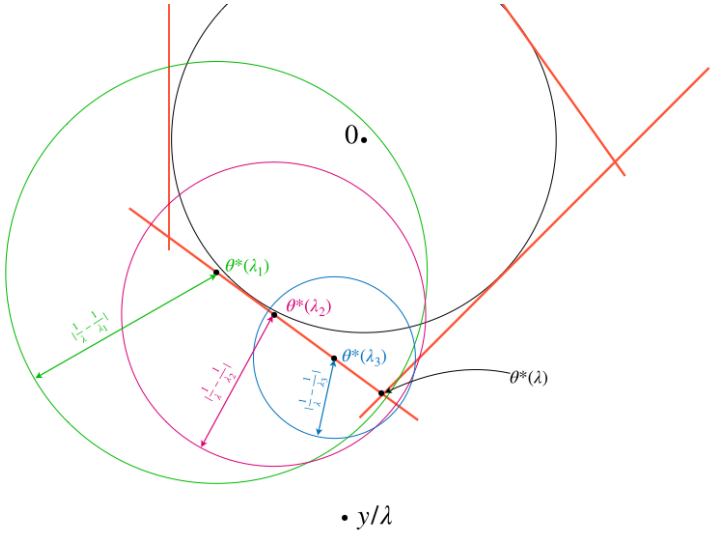

Figure 10: Dual-feasible points constructed with LARS outputs and projection-based regions constructed with these points

5 Putting screening into perspective

From the beginning of this report, we explained techniques allowing to reduce the dimensionality of the LASSO for a fixed regularization parameter in order to make this problem easier to solve and avoid heavy matrix calculations, often used in classical algorithms. In this last section, we will discuss on perspectives for these screening techniques.

5.1 Multi-coefficient screening (Joint screening)

All screening techniques that we exposed before proceed a *one coefficient* screening test in the way that each test proceeded concerns only one coefficient. To screen all the coefficients of x^*, we need n tests (or a bit less using the LARS). Here, we will present a new screening technique called *joint screening* [HD18][HDD19]. The idea of this technique is to screen not one but several coefficients with only one test. This approach is a direct consequence of the following remark. If we denote $\mathcal{G} \in \mathbb{R}^m$, $\mathcal{A} := \{a_i\}_{i=1,\dots,n}$ and $\mathcal{R}$ one of the test region exposed before, then

$$\boxed{\max_{a\in\mathcal{G}} \max_{\theta\in\mathcal{R}} |\theta^T a| < 1 \Rightarrow \max_{\theta\in\mathcal{R}} |\theta^T a_i| < 1 \Rightarrow x_i^* = 0 \quad \forall i : a_i \in \mathcal{G} \cap \mathcal{A}} \tag{31}$$

With this remark, we can derive that if we can construct such a region $\mathcal{G}$, it is possible to screen multiple coefficients at once. Assuming that $\mathcal{R}$ is chosen as a sphere region with parameter q and r, we can rewrite (31) as

$$\max_{a\in\mathcal{G}} |a^T q| < 1 - r \Rightarrow x_i^* = 0 \quad \forall i : a_i \in \mathcal{A} \cap \mathcal{G} \tag{32}$$

In other word, choosing a spherical region for $\mathcal{G}$ and $\mathcal{R}$ allows to screen multiple coefficients with a simple test similar to the ST1 test. As in "classical" screening tests, the choice of $\mathcal{G}$ will either be a sphere $\mathcal{G}^s$ or a dome $\mathcal{G}^d$.

$$\mathcal{G}^s(t,\epsilon) = \{a : \|a - t\|_2 \leq \epsilon\} \qquad \mathcal{G}^d(t,\delta) = \{a : a^T t \geq \delta, \|a\|_2 \leq 1\} \tag{33}$$

These two regions leads to screening test cheap to compute and allowing to screen multiple coefficients at once. These tests are described in appendix. The principle of joint screening will be to chose regions $\{\mathcal{G}_i\}_{i=1,\dots,L}$ and to apply (31) for each region. Obviously, the relevance of these tests relies on the chosen regions $\{\mathcal{G}_i\}_{i=1,\dots,L}$. As in classical screening tests, smaller regions $\mathcal{G}_i$ leads to tests able to screen more coefficients. In [HDD19], two techniques are exposed to choose these regions.

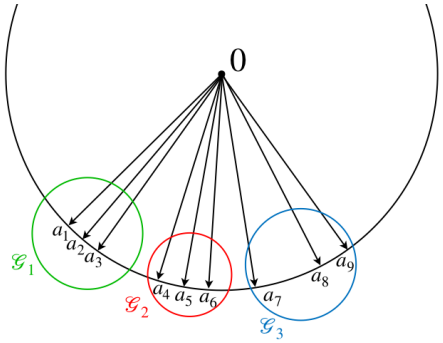

Figure 11: Tree spherical regions able to screen 3 coefficient each with one single joint screening test. If (31) is verified for $\mathcal{G}_1$, then $x_1^* = x_2^* = x_3^* = 0$

5.2 Generalisation of screening to other problems

In the report, we focused on the LASSO problem expressed as

$$\frac{1}{2}\|y - Ax\|_2^2 + \lambda\|x\|_1 = \text{cost}(x) + \text{reg}(x) \tag{34}$$

The aim of the *cost* function is to ensure that the solution of the LASSO is a good approximation of y and the aim of the *reg* function is to ensure that the solution is sparse. A question to be asked is the generalization of screening techniques to problems with other *cost* functions.

