Peeling for L0-Regularized Least-Squares

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L0-Regularized Least-Squares

Sparse linear model

Linear approximation $\longleftrightarrow$ **Sparse solution**

Sparse linear model

Machine Learning
Network design
Signal processing
...

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ℓ_0-regularized least-squares

$$\min_x \quad \frac{1}{2}\|y - Ax\|_2^2 + \lambda\|x\|_0$$

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$$\min_x \quad \frac{1}{2}\|y - Ax\|_2^2 + \lambda\|x\|_0$$
$$\text{s.t.} \quad L \leq x \leq U$$

Bounds needed for numerical tractability

Sparse linear model

Linear approximation ⟷ **Sparse solution**

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$$\min_x \quad \frac{1}{2}\|y - Ax\|_2^2 + \lambda \|x\|_0$$
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Bounds needed for numerical tractability

NP hard

Specialized Branch-and-Bound solvers

Branch-and-Bound

Tree search

$\min_{\mathbf{x} \in [\mathbf{L}, \mathbf{U}]} \ \frac{1}{2}\|\mathbf{y} - \mathbf{A}\mathbf{x}\|_2^2 + \lambda \|\mathbf{x}\|_0$

N^0

Tree search

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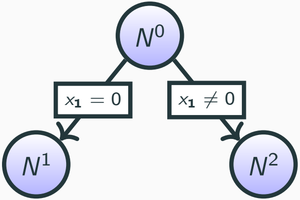

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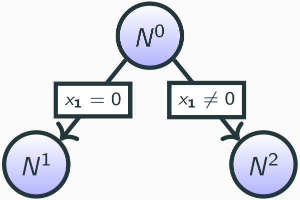

Same problem + constraint $x_1 = 0$

Same problem + constraint $x_1 \neq 0$

Tree search

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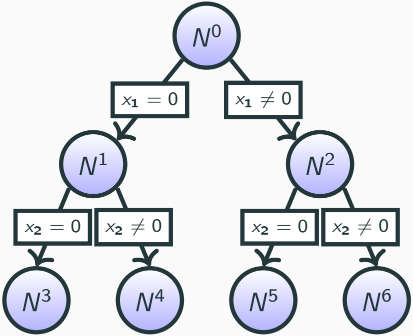

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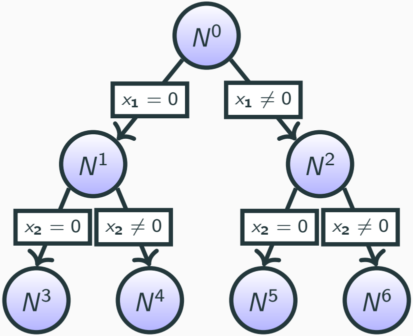

Subproblem
Still NP-hard

Tree search

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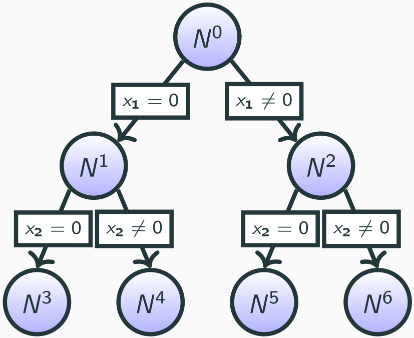

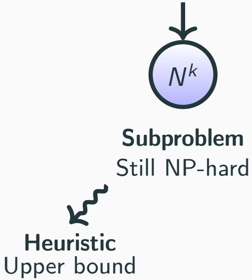

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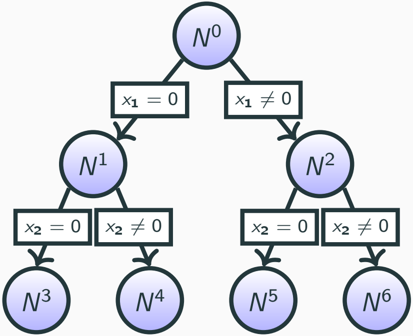

