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Variation of cost functions in integer programming [1]

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Abstract

We study the problem of minimizing $c \cdot x$ subject to $A \cdot x = b$, $x \geqslant 0$ and x integral, for a fixed matrix A. Two cost functions c and c' are considered equivalent if they give the same optimal solutions for each b. We construct a polytope $St(A)$ whose normal cones are the equivalence classes. Explicit inequality presentations of these cones are given by the reduced Gröbner bases associated with A. The union of the reduced Gröbner bases as c varies (called the universal Gröbner basis) consists precisely of the edge directions of $St(A)$. We present geometric algorithms for computing $St(A)$, the Graver basis, and the universal Gröbner basis. © 1997 The Mathematical Programming Society, Inc. Published by Elsevier Science B.V.

0. Introduction

In this paper we study the general integer programming problem

$$IP_{A,c}(b): \quad \text{minimize } c \cdot x \quad \text{subject to } A \cdot x = b \text{ and } x \geqslant 0,\ x \in \mathbb{Z}^n,$$

where $c \in \mathbb{R}^n$, A is a $(d \times n)$ integral matrix of rank d, and $b \in \mathbb{Z}^d$. Let IP_A denote the family of all such programs with fixed matrix A. The convex hull P_b^I of the set of feasible solutions $\{x \in \mathbb{N}^n : Ax = b\}$ is called the *b-fiber* of IP_A. For simplicity we assume that each fiber is bounded. Two cost functions c and c' are *equivalent* if $IP_{A,c}(b)$ and $IP_{A,c'}(b)$ have the same set of optimal solutions, for each right-hand side b. These

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equivalence classes are relatively open polyhedral cones in $\mathbb{R}^n$. They fit together to form a polyhedral fan, called the *Gröbner fan* of A. The study of the Gröbner fan is what we mean by "variation of cost functions in integer programming".

We shall construct a certain $(n-d)$-dimensional polytope $St(A)$ in $\mathbb{R}^n$, called the *state polytope* of A, which has the Gröbner fan as its normal fan. The state polytope can be represented as the *Minkowski integral* $\int_b P_b^I \, db$, where db is any suitable probability measure with support $cone_{\mathbb{N}}(A)$. This provides an integral refinement of polyhedral results in [15,4,6]. The concepts of Gröbner fans and state polytopes were first introduced in a more general algebraic setting by Mora and Robbiano [19] and Bayer and Morrison [2]. The new construction to be given in Section 3 is more explicit, self-contained and custom-tailored to integer programming.

Our discussion assumes familiarity with Gröbner bases as presented in, for example, [1,9,26,27]. We remark that software for computing them is readily available and surprisingly efficient [3]. The *reduced Gröbner basis* of A with respect to c is a finite subset $\mathcal{G}_c$ of $ker_{\mathbb{Z}}(A)$: it is the minimal *test set* (in the sense of [21, §17.3]) for all programs $IP_{A,c}(\cdot)$. The union of all reduced Gröbner bases $\mathcal{G}_c$, as c varies over $\mathbb{R}^n$, is a finite set. It is denoted UGB_A and called the *universal Gröbner basis* of A. One of our main results (Theorem 5.1) states that the elements of UGB_A are precisely the edge directions of all fibers P_b^I.

This paper is organized as follows: In Section 1 we review known results for linear programming whose integer analogues are to be established later. Writing $LP_{A,c}(b)$ for the linear relaxation of $IP_{A,c}(b)$, we say that two cost functions c and c' are *equivalent (for LP_A)* if the linear programs $LP_{A,c}(b)$ and $LP_{A,c'}(b)$ have the same set of optimal solutions for each b. This is the case if and only if the *regular polyhedral subdivisions* Δ_c and $\Delta_{c'}$ coincide (Theorem 1.3). Each equivalence class is the relative interior of a cone in the *secondary fan* of A, which is the normal fan of the *secondary polytope* $\Sigma(A)$ (see [5,6,15]). The *circuits* of A form a universal test set for LP_A, and these are precisely the set of edge directions of $\Sigma(A)$ (Theorem 1.8).

In Section 2 we examine the reduced Gröbner basis $\mathcal{G}_c$ and the universal Gröbner basis UGB_A. The set UGB_A is contained in the *Graver basis* of A, which is the well-known test set introduced in [16] (see also [8,10]). Example 2.11 shows that the Graver basis can be much larger than the universal Gröbner basis. The integer analogue of the regular triangulation Δ_c is the *(initial) monomial ideal* $in_c(I_A)$. We show how the two are related (Theorem 2.4).

The main result in Section 3 is the structure theorem (3.10) for the equivalence classes of IP_A, involving the Gröbner fan and the state polytope. Each equivalence class is shown to be the relative interior of a face of a *Gröbner cone* $\mathcal{K}_c = \{x \in \mathbb{R}^n : g_i \cdot x \geqslant 0,\ g_i \in \mathcal{G}_c\}$. Thus the local variation of cost functions is controlled by the reduced Gröbner basis $\mathcal{G}_c$. In Proposition 3.13 we introduce the *Graver arrangement* of A which is the collection of hyperplanes normal to the Graver basis. This is a natural refinement of the Gröbner fan, and it serves as an approximation to the latter.

The *Lawrence lifting* of A is the enlarged matrix

Table 1

Linear programming		Integer programming
$LP_{A,c}(b)$: Min cx: $Ax = b$, $x \in \mathbb{R}^n_+$	programs	$IP_{A,c}(b)$: Min cx: $Ax = b$, $x \in \mathbb{N}^n$
$P_b = \{x \in \mathbb{R}^n_+ : Ax = b\}$ $\Sigma(A) = \int_b P_b \, \mathrm{d}b$ (secondary polytope)	polytopes $<$	$P_b^I = \mathrm{conv}\{x \in \mathbb{N}^n : Ax = b\}$ $St(A) = \int_b P_b^I \, \mathrm{d}b$ (state polytope)
$\mathcal{N}(\Sigma(A)) =$ secondary fan	normal fans $<$	$\mathcal{N}(St(A)) =$ Gröbner fan
$\wedge$ circuit arrangement	hyperplane arrgts. $<$	$\wedge$ Gröbner arrgt. $<$ Graver arrgt.
circuits $\Vert$ (edges directions of P_b, for all b)	universal test sets $\subseteq$	universal Gröbner basis $\subseteq$ Graver basis $\Vert$ (edge directions of P_b^I for all b)

$$\Lambda(A) = \begin{pmatrix} A & \mathbf{0} \\ \mathbf{1} & \mathbf{1} \end{pmatrix}$$

where $\mathbf{0}$ is the zero $(d \times n)$-matrix and $\mathbf{1}$ is the unit $(n \times n)$-matrix. The Lawrence lifting corresponds to programs in IP_A with upper bound constraints (as in Eq. (1.4)). Our main result in Section 4 states that *every* reduced Gröbner basis of $\Lambda(A)$ coincides with the Graver basis of $\Lambda(A)$. This leads to Algorithms 4.5 and 4.9 for computing the Graver basis of A and the universal Gröbner basis of A.

In Section 5 we give a geometric characterization of the vectors in the universal Gröbner basis: they are the edge directions of the state polytope (and hence of all fibers P_b^I). Algorithm 5.8 gives a geometric method for computing the Gröbner fan, and Theorem 5.9 relates the following three properties of a matrix:

(i) *A unimodular,* (ii) $St(A) = \Sigma(A)$, (iii) $UGB_A = \{$circuits of $A\}$.

The last section deals with the number of facets of a Gröbner cone or equivalently the valency of a vertex of $St(A)$. We conjecture that this number is bounded above by a function in the corank of A. Two examples that give lower bounds for this function are constructed.

Table 1 summarizes the interrelations between the main concepts in this paper. The symbol "$<$" denotes refinement for polyhedral fans and "is Minkowski summand of" for polytopes.

1. Variation of cost functions in linear programming

The results on integer programming to be presented in this paper have known easier analogues in linear programming. In this section we give an exposition of these analogues. The results presented below arise from a geometric perspective of linear programming based on recent results in [5,6,15]. Although proofs are not always given in detail, we hope that the reader will find the results plausible if not familiar. We also use the first two sections to introduce the definitions and notation needed in this paper. For a matrix $A = (a_1, \ldots, a_n) \in \mathbb{Z}^{d\times n}$ of rank d, let $cone(A) = cone(\{a_1, \ldots, a_n\})$ represent the closed convex polyhedral d-cone $\{Ax : x \in \mathbb{R}^n_+\}$ and $ker(A)$ represent the $(n-d)$-dimensional subspace $\{x \in \mathbb{R}^n : Ax = 0\}$. A *polyhedral subdivision* of $cone(A)$ is a collection of subcones of the form $cone(\{a_{i_1}, \ldots, a_{i_k}\})$, called *cells* (or *faces*) of the subdivision, that form a *polyhedral fan* covering $cone(A)$. By a polyhedral fan we mean a family of polyhedral cones such that the intersection of any two is a face of each and is itself in the family. A cell $cone(\{a_{i_1}, \ldots, a_{i_k}\})$ of a subdivision is often abbreviated as $\{i_1, \ldots, i_k\}$ and if it is of dimension k, it is called a *k-cell*. Notice that a polyhedral subdivision is completely specified by listing its maximal cells. A subdivision of $cone(A)$ is a *triangulation* if each d-cell of the complex is simplicial (has exactly d extreme rays). Our starting point is the Basis Decomposition Theorem for Linear Programming in [28].

Theorem 1.1 (Walkup and Wets [28]). *Let $LP_{A,c}(b)$ denote the linear program*

$$\text{minimize } c\cdot x \quad \text{subject to } A\cdot x = b \text{ and } x \geqslant 0, \tag{1.1}$$

where $c \in \mathbb{R}^n$ is fixed and $A = (a_1, a_2, \ldots, a_n)$ is a fixed $d\times n$-matrix of rank d. Then:

(i) *$LP_{A,c}(b)$ is feasible if and only if b lies in $cone(A)$.*

(ii) *$LP_{A,c}(b)$ is bounded for all $b \in cone(A)$ and all $c \in \mathbb{R}^n$ if and only if $ker(A) \cap \mathbb{R}^n_+ = \{0\}$.*

(iii) *If $LP_{A,c}(b)$ is bounded, then there exists a triangulation Δ of $cone(A)$ such that*

  (a) *the d-dimensional cells of Δ have the form $C = cone(\{a_{i_1}, \ldots, a_{i_d}\})$, and*

  (b) *the column vectors $a_{i_1}, \ldots, a_{i_d}$ constitute an optimal basis for all b in cell C.*

This theorem is best understood within the context of *secondary polytopes* (see [5,6,15]). For simplicity we shall assume throughout this paper that $ker(A) \cap \mathbb{R}^n_+ = \{0\}$. We denote by LP_A the family of all linear programs of the form (1.1) with the fixed coefficient matrix A. For every $c \in \mathbb{R}^n$ there is a polyhedral subdivision Δ_c of $cone(A)$ defined as follows: $cone(\{a_{i_1}, \ldots, a_{i_k}\})$ is a cell of Δ_c if and only if there exists a row vector $y \in \mathbb{R}^d$ such that $y\cdot a_j = c_j$ if $j \in \{i_1, \ldots, i_k\}$ and $y \cdot a_j < c_j$ if $j \in \{1, \ldots, n\} \setminus \{i_1, \ldots, i_k\}$. It is customary (and more precise) to say that $\{i_1, \ldots, i_k\}$ is a cell of Δ_c. Subdivisions obtained in this way are called *regular* (or *coherent*). For almost all $c \in \mathbb{R}^n$, the regular subdivision Δ_c is a triangulation, in which case we call c *generic* (with respect to LP_A). See Corollary 1.6 for equivalent definitions.

Part (iii) of the Walkup–Wets Theorem can be proved as follows: If c is generic, then $\Delta = \Delta_c$ is the desired triangulation. If c is not generic, then we may take Δ to be any regular triangulation which refines Δ_c. In other words, we may take $\Delta = \Delta_{c'}$ where c' is generic and very close to c.

