Screen & Relax

Accelerating the resolution of the Elastic-Net by safe identification of the solution support

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Ingredients :

- An observation $y \in \mathbb{R}^m$
- A dictionary $A = [a_i]_{i=1}^n \in \mathbb{R}^{m \times n}$ (columns $\equiv$ atoms)

Sparse-linear problem

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The vector x weights each atom in the approximation.

The Elastic-net problem

Idea : Consider the problem

Elastic-net

$$\mathsf{x}^\star = \operatorname{argmin}_{\mathsf{x}} \left\{ \mathrm{P}(\mathsf{x}) = \frac{1}{2}\|\mathsf{y} - \mathsf{A}\mathsf{x}\|_2^2 + \lambda\|\mathsf{x}\|_1 + \frac{\gamma}{2}\|\mathsf{x}\|_2^2 \right\} \qquad (\mathcal{P})$$

where $\lambda > 0$ and $\gamma > 0$ are two hyperparameters.

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where $\lambda > 0$ and $\gamma > 0$ are two hyperparameters.

Properties :

- Ensures a good approximation
- Induces sparsity
- Good statistical properties
- Convex problem

Screening and relaxing tests

Main idea

Sparse problem :

- Where are zero entries of $x^\star$?
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$\rightarrow$ Spoiler alert : yes !

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Fenchel dual of $(\mathcal{P})$

$$u^\star = \mathrm{argmax}_u \left\{ D(u) = \tfrac{1}{2}\|y\|_2^2 - \tfrac{1}{2}\|y-u\|_2^2 - \tfrac{1}{2\gamma}\|[|A^T u| - \lambda]_+\|_2^2 \right\} \quad (\mathcal{D})$$

Screening and relaxing tests

Optimality conditions :

$$\begin{aligned} |\mathsf{a}_i^{\mathrm{T}}\mathsf{u}^\star| \leq \lambda \quad &\Longleftrightarrow \quad \mathsf{x}^\star(i) = 0 \\ |\mathsf{a}_i^{\mathrm{T}}\mathsf{u}^\star| > \lambda \quad &\Longleftrightarrow \quad \mathsf{x}^\star(i) \neq 0 \end{aligned}$$

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\end{aligned}
$$

$$\rightsquigarrow$$

Relaxed optimality condition : Let $\mathbb{S}(\mathsf{u}, r)$ be a spherical region containing $\mathsf{u}^\star$, then

$$
\begin{aligned}
|\mathsf{a}_i^{\mathrm{T}}\mathsf{u}| + r < \lambda \quad &\Longrightarrow \quad \mathsf{x}^\star(i) = 0 \qquad \text{(screening test)} \\
|\mathsf{a}_i^{\mathrm{T}}\mathsf{u}| - r > \lambda \quad &\Longrightarrow \quad \mathsf{x}^\star(i) \neq 0 \qquad \text{(relaxing test)}
\end{aligned}
$$

Dimensionality reduction

Problem reduction

With screening test

Zero entries of $x^\star$ can be discarded from the problem without changing the objective value.

Problem reduction

With screening test

Zero entries of $x^\star$ can be discarded from the problem without changing the objective value.

With relaxing test

Nonzero entries of $x^\star$ can be expressed as a linear combination of all the other entries.

Problem reformulation

Let $(\mathcal{S}_0, \mathcal{S}_\pm, \mathcal{S}_*)$ be subsets of zero, non-zero and unclassified indices of $\mathsf{x}^\star$:

$$\mathsf{x}^\star = \operatorname{argmin}_{\mathsf{x}} \left\{ \mathrm{P}(\mathsf{x}) = \tfrac{1}{2}\|\mathsf{y} - \mathsf{A}\mathsf{x}\|_2^2 + \lambda\|\mathsf{x}\|_1 + \tfrac{\gamma}{2}\|\mathsf{x}\|_2^2 \right\}$$

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Solve an *n* dimensional problem

$$\Big\downarrow$$

$$\begin{aligned} \mathsf{x}^\star_{\mathcal{S}_*} &= \operatorname{argmin}_{\mathsf{x}} \left\{ \tilde{\mathrm{P}}(\mathsf{x}) = \tfrac{1}{2}\|\tilde{\mathsf{y}} - \tilde{\mathsf{A}}\mathsf{x}\|_2^2 + \lambda\|\mathsf{x}\|_1 + \tfrac{\gamma}{2}\|\mathsf{x}\|_{\mathsf{M}}^2 \right\} \\ \mathsf{x}^\star_{\mathcal{S}_\pm} &= \mathsf{B}\mathsf{x}^\star_{\mathcal{S}_*} + \mathsf{b} \\ \mathsf{x}^\star_{\mathcal{S}_0} &= 0 \end{aligned}$$