LASSO can be seen as a least-square regression with a sparsity constraint on the solution. We can try to apply screening procedures in the case of a *least-square regression with an intercept* or in the case of the *elastic nest problem* [ZH05]. These problems can be respectively expressed as

$$\begin{array}{c} \min_x \left\{ \|Ax + v - y\|_2^2/2 + \lambda\|x\|_1,\ \lambda \geq 0 \right\} \\ \min_x \left\{ \|Ax - y\|_2^2/2 + \lambda\|x\|_1 + \epsilon\|x\|_2^2/2,\ \lambda \geq 0 \right\} \end{array} \tag{35}$$

In the first case, the variable shift $\tilde{A} = A - \bar{A}\mathbb{1}^T$ and $\tilde{y} = y - \bar{y}\mathbb{1}$ (where $\bar{A} = X\mathbb{1}/m$ and $\bar{y} = y^T\mathbb{1}/m$) and in the second case the variable shift $\tilde{A} = (A^T, \sqrt{\epsilon}I)^T$ and $\tilde{y} = (y^T, 0^T)^T$ allows to reformulate these problems under the classical LASSO form so as to proceed screening tests.

In addition to the cases where it is possible to transform the cost function, we can construct screening tests for more general problems. As example, we can mention the *general ℓ_1-regularization* problem :

$$\min_{x_1,x_2} \left\{ \sum_{i=1}^{m} f(a_i^T x_1 + b_i x_2 + c_i) + \lambda\|x_1\|_1,\ \lambda \geq 0 \right\} \tag{36}$$

where $A = [a_1, \ldots, a_m]$, b, c and λ are given and f is a closed convex function. The LASSO is a particular case of this problem when $y = c$, $b = 0$ and f is the 2-norm. The dual of this problem is

$$\max_{\theta} \left\{ c^T\theta - \sum_{i=1}^{m} f^*(\theta_i),\ \theta^T b = 0,\ |\theta^T a_i| \leq 1,\ \forall\ i = 1, \ldots, m \right\} \tag{37}$$

where f^* is the conjugate function of f. In this case, we can still write KKT conditions for the constraints $|\theta^T a_i| \leq 1$ and construct a test based on the fact that $\max_\theta |\theta^T a_i| < 1 \Rightarrow |\theta^{*T} a_i| < 1 \Rightarrow x_i^* = 0$. One particular case of this general ℓ_1-regularized convex problem is the sparse Support Vector Machine problem which is very important in the Machine Learning field. For given y and z, it can be expressed as follows :

$$\min_{x_1,x_2} \left\{ \sum_{i=1}^{m} (1 - y_i(z_i^T x_1 + x_2))_+ + \lambda\|x_1\|_1,\ \lambda \geq 0 \right\} \tag{38}$$

Lastly, screening can we applied in the case of the well known sparse logistic regression problem which can be expressed as

$$\min_{x_1,x_2} \left\{ \sum_{i=1}^{m} \log(1 + \exp(-y_i(z_i^T x_1 + x_2))) + \lambda\|x_1\|_1,\ \lambda \geq 0 \right\} \tag{39}$$

where y and z are given. This problem has an important purpose in the Neural Network field and in the probability field. The dual of this problem can be expressed as

$$\max_{\theta} \left\{ \sum_{i=0}^{m} (\theta_i \log(-\theta_i) - (1 + \theta_i)^T \log(1 + \theta_i)),\ -1 \leq \theta \leq 0,\ \theta^T y = 0,\ |\theta^T a_i| \leq 1,\ \forall\ i = 1, \ldots, m \right\} \tag{40}$$

with $a_i = (y_1 z_1(i), \ldots, y_m z_m(i))$. Again, we can use the constraint $|\theta^T a_i| \leq 1$ to write KKT conditions leading to a screening test.

In summary, if an optimization problem leads to a dual problem with a constraint of the type $|\theta^T a_i| \leq 1$, we can hope to be able to construct screening tests. Of course, this won't be easy or useful in any case. For example, the feasible region can have a bad shape which doesn't allow to construct dual-feasible points.

5.3 Reducing the dimension using other properties

The main principle of LASSO and screening techniques is to take advantage of the property that 0-regularization and 1-regularization problems gives sparse solutions which allows to compress data. These sparse properties also

allows to create screening tests to identify null-coefficients before solving the LASSO problem so as to reduce dimension by removing some of the coordinates of the solutions and some columns of the dictionary. A new technique called *squeezing* was recently introduced [EH19].

This main idea of squeezing is to compress data contained in y, using a dictionary A, onto an other *saturated* vector x (*i.e* lots of coefficients with the same absolute value). In this way, we can only store a vector coding the sign of the saturated coefficients and store once their absolute value instead of storing all the values of theses coefficients. The aim of squeezing is to construct *squeezing tests*, as these created for screening, to identify saturated coefficients of the solution without solving the problem. Instead of solving the LASSO problem, we focus on the ℓ-∞ regularization problem :

$$x^* \in \arg\min_x \left\{ \frac{1}{2}\|y - Ax\|_2^2 + \lambda \|x\|_\infty, \lambda \geq 0 \right\} \tag{41}$$

Different regularization functions can be used to have different properties on the optimal vector of problems in the form of (34) but the ∞-norm is known to "spread" the information contained in y in the "most uniform" way in x (*i.e* produce a saturated solution or *anti-sparse* solution). If we denote $\mathcal{I}_+^* = \{i, +\|x_i^*\|_\infty = x_i^*\}$, $\mathcal{I}_-^* = \{i, -\|x_i^*\|_\infty = x_i^*\}$ and $\mathcal{I}^* = \mathcal{I}_+^* \cup \mathcal{I}_-^*$, then for any $\mathcal{I} = \mathcal{I}_+ \cup \mathcal{I}_-$ with $\mathcal{I}_+ \subset \mathcal{I}_+^*$ and $\mathcal{I}_- \subset \mathcal{I}_-^*$, (41) can be rewritten as

$$(w^*, \tilde{x}^*) \in \arg\min_x \left\{ \frac{1}{2}\|y - A_{\bar{\mathcal{I}}}\tilde{x} - sw\|_2^2 + \lambda w, \lambda \geq 0,\ \tilde{x} \leq w,\ -\tilde{x} \leq w \right\} \tag{42}$$

