Screen & Relax

Accélérer la résolution de l'Elastic-Net par identification du support de la solution

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General context

Sparse problem

Ingredients of the problem

- A target y

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Rough formulation

Problem

Find x sparse such that $y \simeq \text{Model}(Ax)$

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Find x sparse such that $y \simeq \mathrm{Model}(Ax)$

Remark : Entries of x weight each atom in the linear combination.

The Elastic-Net problem

Formulation and properties

Target problem

Solve

$$x^\star = \arg\min_x \left\{ \mathrm{P}(x) = \underbrace{\tfrac{1}{2}\|y - Ax\|_2^2}_{f(Ax)} + \underbrace{\lambda(\sigma\|x\|_1 + \tfrac{1-\sigma}{2}\|x\|_2)}_{\lambda g(x)} \right\} \qquad (P)$$

where $\lambda > 0$ and $\sigma \in]0,1[$ are tuning hyperparameters.

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Properties of (P)

- Convex non-smooth problem
- Least-squares : Ensures a good reconstruction of the target
- ℓ_1-norm : Enforces sparsity
- ℓ_2-norm : Promotes desirable properties

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Solving (P)

- Broad class of solution methods (gradient-based, pivot-based, ...)
- Acceleration strategies (backtracking, screening tests, ...)

Screening and Relaxing tests

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- Can we accelerate a given solution method if we knew some ?

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Primal problem

$$\mathrm{x}^\star = \arg\min_{\mathrm{x}} \mathrm{P}(\mathrm{x}) \qquad (P)$$

$\equiv$

Dual problem

$$\mathrm{u}^\star = \arg\max_{\mathrm{u}} \mathrm{D}(\mathrm{u}) \qquad (D)$$

with optimality conditions linking $\mathrm{x}^\star$ and $\mathrm{u}^\star$

Optimality conditions

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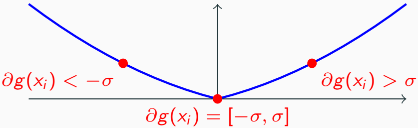

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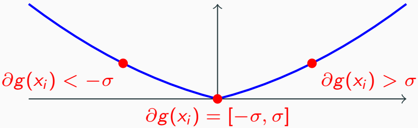

$$|\mathrm{a}_i^T \mathrm{u}^\star| \leq \lambda\sigma \quad \Longleftrightarrow \quad x_i^\star = 0$$
$$|\mathrm{a}_i^T \mathrm{u}^\star| > \lambda\sigma \quad \Longleftrightarrow \quad x_i^\star \neq 0$$

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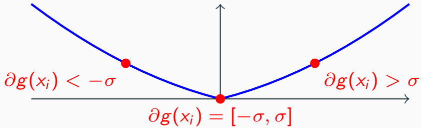

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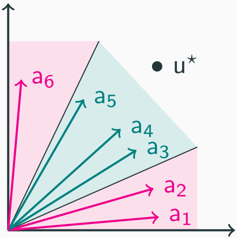

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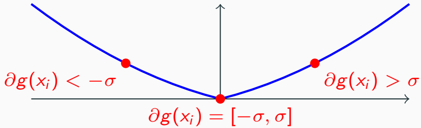

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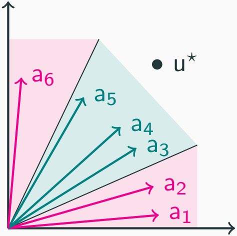

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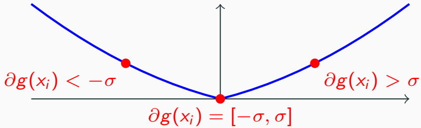

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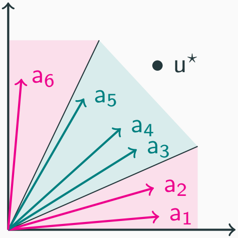

We can identify zeros and non-zeros in $x^\star$

... but we need $u^\star$!

Relaxed optimality condition

Let $u^\star \in \mathbb{S}(c, r)$, then

$$|a_i^T c| + r \leq \lambda\sigma \quad \Longrightarrow \quad x_i^\star = 0 \qquad \text{(screening test)}$$

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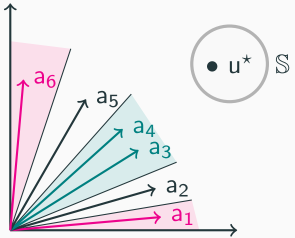

Dimensionality reduction

Leveraging test results

With screening test

Zero entries identified in $x^\star$ can be discarded from the problem without changing the objective value.

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Zero entries identified in $x^\star$ can be discarded from the problem without changing the objective value.

With relaxing test

Non-zero entries identified in $x^\star$ can be expressed as a linear combination of all the other entries.

Problem reformulation

Notations $(\mathcal{S}_0, \mathcal{S}_1, \bar{\mathcal{S}})$: Set of zero/non-zero/unclassified entries in $\mathbf{x}^\star$

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Initial problem

$$\mathsf{x}^\star = \operatorname{argmin}_{\mathsf{x}\in\mathbb{R}^n} \left\{ \mathrm{P}(\mathsf{x}) = \tfrac{1}{2}\|\mathsf{y} - \mathsf{A}\mathsf{x}\|_2^2 + \lambda(\sigma\|\mathsf{x}\|_1 + \tfrac{1-\sigma}{2}\|\mathsf{x}\|_2^2) \right\}$$

n dimensional problem

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n dimensional problem

$$\rightsquigarrow$$

Reduced problem

$$\begin{cases} \mathsf{x}^\star_{\bar{\mathcal{S}}} &= \operatorname{argmin}_{\mathsf{x}\in\mathbb{R}^{|\bar{\mathcal{S}}|}} \left\{ \tilde{\mathrm{P}}(\mathsf{x}) = \tfrac{1}{2}\|\tilde{\mathsf{y}} - \tilde{\mathsf{A}}\mathsf{x}\|_2^2 + \lambda(\sigma\|\mathsf{x}\|_1 + \tfrac{1-\sigma}{2}\|\mathsf{x}\|_{\mathsf{M}}^2) \right\} \\ \mathsf{x}^\star_{\mathcal{S}_1} &= \mathsf{B}\mathsf{x}^\star_{\bar{\mathcal{S}}} + \mathsf{b} \\ \mathsf{x}^\star_{\mathcal{S}_0} &= 0 \end{cases}$$