Example 1.2. Consider the family of linear programs of the form (1.1) defined by the (3×6)-matrix

$$A = \begin{pmatrix} 2 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 2 \end{pmatrix}.$$

The vector $c = (1,0,0,1,0,0)$ is not generic. The corresponding subdivision of $cone(A) = \mathbb{R}^3_+$ consists of two triangular cones and one quadrangular cone:

$$\Delta_c = \{\{1,2,3\},\{2,4,5\},\{2,3,5,6\}\},$$

where i indexes a_i. To get a regular triangulation refining Δ_c we may take $c' = (1,0,0,1,0,1)$. The cost function c' is generic, since

$$\Delta_{c'} = \{\{1,2,3\},\{2,3,5\},\{2,4,5\},\{3,5,6\}\}.$$

Let P_b denote the polytope $\{x \in \mathbb{R}^n_+ : Ax = b\}$. (Since $ker(A) \cap \mathbb{R}^n_+ = \{0\}$, P_b is a polytope for all $b \in cone(A)$.) We call P_b the *b-fiber* of the family LP_A. This terminology is consistent with the usage in [4] and [27]. For $x \in \mathbb{R}^n$ abbreviate

$$supp(x) = \{j \in \{1,\ldots,n\} : x_j \neq 0\}.$$

Theorem 1.3. *Given two cost functions c and c' in $\mathbb{R}^n$, the following are equivalent:*
(i) *The programs $LP_{A,c}(b)$ and $LP_{A,c'}(b)$ have the same set of optimal solutions, for every b.*
(ii) *The cost functions c and c' support the same optimal face in each fiber P_b of LP_A.*
(iii) *The vectors c and c' define the same polyhedral subdivision $\Delta_c = \Delta_{c'}$.*

Proof. The conditions (i) and (ii) are equivalent because the set of optimal solutions of $LP_{A,c}(b)$ is the face of P_b supported by c. The equivalence of (i) and (iii) follows from Lemma 1.4 below. □

Lemma 1.4. *The optimal solutions x to $LP_{A,c}(b)$ are the solutions to the problem:*

Find $x \in \mathbb{R}^n$ such that $A \cdot x = b$, $x \geqslant 0$, and $supp(x)$ is a subset of a cell of Δ_c. (1.2)

Proof. Consider the linear program dual to (1.1):

$$\text{Maximize } y \cdot b \quad \text{subject to } y \cdot A \leqslant c \text{ and } y \in \mathbb{R}^d. \tag{1.3}$$

Let x be an optimal solution of (1.1) and y an optimal solution of (1.3). By complementary slackness, $x_j > 0$ implies $y \cdot a_j = c_j$, which means that $supp(x)$ lies in a face of Δ_c. Conversely, let x be any solution to (1.2). Then there exists $y \in \mathbb{R}^d$ with $supp(x) \subseteq \{j : y \cdot a_j = c_j\}$. This implies $c \cdot x = y \cdot A \cdot x = y \cdot b$ and hence, x is an optimal solution of (1.1). □

We now recall some general facts about convex polytopes and polyhedral fans. If P is any polytope in $\mathbb{R}^n$ and c is any (not necessarily generic) vector in $\mathbb{R}^n$, then we write $face_c(P)$ for the face of P at which c gets minimized. If F is any face of P, then $\mathcal{N}(F;P)$ denotes the cone of (inner) normals, called the *inner normal cone* of P at F. In symbols, $\mathcal{N}(F;P) = \{c \in \mathbb{R}^n : c \cdot x \leqslant c \cdot y \text{ for all } x \in F,\ y \in P\}$. The collection of cones $\mathcal{N}(F;P)$ is denoted $\mathcal{N}(P)$ and called the (inner) *normal fan* of the polytope P. The normal fan of P is a polyhedral fan that covers $\mathbb{R}^n$. We say that two polytopes are *normally equivalent* if they have the same normal fan. Given two polytopes P and Q in $\mathbb{R}^n$, their *Minkowski sum* is the polytope $P + Q = \{p + q : p \in P,\ q \in Q\} \subset \mathbb{R}^n$. The polytopes P and Q are called *Minkowski summands* of $P + Q$. As in the usual extension of addition to integration, the operation of taking Minkowski sums of finitely many polytopes extends naturally to the operation of taking *Minkowski integrals* of infinitely many polytopes. See [4] for details. The common *refinement* of two fans $\mathcal{F}$ and $\mathcal{G}$ in $\mathbb{R}^n$, denoted $\mathcal{F} \cap \mathcal{G}$, is the fan of all intersections of cones from $\mathcal{F}$ and $\mathcal{G}$. We say that $\mathcal{F} \cap \mathcal{G}$ is a refinement of $\mathcal{F}$ (respectively $\mathcal{G}$). The following are two useful facts in this context: (i) for polytopes P and Q in $\mathbb{R}^n$, the fan $\mathcal{N}(P+Q) = \mathcal{N}(P) \cap \mathcal{N}(Q)$ and (ii) the fan $\mathcal{N}(P)$ is a refinement of $\mathcal{N}(Q)$ if and only if λQ is a Minkowski summand of P for some positive real number λ. For a hyperplane $H = \{x \in \mathbb{R}^n : ax = 0\}$ in $\mathbb{R}^n$, let H^+ denote the closed half space $\{x \in \mathbb{R}^n : ax \geqslant 0\}$ and H^- denote $\{x \in \mathbb{R}^n : ax \leqslant 0\}$. A *hyperplane arrangement* in $\mathbb{R}^n$ is the common refinement of finitely many fans of the form $\{H^+, H^-\}$. The arrangement is usually specified by listing the associated hyperplanes. The Minkowski sum of finitely many line segments is called a *zonotope* and by (i) its normal fan is a hyperplane arrangement.

Theorem 1.3 gives rise to a natural equivalence relation on cost functions: two vectors c and c' in $\mathbb{R}^n$ are *equivalent* (*with respect to* LP_A) if the conditions in Theorem 1.3 hold. We have the following structure theorem for the equivalence classes. Theorem 1.5 is a direct translation of results of Gel'fand et al. [15, Ch. 7] and Billera et al. [6].

Theorem 1.5.
(i) *There are only finitely many equivalence classes of cost functions for* LP_A.
(ii) *Each equivalence class is the relative interior of a convex polyhedral cone in* $\mathbb{R}^n$.
(iii) *The collection of these cones defines a polyhedral fan which covers* $\mathbb{R}^n$. *This fan is called the* secondary fan *of* A.
(iv) *Let* db *denote any probability measure with support cone*(A). *Then the Minkowski integral* $\Sigma(A) = \int_b P_b \,\mathrm{d}b$ *is an* $(n-d)$*-dimensional convex polytope, called the* secondary polytope *of* A. *The inner normal fan of* $\Sigma(A)$ *equals the secondary fan of* A.

Example 1.2 (*continued*). The secondary polytope $\Sigma(A)$ is a simple 3-polytope with 14 vertices, 21 edges, and 9 facets; see Fig. 34 in [15] and Fig. 3 in [25]. It is known as the *associahedron* or *Stasheff polytope*. The family LP_A has $45 = 14 + 21 + 9 + 1$ equivalence classes of cost functions. The 14 distinct regular triangulations of *cone*(A) are listed in Section 3.

Corollary 1.6. *For a cost function c in $\mathbb{R}^n$ the following are equivalent:*
(i) *c is generic, i.e., the subdivision Δ_c is a triangulation.*
(ii) *c supports a vertex in each fiber P_b of LP_A.*
(iii) *For every $b \in cone(A)$, the program $LP_{A,c}(b)$ has a unique optimal solution.*
(iv) *c lies in the interior of an n-dimensional cell of the secondary fan of A.*
(v) *c supports a vertex of the secondary polytope $\Sigma(A)$.*

An important tool used in this paper is a "test set" for integer programming (cf. [21,27]). It is instructive to define the following analogue for linear programming: A *test set* for the family $LP_{A,c}(\cdot)$ is any finite subset $\mathcal{T}$ of $ker(A)$ such that, for every $b \in cone(A)$ and every $x \in P_b$, either x is an optimal solution of $LP_{A,c}(b)$ or there exists $t \in \mathcal{T}$ and $\varepsilon > 0$ such that $x - \varepsilon t \geqslant 0$ and $c \cdot t > 0$. A test set is *minimal* if it has minimal cardinality.

For the remainder of this section we shall assume that c is generic, so that Δ_c is a triangulation. We say that $I \subset \{1, \ldots, n\}$ is a *minimal non-face* of Δ_c if I is not a face of Δ_c but every proper subset of I is a face of Δ_c.

Proposition 1.7. *A finite subset $\mathcal{T} \subset ker(A)$ is a minimal test set for $LP_{A,c}(\cdot)$ if and only if for every minimal non-face I of Δ_c there is a unique vector $t \in \mathcal{T}$ such that $I = \{i : t_i > 0\}$.*

Proof. By Lemma 1.4, a feasible solution y of $LP_{A,c}(b)$ is non-optimal if and only if $supp(y)$ contains a minimal non-face of Δ_c. Hence $\mathcal{T} \subset ker(A)$ is a minimal test set for $LP_{A,c}(\cdot)$ if and only if for every minimal non-face I of Δ_c there is a unique vector $t \in \mathcal{T}$ such that $I = \{i : t_i > 0\}$. This property of $\mathcal{T}$ is necessary and sufficient to guarantee the existence of an improving direction in $\mathcal{T}$ for every non-optimal solution to a program in $LP_{A,c}(\cdot)$. □

Example 1.2 (*continued*). Let $c' = (1,0,0,1,0,1)$ as above. The minimal non-faces of the triangulation $\Delta_{c'}$ are $\{1,4\}, \{1,5\}, \{1,6\}, \{2,6\}, \{3,4\}$ and $\{4,6\}$. A minimal test set for $LP_{A,c'}(\cdot)$ is $\mathcal{T} = \{e_1 + e_4 - 2e_2, e_1 + e_5 - e_2 - e_3, e_1 + e_6 - 2e_3, e_2 + e_6 - e_3 - e_5, e_3 + e_4 - e_2 - e_5, e_4 + e_6 - 2e_5\}$. Here and throughout this paper we write e_i for the ith unit vector in $\mathbb{R}^n$.

We define a *universal test set* for the family LP_A to be any finite subset $\mathcal{T}$ of $ker(A)$ such that $\mathcal{T}$ is a test set for $LP_{A,c}(\cdot)$, for every generic $c \in \mathbb{R}^n$.

Theorem 1.8. *For a finite subset $C \subset ker(A)$ the following are equivalent:*
(i) *$C \cup -C$ is a minimal universal test set.*
(ii) *Every edge direction of any fiber P_b has a unique representative in C.*
(iii) *Every edge direction of the secondary polytope $\Sigma(A)$ has a unique representative in C.*
(iv) *Every equivalence class of circuits of A has a unique representative in C.*

Here a *circuit* of A is a non-zero vector of minimal support in $ker(A)$, and two circuits t and t' are equivalent if $t' = \lambda t$. (This is consistent with the usage in matroid theory.) The circuits $\mathcal{C}$ are precisely the possible directions taken by the simplex algorithm when solving any of the programs $LP_{A,c}(b)$. Thus the study of test sets can be viewed as an abstraction of the simplex algorithm.

Example 1.2 (*continued*). The matrix A has nine circuits. With $\mathcal{T}$ as above, we have

$$\mathcal{C} = \mathcal{T} \cup \{e_1 + 2e_5 - 2e_2 - e_6, 2e_2 + e_6 - 2e_3 - e_4, e_1 - e_4 - 2e_3 + 2e_5\}.$$

Each fiber P_b of LP_A is a polytope of dimension at most three and has its edge directions among these nine circuits. A fiber P_b is of dimension three if and only if b lies in the interior of $cone(A)$.

Corollary 1.9. *For every generic $c \in \mathbb{R}^n$, there exists a minimal test set for $LP_{A,c}(\cdot)$ which consists only of edges of certain fibers P_b.*

Proof. For every minimal non-face I of Δ_c there is a circuit t with $I = \{i : t_i > 0\}$. By Proposition 1.7, the collection of such circuits form a minimal test set for $LP_{A,c}(\cdot)$. The result now follows from Theorem 1.8 (ii) and (iv). □

Remark 1.10. If the set of circuits is known, and a generic vector $c \in \mathbb{R}^n$ is given, then the regular triangulation Δ_c can be computed as follows. The faces of Δ_c are those subsets of $\{1, \ldots, n\}$ which do not contain $\{i : t_i > 0\}$ for any $t \in \mathcal{C}$ such that $t \cdot c > 0$.

The *circuit arrangement* of A is the hyperplane arrangement consisting of the hyperplanes in $\mathbb{R}^n$ which are orthogonal to the circuits of A. There are at most $\binom{n}{d+1}$ hyperplanes in the circuit arrangement, and all of them contain the row span of A. It is thus natural to consider the circuit arrangement in the $(n-d)$-dimensional space $ker(A) \simeq \mathbb{R}^n/rowspan(A)$.

A matrix A is of *Lawrence type* if

$$A = \begin{pmatrix} A' & \mathbf{0} \\ \mathbf{1} & \mathbf{1} \end{pmatrix}$$

where $\mathbf{0}$ is a zero matrix of the same format as A', and $\mathbf{1}$ denotes the unit matrix with the same number of columns. The above matrix A is called the *Lawrence lifting* of A'. This construction and terminology stems from the theory of oriented matroids (see Section 9.3 in [7]). For a matrix A of Lawrence type and the right-hand side $b = \binom{b'}{b''}$, the linear program (1.1) takes the special form

$$\text{minimize } c \cdot x \quad \text{subject to } A' \cdot x = b' \text{ and } 0 \leqslant x \leqslant b''. \tag{1.4}$$

The corresponding class of integer programs will a special role in Section 4.

