Algèbre S3

Ex 1:

(E) : $x' + ax = b$ (H) : $x' + ax = 0$

1. Let $f(t) = x(t)\, e^{\int_0^t a(s)\,ds}$
$= x(t)\, e^{F(t)}$ $\quad (F(t) = A(t) - A(0)$ with $A'(t) = a(t))$

One has
$$f'(t) = x'(t)e^{F(t)} + x(t)\left(e^{F(t)}\right)'$$
$$= x'(t)e^{F(t)} + x(t)F'(t)e^{F(t)}$$
$$= x'(t)e^{F(t)} + x(t)a(t)e^{F(t)}$$
$$= e^{F(t)}\big(\underbrace{x'(t) + x(t)a(t)}_{=0 \text{ from } (H)}\big) = 0$$

So $\begin{cases} f'(t) = 0 & \forall t \\ f \in \mathcal{C}^1 \end{cases} \iff f$ is constant $(f(t) = \alpha \ \forall t)$

Thus, $f(t) = x(t)e^{F(t)} = \alpha \iff x(t) = \alpha e^{-F(t)}$

2. Let x_p a particular solution of (E) with $x_p(t) = \lambda(t)e^{-F(t)}$ $\forall t$
$\rightarrow$ We want to find the expression of $\lambda(t)$ $\forall t$

One has $\forall t$, $x_p'(t) = \left(\lambda(t)e^{-F(t)}\right)'$
$$= \lambda'(t)e^{-F(t)} - \lambda(t)F'(t)e^{-F(t)}$$
$$= \lambda'(t)e^{-F(t)} - \lambda(t)a(t)e^{-F(t)}$$

So $\forall t$, $x_p'(t) + a(t)x_p(t) = \lambda'(t)e^{-F(t)} - \lambda(t)a(t)e^{-F(t)} + a(t)\lambda(t)e^{-F(t)}$
$$= \lambda'(t)e^{-F(t)}$$
$$= b(t) \quad \text{as } x_p \text{ sol of (E)}$$

$\rightarrow$ $\lambda'(t) = b(t)e^{F(t)}$
so $\lambda(t) = \int_0^t b(s)e^{F(s)}\,ds$ and $x_p(t) = \left[\int_0^t b(s)e^{F(s)}\,ds\right] e^{-F(t)}$

Solution of (E) : $x^* = x_H + x_p$ with $\begin{cases} x_H \text{ sol of (H)} \\ x_p \text{ sol part of (E)} \end{cases}$

$$\longrightarrow \quad x^*(t) = e^{-F(t)}\left[\alpha + \int_0^t b(s)\, e^{F(s)}\, ds\right]$$

3. Find x st $\begin{cases} x' + ax = b \\ x(t_0) = x_0 \end{cases}$ $\rightarrow$ Cauchy's problem so unique solution

$$(E) : \quad (1+t^2)\, x'(t) + 2t\, x(t) = \frac{1}{1+t^2}$$

$$\Longleftrightarrow \quad x'(t) + \frac{2t}{1+t^2}\, x(t) = \frac{1}{(1+t^2)^2} \qquad (\text{as } 1+t^2 > 0 \ \forall t)$$

$$\Longleftrightarrow \quad x'(t) + a(t)\, x(t) = b(t)$$

Solution of (E) : $x^*(t) = e^{-F(t)}\left[\alpha + \int_0^t b(s)\, e^{F(s)}\, ds\right]$

With $F(t) = \int_0^t \frac{2s}{1+s^2} ds = \int_0^t \frac{u'(s)}{u(s)} ds$ with $u(s) = 1+s^2$

$$= \int_0^t [\ln(u(s))]'\, ds = \left[\ln(u(s))\right]_0^t = \ln(1+t^2)$$

And $\int_0^t b(s)\, e^{F(s)}\, ds = \int_0^t \frac{1}{(1+s^2)^2} \times (1+s^2)\, ds = \int_0^t \frac{1}{1+s^2} ds$

$$= \int_0^t [\arctan(s)]'\, ds = [\arctan(s)]_0^t$$

$$= \arctan(t)$$

So finally, $x^*(t) = \frac{1}{1+t^2}\left[\alpha + \arctan(t)\right]$

If $x^*(1) = 0$, then $\frac{1}{2}(\alpha + \arctan(1)) = 0 \Longleftrightarrow \alpha = -\arctan(1) = -\frac{\pi}{4}$