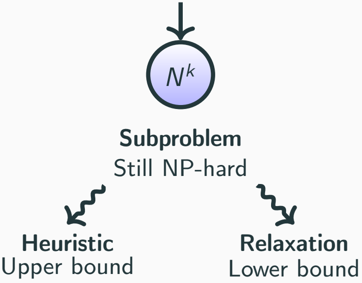

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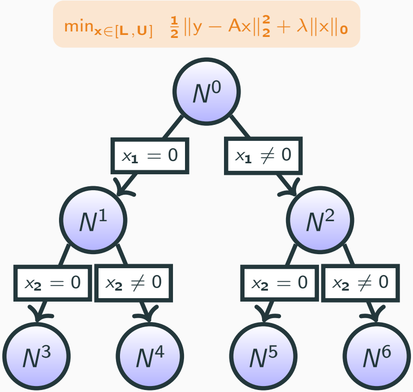

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$x_1 \neq 0$
N^1
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$x_2 \neq 0$
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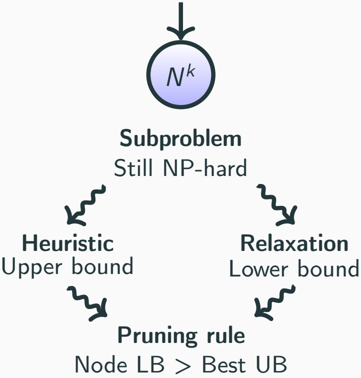
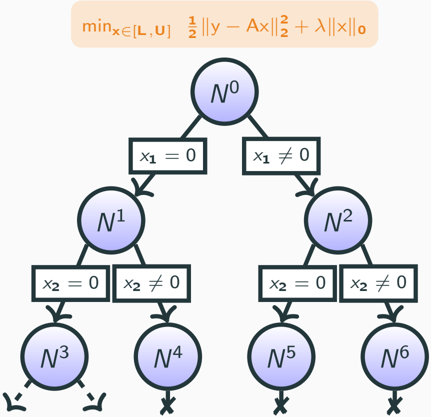
N^3
N^4
N^5
N^6



N^k
Subproblem
Still NP-hard
Heuristic
Upper bound
Relaxation
Lower bound
Pruning rule
Node LB > Best UB

Tree search



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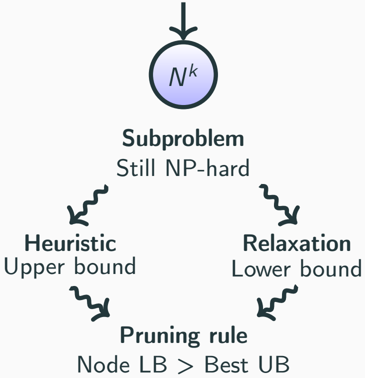

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Relaxation construction

Node problem

$\min_{\mathbf{x} \in [\mathbf{L}, \mathbf{U}]} \quad \frac{1}{2}\|y - Ax\|_2^2 + \lambda\|x\|_0$

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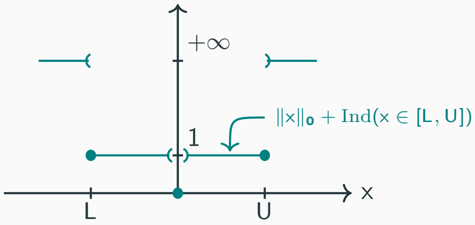

Relaxation construction

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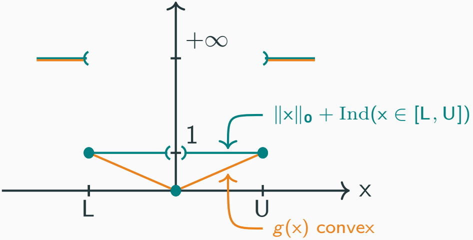

