TD - Algèbre S3

- Mail : theo.guyard@insa-rennes.fr
- TD : 9 séances
  - → 4 sur la 1ère partie
  - → 5 sur la 2ème partie
- CC : Au moins 2 CC portants sur les 4 LHC précédentes

My notations :

- $eig(A) = sp(A)$
- $E_\lambda^m = Ker((A-\lambda I)^m)$

Ex 1 $\quad A \in \mathcal{M}_n(\mathbb{C}) \qquad eig(A) = (\lambda_1, \ldots, \lambda_n)$

a) Proove that $\det(A) = \prod_i \lambda_i$ and $Tr(A) = \sum_i \lambda_i$

- Find relation linking $\det(A)$, $Tr(A)$ and $eig(A)$
- Proove

- $A \in \mathcal{M}_n(\mathbb{C})$ so $\chi_A(t) = \prod_i (\lambda_i - t)$ — Remark slide 22

  Moreover, $\chi_A(t) = (-1)^n t^n + (-1)^{n-1} Tr(A)\, t^{n-1} + \ldots + \det(A)$ — Inc. form.

- Idea : develop and identify with induction

→ Rank n : Assume $\chi_A^n(t) = (-1)^n t^n + (-1)^{n-1} \sum_i \lambda_i t^{n-1} + \ldots + \det(A)$

→ Rank $n+1$:

$$\begin{aligned}
\chi_A^{n+1}(t) &= (\lambda_{n+1} - t)\, \chi_A^n(t) \quad \leftarrow \text{deg } n+1 \\
&= (\lambda_{n+1} - t)\left[(-1)^n t^n + (-1)^{n-1} \sum_{i=1}^{n} \lambda_i t^{n-1} + \underbrace{\ldots}_{\deg < n-1}\right] \\
&= \lambda_{n+1} (-1)^n t^n + (-1)^{n+1} t^{n+1} + (-1)^n \sum_{i=1}^{n} \lambda_i t^n + \underbrace{\ldots}_{\deg < n} \\
&= (-1)^{n+1} t^{n+1} + (-1)^n \underbrace{\sum_{i=1}^{n+1} \lambda_i}_{= Tr(A)} t^n + \underbrace{\ldots}_{\deg < n}
\end{aligned}$$

→ Rank 1 : $A = a \in \mathbb{R}$ so $Tr(A) = a$ and $ax = \lambda x \overset{x \neq 0}{\iff} a = \lambda$

b) Let $A \in \mathcal{M}_n(\mathbb{K})$ with $\sum_j a_{ij} = \lambda$. Show that $\exists X \neq 0,\ AX = \lambda X$

→ Make λ appear!

$$A \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} \sum_{j=1}^{n} a_{ij} \\ \vdots \\ \sum_{j=1}^{n} a_{ij} \end{pmatrix} = \begin{pmatrix} \lambda \\ \vdots \\ \lambda \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \quad \text{so } A\mathbb{1} = \lambda \mathbb{1} \text{ and } \mathbb{1} \neq 0$$

c) Find all eig using problem properties (recall $|eig(A)| \leq 3$)

- A non-invertible so $\det(A) = 0 \implies \prod_i \lambda_i = 0 \implies \lambda_1 = 0$
- $\sum$ line $= a+b+c$ so $\lambda_2 = a+b+c$ wich is $\neq \lambda_1$ cf 2)
- $Tr(A) = \sum_i \lambda_i$ so $3a = \lambda_1 + \lambda_2 + \lambda_3 \implies \lambda_3 = 2a - b - c$

Ex 2 $A \in \mathcal{M}_n(\mathbb{R})$

- $\chi_A(t) = p_n t^n + \dots + p_0$ $\deg(\chi_A) = n$

- Cayley-Hamilton : $\chi_A(A) = p_n A^n + \dots + p_0 I = 0$

$\rightarrow A^{-1}\chi_A(A) = p_n A^{n-1} + \dots + p_0 A^{-1} = 0$

$\Leftrightarrow A^{-1} = -\frac{1}{p_0}\left(p_n A^{n-1} + \dots + p_1 I\right)$ deg $n-1$