Ex 2 $\quad (E) : x'' + x' - 2x = 0$

1. $X = \begin{pmatrix} x \\ x' \end{pmatrix}$ so $X' = \begin{pmatrix} x' \\ x'' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix} = AX$

2. We have to diagonalize A :

$$\begin{cases} \chi_A(t) = (t+2)(t-1) \\ E_1 = \mathrm{Vect}\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\} \\ E_{-2} = \mathrm{Vect}\left\{\begin{pmatrix} 1 \\ -2 \end{pmatrix}\right\} \end{cases} \quad \text{so} \quad P = \begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix}$$

3. Let $U = P^{-1}X$, then $\quad X' = AX \iff PU' = APU$

$\rightsquigarrow (U\,X(t))' = U\,X'(t)$

$\iff U' = (P^{-1}AP)\,U$

$\iff U' = DU$ where $D = \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix}$

The new system is

$$\begin{cases} u_1' = u_1 \\ u_2' = -2u_2 \end{cases} \quad \to \text{two indep. EDO in 1D}$$

We have $\forall t$ $\begin{cases} u_1'(t) = u_1(t) \\ u_2'(t) = -2u_2(t) \end{cases} \iff \begin{cases} u_1(t) = \alpha e^{t} \\ u_2(t) = \beta e^{-2t} \end{cases}$

4. $X = PU$ so $\begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} \alpha e^{t} \\ \beta e^{-2t} \end{pmatrix} = \begin{pmatrix} \alpha e^{t} + \beta e^{-2t} \\ \alpha e^{t} - 2\beta e^{-2t} \end{pmatrix}$

Verif : $x(t) = \alpha e^{t} + \beta e^{-2t} \iff x'(t) = \alpha e^{t} - 2\beta e^{-2t} \rightsquigarrow$ ok !

Ex 4

$$A = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$$

1. $\begin{cases} A \in \mathcal{M}_n(\mathbb{R}) \\ A \text{ symetric} \end{cases} \Longrightarrow$ A is diago.

$$\begin{cases} \chi_A(t) = (-t)(3-t)^2 \\ E_0 = \mathrm{Vect}\left\{\begin{pmatrix}1\\1\\1\end{pmatrix}\right\} \\ E_3 = \mathrm{Vect}\left\{\begin{pmatrix}1\\0\\-1\end{pmatrix}, \begin{pmatrix}0\\1\\-1\end{pmatrix}\right\} \end{cases} \rightsquigarrow \text{So } P = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & -1 & -1 \end{pmatrix} \text{ and } D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

2. We know that $\mathcal{B} = \{ t \mapsto e^{\lambda_i t} V_i \}_{i=1}^{3}$ is a basis of solutions where (λ_i, V_i) are the eigenvalues and eigenvectors of A

$\rightsquigarrow$ Th. 15, only works if A is diago and with an homogenous EDO !

Thus, $X(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix} = \alpha e^{0t}\begin{pmatrix}1\\1\\1\end{pmatrix} + \beta e^{3t}\begin{pmatrix}1\\0\\-1\end{pmatrix} + \gamma e^{3t}\begin{pmatrix}0\\1\\-1\end{pmatrix}$ with $\begin{pmatrix}\alpha\\\beta\\\gamma\end{pmatrix} \in \mathbb{R}^3$

3. We seek a particular solution of $X' = AX + b$ with $b = (1\ -1\ 0)^T$, that is

$$X_p = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \quad \text{with} \quad X_p' = 0 = AX + b$$

$$\iff \begin{cases} 2a - b - c + 1 = 0 \\ -a + 2b - c - 1 = 0 \\ -a - b + 2c = 0 \end{cases}$$

⚠ The sol is not unique since $\det(A) = 0$

$$\iff \begin{cases} b = a + \frac{2}{3} \\ c = a + \frac{1}{3} \end{cases}$$

So any $X_p = a\begin{pmatrix}1\\1\\1\end{pmatrix} + \begin{pmatrix}0\\2/3\\1/3\end{pmatrix}$ with $a \in \mathbb{R}$ is a particular solution of $X' = AX + b$.

The term $a\begin{pmatrix}1\\1\\1\end{pmatrix}$ is already captured in X_H so we can set $a = 0$ in X_p.