Where $s = \sum_{i \in \mathcal{I}_+} a_i - \sum_{i \in \mathcal{I}_-} a_i$. We have $x_{\bar{\mathcal{I}}}^* = \tilde{x}^*$ and

$$x_i^* = \begin{cases} -w^* & \text{if } i \in \mathcal{I}_- \\ +w^* & \text{if } i \in \mathcal{I}_+ \end{cases} \tag{43}$$

In (41), $\dim(A) = n$ whereas in (42), $\dim(A_{\bar{\mathcal{I}}}) = n - card(\mathcal{I}) + 1$. In the limit case where $\mathcal{I} = \mathcal{I}^*$, the dimension of (42) is $n - card(\mathcal{I}^*) + 1$. For a fixed w, if we denote $kruscal(A)$ the Kruskal rank of A, we can even derive that

$$kruscal(A) = m \Longrightarrow card(\mathcal{I}^*) \geq n - m + 1 \tag{44}$$

That is how squeezing is used to reduce the dimension of the problem. More details on squeezing tests are given in appendix.

A Optimization background

A standard way to write a convex optimization problem (or *primal problem*) is under the following form :

$$\begin{cases} \min\limits_{x} & f(x) & & & \\ \text{s.t.} & h_i(x) & \leq & 0 & \forall i \in \{0, \dots, m\} \\ & g_j(x) & = & 0 & \forall j \in \{0, \dots, p\} \end{cases}$$

Where f, $h_1, \dots, h_m$ are convex and $g_1, \dots, g_p$ are affine functions. We call this form the *standard form* of a convex problem. The *Lagrangian* and the *dual function* of the primal problem with the multipliers λ and $\mu \geq 0$ are the following functions :

$$\mathcal{L}(x, \lambda, \mu) = f(x) + \sum_{i=1}^{m} \lambda_i h_i(x) + \sum_{i=1}^{p} \mu_i h_i(x) \qquad q(\lambda, \mu) = \inf_x \mathcal{L}(x, \lambda, \mu)$$

We can then define the *dual problem* as

$$\begin{cases} \max\limits_{\lambda, \mu} & q(\lambda, \mu) & \\ \text{s.t.} & \lambda_i \in \mathbb{R}_+ & \forall i \in \{0, \dots, m\} \\ & \mu_j \in \mathbb{R} & \forall j \in \{0, \dots, p\} \end{cases}$$

In the case of linear programming ($h_1, \dots, h_m$, $g_1, \dots, g_p$ and f are linear), we say that *strong duality holds*. This means that if we denote x^* the solution of the primal and (λ^*, μ^*) the solutions of the dual, then $f(x^*) = q(\lambda^*, \mu^*)$. In such cases, solving the dual problem allows to find the optimal value of the primal problem. It's very useful when the dual (resp. the primal) is easier to solve than the primal (resp. the dual).

Suppose that f, $h_1, \dots, h_m$ and $g_1, \dots, g_p$ are continuously differentiable at the optimal point x^*, we can derive some conditions on the primal and dual solutions. The Karush-Kuhn-Tucker (KKT) conditions gives these conditions on x^* and (λ^*, μ^*):

Stationarity	$-\nabla_x f(x^*) = \sum_{i=1}^{m} \lambda_i \nabla_x h_i(x^*) + \sum_{j=1}^{p} \mu_j \nabla_x h_j(x^*)$
Primal feasibility	$\begin{cases} h_i(x^*) \leq 0, & \forall i \in \{1, \dots, m\} \\ g_j(x^*) = 0, & \forall j \in \{1, \dots, p\} \end{cases}$
Dual feasibility	$\lambda_i \geq 0, \quad \forall i \in \{1, \dots, m\}$
Complementary slackness	$\lambda_i h_i(x^*) = 0, \quad \forall i \in \{1, \dots, m\}$

B Dual LASSO parametrization

In the report, we used a particular parametrization of the dual LASSO problem. We give here a proof of such parametrization. This proof was made without the help of any other paper so as to better understand the properties of the LASSO. In fact, it is very difficult to find this proof in the literature.

We denote a_k the k-th column of A. First, by setting $\tilde{x} = [x_+|x_-]$, $\tilde{A} = [A_+|A_-]$ and $z = \tilde{A}\tilde{x}$, we can rewrite (3) under the following form :

$$\begin{cases} \min_{\tilde{x}\in\mathbb{R}^{2n}} & \frac{1}{2}\|y-z\|_2^2 + \lambda\sum_{i=1}^{2n}\tilde{x_i} \\ \text{s.t.} \quad z & = & \tilde{A}\tilde{x} \\ \tilde{x_i} & \geq & 0 \qquad i = 1,\dots,2n \end{cases} \tag{45}$$

The Lagrangian of this problem is

$$\mathcal{L}(\tilde{x}, z, \mu, \theta) = \frac{1}{2}\|y-z\|_2^2 + \lambda\sum_{i=1}^{2n}\tilde{x_i} - \sum_{i=1}^{2n}\mu_i\tilde{x_i} + \theta^T(z-\tilde{A}\tilde{x}) \tag{46}$$

$$= \frac{1}{2}\|y-z\|_2^2 + \theta^T z + \sum_{i=1}^{2n}\tilde{x_i}(\lambda + \mu_i - \theta^T\tilde{a_i}) \tag{47}$$

With $\mu_i \geq 0$ and $\theta \in \mathbb{R}^m$. If $\mu_i \neq \lambda - \theta^T\tilde{a_i}$, then $\inf_{\tilde{x},z}\mathcal{L}(\tilde{x}, z, \mu, \theta) = -\infty$. Now, suppose that we have $\mu_i = \lambda - \theta^T\tilde{a_i}$ (*i.e* we consider the dual function where it is finite). We can write the dual function as

$$q(\theta) = \inf_{\tilde{x},z}\left\{\mathcal{L}(\tilde{x}, z, \mu, \theta)\right\} = \inf_z\left\{\frac{1}{2}\|y-z\|_2^2 + \theta^T z\right\} = \frac{1}{2}\|y\|_2^2 + \inf_z\left\{\frac{1}{2}(\|z\|_2^2 - 2y^T z) + \theta^T z\right\} \tag{48}$$