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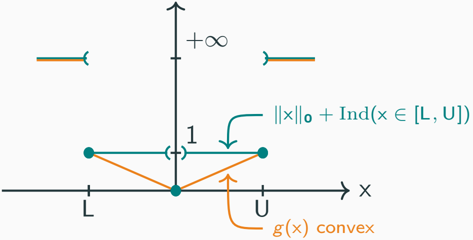
Relaxation

$\min_x \frac{1}{2}\|y - Ax\|_2^2 + \lambda g(x)$



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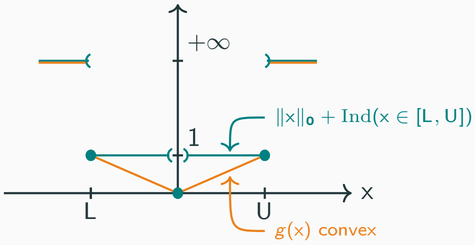

Dilemma in the choice of $[L, U]$

- Wide interval $[L, U]$ to recover the right solution
- Tight interval $[L, U]$ to build strong relaxations

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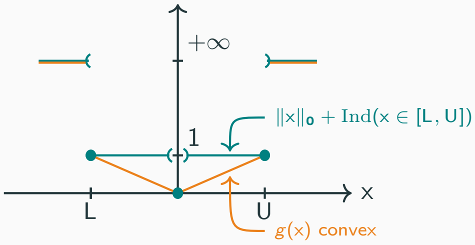

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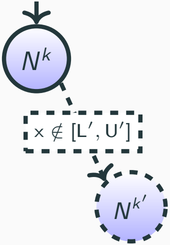
Our solution : Peeling

Peeling

Bound tightening



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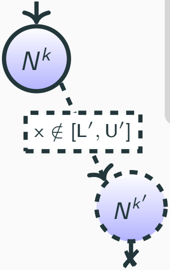
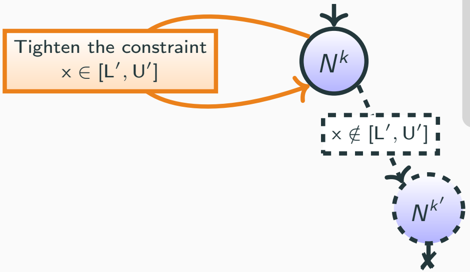


Find some $[L', U'] \subseteq [L, U]$ such that $N^{k'}$ is pruned

Bound tightening



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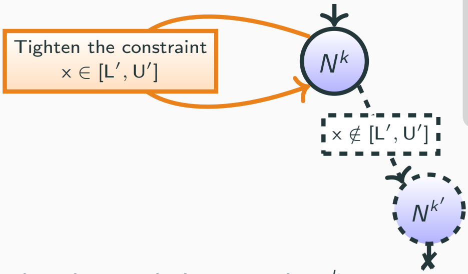

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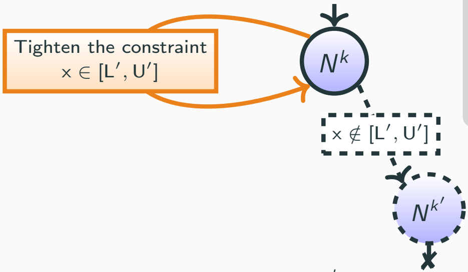

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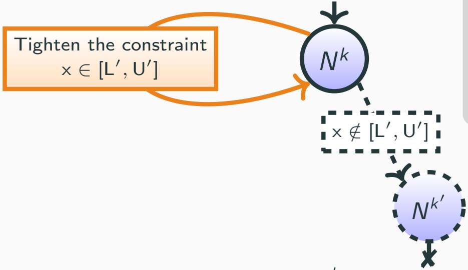

