Discrete Optimization Methods for L0-Penalized Sparse Problems

Theo Guyard with Cedric Herzet, Clement Elvira and Ayse-Nur Arslan
Inria and INSA, Rennes, France
theo.guyard@insa-rennes.fr

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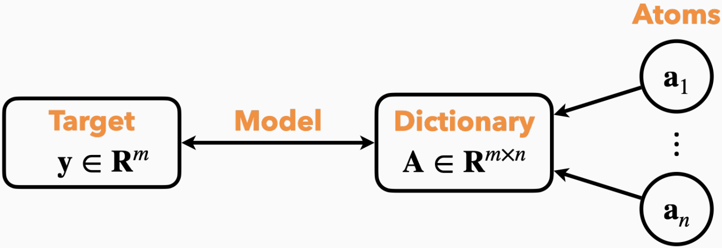
Sparse problems

Sparse problems

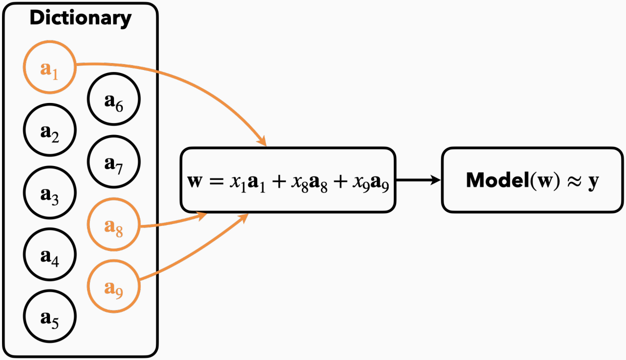
Sparse model components

- Target $y \in \mathbb{R}^m$ to be decomposed
- Atoms $a_i \in \mathbb{R}^m$ as basic components
- Dictionary $A \in \mathbb{R}^{m \times n}$ gathering the atoms
- Model linking y and A



Sparse problems

Objective : Fit the target with a sparse combination of atoms



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Find x sparse such that $y \simeq \mathrm{Model}(Ax)$

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$$p^\star = \min F(\mathrm{y}, \mathrm{A}\mathrm{x}) + \lambda\|\mathrm{x}\|_0 \qquad (P)$$

- F ensures the model fitting
- $\|\cdot\|_0$ enforces sparsity
- $\lambda > 0$ controls the trade-off

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(P) is NP-hard !

Mixed-Integer Program (MIP) formulation

MIP formulations

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- Optimization problem
- Objective and constraints
- Continuous and integer variables

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- Linear and quadratic expressions (GLPK, Cbc, CPLEX, Gurobi, ...)
- Conic expressions (Knitro, Mosek, ...)
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The ℓ_0-norm is not solver friendly !

Specially-Ordered Set (SOS) approach

SOS approach

Idea : Model the nullity in x using a binary variable.

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Let $z \in \{0, 1\}$, then the constraint

$$x(1 - z) = 0 \iff (x, z) \in SOS_1$$

models the nullity in x entries.

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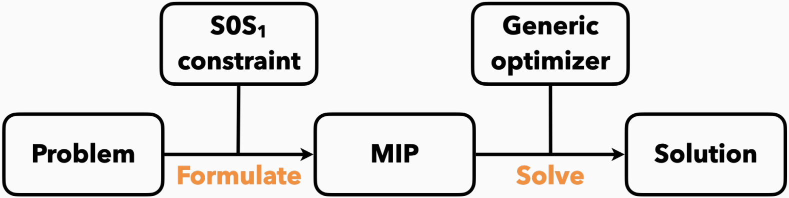
Solver friendly formulation

$$p^\star = \begin{cases} \min & F(\mathrm{y}, \mathrm{Ax}) + \lambda \mathbf{1}^\top \mathrm{z} \\ \text{s.t.} & (\mathrm{x}, \mathrm{z}) \in SOS_1 \\ & \mathrm{x} \in \mathbb{R}^n, \ \mathrm{z} \in \{0, 1\}^n \end{cases} \qquad (P_{sos})$$

SOS approach



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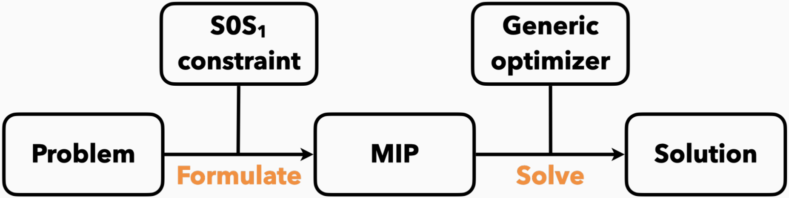