Ex 3

a) Objective is to show that for $X, Y \neq 0$, $AX = \lambda X \Rightarrow P(A)Y = P(\lambda)Y$

- Proove that $A^k X = \lambda^k X$: $A^{k+1} X = AA^k X = \lambda^k AX = \lambda^{k+1} X$ (by induction)
- Compute $P(A)X$: $P(A)X = \left(\sum_{k=0}^{n} p_k A^k\right) X = \sum_{k=0}^{n} p_k A^k X = \sum_{k=0}^{n} p_k \lambda^k X = P(\lambda)X$

b) Find a counter example with $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $P(T) = T^2 + 1$

$P(A) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ so $\chi_{P(A)} = (-t)^2$ and $\mathrm{eig}(P(A)) = \{0, 0\}$

Assume $P(\lambda) \in \mathrm{eig}(P(A))$, then we have to find a $\lambda \in \mathbb{R}$ such that $P(\lambda) = 0 \Leftrightarrow \lambda^2 + 1 = 0$

$\rightarrow$ Impossible as $\lambda \in \mathbb{R}$

so $\nexists X$ st $P(A)X = P(\lambda)X$

c) If $\mathbb{K} = \mathbb{C}$, show that with $\lambda \in \mathrm{eig}(A)$, $P(A)X = P(\lambda)X$ for $X \neq 0$

- $A \in \mathcal{M}_n(\mathbb{C})$ so A is triangularizable $\rightarrow$ Remark p. 61
  - $\rightarrow$ Let B upper-tri st $B = P^{-1}AP$
  - $\rightarrow$ Recall that $\mathrm{eig}(A) = \mathrm{eig}(B)$ $(*)$
  - $\rightarrow$ Recall that $\mathrm{eig}(B) = \mathrm{diag}(B)$ $(**)$
- Compute $P(B)$ $\rightarrow$ $\mathrm{eig}(P(A))$ are on the diag of $P(B)$

$$P(B) = \sum p_k \begin{pmatrix} \lambda_1 & \ast \\ 0 & \lambda_n \end{pmatrix}^k \underset{(*)+(**)}{=} \sum p_k \begin{pmatrix} \lambda_1^k & \ast \\ 0 & \lambda_n^k \end{pmatrix} = \sum \begin{pmatrix} p_k \lambda_1^k & \ast \\ 0 & p_k \lambda_n^k \end{pmatrix} = \begin{pmatrix} P(\lambda_1) & \ast \\ 0 & P(\lambda_n) \end{pmatrix}$$

So $\mathrm{eig}(P(B)) \overset{(**)}{=} P(\lambda) \overset{(*)}{=} \mathrm{eig}(P(A))$

Ex 4 $\quad A = \begin{pmatrix} 0 & 0 & c \\ 0 & b & d \\ a & 0 & 0 \end{pmatrix} \in \mathcal{M}_n(\mathbb{R})$

a) $eig(A) \equiv$ roots of χ_A

$$\chi_A(t) = \begin{vmatrix} -t & 0 & c \\ 0 & b-t & d \\ a & 0 & -t \end{vmatrix} = -t \begin{vmatrix} b-t & d \\ 0 & -t \end{vmatrix} + c \begin{vmatrix} 0 & b-t \\ a & 0 \end{vmatrix} = (b-t)(t^2 - ca)$$

- If $ca > 0$, $eig(A) = \{b, \sqrt{ca}, -\sqrt{ca}\}$
- If $ca < 0$, $eig(A) = \{b, i\sqrt{-ca}, -i\sqrt{-ca}\}$
- If $ca = 0$, $eig(A) = \{b, 0\}$

b) A is triangularizable on $\mathbb{R}$ $\Leftrightarrow$ χ_A splits on $\mathbb{R}$ $\leadsto$ Theorem 3

$\longrightarrow$ only true if $ca \geqslant 0$

c) $A = \begin{pmatrix} 0 & 0 & c \\ 0 & b & 0 \\ a & 0 & 0 \end{pmatrix}$ with $ac > 0$ so $\chi_A(t) = (b-c)(t - \sqrt{ac})(t + \sqrt{ac})$