Proposition 1.11 ([5, Lemma 5.2], see also [7, Exercise 9.3.1]).
(i) *The circuit arrangement of A is a refinement of the secondary fan of A.*
(ii) *If A is of Lawrence type, then the circuit arrangement equals the secondary fan.*

2. Test sets and monomial ideals in integer programming

In this section we review some known *test sets* in integer programming, and we explain their connections with regular triangulations and with monomial ideals. We also use this section to introduce the remaining definitions and notation needed in the paper. A brief summary of results concerning test sets for integer programming, necessary for later sections, is also included. For more details and proofs see [27,23], and the references given there.

Let $IP_{A,c}(b)$ denote the integer program

$$\text{minimize } c \cdot x \quad \text{subject to } A \cdot x = b \text{ and } x \geqslant 0,\ x \in \mathbb{Z}^n, \tag{2.1}$$

where $c \in \mathbb{R}^n$ is fixed and $A = (a_1, a_2, \ldots, a_n)$ is a fixed $(d \times n)$-integer matrix of rank d. We denote by IP_A the family of all integer programs (2.1) for which the coefficient matrix A is fixed. Let P_b^I denote the polytope that is the convex hull of all feasible solutions to $IP_{A,c}(b)$. (Recall our assumption that $ker(A) \cap \mathbb{R}^n_+ = \{0\}$.) We call P_b^I the *b-fiber* of IP_A. A cost function c is said to be *generic* (*with respect to* IP_A) if the optimal solution of $IP_{A,c}(b)$ is a unique vertex of P_b^I for every b for which $IP_{A,c}(b)$ is feasible. In this section we shall assume that c is generic. We remark that any given c can be made generic by refining the partial order on $\mathbb{N}^n$ given by the objective function value by the lexicographic order on $\mathbb{N}^n$. It will be evident later that a cost function that is generic with respect to IP_A is also generic with respect to LP_A but not vice versa.

We introduce the lattice $ker_{\mathbb{Z}}(A) := ker(A) \cap \mathbb{Z}^n$ and the semigroup $cone_{\mathbb{N}}(A) := \{\sum_{i=1}^n m_i a_i : m_i \geqslant 0, m_i \in \mathbb{Z}\}$. A *test set* for the family of integer programs $IP_{A,c}(\cdot)$ is a finite subset $\mathcal{G}$ of $ker_{\mathbb{Z}}(A)$ such that, for every $b \in cone_{\mathbb{N}}(A)$ and every $x \in P_b^I$, either x is the optimal solution of $IP_{A,c}(b)$ or there exists $g \in \mathcal{G}$ such that $x - g \geqslant 0$ and $c \cdot g > 0$. As before, a test set is minimal if it has minimal cardinality. We shall now construct a canonical minimal test set for $IP_{A,c}(\cdot)$.

Lemma 2.1. *There exists a unique minimal set of vectors $\alpha_1, \ldots, \alpha_t$ in $\mathbb{N}^n$ such that the set of all non-optimal solutions to all programs in $IP_{A,c}(\cdot)$ is of the form $\bigcup_{i=1}^t (\alpha_i + \mathbb{N}^n)$.*

Let $\mathcal{G}_c = \{\alpha_i - \beta_i : i := 1, \ldots, t\}$ where β_i is the unique optimal solution of $IP_{A,c}(A\alpha_i)$. Since α_i is a minimal element in the set of non-optimal solutions, it follows that $supp(\alpha_i) \cap supp(\beta_i) = \emptyset$ for each $\alpha_i - \beta_i$ in $\mathcal{G}_c$. (See Lemma 5.3 for the proof of a stronger statement.)

Proposition 2.2 ([27]). *The set $\mathcal{G}_c$ is a minimal test set for the family of integer programs $IP_{A,c}(\cdot)$.*

The set $\mathcal{G}_c$ is called the *reduced Gröbner basis* of $IP_{A,c}(\cdot)$. The reduced Gröbner basis $\mathcal{G}_c$ was first introduced in an algebraic setting by Conti and Traverso [9]. We briefly explain the relationship to our formulation, since general properties of Gröbner bases for polynomial ideals will be called upon repeatedly below. For an introduction to the algebraic theory of Gröbner bases with a view towards integer programming see also in Section 2.8 in [1].

Let k be any field. The matrix $A = (a_1, \ldots, a_n)$ defines a k-algebra homomorphism

$$\phi_A : k[x_1, x_2, \ldots, x_n] \to k[t_1, t_1^{-1}, \ldots, t_d, t_d^{-1}], \quad x_i \mapsto t^{a_i}.$$

Its kernel $I_A := ker(\phi_A)$ is the ideal generated by all "binomials" $x^u - x^v$ such that $Au = Av$, $u, v \in \mathbb{N}^n$, (i.e., u and v lie in the same fiber of IP_A). Here x^u stands for the monomial $x_1^{u_1} x_2^{u_2} \cdots x_n^{u_n}$. We call I_A the *toric ideal* of A. The subset $\{x^{\alpha_i} - x^{\beta_i} : i := 1, \ldots, t\}$, with x^{α_i} as *leading term*, is the reduced Gröbner basis of I_A with respect to c. We identify this set with $\mathcal{G}_c$ by identifying lattice points u in $\mathbb{N}^n$ and monomials x^u in $k[x_1, \ldots, x_n]$. The binomial $x^u - x^v$ is identified with the vector $u - v$. The set $\bigcup_{i=1}^t (\alpha_i + \mathbb{N}^n)$ corresponds to the initial monomial ideal $in_c(I_A) = \langle x^u : u \in \bigcup_{i=1}^t (\alpha_i + \mathbb{N}^n) \rangle$, i.e., a feasible solution u of $IP_{A,c}(b)$ is optimal if and only if x^u does not lie in $in_c(I_A)$. In what follows, we use $in_c(I_A)$ to denote $\bigcup_{i=1}^t (\alpha_i + \mathbb{N}^n)$.

An important motivation for the use of Gröbner bases in integer programming is the existence of computer programs for calculating them. The general purpose package MACAULAY [3] worked well for many non-trivial computations, such as the ones listed in the table in Example 2.12 below. More specialized Gröbner basis software for integer programming is currently being developed by Serkan Hosten at Cornell University [17].

Example 1.2 (*continued*). For $c' = (1, 0, 0, 1, 0, 1)$ the problems $LP_{A,c'}$ and $IP_{A,c'}$ are equivalent, since the triangulation $\Delta_{c'}$ consists of unimodular cones. Unimodularity is reflected by the fact that the minimal LP-test set $\mathcal{T}$ and the reduced Gröbner basis $\mathcal{G}_{c'}$ coincide. Therefore, in binomial notation,

$$\mathcal{G}_{c'} = \{\underline{x_1 x_4} - x_2^2, \underline{x_1 x_5} - x_2 x_3, \underline{x_1 x_6} - x_3^2, \underline{x_2 x_6} - x_3 x_5, \underline{x_3 x_4} - x_2 x_5, \underline{x_4 x_6} - x_5^2\}.$$

The initial ideal $in_{c'}(I_A)$ is generated by the six underlined monomials or equivalently, the α_i in the set $\cup_{i=1}^t (\alpha_i + \mathbb{N}^n)$ are the exponent vectors of the underlined monomials.

Our discussion gives rise to the following integer analogue to the Walkup–Wets Theorem 1.1. The role of the regular triangulation Δ_c in Section 1 is now being played by the initial monomial ideal $in_c(I_A)$. We continue to identify lattice points in $\mathbb{N}^n$ and monomials in $k[x_1, \ldots, x_n]$.

Proposition 2.3.

(i) *The integer program $IP_{A,c}(b)$ is feasible if and only if b lies in the semigroup $cone_{\mathbb{N}}(A)$.*

(ii) *$IP_{A,c}(b)$ is bounded for all $b \in cone_{\mathbb{N}}(A)$ and all $c \in \mathbb{R}^n$ if and only if $ker(A) \cap \mathbb{R}_+^n = \{0\}$.*

(iii) *If* $IP_{A,c}(\cdot)$ *is bounded, then the set of all optimal solutions with respect to c, in the various fibers of* IP_A, *is the complement in* $\mathbb{N}^n$ *of the initial monomial ideal* $in_c(I_A)$.

Theorem 2.4 below shows that the faces of Δ_c can be recovered from the generators $\alpha_1, \ldots, \alpha_t$ of $in_c(I_A)$. This reflects the philosophy that integer programming is an arithmetic refinement of linear programming.

Theorem 2.4. *The faces of the regular triangulation* Δ_c *are those subsets of* $\{1, \ldots, n\}$ *which do not contain* $supp(\alpha_i)$ *for any minimal generator* α_i *of the initial monomial ideal* $in_c(I_A)$.

Sketch of Proof. This result first appeared in Theorem 3.1 of [23]. It can be derived from Remark 1.10 as follows. Every Gröbner basis element $\alpha_i - \beta_i$ can be written as a $\mathbb{Q}$-linear combination of circuits t such that $\{j : t_j > 0\} \subseteq supp(\alpha_i)$. At least one of these circuits satisfies $t \cdot c > 0$, since $(\alpha_i - \beta_i) \cdot c > 0$. On the other hand, for each circuit t with $t \cdot c > 0$ there exists a Gröbner basis element $\alpha_i - \beta_i$ such that $supp(\alpha_i) \subseteq \{j : t_j > 0\}$. □

A set $\mathcal{U} \subseteq ker_{\mathbb{Z}}(A)$ is a *universal test set* for IP_A if $\mathcal{U}$ is a test set for $IP_{A,c}(\cdot)$ for every generic c in $\mathbb{R}^n$. Let UGB_A be the union of all reduced Gröbner bases $\mathcal{G}_c$ as c varies. Then clearly, UGB_A is a uniquely defined universal test set for IP_A. We call it the *universal Gröbner basis* of A.

In Section 1 we defined a circuit of A to be any non-zero vector of minimal support in $ker(A)$. For the rest of this paper we need a more specific definition, which is suitable for integer programming:

Definition 2.5. A *circuit* of A is a non-zero vector u in $ker_{\mathbb{Z}}(A)$ such that its support $supp(u)$ is minimal with respect to inclusion and u is a primitive lattice point, i.e., g.c.d.$(u_1, \ldots, u_n) = 1$.

Lemma 2.6. *The circuits of A are contained in* UGB_A.

Proof. Let $u = u^+ - u^-$ be a circuit of A. Consider the cost function $c := \sum\{e_i : i \notin supp(u)\}$. After refining c lexicographically to be generic we may suppose that $c \cdot u^+ > c \cdot u^-$. Then the monomial x^{u^+} lies in $in_c(I_A)$ and so there exists a binomial $x^{\alpha_i} - x^{\beta_i}$ in $\mathcal{G}_c$ such that x^{α_i} divides x^{u^+}. Since $c \cdot \alpha_i \geqslant c \cdot \beta_i$ and $supp(\alpha_i) \subseteq supp(u^+) \subseteq supp(u)$, we conclude that $supp(\beta_i) \subseteq supp(u)$. Since u is a circuit, these facts imply $u = \alpha_i - \beta_i \in UGB_A$. □

We now describe a universal test set for IP_A due to Graver [16]. For each $\sigma \in \{+, -\}^n$, consider the semigroup $S_\sigma = ker_{\mathbb{Z}}(A) \cap \mathbb{R}^n_\sigma$, where $\mathbb{R}^n_\sigma$ is the orthant with sign pattern σ. Then $cone(S_\sigma)$ is a pointed closed polyhedral $(n-d)$-cone in $\mathbb{R}^n$. Let H_σ denote the unique *Hilbert basis* of S_σ. The Hilbert basis of a polyhedral cone K in $\mathbb{R}^n$

is a minimal subset of $K \cap \mathbb{Z}^n$ such that every integral vector in K can be written as a non-negative integral combination of the elements in the basis. Pointed cones have unique Hilbert bases (see Chapter 16 in [21]). Graver showed that $\mathcal{H} := \bigcup_\sigma H_\sigma \backslash \{0\}$ is a universal test set for A. We call $\mathcal{H}$ the *Graver basis* of A. It is equivalent to the universal test set of IP_A due to Blair and Jeroslow in [8] and under an appropriate transformation, to the universal test set due to Cook, Gerards, Schrijver and Tardos in [10] (see also Section 17.4 in [21]). The following theorem relates the Graver basis and the universal Gröbner basis of A.

Theorem 2.7 ([27]). *The Graver basis of A contains the universal Gröbner basis UGB_A.*

Corollary 2.8. *There exists only finitely many distinct reduced Gröbner bases associated with A as the cost function is varied. In particular, UGB_A is a finite set.*

Definition 2.9. An integral matrix A of full row rank is called *unimodular* if each of its maximal minors is one of $-c$, 0 or c, where c is a positive integral constant.

Theorem 2.10 ([24]). *If A is unimodular, the circuits of A form the universal Gröbner basis of A.*

Proof. Since a circuit u of A lies in $ker_{\mathbb{Z}}(A)$, by Cramer's rule, every coordinate of u is the ratio of two maximal minors of A of which the denominator is necessarily non-zero. Hence if A is unimodular, every coordinate of a circuit is one of $0, 1$ or -1. This implies that the extreme rays of $cone(S_\sigma)$ are generated by $0, \pm 1$ vectors for all $\sigma \in \{+,-\}^n$. Therefore, each H_σ and hence the Graver basis $\mathcal{H}$ does not contain any integral vector that is not a circuit of A. Lemma 2.6 and Theorem 2.7 then imply that the circuits of A form UGB_A. □

The universal Gröbner basis is generally a proper subset of the Graver basis. Since the Graver basis is a symmetric set, its elements will be represented up to sign.