As $f(z) := \frac{1}{2}(\|z\|_2^2 - 2y^T z) + \theta^T z$ is convex and coercive, it reach its minimum where $\nabla_z f(z) = z - y + \theta = 0$. So at the minimum, $z = \theta - y$. Thus, we have $q(\theta) = \frac{1}{2}\|y\|_2^2 - \frac{1}{2}\|y-\theta\|_2^2$. We can now write the dual problem of (45) as

$$\tilde{\mathcal{D}}(\lambda) : \max_\theta\left\{\frac{1}{2}\|y\|_2^2 - \frac{1}{2}\|\theta - y\|_2^2, \quad \mu_i = \lambda - \theta^T\tilde{a_i},\ \mu_i \geq 0,\ i = 1,\dots,n,\ \theta \in \mathbb{R}^m\right\} \tag{49}$$

The constraints can be re-written as $\theta^T\tilde{a_i} \leq \lambda$ or $|\theta^T a_i| \leq \lambda$ as we set $\tilde{A} = [A_+|A_-]$. By applying the variable shift $\theta' = \lambda\theta$, we finally obtain the following dual problem :

$$\mathcal{D}(\lambda) : \max_\theta\left\{\frac{1}{2}\|y\|_2^2 - \frac{\lambda^2}{2}\|\theta - \frac{y}{\lambda}\|_2^2, \quad |\theta^T a_i| \leq 1,\ i = 1,\dots,n\right\} \tag{50}$$

C Sharpening the dome test

We have seen that a dome is the intersection between a test sphere (like ST1) and one of the half spaces created by the constraints of the dual LASSO. We can ask ourselves if it is possible to construct a test involving multiple half spaces so as to reduce the area of the test region. It is important to remember that we have to construct tests which are not only good to screen x^* but also cheap to compute. Involving more half spaces leads to complex test and their accuracy is not far better than the accuracy of the dome test [XWR14].

An other question to ask is can we deduce a smaller sphere which contains θ^* using the dome region. Geometrically, it is quite easy to see that the sphere with center $q_d = q - \psi_d r n$ and radius $r_d = r\sqrt{1-\psi_d^2}$ is a smaller sphere than the one used to construct the dome but it still contains the dome region (so it also contains θ^*). With this consideration, we can iteratively construct smaller and smaller spheres and domes with the following algorithm.

Algorithm 3: Iteratively finding a smaller spherical test region

Data: $\mathcal{S}(q_1, r_1)$, a sphercial test region
Result: A small spherical test region

repeat
  $n_k \leftarrow a_d$
  $c_k \leftarrow 1$
  $\psi_{k+1} \leftarrow (a_d^T q_k - c_k)/r_k$
  $q_{k+1} \leftarrow q_k - \psi_k r_k n_k$
  $r_{k+1} \leftarrow r_k\sqrt{1-\psi_k^2}$
until *It is possible to find* a_d *which solves* (27) *with* n_k, c_k, q_{k+1} *and* r_{k+1}

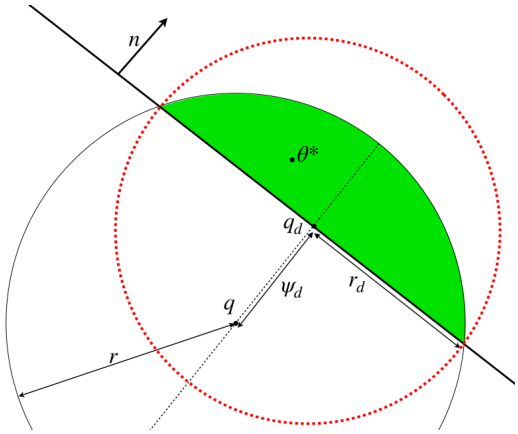

Figure 12: A dome region (in green) used to construct a smaller sphere (in red) also containing θ^*

D Iterative algorithms for the LASSO

In the report, we mention two algorithms used to solve the LASSO : ISTA and LARS. Here, we explain how they works.

D.1 ISTA algorithm

The main idea of this algorithm is to use the convexity of the function $f(x) := \|y - Ax\|_2^2/2 + \lambda\|x\|_1$. As it is convex, it only has one minimum x^* where $\nabla f(x^*) = 0$. The aim is to construct a sequence $(x_k)_{k\in\mathbb{N}}$ which converges to x^*. Starting at an initial point x_0, the points are constructed as follows :

$$x_{k+1} = h_{1/\mu_{max}}(x_k - \alpha_k \nabla f(x_k)), \quad \alpha_k > 0 \tag{51}$$

Where

$$h_\alpha(x_i) = (|x_i| - \alpha) + sign(x_i) \tag{52}$$

and μ_{max} is the largest eigenvalue of $A^T A$.

As the gradient of f is pointing toward the direction of the maximum slope, the points $x_k - \alpha_k \nabla f(x_k)$ constructed will go on the opposite direction, so as to decrease the function and approach its minimum. The *threshold function* h serves to "de-noise" the points computed. The stopping criterion can be either based on $\|x_{k+1} - x_k\| \leq \epsilon$ or $\|\nabla f(x_k)\| \leq \epsilon$. The speed of the algorithm depends on the coefficient α_k. Finding a good α_k is a whole subject of discussion. Once can set $\alpha_k = 1/\mu_{max}$.