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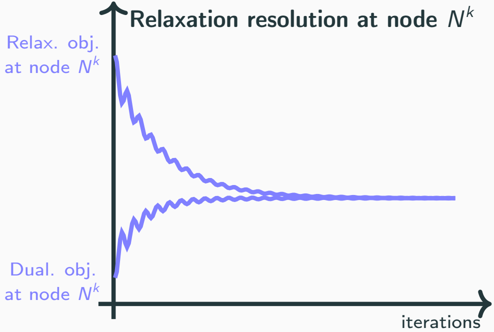
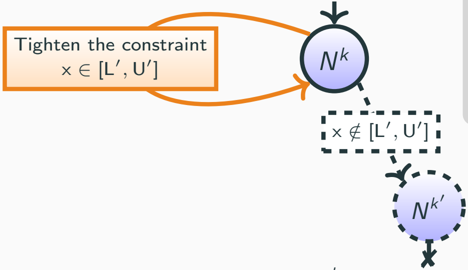


Ingredients

- Already-computed quantities
- Cost-free dual evaluation



Find some
[L′, U′] ⊆ [L, U]
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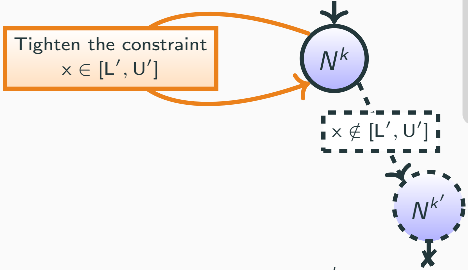
Ingredients

- Already-computed quantities
- Cost-free dual evaluation
- Dual link between nodes

Bound tightening



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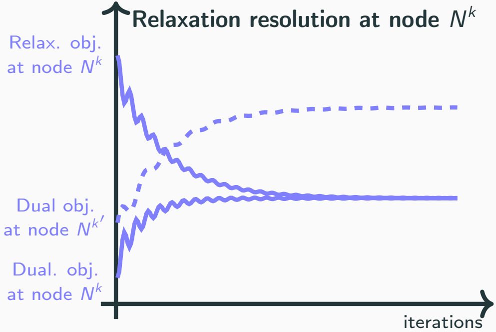

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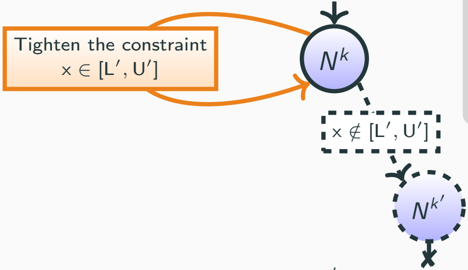

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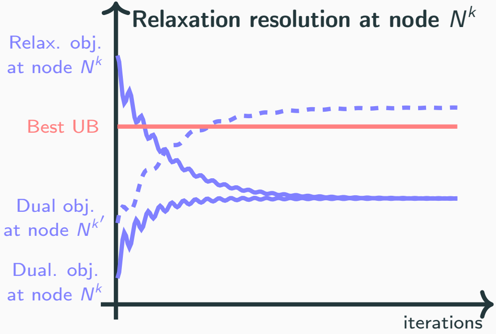

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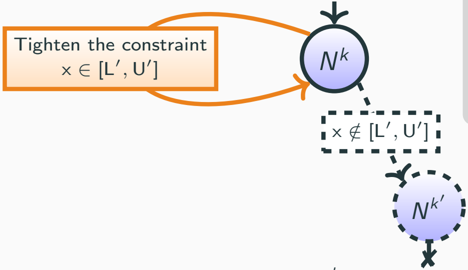

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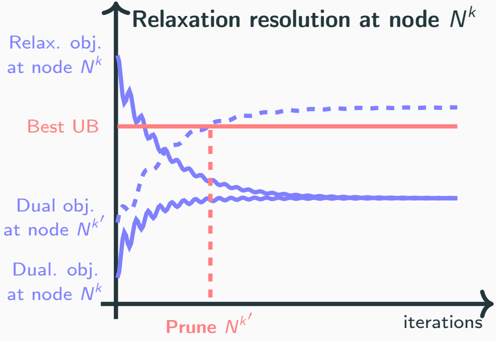