Example 1.2 (*continued*). The universal Gröbner basis UGB_A consists precisely of the nine circuits. The Graver basis consists of the nine circuits plus the one additional non-circuit $e_1 + e_4 + e_6 - e_2 - e_3 - e_5$. (We identify this vector with the binomial $x_1x_4x_6 - x_2x_3x_5$.)

Example 2.11. Consider the $(1 \times n)$-matrix $A = [1, 1, \ldots, 1, D]$ where D is any non-negative integer. The Graver basis consists of $\binom{n-1}{2}$ elements of the form $e_i - e_j$, $1 \leqslant i < j \leqslant n-1$, and $\binom{n+D-2}{D}$ elements of the form $(i_1, i_2, \ldots, i_{n-1}, -1)$ where $i_1 + i_2 + \cdots + i_{n-1} = D$. The universal Gröbner basis consists of the $\binom{n-1}{2}$ elements of the form $e_i - e_j$, $1 \leqslant i < j \leqslant n-1$ and $n-1$ elements of the form $De_k - e_n$, $k = 1, \ldots, n-1$. The ratio of the cardinality of the Graver basis over the cardinality of UGB_A tends to infinity both in n and in D.

Example 2.12. *Knapsack problems* can be modeled using the family of matrices $A_n = [1, 2, 3, \ldots, n]$. The Graver basis of A_n consists of all binomials $x_{i_1}x_{i_2}\cdots x_{i_k} - x_{j_1}x_{j_2}\cdots x_{j_l}$ such that $i_1 + i_2 + \cdots + i_k = j_1 + j_2 + \cdots + j_l$ but no proper subsum of $i_1 + \cdots + i_k$ equals a subsum of $j_1 + \cdots + j_l$. Such binomials are called *primitive partition identities* (ppi). It is proved in [12] that the degree of a ppi (1-norm of the exponent vector of either monomial in the binomial) is at most $n \cdot (n-1)$. We list the number of ppi's for small values of n:

n	2	3	4	5	6	7	8	9	10	11	12	13
#	1	5	15	47	102	276	578	1261	2465	5362	9285	18900

For instance, for $n = 4$ the Graver basis equals $\{x_1^2 - x_2, x_1x_2 - x_3, x_1x_3 - x_2^2, x_1^3 - x_3, x_2^3 - x_3^2, x_1^4 - x_4, x_1^2x_4 - x_3^2, x_1x_4 - x_2x_3, x_1x_4^2 - x_3^3, x_1x_3 - x_4, \underline{x_1^2x_2 - x_4}, x_2x_3^2 - x_4^2, x_2^2 - x_4, x_2x_4 - x_3^2, x_3^4 - x_4^3\}$. The underlined binomial is the unique element of this Graver basis which is not in UGB_{A_4}.

3. The Gröbner fan and the state polytope

Our objective is to study the variation of cost functions in integer programming using Gröbner bases methods. There is a natural equivalence relation on the space of all (not just generic) cost functions with respect to IP_A. It is analogous to the one for linear programming in Theorem 1.3.

Definition 3.1. Two cost vectors c and c' in $\mathbb{R}^n$ are *equivalent* (*with respect to* IP_A) if the integer programs $IP_{A,c}(b)$ and $IP_{A,c'}(b)$ have the same set of optimal solutions for all b in $cone_{\mathbb{N}}(A)$.

The main result in this section is a structure theorem for these equivalence classes (Theorem 3.10). It is the integer analogue to Theorem 1.5. We note that Theorem 3.10 can also be derived from more general results of Mora and Robbiano [19] and Bayer and Morrison [2] on *Gröbner fans* and *state polytopes* for *graded polynomial ideals*. What we present here is an alternative construction for toric ideals, which is self-contained and provides more precise information for integer programming. No knowledge of results in [2] and [19] is assumed.

Recall that a cost vector c is *generic* for IP_A if the optimal solution with respect to c in every fiber P_b^I of IP_A is a unique vertex. Generic equivalence classes are characterized as follows:

Proposition 3.2. *Given two generic cost functions c and c' in $\mathbb{R}^n$, the following are equivalent:*

(i) *For every $b \in cone_{\mathbb{N}}(A)$, the programs $IP_{A,c}(b)$ and $IP_{A,c'}(b)$ have the same optimal solution.*

(ii) *The cost functions c and c' support the same optimal vertex in each fiber P_b^I of IP_A.*
(iii) *The reduced Gröbner bases $\mathcal{G}_c$ and $\mathcal{G}_{c'}$ associated with A are equal.*

Proof. Conditions (i) and (ii) are equivalent since the optimal solution of $IP_{A,c}(b)$ is the vertex of P_b^I supported by c. The set of all non-optimal solutions to the programs $IP_{A,c}(\cdot)$ and $IP_{A,c'}(\cdot)$ are the monomial ideals $in_c(I_A)$ and $in_{c'}(I_A)$ respectively. Then (i) holds if and only if $in_c(I_A) = in_{c'}(I_A)$. This is equivalent to (iii) by Proposition 2.2. □

In what follows we view elements of reduced Gröbner bases as vectors and not as binomials.

Lemma 3.3. *Let $\mathcal{G}_c \subset \mathbb{Z}^n$ be the reduced Gröbner basis of $IP_{A,c}(\cdot)$. Then*

$$span_{\mathbb{Z}}(\mathcal{G}_c) = ker_{\mathbb{Z}}(A).$$

Proof. Every vector $g_i = \alpha_i - \beta_i$ in $\mathcal{G}_c$ lies in $ker_{\mathbb{Z}}(A)$. Hence $span_{\mathbb{Z}}(\mathcal{G}_c) \subseteq ker_{\mathbb{Z}}(A)$. Let $\alpha \in ker_{\mathbb{Z}}(A)$. We can write α uniquely as $\alpha^+ - \alpha^-$ where α^+, α^- are vectors in $\mathbb{N}^n$ with disjoint supports. Further, $A\alpha^+ = A\alpha^-$, and hence α^+ and α^- lie in the same fiber of IP_A. Let β be the unique optimum in this fiber with respect to c. Since $\mathcal{G}_c$ is a test set for $IP_{A,c}(\cdot)$, there exist non-negative integral multipliers n_i and n_i' such that

$$\alpha^+ - \beta = \sum_{g_i \in \mathcal{G}_c} n_i g_i \quad \text{and} \quad \alpha^- - \beta = \sum_{g_i \in \mathcal{G}_c} n_i' g_i.$$

Hence $\alpha = \sum_{g_i \in \mathcal{G}_c} (n_i - n_i') g_i$ which implies that $span_{\mathbb{Z}}(\mathcal{G}_c) = ker_{\mathbb{Z}}(A) \simeq \mathbb{Z}^{n-d}$. □

Let $u = u^+ - u^- \in ker_{\mathbb{Z}}(A)$. Both u^+ and u^- lie in the Au^+-fiber of IP_A, and we may think of u as the line segment $[u^+, u^-]$ in this fiber. We shall refer to the polytope $P_{Au^+}^I$ as *the fiber of u*. By a *Gröbner fiber* of IP_A we mean the fiber of an element $u \in UGB_A$. Let $St(A)$ denote the Minkowski sum of all Gröbner fibers. This is a well-defined polytope in $\mathbb{R}^n$ which we call the *state polytope* of A. Lemma 3.3 implies $\dim(St(A)) = n - d$. The complete polyhedral fan $\mathcal{N}(St(A))$ is called the *Gröbner fan* of A.

Lemma 3.4. *Every fiber of IP_A is a Minkowski summand of $St(A)$.*

Proof. It suffices to show that $\mathcal{N}(St(A))$ is a refinement of $\mathcal{N}(P_b^I)$ for all $b \in cone_{\mathbb{N}}(A)$. Let c be a generic cost function and let $w \neq c$ belong to the interior of the cone $\mathcal{N}\big(face_c(St(A)); St(A)\big)$. Then w lies in $\mathcal{N}\big(\beta_i; P_{A\beta_i}^I\big)$ for each element $\alpha_i - \beta_i$ in the reduced Gröbner basis $\mathcal{G}_c$. This implies $w \cdot \alpha_i > w \cdot \beta_i$ for all i, and therefore $\mathcal{G}_w = \mathcal{G}_c$.

Now consider an arbitrary $b \in cone_{\mathbb{N}}(A)$. Let u be the unique optimum of $IP_{A,c}(b)$. The equality of test sets $\mathcal{G}_w = \mathcal{G}_c$ implies that u is also the unique optimum of $IP_{A,w}(b)$.

Hence w lies in the interior of $\mathcal{N}(u; P_b^I)$. Therefore, $\mathcal{N}\big(face_c(St(A)); St(A)\big) \subseteq \mathcal{N}(u; P_b^I)$, as desired. □

Proposition 3.5. *Let* db *denote any probability measure with support* $cone_{\mathbb{N}}(A)$ *such that* $\int_b b\,\mathrm{d}b$ *is finite. Then the Minkowski integral* $\int_b P_b^I\,\mathrm{d}b$ *is a polytope normally equivalent to* $St(A)$.

Proof. The hypothesis $\int_b b\,\mathrm{d}b < \infty$ guarantees that $\int_b P_b^I\,\mathrm{d}b$ is bounded. By Lemma 3.4, $\int_b P_b^I\,\mathrm{d}b$ is a summand of $St(A)$ and is hence a polytope. However, each Gröbner fiber is a summand of $\int_b P_b^I\,\mathrm{d}b$ and hence $\int_b P_b^I\,\mathrm{d}b$ is an $(n-d)$-polytope in $\mathbb{R}^n$ that has the same normal fan as $St(A)$. □

Corollary 3.6. *There exists only finitely many facet directions among the fibers of* IP_A.

From now on we shall use the term *state polytope* for any polytope normally equivalent to $\int_b P_b^I\,\mathrm{d}b$. We define the *Gröbner cone* associated with $\mathcal{G}_c$ to be the closed convex polyhedral cone

$$\mathcal{K}_c := \{x \in \mathbb{R}^n : g_i \cdot x \geqslant 0, g_i \in \mathcal{G}_c\} \tag{3.1}$$

Observation 3.7. *The Gröbner cone* $\mathcal{K}_c$ *has full dimension n. Its lineality space* $\mathcal{K}_c \cap -\mathcal{K}_c$ *equals* $rowspan(A) \simeq \mathbb{R}^d$.

Proof. We have $\dim(\mathcal{K}_c) = n$ because c lies in the interior of $\mathcal{K}_c$. The lineality space $\mathcal{K}_c \cap -\mathcal{K}_c$ equals the orthogonal complement of $\mathcal{G}_c$ in $\mathbb{R}^n$, which coincides with the row span of A by Lemma 3.3. □

Proposition 3.8. *The Gröbner fan of A is the collection of all Gröbner cones* $\mathcal{K}_c$ *together with their faces, as c varies over all generic cost functions.*

Proof. The argument in the proof of Lemma 3.4 shows that, for c generic, the Gröbner cone $\mathcal{K}_c$ equals $\mathcal{N}\big(face_c(St(A)); St(A)\big)$. □

We remark that each cone in the Gröbner fan has the same lineality space $rowspan(A) \simeq \mathbb{R}^d$ and it is often more convenient to work with its image in

$$ker(A) \simeq \mathbb{R}^n/rowspan(A) \simeq \mathbb{R}^{n-d}.$$

Corollary 3.9. *The equivalence classes of cost functions with respect to* IP_A *(cf. Definition* 3.1*) are precisely the cells of the Gröbner fan.*

Proof. By Proposition 3.2, two cost vectors c and c' are equivalent if and only if they support the same optimal face in each fiber of IP_A. Using Propositions 3.5 and 3.8, it follows that c and c' are equivalent if and only if they lie in the relative interior of the same cell in $\mathcal{N}(St(A))$. □

A cost vector w lies in the interior of a Gröbner cone $\mathcal{K}_c$ if and only if w is generic and equivalent to c. Hence the interiors of the top-dimensional cells in the Gröbner fan are precisely the equivalence classes of generic cost functions. The following theorem summarizes the above discussion.

Theorem 3.10.
(i) *There are only finitely many equivalence classes of cost vectors with respect to* IP_A.
(ii) *Each equivalence class is the relative interior of a convex polyhedral cone in* $\mathbb{R}^n$.
(iii) *The collection of these cones defines a polyhedral fan that covers* $\mathbb{R}^n$. *This fan is called the* Gröbner fan *of* A.
(iv) *Let* $\mathrm{d}b$ *denote any probability measure with support* $\mathrm{cone}_{\mathbb{N}}(A)$ *such that* $\int_b b\,\mathrm{d}b < \infty$. *Then the Minkowski integral* $St(A) = \int_b P_b^I\,\mathrm{d}b$ *is an* $(n-d)$*-dimensional convex polytope, called the* state polytope *of* A. *The normal fan of* $St(A)$ *equals the Gröbner fan of* A.