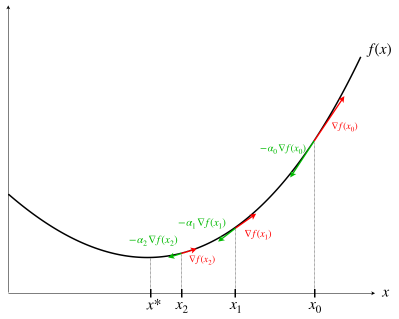

Figure 13: Gradient-descent procedure

The ISTA algorithm is the following :

Algorithm 4: ISTA

Data: y, A, λ
Result: An approximation of x^*, the optimal solution of (3)

Compute μ_{max}, the largest eigenvalue of $A^T A$
Set $x = 0$
repeat
  $x \leftarrow h_{1/\mu_{max}}\left(x - \frac{1}{\mu_{max}} A^T(Ax - y)\right)$
until *the change of* x *is below a certain threshold*

D.2 LARS algorithm

LASSO problem can be interpreted in an interesting way. Suppose that y are some observations and A is a matrix of variables. The problem $\min_x\{\|y - Ax\|_2\}$ the called the *least square model*. It is a standard approach in regression analysis to approximate observations by a linear combination of variables. "Least squares" means that the overall solution minimizes the sum of the squares of the residuals made in the results. Adding the constraints

that we want a sparse solution for this problem leads directly to (1) : $\min_x\{\|y - Ax\|_2, \|x\|_0 \leq t)\}$ for given y, A and t. Historically, LASSO was introduced with this interpretation.

The LARS algorithm [Efr+04] was first introduced to solve LASSO and the method relies on this interpretation. Instead of giving a vector result, the LARS solution consists of a curve denoting the value of the coefficients depending on the solution of the LASSO for different values of the regularization parameter. This allows to graphically see how to set λ, the regularization parameter in the LASSO problem, to have a solution with a certain number of null-coefficients.

We recall that an optimality condition for x^* in the LASSO is

$$\theta^* \in \begin{cases} sign\{x_i^*\} & \text{if } x_i^* \neq 0 \quad (L1) \\ [-1; 1] & \text{if } x_i^* = 0 \quad (L2) \end{cases}$$

The LARS algorithm is the following :

Algorithm 5: LARS

Data: y, A
Result: A finite sequence of breakpoints $(x^*(\lambda_0), \lambda_0), (x^*(\lambda_1), \lambda_1), \ldots$

$\lambda_0 \leftarrow \lambda_{max}$
$x^*(\lambda_0) \leftarrow 0_{\mathbb{R}^n}$
$s_0 \leftarrow 0_{\mathbb{R}^n}$
repeat
  Decrease λ_k until there is a coordinate j of $x^*(\lambda_k)$ which satisfies $x_j^*(\lambda_k) \neq 0$ but not $(L2)$
  Set λ_{k+1} to the stopping value of λ_k
  Yield $(x^*(\lambda_{k+1}), \lambda_{k+1})$
until $\lambda_{k+1} \neq 0$

As the regularization parameter decreases, the coefficients of x^* becomes sequentially non-null one by one. Once a coefficient is become non-null, it won't vanish for smaller λ. Every time a coefficient becomes non-null for a λ_k, the algorithm outputs the *breakpoint* $(x^*(\lambda_k), \lambda_k)$. One thing to notice is that if $(x^*(\lambda_k), \lambda_k)$ and $(x^*(\lambda_{k+1}), \lambda_{k+1})$ are two successive breakpoints, then $\forall\lambda$ such that $\lambda_k \geq \lambda > \lambda_{k+1}$, we have $\|x^*(\lambda_k)\|_0 = \|x^*(\lambda)\|_0 > \|x^*(\lambda_{k+1})\|_0$.

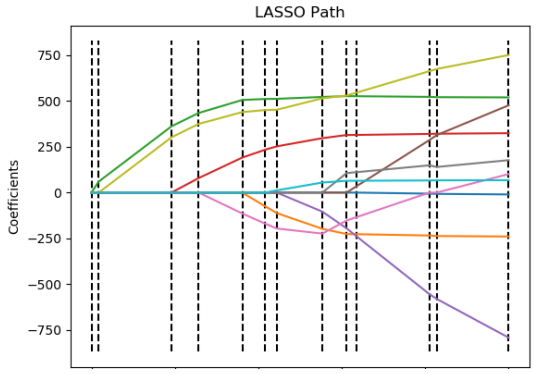

Figure 14: LARS output : Each colour represents a different coefficient of $x^*(\lambda_k)$ as a function of $\|x(\lambda_k)\|_1$. Dashed lines represent breakpoints of the algorithm (Source : Scikit-Learn).

E Joint screening and squeezing tests

Two other techniques inspired by screening were presented above : joint screening and squeezing. Here, we go a bit into details in these two techniques.

E.1 Joint screening

In the report, we mentioned *joint screening*, techniques derived from screening, which allows to screen multiple coefficients at once. We said that they are based on the fact that for $\mathcal{G} \in \mathbb{R}^m$ and $\mathcal{A} = \{a_i, i = 1, \dots, n\}$, we can construct the joint screening test (32). There are two candidates for $\mathcal{G}$: sphere and dome region (33). Those two regions leads respectively to the following tests :

$$t^T q < r - \epsilon\|q\|_2 \Leftrightarrow \max_{a_i \in \mathcal{G}^s(t,\epsilon)} a_i^T q < r \Rightarrow x_i^* = 0 \quad \forall i : a_i \in \mathcal{G} \cap \mathcal{A} \tag{53}$$

$$(t^T q) < r \Leftrightarrow \max_{a \in \mathcal{G}^d(t,\delta)} < r \Rightarrow x_i^* = 0 \quad \forall i : a_i \in \mathcal{G} \cap \mathcal{A} \qquad \text{for} \quad \delta > \frac{t^T q r + \sqrt{\|q\|_2^2 - (t^T q)^2}\sqrt{\|q\|_2^2 - r^2}}{\|q\|_2^2} \tag{54}$$

Knowing that, we can wonder how to choose the number of regions $\{\mathcal{G}_l\}_{l \in 1,\dots,L}$ bounding a part of the $\{a_i\}_i$ and how to apply efficiently joint-screening tests. As a test "only" requires the computation of $t^T q$, the overall complexity to test all the regions $\{\mathcal{G}_l\}_{l=1,\dots,L}$ is L times the complexity of $t^T q$. So the total test complexity is $o(m)$. It remains to find t and q ($i.e$ to choose $\{\mathcal{G}_l\}_{l \in 1,\dots,L}$).