We conclude that $X^* = \alpha\begin{pmatrix}1\\1\\1\end{pmatrix} + \beta e^{3t}\begin{pmatrix}1\\0\\-1\end{pmatrix} + \gamma e^{3t}\begin{pmatrix}0\\1\\-1\end{pmatrix} + \begin{pmatrix}0\\2/3\\1/3\end{pmatrix}$

If in addition, $X(0) = \begin{pmatrix}1\\0\\0\end{pmatrix}$, we must have $\begin{cases} \alpha + \beta = 1 \\ \alpha + \gamma + 2/3 = 0 \\ \alpha - \beta - \gamma + 1/3 = 0 \end{cases} \iff \begin{cases} \alpha = 0 \\ \beta = 1 \\ \gamma = \frac{-2}{3} \end{cases}$

Ex 5

1. $\begin{cases} x' = 3x - y + z \\ y' = 2x + z \\ z' = x - y + 2z \end{cases} \iff X' = AX$ with $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and $A = \begin{pmatrix} 3 & -1 & 1 \\ 2 & 0 & 1 \\ 1 & -1 & 2 \end{pmatrix}$

$\begin{cases} \chi_A(t) = (2-t)^2(1-t) \\ E_1 = \text{Vect}\left\{\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}\right\} \\ E_2 = \text{Vect}\left\{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}\right\} \end{cases}$ so $\dim(E_2) = 1 \neq m_2 \Rightarrow A$ is not diago

2. $E_2^2 = \text{Vect}\left\{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right\}$ so $P = (u, v, w)$ with $\begin{cases} Au = 2u \\ Av = u + 2v \\ Aw = w \end{cases} \Rightarrow \begin{cases} u \in E_2 \\ v \in E_2^2 \setminus E_2 \\ w \in E_1 \end{cases}$

$\begin{cases} \text{Select } v = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \in E_2^2 \setminus E_2 \\ \text{Find } u \text{ as } (A - 2I)v = u = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \\ \text{Select } w = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \end{cases} \rightsquigarrow P = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$

3. Let $U = P^{-1}X$, then $X' = AX \iff U' = (P^{-1}AP)U = TU$ with $T = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

So $\begin{cases} u_1' = 2u_1 + u_2 \\ u_2' = 2u_2 \\ u_3' = u_3 \end{cases} \iff \begin{cases} u_1'(t) = 2u_1(t) + \beta e^{2t} & \forall t \\ u_2(t) = \beta e^{2t} & \forall t \\ u_3(t) = \gamma e^{t} & \forall t \end{cases}$

Solve (E) : $u' = 2u + \beta e^{2t}$

- (H) : $u' = 2u$ donc $u_H(t) = \alpha e^{2t}$ sol homogenous
- Let $u_p(t) = \lambda(t)\alpha e^{2t}$ sol part of (E), then

$$u_p'(t) = \lambda'(t)\alpha e^{2t} + 2\lambda(t)\alpha e^{2t}$$
$$= \lambda'(t)\alpha e^{2t} + 2u_p(t)$$

So $u_p'(t) - 2u_p(t) = \lambda'(t)\alpha e^{2t} = \beta e^{2t}$ as u_p sol of (E)

And therefore $\lambda'(t) = \beta/\alpha$ thus $\lambda(t) = \frac{\beta}{\alpha}t$

So finally $u_p(t) = \beta t e^{2t}$

- Solutions of (E) : $u^*(t) = (\alpha + \beta t)e^{2t}$ $\forall \alpha$ (β is fixed in (E) !)

$\iff \begin{cases} u_1(t) = (\alpha + \beta t)e^{2t} \\ u_2(t) = \beta e^{2t} \\ u_3(t) = \gamma e^{t} \end{cases}$

So finally $\quad X = PU = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} (\alpha+\beta t)e^{2t} \\ \beta e^{2t} \\ \gamma e^{t} \end{pmatrix} = \begin{pmatrix} (\alpha+\beta t)e^{2t} \\ (\alpha+\beta t)e^{2t} + \gamma e^{t} \\ \beta e^{2t} + \gamma e^{t} \end{pmatrix}$

⚠ We could have used directly Th. 15 b) which yield the same result more directly

$$\begin{cases} E_2^2 = \mathrm{Vect}\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\} \\ E_1 = \mathrm{Vect}\left\{ \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\} \end{cases} \xrightarrow[\text{de sol.}]{\text{base}} \begin{cases} e^{2t}\left(I + t(A-2I)\right)\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \\ e^{2t}\left(I + t(A-2I)\right)\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \\ e^{1t}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \end{cases} = \begin{cases} e^{2t}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \\ e^{2t}\left[\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + t\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}\right] \\ e^{t}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \end{cases}$$