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$n - |\mathcal{S}_0| - |\mathcal{S}_1|$ dimensional problem (similar structure)

Some linear algebra operations (negligible computational cost)

Dynamic Screen & Relax principle

Algorithm 1: "Screen & Relax" solving procedure

Input: A, y, λ, σ, $x^{(0)}$

$(\mathcal{S}_0, \mathcal{S}_1, \bar{\mathcal{S}}) \leftarrow (\emptyset, \emptyset, \{1, \ldots, n\})$

while *convergence criterion is not met* **do**

Dynamic Screen & Relax principle

Algorithm 2: "Screen & Relax" solving procedure

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    Update $(\mathcal{S}_0, \mathcal{S}_1, \bar{\mathcal{S}})$ with screening and relaxing tests

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Algorithm 4: "Screen & Relax" solving procedure

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while *convergence criterion is not met* **do**

- Update the current iterate
- Update $(\mathcal{S}_0, \mathcal{S}_1, \bar{\mathcal{S}})$ with screening and relaxing tests
- Reduce the problem

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Algorithm 5: "Screen & Relax" solving procedure

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  Reduce the problem

  if $\bar{\mathcal{S}} = \emptyset$ **then**

    The solution is available in closed form

  end

end

Some numerical results

Experimental setup

Synthetic data generation

- Generate the dictionary A randomly
- Generate a k-sparse vector $x^{\dagger}$
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- Accelerated proximal-gradient algorithm
- Same with screening tests
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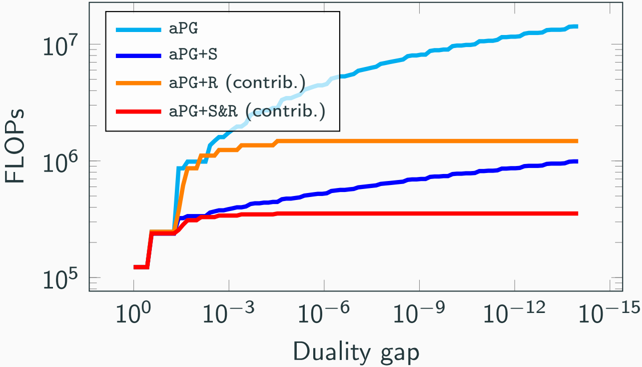
Metrics

- Duality gap : How close is the objective from its optimal value
- FLOPs : Number of linear algebra operations performed

Classical convergence scheme



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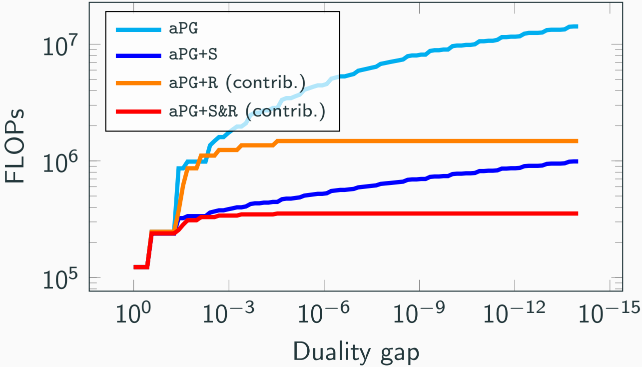

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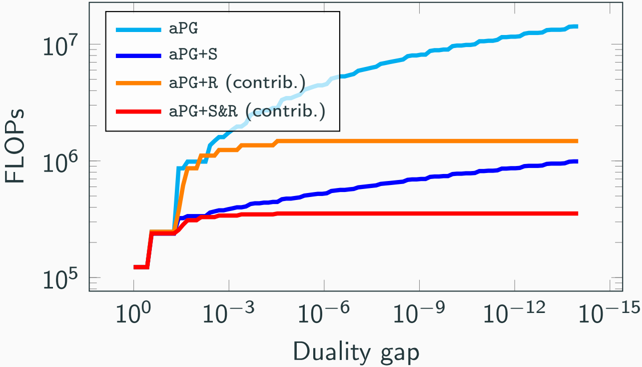

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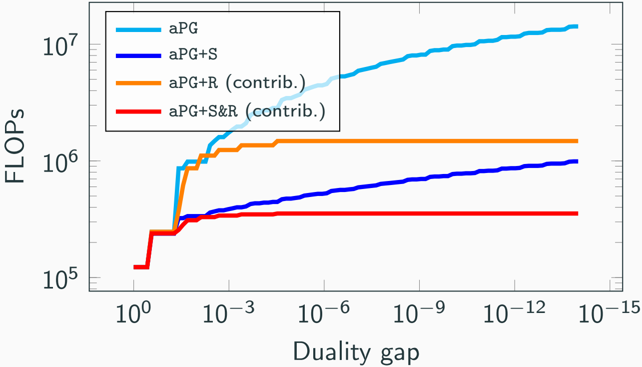

- Complexity reduction with screening and/or relaxing
- Convergence to machine precision at some point
- Gains depend on the sparsity in $x^\star$
    - Too few non-zeros : relaxing has little impact
    - Too many non-zeros : problem updates become binding