We will focus on spherical region so in the following, $\mathcal{G}^s = \mathcal{G}$. Given the value of t, we want to set ϵ so as to construct the largest region passing the joint screening test (so as to screen the maximum number of coefficients of x^*). We can note that (53) is satisfied as soon as

$$\epsilon < \epsilon_{t,q} := \frac{r - t^T q}{\|q\|_2} \tag{55}$$

Setting $\epsilon = \epsilon_{t,q}$ for the region $\mathcal{G}_l$, we can construct the screening test

$$\|a_i - t\|_2 < \epsilon_{t,c} \Rightarrow x_i^* = 0, \quad \forall a_i \in \mathcal{G}_l \tag{56}$$

The values $\{\|a_i - t\|_2, a_i \in \mathcal{G}_l\}$ can be computed and sorted in advance so as to only test the largest value and hopefully screen all the coefficients corresponding to $\mathcal{G}_l$. Adding the sorting operation and the computation of $\epsilon_{t,q}$ leads to a complexity to proceed the L test is $o(m + \log(n))$.

Now, we will focus on the dome region so in the following, $\mathcal{G}^d = \mathcal{G}$. Assuming that t is given, a lower bound on δ such that the region verifies the dome test (54) is $\delta_{t,q}$:

$$\delta > \delta_{t,q} := \frac{t^T q r + \sqrt{\|q\|_2^2 - (t^T q)^2}\sqrt{\|q\|_2^2 - r^2}}{\|q\|_2^2} \tag{57}$$

This leads to the following joint screening test

$$\begin{cases} t^T q < r \\ t^T a_i > \delta_{t,q} \end{cases} \Rightarrow x_i^* = 0 \tag{58}$$

Again, it is possible to compute and short in advance the values $\{t^T a_i, a_i \in \mathcal{G}_l\}$ so as to only test the smallest value. Then, we can deduce that the complexity to proceed the L joint screening dome test is $o(m + \log(n))$.

Compared to the classical screening tests in $o(nm)$, joint screening tests are much more cheaper to proceed but in practice, they are also worst to screen x^*.

E.2 Squeezing

As in screening, SAFE squeezing test are constructed using the following dual problem

$$\theta^* \in \arg\max_{\theta} \left\{ \frac{1}{2}\|y\|_2^2 - \frac{1}{2}\|\theta - y\|_2^2, \quad \|A_{\mathcal{I}}^T \theta\|_1 + s^T \theta \leq \lambda \right\} \tag{59}$$

With $\mathcal{I} \subset \mathcal{I}^*$. We denote $\mathcal{U}_\mathcal{I}$ the feasible region for the given $\mathcal{I}$. As in screening, it is possible to use KKT conditions to construct the ideal test to *squeeze* x^* as $|a_i\theta^*| > 0 \Rightarrow i \in \mathcal{I}^*_{sign(a^T i\theta^*)}$ but the solution of (59) is as complex to obtain as x^*. The idea is the same as this in screening test, we will try to bound θ^* with a region Θ so as to construct the following squeezing tests

$$\begin{array}{rcl} \min_{\theta\in\Theta}\{a_i^T\theta\} > 0 & \Rightarrow & a_i^T\theta^* > 0 \quad \Rightarrow \quad i \in \mathcal{I}^*_+ \\ \max_{\theta\in\Theta}\{a_i^T\theta\} < 0 & \Rightarrow & a_i^T\theta^* < 0 \quad \Rightarrow \quad i \in \mathcal{I}^*_- \end{array} \tag{60}$$

As in screening, we will focus on spherical regions and especially on two particular cases which are similar to ST1 and GAP regions. If θ^* is contained in a spherical region $\mathcal{B}(q, r)$, then we can write the SAFE squeezing test as

$$|a_i^T q| > r \Rightarrow i \in \mathcal{I}^*_{sign(a_i^T q)} \tag{61}$$

In the case of a ST1 squeezing test, the construction of the region relies on a dual-feasible point as in screening. The way to obtain such dual point is dual-scaling. For any dual point θ, a dual-feasible point can be obtained as follows :

$$\theta_f = \begin{cases} \theta & \text{if } \|A_\mathcal{I}^T\theta\|_1 + s^T\theta \leq \lambda \\ \theta/\|A_\mathcal{I}^T\theta\|_1 + s^T\theta & \text{otherwise} \end{cases} \tag{62}$$

It is important to notice that the dual-scaling operation is heavy to compute.

For the GAP sphere, taking any $\mathcal{I} \subset \mathcal{I}^*$ and $(w, \tilde{x}, \theta)$ satisfying $\tilde{x} \leq w,\ -\tilde{x} \leq w,\ \theta \in \mathcal{U}_\mathcal{I}$, it is possible to define the GAP squeezing sphere with center θ and radius $\sqrt{2gap(w, \tilde{x}, \theta)}$ with

$$gap(w, \tilde{x}, \theta) = \frac{1}{2}\|y - A_{\bar{\mathcal{I}}}\tilde{x} - sw\|_2^2 + \lambda w - (\frac{1}{2}\|y\|_2^2 - \frac{1}{2}\|\theta - y\|_2^2) \tag{63}$$

F Interviews

In addition to this report, three interviews of researches from different laboratories were required. These interviews were made in French.

F.1 Clément Elvira, INRIA Rennes

Quel a été ton parcours universitaire ?