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- ✓ Standard formulation
- ✓ Explicitly models sparsity
- ✓ Simple

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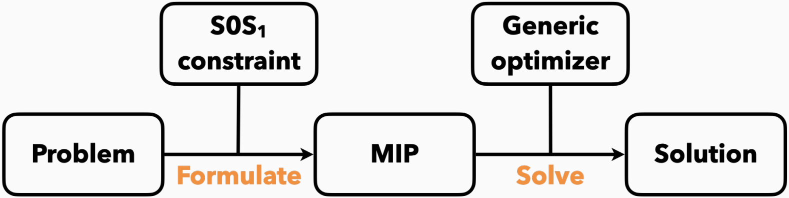

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Cons

- ✗ Twice more variables
- ✗ Handling $\mathtt{SOS}_1$ constraints
- ✗ New layer of complexity

Branch-and-Bound (BnB) approach

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BnB algorithms

- Separation of the search space
- Enumerate all feasible solutions
- Use rules to discard irrelevant candidates

$\rightarrow$ Explore a decision tree and prune uninteresting nodes

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Particularization to (P)

- Node defined as $\nu = (\mathcal{S}_0, \mathcal{S}_1, \bar{\mathcal{S}})$
  - $\mathcal{S}_0$: indices of x fixed to zero
  - $\mathcal{S}_1$: indices of x fixed to non-zero
  - $\bar{\mathcal{S}}$: indices of x not fixed yet
- Decisions about the nullity in the problem variable

BnB approach

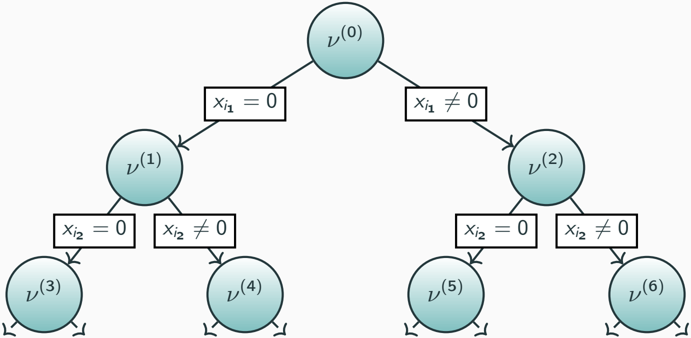
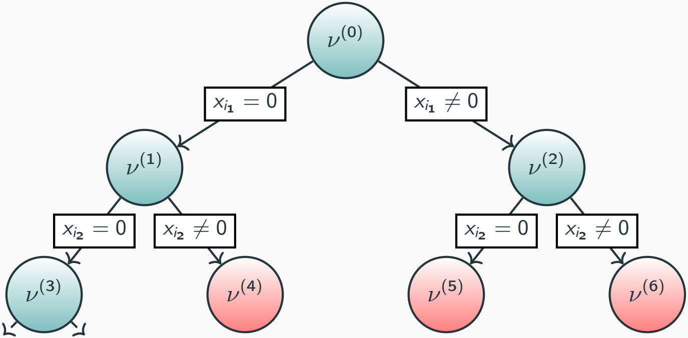


Figure 1: Example of BnB tree.

BnB approach



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Problem at node ν

$$p^\nu = \begin{cases} \min & F(\mathbf{y}, \mathbf{A}\mathbf{x}) + \lambda \|\mathbf{x}\|_0 \\ \text{s.t.} & \mathbf{x}_{\mathcal{S}_0} = 0,\ \mathbf{x}_{\mathcal{S}_1} \neq 0 \end{cases} \tag{P^ν}$$

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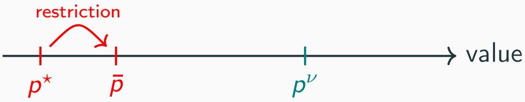

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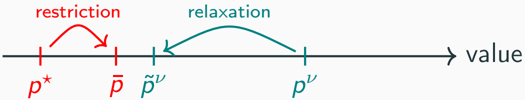

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Finding some upper bound $\bar{p} \geq p^\star$

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Finding some lower bound $\tilde{p}^\nu \leq p^\nu$

- Relax the objective function of (P^ν) into a convex function

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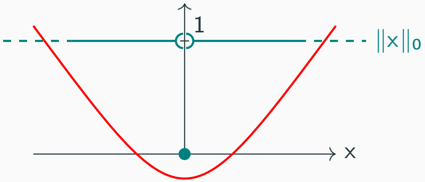

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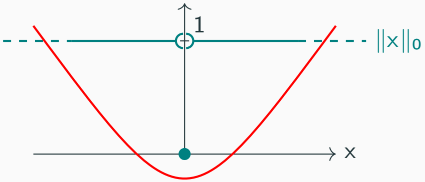

No valid convex relaxation !