So $\quad X' = \alpha e^{2t}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \beta e^{2t}\left[\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + t\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}\right] + \gamma e^{t}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

Ex 7

1. $\begin{cases} x' = 4x - y \\ y' = 3x \end{cases} \iff X' = AX$ with $X = \begin{pmatrix} x \\ y \end{pmatrix}$ and $A = \begin{pmatrix} 4 & -1 \\ 3 & 0 \end{pmatrix}$

2. $\chi_A(t) = (t-1)(t-3)$ $\rightsquigarrow$ distinct eig so A is diago

$\begin{cases} E_1 = \mathrm{Vect}\left\{\begin{pmatrix} 1 \\ 3 \end{pmatrix}\right\} \\ E_3 = \mathrm{Vect}\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\} \end{cases}$ so $P = \begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix}$, $Q = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$ and $P^{-1} = \frac{1}{2}\begin{pmatrix} 3 & -1 \\ -1 & 1 \end{pmatrix}$

3. $e^{tA} = \sum_{k=0}^{\infty} \frac{1}{k!}(tA)^k = \sum_{k=0}^{\infty} \frac{1}{k!} t^k (PQP^{-1})^k = \sum_{k=0}^{\infty} \frac{1}{k!} t^k P Q^k P^{-1} = P e^{tQ} P^{-1}$

$= \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} e^{3t} & 0 \\ 0 & e^t \end{pmatrix}\begin{pmatrix} 3 & -1 \\ -1 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 3e^{3t} - e^t & e^t - e^{3t} \\ 3e^{3t} - 3e^t & 3e^t - e^{3t} \end{pmatrix}$

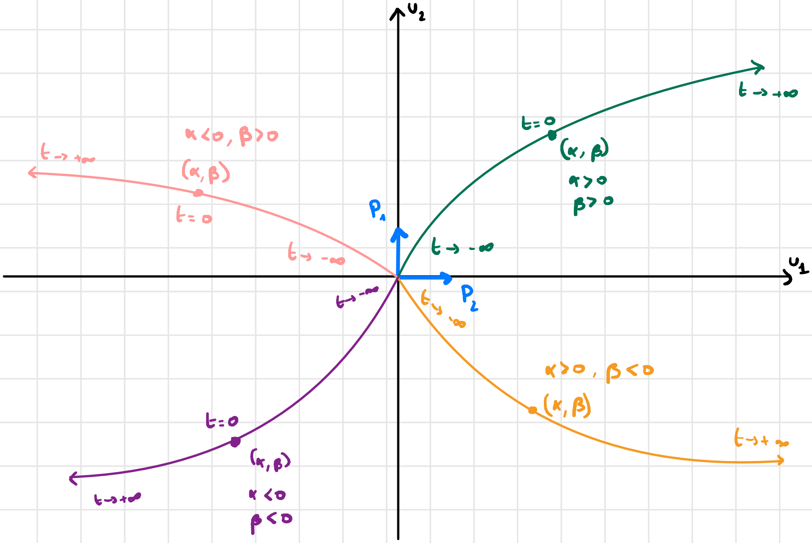
4. Solutions are of the form $X(t) = e^{(t-t_0)A} X_0$ with $X_0 = X(0) = \begin{pmatrix} a \\ b \end{pmatrix}$

$\rightsquigarrow$ Th. 14

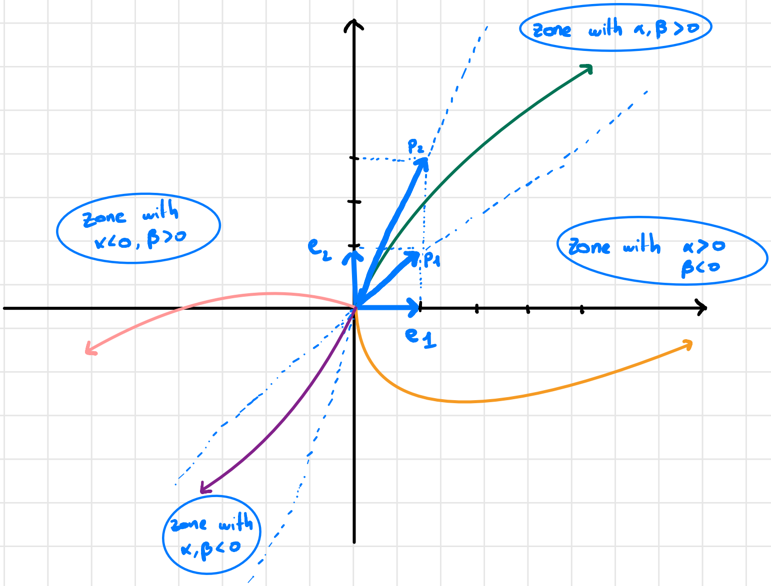
5. In the bases $\mathcal{B} = (P_1, P_2) = \left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \end{pmatrix}\right\}$, one has $U(t) = \begin{pmatrix} \alpha e^{3t} \\ \beta e^t \end{pmatrix}$

with $\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = U(0) = P^{-1} X(0) = \frac{1}{2}\begin{pmatrix} 3 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}$