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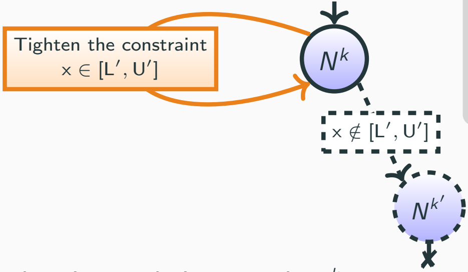

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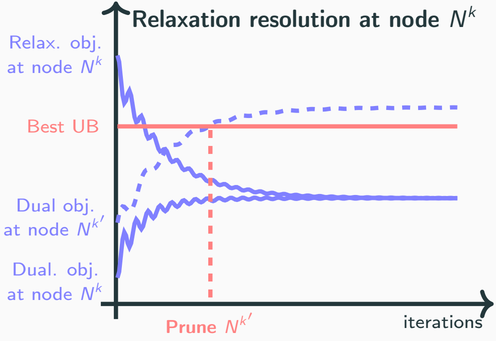

Ingredients

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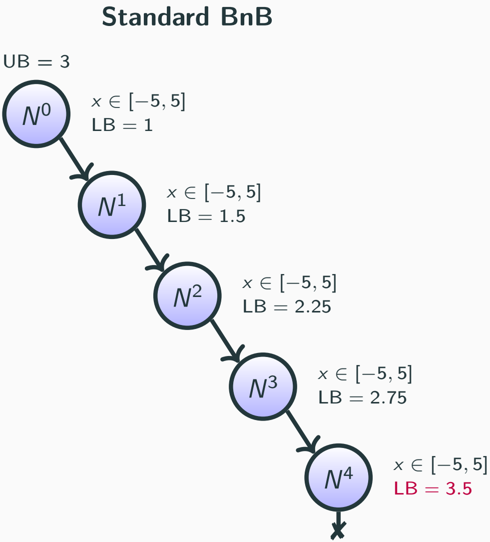
Propagation along branches



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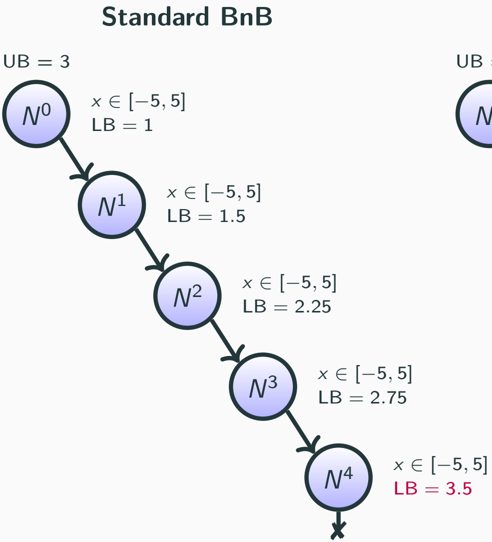
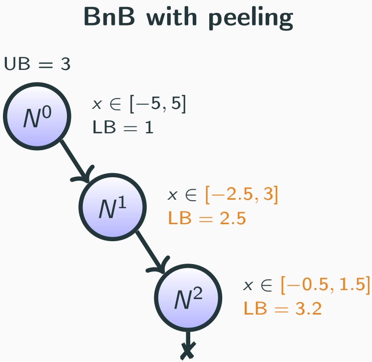
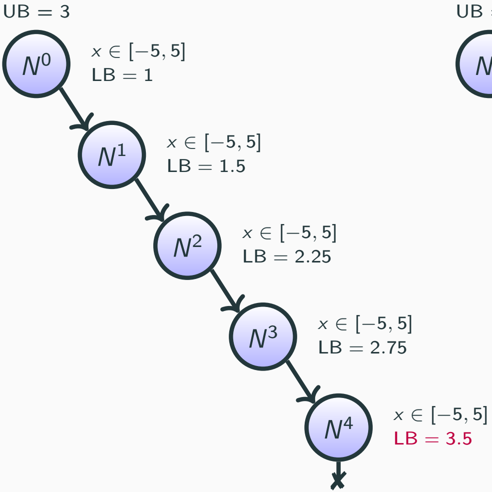