- Observe that $\sqrt{ac} \neq -\sqrt{ac}$
- If in addition if $b^2 \neq ac$, $eig(A)$ are distinct $\Rightarrow$ A is diag

⚠ A is diag $\not\Rightarrow$ $eig(A)$ are distinct

- If $b^2 = ac$, $m_b = 2$ so we must have $\dim(E_b) = 2$ $\leadsto$ Theorem 2

Let $X \in \ker(A - bI)$, then $(A - bI)X = 0 \Leftrightarrow \begin{cases} -b x_1 + c x_3 = 0 \\ a x_1 - b x_3 = 0 \\ x_2 \in \mathbb{R} \end{cases}$

$$\Leftrightarrow \begin{cases} x_1 = \frac{c}{b} x_3 \\ x_3\left(\frac{ca}{b} - b\right) = x_3 \times 0 = 0 \\ x_2 \in \mathbb{R} \end{cases} \Leftrightarrow \begin{cases} x_1 = \frac{c}{b} x_3 \\ x_2, x_3 \in \mathbb{R} \end{cases}$$

$\uparrow b^2 = ac$

So $X \in \mathrm{Vect}\left\{\begin{pmatrix} c/b \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right\} \Rightarrow \dim(E_b) = 2 \Rightarrow A$ is diagonalizable

Ex 5 $\quad A = \begin{pmatrix} 1 & a & a \\ -1 & 1 & -1 \\ 1 & 0 & 2 \end{pmatrix}, \quad a \in \mathbb{R}$

a). eig(A) distinct $\Rightarrow$ A is diagonalizable $\rightarrow$ Prop 5

$\dim(E_{\lambda_i}) = m_i \iff$ A is diagonalizable $\rightarrow$ Theorem 2

. Compute $\chi_A(t) = (1-t)^2(2-t) \longrightarrow \text{eig}(A) = \begin{cases} 1 & \text{with } m_1 = 2 \\ 2 & \text{with } m_2 = 1 \end{cases}$

• Find $\dim(E_2)$: Let $X \in E_2 = \text{Ker}(A - 2I)$, ie $AX = 2X \iff \begin{cases} x_1 = 0 \\ -x_2 = x_3 \\ x_3 \in \mathbb{R} \end{cases}$

$\longrightarrow E_2 = \text{Vect}\left\{\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}\right\}$ so $\dim(E_2) = m_2 = 1$

• Find $\dim(E_1)$: Let $X \in E_1 = \text{Ker}(A - 1I)$, ie $AX = X \iff \begin{cases} a x_1 = a x_2 \\ x_3 = -x_1 \\ x_1 \in \mathbb{R} \end{cases}$

$\rightarrow$ Case $a \neq 0$: $x_1 = x_2 = -x_3$ so $E_1 = \text{Vect}\left\{\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}\right\}$ and $\dim(E_1) = 1 \neq m_1$

$\rightarrow$ Case $a = 0$: $\begin{cases} x_1 = -x_3 \\ x_1, x_2 \in \mathbb{R} \end{cases}$ so $E_1 = \text{Vect}\left\{\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right\}$ and $\dim(E_1) = 2 = m_1$

CNS : $a = 0 \iff$ A is diagonalizable

b) Find P st $\quad P^{-1}AP = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$: $P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{pmatrix}$

$\quad E_1 \quad E_2 \qquad\qquad E_1 \quad E_2$

Ex 6 $\quad A = \begin{pmatrix} 1 & 2 & -2 \\ 2 & -3 & 2 \\ -2 & 2 & 1 \end{pmatrix}$

a) $A \in \mathbb{R}$ + A symmetric $\Rightarrow$ A is diagonalizable $\rightsquigarrow$ Theorem 6

b) P ortho with $P^{T} A P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -5 \end{pmatrix}$: $\mathrm{eig}(A) = \{1, 3, -5\}$

• Find $X \in E_1 \Leftrightarrow (A - 1I)X = 0$

$$\Leftrightarrow \begin{cases} \quad 2x_2 - 2x_3 = 0 \\ 2x_1 - 4x_2 + 2x_3 = 0 \\ -2x_1 + 2x_2 = 0 \end{cases} \Leftrightarrow x_1 = x_2 = x_3$$