$\rightarrow$ different cases depending on the initial conditi (α, β)



In the basis $\mathcal{B} = (e_1, e_2)$, we "tilt" the graph :



Ex 9

1. $\begin{cases} x' = 2x + y \\ y' = 2y + z \\ z' = 2z \end{cases} \iff X' = AX$ with $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and $A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$

$\rightarrow A = D + N$ with $D = 2I$ and $N = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

Moreover, $DN = ND$ so $e^{tA} = e^{t(D+N)} = e^{tD + tN} = e^{tD} e^{tN}$

$\rightarrow e^{tD} = e^{2t} I$

$e^{tN} = \sum_{k=0}^{\infty} \frac{1}{k!} t^k N^k = \sum_{k=0}^{2} \frac{1}{k!} t^k N^k = \begin{pmatrix} 1 & t & t^2/2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}$ as N is nilpotent

$\rightarrow e^{tA} = e^{2t} \begin{pmatrix} 1 & t & t^2/2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}$

2. Solutions are of the form $X(t) = e^{(t-t_0)A} X_0$ with $t_0 = 0$ and $X_0 = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$

So $X(t) = e^{2t} \begin{pmatrix} a + bt + ct^2/2 \\ b + ct \\ c \end{pmatrix}$

Ex 10

1. $\begin{cases} x' = x + z + e^{-t} \\ y' = x + 2y - 2z - e^{-t} \\ z' = -2x - 2y + z + 2e^{-t} \end{cases}$ So $(E): X' = AX + b$ with $A = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 2 & -2 \\ -2 & -2 & 1 \end{pmatrix}$ and $b = e^{-t}\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$

Homogenous sys : $(H) : X' = AX$

$$\begin{cases} \chi_A(t) = (3-t)(1-t)(-t) \\ E_3 = \mathrm{Vect}\left\{\begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix}\right\} \\ E_1 = \mathrm{Vect}\left\{\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}\right\} \\ E_0 = \mathrm{Vect}\left\{\begin{pmatrix} 2 \\ -3 \\ -2 \end{pmatrix}\right\} \end{cases}$$

$\{t \mapsto e^{\lambda_i t} V_i\}_{i=1}^{3}$ is a basis of solutions of (H) so

$X_H(t) = \alpha \begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix} + \beta e^{t}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + \gamma e^{3t}\begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix}$ is a sol. of (H) $\forall \alpha, \beta, \gamma$

2. Let $X_p(t) = e^{-t} V$ with $V = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$

One has $X_p'(t) = AX_p(t) + b \iff -X_p(t) = AX_p(t) + b$

$$\iff -e^{-t}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} - e^{-t} A \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} - e^{-t}\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} = 0 \quad \forall t$$

$$\iff \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} + A\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} = 0 \iff \begin{cases} 2v_1 + v_3 = -1 \\ v_1 + 3v_2 - 2v_3 = 1 \\ -2v_1 - 2v_2 + 2v_3 = -2 \end{cases} \iff \begin{cases} v_1 = 1/4 \\ v_2 = -3/4 \\ v_3 = -3/2 \end{cases}$$

So $X_p(t) = e^{-t}\begin{pmatrix} 1/4 \\ -3/4 \\ -3/2 \end{pmatrix}$ is a particular solution

3. Solutions of (E) : $X(t) = X_H(t) + X_p(t)$

Ex 12

1. $\begin{cases} x'' = x' + y' - y \\ y'' = x' + y' - x \end{cases}$ So $X' = AX$ with $X = \begin{pmatrix} x \\ y \\ x' \\ y' \end{pmatrix}$ and $A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -1 & 1 & 1 \\ -1 & 0 & 1 & 1 \end{pmatrix}$

2. $\chi_A(t) = (1-t)^3(-1-t)$, $E_{-1} = \mathrm{Vect}\left\{\begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}\right\}$, $E_1^3 = \mathrm{Vect}\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}\right\}$