Classe préparatoire MPSI / MP, Centrale Lille (1ère et 2nd année généraliste, spécialisation "analyse de données" en dernière année) et en parallèle de ma troisième année à Centrale, j'ai aussi fait un master recherche en mathématiques appliquées à l'université de Lille 1.

Pourquoi s'orienter vers le métier de chercheur plutôt que vers un autre métier en lien avec les mathématiques (ingénieur en maths appliqués, finance ...)?

Bonne question! Initialement, j'étais motivé par les métiers de la finance (analyste quantitatif, etc). Je n'avais pas de goût particulier pour la finance, mais c'est le seul métier assez "math" dont j'avais entendu parlé. De par le hasard des stages, je me suis retrouvé en R&D chez Schlumberger en 2013. J'ai tout de suite bien accroché mais je me suis rendu compte que les ingénieurs non docteurs étais assez méprisés (boîte américaine oblige) par leurs collègues docteur. C'est ce qui m'a motivé à au moins faire une thèse. Je me suis ensuite rendu compte pendant ma thèse que le monde et la recherche académique me convenait plus.

Pourquoi avoir choisi ce domaine de recherche plutôt qu'un autre ?

Pour l'instant, je n'ai pas vraiment choisi de domaine. J'ai fait une thèse plutôt proba stats. Pour mon post-doc, j'ai eu l'occasion de changer de domaine et j'en ai profité. Disons que pour l'instant, mon choix était plutôt dirigé par l'envie de travailler avec les encadrants plutôt que le sujet en lui-même.

Est-ce que parfois, le choix du sujet de recherche a été contraint/imposé ou le choix du sujet a-t-il toujours été assez libre ?

Pour rebondir avec ce qui a été dit précédemment, tu as toujours un sujet d'imposé quand tu arrives en tant que doctorant / postdoc. Après, sauf exception particulière, les sujets sont toujours suffisamment assez large pour que tu puisses t'exprimer. C'est une forme de "création sous contraintes" : on te dit d'aller dans une direction et d'utiliser certains ingrédients, mais après tu es libre.

Pourquoi avoir choisi un organisme de recherche public plutôt qu'un organisme de R&D privé ?

Dans un organisme privé, tu as toujours la question de la rentabilité de ta recherche. Tu peux bien sûr pratiquer le même genre de logique dans le publique (rentabilité en terme de nombre de publi, de citations etc), les gens se laissent généralement guider par les sujets qui les excitent. De fait, j'apprends presque tous les jours de nouvelles choses.

Est-ce que le fait de devoir enseigner est un avantage ou un inconvénient du métier de chercheur ?

Déjà, tous les chercheurs n'enseignent pas ! De mon côté j'apprécie beaucoup enseigner, mais à petites doses. Je peux te lister quelques avantages :

- visibilité auprès des étudiants pour recruter des projets recherches, doctorants, stagiaires etc.
- ça permet généralement de prendre du recul sur une matière puisque l'on doit la faire comprendre aux autres. On dit généralement que l'on a pas compris un sujet tant qu'on ne l'a pas enseigné.
- financière puisque les enseignements sont rémunérés au titre "d'heures supplémentaires".

Après, il y a aussi quelques inconvénients :

- ça demande du temps pour préparer les cours
- les étudiants ne sont pas toujours intéressés,
- c'est du temps investi qui est dur à valoriser sur un CV.

De mon côté, je trouve que les avantages surpassent largement les inconvénients.

L'INRIA me semble traiter de sujets plus "appliqués" que les laboratoires de recherches classiques en mathématiques (comme l'IRMAR par exemple). Est-ce un aprioris ou est-ce bien le cas ?

Alors non, l'Inria ne traite pas que des sujets appliqués. Je ne sais pas d'où tu tiens cette information. Ensuite, à l'IRMAR, il font des mathématiques appliquées, ce qui est un sujet appliqué par définition ! Enfin, sache pour ta gouverne que Cédric fait aussi partie de l'IRMAR et ce n'est pas le seul chercheur à l'Inria. Disons qu'il y a un peu de tout à l'Inria. Alors effectivement, personne ne fait de mathématiques fondamentales et il y a des équipes qui sont très proches de l'industrie et dont l'aspect "mathématique" est moins poussé. En fait c'est un peu à la discrétion de chaque équipe / chercheur. C'est en partie pour cela que j'apprécie l'Inria : tu peux à la fois pousser des directions très appliquées (ex: interagir avec des médecins sur des applis concrètes) et à la fois pousser vers des résultats assez fondamentaux.

F.2 Jérémy Cohen, CNRS Rennes

Quel a été votre parcours universitaire ?

Après un Bac S option maths, j'ai fais une prépa MPSI - MP*, puis l'école centrale de Lyon. Durant l'ECL, j'ai suivi une licence de mathématiques à l'université de Lyon (j'ai juste fait les examens en fait), puis un master en télécomunications. J'ai effectué mon stage de fin d'études et mon stage de master recherche (2 en 1) à l'université de Tokyo sur un sujet d'information quantique. J'ai effectué ma thèse au Gipsa-lab à Grenoble, avec Pierre Comon, sur du traitement de données par décompositions tensorielles.

Pourquoi s'orienter vers le métier de chercheur plutôt que vers un autre métier en lien avec les mathématiques (ingénieur en maths appliqués, finance ...) ?

Au départ c'était plus le hasard qu'autre chose; mais maintenant je pense pouvoir dire à postériori que le métier de chercheur permet d'explorer dans des directions très différentes suivant ses goûts du moment et permet de creuser sa propre voie.

Pourquoi avoir choisi ce domaine de recherche plutôt qu'un autre ?

Je n'ai pas vraiment choisi. Mais je pense que n'importe quel sujet de recherche mettant en oeuvre des mathématiques et une forme d'application à des problèmes concrets m'aurait convenu. Aujourd'hui je choisi les sujets en fonction de ce que je connais, ce qui me questionne et de mes envies. Je ne crois pas qu'il existe un choix intrinsèque du meilleur sujet.