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Perturbated ℓ_0-penalized problem

Rather consider

$$p^\star = \min F(y, Ax) + \lambda(\|x\|_0 + G(x)) \qquad (P_G)$$

where $G(\cdot)$ is a convex function.

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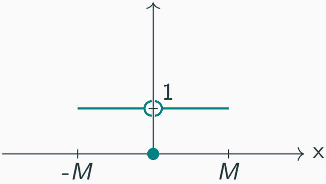

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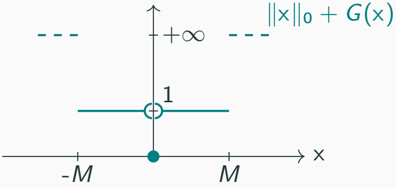

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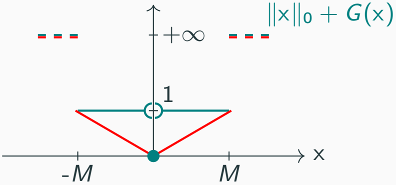

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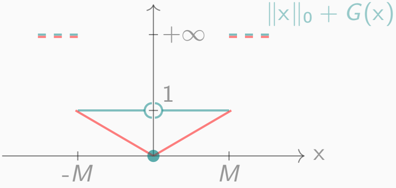

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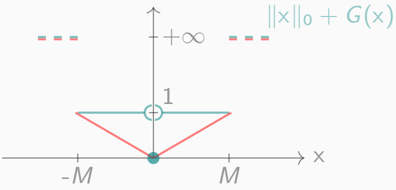

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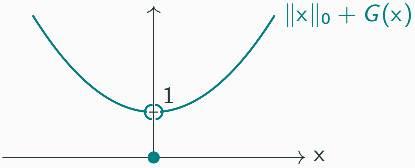

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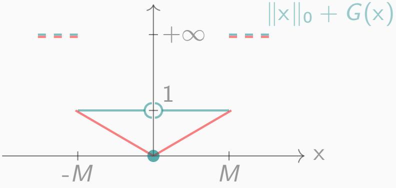

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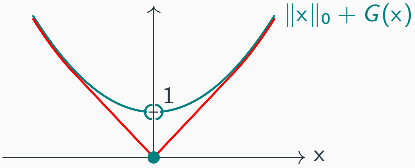

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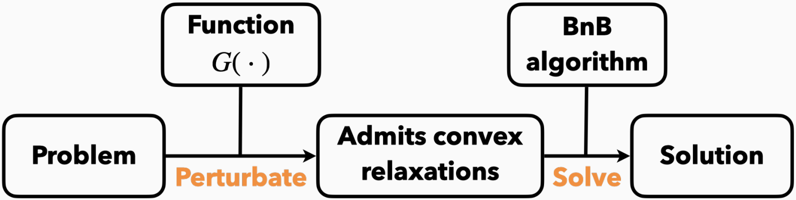

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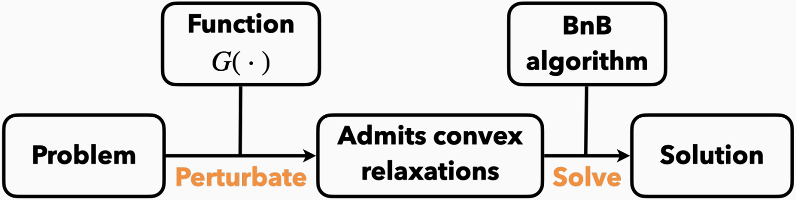

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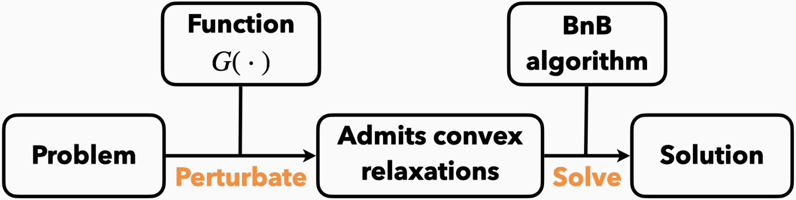

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- ✗ New hyperparameters to tune

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- Using a Branch-and-Bound procedure
  - ✓ Specialized solution method
  - ✓ Efficient to solve the problem
  - ✗ Have to perturb the problem

Advertizing time

https://github.com/TheoGuyard/El0ps.jl