$(A-I) = \begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & -1 & 0 & 1 \\ -1 & 0 & 1 & 0 \end{pmatrix}$ $(A-I)^2 = \begin{pmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$ $(A-I)^3 = (-2)(A-I)^2$

A is triangularizable so a basis of solutions of the homogenous system is

$$B = \left\{ t \mapsto e^{\lambda_i t} \sum_{k=0}^{m_i - 1} \frac{t^k}{k!} (A - \lambda_i I)^k V_{ij} \right\}_{\substack{1 \leq i \leq 2 \\ 1 \leq j \leq m_i}} \quad \rightsquigarrow \text{Th 15. b)}$$

So $B = (b_1 \ b_2 \ b_3 \ b_4)$ with

$$\begin{cases} b_1(t) = e^t \displaystyle\sum_{k=0}^{2} \frac{t^k}{k!}(A-I)^k \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} = e^t \left(\begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} + t \begin{pmatrix} -1 \\ -1 \\ -1 \\ -1 \end{pmatrix} \right) \\ b_2(t) = e^t \displaystyle\sum_{k=0}^{2} \frac{t^k}{k!}(A-I)^k \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} = e^t \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} \\ b_3(t) = e^t \displaystyle\sum_{k=0}^{2} \frac{t^k}{k!}(A-I)^k \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} = e^t \left(\begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} + t \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \right) \\ b_4(t) = e^{-t} \displaystyle\sum_{k=0}^{0} \frac{t^k}{k!}(A+I)^k \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix} = e^{-t} \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix} \end{cases}$$

Therefore, any solution of the homogenous problem has the form

$$X(t) = \alpha b_1(t) + \beta b_2(t) + \gamma b_3(t) + \delta b_4(t) = e^t \begin{pmatrix} \alpha + t(-\alpha + \gamma) \\ \beta + t(-\alpha + \gamma) \\ \gamma + t(-\alpha + \gamma) \\ -\alpha + \beta + \gamma + t(-\alpha + \gamma) \end{pmatrix} + e^{-t} \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}$$

Ex 15

(E) : $x''' + x'' + x' + x = \sin t$

(H) : $x''' + x'' + x' + x = 0$

$\chi_H(t) = t^3 + t^2 + t + 1 = (-t-1)(t^2+1)$ is not splitted on $\mathbb{R}$

Method : Solve (E) in $\mathbb{C}$ and deduce solutions in $\mathbb{R}$

1. Solve (E) in $\mathbb{C}$

• Homogenous solution

$\chi_H(t) = (-1-t)(i-t)(-i-t)$ so $\tilde{B} = (\tilde{\varphi}_1, \tilde{\varphi}_2, \tilde{\varphi}_3)$ with $\tilde{\varphi}_1(t) = e^{-t}$, $\tilde{\varphi}_2(t) = e^{it}$ and $\tilde{\varphi}_3(t) = e^{-it}$ is a basis of sol. of (H) $\rightsquigarrow$ Th 17

Thus, the homogenous sol. writes $\tilde{x}_H(t) = \alpha e^{-t} + \beta e^{it} + \gamma e^{-it}$

• Particular solution

One has $\sin(t) = \mathrm{Im}(e^{it}) = \mathrm{Im}(e^{it} Q(t))$ where Q is the polynom $Q(t) = 1$.
Hence, we can seek for a particular solution of the form

$\tilde{x}_p(t) = e^{it} P(t)$ with $\deg(P) \leq m_i + \deg(Q) = 1$ $\rightsquigarrow$ Prop. 12

(m_i : multipl. of the vp i)

As $\deg(P) \leq 1$, we can write $P(t) = \delta t + \varepsilon$.
Note that we can choose $\varepsilon = 0$ since $e^{it}(\delta t + \varepsilon) = e^{it}\delta t + \underbrace{e^{it}\varepsilon}_{\text{already captured in } \tilde{x}_H}$
So we let $\tilde{x}_p(t) = \delta e^{it} t$ and it remains to find δ.

$$\begin{cases} \tilde{x}_p(t) = \delta t e^{it} \\ \tilde{x}_p'(t) = \delta(1+it)e^{it} \\ \tilde{x}_p''(t) = \delta(2i-t)e^{it} \\ \tilde{x}_p'''(t) = \delta(-3-it)e^{it} \end{cases}$$