Est-ce que parfois, le choix du sujet de recherche a été contraint/imposé ou le choix du sujet a-t-il toujours été assez libre ?

Le sujet pour la thèse était globalement imposé. De plus, les sujets que je traite sont toujours guidés par ce que je pense pouvoir accomplir en un temps raisonable. Sinon on ne m'impose pas de contraintes. Je dirai donc que le choix et libre mais sous contraintes exogènes.

Pourquoi avoir choisi un organisme de recherche public plutôt qu'un organisme de R&D privé ?

Pas de raison particulière. Et rien ne dit que celà ne changera pas, tant que l'institut privé donne la liberté de poursuivre des travaux originaux.

Est-ce que le fait de devoir enseigner est un avantage ou un inconvénient du métier de chercheur ?

Un avantage à mon avis, tant que la charge (i.e. le nombre d'heures) est décidé/maîtrisé par le chercheur. Ce n'est souvent pas le cas pour les maîtres de conférences par contre...

L'INRIA me semble traiter de sujets plus "appliqués" que les laboratoires de recherches classiques en mathématiques (comme l'IRMAR par exemple). Est-ce un aprioris ou est-ce bien le cas ?

Je ne pense pas. L'Inria c'est très grand, il s'y fait de l'informatique théorique comme des choses très pratiques.

F.3 Frédéric Guyard, Orange Labs Sophia-Antipolis

Pourquoi avoir choisi le métier de chercheur ?

D'aussi loin que je me souvienne, j'ai toujours été intéressé par les sciences. Puis à l'école et dans la suite de mes études, il s'est avéré que j'avais plutôt des facilités pour les mathématiques et pour les sciences en général qui me passionnaient toujours autant. C'est donc essentiellement par passion que j'ai tout naturellement poursuivi des études de sciences à l'université et que je suis devenu chercheur. A vrai dire, je n'ai jamais sérieusement considéré un autre métier et j'ai énormément de chance de pouvoir vivre ma passion, encore aujourd'hui.

Pourquoi travailler sur le Deep Learning plutôt qu'un autre sujet ?

Le deep learning est au carrefour des mathématiques, de l'informatique, de la physique, de la biologie, de la neurologie etc. tout en portant de fortes questions philosophiques, humaines, sociologiques, sociétales et éthiques. Lorsque l'on est passionné par les sciences, il constitue un sujet des plus attirants. Pour mon travail à Orange Labs, je suis appelé à m'intéresser, aussi bien aux aspects théoriques qu'aux nombreuses applications potentielles des réseaux de neurones. Les performances étonnantes des réseaux de neurones profonds sont encore largement incomprises, posant de nombreuses questions théoriques passionnantes. Quant à leurs applications, elles n'en sont vraisemblablement qu'à leur début et tout reste encore à imaginer.

Est-ce que le choix du sujet de recherche a toujours été plutôt libre ou a-t-il parfois été contraint ?

Etant chercheur pour une entreprise privée, les sujets de recherche auxquels je m'intéresse sont bien entendu légitimement conditionnés par leur intérêt pour l'entreprise. L'équipe de recherche dans laquelle je travaille focalise ses activités sur l'apprentissage machine et sur la valorisation des données. Mes sujets sont donc contraints par cette thématique. Cette thématique étant large, j'ai cependant une assez grande latitude sur le choix précis de mes sujets et de mes collaborations, ce qui me permettant d'aborder à la fois des sujets théoriques et des sujets plus appliqués.

Quelle est la journée type d'un chercheur ?

Pas de journée qui ne débute bien sans un café avec les collègues. Puis chacun s'en va à ces occupations. Dans mon cas, mon activité est régie par l'ensemble des projets en cours avec, une fois tous les 15 jours une réunion d'équipe ou chacun présente un point sur ses activités. Mes projets s'organisent autour de deux axes. Approximativement 30% de mon temps annuel est dédié aux projets dits de « delivery », les 70% restant étant dédiés aux projets de recherche. Les projets de « delivery » consistent en des études de data science pour nos « clients » (des entités internes à Orange ; nous ne travaillons pas pour les clients externes à Orange) ou des réalisations de services ou programmes informatiques mettant en application nos résultats de recherche. Pour les activités de recherche, nous travaillons soit seuls, soit avec les membres de l'équipes ou d'autres équipes de Orange, soit encore avec des chercheurs externes : université, INRIA, CNRS, projets de recherche ANR (Agence Nationale de la Recherche) ou Européens. Une partie des activités de recherche consiste également en encadrement de stagiaires ingénieurs ou de doctorants. Pour répondre plus directement à la question, il n'y a pas vraiment de journée « type ». Il m'arrive de travailler l'ensemble de la journée pendant plusieurs semaines d'affilée sur un projet de recherche ou sur une application. Il m'arrive également de travailler sur 2 ou 3 projets différents au cours d'une même journée.

Pourquoi avoir choisi le secteur privé plutôt que le secteur public ?

Après un doctorat, j'ai passé presque une dizaine d'années en postdoc dans des instituts de recherche ou dans des universités étrangères. De retour en France au début des années 2000, il était alors très difficile d'obtenir des postes dans le milieu de la recherche publique à l'université ou au CNRS et je me suis donc résigné à chercher des emplois dans le domaine privé. J'ai alors eu la très grande chance d'obtenir un poste de chercheur à Orange Labs, poste que j'occupe encore et dans lequel les conditions de travail sont très proches de celles que j'avais dans le secteur public. L'équilibre qui m'est demandé à Orange Labs entre la recherche appliquée et la recherche théorique est en fait très plaisant et me permet de m'intéresser à des sujets probablement plus variés que ceux auxquels mes penchants naturels m'auraient entrainés à étudier si j'étais resté dans le secteur de la recherche publique.

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