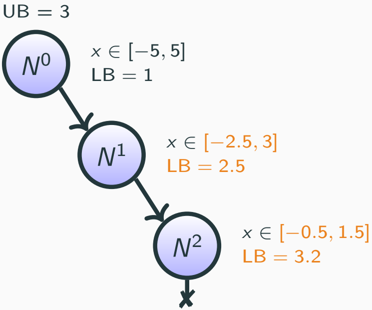


Propagation along branches

Standard BnB



BnB with peeling



- ✔ Branches pruned early-on
- ✔ Less nodes explored
- ✔ Reduce solve time

Numerical results

Numerical results

$$\min_{x \in [L,U]} \ \frac{1}{2}\|y - Ax\|_2^2 + \lambda\|x\|_0$$

Data A **and** y : Sparse regression

Bounds L **and** U : $-M \leq x \leq M$

Parameter λ : Set statistically

Bounds spead : $M = \gamma\|x^\star\|_\infty$

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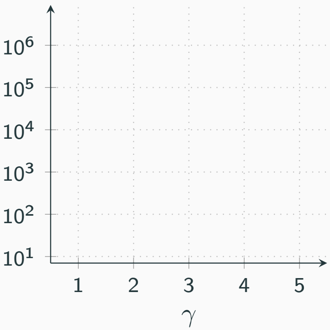
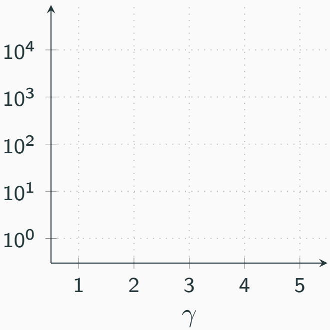
Node count



Solve time (sec.)



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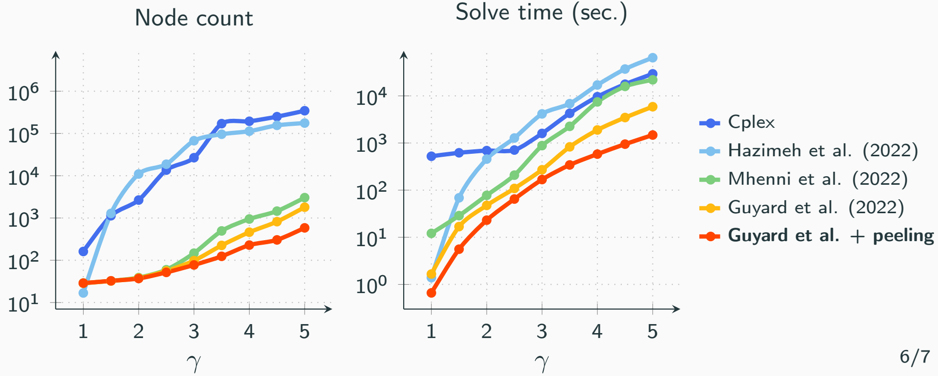

Take-home message

ℓ_0-regularized least-squares

$$\begin{aligned} \min_x \quad & \tfrac{1}{2}\|y - Ax\|_2^2 + \lambda\|x\|_0 \\ \text{s.t.} \quad & L \leq x \leq U \end{aligned}$$

Wide interval $[L, U]$ to recover the solution

Tight interval $[L, U]$ to build strong relaxations

Peeling

Start with wide interval $[L, U]$ and refine it along the BnB branches

Question time

🤓