So $\tilde{x}_p''' + \tilde{x}_p'' + \tilde{x}_p' + \tilde{x}_p = e^{it}$

$\iff \delta(-2+2i)e^{it} = e^{it}$

So $\delta = \dfrac{1}{-2+2i} = \dfrac{-2-2i}{8} = -\dfrac{1}{2\sqrt{2}}\left(\dfrac{1}{\sqrt{2}} + \dfrac{1}{\sqrt{2}}i\right) = -\dfrac{1}{2\sqrt{2}} e^{i\frac{\pi}{4}}$

And $\tilde{x}_p(t) = -\dfrac{1}{2\sqrt{2}} t e^{i(t+\frac{\pi}{4})}$

2. Solve (E) in $\mathbb{R}$

- Homogenous solution :

In $\mathbb{C}$, the basis of $\tilde{x}_H$ is $\tilde{B} = \{\tilde{\varphi}_1, \tilde{\varphi}_2, \tilde{\varphi}_3\}$

So in $\mathbb{R}$, the basis of x_H is $B = \{\tilde{\varphi}_1, Re(\tilde{\varphi}_2), Im(\tilde{\varphi}_2)\} = \{\tilde{\varphi}_1, Re(\tilde{\varphi}_3), Im(\tilde{\varphi}_3)\}$

⇝ Rem. 7

Thus, x_H writes $x_H(t) = \alpha e^{-t} + \beta\cos(t) + \gamma\sin(t)$

- Particular solution :

In $\mathbb{C}$, we have a solution $\tilde{x}_p$ for the RHS e^{it}

In $\mathbb{R}$, we want a solution x_p for the RHS $Im(e^{it})$, that is

$$x_p(t) = Im(\tilde{x}_p(t)) = -\frac{t}{2\sqrt{2}}\sin\left(t + \frac{\pi}{4}\right)$$

- Full solution

$$x^*(t) = x_H(t) + x_p(t) = \alpha e^{-t} + \beta\cos(t) + \gamma\sin(t) - \frac{t}{2\sqrt{2}}\sin\left(t + \frac{\pi}{4}\right)$$

First line of X_H, the only one that matters

- Check the solution :

$$\begin{cases} x^*(t) = \alpha e^{-t} + \beta\cos(t) + \gamma\sin(t) - \frac{1}{2\sqrt{2}} t\sin\left(t+\frac{\pi}{4}\right) \\ (x^*)'(t) = -\alpha e^{-t} - \beta\sin(t) + \gamma\cos(t) - \frac{1}{2\sqrt{2}}\sin\left(t+\frac{\pi}{4}\right) - \frac{1}{2\sqrt{2}} t\cos\left(t+\frac{\pi}{4}\right) \\ (x^*)''(t) = \alpha e^{-t} - \beta\cos(t) - \gamma\sin(t) - \frac{1}{\sqrt{2}}\cos\left(t+\frac{\pi}{4}\right) + \frac{1}{2\sqrt{2}} t\sin\left(t+\frac{\pi}{4}\right) \\ (x^*)'''(t) = -\alpha e^{-t} + \beta\sin(t) - \gamma\cos(t) + \frac{3}{2\sqrt{2}}\sin\left(t+\frac{\pi}{4}\right) + \frac{1}{2\sqrt{2}} t\cos\left(t+\frac{\pi}{4}\right) \end{cases}$$

$$(x^*)'''(t) + (x^*)''(t) + (x^*)'(t) + x^*(t) = \frac{1}{\sqrt{2}}\left(\sin\left(t+\frac{\pi}{4}\right) - \cos\left(t+\frac{\pi}{4}\right)\right) = \sin(t)$$

Ex 19 : $(E) : x'' + x = \dfrac{2}{\cos^3(t)}$ in $]-\frac{\pi}{2}, \frac{\pi}{2}[$

- **Homogenous solution :** $(H) : x'' + x = 0$, $\chi_H(t) = t^2 + 1 = (t-i)(t+i)$

In $\mathbb{C}$, a basis of sol. is $\tilde{B} = (\tilde{\varphi}_1, \tilde{\varphi}_2)$ with $\tilde{\varphi}_1(t) = e^{it}$ and $\tilde{\varphi}_2(t) = e^{-it}$

In $\mathbb{R}$, a basis of sol. is $B = (Re(\tilde{\varphi}_1), Im(\tilde{\varphi}_1)) = (Re(\tilde{\varphi}_2), Im(\tilde{\varphi}_2))$