So $X = x_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ and $\mathrm{Vect}(E_1) = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$

• Find $X \in E_3 \Leftrightarrow (A - 3I)X = 0$

$$\Leftrightarrow \begin{cases} -2x_1 + 2x_2 - 2x_3 = 0 \\ 2x_1 - 6x_2 + 2x_3 = 0 \\ -2x_1 + 2x_2 - 2x_3 = 0 \end{cases} \Leftrightarrow \begin{cases} x_2 = 0 \\ x_1 = -x_3 \end{cases}$$

So $X = x_1 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ and $\mathrm{Vect}(E_3) = \left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\}$

• Find $X \in E_{-5} \Leftrightarrow (A + 5I) = 0$

$$\Leftrightarrow \begin{cases} 6x_1 + 2x_2 - 2x_3 = 0 \\ 2x_1 + 2x_2 + 2x_3 = 0 \\ -2x_1 + 2x_2 + 6x_3 = 0 \end{cases} \Leftrightarrow \begin{cases} x_2 = -2x_1 \\ x_2 = -2x_3 \end{cases}$$

So $X = x_1 \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$ and $\mathrm{Vect}(E_{-5}) = \left\{ \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \right\}$

- $P = \begin{pmatrix} Vect(E_1) & Vect(E_3) & Vect(E_{-5}) \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & -2 \\ 1 & -1 & 1 \end{pmatrix}$

  $\langle p_i, p_j \rangle = 0 \quad i \neq j$ so P orthogonal

- To have $P^{-1} = P^T$, we must normalize P, ie $p_{ij} \longrightarrow \dfrac{p_{ij}}{\|p_{\cdot j}\|_2}$

  $\tilde{P} = \begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \end{pmatrix} \rightarrow$ we have $P^T = P^{-1}$ (optional to check)

Remark: $\|x\|_2 = \sqrt{\sum_{i=1}^{n} x_i^2}$

Ex 7: $A = \begin{pmatrix} 3 & -4/3 \\ 3 & -2 \end{pmatrix}$

1) • Compute $\chi_A(t) = \begin{vmatrix} 3-t & -4/3 \\ 3 & -2-t \end{vmatrix} = (t+1)(t-2)$

• We want $R_n(T)$ s.t $\underbrace{T^n}_{\deg n} = Q_n(T)\,\underbrace{\chi_A(T)}_{\deg 2} + \underbrace{R_n(T)}_{\deg \leq 1}$ $\rightsquigarrow a_n T + b_n$

• Property of χ_A : $\chi_A(2) = \chi_A(-1) = 0$

$$\begin{cases} 2^n = 0 + R_n(2) = 2a_n + b_n \\ (-1)^n = 0 + R_n(-1) = -a_n + b_n \end{cases}$$

$$\Leftrightarrow \begin{cases} a_n = 2^n - (-1)^n/3 \\ b_n = 2^n + 2(-1)^n/3 \end{cases} \quad \text{so} \quad R_n(t) = \left(\frac{2^n - (-1)^n}{3}\right)t + \left(\frac{2^n + 2(-1)^n}{3}\right)$$

2) Recall $\chi_A(A) = 0$ so $T \equiv A \rightsquigarrow A^n = 0 + R_n(A)$

And $R_n(A) = \left(\frac{2^n - (-1)^n}{3}\right) A + \left(\frac{2^n + 2(-1)^n}{3}\right) I$

Ex 8

$$A = \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix}$$

a) $A \in \mathcal{M}_n(\mathbb{R})$ and A is symmetric $\Longrightarrow$ A is diagonalizable

b) P_A divides χ_A Theorem 8

- Compute $\chi_A(t) = \begin{vmatrix} 1-t & -1 & -1 \\ -1 & 1-t & -1 \\ -1 & -1 & 1-t \end{vmatrix} = \begin{vmatrix} 1-t & -1 & 0 \\ -1 & 1-t & t-2 \\ -1 & -1 & 2-t \end{vmatrix} = (-1-t)(t-2)^2$
- P_1 divides P_2 $\Rightarrow$ $\text{roots}(P_1) \subseteq \text{roots}(P_2)$ $\Rightarrow$ $P_2 = P_1 \times \prod (t-\lambda_i)$
- Polynomes dividing χ_A :