The secondary polytope of A was defined in Section 1 as the Minkowski integral of all fibers of LP_A. Using Theorem 2.4 we obtain the following corollaries.

Corollary 3.11. *The Gröbner fan of* A *is a refinement of the secondary fan of* A.

Corollary 3.12. *The secondary polytope of* A *is a summand of the state polytope of* A.

The Graver basis $\mathcal{H}$ introduced in Section 2 gives rise to a natural refinement of the Gröbner fan. The *Graver arrangement* of A is the arrangement consisting of the hyperplanes in $\mathbb{R}^n$ which are orthogonal to the elements in the Graver basis $\mathcal{H}$. The following proposition is a direct consequence of Theorem 2.7 and Lemma 2.6.

Proposition 3.13.
(i) *The Graver arrangement of* A *is a refinement of the Gröbner fan of* A.
(ii) *The Graver arrangement of* A *is a refinement of the circuit arrangement of* A.

Example 1.2 (*continued*). This matrix has seven distinct Gröbner fibers. For the sake of clarity we express the elements of UGB_A as binomials in the variables a, b, c, d, e, f. In the notation used thus far, the variables should be identified as follows: $a = x_1$, $b = x_2$, $c = x_3$, $d = x_4$, $e = x_5$ and $f = x_6$. Hence $UGB_A = \{ad - b^2, af - c^2, df - e^2, ae - bc, be - cd, bf - ce, b^2f - ae^2, b^2f - c^2d, c^2d - ae^2\}$. Each binomial of degree two comes from a one-dimensional fiber, the only two lattice points in which are the exponent vectors of the two monomials in the binomial. All three of the degree three elements in UGB_A come from a single three-dimensional fiber (see Fig. 1). The lattice points in the fiber are indexed by the associated monomials. The state polytope $St(A)$ which is the Minkowski sum of the seven Gröbner fibers, is hence a three-dimensional polytope with 29 vertices, 45 edges and 18 facets. Therefore, there are $1 + 29 + 45 + 18 = 93$ equivalence classes of cost functions with respect to IP_A. Since the vertices of $St(A)$ are in bijection with the distinct reduced Gröbner bases of I_A, there are 29 distinct initial

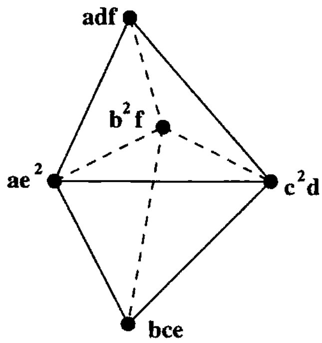

Fig. 1. The three-dimensional Gröbner fiber of IP_A.

ideals for I_A, each corresponding to one of the 14 regular triangulations of $cone(A)$. Each initial ideal is listed in Table 2 along with its associated regular triangulation and a representative cost vector. Theorem 2.4 may be verified using this example.

Table 2

	Cost vector	Initial ideal	Regular triangulation
1.	(1,0,0,1,0,1)	$\langle df, cd, bf, af, ae, ad\rangle$	{1,2,3}, {2,3,5}, {2,4,5}, {3,5,6}
2.	(10,0,6,10,6,7)	$\langle df, ce, ad, af, ae, cd\rangle$	{1,2,3}, {2,3,6}, {2,4,5}, {2,5,6}
3.	(1,0,0,0,0,0)	$\langle e^2, ce, cd, af, ae, ad\rangle$	{1,2,3}, {2,3,6}, {2,4,6}
4.	(7,1,1,0,15,0)	$\langle e^2, ce, c^2d, be, af, ae, ad\rangle$	{1,2,3}, {2,3,6}, {2,4,6}
5.	(49,24,0,14,70,26)	$\langle e^2, ce, be, b^2f, af, ae, ad\rangle$	{1,2,3}, {2,3,4}, {3,4,6}
6.	(17,6,0,10,14,13)	$\langle e^2, bf, be, af, ae, ad\rangle$	{1,2,3}, {2,3,4}, {3,4,6}
7.	(10,6,0,7,6,10)	$\langle df, bf, be, af, ae, ad\rangle$	{1,2,3}, {2,3,4}, {3,4,5}, {3,5,6}
8.	(7,6,6,10,0,10)	$\langle df, cd, bf, bc, af, ad\rangle$	{1,2,5}, {2,4,5}, {1,3,5}, {3,5,6}
9.	(79,23,77,27,181,0)	$\langle e^2, ce, c^2, be, ae, ad\rangle$	{1,2,6}, {2,4,6}
10.	(1,0,1,1,1,1)	$\langle e^2, ce, cd, c^2, ae, ad\rangle$	{1,2,6}, {2,4,6}
11.	(27,23,181,79,77,0)	$\langle e^2, ce, cd, c^2, bc, ad\rangle$	{1,2,6}, {2,4,6}
12.	(0,0,0,1,0,0)	$\langle df, ce, cd, c^2, bc, ae^2, ad\rangle$	{1,2,6}, {2,4,5}, {2,5,6}
13.	(26,0,22,29,16,18)	$\langle df, ce, cd, c^2, ae, ad\rangle$	{1,2,6}, {2,4,5}, {2,5,6}
14.	(14,24,70,49,0,26)	$\langle df, ce, cd, c^2, bc, b^2f, ad\rangle$	{1,2,5}, {2,4,5}, {1,5,6}
15.	(5,4,6,8,0,7)	$\langle df, cd, c^2, bf, bc, ad\rangle$	{1,2,5}, {2,4,5}, {1,5,6}
16.	(0,0,0,0,1,0)	$\langle e^2, ce, c^2, be, b^2, ae\rangle$	{1,4,6}
17.	(1,1,1,0,1,1)	$\langle e^2, ce, c^2, be, bc, b^2\rangle$	{1,4,6}
18.	(0,1,0,0,0,0)	$\langle e^2, c^2, bf, be, bc, b^2\rangle$	{1,4,6}
19.	(0,0,1,0,0,0)	$\langle e^2, ce, cd, c^2, bc, b^2\rangle$	{1,4,6}
20.	(0,181,77,27,23,29)	$\langle df, c^2, bf, be, bc, b^2\rangle$	{1,4,5}, {1,5,6}
21.	(2,6,16,10,0,5)	$\langle df, ce, cd, c^2, bc, b^2\rangle$	{1,4,5}, {1,5,6}
22.	(1,1,1,1,0,1)	$\langle df, cd, c^2, bf, bc, b^2\rangle$	{1,4,5}, {1,5,6}
23.	(10,6,0,2,16,5)	$\langle e^2, ce, be, b^2, af, ae\rangle$	{1,3,4}, {3,4,6}
24.	(1,1,0,1,1,1)	$\langle e^2, bf, be, b^2, af, ae\rangle$	{1,3,4}, {3,4,6}
25.	(5,16,0,2,6,10)	$\langle e^2, bf, be, bc, b^2, af\rangle$	{1,3,4}, {3,4,6}
26.	(7,6,0,5,4,8)	$\langle df, bf, be, b^2, af, ae\rangle$	{1,3,4}, {3,4,5}, {3,5,6}
27.	(26,70,0,14,24,49)	$\langle df, bf, be, bc, b^2, af, ae^2\rangle$	{1,3,4}, {3,4,5}, {3,5,6}
28.	(0,15,1,0,1,7)	$\langle df, c^2d, bf, be, bc, b^2, af\rangle$	{1,3,5}, {1,4,5}, {3,5,6}
29.	(0,0,0,0,0,1)	$\langle b^2, af, df, cd, bf, bc\rangle$	{1,3,5}, {1,4,5}, {3,5,6}

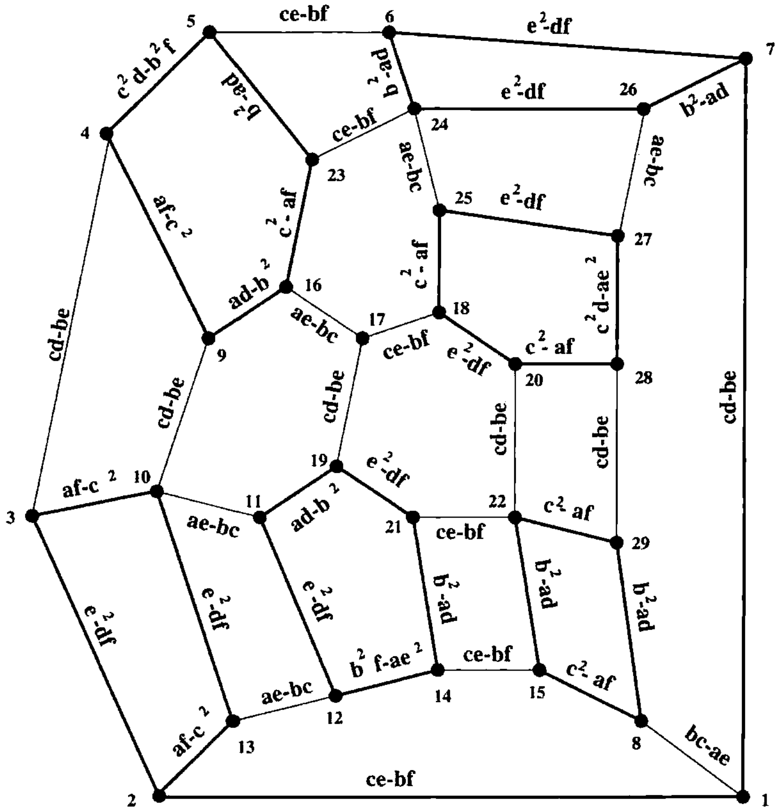

Fig. 2. Schlegel diagram of the state polytope.

Fig. 2 is a Schlegel diagram of $St(A)$ where the numbers on the nodes tally with the numbers of the initial ideals. Informally, a Schlegel diagram of a polytope is the view of the polytope from beyond any one of its facets. It will be seen in Section 5 that each edge direction of $St(A)$ is given by an element in UGB_A. The binomials on the edges denote this correspondence. The polytope $St(A)$ has an axis of symmetry through the vertices 1 and 17. The thin edges in Fig. 2 are contracted when passing to the secondary polytope of A.

4. Computing the Graver basis and universal Gröbner basis

In this section we present algorithms for computing the Graver basis $\mathcal{H}$ and the universal Gröbner basis UGB_A of a matrix $A \in \mathbb{Z}^{d \times n}$ of rank d. Recall the definition of the Lawrence lifting $\Lambda(A)$ of the matrix A given in the paragraph preceding Eq. (1.4). The matrices A and $\Lambda(A)$ have isomorphic kernels: $ker_{\mathbb{Z}}(\Lambda(A)) = \{(u, -u) : u \in ker_{\mathbb{Z}}(A)\}$. The toric ideal $I_{\Lambda(A)}$ is the homogeneous prime ideal

$$I_{\Lambda(A)} = \langle x^{\alpha} y^{\beta} - x^{\beta} y^{\alpha} : \alpha, \beta \in \mathbb{N}^n, A\alpha = A\beta \rangle$$

in the polynomial ring $k[x_1, \ldots, x_n, y_1, \ldots, y_n]$.

We say that a one-dimensional fiber of IP_A is *primitive* if its two vertices have disjoint supports and are relatively prime. Clearly all one-dimensional Gröbner fibers are primitive. On the other hand, if the segment $[\alpha, \beta]$ is a primitive fiber of IP_A, then $x^\alpha - x^\beta$ (with either term as leading term) belongs to every reduced Gröbner basis of A and hence to UGB_A. In general, the set of primitive one-dimensional fibers of A is a proper subset of the set of Gröbner fibers of A. We call the fiber of an element in the Graver basis of A a *Graver fiber* of IP_A.