Problem reformulation

Let $(\mathcal{S}_0, \mathcal{S}_\pm, \mathcal{S}_*)$ **be subsets of zero, non-zero and unclassified indices of** $x^\star$ **:**

$$x^\star = \operatorname{argmin}_x \left\{ \mathrm{P}(x) = \tfrac{1}{2}\|y - Ax\|_2^2 + \lambda\|x\|_1 + \tfrac{\gamma}{2}\|x\|_2^2 \right\}$$

Solve an n dimensional problem

$$\begin{aligned}
x^\star_{\mathcal{S}_*} &= \operatorname{argmin}_x \left\{ \tilde{\mathrm{P}}(x) = \tfrac{1}{2}\|\tilde{y} - \tilde{A}x\|_2^2 + \lambda\|x\|_1 + \tfrac{\gamma}{2}\|x\|_M^2 \right\} \\
x^\star_{\mathcal{S}_\pm} &= Bx^\star_{\mathcal{S}_*} + b \\
x^\star_{\mathcal{S}_0} &= 0
\end{aligned}$$

- Compute $\tilde{y}$, $\tilde{A}$, M, B and b (linear algebra operations)
- Solve an $n - |\mathcal{S}_0| - |\mathcal{S}_\pm|$ dimensional problem

Dynamic Screen & Relax principle

Algorithm 1: "Screen & Relax" solving procedure

Input: $\mathrm{x}^{(0)}$, A, y, λ, γ

$(\mathcal{S}_0, \mathcal{S}_\pm, \mathcal{S}_*) \leftarrow (\emptyset, \emptyset, \{1, \ldots, n\})$

while *convergence criterion is not met* **do**
    Update the current iterate
    Compute a new safe sphere
    Update $(\mathcal{S}_0, \mathcal{S}_\pm, \mathcal{S}_*)$ with screening and relaxing tests
    Update the problem data
    if $\mathcal{S}_* = \emptyset$ **then**
        The solution is available in closed form
    end
end

Some numerical results

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Setup : Percentage of instances solved up to a given accuracy for a fixed FLOPs budget.

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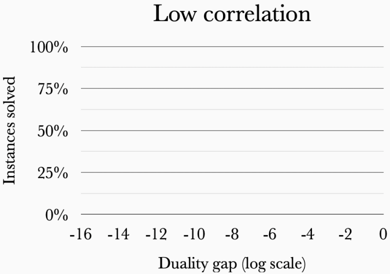

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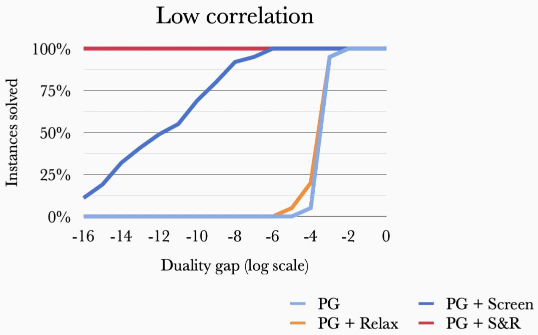

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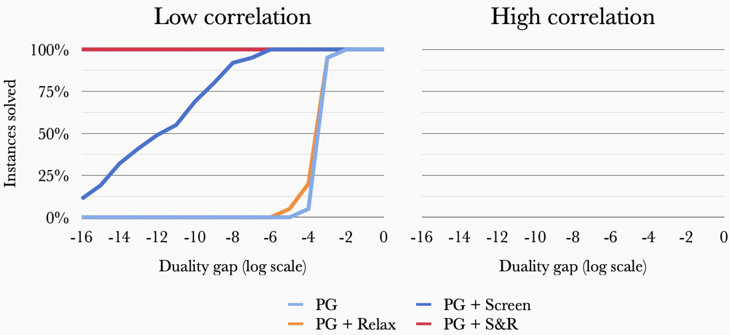

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