$\rightarrow$ $P_1 = -T-1$: $P_1(A) = -A - I \neq 0$

$\rightarrow$ $P_2 = T-2$: $P_2(A) = A - 2I \neq 0$

$\rightarrow$ $P_3 = (-T-1)(T-2)$: $P_3(A) = 0$ with deg 2 $\leftarrow$ this is P

$\rightarrow$ $P_4 = (-T-1)(T-2)^2$: $P_4(A) = 0$ with deg 3

c) $\deg(P) = 2$ so $T^n = Q_n(T)\, P(T) + R_n(T)$ deg ≤ 1 $R_n = a_n T + b_n$

$$\begin{cases} P(2) = 0 \\ P(-1) = 0 \end{cases} \Rightarrow \begin{cases} 2^n = 2a_n + b_n \\ (-1)^n = -a_n + b_n \end{cases} \Longrightarrow \text{find } a_n \text{ and } b_n \text{ as in ex 7}$$

Moreover, $P(A) = 0$ so $A^n = a_n A + b_n I$

Ex 10

a) $\chi_A(t) = 3 - 3t + t^2 \longrightarrow$ not splitted so A is not triangularizable on $\mathbb{R}$

b) $\chi_B(t) = (1-t)^2(2-t)$

- $Bx = 2x \iff \begin{cases} x_1 = x_3 \\ x_2 = 0 \end{cases}$ so $E_2 = \text{Vect}\left\{\begin{pmatrix}1\\0\\1\end{pmatrix}\right\} \longrightarrow$ 1 bloc

  $Bx = x \iff \begin{cases} x_1 = x_2 \\ x_3 = 0 \end{cases}$ $E_1 = \text{Vect}\left\{\begin{pmatrix}1\\1\\0\end{pmatrix}\right\} \longrightarrow$ 1 bloc

- $\begin{array}{cc} \begin{matrix} Bu & Bv & Bw \end{matrix} & \\ \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} & \begin{matrix} u \\ v \\ w \end{matrix} \end{array} \rightsquigarrow \begin{cases} Bu = 2u \\ Bv = v \\ Bw = v + w \end{cases}$

  $P = (u \ \ v \ \ w)$ with $\begin{cases} u \in E_2 \rightsquigarrow u = (1 \ 0 \ 1) \\ v \in E_1 \\ w \in E_1^2 \setminus E_1 \quad \text{since otherwise } v \,/\!/\, w \end{cases}$

- $x \in \text{Ker}((B-I)^2) \iff \begin{cases} x_1 = x_2 \\ x_3 \in \mathbb{R} \end{cases}$ so $\text{Ker}((B-I)^2) = \text{Vect}\left\{\begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}\right\}$

  ? w ?

  $\to$ We must have $w \notin \text{Ker}(B-I)$

  $(B-I)\begin{pmatrix}1\\1\\0\end{pmatrix} = 0 \qquad (B-I)\begin{pmatrix}0\\0\\1\end{pmatrix} \neq 0$ so we set $w = (0 \ 0 \ 1)$

  $\to$ Compute v as $v = Bw - w = (1 \ 1 \ 0)$

- $P = (u \ \ v \ \ w) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$ and $P^{-1} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ -1 & 1 & 0 \end{pmatrix}$

c) $\chi_C(t) = -t(t-2)^2$

- $Cx = 0x \iff \begin{cases} x_1 = -x_2 \\ x_1 = x_3 \end{cases}$ so $E_0 = \text{Vect}\left\{\begin{pmatrix}1\\-1\\1\end{pmatrix}\right\} \longrightarrow$ 1 block

  $Cx = 2x \iff \begin{cases} x_1 = x_2 \\ x_1 = -x_3 \end{cases}$ so $E_2 = \text{Vect}\left\{\begin{pmatrix}1\\1\\-1\end{pmatrix}\right\} \longrightarrow$ 1 block