Theorem 4.1. *For a matrix $\Lambda(A)$ of Lawrence type, the following sets coincide:*
(i) *the Graver basis of $\Lambda(A)$,*
(ii) *the universal Gröbner basis of $\Lambda(A)$,*
(iii) *any reduced Gröbner basis of $I_{\Lambda(A)}$,*
(iv) *any minimal generating set of $I_{\Lambda(A)}$ (up to scalar multiples), and*
(v) *the set of binomials $x^\alpha y^\beta - x^\beta y^\alpha$ supported on primitive one-dimensional fibers $[(\alpha, \beta), (\beta, \alpha)]$.*

Proof. Let $\mathcal{H}$ be the Graver basis of A, and let $\mathcal{H}'$ be the Graver basis of $\Lambda(A)$. These two sets of binomials are related as follows: $\mathcal{H}' = \{ x^\alpha y^\beta - x^\beta y^\alpha : \alpha, \beta \in \mathbb{N}^n, x^\alpha - x^\beta \in \mathcal{H} \}$. Since $\mathcal{H}'$ is the Graver basis of $\Lambda(A)$, it is a generating set of $I_{\Lambda(A)}$ and by Theorem 2.7, it is a Gröbner basis of $I_{\Lambda(A)}$ (not necessarily reduced), with respect to every generic cost function. Notice that it suffices to show that $\mathcal{H}'$ is the unique minimal generating set of $I_{\Lambda(A)}$ in order to prove the equality of the sets in (i),(ii),(iii) and (iv). This is because of Theorem 2.7, the definition of $UGB_{\Lambda(A)}$, and the fact that every reduced Gröbner basis of $I_{\Lambda(A)}$ contains a minimal generating set for $I_{\Lambda(A)}$. We show below that the sets in (i) and (iv) coincide. Choose any element $g := x^\alpha y^\beta - x^\beta y^\alpha$ of $\mathcal{H}'$, and fix $\sigma \in \{-,+\}^n$ such that $\alpha - \beta$ lies in $S_\sigma = ker_{\mathbb{Z}}(A) \cap \mathbb{R}^n_\sigma$. Let $\mathcal{B}$ be the set of all binomials $x^\gamma y^\delta - x^\delta y^\gamma$ in $I_{\Lambda(A)}$ except g. Suppose that $\mathcal{B}$ generates $I_{\Lambda(A)}$. Then $x^\alpha y^\beta - x^\beta y^\alpha$ can be written as a linear combination of elements in $\mathcal{B}$. But this is only possible if there exists a binomial $x^\gamma y^\delta - x^\delta y^\gamma$ in $\mathcal{B}$ such that $x^\gamma y^\delta$ divides $x^\alpha y^\beta$. This implies that $\gamma - \delta$ lies in the semigroup S_σ. Moreover, since $\gamma \leqslant \alpha$ and $\delta \leqslant \beta$, the non-zero vector $(\alpha - \beta) - (\gamma - \delta)$ lies in S_σ as well. Therefore $\alpha - \beta$ cannot be an element in the Hilbert basis of S_σ. This is a contradiction, and we conclude that every minimal generating set of $I_{\Lambda(A)}$ requires (a scalar multiple of) the binomial g.

For the equality of (i) and (v) we shall prove that every Graver fiber contains precisely two lattice points. Let $g \in \mathcal{H}'$ as above. Suppose that the common fiber of (α, β) and (β, α) contains a third point $(\gamma, \delta) \in \mathbb{N}^{2n}$. Then $\alpha, \beta, \gamma, \delta \in \mathbb{N}^n$ all lie in the same fiber of IP_A and $\alpha + \beta = \gamma + \delta$. This implies that $\alpha - \beta = (\gamma - \beta) + (\delta - \beta)$. We will show that the non-zero vectors $\gamma - \beta$ and $\delta - \beta$ are sign compatible with $\alpha - \beta$. This contradicts $\alpha - \beta \in \mathcal{H}$ and thus completes the proof. Let $j \in \{1, \ldots, n\}$. If $\alpha_j > 0$ then $\beta_j = 0$, and this implies $(\alpha - \beta)_j = \alpha_j > 0$, $(\gamma - \beta)_j = \gamma_j \geqslant 0$, and $(\delta - \beta)_j = \delta_j \geqslant 0$. If $\alpha_j = 0$ then $(\alpha - \beta)_j = -\beta_j \leqslant 0$, $(\gamma - \beta)_j = \gamma_j - \beta_j = -\delta_j \leqslant 0$, and $(\delta - \beta)_j = \delta_j - \beta_j = -\gamma_j \leqslant 0$. $\square$

Corollary 4.2. *The state polytope of $\Lambda(A)$ is a zonotope (i.e., a Minkowski sum of line segments). The Gröbner fan of $\Lambda(A)$ coincides with the Graver arrangement of $\Lambda(A)$.*

Corollary 4.3. *If the matrix A is unimodular and of Lawrence type, then its secondary fan, circuit arrangement, Gröbner fan and Graver arrangement all coincide.*

Example 4.4. Let A_G be the vertex-edge incidence matrix of a directed graph $G = (V, E)$, and consider the *capacitated transshipment problem*:

$$\text{minimize } c \cdot x \quad \text{subject to } A_G \cdot x = b \text{ and } 0 \leqslant x \leqslant b',\ x \in \mathbb{Z}^E.$$

When rewriting this integer program in the form (2.1), we get the enlarged coefficient matrix

$$\Lambda(A_G) = \begin{pmatrix} A_G & \mathbf{0} \\ \mathbf{1} & \mathbf{1} \end{pmatrix}.$$

This matrix has format $(|E| + |V|) \times 2|V|$ and it is unimodular and of Lawrence type. Hence, for the family of flow problems $IP_{\Lambda(A_G)}$, the secondary fan, the circuit arrangement, the Gröbner fan and the Graver arrangement all coincide.

Algorithm 4.5 (*How to compute the Graver basis of A*).
1. Compute the reduced Gröbner basis $\mathcal{G}$ of $I_{\Lambda(A)}$ with respect to any term order.
2. The Graver basis $\mathcal{H}$ of A consists of all elements $\alpha - \beta$ such that $x^\alpha y^\beta - x^\beta y^\alpha$ appears in $\mathcal{G}$.

The correctness of this algorithm is a corollary of Theorem 4.1. We found Algorithm 4.5 to be very useful for explicit computations. The main point is that, in order to compute the Graver basis of A, one only needs to compute a single reduced Gröbner basis for its Lawrence lifting $\Lambda(A)$. Algorithm 4.5 was first found in collaboration with Persi Diaconis. It can be applied to the problem of "sampling in the presence of prescribed zeros" as discussed in [11].

By Theorem 2.7, the universal Gröbner basis UGB_A is contained in the Graver basis $\mathcal{H}$. We now present a geometric characterization of those binomials that belong to some reduced Gröbner basis of A. This gives an algorithm for computing UGB_A from $\mathcal{H}$. In Section 5 we shall provide yet another geometric characterization and algorithm for the universal Gröbner basis.

Let $\alpha, \beta \in \mathbb{N}^n$ such that $\alpha - \beta \in ker_{\mathbb{Z}}(A)$ and $supp(\alpha) \cap supp(\beta) = \emptyset$. Define $C_+ := \{w \in \mathbb{R}^n : \alpha - \beta \in \mathcal{G}_w\}$ and $C_- := \{w \in \mathbb{R}^n : \beta - \alpha \in \mathcal{G}_w\}$, where $\mathcal{G}_w$ is the reduced Gröbner basis of $IP_{A,w}(\cdot)$. Then $\alpha - \beta$ lies in UGB_A if and only if $C_+ \cup C_- \neq \emptyset$. For $u \in \mathbb{N}^n$, denote by $\mathcal{M}(u)$ the interior of the inner normal cone of the Au-fiber of IP_A at u. In symbols, $\mathcal{M}(u) = int\,\mathcal{N}(u; P^I_{Au})$. Thus $\mathcal{M}(u)$ is empty unless u is a vertex of P^I_{Au}.

Proposition 4.6. *The set C_+ equals the intersection $\mathcal{M}(\beta) \cap \bigcap_{i \in supp(\alpha)} \mathcal{M}(\alpha - e_i)$.*

Proof. A cost vector $w \in \mathbb{R}^n$ belongs to C_+ if and only if $x^\alpha - x^\beta$ appears with leading term x^α in the reduced Gröbner basis $\mathcal{G}_w$ if and only if $x^\beta \notin in_w(I_A)$, $x^\alpha \in in_w(I_A)$, and no proper factor of x^α is in $in_w(I_A)$. This is equivalent to $w \in \mathcal{M}(\beta)$ and $w \in \mathcal{M}(\alpha - e_i)$ for all $i \in supp(a)$. $\square$

Corollary 4.7. *The set of all cost functions which has a fixed binomial with fixed leading term in its reduced Gröbner basis is an open convex cone.*

Using the Graver basis $\mathcal{H}$, we get the following explicit inequality presentation for the cone $\mathcal{M}(u)$.

Lemma 4.8. *The interior of the inner normal cone of the Au-fiber of IP_A at $u \in \mathbb{N}^n$ equals*

$$\mathcal{M}(u) = \{w \in \mathbb{R}^n : wd > we \text{ for all } x^d - x^e \in \mathcal{H} \text{ such that } x^e \text{ divides } x^u\}.$$

Proof. We have $w \in \mathcal{M}(u)$ if and only if x^u does not lie in

$$in_w(I_A) = \langle in_w(h) : h \in \mathcal{H} \rangle. \qquad \square$$

Algorithm 4.9 (*How to compute the universal Gröbner basis UGB_A*).
1. Compute the Graver basis $\mathcal{H}$ using Algorithm 4.5.
2. For each element $x^\alpha - x^\beta$ of $\mathcal{H}$:
   2.1. Compute the cones C_+ and C_- using Proposition 4.6 and Lemma 4.8.
   2.2. The binomial $x^\alpha - x^\beta$ is in UGB_A if and only if $C_+ \cup C_-$ is non-empty.

Example 1.2 (*continued*). The Graver basis equals $\mathcal{H} = \{ad - b^2, ae - bc, af - c^2, bf - ce, cd - be, df - e^2, ae^2 - b^2 f, b^2 f - c^2 d, ae^2 - c^2 d, adf - bce\}$. We shall prove that $\mathcal{H} \backslash UGB_A = \{adf - bce\}$. It suffices to show that $adf - bce$ is not in UGB_A, because the other nine binomials in $\mathcal{H}$ are all circuits (cf. Lemma 2.6).

Let $\alpha = (1,0,0,1,0,1)$ and $\beta = (0,1,1,0,1,0)$. By Lemma 4.8, we have $\mathcal{M}(\beta) = \{w \in \mathbb{R}^6 : w_1 + w_5 > w_2 + w_3, w_2 + w_6 > w_3 + w_5, w_3 + w_4 > w_2 + w_5\}$, $\mathcal{M}(\alpha - e_1) = \{w \in \mathbb{R}^6 : 2w_5 > w_4 + w_6\}$, $\mathcal{M}(\alpha - e_4) = \{w \in \mathbb{R}^6 : 2w_3 > w_1 + w_6\}$ and $\mathcal{M}(\alpha - e_6) = \{w \in \mathbb{R}^6 : 2w_2 > w_1 + w_4\}$. The intersection of these four cones is easily seen to be empty, so that $C_+ = \emptyset$. Reversing the roles of α and β we similarly find that $C_- = \emptyset$. Therefore $adf - bce \notin UGB_A$.

5. The geometry of the universal Gröbner basis

The main result in this section is a geometric characterization of the universal Gröbner basis.

Theorem 5.1. *A vector $\alpha - \beta \in ker_{\mathbb{Z}}(A)$ lies in the universal Gröbner basis UGB_A if and only if $\alpha - \beta$ is primitive and the line segment $[\alpha, \beta]$ is an edge of the $A\alpha$-fiber of IP_A.*

We first recall a general fact about Minkowski sums of polytopes.

Lemma 5.2. *Let P be the Minkowski sum of the polytopes $P_1, \ldots, P_k$. Then the set of edge directions of P is the union of the sets of edge directions of P_i for $i = 1, \ldots, k$.*

In view of Proposition 3.5, this says that the edge directions of the fibers of IP_A are precisely the edge directions of the state polytope. If $[\alpha, \beta]$ is the primitive representative of an edge direction, then $[\alpha, \beta]$ is an edge of the $A\alpha$-fiber of IP_A. Therefore, Theorem 5.1 is equivalent to the following assertion: *the universal Gröbner basis consists of the edge directions of the state polytope.*

Proof of Theorem 5.1 (*if*). Suppose $g = \alpha - \beta$ is primitive and defines an edge direction of the state polytope $St(A)$. Then g is the normal vector to a facet of a maximal cone $\mathcal{K}_c$ in the Gröbner fan $\mathcal{N}(St(A))$. Therefore g appears in the inequality presentation of $\mathcal{K}_c$ given in (3.1). In other words, g is equal to one of the elements g_i of the reduced Gröbner basis $\mathcal{G}_c$. □

For the proof of the only-if direction we need two lemmas.