$= (\varphi_1, \varphi_2)$ with $\begin{cases} \varphi_1(t) = \cos(t) \\ \varphi_2(t) = \sin(t) \end{cases}$

So in $]-\frac{\pi}{2}, \frac{\pi}{2}[\subset \mathbb{R}$, the homo. sol. writes $x_H(t) = \alpha \cos(t) + \beta \sin(t)$

- **Particular solution :**

We seek a particular solution of the form $x_p(t) = \alpha(t)\cos(t) + \beta(t)\sin(t)$

We impose $\alpha'(t)\cos(t) + \beta'(t)\sin(t) = \dfrac{2}{\cos^3(t)}$ $(*)$ ⇝ Cf. Slide 152

$$\begin{cases} x_p(t) = \alpha(t)\cos(t) + \beta(t)\sin(t) \\ x_p'(t) = -\alpha(t)\sin(t) + \beta(t)\cos(t) + \overbrace{\alpha'(t)\cos(t) + \beta'(t)\sin(t)}^{} = \dfrac{2}{\cos^3(t)} \\ x_p''(t) = -x_p(t) - \alpha'(t)\sin(t) + \beta'(t)\cos(t) + \dfrac{2}{\cos^3(t)} \times 3\tan(t) \qquad (**) \end{cases}$$

$2/\cos^3(t)$

So $$\begin{cases} \alpha'(t)\cos(t) + \beta'(t)\sin(t) = \dfrac{2}{\cos^3(t)} & (*) \\ \alpha'(t)\sin(t) - \beta'(t)\cos(t) = \dfrac{2}{\cos^3(t)}\left(3\tan(t) - 1\right) & (**) \end{cases}$$

$\cos \neq 0$ $\iff$ on $]-\frac{\pi}{2}, \frac{\pi}{2}[$
$$\begin{cases} \alpha'(t) = \dfrac{2}{\cos^4(t)} - \beta'(t)\tan(t) & (***) \\ 2\dfrac{\sin(t)}{\cos^4(t)} - \beta'(t)\tan(t)\sin(t) - \beta'(t)\cos(t) = \dfrac{2}{\cos^3(t)}\left(3\tan(t) - 1\right) \end{cases}$$

$\iff$ $\times \cos^3$
$$\begin{cases} \quad '' \\ -\beta'(t)\cos^2(t) = 4\tan(t) - 2 \end{cases}$$

Find β: $\beta'(t) = -\dfrac{4\sin(t)}{\cos^3(t)} + \dfrac{2}{\cos^2(t)}$ so $\beta(t) = \dfrac{-2}{\cos^2(t)} + 2\tan(t)$

Find α: $(**)$: $\alpha'(t) = \dfrac{2}{\cos^4(t)} + \dfrac{4\sin^2(t)}{\cos^4(t)} - 2\dfrac{\sin(t)}{\cos^3(t)} = 2\underbrace{\left(\dfrac{1+2\sin^2(t)}{\cos^4(t)}\right)} - 2\dfrac{\sin(t)}{\cos^3(t)}$

So $\alpha(t) = \dfrac{2\tan(t)}{\cos^2(t)} - \dfrac{1}{\cos^2(t)} = \dfrac{1}{\cos^2(t)}\left(2\tan(t) - 1\right)$

So finally, $x_p(t) = \alpha(t)\cos(t) + \beta(t)\sin(t)$

$$= \frac{1}{\cos(t)}\left(2\tan(t) - 1\right) + \frac{1}{\cos(t)}\left(-2\tan(t) + 2\sin^2(t)\right)$$

$$= \frac{1}{\cos(t)}\left(2\sin^2(t) - 1\right) = \boxed{\frac{-\cos(2t)}{\cos(t)}}$$

Solution of (E): $x^*(t) = \alpha\cos(t) + \beta\sin(t) - \dfrac{\cos(2t)}{\cos(t)}$

Integrate $f(t) = \dfrac{1+2\sin^2(t)}{\cos^4(t)}$

$$\rightarrow f(t) = \frac{\frac{1}{\cos^2(t)}\cos^2(t) + 2\cos(t)\sin(t)\frac{\sin(t)}{\cos(t)}}{(\cos^2(t))^2}$$

$$= \frac{\tan'(t) \times \cos^2(t) - (\cos^2)'(t)\tan(t)}{(\cos^2(t))^2}$$

$$= \frac{u'v - v'u}{v^2} \quad \text{so} \quad \int f = \frac{u}{v} = \frac{\tan(t)}{\cos^2(t)}$$