- $\begin{array}{cc} \begin{matrix} Cu & Cv & Cw \end{matrix} & \\ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix} & \begin{matrix} u \\ v \\ w \end{matrix} \end{array} \rightsquigarrow \begin{cases} Cu = 0 \\ Cv = 2v \\ Cw = v + 2w \end{cases}$ and $P = (u \ \ v \ \ w)$ with $\begin{cases} u \in E_0 \rightsquigarrow u = \begin{pmatrix}1\\-1\\1\end{pmatrix} \\ v \in E_2 \\ w \in E_2^2 \setminus E_2 \end{cases}$

- $x \in \text{Ker}((C-2I)^2) \iff x_3 = -2x_1 + x_2$ so $E_2^2 = \text{Vect}\left\{\begin{pmatrix}1\\0\\-2\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}\right\}$

  ? w ?

→ We must have $w \notin E_2$: $(C-2I)\begin{pmatrix}1\\0\\-2\end{pmatrix} \neq 0$ so we can choose $w = (1 \;\; 0 \;\; -2)$

→ Compute v as $v = (C-2I)\,w = (-1 \;\; -1 \;\; 1)$

- $P = (u \;\; v \;\; w) = \begin{pmatrix} 1 & -1 & 1 \\ -1 & -1 & 0 \\ 1 & 1 & -2 \end{pmatrix}$ and $P^{-1} = \begin{pmatrix} 1/2 & -1/4 & 1/4 \\ -1/2 & -3/4 & -1/4 \\ 0 & -2/4 & -2/4 \end{pmatrix}$

Ex 12

a) • $\chi_A = \chi_B \Rightarrow A \overset{\text{similar}}{\sim} B$

$\rightarrow \chi_A(t) = (3-t)\left[(4-t)(2-t)+1\right] = (3-t)^3$

$\rightarrow \chi_B(t) = (3-t)^3$ because upper-tri

• $\begin{array}{cccc} Au & Av & Aw & \\ \end{array}$

$$\begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{pmatrix}\begin{array}{c} u \\ v \\ w \end{array} \rightsquigarrow \begin{cases} Au = 3u \\ Av = u + 3v \\ Aw = v + 3w \end{cases} \Rightarrow \begin{cases} (A-3I)u = 0 \\ (A-3I)^2 v = 0 \\ (A-3I)^3 w = 0 \end{cases}$$

On cherche (u,v,w) base donc $\begin{cases} u \in E_3 \\ v \in E_3^2 \setminus E_3 \\ w \in E_3^3 \setminus E_3^2 \end{cases}$

• $(A-3I)x = 0 \iff x \in \mathrm{Vect}\left\{\begin{pmatrix}0\\1\\-1\end{pmatrix}\right\} \longrightarrow E_3 = \mathrm{Vect}\left\{\begin{pmatrix}0\\1\\-1\end{pmatrix}\right\}$

$(A-3I)^2 x = 0 \iff x \in \mathrm{Vect}\left\{\begin{pmatrix}0\\1\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix}\right\} \longrightarrow E_3^2 = \mathrm{Vect}\left\{\begin{pmatrix}0\\1\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix}\right\}$

$(A-3I)^2 x = 0 \iff x \in \mathrm{Vect}\left\{\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix}\right\} \longrightarrow E_3^3 = \mathrm{Vect}\left\{\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix}\right\}$

? ? w ?

• $w \in E_3^3 \setminus E_3^2$ so we must choose $w = (1\ 0\ 0)$

• Compute $\begin{cases} v \text{ as } & v = (A-3I)w = (0\ 2\ -1) \\ u \text{ as } & u = (A-3I)v = (0\ 1\ -1) \end{cases}$

• $P = (u\ v\ w) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 2 & 0 \\ -1 & -1 & 0 \end{pmatrix} \qquad P^{-1} = \begin{pmatrix} 0 & -1 & -2 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}$

b) $A = P^{-1}BP \rightarrow A^n = P^{-1}B^nP$

Find B^n for $B = \begin{pmatrix} \ddots & 1 & 0 \\ & \ddots & \ddots \\ 0 & & 1 \end{pmatrix}$