Lemma 5.3. *Let x^α be a minimal generator of the initial monomial ideal $in_c(I_A)$, and let δ be any lattice point in the $A\alpha$-fiber of IP_A such that $c \cdot \alpha \geqslant c \cdot \delta$. Then $supp(\delta) \cap supp(\alpha) = \emptyset$.*

Proof. Suppose $k \in supp(\alpha) \cap supp(\delta)$ for a lattice point δ in the $A\alpha$-fiber of IP_A for which $c \cdot \alpha \geqslant c \cdot \delta$. Then $\alpha - e_k$ and $\delta - e_k$ are lattice points in the same fiber of IP_A and $c \cdot (\alpha - e_k) \geqslant c \cdot (\delta - e_k)$. This implies that x^α / x_k lies in the initial monomial ideal $in_c(I_A)$, which is a contradiction to x^α being a minimal generator. □

Lemma 5.4. *For an element $\alpha - \beta$ of UGB_A, both α and β are vertices in the $A\alpha$-fiber of IP_A.*

Proof. By definition, β is the optimal vertex with respect to some cost function c in the $A\alpha$-fiber of IP_A. Recall our assumption that the integer programs $IP_{A,c}(b)$ are bounded. This implies the existence of an integral vector M with all coordinates positive in the row space of A. After replacing M by a multiple if necessary, we may assume that $M - c$ has all coordinates positive. Clearly, the cost function $\omega := M - c$ attains its *maximum* over $P^I_{A\alpha}$ at β. Let υ denote the restriction of ω to the support of α (i.e, $\upsilon_i = w_i$ if $\alpha_i > 0$ and $\upsilon_i = 0$ if $\alpha_i = 0$). We claim that υ attains a unique maximum over $P^I_{A\alpha}$ at α. If not, then there exists another lattice point δ in $P^I_{A\alpha}$ with $\upsilon \cdot \delta \geqslant \upsilon \cdot \alpha$. Since $\upsilon \cdot \alpha > 0$, the set $supp(\upsilon) \cap supp(\delta) = supp(\alpha) \cap supp(\delta)$ is not empty. By Lemma 5.3, this implies

$c \cdot \alpha < c \cdot \delta$. In view of $M \cdot \alpha = M \cdot \delta$, we conclude that $v \cdot \alpha = w \cdot \alpha > w \cdot \delta \geqslant v \cdot \delta$, as desired. □

Proof of Theorem 5.1 (*only-if*). Let $\alpha - \beta \in UGB_A$ and choose $w, v \in \mathbb{N}^n$ as in the proof of Lemma 5.4. Consider the cost vector $u := (v \cdot (\alpha - \beta))w + (w \cdot (\beta - \alpha))v \in \mathbb{N}^n$. We have $u \cdot \alpha = u \cdot \beta$. It suffices to show that $u \cdot \alpha > u \cdot \gamma$ for all lattice points γ other than α and β in the $A\alpha$-fiber of IP_A. If $supp(\gamma) \cap supp(\alpha) = \emptyset$, then $supp(\gamma) \cap supp(v) = \emptyset$ which implies that $u \cdot \beta = (v \cdot (\alpha - \beta))(w \cdot \beta) > (v \cdot (\alpha - \beta))(w \cdot \gamma)$. If $supp(\gamma) \cap supp(\alpha) \neq \emptyset$, then by Lemma 5.3, $w \cdot \alpha > w \cdot \gamma$. This implies that

$$\begin{aligned} u \cdot \alpha &= (v \cdot (\alpha - \beta))(w \cdot \alpha) + (w \cdot (\beta - \alpha))(v \cdot \alpha) \\ &> (v \cdot (\alpha - \beta))(w \cdot \gamma) + (w \cdot (\beta - \alpha))(v \cdot \gamma) = u \cdot \gamma. \end{aligned}$$

Therefore the line segment $[\alpha, \beta]$ is an edge of the $A\alpha$-fiber of IP_A with outer normal vector u. □

Theorem 5.1 implies several interesting corollaries.

Corollary 5.5. *For an element $\alpha - \beta$ in UGB_A, there exists two cost functions c and c' in $\mathbb{R}^n$ such that $\alpha - \beta \in \mathcal{G}_c$ and $\beta - \alpha \in \mathcal{G}_{c'}$.*

Proof. Every element in UGB_A appears as a facet normal of some cell in the Gröbner fan. Take as $\mathcal{G}_c$ and $\mathcal{G}_{c'}$ the Gröbner bases associated with the two Gröbner cones that share this facet. □

Corollary 5.6. *For every generic cost function $c \in \mathbb{R}^n$, the reduced Gröbner basis of $IP_{A,c}(\cdot)$ consists only of edges of certain fibers P_b^I.*

This is the integer programming analogue to Corollary 1.9. Theorem 5.1 implies that we can trace a monotone edge path from every non-optimal vertex of $IP_{A,c}(b)$ to the optimal vertex, using only elements in UGB_A. Thus reduction with respect to the universal Gröbner basis can be viewed as an integer analogue to the simplex method for linear programming.

Theorem 5.1 gives rise to the following algorithm for computing the universal Gröbner basis.

Algorithm 5.7 (*How to compute the universal Gröbner basis UGB_A*).
1. Compute the Graver basis $\mathcal{H}$ using Algorithm 4.5.
2. For each element $x^\alpha - x^\beta$ of $\mathcal{H}$ decide whether $[\alpha, \beta]$ is an edge of its fiber.

When applying Algorithm 5.7 to examples, we often found the following method sufficient. Given $\alpha - \beta \in \mathcal{H}$, we first list *all* feasible solutions to the integer program $IP_{A,w}(A\alpha)$ where w is any cost function. This can be done by a *reverse search* method starting at the optimum of $IP_{A,w}(A\alpha)$, using the Gröbner basis $\mathcal{G}_w$. See §3.1 in [26] for details. We then check whether there exists $c \in \mathbb{R}^n$ such that $c \cdot \alpha = c \cdot \beta = 0$ and $c \cdot \gamma \geqslant 1$ for all lattice points γ different from α and β in the $A\alpha$-fiber. This is the case if and only if the segment $[\alpha, \beta]$ is an edge of its fiber.

Example 1.2 (*continued*). The fiber of the binomial $adf - bce \in \mathcal{H} \backslash UGB_A$ contains five lattice points. They form the vertices of the 3-dimensional bipyramid in Fig. 1. The line segment $[(1,0,0,1,0,1), (0,1,1,0,1,0)]$ is the diagonal of this bipyramidal fiber.

Example 2.12 (*continued*). In contrast to Lemma 5.4, it can happen that neither term of a Graver basis element corresponds to a vertex of its fiber. The ppi $x_2^2 x_7 x_9 - x_5^2 x_{10}$ in $\mathcal{H} \backslash UGB_{A_{10}}$ has this property. To see this note $2e_2 + e_7 + e_9 \in conv\{3e_2 + 2e_7, e_2 + 2e_9\}$ and $2e_5 + e_{10} \in conv\{4e_5, 2e_{10}\}$.

The universal Gröbner basis of A can be used to devise a geometric method to construct the Gröbner fan of A and hence the state polytope of A. The *Gröbner arrangement* of A, denoted $Gr(A)$, is the arrangement consisting of the hyperplanes in $\mathbb{R}^n$ that are orthogonal to the elements in UGB_A. The Gröbner arrangement of A is a refinement of the Gröbner fan of A, since a hyperplane is in $Gr(A)$ if and only if it is the linear span of a facet of some Gröbner cone. The Graver arrangement of A is a refinement of the Gröbner arrangement of A.

Algorithm 5.8 (*A geometric construction of the Gröbner fan*).
Input: The universal Gröbner basis UGB_A
Output: The maximal cells in the Gröbner fan $\mathcal{N}(St(A))$.

Compute the Gröbner arrangement $Gr(A)$ from its set UGB_A of normals (cf. [13]). Let $G(A)$ be the set of maximal cells in $Gr(A)$, each represented by a vector in its interior.

While $G(A) \neq \emptyset$ **do**

Select a cell $C \in G(A)$.
Let $I_C = \{\underline{x^\alpha} - x^\beta \in UGB_A : C \subseteq \{c \in \mathbb{R}^n : \alpha \cdot c \geqslant \beta \cdot c\}\}$. Auto-reduce the set I_C with respect to the underlined leading terms to get the reduced Gröbner basis $\mathcal{G}_C$.
Let C' be the new cell obtained by erasing all facets of C whose normals are not in $\mathcal{G}_C$. Remove from $G(A)$ all cells which are contained in C'. Output C'.

Proof of correctness. If $c \in int(C')$ then the cone C' coincides with the Gröbner cone $\mathcal{K}_c$ in (3.1). Therefore each cell output by Algorithm 5.8 lies in the Gröbner fan. Conversely, each maximal cell in the Gröbner fan will eventually be generated in the While-loop since $Gr(A)$ covers $\mathbb{R}^n$. $\square$

We now relate certain properties of a matrix that have been discussed in earlier sections.

Theorem 5.9. *Conder the following properties of a matrix $A \in \mathbb{Z}^{d\times n}$ of maximal row rank:*

(i) *A is unimodular.*

(ii) *The state polytope $St(A)$ coincides with the secondary polytope $\Sigma(A)$.*

(iii) *The circuits of A constitute the universal Gröbner basis UGB_A.*

Then (i) $\Rightarrow$ (ii) *and* (ii) $\Rightarrow$ (iii) *but* (iii) $\not\Rightarrow$ (ii) *and* (ii) $\not\Rightarrow$ (i).

Proof. If A is unimodular then the integer programming fiber P_b^I coincides with the linear programming fiber P_b for all $b \in cone_{\mathbb{N}}(A)$ ([21], Theorem 19.2). Moreover, if $b \in cone(A) \setminus cone_{\mathbb{N}}(A)$, then there exists $b' \in cone_{\mathbb{N}}(A)$ such that P_b and $P_{b'}$ are normally equivalent. Therefore the Minkowski integrals in Theorems 1.5(iv) and 3.10(iv) coincide, which proves that (i) $\Rightarrow$ (ii).

By Theorem 5.1 the edge directions of $St(A)$ are the elements of UGB_A and by Theorem 1.8 the edges of $\Sigma(A)$ are the circuits of A. Hence if $St(A) = \Sigma(A)$, then the circuits of A constitute UGB_A. This proves the implication (ii) $\Rightarrow$ (iii).

To see that (iii) $\not\Rightarrow$ (ii) consider our running Example 1.2. In the end of Section 4 we proved that the circuits constitute the universal Gröbner basis. However, the secondary polytope $\Sigma(A)$ has 14 vertices (it is the 3-dimensional associahedron) while the state polytope $St(A)$ has 29 vertices (it is depicted in Fig. 2).

The fact that (ii) $\not\Rightarrow$ (i) is shown by the example $A = [1,2]$. This matrix is not unimodular, but $\Sigma(A)$ and $St(A)$ are identical line segments parallel to $ker(A) = span\{(-2,1)\}$. $\square$

Example 5.10. The example in the proof of (ii) $\not\Rightarrow$ (i) is trivial since $St(A) = \Sigma(A)$ for every matrix A of corank 1. For an example with corank 2 consider the matrix

$$A = \begin{bmatrix} 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

This matrix is of Lawrence type. Its Graver basis consists precisely of the four circuits. There are eight distinct reduced Gröbner bases associated with this matrix each of which corresponds to a distinct triangulation. This implies that the state and secondary polytopes coincide. However, A is not unimodular since it has maximal minors of absolute value zero, one and two.

Recall that the fiber containing an element of the universal Gröbner basis UGB_A was called a Gröbner fiber of IP_A and the fiber containing an element of the Graver basis a Graver fiber of IP_A. By Theorem 2.7, the set of Gröbner fibers of A is contained in the set of Graver fibers of A. The following example shows that this containment may be strict.

Example 5.11 (*Graver fibers versus Gröbner fibers*). For any non-negative integer i consider the (2×5)-matrix

$$A_i = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 3+6i & 4+6i & 6+6i \end{bmatrix}.$$

Then the binomials $x_1^2 x_5^{2i+k+1} - x_2 x_3^{2i-2k+3} x_4^{3k-1}$, $k = 1, \ldots, i+1$, are contained in $\mathcal{H} \backslash UGB_{A_i}$. Each element comes from a distinct fiber of IP_{A_i}, and it can be shown that none of these is a Gröbner fiber. (The proof is a lengthy case analysis and will be omitted.) We conclude that the matrix A_i has at least $i+1$ Graver fibers that are not Gröbner fibers.

Gröbner fibers can have arbitrarily many vertices even for (1×4)-matrices:

Example 5.12 (*Gröbner fibers with many vertices*). This example is based on Remark 18.1 in [21]. Let ϕ_k denote the kth Fibonacci number, and consider the (1×4)-matrix $A_k := [\phi_{2k}, \phi_{2k+1}, 1, \phi_{2k+1}^2 - 1]$. Consider the fiber of A_k over $b_k = \phi_{2k+1}^2 - 1$. This is a Gröbner fiber because it is the fiber of the circuit $(0, 0, 1 - \phi_{2k+1}^2, 1)$. The set of points with last coordinate zero is a facet of this fiber. It is a polygon isomorphic to the convex hull of all non-negative lattice points (x, y) with $\phi_{2k} \cdot x + \phi_{2k+1} \cdot y \leqslant \phi_{2k+1}^2 - 1$. This lattice polygon has $k+3$ vertices. We conclude that the b_k-fiber of A_k is a Gröbner fiber with at least $k+4$ vertices.

The encoding of the lattice polygon as a facet of a 3-polytope in Example 5.12 is a special case of the following general construction.