$B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} + 3I = N + 3I$ and $N^k = 0 \ \forall k \geqslant 3$

As N and $3I$ commute, $\quad C_n^k = \dfrac{n!}{k!(n-k)!}$

$$(N+3I)^n = \sum_{k=0}^{n} C_n^k N^k (3I)^{n-k} = C_n^0 N^0 (3I)^n + C_n^1 N (3I)^{n-1} + C_n^2 N^2 (3I)^{n-2}$$

$$= 3^n I + 3^{n-1} n N + 3^{n-2} \frac{n(n-1)}{2} N^2$$

Ex 14

a) A diagonalizable $\Leftrightarrow \forall i, \quad \dim(E_{\lambda_i}) = m_i$

⚠ χ_A not splitted but A is potentially diagonalizable

- $\chi_A(t) = (-t)^2(t+1)(t-1)$
- Let $X \neq 0$ st $X \in \ker(A + 1I)$,

$\rightarrow (A+I)X = 0 \Leftrightarrow \begin{cases} x_1 + x_4 = 0 \\ x_2 + x_3 = 0 \\ x_2 + x_3 = 0 \\ x_4 = 0 \end{cases} \Leftrightarrow \begin{cases} x_1 = x_4 = 0 \\ x_2 = -x_3 \end{cases}$

$\rightarrow X \in \mathrm{Vect}\left\{\begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}\right\}$ so $\dim(E_1) = 1 \neq 2$

$\rightarrow$ A is non diag (A is triangularizable since χ_A splits)

b) Matrices : ☐ B_1 ☐ B_2 ☒ B_3

☐ B_4 ☒ B_5

- **B_1 and B_4**: Triangular so eigenvalues on the diagonal

$\mathrm{eig}(B_1) = \{0, 1, 1, -1\} \neq \mathrm{eig}(A) = \{0, 0, 1, -1\}$

$\mathrm{eig}(B_4) = \{0, 0, 0, -1\} \neq \mathrm{eig}(A) = \{0, 0, 1, -1\}$

$\rightarrow \nexists U$ such that $U^{-1}AU = B_1$ or $U^{-1}AU = B_4$

- **B_2**: Assume that $\exists U$ such that $U^{-1}AU = B_2$

Then A is diagonalizable

$\rightarrow$ Contradiction with a)

So $\nexists U$ such that $U^{-1}AU = B_2$

- **B_3**: Yes, it is the Jordan's form which can always be constructed if A is triangularizable

• B_5: ⚠ A is triangularizable but not in a unique way! Why not try to find U such that $U^{-1}AU = B_5$?

(*) A is triangularizable so $\exists\, U_1$ such that $U_1^{-1} A U_1 = T_A$ where T_A is a Jordan matrix

(**) B_5 is triangularizable so $\exists\, U_2$ such that $U_2^{-1} B_5 U_2 = T_B$ where T_B is a Jordan matrix

If $T_A = T_B$, then $U_1^{-1} A U_1 = U_2^{-1} B_5 U_2$

$$\Leftrightarrow \underbrace{U_2 U_1^{-1}}_{U_3^{-1}} A \underbrace{U_1 U_2^{-1}}_{U_3} = B_5$$
$$\Leftrightarrow U_3^{-1} A U_3 = B_5$$

So $\exists\, U$ such that $U^{-1} A U = B_5$

• Now, lets prove that $T_A = T_B$

$T_A = T_B \Leftrightarrow \dim(E_\lambda)$ are the same for A and B

↳ Same # of blocks per eigenvalues so $T_A = T_B$

→ For A: $\begin{cases} \dim(E_0) = 1 \\ \dim(E_1) = 1 \\ \dim(E_{-1}) = 1 \end{cases}$ so $T_A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$

→ For B: $X \in E_0 \Leftrightarrow \begin{cases} x_2 = 0 \\ x_3 = 0 \\ -x_4 = 0 \end{cases}$ so $E_0 = \text{Vect}\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}\right\}$ and $\dim(E_0) = 1$

Moreover, $\dim(E_1) = \dim(E_{-1}) = 1$

so $T_B = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$

So $T_A = T_B$