Proposition 5.13. *Every lattice polytope appears as a facet of some Gröbner fiber.*

Proof. Every $(n-d)$-dimensional lattice polytope can be written as a fiber P_b^I for some $A \in \mathbb{Z}^{d \times n}$ of maximal row rank and some $b \in \mathbb{Z}^d$. This polytope is isomorphic to the facet of points with zero last coordinate in the b-fiber of $IP_{[A,b]}$ where $[A, b] \in \mathbb{Z}^{d \times (n+1)}$. Moreover, the b-fiber of $IP_{[A,b]}$ is a Gröbner fiber since $x_{n+1} - x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_n^{\lambda_n}$ lies in $UGB_{[A,b]}$ for every vertex λ of P_b^I. □

6. On the complexity of Gröbner cones

One direct application of the reduced Gröbner basis $\mathcal{G}_c$ is that it provides an inequality presentation for the equivalence class of cost functions containing c. This equivalence class is the interior of the normal cone of a vertex of the state polytope, or, equivalently, the interior of the Gröbner cone:

$$int\, \mathcal{K}_c = \{ w \in \mathbb{R}^n : \alpha_i \cdot w > \beta_i \cdot w \text{ for all } \alpha_i - \beta_i \in \mathcal{G}_c \}. \tag{6.1}$$

While the Gröbner basis $\mathcal{G}_c$ can have arbitrarily many elements for fixed d and n, we observed in a large number of computations that a vast majority of the inequalities in

(6.1) is redundant. Based on this experimental evidence we make the following two conjectures.

Conjecture 6.1. *There exists a function $\varphi : \mathbb{N} \longrightarrow \mathbb{N}$ such that, for every matrix $A \in \mathbb{Z}^{d\times n}$ of rank d, every Gröbner cone $\mathcal{K}_c$ associated with A has at most $\varphi(n-d)$ facets.*

Conjecture 6.2. *For a matrix A of corank three, every vertex of $St(A)$ has at most four neighbors.*

Obviously we have $\varphi(2) = 2$. Conjecture 6.2 asserts that $\varphi(3) = 4$, or, equivalently, that every vertex of a 3-dimensional state polytope is either 3-valent or 4-valent. A proof of Conjecture 6.2 for matrices of format 1×4 has been announced by Imre Barany (personal communication). His proof relies on a description of the "neighbors of the origin" as defined by Scarf [20,22].

In this section we present two constructions which provide lower bounds for the function φ (assuming it exists). These constructions show in particular that $\varphi(4) \geqslant 8, \varphi(5) \geqslant 12, \varphi(6) \geqslant 18, \varphi(7) \geqslant 30$, and that $\varphi(n-d)$ is bounded below by an exponential function in $n-d$. To study the facets of a Gröbner cone we shall use the following general lemma about facets of polyhedra.

Lemma 6.3. *Let $\mathcal{G}_c = \{\alpha_i - \beta_i : i = 1, \ldots, t\}$. Then $\alpha_j - \beta_j$ defines a facet of the Gröbner cone $\mathcal{K}_c$ if and only if the system $\{\alpha_i \cdot x > \beta_i \cdot x, i \in \{1, \ldots, t\} \setminus \{j\}\} \cup \{\beta_j \cdot x > \alpha_j \cdot x\}$ is consistent.*

We say that a binomial $x^{\alpha_j} - x^{\beta_j}$ in $\mathcal{G}_c$ can be *flipped* if $\alpha_j - \beta_j$ defines a facet of the Gröbner cone $\mathcal{K}_c$. If $x^{\alpha_j} - x^{\beta_j}$ can be flipped and w is a solution of the linear system in Lemma 6.3, then the vertices of $St(A)$ in directions c and w are connected by an edge parallel to $\alpha_j - \beta_j$. Our first result concerns the family of knapsack problems in Example 2.12. We show that the complexity of their Gröbner cones grows at least quadratically in the dimension.

Proposition 6.4. *The state polytope of the $(1\times n)$-matrix $A_n = [1, 2, 3, \ldots, n]$ possesses a vertex with $\binom{n}{2} - \lfloor \frac{n}{2} \rfloor$ neighboring vertices. Hence $\varphi(n-1) \geqslant \binom{n}{2} - \lfloor \frac{n}{2} \rfloor$.*

Proof. Consider the cost vector $c = e_1 + e_2 + \cdots + e_{n-1}$. The corresponding integer program takes the form

$$\begin{array}{ll} \text{minimize} & \gamma_1 + \gamma_2 + \ldots + \gamma_{n-1} \\ \text{subject to} & \gamma_1 + 2\gamma_2 + \cdots + (n-1)\gamma_{n-1} + n\gamma_n = \beta. \end{array}$$

The vector c is generic because, for every positive integer β, there is a unique optimal solution γ^*: if n divides β then $\gamma^* = (\beta/n) \cdot e_n$, otherwise $\gamma^* = \lfloor \beta/n \rfloor \cdot e_n + e_i$, where $i \equiv \beta \pmod n$.

The corresponding Gröbner basis has $\binom{n}{2}$ elements:

$$\mathcal{G}_c = \{\underline{x_i x_j} - x_{i+j} : 1 \leqslant i \leqslant j \leqslant n-i\} \cup \{\underline{x_i x_j} - x_{i+j-n}x_n : n-j < i \leqslant j \leqslant n\}.$$

Here the leading terms are underlined. Thus the initial ideal of the toric ideal equals

$$in_c(I_A) = \langle x_1, x_2, \ldots, x_{n-1}\rangle^2.$$

We shall prove the proposition by establishing the following two claims:

Claim 1. *The "diagonal" elements $\underline{x_i x_{n-i}} - x_n, i = 1, \ldots, \lfloor \frac{n}{2} \rfloor$, cannot be flipped.*

Claim 2. *All other elements of $\mathcal{G}_c$ can be flipped.*

To prove Claim 1 we assume on the contrary that $x_i x_{n-i} - x_n$ can be flipped in $\mathcal{G}_c$. That means there exists $\omega \in \mathbb{R}^n$ such that $\omega_i + \omega_{n-i} < \omega_n$ but ω selects the underlined leading term for all other binomials in $\mathcal{G}_c$.

Case (a): $i < n/2$. The binomial $\underline{x_{n-i}x_{2i}} - x_n x_i$ implies $\omega_{n-i} + \omega_{2i} - \omega_i - \omega_n > 0$, and the binomial $\underline{x_i^2} - x_{2i}$ implies $2\omega_i - \omega_{2i} > 0$. Summing both inequalities gives a contradiction.

Case (b): $i = n/2$. The binomial $\underline{x_{n/2}x_{(n+2)/2}} - x_1 x_n$ implies $\omega_{n/2} + \omega_{(n+2)/2} - \omega_1 - \omega_n > 0$, and $\underline{x_{n/2}x_1} - x_{n+2/2}$ implies $\omega_{n/2} + \omega_1 - \omega_{(n+2)/2} > 0$. Summing both inequalities gives a contradiction.

In our proof of Claim 2 we distinguish four cases.

Case (c): To flip a binomial $\underline{x_i x_j} - x_{i+j}$ with $i \neq j$ we take the cost function

$$\omega = 9e_i + 9e_j + 19e_{i+j} + \sum\{10e_k : k \in \{1, \ldots, n-1\} \setminus \{i, j, i+j\}\}.$$

Case (d): To flip a binomial $\underline{x_i^2} - x_{2i}$ we take the cost function

$$\omega = 9e_i + 19e_{2i} + \sum\{10e_k : k \in \{1, \ldots, n-1\} \setminus \{i, 2i\}\}.$$

Case (e): To flip a binomial $\underline{x_i x_j} - x_n x_{i+j-n}$ with $i \neq j$ we take the cost function

$$\omega = 12e_{i+j-n} + 9e_i + 9e_j + 7e_n + \sum\{10e_k : k \in \{1, \ldots, n-1\} \setminus \{i, j, i+j-n\}\}.$$

Case (f): To flip a binomial $\underline{x_i^2} - x_n x_{2i-n}$ we take the cost function

$$\omega = 12e_{2i-n} + 9e_i + 7e_n + \sum\{10e_k : k \in \{1, \ldots, n-1\} \setminus \{i, 2i-n\}\}.$$

In each case the cost functions c and ω select the same leading terms for all binomials in $\mathcal{G}_c$ except for the one which is to be flipped. By Lemma 6.3, this completes the proof. □

We conclude this section with an example in which the number of facets of a Gröbner cone is exponential in the corank of the matrix. This example uses a construction which was shown to us by Eric Babson. We first need to recall some general definitions and results from [6].

Definition 6.5. Given a matrix $A \in \mathbb{Z}^{d\times n}$, then the *chamber complex* $\Gamma(A)$ is the coarsest polyhedral complex that covers *cone*(A) and refines all triangulations of A.

Proposition 6.6 ([6]). *Let $B \in \mathbb{Z}^{(n-d)\times n}$ be a Gale transform of the matrix $A \in \mathbb{Z}^{d\times n}$. Then the boundary complex of the secondary polyhedron $\Sigma(A)$ is antiisomorphic to the chamber complex $\Gamma(B)$ and the boundary complex of the secondary polyhedron $\Sigma(B)$ is antiisomorphic to $\Gamma(A)$.*

This shows that every chamber complex is a secondary fan and conversely. It is known that a matrix A is unimodular if and only if its Gale transform B is unimodular. In this case the Gröbner fan of A coincides with the secondary fan of A (cf. Theorem 5.9) and hence also with the chamber complex of B. To find a Gröbner cone of A with many facets, it therefore suffices to construct a chamber with many facets in the chamber complex of a unimodular matrix B.

Let B be the node-edge incidence matrix of the complete bipartite graph $K_{n,m}$ where $n = 2k-1$ and $m = 2k+1$. The columns of B are the vertices of the product of a regular $(n-1)$-simplex and a regular $(m-1)$-simplex. It is well known that B is unimodular [21, §19.3].

Proposition 6.7 (E. Babson). *There exists a chamber in $\Gamma(B)$ with at least $\binom{n}{k}\binom{m}{k+1}$ facets.*

Proof. The cone *cone*(B) has codimension 1 in $\mathbb{R}^n \times \mathbb{R}^m$. It consists of all non-negative vectors $(u_1,\dots,u_n)\times(v_1,\dots,v_m)$ such that $u_1+\cdots+u_n = v_1+\cdots+v_m$. By the *central ray* in *cone*(B) we mean the one-dimensional cone generated by $(1/n,\dots,1/n)\times(1/m,\dots,1/m)$. The fact that n and m are relatively prime implies that the central ray lies in the interior of a chamber in $\Gamma(B)$. (Reason: it lies on none of the hyperplanes (6.2) below.) It is called the *central chamber* of *cone*(B).

We will show that the central chamber has at least $\binom{n}{k}\binom{m}{k+1}$ facets. A facet of a chamber corresponds to a cut $(C_+,C_-;D_+,D_-)$ in $K_{n,m}$. Here (C_+,C_-) is a partition of $\{1,\dots,n\}$ and (D_+,D_-) is a partition of $\{1,\dots,m\}$. The hyperplane spanned by this facet is defined by

$$\sum_{i\in C_+} u_i - \sum_{i\in C_-} u_i - \sum_{j\in D_+} v_j + \sum_{j\in D_-} v_j = 0. \tag{6.2}$$

We call $\big(\mathrm{card}(C_+),\mathrm{card}(D_+)\big)$ the *type* of this hyperplane.

The product of symmetric groups $S_n \times S_m$ fixes the central chamber and it acts transitively on the set of hyperplanes of fixed type. In what follows we shall determine which type(s) of hyperplanes appear(s) as facets of the central chamber. Starting at the point $(1/n,\dots,1/n)\times(1/m,\dots,1/m)$ on the central ray we move in the direction

$(-1,-1,\ldots,-1,n-1)\times(0,0,\ldots,0)$ to a generic point $(a,a,\ldots,a,a+1-na)\times(1/m,\ldots,1/m)$. By (6.2), we cross a facet of type (r,s) with $n\in C_-$ when

$$r\cdot a-(n-r-1)\cdot a-(a+1-na)-s/m+(m-s)/m=2\cdot r\cdot a-2\cdot s/m=0.$$

Solving for a, we get $a=s/mr<1/n$. We need to find $r\in\{1,\ldots,n\}$ and $s\in\{1,\ldots,m\}$ such that $1/n-s/mr$ is positive and as small possible. Equivalently, we wish to minimize the positive integer $m\cdot r-n\cdot s$. The unique solution that minimizes this expression is $r=k$ and $s=k+1$. We conclude that *every* hyperplane of type (r,s) is a facet of the central chamber. There exists $\binom{n}{k}\binom{m}{k+1}$ such facets. □

Corollary 6.8. *For all positive integers k we have* $\varphi(4k-1)\geqslant\binom{2k-1}{k}\binom{2k+1}{k+1}>2^k$.

Proof. The matrix B has rank $m+n-1$. Its Gale transform A is a matrix of format $(m-1)(n-1)\times mn$ with maximal row rank. The kernel of A has dimension $mn-(m-1)(n-1)=m+n-1=4k-1$. By Proposition 6.6, the central chamber of $\Gamma(B)$ appears in the Gröbner fan of A. Corollary 6.8 now follows immediately from Proposition 6.7. □

We remark that the integer programs IP_A corresponding to the above Gale transform A are not bounded. This can be remedied by adding an extra column to B which is the negative of the sum of the other columns. This operation does not change $\mathrm{rank}(B)=\mathrm{corank}(A)$, and Corollary 6.8 holds for bounded integer programs as well.

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